# Homework 4 : Solutions
# STAT 150

**Problem 5.3.5 :**

It is easy to see that :

$$\begin{aligned}\mathbb{P}(X(T) \geq k) &= \mathbb{P}(W_k \leq T) = \mathbb{P}(W_{k-1} \leq T)\mathbb{P}(W_k \leq T | W_{k-1} \leq T) \\ &= \frac{\lambda}{\lambda+\theta}\mathbb{P}(X(T) \geq k-1)\end{aligned}$$

using the memoryless property of exponential distribution. Thus we get :

$$\mathbb{P}(X(T) = k) = (\tfrac{\theta}{\lambda+\theta})(\tfrac{\lambda}{\lambda+\theta})^k$$

**Problem 5.3.6 :**

We know that $T = W_Q$, thus $\mathbb{E}[T] = \frac{Q}{\lambda}$. Now we can decompose our integral in this way :

$$\begin{aligned}\mathbb{E}[\int_0^T N(t)dt] &= \mathbb{E}[\sum_{k=0}^{Q-1} \int_{W_k}^{W_{k+1}} N(t)dt] \\ &= \mathbb{E}[\sum_{k=0}^{Q-1} k(W_{k+1} - W_k)dt] \\ &= \sum_{k=0}^{Q-1} k\mathbb{E}[W_{k+1} - W_k] \\ &= \frac{Q(Q-1)}{2\lambda}\end{aligned}$$

**Problem 5.4.2 :**

It is clear that $N(t) = X(t) + Y(t)$, note that given $N(t) = n$, then $X(t) = \sum_{k=1}^n 1_{\{U_k+Z_k>t\}}$, where $U$ and $Z$'s are independent random variables, distributed as uniform on $[0,t]$ and having a cdf $G$ respectively. Thus it is easy to that $X(t)$ and $Y(t)$ are both independent Poisson random variables with rates :

$$\lambda \int_0^t [1-G(s)]ds \text{ and } \lambda \int_0^t G(s)ds \text{ respectively}$$

**Problem 5.4.4 :**

Given $X(t) = n$, then $Z(t) > z$ if and only if $U_i + \xi_i > z$ for every $i$ in $[n]$, with $U_i$ and $\xi$'s independent and having distribution $U[0,t]$ and with cdf $F$ respectively. Thus we have :

$$\begin{aligned}
\mathbb{P}(Z(t) > z) &= \sum_{n=0}^{\infty} \mathbb{P}(U + \xi > z)^n \mathbb{P}(X(t) = n) \\
&= e^{-\lambda t \mathbb{P}(U+\xi>z)} \\
&= \exp(-\lambda \int_{z-t}^{z} F(s)ds)
\end{aligned}$$

Hence, by letting $t \to \infty$, we get that $\mathbb{P}(Z > z) = \exp(-\lambda \int_{-\infty}^{z} F(s)ds)$

**Problem 2.38:**

By conditioning on the value of $N_t$ we get :

$$\mathbb{E}[S_t] = S_0 \mathbb{E}[\mu^{N(t)}] = S_0 e^{\lambda t(\mu - 1)}$$

$$\text{Var}(S_t) = \mathbb{E}[S_t^2] - (\mathbb{E}[S_t])^2 = S_0^2 \mathbb{E}[(\sigma^2 + \mu^2)^{N(t)}] - S_0^2 e^{2\lambda t(\mu-1)}.$$

Thus :

$$\text{Var}(S_t) = S_0^2 [e^{\lambda t(\sigma^2 + \mu^2 - 1)} - e^{2\lambda t(\mu - 1)}]$$

**Problem 2.53:**

As usual given $N_t = n$ we have $X_t = \sum_{k=1}^{n} 1_{\{U_k + Z_k > t\}}$ for $U$ uniformly distributed on $[0,t]$ and $Z$ with cdf $F$. Then :

$$\mathbb{P}[X_t = k] = \frac{e^{-\lambda t p_t}(\lambda t p_t)^k}{k!} \text{ with}$$

$$p_t = \mathbb{P}[U + Z > t] = \frac{1}{t} \int_0^t (1 - F(u))du$$

Hence $X_t$ is Poisson with parameter $\lambda t p_t = \lambda \int_0^t (1 - F(u))du$

Thus when $t \to \infty$, the limiting distribution is Poisson with parameter $\lambda \int_0^\infty (1 - F(u))du = \lambda \int_0^\infty \mathbb{P}[Z > u]du = \lambda\mu$

**Problem 1 :**

(a) $X_5$ has a Poisson distribution of parameter $5\lambda$, thus :

$$\mathbb{E}[X_5] = 5\lambda.$$

(b) $W_3$ has a Gamma distribution with parameters $(3, \frac{1}{\lambda})$ thus :

$$\mathbb{E}[W_3] = \tfrac{3}{\lambda}.$$

(c) $\mathbb{P}(X_5 < 3) = \sum_{k=0}^{2} e^{-5\lambda} \frac{(5\lambda)^k}{k!}$.

(d) $\mathbb{P}(W_3 > 5) = \mathbb{P}(X_5 < 3) = \sum_{k=0}^{2} e^{-5\lambda} \frac{(5\lambda)^k}{k!}$.

(e)

$$\begin{aligned}\mathbb{P}(W_3 > 5 | X_2 = 1) &= \mathbb{P}(X_5 < 3 | X_2 = 1) \\ &= \mathbb{P}(X_3 < 2) \\ &= \sum_{k=0}^{1} e^{-3\lambda} \frac{(3\lambda)^k}{k!}.\end{aligned}$$

**Problem 2 :**

(a) The process $\{T_t, t \geq 0\}$ has clearly independent increments and for every $0 \leq s \leq t$, we have :

$$T_t - T_s \sim \text{Poisson}((\lambda_R + \lambda_B)(t - s))$$

Hence, it is a Poisson process with rate $\lambda_R + \lambda_B$.

(b) The probability that the first bird to arrive is a robin is equal to :

$$\mathbb{P}(W_1^R < W_1^B) = \int_0^\infty \lambda_R e^{-\lambda_R x}\mathbb{P}(x < W_1^B)dx = \int_0^\infty \lambda_R e^{-\lambda_R x} e^{-\lambda_B x} dx.$$

Thus the probability is equal to $\frac{\lambda_R}{\lambda_R+\lambda_B}$.

(c) *First method* : We have $\mathbb{P}(R_t = k | T_t = n) = \frac{\mathbb{P}(R_t=k, B_t=n-k)}{\mathbb{P}(T_t=n)} = \binom{n}{k}(\frac{\lambda_R}{\lambda_R+\lambda_B})^k(\frac{\lambda_B}{\lambda_R+\lambda_B})^{n-k}$

*Second method* : Given that $T_t = n$, there is $n$ birds that arrives up to time $t$, and using the fact each bird that arrives has a probability $\frac{\lambda_R}{\lambda_R+\lambda_B}$ of being a robin, then there is $\binom{n}{k}$ ways to choose the $k$ robin arrival times from the $n$ overall ones, and then at each time it is robin with probability $\frac{\lambda_R}{\lambda_R+\lambda_B}$, and blackbird with probability $\frac{\lambda_B}{\lambda_R+\lambda_B}$ which gives us the desired formula.