

Announcements

- Midterm 1: Grading should be complete
- Homework 7: Due *Wednesday Oct 26 11:59pm*
- Project 4: Due *Friday Oct 28 5:00pm*
- Homework 8: Due *Monday Oct 31 11:59pm*
- Midterm 2: *Wednesday 11/9 7-9pm*

CS 188: Artificial Intelligence

Hidden Markov Models



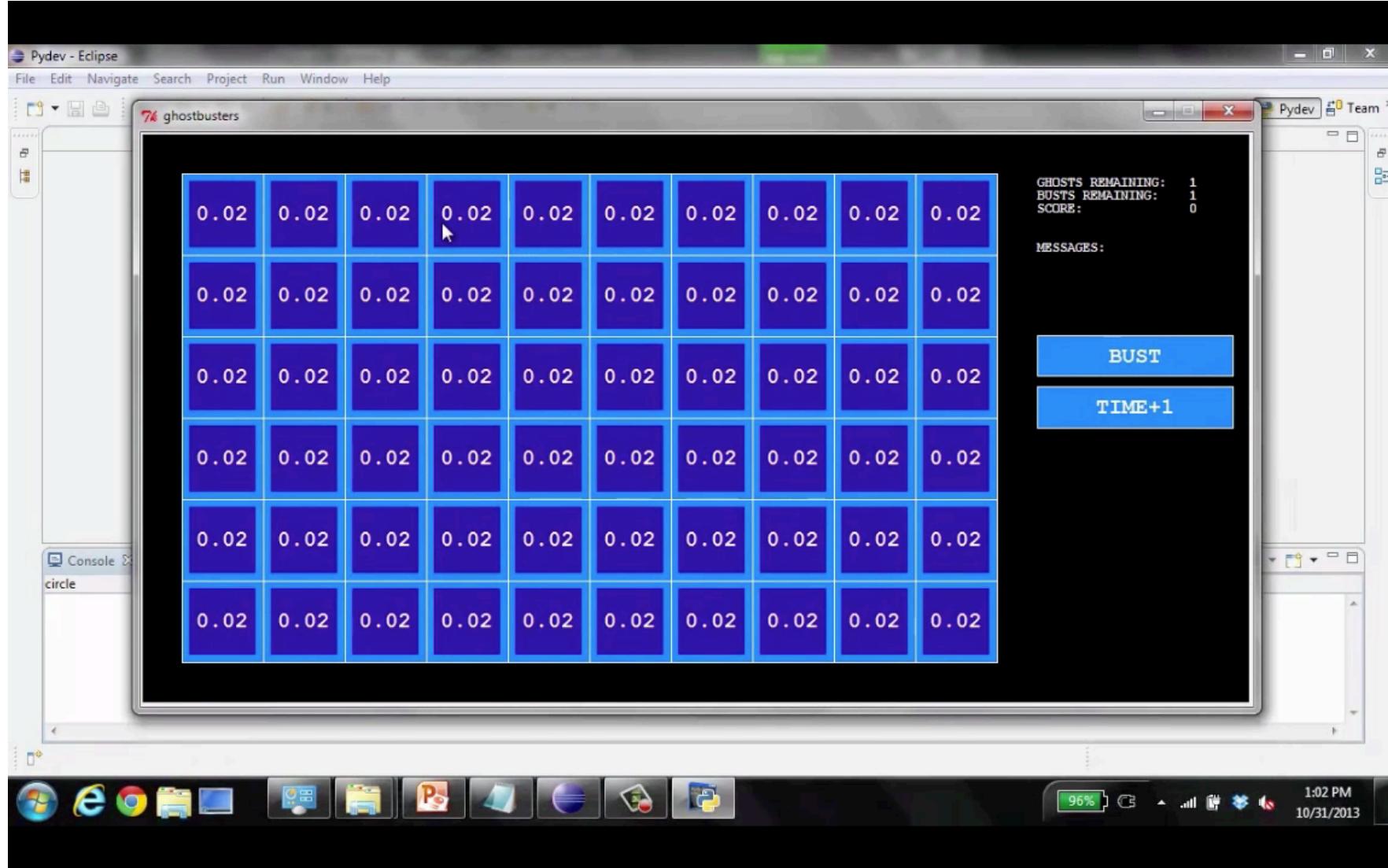
Instructors: Josh Hug & Adam Janin --- University of California, Berkeley

[These slides were created by Dan Klein, Pieter Abbeel, Anca Dragan for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at <http://ai.berkeley.edu>.]

Reasoning over Time or Space

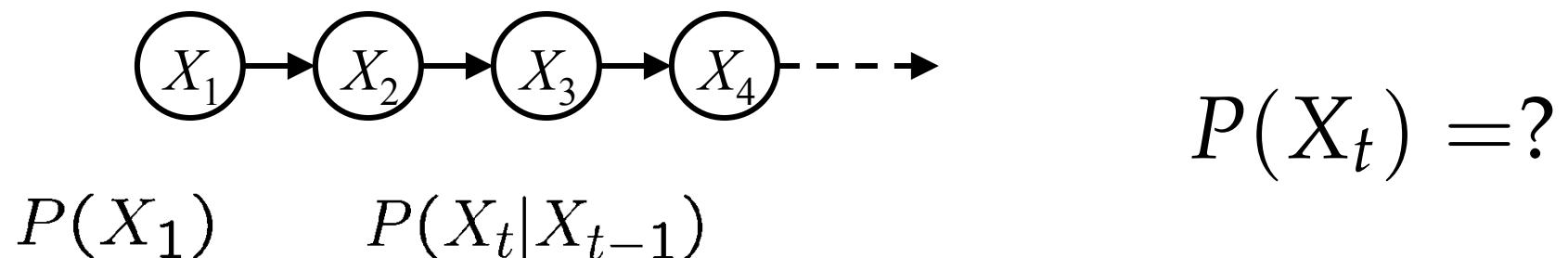
- Often, we want to **reason about a sequence of observations**
 - Speech recognition
 - Robot localization
 - Medical monitoring
 - User attention
- Need to introduce time (or space) into our models

Video of Demo Ghostbusters Circular Dynamics



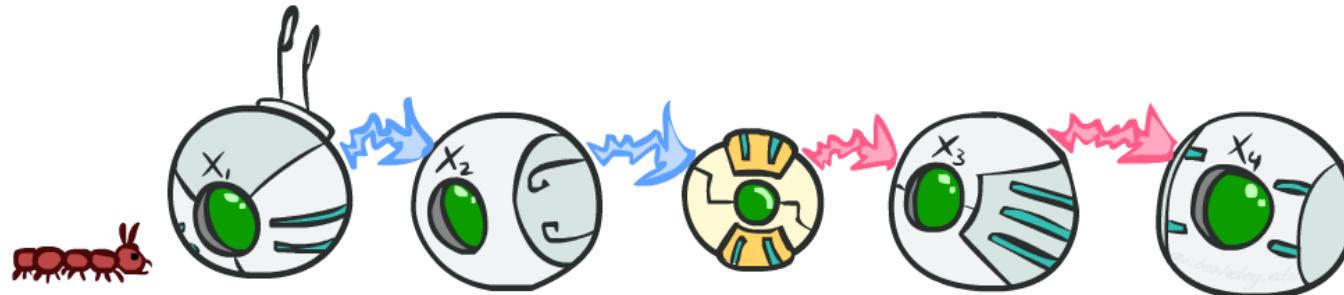
Markov Models

- Value of X at a given time is called the **state**



- Parameters: called **transition probabilities** or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action
- A (growable) BN: We can always use generic BN reasoning on it if we truncate the chain at a fixed length

Markov Assumption: Conditional Independence

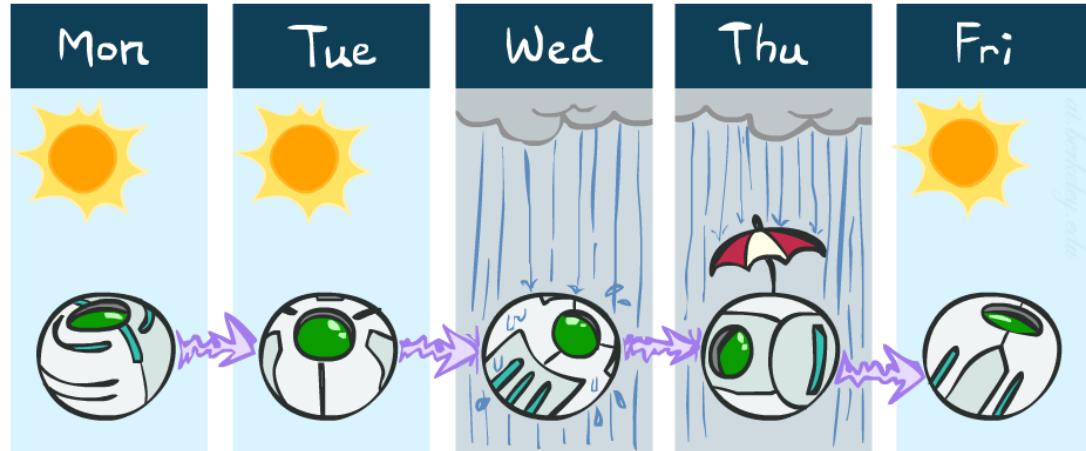


- Basic conditional independence:
 - Past and future independent given the present
 - Each time step only depends on the previous
 - This is called the (first order) Markov property

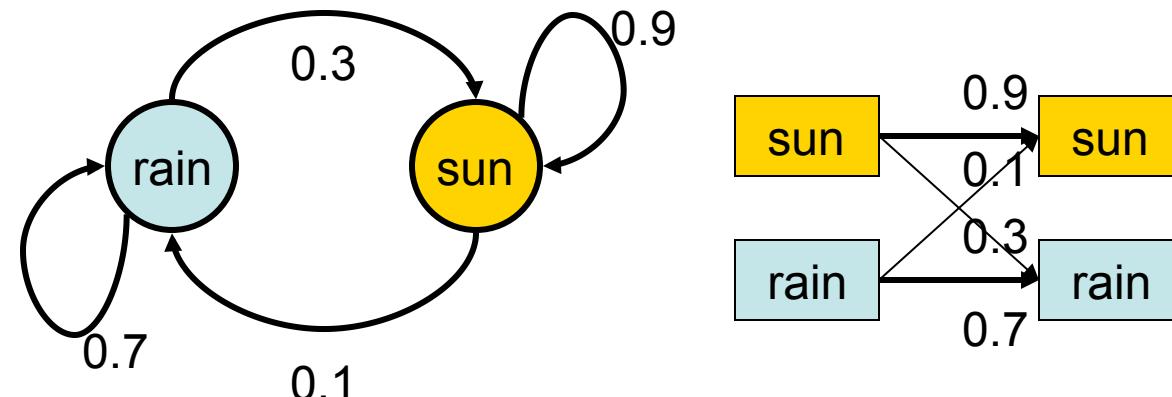
Example Markov Chain: Weather

- States: $X = \{\text{rain, sun}\}$
- Initial distribution: 1.0 sun
- CPT $P(X_t | X_{t-1})$:

X_{t-1}	X_t	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

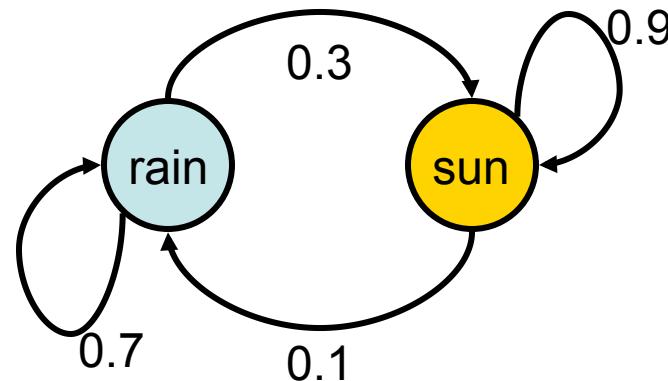


Two new ways of representing the same CPT



Example Markov Chain: Weather

- Initial distribution: 1.0 sun



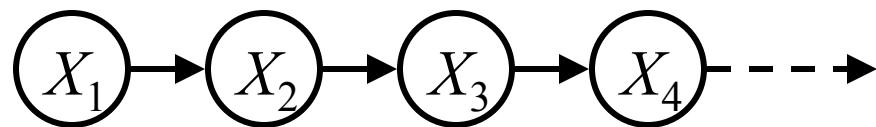
- What is the probability distribution after one step?

$$P(X_2 = \text{sun}) = \sum_{x_1} P(x_1, X_2 = \text{sun}) = \sum_{x_1} P(X_2 = \text{sun}|x_1)P(x_1)$$

$$\begin{aligned} P(X_2 = \text{sun}) &= P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + \\ &\quad P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain}) \\ &0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9 \end{aligned}$$

Mini-Forward Algorithm

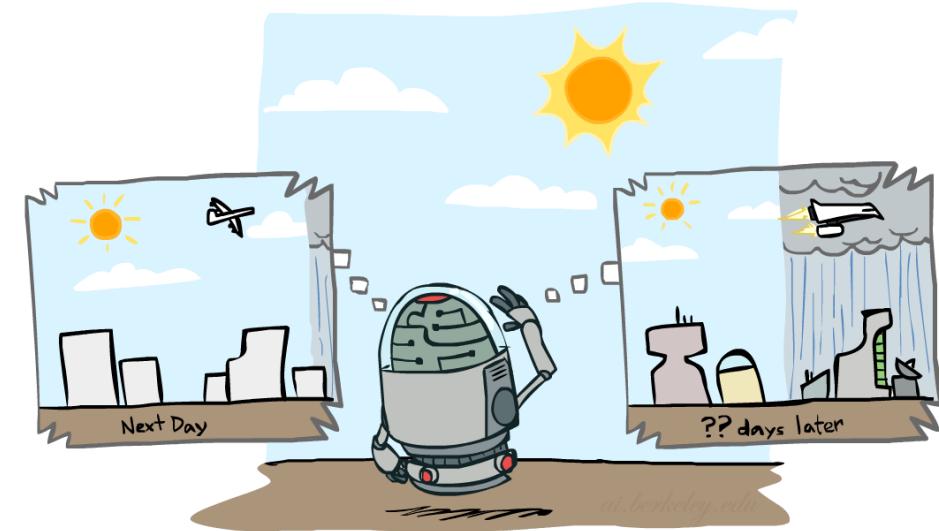
- Question: What's $P(X)$ on some day t ?



$P(x_1)$ = known

$$\begin{aligned} P(x_t) &= \sum_{x_{t-1}} P(x_{t-1}, x_t) \\ &= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1}) \end{aligned}$$

Forward simulation



Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\begin{array}{ccccc} \left\langle \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.9 \\ 0.1 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.84 \\ 0.16 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.804 \\ 0.196 \end{array} \right\rangle & \xrightarrow{\hspace{1cm}} \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\ P(X_1) & P(X_2) & P(X_3) & P(X_4) & P(X_\infty) \end{array}$$

- From initial observation of rain

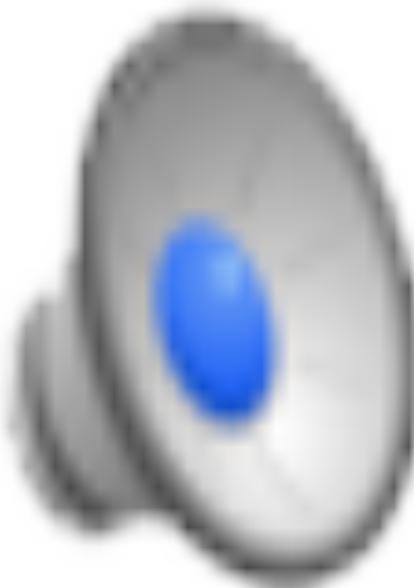
$$\begin{array}{ccccc} \left\langle \begin{array}{c} 0.0 \\ 1.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.3 \\ 0.7 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.48 \\ 0.52 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.588 \\ 0.412 \end{array} \right\rangle & \xrightarrow{\hspace{1cm}} \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\ P(X_1) & P(X_2) & P(X_3) & P(X_4) & P(X_\infty) \end{array}$$

- From yet another initial distribution $P(X_1)$:

$$\begin{array}{ccc} \left\langle \begin{array}{c} p \\ 1 - p \end{array} \right\rangle & \dots & \xrightarrow{\hspace{1cm}} \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\ P(X_1) & & P(X_\infty) \end{array}$$

[Demo: L13D1,2,3]

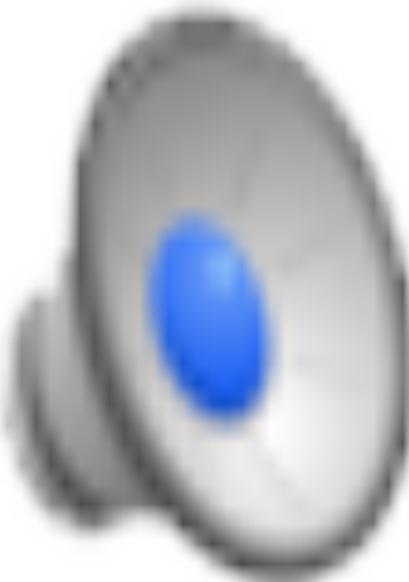
Video of Demo Ghostbusters Basic Dynamics



Video of Demo Ghostbusters Circular Dynamics



Video of Demo Ghostbusters Whirlpool Dynamics



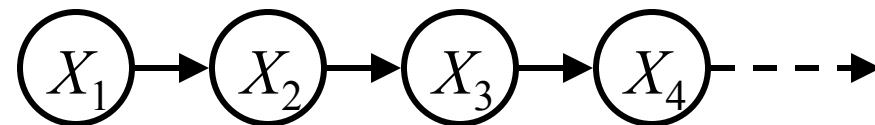
Stationary Distributions

- For most chains:
 - Influence of the initial distribution gets less and less over time.
 - The distribution we end up in is independent of the initial distribution
- Stationary distribution:
 - The distribution we end up with is called the **stationary distribution** of the chain
 - It satisfies
$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$



Example: Stationary Distributions

- Question: What's $P(X)$ at time $t = \text{infinity}$?



$$P_{\infty}(\text{sun}) = P(\text{sun}|\text{sun})P_{\infty}(\text{sun}) + P(\text{sun}|\text{rain})P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = P(\text{rain}|\text{sun})P_{\infty}(\text{sun}) + P(\text{rain}|\text{rain})P_{\infty}(\text{rain})$$

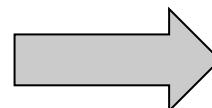
$$P_{\infty}(\text{sun}) = 0.9P_{\infty}(\text{sun}) + 0.3P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = 0.1P_{\infty}(\text{sun}) + 0.7P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{sun}) = 3P_{\infty}(\text{rain})$$

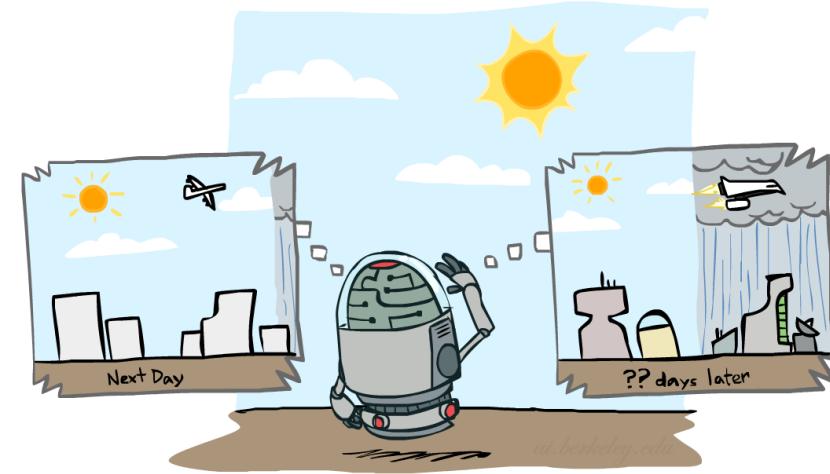
$$P_{\infty}(\text{rain}) = 1/3P_{\infty}(\text{sun})$$

$$\text{Also: } P_{\infty}(\text{sun}) + P_{\infty}(\text{rain}) = 1$$



$$P_{\infty}(\text{sun}) = 3/4$$

$$P_{\infty}(\text{rain}) = 1/4$$

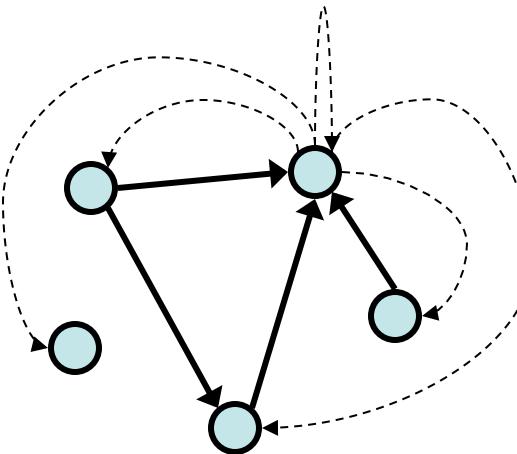


X_{t-1}	X_t	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Application of Stationary Distribution: Web Link Analysis

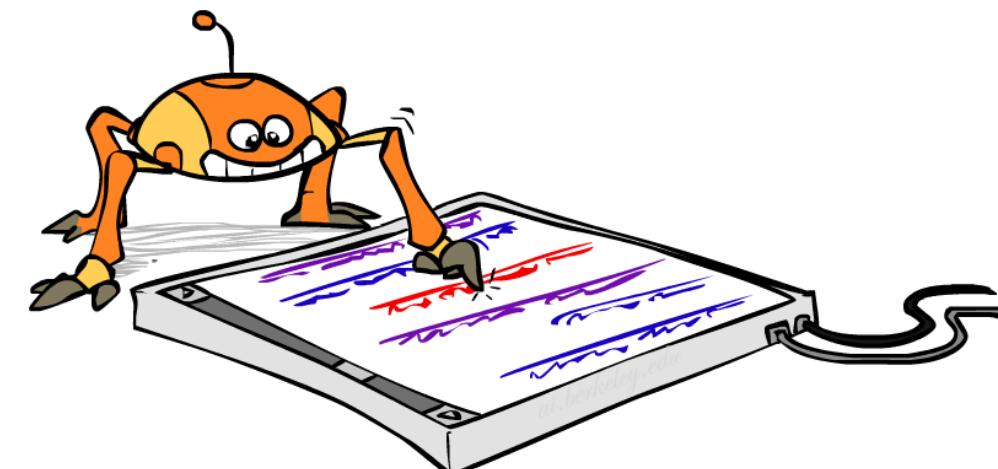
■ PageRank over a web graph

- Each web page is a possible value of a state
- Initial distribution: uniform over pages
- Transitions:
 - With prob. c , uniform jump to a random page (dotted lines, not all shown)
 - With prob. $1-c$, follow a random outlink (solid lines)



■ Stationary distribution

- Will spend more time on highly reachable pages
- E.g. many ways to get to the Acrobat Reader download page
- Somewhat robust to link spam.
- Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



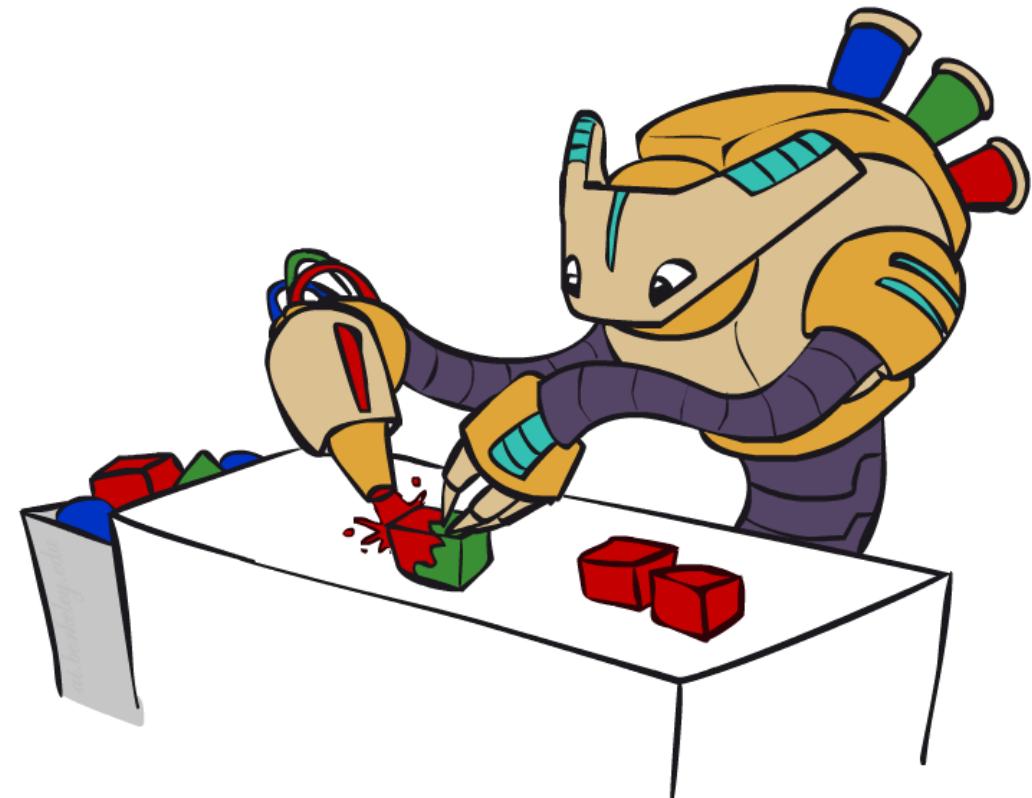
Application of Stationary Distributions: Gibbs Sampling*

- Each joint instantiation over all hidden and query variables is a state: $\{X_1, \dots, X_n\} = H \cup Q$

- Transitions:
 - With probability $1/n$ resample variable X_j according to

$$P(X_j | x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, e_1, \dots, e_m)$$

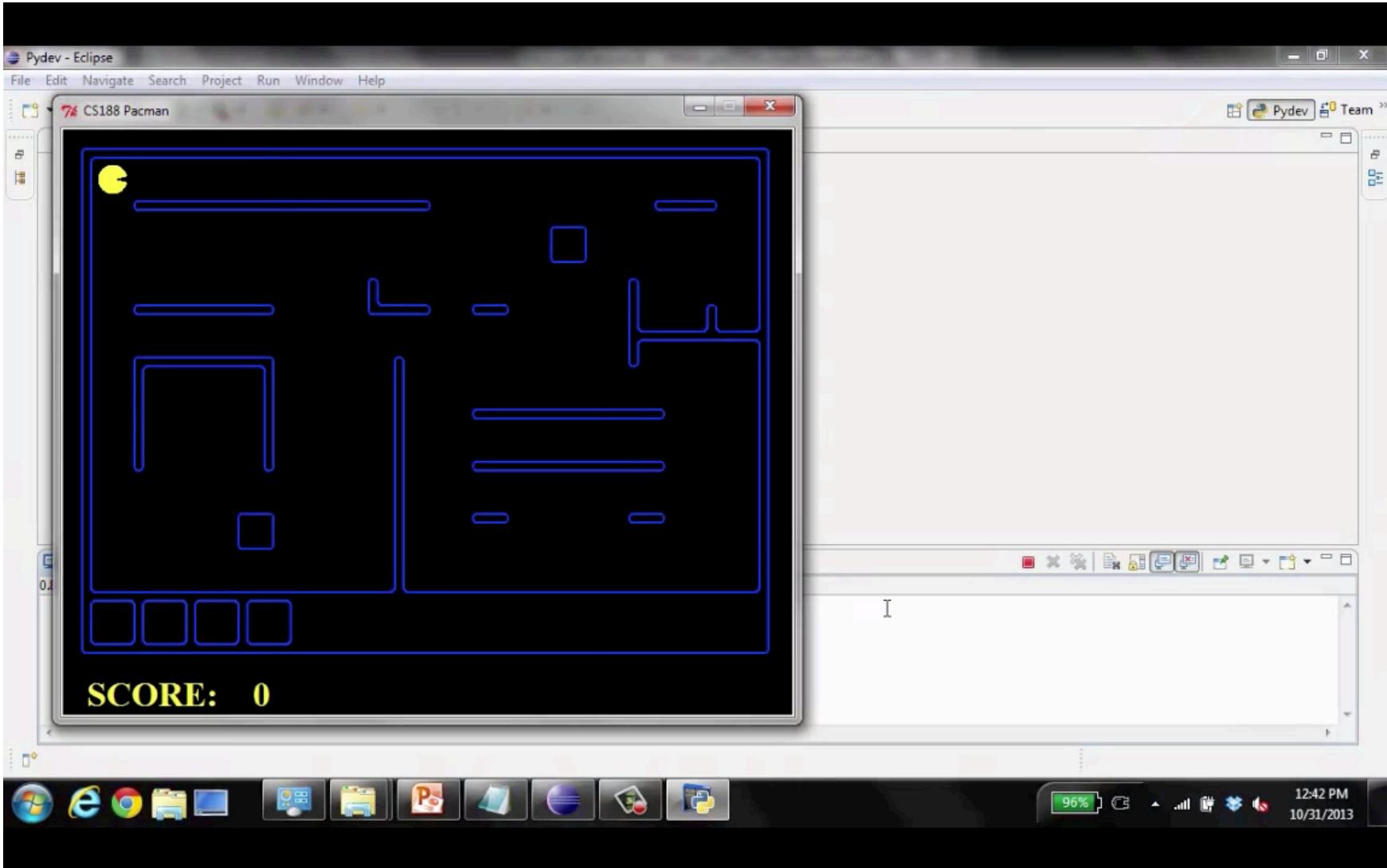
- Stationary distribution:
 - Conditional distribution $P(X_1, X_2, \dots, X_n | e_1, \dots, e_m)$
 - Means that when running Gibbs sampling long enough we get a sample from the desired distribution
 - Requires some proof to show this is true!



Hidden Markov Models

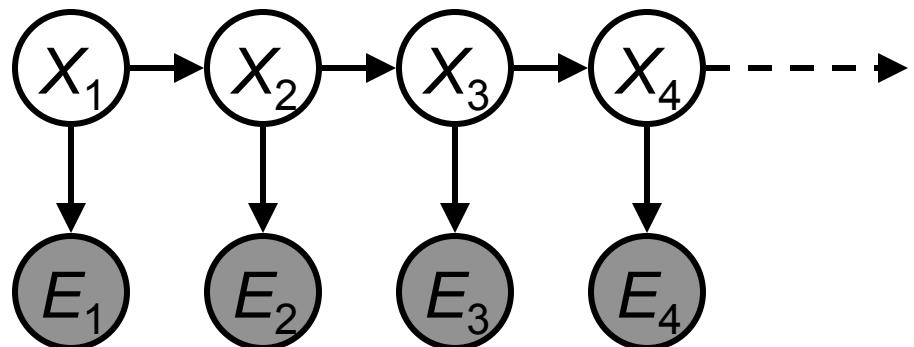


Video of Demo Pacman – Sonar (no beliefs)

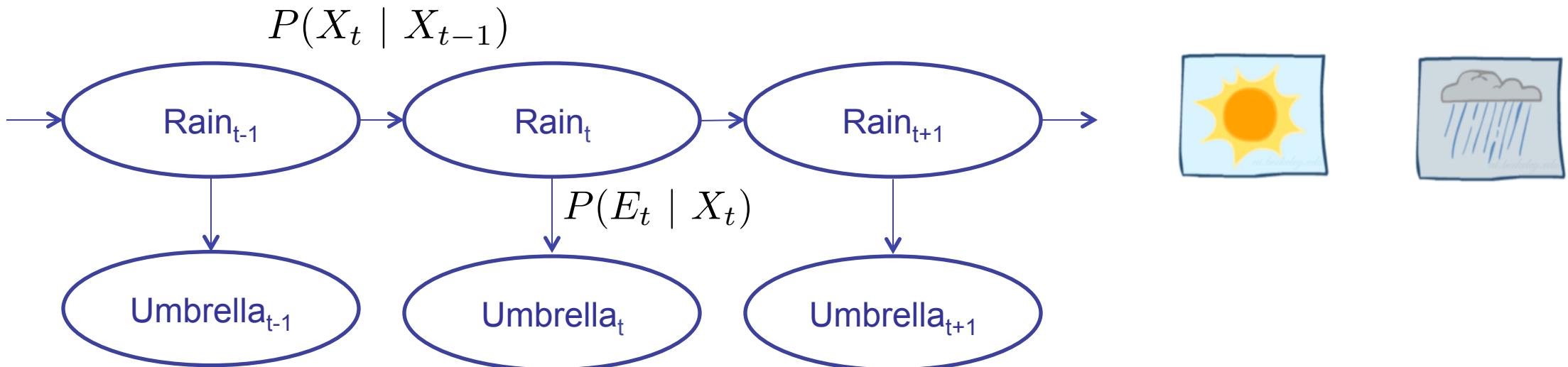


Hidden Markov Models

- Markov chains not so useful for most agents
 - Need observations to update your beliefs
- Hidden Markov models (HMMs)
 - Underlying Markov chain over states X
 - You observe outputs (effects) at each time step



Example: Weather HMM



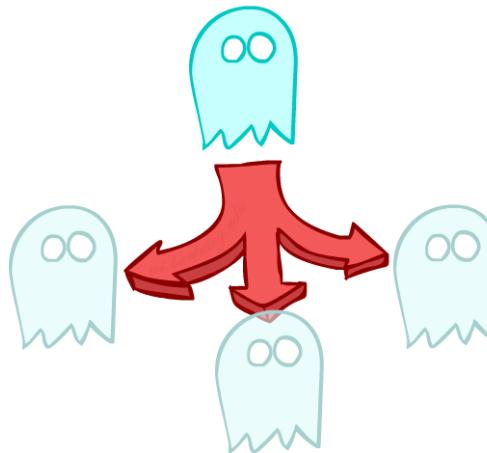
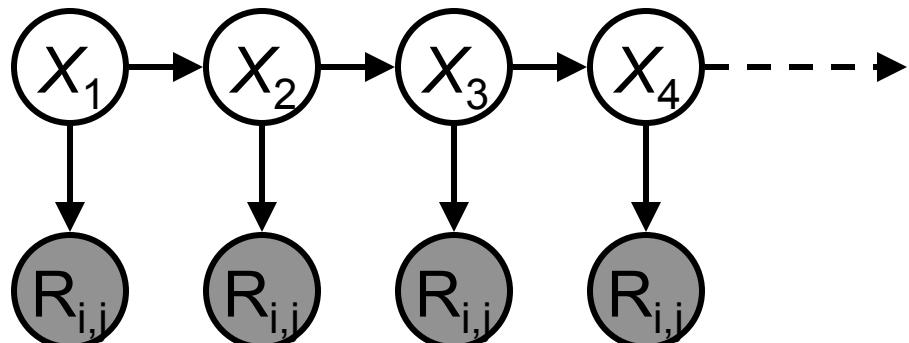
- An HMM is defined by:
 - Initial distribution: $P(X_1)$
 - Transitions: $P(X_t \mid X_{t-1})$
 - Emissions: $P(E_t \mid X_t)$

R_{t-1}	R_t	$P(R_t R_{t-1})$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

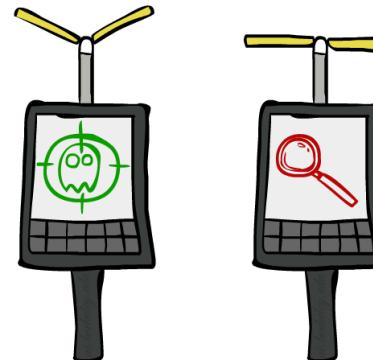
Example: Ghostbusters HMM

- $P(X_1) = \text{uniform}$
- $P(X|X') = \text{usually move clockwise, but sometimes move in a random direction or stay in place}$
- $P(R_{ij}|X) = \text{same sensor model as before: red means close, green means far away.}$



1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$P(X_1)$



1/6	1/6	1/2
0	1/6	0
0	0	0

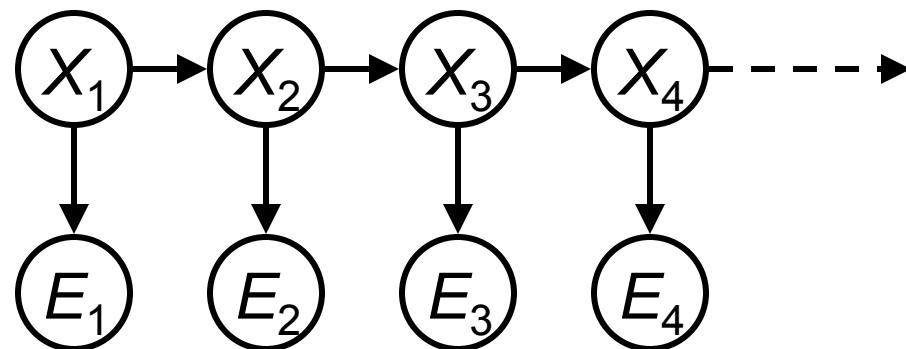
$P(X|X'=<1,2>)$

Video of Demo Ghostbusters – Circular Dynamics -- HMM



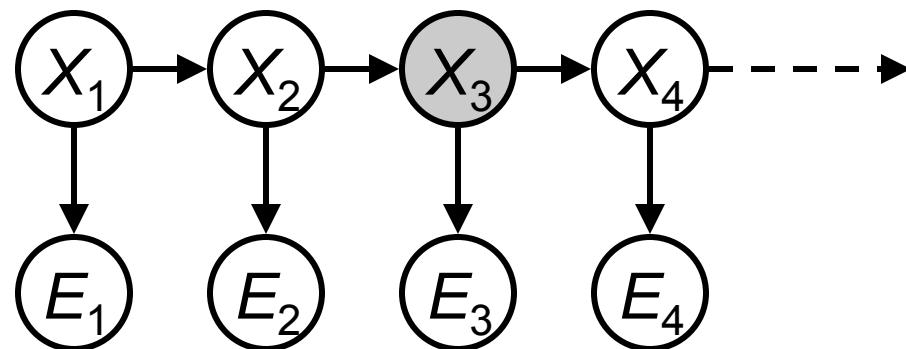
Conditional Independence

- HMMs have two important independence properties:
 - Markov hidden process: future depends on past via the present
 - Current observation independent of all else given current state



Conditional Independence

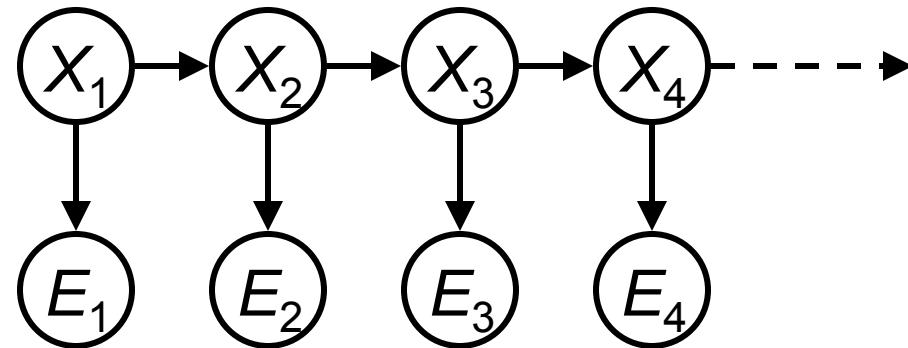
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Conditional Independence



- HMMs have two important independence properties:
 - Markov hidden process: future depends on past via the present
 - Current observation independent of all else given current state



- Does this mean that evidence variables are guaranteed to be independent?
No. They tend to be correlated by the hidden state.

Real HMM Examples

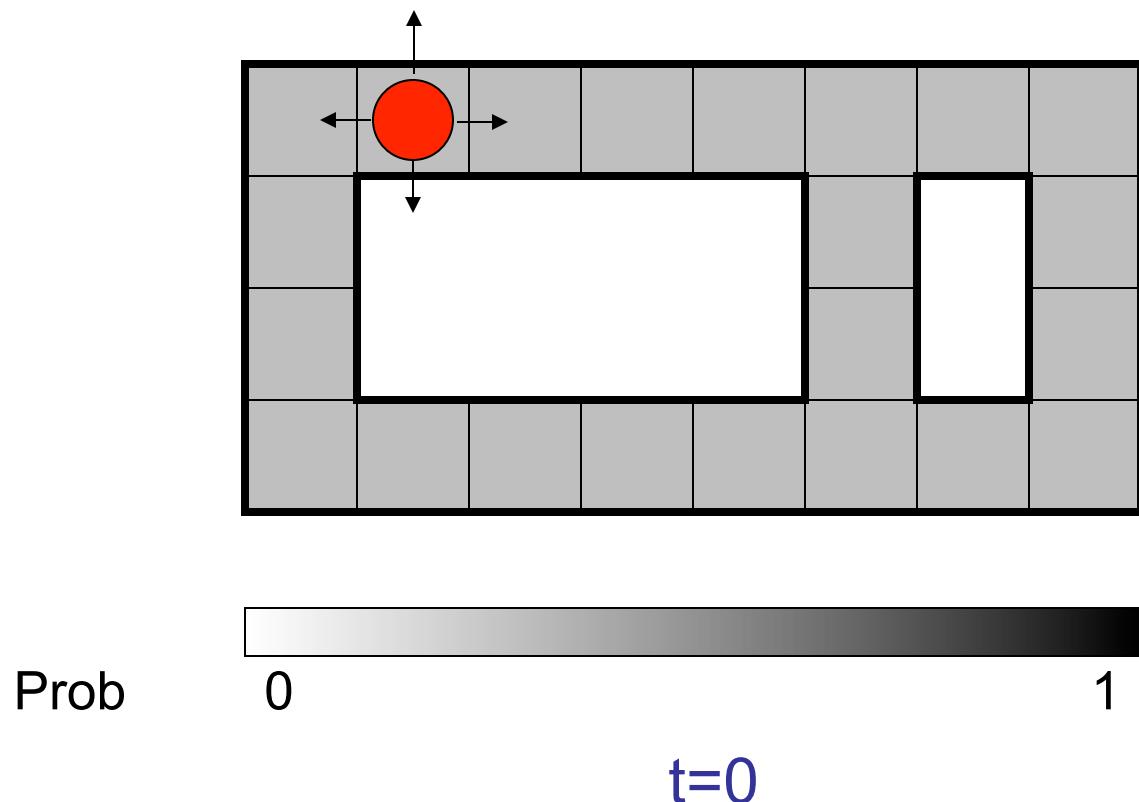
- Robot tracking:
 - Observations are range readings (continuous)
 - States are positions on a map (continuous)
- Speech recognition HMMs:
 - Observations are acoustic signals (continuous valued)
 - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
 - Observations are words (tens of thousands)
 - States are translation options

Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t | e_1, \dots, e_t)$ (the belief state) over time
- We start with $B_1(X)$ in an initial setting, usually uniform
- As time passes, or we get observations, we update $B(X)$
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

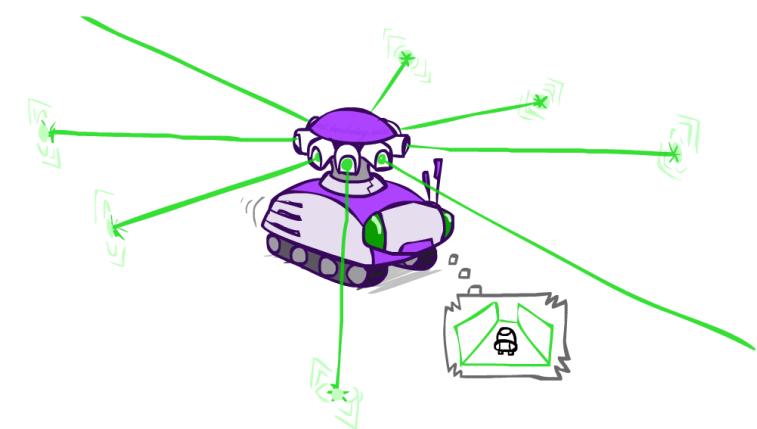
Example: Robot Localization

*Example from
Michael Pfeiffer*

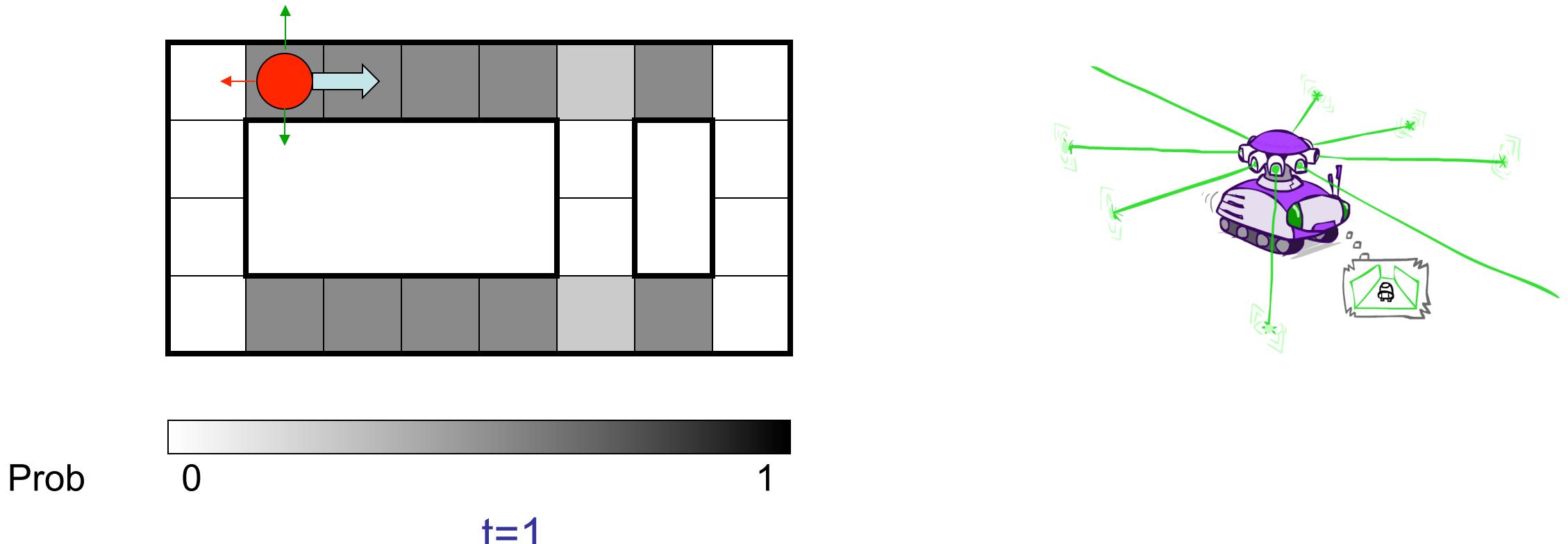


Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.

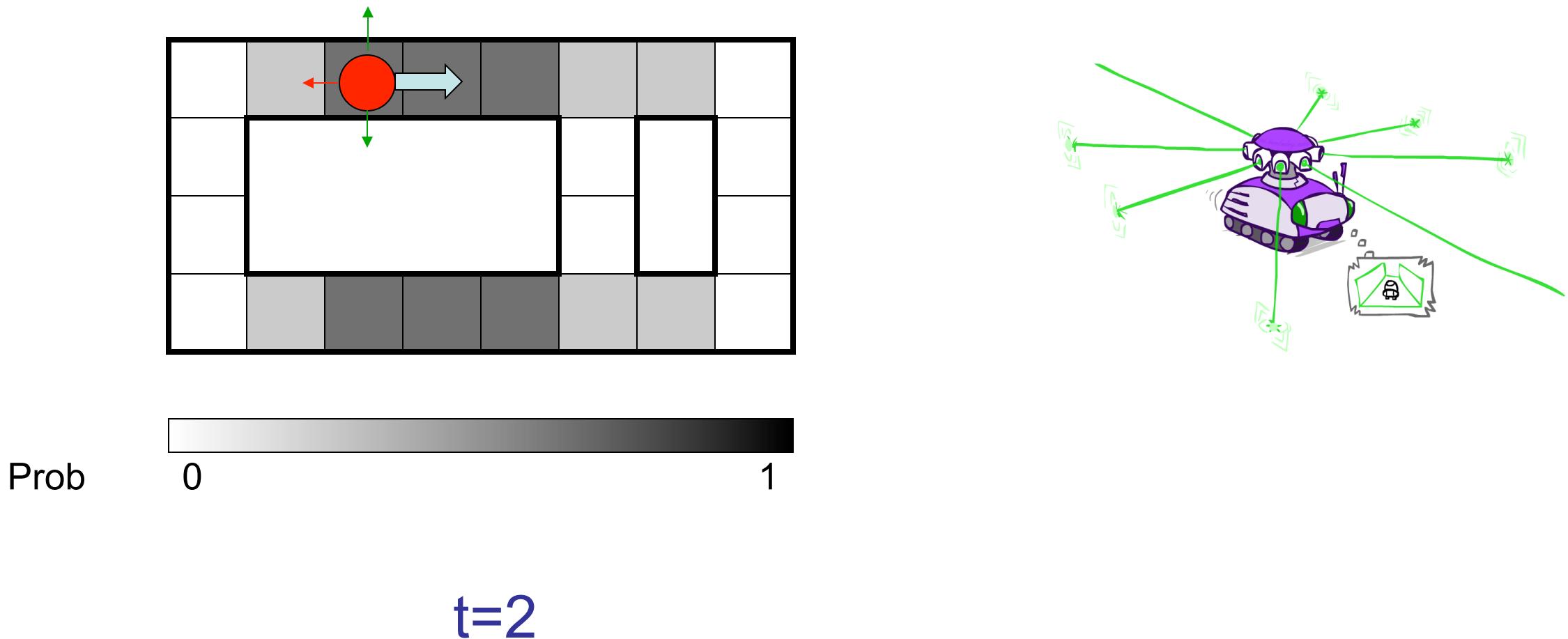


Example: Robot Localization

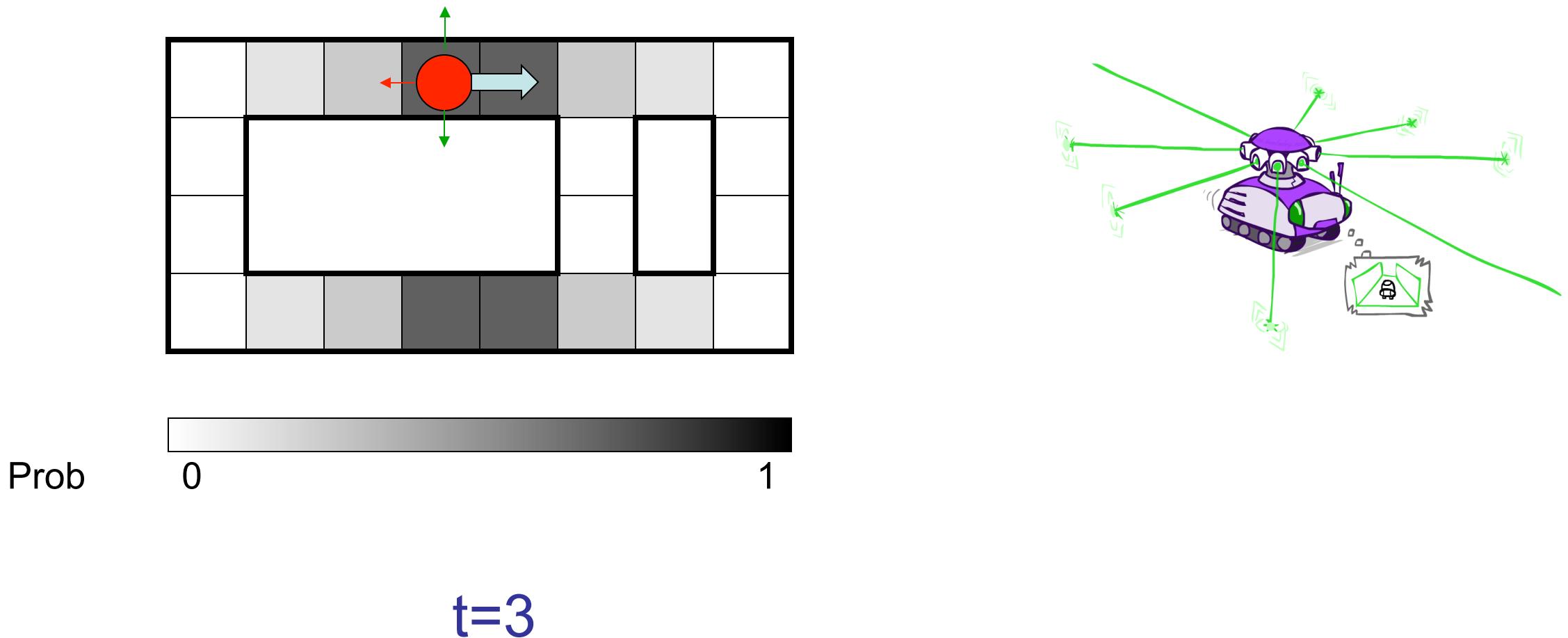


Lighter grey: was possible to get the reading, but less likely b/
c required 1 mistake

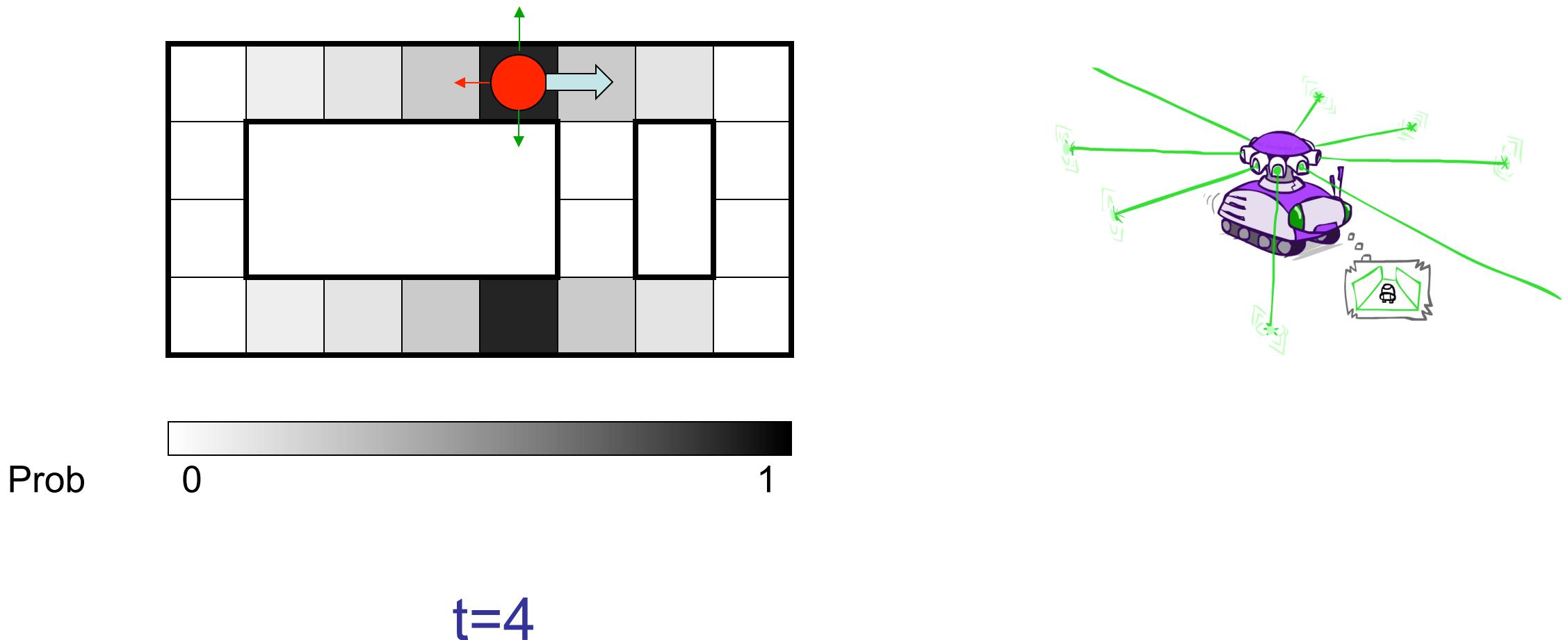
Example: Robot Localization



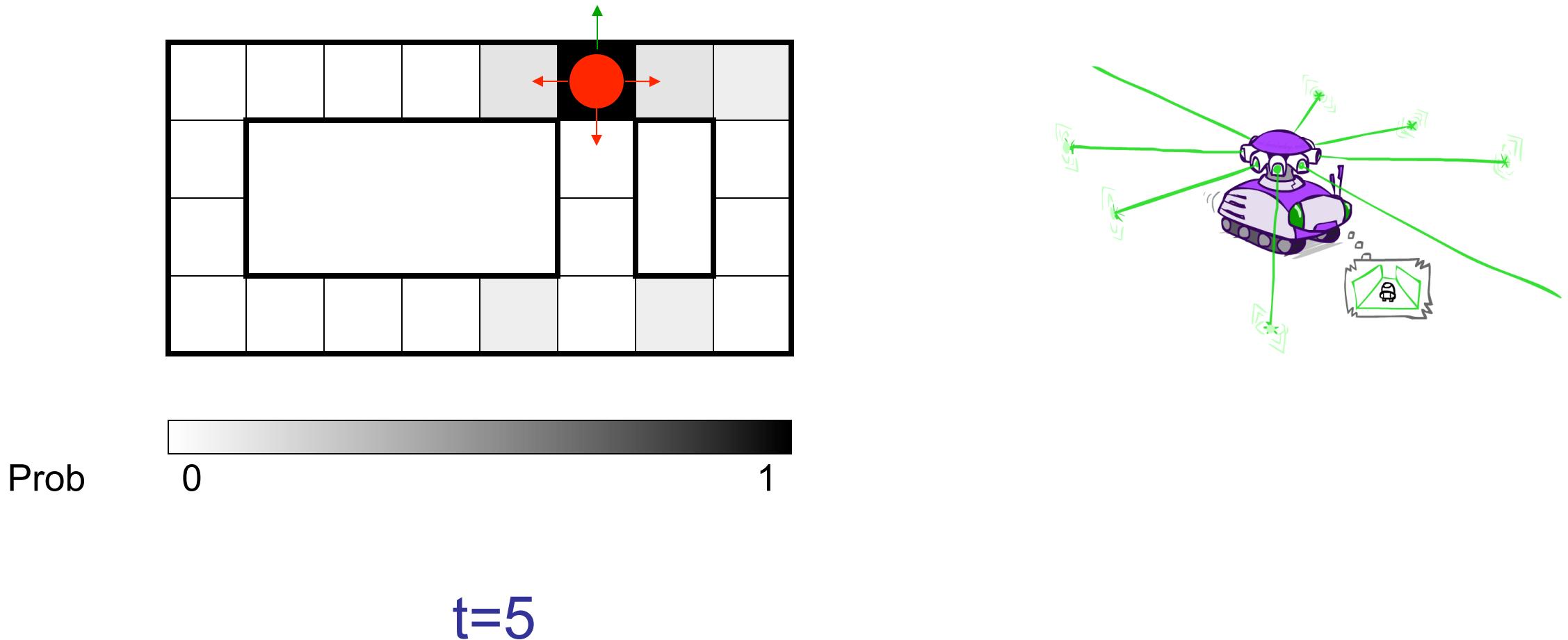
Example: Robot Localization



Example: Robot Localization



Example: Robot Localization



Inference: Find State Given Evidence

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t | e_{1:t})$$

- Idea: start with $P(X_1)$ and derive B_t in terms of B_{t-1}
 - equivalently, derive B_{t+1} in terms of B_t

Two Steps: Passage of Time + Observation

$$B(X_t) = P(X_t|e_{1:t}) \quad B'(X_{t+1})$$

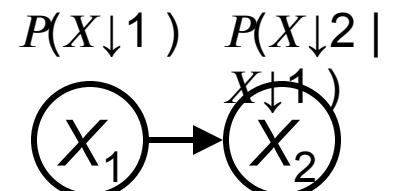
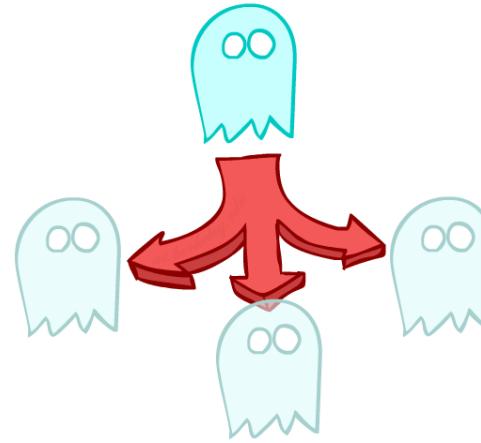
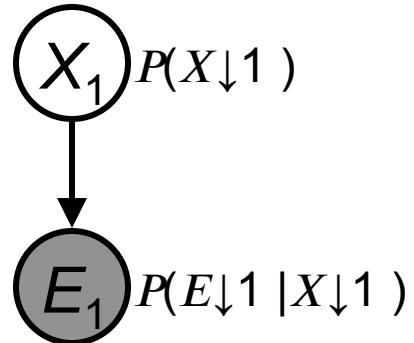
```
graph TD; X1((X1)) --> X2((X2)); X2 --> X3((X3)); X3 --> X4((X4)); X1 --> E1((E1)); X2 --> E2((E2)); X3 --> E3((E3)); X4 --> E4((E4));
```

The diagram illustrates a sequence of hidden states X_1, X_2, X_3, X_4 arranged horizontally. Arrows point from X_1 to X_2 , X_2 to X_3 , and X_3 to X_4 . Below each hidden state X_i is an observation E_i , represented by a circle. Arrows also point from each X_i to its corresponding E_i .

Inference: Base Cases



$$P(X_1|e_1)$$



$$P(X_{\downarrow 1} | e_{\downarrow 1}) = P(X_{\downarrow 1}, e_{\downarrow 1}) / P(e_{\downarrow 1})$$

$$\propto_{\downarrow X_{\downarrow 1}} P(X_{\downarrow 1}, e_{\downarrow 1})$$

$$= P(X_{\downarrow 1}) P(e_{\downarrow 1} | X_{\downarrow 1})$$

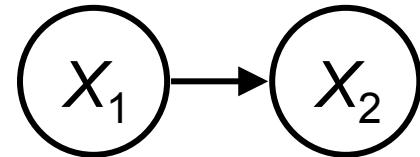
$$P(X_2) = \sum_{x_1} P(x_1, X_2)$$

$$P(X_2) = \sum_{x_1} P(X_2 | x_1) P(x_1)$$

Passage of Time

- Assume we have current belief $P(X | \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$



- Then, after one time step passes:

$$\begin{aligned} P(X_{t+1} | e_{1:t}) &= \sum_{x_t} P(X_{t+1}, x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t, e_{1:t}) P(x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t}) \end{aligned}$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X' | x_t) B(x_t)$$

- Basic idea: beliefs get “pushed” through the transitions

- With the “B” notation, we have to be careful about what time step t the belief is about, and what evidence it includes

Example: Passage of Time

- As time passes, uncertainty “accumulates”

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	1.00	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

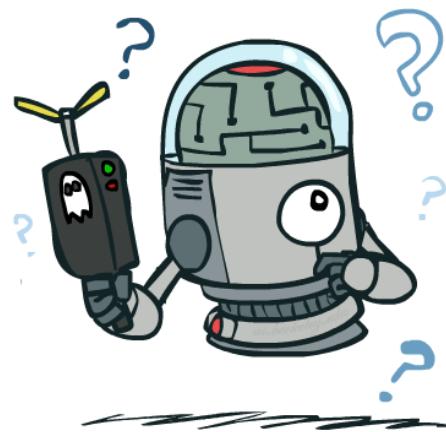
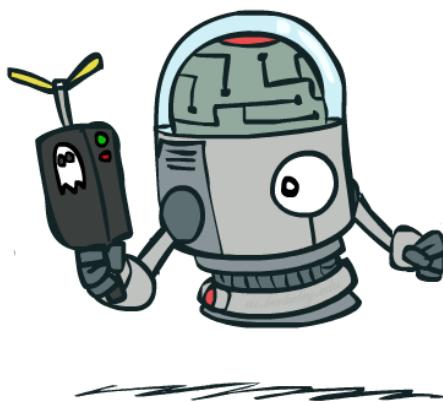
T = 1

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01
<0.01	0.76	0.06	0.06	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01

T = 2

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

T = 5



(Transition model: ghosts usually go clockwise)

Observation

- Assume we have current belief $P(X | \text{previous evidence})$:

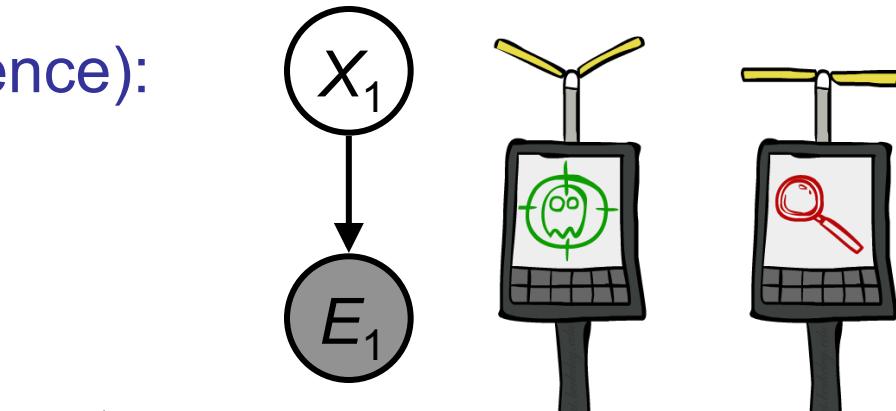
$$B'(X_{t+1}) = P(X_{t+1} | e_{1:t})$$

- Then, after evidence comes in:

$$\begin{aligned} P(X_{t+1} | e_{1:t+1}) &= P(X_{t+1}, e_{t+1} | e_{1:t}) / P(e_{t+1} | e_{1:t}) \\ &\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | e_{1:t}, X_{t+1}) P(X_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | X_{t+1}) B'(X_{t+1}) \end{aligned}$$

- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} | X_{t+1}) B'(X_{t+1})$$



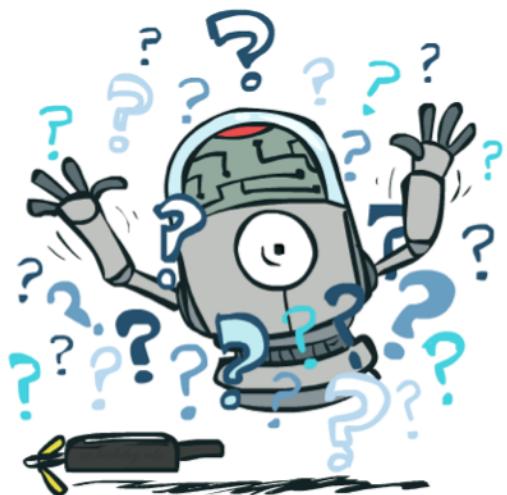
- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we

Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

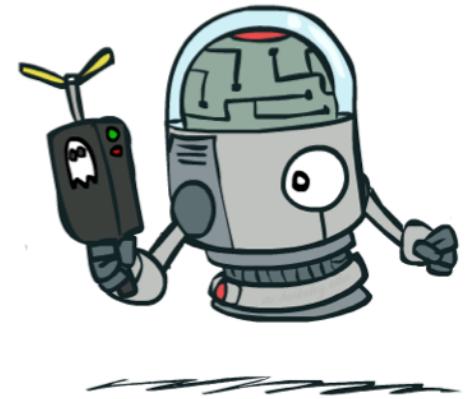
Before observation



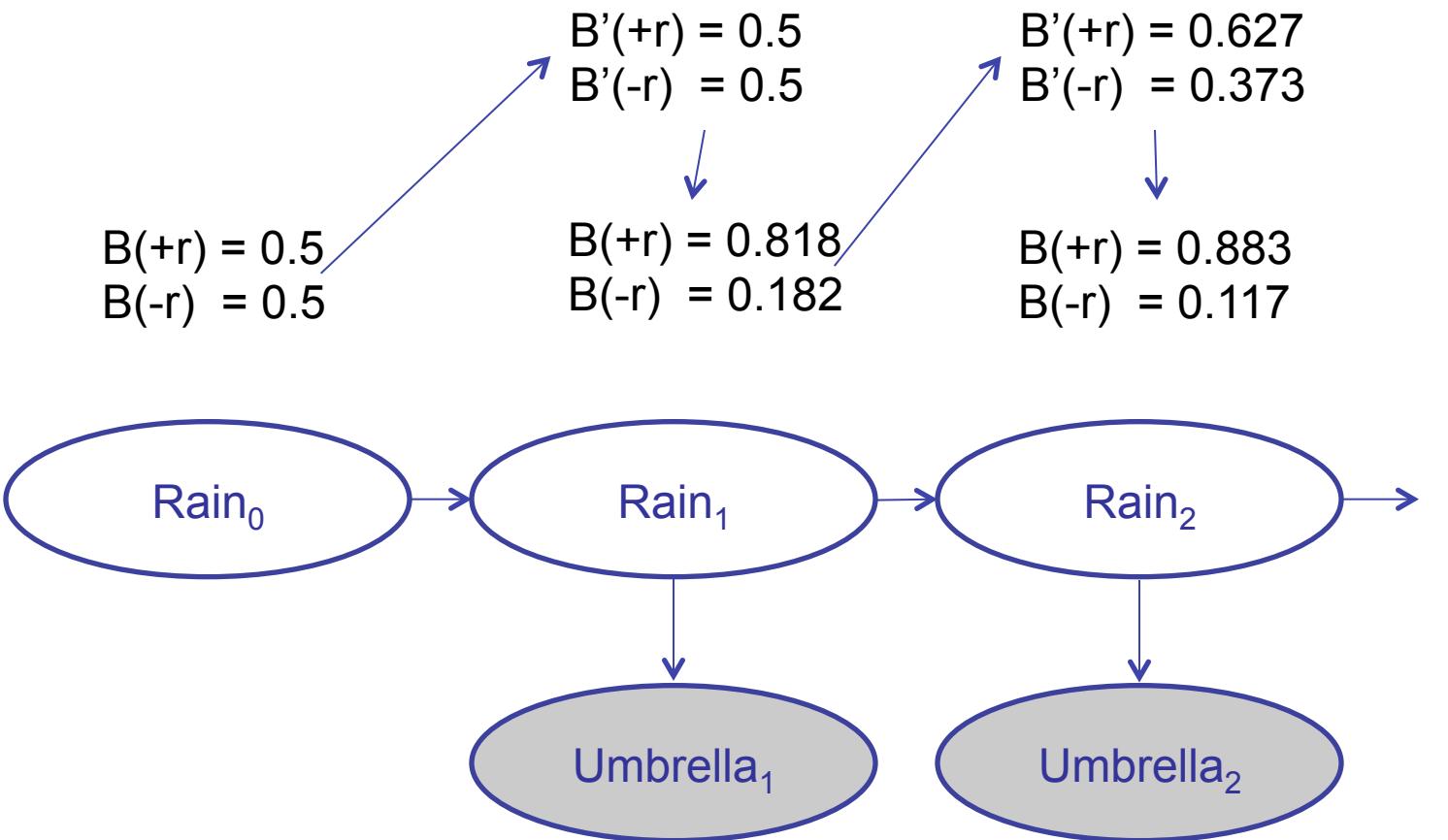
<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation

$$B(X) \propto P(e|X)B'(X)$$



Example: Weather HMM



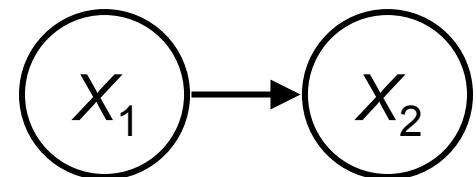
R_t	R_{t+1}	$P(R_{t+1} R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Online Belief Updates

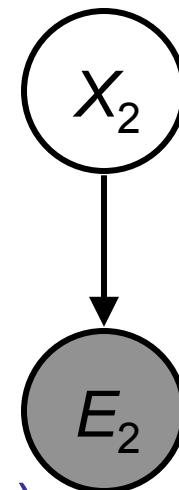
- Every time step, we start with current $P(X | \text{evidence})$
- We update for time:

$$P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$$



- We update for evidence:

$$P(x_t | e_{1:t}) \propto_X P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$$



- The forward algorithm does both at once (and doesn't normalize)

The Forward Algorithm

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t|e_{1:t})$$

- We can derive the following updates

$$P(x_t|e_{1:t}) \propto_X P(x_t, e_{1:t})$$

We can normalize as we go if we want to have $P(x|e)$ at each time step, or just once at the end...

$$= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})$$

$$= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t)$$

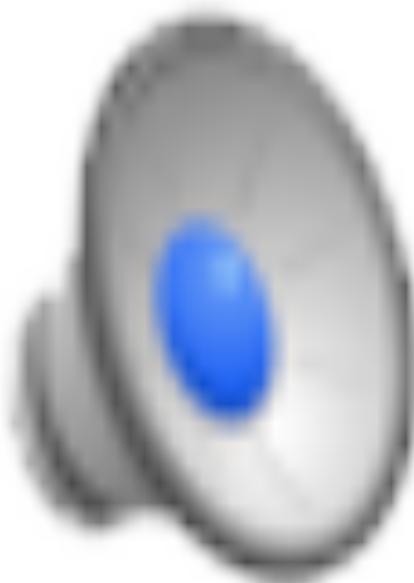
$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1})$$

Pacman – Sonar (P4)



[Demo: Pacman – Sonar – No Beliefs(L14D)]

Video of Demo Pacman – Sonar (with beliefs)



Next Time: Particle Filtering and Applications of HMMs
