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Submission: 06/02/2018
(2 days late)

# CS280 - Homework #1

## Problem 1 - Perspective Projection

1. Consider a plane based on two non-parallel lines $l_1$ & $l_2$:

$$l_1: \quad X(\lambda) = A_1 + \lambda D_1$$
$$l_2: \quad X(\lambda) = A_2 + \lambda D_2$$

without the loss of generality, we can take

$$D_1 = \begin{bmatrix} D_{1x} \\ D_{1y} \\ 1 \end{bmatrix} \qquad D_2 = \begin{bmatrix} D_{2x} \\ D_{2y} \\ 1 \end{bmatrix}$$

From the lecture, the two vanishing points are:

$$P_1 = \left( \frac{f D_{1x}}{\cancel{D_{1z}}\,1}, \frac{f D_{1y}}{\cancel{D_{1z}}\,1} \right) \qquad P_2 = \left( \frac{f D_{2x}}{\cancel{D_{2z}}\,1}, \frac{f D_{2y}}{\cancel{D_{2z}}\,1} \right)$$

i.e

$$P_1 = (f D_{1x}, f D_{1y}) \qquad P_2 = (f D_{2x}, f D_{2y})$$

$$(\text{since } D_{1z} = D_{2z} = 1)$$

Now we prove that any line on the plane has the vanishing point on the line $P_1 P_2$

Consider an arbitrary line $l_3$ on the plane. $l_3$ has the directional vector $D_3$

Since $D_1$, $D_2$ & $D_3$ are parallel to the plane so we can express $D_3$ in terms of $D_1$ & $D_2$

$$D_3 = \alpha D_1 + \beta D_2 = \begin{bmatrix} \alpha D_{1x} + \beta D_{2x} \\ \alpha D_{1y} + \beta D_{2y} \\ \alpha + \beta \end{bmatrix}$$

Again, we can set $D_{3z} = 1$ by dividing the coordinates by $\alpha+\beta$.

$$D_3 = \begin{bmatrix} \frac{\alpha}{\alpha+\beta} D_{1x} + \frac{\beta}{\alpha+\beta} D_{2x} \\ \frac{\alpha}{\alpha+\beta} D_{1y} + \frac{\beta}{\alpha+\beta} D_{2y} \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha' D_{1x} + \beta' D_{2x} \\ \alpha' D_{1y} + \beta' D_{2y} \\ 1 \end{bmatrix} \qquad \alpha'+\beta'=1$$

Omidding "1" we have:

$$D_3 = \begin{bmatrix} \alpha D_{1x} + \beta D_{2x} \\ \alpha D_{1y} + \beta D_{2y} \\ 1 \end{bmatrix} \qquad \text{where } \alpha+\beta=1$$

The vanishing point of $l_3$ is:

$$P_3 = \left( f(\alpha D_{1x} + \beta D_{2x}), f(\alpha D_{1y} + \beta D_{2y}) \right)$$

We prove that $P_1$, $P_2$ & $P_3$ are colinear:

Consider 2 vector $\vec{P_1P_2}$ and $\vec{P_1P_3}$

$$\vec{P_1P_2} = P_2 - P_1 = \left( f(D_{2x} - D_{1x}), f(D_{2y} - D_{1y}) \right)$$

$$\vec{P_1P_3} = P_3 - P_1 = \left( f(\alpha D_{1x} + \beta D_{2x} - D_{1x}), f(\alpha D_{1y} + \beta D_{2y} - D_{1y}) \right)$$

$$= \left( f(-\beta D_{1x} + \beta D_{2x}), f(-\beta D_{1y} + \beta D_{2y}) \right)$$

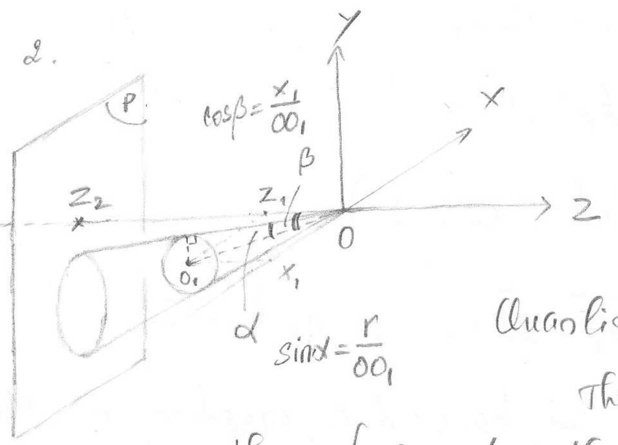
$$= -\beta \left( f(D_{2x} - D_{1x}), f(D_{2y} - D_{1y}) \right)$$

$$= -\beta \vec{P_1P_2}$$

$\Rightarrow$ $\vec{P_1P_2} \parallel \vec{P_1P_3}$ or $P_1$, $P_2$, $P_3$ are colinear

$\Rightarrow$ $P_3$ is on the vanishing line $P_1P_2$

2.




The sphere centered at $O_1(X_1, 0, Z_1)$
The image plane is the XY plane at $Z = Z_2$

Quanlitatively :

The light source at origin O projects the sphere onto the image plane (P). This creates a conic section. Since the sphere centered on XZ plance does not intersect with Y axis, the light rays from O through its edge are not parallel to Y axis, thus intersecting the plane P (assuming $r < Z_1$)
The plane P cut all the conic surface → the sihouette is an ellipse.

Qualitatively :

The equation of a line $l$ going through a point $A(x_0, y_0, z_0)$ with the direction vector $\vec{d}(a, b, c)$ takes the form:

$$\frac{X - x_0}{a} = \frac{Y - y_0}{b} = \frac{Z - z_0}{c}$$

The line $l_1$ going through O and $O_1$ has the equation.

$$\frac{X}{X_1} = \frac{Z}{Z_1} \qquad Y = 0$$

The lines $l_n$ going through O and tangent w/ the sphere make the conic surface, They have the directional vector $\vec{d_n}(a_n, b_n, c_n)$ making w/ the line $l_1$ $(OO_1)$ an angel $\alpha$

$$\Rightarrow \quad \vec{d_n} \cdot \vec{OO_1} = |\vec{d_n}|\,|\vec{OO_1}| \cos\alpha = a_n X_1 + b_n \cdot 0 + c_n Z_1$$

$$\sqrt{a_n^2 + b_n^2 + c_n^2}\,\sqrt{X_1^2 + Z_1^2}\,\sqrt{1 - \frac{r^2}{OO_1^2}} = a_n X_1 + c_n Z_1$$

$X_1^2 + Z_1^2$

$$\sqrt{a_n^2+b_n^2+c_n^2}\,\sqrt{X_1^2+Z_1^2-r^2} = a_n X_1 + c_n Z_1$$

we have the equation of the lines $l_n$ is :

$$\frac{X}{a_n} = \frac{Y}{b_n} = \frac{Z}{1} \qquad (\text{we can take } c_n = 1)$$

subject to : $$\sqrt{a_n^2+b_n^2+1}\,\sqrt{X_1^2+Z_1^2-r^2} = a_n X_1 + Z_1$$

$l_n$ cut (P) at $Z = Z_2$, thus we have the equation of the contour of the silouhette is

$$\frac{X}{a_n} = \frac{Y}{b_n} = Z_2 \qquad Z = Z_2 \qquad (1)$$

subjectto $$\sqrt{a_n^2+b_n^2+1}\,\sqrt{X_1^2+Z_1^2-r^2} = a_n X_1 + Z_1 \qquad (2)$$

From (1) : $a_n = \frac{X}{Z_2}$ $\quad b_n = \frac{Y}{Z_2}$ , substituting them into (2) gives :

$$\sqrt{\frac{X^2}{Z_2^2}+\frac{Y^2}{Z_2^2}+1}\,\sqrt{X_1^2+Z_1^2-r^2} = \frac{X}{Z_2}X_1 + Z_1$$

$$\cancel{\frac{X^2}{Z_2^2}X_1^2} + \frac{X^2}{Z_2^2}Z_1^2 - \frac{X^2}{Z_2^2}r^2 + \frac{Y^2}{Z_2^2}X_1^2 + \frac{Y^2}{Z_2^2}Z_1^2 - \frac{Y^2}{Z_2^2}r^2 + X_1^2 + \cancel{Z_1^2} - r^2$$

$$= \cancel{\frac{X^2}{Z_2^2}X_1^2} + 2\frac{X}{Z_2}X_1 Z_1 + \cancel{Z_1^2}$$

$$X^2(Z_1^2-r^2) + Y^2(X_1^2+Z_1^2-r^2) - 2XX_1Z_1Z_2 + (X_1^2-r^2)Z_2^2 = 0$$

$$\left(X\sqrt{Z_1^2-r^2} - \frac{X_1Z_1Z_2}{\sqrt{Z_1^2-r^2}}\right)^2 + Y^2(X_1^2+Z_1^2-r^2) = \frac{X_1^2Z_1^2Z_2^2}{Z_1^2-r^2} - (X_1^2-r^2)Z_2^2$$

This is the equation of ellipse in 2D-XY if satisfied :

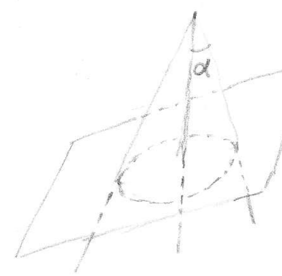
$Z_1 > r$ and $X_1^2 + Z_1^2 > r^2$ (then RHS $> 0$ automatically)

Thus the silouhette can be an ellipse, a hyperbola or a parabola:

+ ellipse : $z_1 > r$ $\quad x_1^2 + z_1^2 > r^2$.
+ parabola : $z_1 = r$
+ hyperbola : $z_1 < r$ $\quad x_1^2 + z_1^2 > r^2$.

The eccentricity :

$$e = \frac{\cos\beta}{\cos\alpha}$$




where $\alpha$ is the angel of conic section :

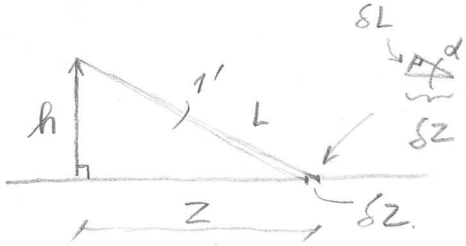
$$\cos\alpha = \sqrt{1 - \frac{r^2}{OO_1^2}} = \sqrt{\frac{x_1^2 + z_1^2 - r^2}{x_1^2 + z_1^2}}$$

$\beta$ is the angel of the cutting plane (P) and the axis $OO_1$ of the conic section :

$$\cos\beta = \frac{x_1}{OO_1} = \frac{x_1}{\sqrt{x_1^2 + z_1^2}}$$

$$\Rightarrow e = \frac{x_1}{\sqrt{x_1^2 + z_1^2 - r^2}}$$

3.




$$\delta z = \frac{\delta L}{\sin\alpha} \qquad \sin\alpha = \frac{h}{L}$$

$$\delta L \simeq L \times \mathrm{rad}(1')$$

$$\Rightarrow \delta z = \frac{L \times \mathrm{rad}(1')}{h/L} = \frac{L^2\,\mathrm{rad}(1')}{h}$$

$$\delta z = \frac{(h^2 + z^2) \times 0.0002909}{h}$$

Problem 2 - Rotations

1.

$$\hat{s} \times \vec{b} = S\vec{b}$$

$$\hat{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

2.

$$R = e^{\phi S} = I + \phi S + \frac{(\phi S)^2}{2!} + \frac{(\phi S)^3}{3!} + \frac{(\phi S)^4}{4!} + \dots$$

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} = \begin{bmatrix} -s_3^2 - s_2^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & -s_3^2 - s_1^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & -s_2^2 - s_1^2 \end{bmatrix}$$

$$S^3 = S^2 \cdot S = \begin{bmatrix} -s_3^2 - s_2^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & -s_3^2 - s_1^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & -s_2^2 - s_1^2 \end{bmatrix} \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & s_3^3 + s_2^2 s_3 + s_1^2 s_3 & -s_2 s_3^2 - s_2^3 - s_1^2 s_2 \\ -s_3^3 - s_1^2 s_3 - s_2^2 s_3 & 0 & s_1 s_2^2 + s_1 s_3^2 + s_1^3 \\ s_2 s_3^2 + s_2^3 + s_1^2 s_2 & -s_1 s_3^2 - s_1 s_2^2 - s_1^3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{bmatrix} = -S \qquad (\text{since } s_1^2 + s_2^2 + s_3^2 = 1)$$

$$\Rightarrow \quad S^4 = -S^2 \qquad S^5 = -SS^2 = S \qquad S^6 = S^2 \qquad S^7 = -S \quad \dots$$

$$\Rightarrow \quad R = I + \phi S + \frac{\phi^2 S^2}{2!} - \frac{\phi^3 S}{3!} - \frac{\phi^4 S^2}{4!} + \frac{\phi^5 S}{5!} + \frac{\phi^6 S^2}{6!} - \frac{\phi^7 S}{7!} - \dots$$

p.T

$$= I + \underbrace{\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \dots\right)}_{\sin\phi} S + \underbrace{\left(\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + \dots\right)}_{1-\cos\phi} S^2$$

$$\Rightarrow \quad R = I + \sin\phi\, S + (1-\cos\phi) S^2$$

3. Plot : See Appendix 1

Coding : See appendix 3

4. Coding : See appendix 3

Analytical Verification :

$s = [0.6, 0.8, 0]$ $u = [-0.8, 0.6, 0]$ $v = [0, 0, 1]$

Point : $p = [0, 1, 0]$

$\phi = 0$ :

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since R is the identity matrix, we can say immediately :

+ eigenvalues = 1 , 1 , 1
+ eigenvector = $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ , $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\underset{0.6s - 0.8u}{\swarrow} \qquad \underset{0.8s + 0.6u}{\downarrow} \qquad \underset{v}{\downarrow}$$

$\phi = \frac{\pi}{12}$ :

$$R = \begin{bmatrix} 0.978 & 0.016 & 0.207 \\ 0.016 & 0.988 & -0.155 \\ -0.207 & 0.155 & 0.969 \end{bmatrix}$$

$$\det(R - \lambda I) = 0$$

$$\det\left( \begin{bmatrix} 0.978 - \lambda & 0.016 & 0.207 \\ 0.016 & 0.98 - \lambda & -0.155 \\ -0.207 & 0.155 & 0.969 - \lambda \end{bmatrix} \right) = 0$$

$$\Rightarrow \lambda = 0.966 + 0.259i \ , \ 0.966 - 0.259i \ , \ 1$$

$\lambda = 0.966 + 0.259i$ : $\quad Rv = \lambda v$

$\Rightarrow \quad v = [0.566i, -0.566i, 0.6]^T$

$= 0.113i \times s - 0.792i \times u + 0.6v$

$\lambda = 0.966 - 0.259i$ : $\quad Rv = \lambda v$

$\Rightarrow \quad v = [-0.424i, 0.424i, 0.8]^T$

$= 0.085i \cdot s + 0.594i \cdot u + 0.8v$

$\lambda = 1$ : $\quad Rv = \lambda v$

$\Rightarrow \quad v = [-0.707, -0.707, 0]$

$= -0.99s + 0.14u$

$\phi = \frac{\pi}{8}$ : $\quad R = \begin{bmatrix} 0.951 & 0.037 & 0.306 \\ 0.037 & 0.973 & -0.23 \\ -0.306 & 0.23 & 0.924 \end{bmatrix}$

$\det(R - \lambda I) = 0$

$\det\left(\begin{bmatrix} 0.951-\lambda & 0.037 & 0.306 \\ 0.037 & 0.973-\lambda & -0.23 \\ -0.306 & 0.23 & 0.924-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow \lambda = 0.934 + 0.383i, \quad 0.934 - 0.383i, \quad 1$

$\lambda = 0.934 + 0.383i$ : $\quad Rv = \lambda v$

$\Rightarrow \quad v = [-0.566i, 0.566i, 0.6]^T$

$= 0.113i \cdot s - 0.792i \cdot u + 0.6v$

$\lambda = 0.934 - 0.383i$ : $\quad Rv = \lambda v$

$\Rightarrow \quad v = [0.424i, -0.424i, 0.8]^T$

$= -0.085i \cdot s - 0.594i \cdot u + 0.8v$

$\lambda = 1$ : $\quad Rv = \lambda v$

$\Rightarrow \quad v = [0.707, 0.707, 0]^T$

$= 0.99s - 0.14u$

$\phi = \frac{\pi}{6}$ : $R = \begin{bmatrix} 0.914 & 0.064 & 0.4 \\ 0.064 & 0.952 & -0.3 \\ -0.4 & 0.3 & 0.866 \end{bmatrix}$

$\det(R - \lambda I) = 0$

$\det\left(\begin{bmatrix} 0.914-\lambda & 0.064 & 0.4 \\ 0.064 & 0.952-\lambda & -0.3 \\ -0.4 & 0.3 & 0.866-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow \lambda = 0.866 + 0.5i,\ 0.866 - 0.5i,\ 1$

$\lambda = 0.866 + 0.5i$ : $Rv = \lambda v$

$\Rightarrow v = [0.566i, -0.566i, 0.6]^T$

$= -0.113i \cdot s - 0.292i \cdot u + 0.6v$

$\lambda = 0.866 - 0.5i$ : $Rv = \lambda v$

$\Rightarrow v = [-0.424i, 0.424i, 0.8]^T$

$= 0.085i\, s + 0.594i \cdot u + 0.8v$

$\lambda = 1$ : $Rv = \lambda v$

$\Rightarrow v = [-0.707, -0.707, 0]$

$= -0.99s + 0.14u$

$\phi = \frac{\pi}{4}$ : $R = \begin{bmatrix} 0.813 & 0.14 & 0.566 \\ 0.14 & 0.895 & -0.424 \\ -0.566 & 0.424 & 0.707 \end{bmatrix}$

$\det(R - \lambda I) = 0$

$\det\left(\begin{bmatrix} 0.813-\lambda & 0.14 & 0.566 \\ 0.14 & 0.895-\lambda & -0.424 \\ -0.566 & 0.424 & 0.707-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow \lambda = 0.707 + 0.707i,\ 0.707 - 0.707i,\ 1$

$\lambda = 0.707 + 0.707i \quad : \quad Rv = \lambda v$

$\Rightarrow \quad v = [0.566i \, , \; -0.566i \, , \; 0.6]$

$= -0.1131i \cdot s \; - 0.792i \cdot u + 0.6v$

$\lambda = 0.707 - 0.707i \quad : \quad Rv = \lambda v$

$\Rightarrow \quad v = [-0.424i \, , \; 0.424i \, , \; 0.8]^T$

$= 0.085i \cdot s + 0.594i \cdot u + 0.8v$

$\lambda = 1 \quad : \quad Ru = \lambda u$

$\Rightarrow \quad v = [-0.707 \, , \; -0.707 \, , \; 0$

$= -0.99s + 0.14u$

$\phi = \frac{\pi}{2} \quad : \quad R = \begin{bmatrix} 0.36 & 0.48 & +0.8 \\ 0.48 & 0.64 & -0.6 \\ -0.8 & 0.6 & 0 \end{bmatrix}$

$\det(R - \lambda I) = 0$

$\det \left( \begin{bmatrix} 0.36-\lambda & 0.48 & 0.8 \\ +0.48 & 0.64-\lambda & -0.6 \\ -0.8 & 0.6 & 0-\lambda \end{bmatrix} \right) = 0$

$\Rightarrow \quad \lambda = i \, , \; -i \, , \; 1$

$\lambda = i \quad : \quad Rv = \lambda v$

$\Rightarrow \quad v = [-0.566i \, , \; 0.566i \, , \; 0.6]^T$

$= 0.113i \cdot s - 0.792i \cdot u + 0.6v$

$\lambda = -i \quad : \quad Rv = \lambda v$

$\Rightarrow \quad v = [0.424i \, , \; -0.424i \, , \; 0.8]^T$

$= -0.085i \cdot s - 0.594i \cdot u + 0.8v$

$\lambda = 1 \quad : \quad Rv = \lambda v$

$\Rightarrow \quad v = [0.707 \, , \; 0.707 \, , \; 0]^T$

$= 0.99s - 0.14u$

$\phi = \pi$ : $R = \begin{bmatrix} -0.28 & 0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\det(R - \lambda I) = 0$

$\det\left(\begin{bmatrix} -0.28-\lambda & 0.96 & 0 \\ 0.96 & 0.28-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}\right) = 0$

$\Rightarrow \lambda = -1, 1, -1$

$\lambda = -1$ : $Rv = \lambda v$

$\Rightarrow v = [-0.8 \quad 0.6 \quad 0.386]^T$

$= u + 0.386u.$

$\lambda = 1$ : $Rv = \lambda u$

$\Rightarrow v = [0.6 \quad 0.8 \quad -0.29]^T$

$= s - 0.29v.$

$\lambda = -1$ : $Rv = \lambda u$

$\Rightarrow v = [0 \quad 0 \quad 0.876]$

$= 0.876 v.$

$\phi = \frac{3\pi}{4}$ : $R = \begin{bmatrix} 0.36 & 0.48 & -0.8 \\ 0.48 & 0.64 & 0.6 \\ 0.8 & -0.6 & 0 \end{bmatrix}$

$\det(R - \lambda I) = 0$

$\det\left(\begin{bmatrix} 0.36-\lambda & 0.48 & -0.8 \\ 0.48 & 0.64-\lambda & 0.6 \\ 0.8 & -0.6 & -\lambda \end{bmatrix}\right) = 0$

$\Rightarrow \quad \lambda = i, -i, 1$

eigen vectors & their suv representation is same as

$\phi = \frac{\pi}{4}$

5. From 2.:

$$R = I + \sin\phi\, S + (1-\cos\phi) S^2$$

$$\Rightarrow tr(R) = \overbrace{tr(I)}^{3} + \sin\phi\, tr(S) + (1-\cos\phi)\, tr(S^2)$$

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} \Rightarrow tr(S) = 0$$

$$S^2 = \begin{bmatrix} -s_3^2 - s_2^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & -s_3^2 - s_1^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & -s_2^2 - s_1^2 \end{bmatrix} \Rightarrow tr(S^2) = -s_3^2 - s_2^2 - s_3^2 - s_1^2 - s_2^2 - s_1^2 = -2$$

since $s_1^2 + s_2^2 + s_3^2 = 1$

$$\Rightarrow tr(R) = 3 + 0 + (1-\cos\phi)(-2) = 1 + 2\cos\phi$$

$$\Rightarrow \cos\phi = \frac{1}{2}(tr(R) - 1)$$

6. Coding of function rot_to_ax_phi.py : See Appendix 3

Basically, given:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\cos\phi = \frac{1}{2}(trace(R) - 1) \Rightarrow \phi = \arccos(\cos\phi)$$

$$\hat{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \frac{1}{2\sin\phi} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Problem 3

$$\vec{v}_i = \begin{bmatrix} v_{ix} \\ v_{iy} \\ 1 \end{bmatrix} = \begin{bmatrix} V_{ix}/V_{iz} \\ V_{iy}/V_{iz} \\ 1 \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} V_{ix} \\ V_{iy} \\ V_{iz} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ 1 \end{bmatrix} = H\vec{u}_i$$

(*)

1. $H^* = \text{argmin}_H \sum_{i=1}^{4} \| T(\vec{u}_i) - \vec{v}_i \|^2$ where $\vec{v}_i = T(\vec{u}_i)$

(a) 2D affine transformation

$$\vec{v} = A\vec{u} + \vec{t}$$

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

We can re-write in the form:

$$\begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix}$$

Comparing this one to (*) we see that we can set $h_{31} = h_{32} = 0$, and thus $V_{iz} = 1$. The equation for affine transformation reduces to:

$$\begin{bmatrix} v_{ix} \\ v_{iy} \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ 1 \end{bmatrix}$$

$$h_{11} u_{ix} + h_{12} u_{iy} + h_{13} = v_{ix}$$

$$h_{21} u_{ix} + h_{22} u_{iy} + h_{23} = v_{iy}$$

In the matrix form:

$$\underbrace{\begin{bmatrix} u_{ix} & u_{iy} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{ix} & u_{iy} & 1 \end{bmatrix}}_{X_i} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} v_{ix} \\ v_{iy} \end{bmatrix}}_{y_i}$$

We have 6 unknowns $h$'s $\rightarrow$ need 6 eqns $\rightarrow$ need 3 points.
As long as we have 3 points, we can form 6 eqns, stack them on top of each other to get the form

$$Xw = y$$

The problem reduces to $w^* = \text{argmin}_w \|Xw - y\|^2$

Solution: $w^* = (X^TX)^{-1}X^Ty$

(b) Homography

$$\begin{bmatrix} V_{ix} \\ V_{iy} \\ V_{iz} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} h_{11} u_{ix} + h_{12} u_{iy} + h_{13} \\ h_{21} u_{ix} + h_{22} u_{iy} + h_{23} \\ h_{31} u_{ix} + h_{32} u_{iy} + 1 \end{bmatrix}$$

We have:

$$\vec{v}_i = \begin{bmatrix} v_{ix} \\ v_{iy} \\ 1 \end{bmatrix} = \begin{bmatrix} V_{ix}/V_{iz} \\ V_{iy}/V_{iz} \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad v_{ix} = \frac{h_{11} u_{ix} + h_{12} u_{iy} + h_{13}}{h_{31} u_{ix} + h_{32} u_{iy} + 1}$$

$$v_{iy} = \frac{h_{21} u_{ix} + h_{22} u_{iy} + h_{23}}{h_{31} u_{ix} + h_{32} u_{iy} + 1}$$

Re-arrange the two eqns, we get:

$$h_{11} u_{ix} + h_{12} u_{iy} + h_{13} - h_{31} u_{ix} v_{ix} - h_{32} u_{iy} v_{ix} = v_{ix}$$

$$h_{21} u_{ix} + h_{22} u_{iy} + h_{23} - h_{31} u_{ix} v_{iy} - h_{32} u_{iy} v_{iy} = v_{iy}$$

Or in matrix form:

$$\underbrace{\begin{bmatrix} u_{ix} & u_{iy} & 1 & 0 & 0 & 0 & -u_{ix} v_{ix} & -u_{iy} v_{ix} \\ 0 & 0 & 0 & u_{ix} & u_{iy} & 1 & -u_{ix} v_{iy} & -u_{iy} v_{iy} \end{bmatrix}}_{X_i} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} v_{ix} \\ v_{iy} \end{bmatrix}}_{y_i}$$

We have 8 unknowns $h$'s $\rightarrow$ need 8 eqns $\rightarrow$ need 4 points.
we can stack the eqns on top of each other to get the form

$$Xw = y$$

The problem reduces to $w^* = \arg\min_w \|Xw - y\|^2$

Solution: $w^* = (X^T X)^{-1} X^T y$

2. The constraints for affine transform is

$$h_{31} = h_{32} = 0$$

and the other elements of $H$ satisfy

$$h^* = (U^T U)^{-1} U^T V \quad \text{for 3 points}$$

where $U$ is the matrix formed by 3 input points
$V$ is the flattened vector of 3 output points

The constraint for homography transformation is

$$h^* = (U^T U)^{-1} U^T V \quad \text{for 4 points}$$

where $U$ is the matrix formed by 4 input points
$V$ is the flattened vector of 3 output points

3. The affine transform is <u>NOT</u> able to exactly transform the points from one image to another because it conserves the parallelism while different images have different perspectives
(parallel lines in one image may not be parallel in the others)
The homography is able. It deforms the perspective in one image to adapt the others.

4. Images : See Appendix 3

Coding : See Appendix 4

Observations : the affine transform produces the parallelogram output that does not fit different surfaces in the target images. The homography transform produces the deformed output that perfectly fit different surfaced in the target images.

5. Images : See Appendix 3

Coding : See Appendix 4