Student: NINH DO
SID # 25949105

CS287 - Homework #2

1 (a)
$$J_{i+1}(x) = \min_{x,u} x^T Q x + u^T R u + \mathbb{E}[x_T^T Q x_T]$$

(dynamic programming, instead of $\min_{x,u} \mathbb{E}\left[\sum_0^{T-1} x_t^T Q x_t + u_t^T R u_t\right] + \mathbb{E}[x_T^T Q x_T]$)

Initialize $J_0(x) = x^T P_0 x$

$$J_1(x) = \min_{x,u} x^T Q x + u^T R u + \mathbb{E}[x_T^T Q x_T]$$

$$\mathbb{E}[x_T^T Q x_T] = \mathbb{E}[(Ax + Bu + w)^T P_0 (Ax + Bu + w)]$$

$$= \mathbb{E}[x^T A^T P_0 A x + 2x^T A^T P_0 B u + 2u^T B^T P_0 w + 2x^T A^T P_0 w + u^T B^T P_0 B u + w^T P_0 w]$$

$$= x^T A^T P_0 A x + 2x^T A^T P_0 B u + \cancel{2u^T B^T P_0 \mathbb{E}[w]}^{\,0} + \cancel{2x^T A^T P_0 \mathbb{E}[w]}^{\,0} + u^T B^T P_0 B u + \mathbb{E}[w^T P_0 w]$$

$$= x^T A^T P_0 A x + 2x^T A^T P_0 B u + u^T B^T P_0 B u + \mathbb{E}[w^T P_0 w] \quad (*)$$

$$J_1(x) = \min_{x,u} [x^T Q x + u^T R u + (*)]$$

$$\nabla_u[\dots] = 2Ru + 2B^T P_0 A x + 2B^T P_0 B u = 0$$

$$\Rightarrow \quad u = -(R + B^T P_0 B)^{-1} B^T P_0 A x$$

Plug u into $J_1(x)$:

$$J_1(x) = x^T P_1 x + \mathbb{E}[w^T P_0 w]$$

for
$$P_1 = Q + K_1^T R K + (A + BK_1)^T P_0 (A + BK_1)$$
$$K_1 = -(R + B^T P_0 B)^{-1} B^T P_0 A$$

Iteratively, we have:

$$K_i = -(R + B^T P_{i-1} B)^{-1} B^T P_{i-1} A$$

$$P_i = Q + K_i^T R K_i + (A + BK_i)^T P_{i-1} (A + BK_i)$$

The optimal feedback controller:

$$u_i = K_i x_i$$

The cost-to-go function:

$$J_i(x) = x^T P_i x + \mathbb{E}[w^T P_i w]$$

compared to the case when there is no noise, the excess of the expected cost is $E[w^T P_i w]$

(b)

$$J_{i+1}(x) = \min_{x,u} \mathbb{E}[x_T^T Q x] \qquad \text{cost-to-go}$$

Initialize $J_0(x) = x^T P_0 x$

$$J_1(x) = \min_{x,u} \mathbb{E}[x_T^T Q x_T]$$

$$\mathbb{E}[x_T^T Q x] = \mathbb{E}[(Ax + (B+W)u)^T P_0 (Ax + (B+W)u)]$$

$$= \mathbb{E}[x^T A^T P_0 A x + 2x^T A^T P_0 B u + 2x^T A^T P_0 W u + u^T B^T P_0 B u + 2u^T B^T P_0 W u + u^T W^T P_0 W u]$$

$$= x^T A^T P_0 A x + 2x^T A^T P_0 B u + 2x^T A^T P_0 \cancelto{0}{\mathbb{E}[W]} u + u^T B^T P_0 B u + 2u^T B^T P_0 \cancelto{0}{\mathbb{E}[W]} u + u^T \mathbb{E}[W^T P_0 W] u$$

$$= \underbrace{x^T A^T P_0 A x + 2x^T A^T P_0 B u + u^T (B^T P_0 B + \mathbb{E}[W^T P_0 W]) u}$$

$$J_1(x) = \min_{x,u} [\ \downarrow\ ]$$

$$\nabla_u [\cdots] = 2B^T P_0 A x + 2(B^T P_0 B + \mathbb{E}[W^T P_0 W]) u$$

$$\Rightarrow u = -(B^T P_0 B + \mathbb{E}[W^T P_0 W])^{-1} B^T P_0 A x$$

plug u into $J_1(x)$:

$$J_1(x) = x^T P_1 x$$

for $P_1 = (A + (B+W_1)K_1)^T P_0 (A + (B+W_1)K_1)$

$$K_1 = -(B^T P_0 B + \mathbb{E}[W^T P_0 W])^{-1} B^T P_0 A$$

Iteratively, we have

$$K_i = -\left(B^T P_{i-1} B + \mathbb{E}[W^T P_0 W] \right)^{-1} B^T P_{i-1} A$$

$$P_i = \left(A + (B + W_i) K_i \right)^T P_{i-1} \left(A + (B + W_i) K_i \right)$$

the optimal feedback controller

$$u_i = K_i x_i$$

The cost-to-go function:

$$J_i(x) = \mathbb{E}[x^T P_i x]$$