Homework 1 : Solutions
STAT 150

Problem 1.3.11:

Let $k, l \in \mathbb{N}$ two nonnegative integers. Then if $l \neq 0$ we get :

$$\begin{aligned}\mathbb{P}(U = k, W = l) &= \mathbb{P}(\min(X,Y) = k, \max(X,Y) = k + l)\\ &= 2\mathbb{P}(X = k, Y = k + l)\\ &= 2(1-\pi)\pi^k(1-\pi)\pi^{k+l}\\ &= 2(1-\pi)^2\pi^{2k}\pi^l\end{aligned}$$

However if $l = 0$ then $\mathbb{P}(U = k, W = l) = \mathbb{P}(X = k, Y = k) = (1-\pi)^2\pi^{2k}$.

However, as we have for $l \neq 0$:

$$\begin{aligned}\mathbb{P}(W = l) &= \sum_{k=0}^{\infty} \mathbb{P}(W = l, U = k)\\ &= 2(1-\pi)^2\pi^l \sum_{k=0}^{\infty} (\pi^2)^k\\ &= 2(1-\pi)^2\pi^l \frac{1}{1-\pi^2}\\ &= 2\pi^l \frac{1-\pi}{1+\pi}\end{aligned}$$

And $\mathbb{P}(W = 0) = \sum_{k=0}^{\infty} \mathbb{P}(W = 0, U = k) = \sum_{k=0}^{\infty} (1-\pi)^2\pi^{2k} = \frac{1-\pi}{1+\pi}$.

On the other hand, it is easy to see that :

$$\begin{aligned}\mathbb{P}(U = k) &= \sum_{l=0}^{\infty} \mathbb{P}(U = k, W = l)\\ &= (1-\pi)^2\pi^{2k} + \sum_{l=1}^{\infty} 2(1-\pi)^2\pi^{2k}\pi^l\\ &= (1-\pi)^2\pi^{2k} + 2(1-\pi)^2\pi^{2k}\frac{\pi}{1-\pi}\\ &= (1-\pi^2)\pi^{2k}\end{aligned}$$

Thus, it is clear that for every k, l we have the relation :
$\mathbb{P}(U = k, W = l) = \mathbb{P}(U = k)\mathbb{P}(W = l)$, which means that W and U are independent.

Problem 1.4.3:

Let $x \in \mathbb{R}$, then as the random variable W takes values in $[-1, 1]$, we have :

$$\mathbb{P}(W \leq x) = \mathbb{P}(X \leq Y + x) = \int \mathbb{P}(X \leq y + x | Y = y) 1_{\theta - \frac{1}{2} \leq y \leq \theta + \frac{1}{2}} \mathrm{dy}$$

Using the fact that X and Y are independent, and that $U = X - (\theta - \frac{1}{2})$ and $Y - (\theta - \frac{1}{2})$ are both uniform on $[0, 1]$, we get that :
$\mathbb{P}(W \leq x) = \displaystyle\int_0^1 \mathbb{P}(U \leq y + x)\mathrm{dy}$.
It is clear that if $|x| > 1$, then $\mathbb{P}(W \leq x) = 0$.
If $0 \leq x \leq 1$: $\mathbb{P}(W \leq x) = \displaystyle\int_0^{1-x} (y + x)\mathrm{dy} + \int_{1-x}^1 \mathrm{dy} = \frac{1 - x^2}{2} + x$.
If $-1 \leq x \leq 0$: $\mathbb{P}(W \leq x) = \displaystyle\int_0^{-x} 0\mathrm{dy} + \int_{-x}^1 (y + x)\mathrm{dy} = \frac{1 + x^2}{2} + x$

Hence by differentiating the two expressions we obtain the density function desired.

Problem 2.1.2 :

Let $y \leq x \leq N$. Using Bayes formula we have :

$$\begin{aligned}
\mathbb{P}(X = x | Y = y) &= \frac{\mathbb{P}(Y = y | X = x)\mathbb{P}(X = x)}{\mathbb{P}(Y = y)} \\
&= \frac{\mathbb{P}(Y = y | X = x)\mathbb{P}(X = x)}{\displaystyle\sum_{k=y}^{N} \mathbb{P}(Y = y | X = k)\mathbb{P}(X = k)} \\
&= \frac{1}{x(\displaystyle\sum_{k=y}^{N} \frac{1}{k})}
\end{aligned}$$

Problem 2.3.3:

(a) Lets fix an integer n, and lets denote $S_n = \sum_{k=1} n\xi_k$, then it is clear that $\mathbb{E}[S_n] = 0$ and $\mathbb{E}[S_n^2] = Var(S_n) = n$, hence :

$$\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[Z|N]] = 0, \text{ and } \mathrm{Var}(Z) = \mathbb{E}[\mathbb{E}[Z^2|N]] = \mathbb{E}[N] = \tfrac{1-\alpha}{\alpha}$$

(b) By the same way we have $\mathbb{E}[S_n^3] = 0$ because the ξ have a symmetric distribution, and we have

$$\mathbb{E}[S_n^4] = \mathbb{E}[(S_{n-1} + \xi_n)^4] = \mathbb{E}[S_{n-1}^4] + 6(n-1) + 1$$

hence $\mathbb{E}[S_n^4] = 3n^2 - 2n$, thus $\mathbb{E}[Z^4] = \mathbb{E}[3N^2 - 2N] = \frac{(1-\alpha)(6-5\alpha)}{\alpha^2}$

Problem 2.4.6:

Let F be the distribution function of the common law of the X_i's. The probability mass function of N is :

$$\begin{aligned}
\mathbb{P}(N = k) &= \mathbb{P}(X_1 \leq X_0, X_2 \leq X_0, .., X_{k-1} \leq X_0, X_k > X_0) \\
&= \mathbb{E}[\mathbb{P}(X_1 \leq X_0, X_2 \leq X_0, .., X_{k-1} \leq X_0, X_k > X_0|X_0)] \\
&= \mathbb{E}[F(X_0)^{k-1}(1 - F(X_0))] \\
&= \int_0^\infty F(x)^{k-1}(1 - F(x))F^{'}(x)dx \\
&= [\frac{F(x)^k}{k} - \frac{F(x)^{k+1}}{k+1}]_0^\infty \\
&= \frac{1}{k(k+1)}
\end{aligned}$$

Thus, $\mathbb{E}[N] = \sum_{k=1}^{\infty} k\frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty$

Problem 3.2.4:

By ordering the states in the following order $(0,0)$,$(0,1)$,$(1,0)$ and then $(1,1)$ we get that the transition probability matrix is :

$$P = \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \\ \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \end{bmatrix}$$

Problem 3.3.5:

By ordering the states in the following order 0 ,H,HH and then HHT we get that the transition probability matrix is :

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3.3.4:

The probability that at a certain time k the customer being served will complete his service is :

$$\mathbb{P}(Z = k | Z \geq k) = \alpha$$

Hence, the transition probabilities are :

$$P_{0,1} = \beta, P_{0,0} = 1 - \beta$$
$$P_{i,i-1} = \alpha(1-\beta), P_{i,i+1} = (1-\alpha)\beta, P_{i,i} = (1-\beta)(1-\alpha) + \alpha\beta$$
$$\text{for } i \geq 1$$

Problem 3.4.1:

Let's consider the Markov Chain that will take as value the latest pattern that agree with the beginning of the pattern we want to achieve (HHT and HTH respectively) and 0 if the length of this pattern is null. Also, the chain get absorbed once we reach for the first time the pattern.

For the first case (HHT) :

We have four states $\{0, H, HH, HHT\}$, and the transition matrix is :

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Our goal is to find the expectation of the time T to reach the absorbing state starting from 0. If we denote $v_i = \mathbb{E}[T|X_0 = 0]$, then we have the following equations from the first step analysis :

$$\begin{aligned} v_0 &= 1 + \tfrac{1}{2}v_0 + \tfrac{1}{2}v_1 \\ v_1 &= 1 + \tfrac{1}{2}v_0 + \tfrac{1}{2}v_2 \\ v_2 &= 1 + \tfrac{1}{2}v_2 \end{aligned}$$

and we get $v_0 = 8$. Using the same technique for the pattern HTH gives us : $v_0 = 10$.

Intuitively, the reason behind this difference in the two patterns, is that once you get your first H, to get HTH you need to have TH for your two following tosses (this event has a probability of $\frac{1}{4}$), however if you get another H (with probability $\frac{1}{2}$), then we are sure to get HHT before HTH, as any following T that will appear in the sequence will achieve this. Hence it is more frequent to reach HHT before HTH which explains why the expected time for HHT is smaller than the one for HTH.

Problem 3.4.5:

The transition matrix of the Markov chain has the following form:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For $i \in \{1, 2, 4, 5, 6\}$ let p_i be the probability to reach the state 3 before 7. The first step analysis gives us the following equations :

$$\begin{gathered} p_1 = \tfrac{1}{2}p_2 + \tfrac{1}{2}p_4 \ ; \ p_2 = \tfrac{1}{3}p_1 + \tfrac{1}{3}p_3 + \tfrac{1}{3}p_5 \\ p_3 = 1 \ ; \ p_4 = \tfrac{1}{3}p_1 + \tfrac{1}{3}p_5 + \tfrac{1}{3}p_7 \\ p_5 = \tfrac{1}{3}p_2 + \tfrac{1}{3}p_4 + \tfrac{1}{3}p_6 \end{gathered}$$

$$p_6 = \frac{1}{2}p_3 + \frac{1}{2}p_5 \ ; \ p_7 = 0$$

Solving this system gives us : $p_4 = \frac{5}{12}$.

Problem 3.4.15:

If we consider the Markov chain that tracks the number of people knowing the rumor, then we start from 1 and we want to reach 5. The probability transition matrix is as follows :

$$P = \begin{bmatrix} 0.96 & 0.04 & 0 & 0 & 0 \\ 0 & 0.94 & 0.06 & 0 & 0 \\ 0 & 0 & 0.94 & 0.06 & 0 \\ \frac{1}{3} & 0 & 0 & 0.96 & 0.04 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The probabilities can be easily found this way (if only one person knows the rumor, the only case when the rumor will be spread in the next step to another person is if the first person is picked among the two persons selected which has a probability of 0.4 to happen and if the rumor is transmitted with probability 0.1.

By first step analysis we get that the mean time it takes for everyone to hear the rumor is equal to $v_1 = \frac{133}{3}$.

Problem 1.1:

It suffices to see that :

$$\mathbb{P}(X_1 = 1, X_2 = 1, X_3 = 1) = \mathbb{P}(Y_0 = 0, Y_1 = 1, Y_2 = 0, Y_3 = 1) + \mathbb{P}(Y_0 = 0, Y_1 = 1, Y_2 = 0, Y_3 = 1) = \frac{1}{8}$$

and that :

$$\mathbb{P}(X_1 = 0, X_2 = 1, X_3 = 1) = \mathbb{P}(Y_0 = 0, Y_1 = 0, Y_2 = 1, Y_3 = 0) = \frac{1}{16}$$

Problem 1.2:

The state space is : $\{0, 1, 2, 3, 4, 5\}$ and the transition matrix of the Markov chain has the following form:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/25 & 8/25 & 16/25 & 0 & 0 & 0 \\ 0 & 4/25 & 12/25 & 9/25 & 0 & 0 \\ 0 & 0 & 9/25 & 12/25 & 4/25 & 0 \\ 0 & 0 & 0 & 16/25 & 8/25 & 1/25 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Problem 1.7:

(a) The value of X_n clearly depends only on the value of X_{n-1}, thus it is a Markov chain. If we denote the states in this order RR, RS, SR and SS then the probability transition matrix is equal to :

$$P = \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{bmatrix}$$

(b) The two step transition matrix is simply equal to P^2 thus :

$$P^2 = \begin{bmatrix} 0.36 & 0.24 & 0.24 & 0.16 \\ 0.36 & 0.24 & 0.12 & 0.28 \\ 0.36 & 0.24 & 0.24 & 0.16 \\ 0.18 & 0.12 & 0.21 & 0.49 \end{bmatrix}$$

(c) We have to compute the probability that $X_2 \in \{SR, RR\}$ given that $X_0 = SS$, which is equal to :

$$P^2(SS, SR) + P^2(SS, RR) = P^2(4,3) + P^2(4,1) = 0.21 + 018 = 0.39.$$