Homework 2 : Solutions
STAT 150

Problem 3.5.1:

Using the first step analysis for v_i being the expected to reach the state 3 starting from i we get :

$$\begin{aligned} v_0 &= 1 + a_0 v_0 + a_1 v_1 + a_2 v_2 + a_3 v_3 \\ v_1 &= 1 + (a_0 + a_1) v_1 + a_2 v_2 + a_3 v_3 \\ v_2 &= 1 + (a_0 + a_1 + a_2) v_2 + a_3 v_3 \\ v_3 &= 0 \end{aligned}$$

Hence, we get that $v_2 = \frac{1}{1-a_0-a_1-a_2} = \frac{1}{a_3}$, and then $v_1 = \frac{1+\frac{a_2}{a_3}}{a_2+a_3} = \frac{1}{a_3}$ and the same goes for $v_0 = \frac{1}{a_3}$.

Problem 3.6.7

Using the first step analysis for a random walk with two absorbing states at 0 an 3 given in the equation (3.66) we have for $p_1 = 0.7$, and $p_2 = 0.9$:

$v_1 = 0.3 + 0.7(1 + v_2) = 1 + 0.7v_2$ and we have $v_2 = 0.9 + 0.1(1 + v_1) = 1 + 0.1v_1$, thus we have $v_1 = 1 + 0.7(1 + 0.1v_1) = 1.7 + 0.07v_1$ thus $0.93v_1 = 1.7$ hence $v_1 = 1.83$.

Problem 3.9.5 :

(a) It is easy to see that the probability that there is only red cells at time n is equal to :

$$\mathbb{P}(\text{all reds}) = (\tfrac{1}{4})(\tfrac{1}{4})^2..(\tfrac{1}{4})^{2^{n-1}} = (\tfrac{1}{4})^{2^n-1}$$

(b) The probability that the entire culture dies out entirely is the probability that the red cells dies out. However, as the number of red cells is a branching process with the following generating function for the offspring distribution :

$$\phi(s) = \frac{1}{12} + \frac{2}{3}s + \frac{1}{4}s^2$$

The smallest solution is equal to $u_\infty = \frac{1}{3}$

Problem 3.9.6 :

Using the fact that $a + b + c = 1$, our equation becomes :

$$as^2 - (a + c)s + c = 0$$

whose smallest solution is equal to $u_\infty = \frac{c}{a} < 1$ if $c < a$.

Problem 2:

(a) When the chain leaves its first position, the problem can be seen as a simple random walk that has two absorbing states either 0 or 12. Thus the expected number of steps to return to the first position is : $1 + \mathbb{E}_1[T]$ where T is the absorption time. From the results of 3.6.1 we have that this is equal to $1 + v_1 = 1 + 1 * (12 - 1) = 12$.

(b) By the same idea, once we reach either 1 or 11, we have to compute the probability of reaching the further absorbing state (11 in the first case and 1 and in the second case), by symmetry those two probabilities are equal. So if we let u_k be the probability to reach state 11 before 0 starting from k, we are looking for u_1. We have $u_0 = 0$ and $u_{11} = 1$ and $u_k = \frac{1}{2}u_{k+1} + \frac{1}{2}u_{k-1}$ for every $1 \leq k \leq 10$. By easy computation we get : $u_k = \frac{k}{11}$, thus $u_1 = \frac{1}{11}$.

(c) Using the notations of 3.6.1 :

$$\begin{aligned}\mathbb{E}_6[T] &= W_{66} \\ &= \frac{(1-\theta^6)^2}{p(1-\theta)(1-\theta^{12})}\end{aligned}$$

Problem 3:

The simple random walk is obviously irreducible so it suffices to only show the null-recurrence for the state 0. let $f(x) = \mathbb{E}_x[T_0]$.

We have : $f(x) = 1 + \frac{1}{2}f(x+1) + \frac{1}{2}f(x-1)$ for $x \geq 2$, hence for every $x \in \mathbb{Z}$ we have $2 + f(x+1) - f(x) = f(x) - f(x-1)$, hence if we let $d(x) = f(x) - f(x-1)$, we get that $d(x+1) = d(x) - 2$. Now, if we suppose that $d(2)$ is finite, then for any large x, $d(x)$ will be less than -1 and thus f will be strictly decreasing, which means that at a certain point f will be negative, this is not possible. Hence $f(2) = f(1) + d(2) = \infty$ so $f(2) = \infty$ and hence because $f(1) = \frac{1}{2} + \frac{1}{2}(1 + f(2)) = \infty$ and thus $f(0) = \frac{1}{2}(f(1) + f(-1)) = \infty$ hence 0 is null-recurrent.

Problem 4:

It is easy to see that the order in which we flip the coins on the table after getting a Tail doesn't matter in our long run process for getting only Tails at the end. Now, suppose we start with one Head on the table, we will do the following procedure : we will flip this coin on the table until it turns Tail, and every time we gets a Head we add a Head to the table. Afterwards, we repeat the same operation with all the new coins on the table.

Theses two methodologies gives birth to the same number of Head coins on the table, and it is clear that for the second one, the new number of new coins simply represent the number of children of the first Head, thus this is a branching process with offspring distribution, the number of flips to turn a coin to a Tail (note that for the coins that are already Tail, they have no children), which is geometric with parameter p. Thus the probability of extinction is equal to the smallest solution of :

$$\phi(s) = \sum_{k=0}^{\infty} p(1-p)^k s^k = \frac{p}{1-(1-p)s} = s$$

Which is equal to : $\frac{p}{1-p}$ if $p < \frac{1}{2}$ and 1 otherwise.

Problem 5:

(a) It is clear that every children from the first generation generates a new branching process that has the same offspring distribution than the original one and all those branching processes emerging from each children have the same distribution so we can write :

$$N_n = 1 + \sum_{k=1}^{X_1} N_{n-1,k}$$

where $N_{n-1,k}$ are i.i.d, have the distribution as N_{n-1} and independent from X_1 thus we have :

$$\begin{aligned}
\psi_n(s) &= \mathbb{E}[s^{N_n}] \\
&= \mathbb{E}[s^{1+\sum_{k=1}^{X_1} N_{n-1,k}}] \\
&= s\mathbb{E}[\psi_{n-1}(s)_1^X] = s\phi(\psi_{n-1}(s))
\end{aligned}$$

As $\psi_n(s) \to \psi(s)$ as n goes to infinity (because $N_n \to N$) then using the continuity of ϕ we get that $\psi(s) = s\phi(\psi(s))$

(b) We differentiate the last equation to get that : $\psi'(s) = \phi(\psi(s)) + s\phi'(\psi(s))\psi'(s)$ so by plugging $s = 1$ and noticing that $\mathbb{E}[N] = \psi'(1)$ and $\mu = \phi'(1)$ then we get that : $\mathbb{E}[N] = 1 + \mu\mathbb{E}[N]$ hence : $\mathbb{E}[N] = \frac{1}{1-\mu}$.