Homework 8 : Solutions
STAT 150

Problem 7.1.3 :

Simply notice that $\gamma_t = W_{N(t)+1} - t$.

Problem 7.3.1 :

Using the fact that $W_{N(t)+1}$ and $N(t)$ are independent, then knowing the two facts :

$$\mathbb{E}[\frac{1}{N(t)+1}] = \sum_{n=0}^{\infty} \frac{1}{n+1} e^{-\lambda t} \frac{(\lambda t)^n}{n!} = \frac{1}{\lambda t}[1 - e^{-\lambda t}]$$

and

$$\mathbb{E}[W_{N(t)+1}] = \mathbb{E}[X_1][M(t)+1] = \frac{1}{\lambda}[\lambda t + 1]$$

By multiplying both the equations we get the desired result.

Problem 7.3.5 :

At every point t, let $D(t)$ be the distance to the nearest bird. Hence : $\mathbb{P}\{D(t) > x\} = \mathbb{P}\{N(t-x, t+x) = 0\} = \begin{cases} e^{-2\lambda x} & \text{if } 0 < x < t \\ e^{-\lambda(x+t)} & \text{if } x \geq t \end{cases}$

Hence the density function is : $f_t(x) = \begin{cases} 2\lambda e^{-2\lambda x} & \text{if } 0 < x < t \\ \lambda e^{-\lambda(x+t)} & \text{if } x \geq t \end{cases}$

and the mean is equal to : $\mathbb{E}[D(t)] = \frac{1}{2\lambda}(1 + e^{-2\lambda t})$.

Problem 7.4.1 :

As : $\frac{1}{t}M(t) \to \frac{1}{\mu}$ and $M(t) - \frac{t}{\mu} \to \frac{\sigma^2 - \mu^2}{2\mu^2}$ as $t \to \infty$ we get easily : $\mu = 1$ and $\sigma^2 = 3$.

Problem 7.4.4 :

The increments of the renewal process are the number of children for each family, hence it is clear that $N(t)$ represents in this case the number of females when the population is around t for t large enough (as each family has only one female), then by the elementary renewal theorem we have :

$$\text{Long run fraction of females} = \lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu} = \frac{1}{2}$$

as the increments have geometric distribution with parameter $\frac{1}{2}$

Problem 7.4.5 :

(a) Let m_i be the mean duration of a sojourn time at state i, then we have by the first step analysis : $m_0 = 1 + 0.3m_0$, hence $m_0 = \frac{10}{7}$, the same for m_2 gives : $m_2 = 2$.

(b) The increments are the sojourn times away from 1 (we count the stay at 1), thus they have mean equal to $m = 1 + 0.6m_0 + 0.4m_2 = \frac{8}{10} + \frac{6}{7} = \frac{93}{35}$, hence the long run fraction of time that the process is in state 1 is by the elementary renewal theorem : $\frac{35}{93}$.

Problem 2:

(a) We have $M(t) = \mathbb{E}[N(t)] = \displaystyle\int_0^\infty \mathbb{E}[N(t)|X_1 = x]dF(x)$, however as $N(t) = \sup\,\{n : S_n = X_1 + \displaystyle\sum_{k=2}^{n} X_k \leq t\}$, then conditionally on $X_1 = x$, $N(t) = N'(t-x) + 1$ for N' is the renewal process for the increments $X_2, X_3, ..$ and hence independent from X_1. Thus if $t \geq x$, we have : $\mathbb{E}[N(t)|X_1 = x] = 1 + \mathbb{E}[N'(t-x)] = 1 + M(t-x)$, and otherwise $\mathbb{E}[N(t)|X_1 = x]$. Hence :

$$M(t) = \int_0^t (1 + M(t-x))dF(x) = F(t) + \int_0^t M(t-x)dF(x)$$

(b) If $X_i \sim \mathcal{U}([0,1])$, then for $t \leq 1$ we have :

$$M(t) = t + \int_0^t (1 + M(t-x))dx = t + \int_0^t M(x)dx$$

By differentiating, we get $M'(t) = 1 + M(t)$, hence $M(t) + 1 = \lambda e^t$ for some constant λ, however as $M(0) = 0$ we get that $M(t) = e^t - 1$.

(c) The random variable N is also equal to $N(1)+1$ for a renewal process N with increments uniform on $[0,1]$, as $N(1)$ is the last index n such that $\sum_{k=1}^{n} U_k \leq 1$, so N must be the next one. Hence :

$$\mathbb{E}[N] = 1 + \mathbb{E}[N(1)] = 1 + M(1) = 1 + e^1 - 1 = e$$