Tutorial 4

STAT 150

Exercise 1

A Markov chain has five states $0, 1, 2, 3, 4$. The transitions are given as follows: when at $i \geq 1$ the next step chooses a uniform state in $0, 1, ..., i-1$. When at $i = 0$ the next step is 4.
1. Is the Markov chain irreducible? Aperiodic?
2. What is the stationary distribution?

Exercise 2

A computer game has 10 levels. If in one round you are at level i you succeed with probability $0 < p_i < 1$ and move up to level $i+1$ or fail with probability $1 - p_i$ and reattempt level i in the next round. You win when you succeed in level 10 and after winning you go back to level 1 and start again. Let X_n be the level you are at in round n.

(a) Starting at level i find the expected number of rounds until you win.

(b) Find the stationary distribution of (X_n).

(c) Over the long run what fraction of times do you win.

Exercise 3

Let Q be any Markov chain and μ a probability distribution. We?ll construct a new Markov chain whose transitions are as follows: from a state x choose a new state y according to the the transition matrix Q. With probability :

$$A_{xy} = \min\{1, \frac{\mu_y Q_{yx}}{\mu_x Q_{xy}}\}$$

accept the move and move to y. Otherwise reject it and stay at x.

(a) Write a formula for the transition matrix P.

(b) Show that μ is the stationary distribution of P.

(c) On a graph G let μ be the uniform distribution and let Q be the transition matrix for the simple random walk on G. Describe the transitions of the new Markov chain P in this case.

Exercise 4 Prove that if $p_{ij} > 0$ for all i and j then a necessary and sufficient condition for the existence of a reversible stationary distribution is :

$$p_{ij}p_{jk}p_{ki} = p_{ik}p_{kj}p_{ji} \text{ for all } i, j, k$$

Hint: fix i and take $\pi_j = c\frac{p_{ij}}{p_{ji}}$.