CS188 Fall 2017 Section 12: Perceptrons / Neural Networks

1 Perceptron

We would like to use a perceptron to train a classifier with 2 features per point and labels +1 or -1. Consider the following labeled training data:

Features (x_1, x_2)	Label y^*
$(-1, 2)$	1
$(3, -1)$	-1
$(1, 2)$	-1
$(3, 1)$	1

1. Our two perceptron weights have been initialized to $w_1 = 2$ and $w_2 = -2$. After processing the first point with the perceptron algorithm, what will be the updated values for these weights?

   For the first point, $y = g(w_1 x_1 + w_2 x_2) = g(2 \cdot -1 + -2 \cdot 2) = g(-5) = -1$, which is incorrectly classified. To updated the weights, we add the first data point: $w_1 = 2 + (-1) = 1$ and $w_2 = -2 + 2 = 0$.

2. After how many steps will the perceptron algorithm converge? Write "never" if it will never converge.

   Note: one step means processing one point. Points are processed in order and then repeated, until convergence.

   The data is not separable, so it will never converge.

Perceptron $\rightarrow$ Neural Nets

Instead of the standard perceptron algorithm, we decide to treat the perceptron as a single node neural network and update the weights using gradient descent on the loss function.

The loss function for one data point is $Loss(y, y^*) = \frac{1}{2}(y - y^*)^2$, where y^* is the training label for a given point and y is the output of our single node network for that point. We will compute a score $z = w_1 x_1 + w_2 x_2$, and then predict the output using an activation function g: $y = g(z)$.

1. Given a general activation function $g(z)$ and its derivative $g'(z)$, what is the derivative of the loss function with respect to w_1 in terms of $g, g', y^*, x_1, x_2, w_1$, and w_2?

$$\begin{aligned}
\frac{\partial Loss}{\partial w_1} &= \frac{\partial}{\partial w_1} \frac{1}{2}(g(w_1 x_1 + w_2 x_2) - y^*)^2 \\
&= (g(w_1 x_1 + w_2 x_2) - y^*) * \frac{\partial}{\partial w_1} g(w_1 x_1 + w_2 x_2) \\
&= (g(w_1 x_1 + w_2 x_2) - y^*) * g'(w_1 x_1 + w_2 x_2) * \frac{\partial}{\partial w_1}(w_1 x_1 + w_2 x_2) \\
&= (g(w_1 x_1 + w_2 x_2) - y^*) * g'(w_1 x_1 + w_2 x_2) * x_1
\end{aligned}$$

2. For this question, the specific activation function that we will use is

$$g(z) = 1 \text{ if } z \geq 0 \text{ , or } -1 \text{ if } z < 0$$

   Given the gradient descent equation $w_i \leftarrow w_i - \alpha \frac{\partial Loss}{\partial w_1}$, update the weights for a single data point. With initial weights of $w_1 = 2$ and $w_2 = -2$, what are the updated weights after processing the first point?

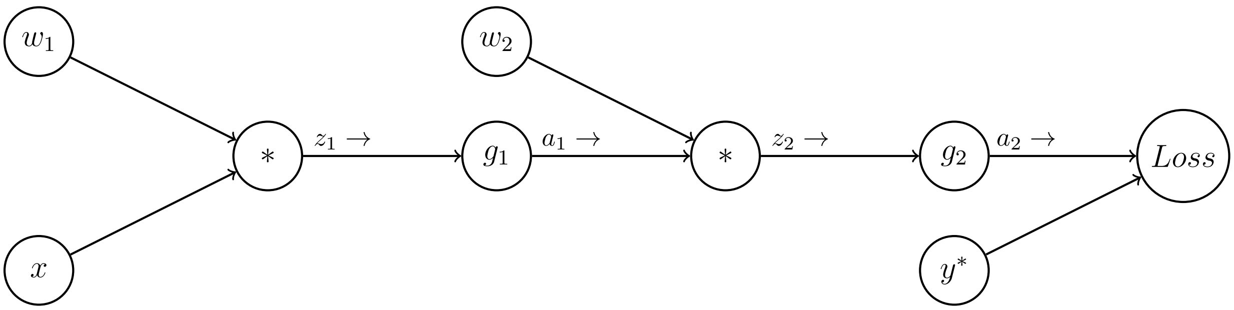
   Because the derivative of g is always zero, $g'(z) = 0$ (although it has two pieces, both pieces are constant and so have no slope), $\frac{\partial Loss}{\partial w_1}$ will be zero, and so the weights will stay $w_1 = 2$ and $w_2 = -2$.

3. What is the most critical problem with this gradient descent training process with that activation function?

   The gradient of that activation function is zero, so the weights will not update.

2 Neural Nets

Consider the following computation graph for a simple neural network for binary classification. Here x is a single real-valued input feature with an associated class y^* (0 or 1). There are two weight parameters w_1 and w_2, and non-linearity functions g_1 and g_2 (to be defined later, below). The network will output a value a_2 between 0 and 1, representing the probability of being in class 1. We will be using a loss function $Loss$ (to be defined later, below), to compare the prediction a_2 with the true class y^*.



1. Perform the forward pass on this network, writing the output values for each node z_1, a_1, z_2 and a_2 in terms of the node's input values:

$$
\begin{aligned}
z_1 &= x * w_1 \\
a_1 &= g_1(z_1) \\
z_2 &= a_1 * w_2 \\
a_2 &= g_2(z_2)
\end{aligned}
$$

2. Compute the loss $Loss(a_2, y^*)$ in terms of the input x, weights w_i, and activation functions g_i:

   Recursively substituting the values computed above, we have:

$$Loss(a_2, y^*) = Loss(g_2(w_2 * g_1(w_1 * x)), y^*)$$

3. Now we will work through parts of the backward pass, incrementally. Use the chain rule to derive $\frac{\partial Loss}{\partial w_2}$. Write your expression as a product of partial derivatives at each node: i.e. the partial derivative of the node's output with respect to its inputs. (Hint: the series of expressions you wrote in part 1 will be helpful; you may use any of those variables.)

$$\frac{\partial Loss}{\partial w_2} = \frac{\partial Loss}{\partial a_2} \frac{\partial a_2}{\partial z_2} \frac{\partial z_2}{\partial w_2}$$

4. Suppose the loss function is quadratic, $Loss(a_2, y^*) = \frac{1}{2}(a_2 - y^*)^2$, and g_1 and g_2 are both sigmoid functions $g(z) = \frac{1}{1+e^{-z}}$ (note: it's typically better to use a different type of loss, *cross-entropy*, for classification problems, but we'll use this to make the math easier).

   Using the chain rule from Part 3, and the fact that $\frac{\partial g(z)}{\partial z} = g(z)(1 - g(z))$ for the sigmoid function, write $\frac{\partial Loss}{\partial w_2}$ in terms of the values from the forward pass, y^*, a_1, and a_2:

   First we'll compute the partial derivatives at each node:

$$\begin{aligned}
\frac{\partial Loss}{\partial a_2} &= (a_2 - y^*) \\
\frac{\partial a_2}{\partial z_2} &= \frac{\partial g_2(z_2)}{\partial z_2} = g_2(z_2)(1 - g_2(z_2)) = a_2(1 - a_2) \\
\frac{\partial z_2}{\partial w_2} &= a_1
\end{aligned}$$

   Now we can plug into the chain rule from part 3:

$$\begin{aligned}
\frac{\partial Loss}{\partial w_2} &= \frac{\partial Loss}{\partial a_2} \frac{\partial a_2}{\partial z_2} \frac{\partial z_2}{\partial w_2} \\
&= (a_2 - y^*) * a_2(1 - a_2) * a_1
\end{aligned}$$

5. Now use the chain rule to derive $\frac{\partial Loss}{\partial w_1}$ as a product of partial derivatives at each node used in the chain rule:

$$\frac{\partial Loss}{\partial w_1} = \frac{\partial Loss}{\partial a_2} \frac{\partial a_2}{\partial z_2} \frac{\partial z_2}{\partial a_1} \frac{\partial a_1}{\partial z_1} \frac{\partial z_1}{\partial w_1}$$

6. Finally, write $\frac{\partial Loss}{\partial w_1}$ in terms of x, y^*, w_i, a_i, z_i: The partial derivatives at each node (in addition to the ones we computed in Part 4) are:

$$\begin{aligned}
\frac{\partial z_2}{\partial a_1} &= w_2 \\
\frac{\partial a_1}{\partial z_1} &= \frac{\partial g_1(z_1)}{\partial z_1} = g_1(z_1)(1 - g_1(z_1)) = a_1(1 - a_1) \\
\frac{\partial z_1}{\partial a_1} &= x
\end{aligned}$$

   Plugging into the chain rule from Part 5 gives:

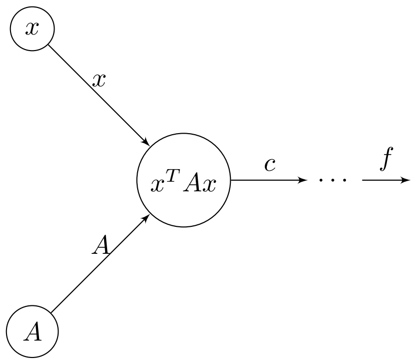
$$\begin{aligned}
\frac{\partial Loss}{\partial w_1} &= \frac{\partial Loss}{\partial a_2} \frac{\partial a_2}{\partial z_2} \frac{\partial z_2}{\partial a_1} \frac{\partial a_1}{\partial z_1} \frac{\partial z_1}{\partial w_1} \\
&= (a_2 - y^*) * a_2(1 - a_2) * w_2 * a_1(1 - a_1) * x
\end{aligned}$$

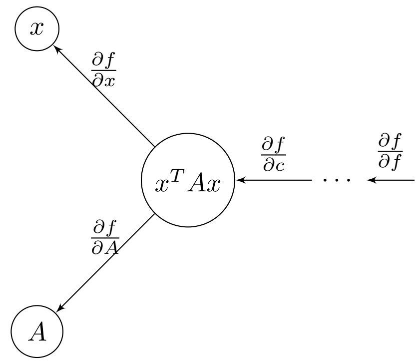
7. What is the gradient descent update for w_1 with step-size α in terms of the values computed above?

$$w_1 \leftarrow w_1 - \alpha(a_2 - y^*) * a_2(1 - a_2) * w_2 * a_1(1 - a_1) * x$$

3 Vectorized Gradients

Let's compute the backward step for a node that computes $x^T Ax$, where x is a vector with m values, and A is a matrix with shape $m \times m$. Thus, $c = \sum_{i=1}^m x_i \sum_{j=1}^m A_{ij}x_j = \sum_{i=1}^m \sum_{j=1}^m A_{ij}x_i x_j = \sum_{j=1}^m x_j \sum_{i=1}^m A_{ij}x_i$.





1. What is $\frac{\partial f}{\partial A_{ij}}$?

   $\frac{\partial f}{\partial A_{ij}} = \frac{\partial f}{\partial c}\frac{\partial c}{\partial A_{ij}} = \frac{\partial f}{\partial c}x_i x_j$

2. What is $\frac{\partial f}{\partial A}$?

   $\frac{\partial f}{\partial A} = \frac{\partial f}{\partial c}xx^T$

3. What is $\frac{\partial f}{\partial x_k}$?

   Use the Product Rule:

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial c}\frac{\partial c}{\partial x_k} = \frac{\partial f}{\partial c}(\frac{d}{dx_k}\sum_{i=1}^m\sum_{j=1}^m A_{ij}x_i x_j) = \frac{\partial f}{\partial c}((\frac{d}{dx_k}x_k)\sum_{j=1}^m A_{kj}x_j + (\frac{d}{dx_k}x_k)\sum_{i=1}^m A_{ik}x_i)$$

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial c}(\sum_{j=1}^m A_{kj}x_j + \sum_{i=1}^m A^T_{ki}x_i) = \frac{\partial f}{\partial c}(\sum_{j=1}^m (A_{kj} + A^T_{kj})x_j)$$

4. What is $\frac{\partial f}{\partial x}$?

   $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial c}(A + A^T)x$