4 The Long Run Behavior of Markov Chains

4.1 Regular Transition Probability Matrices

Suppose that a transition probability matrix $\mathbf{P} = \|P_{ij}\|$ on a finite number of states labeled $0, 1, \ldots, N$ has the property that when raised to some power k, the matrix $\mathbf{P}^k$ has all of its elements strictly positive. Such a transition probability matrix, or the corresponding Markov chain, is called *regular*. The most important fact concerning a regular Markov chain is the existence of a *limiting probability distribution* $\pi = (\pi_0, \pi_1, \ldots, \pi_N)$, where $\pi_j > 0$ for $j = 0, 1, \ldots, N$ and $\Sigma_j \pi_j = 1$, and this distribution is independent of the initial state. Formally, for a regular transition probability matrix $\mathbf{P} = \|P_{ij}\|$, we have the convergence

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j > 0 \quad \text{for } j = 0, 1, \ldots, N,$$

or, in terms of the Markov chain $\{X_n\}$,

$$\lim_{n\to\infty} \Pr\{X_n = j | X_0 = i\} = \pi_j > 0 \quad \text{for } j = 0, 1, \ldots, N.$$

This convergence means that, in the long run ($n \to \infty$), the probability of finding the Markov chain in state j is approximately π_j no matter in which state the chain began at time 0.

Example The Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 1-a & a \\ b & 1-b \end{array} \right\| \end{array} \tag{4.1}$$

is regular when $0 < a, b < 1$, and in this case, the limiting distribution is $\pi = (b/(a+b), a/(a+b))$. To give a numerical example, we will suppose that

$$\mathbf{P} = \left\| \begin{array}{cc} 0.33 & 0.67 \\ 0.75 & 0.25 \end{array} \right\|.$$

An Introduction to Stochastic Modeling. DOI: 10.1016/B978-0-12-381416-6.00004-6

The first several powers of P are given as follows:

$$\mathbf{P}^2 = \begin{Vmatrix} 0.6114 & 0.3886 \\ 0.4350 & 0.5650 \end{Vmatrix}, \quad \mathbf{P}^3 = \begin{Vmatrix} 0.4932 & 0.5068 \\ 0.5673 & 0.4327 \end{Vmatrix},$$

$$\mathbf{P}^4 = \begin{Vmatrix} 0.5428 & 0.4572 \\ 0.5117 & 0.4883 \end{Vmatrix}, \quad \mathbf{P}^5 = \begin{Vmatrix} 0.5220 & 0.4780 \\ 0.5350 & 0.4560 \end{Vmatrix},$$

$$\mathbf{P}^6 = \begin{Vmatrix} 0.5307 & 0.4693 \\ 0.5253 & 0.4747 \end{Vmatrix}, \quad \mathbf{P}^7 = \begin{Vmatrix} 0.5271 & 0.4729 \\ 0.5294 & 0.4706 \end{Vmatrix}.$$

By $n = 7$, the entries agree row-to-row to two decimal places. The limiting probabilities are $b/(a+b) = 0.5282$ and $a/(a+b) = 0.4718$.

Example Sociologists often assume that the social classes of successive generations in a family can be regarded as a Markov chain. Thus, the occupation of a son is assumed to depend only on his father's occupation and not on his grandfather's. Suppose that such a model is appropriate and that the transition probability matrix is given by

$$\begin{array}{ll} & \\ & \\ & \text{Lower} \\ \text{Father's} & \text{Middle} \\ \text{class} & \text{Upper} \end{array} \begin{array}{c} \text{Son's class} \\ \begin{array}{ccc} \text{Lower} & \text{Middle} & \text{Upper} \end{array} \\ \begin{Vmatrix} 0.40 & 0.50 & 0.10 \\ 0.05 & 0.70 & 0.25 \\ 0.05 & 0.50 & 0.45 \end{Vmatrix} \end{array}.$$

For such a population, what fraction of people are middle class in the long run?

For the time being, we will answer the question by computing sufficiently high powers of $\mathbf{P}^n$. A better method for determining the limiting distribution will be presented later in this section.

We compute

$$\mathbf{P}^2 = \mathbf{P} \times \mathbf{P} = \begin{Vmatrix} 0.1900 & 0.6000 & 0.2100 \\ 0.0675 & 0.6400 & 0.2925 \\ 0.0675 & 0.6000 & 0.3325 \end{Vmatrix},$$

$$\mathbf{P}^4 = \mathbf{P}^2 \times \mathbf{P}^2 = \begin{Vmatrix} 0.0908 & 0.6240 & 0.2852 \\ 0.0758 & 0.6256 & 0.2986 \\ 0.0758 & 0.6240 & 0.3002 \end{Vmatrix},$$

$$\mathbf{P}^8 = \mathbf{P}^4 \times \mathbf{P}^4 = \begin{Vmatrix} 0.0772 & 0.6250 & 0.2978 \\ 0.0769 & 0.6250 & 0.2981 \\ 0.0769 & 0.6250 & 0.2981 \end{Vmatrix}.$$

Note that we have not computed $\mathbf{P}^n$ for consecutive values of n but have speeded up the calculations by evaluating the successive squares $\mathbf{P}^2, \mathbf{P}^4, \mathbf{P}^8$.

In the long run, approximately 62.5% of the population are middle class under the assumptions of the model.

Computing the limiting distribution by raising the transition probability matrix to a high power suffers from being inexact, since $n=\infty$ is never attained, and it also requires more computational effort than is necessary. Theorem 4.1 provides an alternative computational approach by asserting that the limiting distribution is the unique solution to a set of linear equations. For this social class example, the exact limiting distribution, computed using the method of Theorem 4.1, is $\pi_0 = \frac{1}{13} = 0.0769$, $\pi_1 = \frac{5}{8} = 0.6250$, and $\pi_2 = \frac{31}{104} = 0.2981$.

If a transition probability matrix $\mathbf{P}$ on N states is regular, then $\mathbf{P}^{N^2}$ will have no zero elements. Equivalently, if $\mathbf{P}^{N^2}$ is not strictly positive, then the Markov chain is not regular. Furthermore, once it happens that $\mathbf{P}^k$ has no zero entries, then every higher power $\mathbf{P}^{k+n}$, $n = 1, 2, \ldots$, will have no zero entries. Thus, it suffices to check the successive squares $\mathbf{P}, \mathbf{P}^2, \mathbf{P}^4, \mathbf{P}^8, \ldots$.

Finally, to determine whether or not the square of a transition probability matrix has only strictly positive entries, it is not necessary to perform the actual multiplication, but only to record whether or not the product is nonzero.

Example Consider the transition probability matrix

$$\mathbf{P} = \begin{Vmatrix} 0.9 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0.9 & 0 & 0 & 0 & 0 & 0 & 0.1 \\ 0.9 & 0 & 0 & 0 & 0 & 0 & 0.1 \end{Vmatrix}.$$

We recognize this as a success runs Markov chain. We record the nonzero entries as $+$ and write $\mathbf{P} \times \mathbf{P}$ in the form

$$\mathbf{P} \times \mathbf{P} = \begin{Vmatrix} + & + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & + & 0 & 0 & 0 & 0 \\ + & 0 & 0 & + & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & + & 0 \\ + & 0 & 0 & 0 & 0 & 0 & + \\ + & 0 & 0 & 0 & 0 & 0 & + \end{Vmatrix}$$

$$\times \begin{Vmatrix} + & + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & + & 0 & 0 & 0 & 0 \\ + & 0 & 0 & + & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & + & 0 \\ + & 0 & 0 & 0 & 0 & 0 & + \\ + & 0 & 0 & 0 & 0 & 0 & + \end{Vmatrix}$$

$$= \left\| \begin{matrix} + & + & + & 0 & 0 & 0 & 0 \\ + & + & 0 & + & 0 & 0 & 0 \\ + & + & 0 & 0 & + & 0 & 0 \\ + & + & 0 & 0 & 0 & + & 0 \\ + & + & 0 & 0 & 0 & 0 & + \\ + & + & 0 & 0 & 0 & 0 & + \\ + & + & 0 & 0 & 0 & 0 & + \end{matrix} \right\| = \mathbf{P}^2,$$

$$\mathbf{P}^4 = \left\| \begin{matrix} + & + & + & + & + & 0 & 0 \\ + & + & + & + & 0 & + & 0 \\ + & + & + & + & 0 & 0 & + \\ + & + & + & + & 0 & 0 & + \\ + & + & + & + & 0 & 0 & + \\ + & + & + & + & 0 & 0 & + \\ + & + & + & + & 0 & 0 & + \end{matrix} \right\|,$$

$$\mathbf{P}^8 = \left\| \begin{matrix} + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \\ + & + & + & + & + & + & + \end{matrix} \right\|.$$

We see that $\mathbf{P}^8$ has all strictly positive entries, and therefore, $\mathbf{P}$ is regular. The limiting distribution for a similar matrix is computed in Section 4.2.2.

Every transition probability matrix on the states $0, 1, \ldots, N$ that satisfies the following two conditions is regular:

1. For every pair of states i, j there is a path $k_1, \ldots, k_r$ for which $P_{ik_1} P_{k_1 k_2} \cdots P_{k_r j} > 0$.
2. There is at least one state i for which $P_{ii} > 0$.

Theorem 4.1. *Let* $\mathbf{P}$ *be a regular transition probability matrix on the states* $0, 1, \ldots, N$. *Then the limiting distribution* $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_N)$ *is the unique nonnegative solution of the equations*

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}, \quad j = 0, 1, \ldots, N, \tag{4.2}$$

$$\sum_{k=0}^{N} \pi_k = 1. \tag{4.3}$$

Proof. Because the Markov chain is regular, we have a limiting distribution, $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$, for which $\sum_{k=0}^{N} \pi_k = 1$. Write $\mathbf{P}^n$ as the matrix product $\mathbf{P}^{n-1}\mathbf{P}$

in the form

$$P_{ij}^{(n)} = \sum_{k=0}^{N} P_{ik}^{(n-1)} P_{kj}, \quad j = 0, \ldots, N, \tag{4.4}$$

and now let $n \to \infty$. Then, $P_{ij}^{(n)} \to \pi_j$, while $P_{ik}^{(n-1)} \to \pi_k$, and (4.4) passes into $\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}$ as claimed.

It remains to show that the solution is unique. Suppose that $x_0, x_1, \ldots, x_N$ solves

$$x_j = \sum_{k=0}^{N} x_k P_{kj} \quad \text{for } j = 0, \ldots, N \tag{4.5}$$

and

$$\sum_{k=0}^{N} x_k = 1. \tag{4.6}$$

We wish to show that $x_j = \pi_j$, the limiting probability. Begin by multiplying (4.5) on the right by P_{jl} and then sum over j to get

$$\sum_{j=0}^{N} x_j P_{jl} = \sum_{j=0}^{N} \sum_{k=0}^{N} x_k P_{kj} P_{jl} = \sum_{k=0}^{N} x_k P_{kl}^{(2)}. \tag{4.7}$$

But by (4.5), we have $x_l = \sum_{j=0}^{N} x_j P_{jl}$, whence (4.7) becomes

$$x_l = \sum_{k=0}^{N} x_k P_{kl}^{(2)} \quad \text{for } l = 0, \ldots, N.$$

Repeating this argument n times we deduce that

$$x_l = \sum_{k=0}^{N} x_k P_{kl}^{(n)} \quad \text{for } l = 0, \ldots, N,$$

and then passing to the limit in n and using that $P_{kl}^{(n)} \to \pi_l$, we see that

$$x_l = \sum_{k=0}^{N} x_k \pi_l, \quad l = 0, \ldots, N.$$

But by (4.6), we have $\Sigma_k x_k = 1$, whence $x = \pi_l$ as claimed. ■

Example For the social class matrix

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\left\| \begin{array}{ccc} 0.40 & 0.50 & 0.10 \\ 0.05 & 0.70 & 0.25 \\ 0.05 & 0.50 & 0.45 \end{array} \right\|
\end{array},
$$

the equations determining the limiting distribution (π_0, π_1, π_2) are

$$0.40\pi_0 + 0.05\pi_1 + 0.05\pi_2 = \pi_0, \tag{4.8}$$
$$0.50\pi_0 + 0.70\pi_1 + 0.50\pi_2 = \pi_1, \tag{4.9}$$
$$0.10\pi_0 + 0.25\pi_1 + 0.45\pi_2 = \pi_2, \tag{4.10}$$
$$\pi_0 + \pi_1 + \pi_2 = 1. \tag{4.11}$$

One of the equations (4.8), (4.9), and (4.10) is redundant because of the linear constraint $\Sigma_k P_{ik} = 1$. We arbitrarily strike out (4.10) and simplify the remaining equations to get

$$-60\pi_0 + 5\pi_1 + 5\pi_2 = 0, \tag{4.12}$$
$$5\pi_0 - 3\pi_1 + 5\pi_2 = 0, \tag{4.13}$$
$$\pi_0 + \pi_1 + \pi_2 = 1. \tag{4.14}$$

We eliminate π_2 by subtracting (4.12) from (4.13) and five times (4.14) to reduce the system to

$$
\begin{aligned}
65\pi_0 - 8\pi_1 &= 0, \\
65\pi_0 \qquad &= 5.
\end{aligned}
$$

Then, $\pi_0 = \frac{5}{65} = \frac{1}{13}, \pi_1 = \frac{5}{8}$, and then $\pi_2 = 1 - \pi_0 - \pi_1 = \frac{31}{104}$, as given earlier.

4.1.1 Doubly Stochastic Matrices

A transition probability matrix is called *doubly stochastic* if the columns sum to one as well as the rows. Formally, $\mathbf{P} = \|P_{ij}\|$ is doubly stochastic if

$$P_{ij} \geq 0 \quad \text{and} \quad \sum_k P_{ik} = \sum_k P_{kj} = 1 \quad \text{for all } i, j.$$

Consider a doubly stochastic transition probability matrix on the N states 0, $1, \ldots, N-1$. If the matrix is regular, then the unique limiting distribution is the uniform distribution $\pi = (1/N, \ldots, 1/N)$. Because there is only one solution to $\pi_j = \Sigma_k \pi_k P_{kj}$ and $\Sigma_k \pi_k = 1$ when P is regular, we need only to check that $\pi = (1/N, \ldots, 1/N)$ is a solution where $\mathbf{P}$ is doubly stochastic in order to establish the claim. By using the doubly stochastic feature $\Sigma_j P_{jk} = 1$, we verify that

$$\frac{1}{N} = \sum_j \frac{1}{N} P_{jk} = \frac{1}{N}.$$

As an example, let Y_n be the sum of n independent rolls of a fair die and consider the problem of determining with what probability Y_n is a multiple of 7 in the long run. Let X_n be the remainder when Y_n is divided by 7. Then, X_n is a Markov chain on the states $0, 1, \ldots, 6$ with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left\| \begin{array}{ccccccc} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \end{array} \right\| \end{array}.$$

The matrix is doubly stochastic, and it is regular ($\mathbf{P}^2$ has only strictly positive entries), hence the limiting distribution is $\pi = \left(\frac{1}{7}, \ldots, \frac{1}{7}\right)$. Furthermore, Y_n is a multiple of 7 if and only if $X_n = 0$. Thus, the limiting probability that Y_n is a multiple of 7 is $\frac{1}{7}$.

4.1.2 Interpretation of the Limiting Distribution

Given a regular transition matrix $\mathbf{P}$ for a Markov process $\{X_n\}$ on the $N+1$ states $0, 1, \ldots, N$, we solve the linear equations

$$\pi_i = \sum_{k=0}^{N} \pi_k P_{ki} \quad \text{for } i = 0, 1, \ldots, N$$

and

$$\pi_0 + \pi_1 + \cdots + \pi_N = 1.$$

The primary interpretation of the solution $(\pi_0, \ldots, \pi_N)$ is as the limiting distribution

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} = \lim_{n\to\infty} \Pr\{X_n = j | X_0 = i\}.$$

In words, after the process has been in operation for a long duration, the probability of finding the process in state j is π_j, irrespective of the starting state.

There is a second interpretation of the limiting distribution $\pi = (\pi_0, \pi_1, \ldots, \pi_N)$ that plays a major role in many models. We claim that π_j also gives the long *run* mean fraction of time that the process $\{X_n\}$ is in state j. Thus, if each visit to state j incurs a "cost" of c_j, then the long run mean cost per unit time associated with this Markov chain is

$$\text{Long run mean cost per unit time} = \sum_{j=0}^{N} \pi_j c_j.$$

To verify this interpretation, recall that if a sequence $a_0, a_1, \ldots$ of real numbers converges to a limit a, then the averages of these numbers also converge in the manner

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} a_k = a.$$

We apply this result to the convergence $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$ to conclude that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} P_{ij}^{(k)} = \pi_j.$$

Now, $(1/m)\sum_{k=0}^{m-1} P_{ij}^{(k)}$ is exactly the mean fraction of time during steps $0, 1, \ldots, m-1$ that the process spends in state j. Indeed, the actual (random) fraction of time in state j is

$$\frac{1}{m} \sum_{k=0}^{m-1} \mathbf{1}\{X_k = j\},$$

where

$$\mathbf{1}\{X_k = j\} = \begin{cases} 1 & \text{if } X_k = j, \\ 0 & \text{if } X_k \neq j. \end{cases}$$

Therefore, the *mean* fraction of visits is obtained by taking expected values according to

$$\begin{aligned} E\left[\frac{1}{m} \sum_{k=0}^{m-1} \mathbf{1}\{X_k = i\} | X_0 = j\right] &= \frac{1}{m} \sum_{k=0}^{m-1} E[\mathbf{1}\{X_k = j\} | X_0 = i] \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \Pr\{X_k = j | X_0 = i\} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} P_{ij}^{(k)}. \end{aligned}$$

Because $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$, the long run mean fraction of time that the process spends in state j is

$$\lim_{m\to\infty} E\left[\frac{1}{m}\sum_{k=0}^{m-1} \mathbf{1}\{X_k = j\} | X_0 = i\right] = \lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^{m-1} P_{ij}^{(k)} = \pi_j,$$

independent of the starting state i.

Exercises

4.1.1 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{Vmatrix} \end{array}.$$

Determine the limiting distribution.

4.1.2 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{Vmatrix} \end{array}.$$

Determine the limiting distribution.

4.1.3 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{Vmatrix} \end{array}.$$

What fraction of time, in the long run, does the process spend in state 1?

4.1.4 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{Vmatrix} \end{array}.$$

Every period that the process spends in state 0 incurs a cost of \$2. Every period that the process spends in state 1 incurs a cost of \$5. Every period that the process spends in state 2 incurs a cost of \$3. What is the long run cost per period associated with this Markov chain?

4.1.5 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 0.1 & 0.5 & 0 & 0.4 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 \end{array}.$$

Determine the limiting distribution for the process.

4.1.6 Compute the limiting distribution for the transition probability matrix

$$\mathbf{P} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 2 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{array}.$$

4.1.7 A Markov chain on the states $0, 1, 2, 3$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ 1 & 0 & 0.3 & 0.3 & 0.4 \\ 2 & 0 & 0 & 0.6 & 0.4 \\ 3 & 1 & 0 & 0 & 0 \end{array}.$$

Determine the corresponding limiting distribution.

4.1.8 Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

.

What fraction of families are upper class in the long run?

4.1.9 Determine the limiting distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{array} \right\| \end{array}.$$

4.1.10 A bus in a mass transit system is operating on a continuous route with intermediate stops. The arrival of the bus at a stop is classified into one of three states, namely

1. Early arrival;
2. On-time arrival;
3. Late arrival.

Suppose that the successive states form a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{ccc} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{array} \right\| \end{array}.$$

Over a long period of time, what fraction of stops can be expected to be late?

Problems

4.1.1 Five balls are distributed between two urns, labeled A and B. Each period, an urn is selected at random, and if it is not empty, a ball from that urn is removed and placed into the other urn. In the long run what fraction of time is urn A empty?

4.1.2 Five balls are distributed between two urns, labeled A and B. Each period, one of the five balls is selected at random, and whichever urn it's in, it is moved to the other urn. In the long run, what fraction of time is urn A empty?

4.1.3 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right\| \end{array},$$

where $\alpha_i \geq 0, i = 1, \ldots, 6$, and $\alpha_1 + \cdots + \alpha_6 = 1$. Determine the limiting probability of being in state 0.

4.1.4 A finite-state regular Markov chain has transition probability matrix $\mathbf{P} = \|P_{ij}\|$ and limiting distribution $\pi = \|\pi_i\|$. In the long run, what fraction of the *transitions* are from a prescribed state k to a prescribed state m?

4.1.5 The four towns A, B, C, and D are connected by railroad lines as shown in the following diagram:

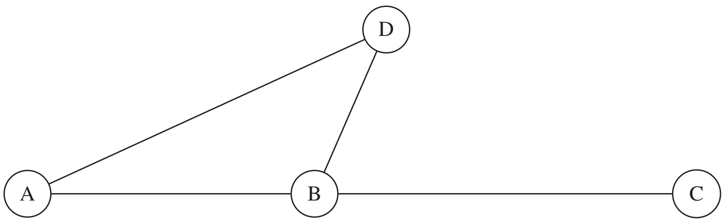

Figure 4.1 A graph whose nodes represent towns and whose arcs represent railroad lines.

Each day, in whichever town it is in, a train chooses one of the lines out of that town at random and traverses it to the next town, where the process repeats the next day. In the long run, what is the probability of finding the train in town D?

4.1.6 Determine the following limits in terms of the transition probability matrix $\mathbf{P} = \|P_{ij}\|$ and limiting distribution $\pi = \|\pi_j\|$ of a finite-state regular Markov chain $\{X_n\}$:

(a) $\lim_{n\to\infty} \Pr\{X_{n+1} = j | X_0 = i\}$.

(b) $\lim_{n\to\infty} \Pr\{X_n = k, X_{n+1} = j | X_0 = i\}$.

(c) $\lim_{n\to\infty} \Pr\{X_{n-1} = k, X_n = j | X_0 = i\}$.

4.1.7 Determine the limiting distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 0 \end{array} \right\| \end{array}.$$

4.1.8 Show that the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{ccccc} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{array} \right\| \end{array}$$

is regular and compute the limiting distribution.

4.1.9 Determine the long run, or limiting, distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \right\| \end{array}.$$

4.1.10 Consider a Markov chain with transition probability matrix

$$\mathbf{P} = \left\| \begin{array}{ccccc} p_0 & p_1 & p_2 & \cdots & p_N \\ p_N & p_0 & p_1 & \cdots & p_{N-1} \\ p_{N-1} & p_N & p_0 & \cdots & p_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_0 \end{array} \right\|,$$

where $0 < p_0 < 1$ and $p_0 + p_1 + \cdots + p_N = 1$. Determine the limiting distribution.

4.1.11 Suppose that a production process changes state according to a Markov process whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.3 & 0.5 & 0 & 0.2 \\ 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \end{array} \right\| \end{array}.$$

It is known that $\pi_1 = \frac{119}{379} = 0.3140$ and $\pi_2 = \frac{81}{379} = 0.2137$.

(a) Determine the limiting probabilities π_0 and π_3.

(b) Suppose that states 0 and 1 are "In-Control" while states 2 and 3 are deemed "Out-of-Control." In the long run, what fraction of time is the process Out-of-Control?

(c) In the long run, what *fraction of transitions* are from an In-Control state to an Out-of-Control state?

4.1.12 Let **P** be the transition probability matrix of a finite-state regular Markov chain, and let Π be the matrix whose rows are the stationary distribution π. Define $\mathbf{Q} = \mathbf{P} - \mathbf{\Pi}$.

(a) Show that $\mathbf{P}^n = \mathbf{\Pi} + \mathbf{Q}^n$.

(b) When

$$\mathbf{P} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix}$$

obtain an explicit expression for $\mathbf{Q}^n$ and then for $\mathbf{P}^n$.

4.1.13 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & \;\;1 & \;\;2 \end{array} \\ \begin{Vmatrix} 0.4 & 0.4 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0.4 \end{Vmatrix} \end{array}.$$

After a long period of time, you observe the chain and see that it is in state 1. What is the conditional probability that the previous state was state 2? That is, find

$$\lim_{n\to\infty} \Pr\{X_{n-1} = 2 | X_n = 1\}.$$

4.2 Examples

Markov chains arising in meteorology, reliability, statistical quality control, and management science are presented next, and the long run behavior of each Markov chain is developed and interpreted in terms of the phenomenon under study.

4.2.1 Including History in the State Description

Often a phenomenon that is not naturally a Markov process can be modeled as a Markov process by including part of the history in the state description. To illustrate this technique, we suppose that the weather on any day depends on the weather conditions for the previous 2 days. To be exact, we suppose that if it was sunny today and yesterday, then it will be sunny tomorrow with probability 0.8; if it was sunny today but cloudy yesterday, then it will be sunny tomorrow with probability 0.6; if it was cloudy today but sunny yesterday, then it will be sunny tomorrow with probability 0.4; if it was cloudy for the last 2 days, then it will be sunny tomorrow with probability 0.1.

Such a model can be transformed into a Markov chain, provided we say that the state at any time is determined by the weather conditions during both that day and the previous day. We say the process is in

State (S, S) if it was sunny both today and yesterday,

State (S, C) if it was sunny yesterday but cloudy today,

State (C, S) if it was cloudy yesterday but sunny today,

State (C, C) if it was cloudy both today and yesterday.

Then, the transition probability matrix is

$$\begin{array}{ll} & \text{Today's state} \\ & \begin{array}{cccc} (S,S) & (S,C) & (C,S) & (C,C) \end{array} \\ \begin{array}{lr} & (S,S) \\ \text{Yesterday's} & (S,C) \\ \text{state} & (C,S) \\ & (C,C) \end{array} & \left\| \begin{array}{cccc} 0.8 & 0.2 & & \\ & & 0.4 & 0.6 \\ 0.6 & 0.4 & & \\ & & 0.1 & 0.9 \end{array} \right\|. \end{array}$$

The equations determining the limiting distribution are

$$\begin{aligned} 0.8\pi_0 \qquad\quad + 0.6\pi_2 \qquad\qquad &= \pi_0, \\ 0.2\pi_0 \qquad\quad + 0.4\pi_2 \qquad\qquad &= \pi_1, \\ 0.4\pi_1 \qquad\quad + 0.1\pi_3 &= \pi_2, \\ 0.6\pi_1 \qquad\quad 0.9\pi_3 &= \pi_3, \\ \pi_0 + \quad \pi_1 + \quad \pi_2 + \quad \pi_3 &= 1. \end{aligned}$$

Again, one of the top four equations is redundant. Striking out the first equation and solving the remaining four equations gives $\pi_0 = \frac{3}{11}, \pi_1 = \frac{1}{11}, \pi_2 = \frac{1}{11}$, and $\pi_3 = \frac{6}{11}$.

We recover the fraction of days, in the long run, on which it is sunny by summing the appropriate terms in the limiting distribution. It can be sunny today in conjunction with either being sunny or cloudy tomorrow. Therefore, the long run fraction of days in which it is sunny is $\pi_0 + \pi_1 = \pi(S, S) + \pi(S, C) = \frac{4}{11}$. Formally, $\lim_{n\to\infty} \Pr\{X_n = S\} = \lim_{n\to\infty}[\Pr\{X_n = S, X_{n+1} = S\} + \Pr\{X_n = S, X_{n+1} = C\}] = \pi_0 + \pi_1$.

4.2.2 Reliability and Redundancy

An airline reservation system has two computers, only one of which is in operation at any given time. A computer may break down on any given day with probability p.

There is a single repair facility that takes 2 days to restore a computer to normal. The facilities are such that only one computer at a time can be dealt with. Form a Markov chain by taking as states the pairs (x, y), where x is the number of machines in operating condition at the end of a day and y is 1 if a day's labor has been expended on a machine not yet repaired and 0 otherwise. The transition matrix is

$$\begin{array}{l} \text{To state} \\ \hline \text{From state} \\ \quad\downarrow \end{array} \rightarrow \begin{array}{cccc} (2,0) & (1,0) & (1,1) & (0,1) \end{array}$$

$$\mathbf{P} = \begin{array}{c} (2,0) \\ (1,0) \\ (1,1) \\ (0,1) \end{array} \left\| \begin{array}{cccc} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right\|,$$

where $p + q = 1$.

We are interested in the long run probability that both machines are inoperative. Let $(\pi_0, \pi_1, \pi_2, \pi_3)$ be the limiting distribution of the Markov chain. Then, the long run probability that neither computer is operating is π_3, and the availability, the probability that at least one computer is operating, is $1 - \pi_3 = \pi_0 + \pi_1 + \pi_2$.

The equations for the limiting distributions are

$$\begin{aligned} q\pi_0 \qquad\quad + q\pi_2 \qquad\quad &= \pi_0, \\ p\pi_0 \qquad\quad + p\pi_2 + \pi_3 &= \pi_1, \\ q\pi_1 \qquad\qquad\qquad &= \pi_2, \\ p\pi_1 \qquad\qquad\qquad &= \pi_3 \end{aligned}$$

and

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1.$$

The solution is

$$\pi_0 = \frac{q^2}{1+p^2}, \quad \pi_2 = \frac{qp}{1+p^2},$$

$$\pi_1 = \frac{p}{1+p^2}, \quad \pi_3 = \frac{p^2}{1+p^2}.$$

The availability is $R_1 = 1 - \pi_3 = 1/\left(1+p^2\right)$.

In order to increase the system availability, it is proposed to add a duplicate repair facility so that both computers can be repaired simultaneously. The corresponding transition matrix is now

$$
\begin{array}{ll}
 & \text{To state} \rightarrow \quad (2,0) \quad (1,0) \quad (1,1) \quad (0,1) \\
 & \text{From state} \downarrow \\
\mathbf{P} = & \begin{array}{c} (2,0) \\ (1,0) \\ (1,1) \\ (0,1) \end{array} \left\| \begin{array}{cccc} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right\|,
\end{array}
$$

and the limiting distribution is

$$
\pi_0 = \frac{q}{1+p+p^2}, \quad \pi_2 = \frac{p}{1+p+p^2},
$$

$$
\pi_1 = \frac{p}{1+p+p^2}, \quad \pi_3 = \frac{p^2}{1+p+p^2}.
$$

Thus, availability has increased to $R_2 = 1 - \pi_3 = (1+p)/\left(1+p+p^2\right)$.

4.2.3 A Continuous Sampling Plan

Consider a production line where each item has probability p of being defective. Assume that the condition of a particular item (defective or nondefective) does not depend on the conditions of other items. The following sampling plan is used.

Initially every item is sampled as it is produced; this procedure continues until i consecutive nondefective items are found. Then, the sampling plan calls for sampling only one out of every r items at random until a defective one is found. When this happens the plan calls for reverting to 100% sampling until i consecutive nondefective items are found. The process continues in the same way.

State $E_k (k = 0, 1, \ldots, i-1)$ denotes that k consecutive nondefective items have been found in the 100% sampling portion of the plan, while state E_i denotes that the plan is in the second stage (sampling one out of r). Time m is considered to follow the mth item, whether sampled or not. Then, the sequence of states is a Markov chain with

$$
\begin{aligned}
P_{jk} &= \Pr\{\text{in state } E_k \text{ after } m+1 \text{ items} | \text{in state } E_j \text{ after } m \text{ items}\} \\
&= \begin{cases} p & \text{for } k = 0, 0 \le j < i, \\ 1-p & \text{for } k = j+1 \le i, \\ \dfrac{p}{r} & \text{for } k = 0, j = i, \\ 1 - \dfrac{p}{r} & \text{for } k = j = i, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Let π_k be the limiting probability that the system is in state E_k for $k = 0, 1, \ldots, i$. The equations determining these limiting probabilities are

$$\begin{array}{llllll}
(0) & p\pi_0 + & p\pi_1 + \cdots + & p\pi_{i-1} + & (p/r)\pi_i & = \pi_0, \\
(1) & (1-p)\pi_0 & & & & = \pi_1, \\
(2) & & (1-p)\pi_1 & & & = \pi_2, \\
\vdots & & & & & \\
(i) & & & (1-p)\pi_{i-1} + & (1-p/r)\pi_i & = \pi_i
\end{array}$$

together with

$$(*) \qquad \pi_0 + \pi_1 + \cdots + \pi_i = 1.$$

From equations (1) through (i), we deduce that $\pi_k = (1-p)\pi_{k-1}$ so that $\pi_k = (1-p)^k\pi_0$ for $k = 0, \ldots, i-1$, while equation (i) yields $\pi_i = (r/p)(1-p)\pi_{i-1}$ or $\pi_i = (r/p)(1-p)^i\pi_0$. Having determined π_k in terms of π_0 for $k = 0, \ldots, i$, we place these values in (*) to obtain

$$\left\{[1 + \cdots + (1-p)^{i-1}] + \frac{r}{p}(1-p)^i\right\}\pi_0 = 1.$$

The geometric series simplifies, and after elementary algebra, the solution is

$$\pi_0 = \frac{p}{1 + (r-1)(1-p)^i},$$

whence

$$\pi_k = \frac{p(1-p)^k}{1 + (r-1)(1-p)^i} \qquad \text{for } k = 0, \ldots, i-1,$$

while

$$\pi_i = \frac{r(1-p)^i}{1 + (r-1)(1-p)^i}.$$

Let AFI (Average Fraction Inspected) denote the long run fraction of items that are inspected. Since each item is inspected while in states $E_0, \ldots, E_{i-1}$ but only one out of r is inspected in state E_i, we have

$$\begin{aligned}
\text{AFI} &= (\pi_0 + \cdots + \pi_{i-1}) + (1/r)\pi_i \\
&= (1 - \pi_i) + (1/r)\pi_i \\
&= \frac{1}{1 + (r-1)(1-p)^i}.
\end{aligned}$$

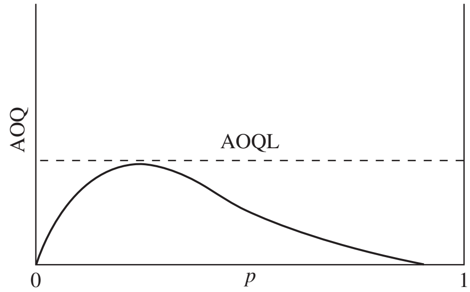

Figure 4.2 The average outgoing quality (AOQ) as a function of the input quality level p.

Figure 4.3 Black-box picture of a continuous inspection scheme as a method of guaranteeing outgoing quality.

Let us assume that each item found to be defective is replaced by an item known to be good. The average outgoing quality (AOQ) is defined to be the fraction of defectives in the output of such an inspection scheme. The average fraction not inspected is

$$1 - \text{AFI} = \frac{(r-1)(1-p)^i}{1+(r-1)(1-p)^i},$$

and of these on the average p are defective. Hence

$$\text{AOQ} = \frac{(r-1)(1-p)^i p}{1+(r-1)(1-p)^i}.$$

This average outgoing quality is zero if $p = 0$ or $p = 1$, and rises to a maximum at some intermediate value, as shown in Figure 4.2. The maximum AOQ is called the *average outgoing quality limit* (AOQL), and it has been determined numerically and tabulated as a function of i and r.

This quality control scheme guarantees an output quality better than the AOQL regardless of the input fraction defective, as shown in Figure 4.3.

4.2.4 Age Replacement Policies

A component of a computer has an active life, measured in discrete units, that is a random variable T, where $\Pr[T = k] = a_k$ for $k = 1, 2, \ldots$. Suppose one starts with a fresh component, and each component is replaced by a new component upon failure. Let X_n be the age of the component in service at time n. Then, (X_n) is a success runs Markov chain. (See Chapter 3, Section 3.5.4.)

In an attempt to replace components before they fail in service, an *age replacement* policy is instituted. This policy calls for replacing the component upon its failure or upon its reaching age N, whichever occurs first. Under this age replacement policy, the Markov chain $\{X_n\}$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & & N-1 \end{array} \\ \left\| \begin{array}{cccccc} p_0 & 1-p_0 & 0 & 0 & \cdots & 0 \\ p_1 & 0 & 1-p_1 & 0 & & 0 \\ p_2 & 0 & 0 & 1-p_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 & & 0 \end{array} \right\| \end{array},$$

where

$$p_k = \frac{a_{k+1}}{a_{k+1} + a_{k+2} + \cdots} \quad \text{for } k = 0, 1, \ldots, N-2.$$

State 0 corresponds to a new component, and therefore, the limiting probability π_0 corresponds to the long run probability of replacement during any single time unit, or the long run replacement per unit time. Some of these replacements are planned or age replacements, and some correspond to failures in service. A planned replacement occurs in each period for which $X_n = N-1$, and therefore, the long run planned replacements per unit time is the limiting probability π_{N-1}. The difference $\pi_0 - \pi_{N-1}$ is the long run rate of failures in service. The equations for the limiting distribution $\pi = (\pi_0, \pi_1, \ldots, \pi_{N-1})$ are

$$\begin{array}{lllll} p_0\pi_0 + & p_1\pi_1 + \cdots + & p_{N-2}\pi_{N-2} + & \pi_{N-1} & = \pi_0, \\ (1-p_0)\pi_0 & & & & = \pi_1, \\ & (1-p_1)\pi_1 & & & = \pi_2, \\ & & & & \vdots \\ & & (1-p_{N-2})\pi_{N-2} & & = \pi_{N-1}, \\ \pi_0 + & \pi_1 + \cdots & & +\pi_{N-1} & = 1. \end{array}$$

Solving in terms of π_0, we obtain

$$\begin{aligned} \pi_0 &= \pi_0, \\ \pi_1 &= (1-p_0)\pi_0, \\ \pi_2 &= (1-p_1)\pi_1 = (1-p_1)(1-p_0)\pi_0, \\ &\vdots \\ \pi_k &= (1-p_{k-1})\pi_{k-1} = (1-p_{k-1})(1-p_{k-2})\cdots(1-p_0)\pi_0, \\ &\vdots \\ \pi_{N-1} &= (1-p_{N-2})\pi_{N-2} = (1-p_{N-2})(1-p_{N-3})\cdots(1-p_0)\pi_0, \end{aligned}$$

and since $\pi_0 + \pi_1 + \cdots + \pi_{N-1} = 1$, we have

$$1 = [1 + (1 - p_0) + (1 - p_0)(1 - p_1) + \cdots \\ + (1 - p_0)(1 - p_1)\cdots(1 - p_{N-2})]\pi_0,$$

or

$$\pi_0 = \frac{1}{1 + (1 - p_0) + (1 - p_0)(1 - p_1) + \cdots + (1 - p_0)(1 - p_1)\cdots(1 - p_{N-2})}.$$

If $A_j = a_j + a_{j+1} + \cdots$ for $j = 1, 2, \ldots,$ where $A_1 = 1$, then $p_k = a_{k+1}/A_{k+1}$ and $1 - p_k = A_{k+2}/A_{k+1}$, which simplifies the expression for π_0 to

$$\pi_0 = \frac{1}{A_1 + A_2 + \cdots + A_N},$$

and then

$$\pi_{N-1} = A_N \pi_0 = \frac{A_N}{A_1 + A_2 + \cdots + A_N}.$$

In practice, one determines the cost C of a replacement and the additional cost K that is incurred when a failure in service occurs. Then, the long run total cost per unit time is $C\pi_0 + K(\pi_0 - \pi_{N-1})$, and the replacement age N is chosen so as to minimize this total cost per unit time.

Observe that

$$\begin{aligned}\frac{1}{\pi_0} = A_1 + A_2 + \cdots + A_N &= \sum_{j=1}^{N} \Pr\{T \geq j\} = \sum_{k=0}^{N-1} \Pr\{T > k\} \\ &= \sum_{k=0}^{\infty} \Pr\{\min\{T, N\} > k\} = E[\min\{T, N\}].\end{aligned}$$

In words, the reciprocal of the mean time between replacements $E[\min\{T, N\}]$ yields the long run replacements per unit time π_0. This relation will be further explored in the chapter on renewal processes.

4.2.5 Optimal Replacement Rules

A common industrial activity is the periodic inspection of some system as part of a procedure for keeping it operative. After each inspection, a decision must be made whether or not to alter the system at that time. If the inspection procedure and the ways of modifying the system are fixed, an important problem is that of determining, according to some cost criterion, the optimal rule for making the appropriate decision. Here, we consider the case in which the only possible act is to replace the system with a new one.

Suppose that the system is inspected at equally spaced points in time and that after each inspection it is classified into one of the $L+1$ possible states $0, 1, \ldots, L$. A system is in state 0 if and only if it is new and is in state L if and only if it is inoperative. Let the inspection times be $n = 0, 1, \ldots,$ and let X_n denote the observed state of the system at time n. In the absence of replacement, we assume that $\{X_n\}$ is a Markov chain with transition probabilities $p_{ij} = \Pr\{X_{n+1} = j | X_n = i\}$ for all i, j, and n.

It is possible to replace the system at any time before failure. The motivation for doing so may be to avoid the possible consequences of further deterioration or of failure of the system. A replacement rule, denoted by R, is a specification of those states at which the system will be replaced. Replacement takes place at the next inspection time. A replacement rule R modifies the behavior of the system and results in a modified Markov chain $\{X_n(R); n = 0, 1, \ldots\}$. The corresponding modified transition probabilities $p_{ij}(R)$ are given by

$$p_{ij}(R) = p_{ij} \quad \text{if the system is not replaced at state } i,$$
$$p_{i0}(R) = 1,$$

and

$$p_{ij}(R) = 0, \quad j \neq 0 \quad \text{if the system is replaced at state } i.$$

It is assumed that each time the equipment is replaced, a replacement cost of K units is incurred. Further it is assumed that each unit of time the system is in state j incurs an operating cost of a_j. Note that a_L may be interpreted as failure (inoperative) cost. This interpretation leads to the one period cost function $c_i(R)$ given for $i = 0, \ldots, L$ by

$$c_i(R) = \begin{cases} a_i & \text{if } p_{i0}(R) = 0, \\ K + a_i & \text{if } p_{i0}(R) = 1. \end{cases}$$

We are interested in replacement rules that minimize the expected long run time average cost. This cost is given by the expected cost under the limiting distribution for the Markov chain $\{X_n(R)\}$. Denoting this average cost by $\phi(R)$, we have

$$\phi(R) = \sum_{i=0}^{L} \pi_i(R) c_i(R),$$

where $\pi_i(R) = \lim_{n\to\infty} \Pr\{X_n(R) = i\}$. The limiting distribution $\pi_i(R)$ is determined by the equations

$$\pi_i(R) = \sum_{k=0}^{L} \pi_k(R) p_{ki}(R), \quad i = 0, \ldots, L,$$

and

$$\pi_0(R) + \pi_1(R) + \cdots + \pi_L(R) = 1.$$

We define a control limit rule to be a replacement rule of the form

"Replace the system if and only if $X_n \geq k$,"

where k, called the control limit, is some fixed state between 0 and L. We let R_k denote the control limit rule with control limit equal to k. Then, R_0 is the rule "Replace the system at every step," and R_L is the rule "Replace only upon failure (State L)."

Control limit rules seem reasonable provided that the states are labeled monotonically from best (0) to worst (L) in some sense. Indeed, it can be shown that a control limit rule is optimal whenever the following two conditions hold:

1. $a_0 \leq a_1 \leq \cdots \leq a_L$.

2. If $i \leq j$, then $\sum_{m=k}^{L} p_{im} \leq \sum_{m=k}^{L} p_{jm}$ for every $k = 0, \ldots, L$.

Condition (1) asserts that the one-stage costs are higher in the "worse" states. Condition (2) asserts that further deterioration is more likely in the "worse" states.

Let us suppose that conditions (1) and (2) prevail. Then, we need only check the $L+1$ control limit rules $R_0, \ldots, R_L$ in order to find an optimal rule. Furthermore, it can be shown that a control limit k^* satisfying $\phi(R_{k^*-1}) \geq \phi(R_{k^*}) \leq \phi(R_{k^*+1})$ is optimal so that not always do all $L+1$ control limit rules need to be checked.

Under control limit k, we have the cost vector

$$c(R_k) = (a_0, \ldots, a_{k-1}, K + a_k, \ldots, K + a_L)$$

and the transition probabilities

$$\mathbf{P}(R_k) = \begin{array}{c} \\ 0 \\ 1 \\ \\ k-1 \\ k \\ \vdots \\ L \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & & k-1 & k & & L \end{array} \\ \left\| \begin{array}{ccccccc} 0 & p_{01} & \cdots & p_{0,k-1} & p_{0k} & \cdots & p_{0L} \\ 0 & p_{11} & \cdots & p_{1,k-1} & p_{1k} & \cdots & p_{1L} \\ & & & & & & \\ 0 & p_{k-1,1} & \cdots & p_{k-1,k-1} & p_{k-1,k} & \cdots & p_{k-1,L} \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} \right\| \end{array}.$$

To look at a numerical example, we will find the optimal control limit k^* for the following data: $L = 5$ and

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.3 \\ 0 & 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0.1 & 0.4 & 0.5 \\ 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array},$$

$a_0 = \cdots = a_{L-1} = 0, a_L = 5$, and $K = 3$. When $k = 1$, the transition matrix is

$$\mathbf{P}(R_1) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\|\begin{array}{cccccc} 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right\| \end{array},$$

which implies the following equations for the stationary distribution:

$$\begin{aligned} \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 &= \pi_0, \\ 0.2\pi_0 &= \pi_1, \\ 0.2\pi_0 &= \pi_2, \\ 0.2\pi_0 &= \pi_3, \\ 0.2\pi_0 &= \pi_4, \\ 0.2\pi_0 &= \pi_5, \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 &= 1. \end{aligned}$$

The solution is $\pi_0 = 0.5$ and $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 0.1$. The average cost associated with $k = 1$ is

$$\begin{aligned} \phi_1 &= 0.5(0) + 0.1(3) + 0.1(3) + 0.1(3) + 0.1(3) + 0.1(3+5) \\ &= 2.0. \end{aligned}$$

When $k = 2$, the transition matrix is

$$\mathbf{P}(R_2) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\|\begin{array}{cccccc} 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right\| \end{array},$$

and the associated stationary distribution is $\pi_0 = 0.450, \pi_1 = 0.100, \pi_2 = \pi_3 = \pi_4 = 0.110, \pi_5 = 0.120$. We evaluate the average cost to be

$$\begin{aligned} \phi_2 &= 0.45(0) + 0.10(0) + 0.11(3) + 0.11(3) + 0.11(3) + 0.12(8) \\ &= 1.95. \end{aligned}$$

Continuing in this manner, we obtain the following table:

Control Limit	**Stationary Distribution**						**Average Cost**
k	π_0	π_1	π_2	π_3	π_4	π_5	ϕ_k
1	0.5000	0.1000	0.1000	0.1000	0.1000	0.1000	2.0000
2	0.4500	0.1000	0.1100	0.1100	0.1100	0.1200	1.9500
3	0.4010	0.0891	0.1089	0.1198	0.1307	0.1505	1.9555
4	0.3539	0.9786	0.0961	0.1175	0.1623	0.1916	2.0197
5	0.2785	0.06189	0.0756	0.0925	0.2139	0.2785	2.2280

The optimal control limit is $k^* = 2$, and the corresponding minimum average cost per unit time is $\phi_2 = 1.95$.

Exercises

4.2.1 On a southern Pacific island, a sunny day is followed by another sunny day with probability 0.9, whereas a rainy day is followed by another rainy day with probability 0.2. Supposing that there are only sunny or rainy days, in the long run on what fraction of days is it sunny?

4.2.2 In the reliability example of Section 4.2.2, what fraction of time is the repair facility idle? When a second repair facility is added, what fraction of time is each facility idle?

4.2.3 Determine the average fraction inspected, AFI, and the average outgoing quality, AOQ, of Section 4.2.3 for $p = 0, 0.05, 0.10, 0.15, \ldots, 0.50$ when

(a) $r = 10$ and $i = 5$.

(b) $r = 5$ and $i = 10$.

4.2.4 Section 4.2.2 determined the availability R of a certain computer system to be

$$R_1 = \frac{1}{1+p^2} \qquad \text{for one repair facility,}$$

$$R_2 = \frac{1+p}{1+p+p^2} \qquad \text{for two repair facilities,}$$

where p is the computer failure probability on a single day. Compute and compare R_1 and R_2 for $p = 0.01, 0.02, 0.05$, and 0.10.

4.2.5 From purchase to purchase, a particular customer switches brands among products A, B, and C according to a Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \end{array} \begin{array}{c} \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \left\| \begin{array}{ccc} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{array} \right\| \end{array}.$$

In the long run, what fraction of time does this customer purchase brand A?

4.2.6 A component of a computer has an active life, measured in discrete units, that is a random variable T where

$$\Pr\{T=1\}=0.1, \quad \Pr\{T=3\}=0.3,$$
$$\Pr\{T=2\}=0.2, \quad \Pr\{T=4\}=0.4.$$

Suppose one starts with a fresh component, and each component is replaced by a new component upon failure. Determine the long run probability that a failure occurs in a given period.

4.2.7 Consider a machine whose condition at any time can be observed and classified as being in one of the following three states:

State 1: Good operating order
State 2: Deteriorated operating order
State 3: In repair

We observe the condition of the machine at the end of each period in a sequence of periods. Let X_n denote the condition of the machine at the end of period n for $n=1,2,\ldots$. Let X_0 be the condition of the machine at the start. We assume that the sequence of machine conditions is a Markov chain with transition probabilities

$$\begin{array}{lll} P_{11}=0.9, & P_{12}=0.1, & P_{13}=0, \\ P_{21}=0, & P_{22}=0.9, & P_{23}=0.1, \\ P_{31}=1, & P_{32}=0, & P_{33}=0, \end{array}$$

and that the process starts in state $X_0=1$.

(a) Find $\Pr\{X_4=1\}$.
(b) Calculate the limiting distribution.
(c) What is the long run rate of repairs per unit time?

4.2.8 At the end of a month, a large retail store classifies each receivable account according to

0: Current
1: 30–60 days overdue
2: 60–90 days overdue
3: Over 90 days

Each such account moves from state to state according to a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.95 & 0.05 & 0 & 0 \\ 0.50 & 0 & 0.50 & 0 \\ 0.20 & 0 & 0 & 0.80 \\ 0.10 & 0 & 0 & 0.90 \end{array} \right\| \end{array}.$$

In the long run, what fraction of accounts are over 90 days overdue?

Problems

4.2.1 Consider a discrete-time periodic review inventory model (see Chapter 3, Section 3.3.1), and let ξ_n be the total demand in period n. Let X_n be the inventory quantity on hand at the end of period n. Instead of following an (s,S) policy, a (q,Q) policy will be used: If the stock level at the end of a period is less than or equal to $q=2$ units, then $Q=2$ additional units will be ordered and will be available at the beginning of the next period. Otherwise, no ordering will take place. This is a (q,Q) policy with $q=2$ and $Q=2$. Assume that demand that is not filled in a period is lost (no back ordering).

(a) Suppose that $X_0=4$ and that the period demands turn out to be $\xi_1=3$, $\xi_2=4, \xi_3=0, \xi_4=2$. What are the end-of-period stock levels for periods $n=1,2,3,4$?

(b) Suppose that $\xi_1,\xi_2,\ldots$ are independent random variables, each having the probability distribution where

$k =$	0	1	2	3	4
$\Pr\{\xi=k\}=$	0.1	0.3	0.3	0.2	0.1

Then, $X_0,X_1,\ldots$ is a Markov chain. Determine the transition probability distribution and the limiting distribution.

(c) In the long run, during what fraction of periods are orders placed?

4.2.2 A system consists of two components operating *in parallel*: The system functions if at least one of the components is operating. In any single period, if both components are operating at the beginning of the period, then each will fail, independently, during the period with probability α. When one component has failed, the remaining component fails during a period with a higher probability β. There is a single repair facility, and it takes two periods to repair a component.

(a) Define an appropriate set of states for the system in the manner of the *Reliability and Redundancy* example and specify the transition probabilities in terms of α and β.

(b) When $\alpha=0.1$ and $\beta=0.2$, in the long run what fraction of time is the system operating?

4.2.3 Suppose that a production process changes state according to a Markov process whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} 0 & \;\;1 & \;\;2 & \;\;3 \end{matrix} \\ \left\| \begin{matrix} 0.2 & 0.2 & 0.4 & 0.2 \\ 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \end{matrix} \right\| \end{array}.$$

(a) Determine the limiting distribution for the process.
(b) Suppose that states 0 and 1 are "In-Control," while states 2 and 3 are deemed "Out-of-Control." In the long run, what fraction of time is the process Out-of-Control?
(c) In the long run, what fraction of transitions are from an In-Control state to an Out-of-Control state?

4.2.4 A component of a computer has an active life, measured in discrete units, that is a random variable ξ, where

$k =$	1	2	3	4
$\Pr\{\xi = k\} =$	0.1	0.3	0.2	0.4

Suppose that one starts with a fresh component, and each component is replaced by a new component upon failure. Let X_n be the *remaining life* of the component in service at the *end* of period n. When $X_n = 0$, a new item is placed into service at the *start* of the next period.
(a) Set up the transition probability matrix for $\{X_n\}$.
(b) By showing that the chain is regular and solving for the limiting distribution, determine the long run probability that the item in service at the end of a period has no remaining life and therefore will be replaced.
(c) Relate this to the mean life of a component.

4.2.5 Suppose that the weather on any day depends on the weather conditions during the previous 2 days. We form a Markov chain with the following states:

State (S, S) if it was sunny both today and yesterday,

State (S, C) if it was sunny yesterday but cloudy today,

State (C, S) if it was cloudy yesterday but sunny today,

State (C, C) if it was cloudy both today and yesterday,

and transition probability matrix

$$\begin{array}{c} \\ \\ \mathbf{P} = \end{array}
\begin{array}{r}
 \\
(S,S) \\ (S,C) \\ (C,S) \\ (C,C)
\end{array}
\begin{array}{c}
\text{Today's state} \\
\begin{array}{cccc} (S,S) & (S,C) & (C,S) & (C,C) \end{array} \\
\left\|\begin{array}{cccc}
0.7 & 0.3 & 0 & 0 \\
0 & 0 & 0.4 & 0.6 \\
0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0.2 & 0.8
\end{array}\right\|.
\end{array}$$

(a) Given that it is sunny on days 0 and 1, what is the probability it is sunny on day 5?
(b) In the long run, what fraction of days are sunny?

4.2.6 Consider a computer system that fails on a given day with probability p and remains "up" with probability $q = 1 - p$. Suppose the repair time is a random variable N having the probability mass function $p(k) = \beta(1-\beta)^{k-1}$ for $k = 1, 2, \ldots$, where $0 < \beta < 1$. Let $X_n = 1$ if the computer is operating on day n and

$X_1 = 0$ if not. Show that $\{X_n\}$ is a Markov chain with transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\|\begin{array}{cc} \alpha & \beta \\ p & q \end{array}\right\| \end{array}$$

and $\alpha = 1 - \beta$. Determine the long run probability that the computer is operating in terms of α, β, p, and q.

4.2.7 Customers arrive for service and take their place in a waiting line. There is a single service facility, and a customer undergoing service at the beginning of a period will complete service and depart at the end of the period with probability β and will continue service into the next period with probability $\alpha = 1 - \beta$, and then the process repeats. This description implies that the service time η of an individual is a random variable with the geometric distribution,

$$\Pr\{\eta = k\} = \beta\alpha^{k-1} \quad \text{for } k = 1, 2, \ldots,$$

and the service times of distinct customers are independent random variables.

At most a single customer can arrive during a period. We suppose that the actual number of arrivals during the nth period is a random variable ξ_n taking on the values 0 or 1 according to

$$\Pr\{\xi_n = 0\} = p$$

and

$$\Pr\{\xi_n = 1\} = q = 1 - p \quad \text{for } n = 0, 1, \ldots.$$

The state X_n of the system at the start of period n is defined to be the number of customers in the system, either waiting or being served. Then, $\{X_n\}$ is a Markov chain. Specify the following transition probabilities in terms of α, β, p, and $q: P_{00}, P_{01}, P_{02}, P_{10}, P_{11}$, and P_{12}. State any additional assumptions that you make.

4.2.8 An airline reservation system has a single computer, which breaks down on any given day with probability p. It takes 2 days to restore a failed computer to normal service. Form a Markov chain by taking as states the pairs (x, y), where x is the number of machines in operating condition at the end of a day and y is 1 if a day's labor has been expended on a machine, and 0 otherwise. The transition probability matrix is

$$\begin{array}{ll} & \begin{array}{l}\text{To state} \\ \hline \end{array} \rightarrow \begin{array}{ccc} (1,0) & (0,0) & (0,1) \end{array} \\ & \text{From state} \\ & \quad\quad\downarrow \\ \mathbf{P} = & \begin{array}{c} (1,0) \\ (0,0) \\ (0,1) \end{array} \left\|\begin{array}{ccc} q & p & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right\|. \end{array}$$

Compute the system availability $\pi_{(1,0)}$ for $p = 0.01, 0.02, 0.05$, and 0.10.

4.3 The Classification of States

Not all Markov chains are regular. We consider some examples.

The Markov chain whose transition probability matrix is the identity matrix

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right\| \end{array}$$

remains always in the state in which it starts. State trivially $\mathbf{P}^n = \mathbf{P}$ for all n, the Markov chain X_n has a limiting distribution, but it obviously depends on the initial state.

The Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right\| \end{array}$$

oscillates deterministically between the two states. The Markov chain is *periodic*, and no limiting distribution exists. When n is an odd number, then $\mathbf{P}^n = \mathbf{P}$, but when n is even, then $\mathbf{P}^n$ is the 2×2 identity matrix.

When $\mathbf{P}$ is the matrix

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{array} \right\| \end{array},$$

$\mathbf{P}^n$ is given by

$$\mathbf{P}^n = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} \left(\frac{1}{2}\right)^n & 1 - \left(\frac{1}{2}\right)^n \\ 0 & 1 \end{array} \right\| \end{array},$$

and the limit is

$$\lim_{n \to \infty} \mathbf{P}^n = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right\| \end{array}.$$

Here, state 0 is *transient*; after the process starts from state 0, there is a positive probability that it will never return to that state.

The three matrices just presented illustrated three distinct types of behavior in addition to the convergence exemplified by a regular Markov chain. Various and more elaborate combinations of these behaviors are also possible. Some definitions and classifications of states and matrices are needed in order to sort out the variety of possibilities.

4.3.1 Irreducible Markov Chains

j is said to be *accessible* from state i if $P_{ij}^{(n)} > 0$ for some integer $n \geq 0$; i.e., state j is accessible from state i if there is positive probability that state j can be reached starting from state i in some finite number of transitions. Two states i and j, each accessible to the other, are said to *communicate*, and we write $i \leftrightarrow j$. If two states i and j do not communicate, then either

$$P_{ij}^{(n)} = 0 \quad \text{for all } n \geq 0$$

or

$$P_{ji}^{(n)} = 0 \quad \text{for all } n \geq 0$$

or both relations are true. The concept of communication is an equivalence relation:

1. $i \leftrightarrow i$ (reflexivity), a consequence of the definition of

$$P_{ij}^{(0)} = \delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

2. If $i \leftrightarrow j$, then $j \leftrightarrow i$ (symmetry), from the definition of communication.

3. If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$ (transitivity).

The proof of transitivity proceeds as follows: $i \leftrightarrow j$ and $j \leftrightarrow k$ imply that there exist integers n and m such that $P_{ij}^{(n)} > 0$ and $P_{jk}^{(m)} > 0$. Consequently, by the nonnegativity of each $P_{rs}^{(t)}$, we conclude that

$$P_{ik}^{(n+m)} = \sum_{r=0}^{\infty} P_{ir}^{(n)} P_{rk}^{(m)} \geq P_{ij}^{(n)} P_{jk}^{(m)} > 0.$$

A similar argument shows the existence of an integer ν such that $P_{ki}^{(\nu)} > 0$, as desired.

We can now partition the totality of states into equivalence classes. The states in an equivalence class are those that communicate with each other. It may be possible starting in one class to enter some other class with positive probability; if so, however, it is clearly not possible to return to the initial class, or else the two classes would together form a single class. We say that the Markov chain is irreducible if the equivalence relation induces only one class. In other words, a process is irreducible if all states communicate with each other.

To illustrate this concept, we consider the transition probability matrix

$$\mathbf{P} = \left\| \begin{array}{cc:ccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{array} \right\| = \left\| \begin{array}{cc} \mathbf{P}_1 & 0 \\ 0 & \mathbf{P}_2 \end{array} \right\|,$$

where $\mathbf{P}_1$ is an abbreviation for the matrix formed from the initial two rows and columns of $\mathbf{P}$, and similarly for $\mathbf{P}_2$. This Markov chain clearly divides into the two classes composed of states {1, 2} and states {3, 4, 5}.

If the state of X_0 lies in the first class, then the state of the system thereafter remains in this class, and for all purposes the relevant transition matrix is $\mathbf{P}_1$. Similarly, if the initial state belongs to the second class, then the relevant transition matrix is P_2. This is a situation where we have two completely unrelated processes labeled together.

In the random walk model with transition matrix

$$\begin{array}{cc} & \text{states} \\ \mathbf{P} = \left\| \begin{array}{cccccccc} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ \vdots & & & & & \vdots & \vdots & \vdots \\ 0 & & & \cdots & & q & 0 & p \\ 0 & & & \cdots & & 0 & 0 & 1 \end{array} \right\| & \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ a-1 \\ a \end{array} \end{array}, \tag{4.15}$$

we have the three classes $\{0\}, \{1, 2, \ldots, a-1\}$, and $\{a\}$. In this example, it is possible to reach the first class or third class from the second class, but it is not possible to return to the second class from either the first or the third class.

4.3.2 Periodicity of a Markov Chain

We define the *period* of state i, written $d(i)$, to be the greatest common divisor (g.c.d.) of all integers $n \geq 1$ for which $P_{ii}^{(n)} > 0$. (If $P_{ii}^{(n)} = 0$ for all $n \geq 1$, define $d(i) = 0$.) In a random walk (4.15), every transient state $1, 2, \ldots, \alpha - 1$ has period 2. If $P_{ii} > 0$ for some single state i, then that state now has period 1, since the system can remain in this state any length of time.

In a finite Markov chain of n states with transition matrix

$$\mathbf{P} = \overbrace{\left\| \begin{array}{cccccc} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & & & \cdots & 1 \\ 1 & 0 & 0 & & \cdots & 0 \end{array} \right\|}^{n},$$

each state has period n.

Consider the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array} \right\| \end{array}.$$

We evaluate $P_{00} = 0, P_{00}^{(2)} = 0, P_{00}^{(3)} = 0, P_{00}^{(4)} = \frac{1}{2}, P_{00}^{(5)} = 0, P_{00}^{(6)} = \frac{1}{4}$. The set of integers $n \geq 1$ for which $P_{00}^{(n)} > 0$ is $\{4, 6, 8, \ldots\}$. The period of state 0 is $d(0) = 2$, the greatest common divisor of this set.

Example Suppose that the precipitation in a certain locale depends on the season (Wet or Dry) as well as on the precipitation level (High or Low) during the preceding season. We model the process as a Markov chain whose states are of the form (x, y), where x denotes the season ($W =$ Wet, $D =$ Dry) and y denotes the precipitation level ($H =$ High, $L =$ Low). Suppose the transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ (W,H) \\ (W,L) \\ (D,H) \\ (D,L) \end{array} \begin{array}{c} \begin{array}{cccc} (W,H) & (W,L) & (D,H) & (D,L) \end{array} \\ \left\| \begin{array}{cccc} 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0.4 & 0.6 \\ 0.7 & 0.3 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \end{array} \right\| \end{array}.$$

All states are periodic with period $d = 2$.

A situation in which the demand for an inventory item depends on the month of the year as well as on the demand during the previous month would lead to a Markov chain whose states had period $d = 12$.

The random walk on the states $0, \pm 1, \pm 2, \ldots$ with probabilities $P_{i,i+1} = p, P_{i,i-1} = q = 1 - p$ is periodic with period $d = 2$.

We state, without proof, three basic properties of the period of a state:

1. If $i \leftrightarrow j$, then $d(i) = d(j)$.

This assertion shows that the period is a constant in each class of communicating states.

2. If state i has period $d(i)$, then there exists an integer N depending on i such that for all integers $n \geq N$,

$$P_{ii}^{(nd(i))} > 0.$$

This asserts that a return to state i can occur at all sufficiently large multiples of the period $d(i)$.

3. If $P_{ji}^{(m)} > 0$, then $P_{ji}^{(m+nd(i))} > 0$ for all n (a positive integer) sufficiently large.

A Markov chain in which each state has period 1 is called *aperiodic*. The vast majority of Markov chain processes we deal with are aperiodic. Results will be developed for the aperiodic case, and the modified conclusions for the general case will be stated, usually without proof.

4.3.3 Recurrent and Transient States

Consider an arbitrary, but fixed, state i. We define, for each integer $n \geq 1$,

$$f_{ii}^{(n)} = \Pr\{X_n = i, X_\nu \neq i, \nu = 1, 2, \ldots, n-1 | X_0 = i\}.$$

In other words, $P_{ii}^{(n)}$ is the probability that starting from state i, the first return to state i occurs at the nth transition. Clearly, $f_{ii}^{(1)} = P_{ii}$, and $f_{ii}^{(n)}$ may be calculated recursively according to

$$P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)}, \quad n \geq 1, \tag{4.16}$$

where we define $f_{ii}^{(0)} = 0$ for all i. Equation (4.16) is derived by decomposing the event from which $P_{ii}^{(n)}$ is computed according to the time of the first return to state i. Indeed, consider all the possible realizations of the process for which $X_0 = i, X_n = i$, and the first return to state i occurs at the kth transition. Call this event E_k. The events $E_k (k = 1, 2, \ldots, n)$ are clearly mutually exclusive. The probability of the event that the first return is at the kth transition is by definition $f_{ii}^{(k)}$. In the remaining $n-k$ transitions, we are dealing only with those realizations for which $X_n = i$. Using the Markov property, we have

$$\begin{aligned}\Pr\{E_k\} &= \Pr\{\text{first return is at } k\text{th transition} | X_0 = i\} \Pr\{X_n = i | X_k = i\} \\ &= f_{ii}^{(k)} P_{ii}^{(n-k)}, \quad 1 \leq k \leq n\end{aligned}$$

(recall that $P_{ii}^{0} = 1$). Hence,

$$\Pr\{X_n = i | X_0 = i\} = \sum_{k=1}^{n} \Pr\{E_k\} = \sum_{k=1}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)},$$

since by definition $f_{ii}^{(0)} = 0$. The verification of (4.16) is now complete.

When the process starts from state i, the probability that it returns to state i at some time is

$$f_{ii} = \sum_{n=0}^{\infty} f_{ii}^{(n)} = \lim_{N \to \infty} \sum_{n=0}^{N} f_{ii}^{(n)}. \tag{4.17}$$

We say that a state i is *recurrent* if $f_{ii} = 1$. This definition says that a state i is recurrent if and only if, after the process starts from state i, the probability of its returning to state i after some finite length of time is one. A nonrecurrent state is said to be *transient*.

Consider a transient state i. Then, the probability that a process starting from state i returns to state i at least once is $f_{ii} < 1$. Because of the Markov property, the probability that the process returns to state i at least twice is $(f_{ii})^2$, and repeating the argument, we see that the probability that the process returns to i at least k times is $(f_{ii})^k$ for $k = 1, 2, \ldots$. Let M be the random variable that counts the number of times that the process returns to i. Then, we have shown that M has the geometric distribution in which

$$\Pr\{M \geq k | X_0 = i\} = (f_{ii})^k \quad \text{for } k = 1, 2, \ldots \tag{4.18}$$

and

$$E[M|X_0 = i] = \frac{f_{ii}}{1 - f_{ii}}. \tag{4.19}$$

Theorem 4.2 establishes a criterion for the recurrence of a state i in terms of the transition probabilities $P_{ii}^{(n)}$.

Theorem 4.2. *A state i is recurrent if and only if*

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty.$$

Equivalently, state i is transient if and only if $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$.

Proof. Suppose first that state i is transient so that, by definition, $f_{ii} < 1$, and let M count the total number of returns to state i. We write M in terms of indicator random variables as

$$M = \sum_{n=1}^{\infty} 1\{X_n = i\},$$

where

$$\mathbf{1}\{X_n = i\} = \begin{cases} 1 & \text{if } X_n = i, \\ 0 & \text{if } X_n \neq i. \end{cases}$$

Now, equation (3.5) shows that $E[M|X_0 = i] < \infty$ when i is transient. But then

$$\begin{aligned} \infty > E[M|X_0 = i] &= \sum_{n=1}^{\infty} E[\mathbf{1}\{X_n = i\}|X_0 = i] \\ &= \sum_{n=1}^{\infty} P_{ii}^{(n)}, \end{aligned}$$

as claimed. ∎

Conversely, suppose $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$. Then, M is a random variable whose mean is finite, and thus, M must be finite. That is, starting from state i, the process returns to state i only a finite number of times. Then, there must be a positive probability that, starting from state i, the process never returns to that state. In other words, $1 - f_{ii} > 0$ or $f_{ii} < 1$, as claimed.

Corollary 4.1. *If $i \leftrightarrow j$ and if i is recurrent, then j is recurrent.*

Proof. Since $i \leftrightarrow j$, there exists $m, n \geq 1$ such that

$$P_{ij}^{(n)} > 0 \quad \text{and} \quad P_{ji}^{(m)} > 0.$$

Let $\nu > 0$. We obtain, by the usual argument (see Section 4.3.1), $P_{jj}^{(m+n+\nu)} \geq P_{ji}^{(m)} P_{ii}^{(\nu)} P_{ij}^{(n)}$ and, on summing,

$$\sum_{\nu=0}^{\infty} P_{jj}^{(m+n+\nu)} \geq \sum_{\nu=0}^{\infty} P_{ji}^{(m)} P_{ii}^{(\nu)} P_{ij}^{(n)} = P_{ji}^{(m)} P_{ij}^{(n)} \sum_{\nu=0}^{\infty} P_{ii}^{(\nu)}.$$

Hence, if $\sum_{\nu=0}^{\infty} P_{ii}^{(\nu)}$ diverges, then $\sum_{\nu=0}^{\infty} P_{jj}^{(\nu)}$ also diverges. ■

This corollary proves that recurrence, like periodicity, is a class property; that is, all states in an equivalence class are either recurrent or nonrecurrent.

Example Consider the one-dimensional random walk on the positive and negative integers, where at each transition the particle moves with probability p one unit to the right and with probability q one unit to the left $(p + q = 1)$. Hence,

$$P_{00}^{(2n+1)} = 0, \quad n = 0, 1, 2, \ldots,$$

and

$$P_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!\,n!} p^n q^n. \tag{4.20}$$

We appeal now to Stirling's formula (see Chapter 1, (1.60)),

$$n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi}. \tag{4.21}$$

Applying (4.21) to (4.20), we obtain

$$P_{00}^{(2n)} \sim \frac{(pq)^n 2^{2n}}{\sqrt{\pi n}} = \frac{(4pq)^n}{\sqrt{\pi n}}.$$

It is readily verified that $p(1-p) = pq \le \frac{1}{4}$, with equality holding if and only if $p = q = \frac{1}{2}$. Hence, $\sum_{n=0}^{\infty} P_{00}^{(n)} = \infty$ if and only if $p = \frac{1}{2}$. Therefore, from Theorem 4.2, the one-dimensional random walk is recurrent if and only if $p = q = \frac{1}{2}$. Remember that recurrence is a class property. Intuitively, if $p \ne q$, there is positive probability that a particle initially at the origin will drift to $+\infty$ if $p > q$ (to $-\infty$ if $p < q$) without ever returning to the origin.

Exercises

4.3.1 A Markov chain has a transition probability matrix

$$\begin{array}{c|cccccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
4 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}.$$

Find the equivalence classes. For which integers $n = 1, 2, \ldots, 20$, is it true that

$$P_{00}^{(n)} > 0?$$

What is the period of the Markov chain?

Hint: One need not compute the actual probabilities. See Section 4.1.1.

4.3.2 Which states are transient and which are recurrent in the Markov chain whose transition probability matrix is

$$\begin{array}{c|cccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\
1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\
4 & 0 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}?$$

4.3.3 A Markov chain on states $\{0, 1, 2, 3, 4, 5\}$ has transition probability matrix

$$\text{(a)} \quad \begin{Vmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{Vmatrix}.$$

$$\text{(b)} \quad \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{7}{8} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & \frac{3}{8} & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

Find all communicating classes; which classes are transient and which are recurrent?

4.3.4 Determine the communicating classes and period for each state of the Markov chain whose transition probability matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{Vmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{Vmatrix} \end{array}.$$

Problems

4.3.1 A two-state Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|cc|} & 0 & 1 \\ \hline 0 & 1-a & a \\ 1 & b & 1-b \end{array}.$$

(a) Determine the first return distribution

$$f_{00}^{(n)} = \Pr\{X_1 \neq 0, \ldots, X_{n-1} \neq 0, X_n = 0 | X_0 = 0\}.$$

(b) Verify equation (4.16) when $i = 0$. (Refer to Chapter 3, (4.40).)

4.3.2 Show that a finite-state aperiodic irreducible Markov chain is regular and recurrent.

4.3.3 Recall the first return distribution (Section 4.3.3),

$$f_{ii}^{(n)} = \Pr\{X_1 \neq i, X_2 \neq j \ldots, X_{n-1} \neq i, X_n = i | X_0 = i\} \quad \text{for } n = 1, 2, \ldots,$$

with $f_{ii}^{(0)} = 0$ by convention. Using equation (4.16), determine $f_{00}^{(n)}, n = 1, 2, 3, 4, 5$, for the Markov chain whose transition probability matrix is

$$\begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}.$$

4.4 The Basic Limit Theorem of Markov Chains

Consider a recurrent state i. Then,

$$f_{ii}^{(n)} = \Pr\{X_n = i, X_\nu \neq i \quad \text{for} \quad \nu = 1, \ldots, n-1 | X_0 = i\} \tag{4.22}$$

is the probability distribution of the *first return time*

$$R_i = \min\{n \geq 1; X_n = i\}. \tag{4.23}$$

This is

$$f_{ii}^{(n)} = \Pr\{R_i = n | X_0 = i\} \quad \text{for } n = 1, 2, \ldots. \tag{4.24}$$

Since state i is recurrent by assumption, then $f_{ii} = \sum_{n=1}^{\infty} = f_{ii}^{(n)} = 1$, and R_i is a finite-valued random variable. The mean duration between visits to state i is

$$m_i = E[R_i|X_0 = i] = \sum_{n=1}^{\infty} n f_{ii}^{(n)}. \tag{4.25}$$

After starting in i, then, on the average, the process is in state i once every $m_i = E[R_i|X_0 = i]$ units of time. The basic limit theorem of Markov chains states this result in a sharpened form.

Theorem 4.3. *The basic limit theorem of Markov chains*

(a) Consider a recurrent irreducible aperiodic Markov chain. Let $P_{ii}^{(n)}$ be the probability of entering state i at the nth transition, $n = 0, 1, 2, \ldots,$ given that $X_0 = i$ (the initial state is i). By our earlier convention $P_{ii}^{(0)} = 1$. Let $f_{ii}^{(n)}$ be the probability of first returning to state i at the nth transition, $n = 0, 1, 2, \ldots,$ where $f_{ii}^{(0)} = 0$. Then,

$$\lim_{n\to\infty} P_{ii}^{(n)} = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^{(n)}} = \frac{1}{m_i}. \tag{4.26}$$

(b) under the same conditions as in (a), $\lim_{n\to\infty} P_{ji}^{(n)} = \lim_{n\to\infty} P_{ii}^{(n)}$ for all states j.

Remark Let C be a recurrent class. Then, $P_{ij}^{(n)} = 0$ for $i \in C, j \notin C$, and every n. Hence, once in C, it is not possible to leave C. It follows that the submatrix $\|P_{ij}\|, i, j \in C$, is a transition probability matrix and the associated Markov chain is irreducible and recurrent. The limit theorem, therefore, applies verbatim to any aperiodic recurrent class.

If $\lim_{n\to\infty} P_{ii}^{(n)} > 0$ for one i in an aperiodic recurrent class, then $\pi_j > 0$ for all j in the class of i. In this case, we call the class *positive recurrent* or strongly ergodic. If each $\pi_i = 0$ and the class is recurrent, we speak of the class as *null recurrent* or weakly ergodic. In terms of the first return time $R_i = \min\{n \geq 1; X_n = i\}$, state i is positive recurrent if $m_i = E[R_i|X_0 = i] < \infty$ and null recurrent if $m_i = \infty$. This statement is immediate from the equality $\lim_{n\to\infty} P_{ii}^{(n)} = \pi_i = 1/m_i$. An alternative method for determining the limiting distribution π_i for a positive recurrent aperiodic class is given in Theorem 4.4.

Theorem 4.4. *In a positive recurrent aperiodic class with states $j = 0, 1, 2, \ldots,$*

$$\lim_{n\to\infty} P_{jj}^{(n)} = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad \sum_{i=0}^{\infty} \pi_i = 1,$$

and the π's are uniquely determined by the set of equations

$$\pi_i \geq 0, \quad \sum_{i=0}^{\infty} \pi_i = 1, \quad \textit{and} \quad \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad \textit{for } j = 0, 1, \ldots. \tag{4.27}$$

Any set $(\pi_i)_{i=0}^{\infty}$ satisfying (4.27) is called a *stationary probability distribution* of the Markov chain. The term "stationary" derives from the property that a Markov chain started according to a stationary distribution will follow this distribution at all points of time. Formally, if $\Pr\{X_0 = i\} = \pi_i$, then $\Pr\{X_n = i\} = \pi_i$ for all $n = 1, 2, \ldots$. We check this for the case $n = 1$; the general case follows by induction. We write

$$\begin{aligned}\Pr\{X_1 = i\} &= \sum_{k=0}^{\infty} \Pr\{X_0 = k\} \Pr\{X_1 = i | X_0 = k\} \\ &= \sum_{k=0}^{\infty} \pi_k P_{ki} = \pi_i,\end{aligned}$$

where the last equality follows because $\pi = (\pi_0, \pi_1, \ldots)$ is a stationary distribution. When the initial state X_0 is selected according to the stationary distribution, then the joint probability distribution of (X_n, X_{n+1}) is given by

$$\Pr\{X_n = i, X_{n+1} = j\} = \pi_i P_{ij}.$$

The reader should supply the proof.

A limiting distribution, when it exists, is always a stationary distribution, but the converse is not true. There may exist a stationary distribution but no limiting distribution. For example, there is no limiting distribution for the periodic Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix},$$

but $\pi = (\frac{1}{2}, \frac{1}{2})$ is a stationary distribution, since

$$\left(\frac{1}{2}, \frac{1}{2}\right) \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Example Consider the class of random walks whose transition matrices are given by

$$\mathbf{P} = \|P_{ij}\| = \begin{Vmatrix} 0 & 1 & 0 & & \cdots \\ q_1 & 0 & p_1 & & \cdots \\ 0 & q_2 & 0 & p_2 & \cdots \\ \vdots & & & & \end{Vmatrix}.$$

This Markov chain has period 2. Nevertheless, we investigate the existence of a stationary probability distribution; that is, we wish to determine the positive solutions of

$$x_i = \sum_{j=0}^{\infty} x_j P_{ji} = p_{i-1} x_{i-1} + q_{i+1} x_{i+1}, \quad i = 0, 1, \ldots, \tag{4.28}$$

under the normalization

$$\sum_{i=0}^{\infty} x_i = 1,$$

where $p_{-1} = 0$ and $p_0 = 1$, and thus, $x_0 = q_1 x_l$. Using equation (4.28) for $i = 1$, we could determine x_2 in terms of x_0. Equation (4.28) for $i = 2$ determines x_3 in terms of x_0, and so forth. It is immediately verified that

$$x_i = \frac{p_{i-1}p_{i-2}\cdots p_1}{q_i q_{i-1}\cdots q_1} x_0 = x_0 \prod_{k=0}^{i-1} \frac{p_k}{q_{k+1}}, \quad i \geq 1,$$

is a solution of (4.28), with x_0 still to be determined. Now, since

$$1 = x_0 + \sum_{i=1}^{\infty} x_0 \prod_{k=0}^{i-1} \frac{p_k}{q_{k+1}},$$

we have

$$x_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \prod_{k=0}^{1-1} \frac{p_k}{q_{k+1}}},$$

and so

$$x_0 > 0 \text{ if and only if } \sum_{i=1}^{\infty} \prod_{k=0}^{i-1} \frac{p_k}{q_{k+1}} < \infty.$$

In particular, if $p_k = p$ and $q_k = q = 1 - p$ for $k \geq 1$, the series

$$\sum_{i=1}^{\infty} \prod_{k=0}^{i-1} \frac{p_k}{q_{k+1}} = \frac{1}{p} \sum_{i=1}^{\infty} \left(\frac{p}{q}\right)^i$$

converges only when $p < q$, and then

$$\frac{1}{p} \sum_{i=1}^{\infty} \left(\frac{p}{q}\right)^i = \frac{1}{p} \frac{p/q}{1 - p/q} = \frac{1}{q - p},$$

and

$$x_0 = \frac{1}{1 + 1/(q-p)} = \frac{q-p}{1+q-p} = \frac{1}{2}\left(1 - \frac{p}{q}\right),$$

$$x_k = \frac{1}{p}\left(\frac{p}{q}\right)^k x_0 = \frac{1}{2p}\left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^k \quad \text{for } k = 1, 2, \ldots.$$

Example Consider now the Markov chain that represents the success runs of binomial trials. The transition probability matrix is

$$\begin{Vmatrix} p_0 & 1-p_0 & 0 & 0 & \cdots \\ p_1 & 0 & 1-p_1 & 0 & \cdots \\ p_1 & 0 & 0 & 1-p_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{Vmatrix} \qquad (0 < p_k < 1).$$

The states of this Markov chain all belong to the same equivalence class (any state can be reached from any other state). Since recurrence is a class property (see Corollary 4.1), we will investigate recurrence for the zeroth state.

Let $R_0 = \min\{n \geq 1; X_n = 0\}$ be the time of first return to state 0. It is easy to evaluate

$$\begin{aligned} \Pr\{R_0 > 1|X_0 = 0\} &= (1-p_0), \\ \Pr\{R_0 > 2|X_0 = 0\} &= (1-p_0)(1-p_1), \\ \Pr\{R_0 > 3|X_0 = 0\} &= (1-p_0)(1-p_1)(1-p_2), \\ &\vdots \\ \Pr\{R_0 > k|X_0 = 0\} &= (1-p_0)(1-p_1)\cdots(1-p_{k-1}) = \prod_{i=0}^{k-1}(1-p_i). \end{aligned}$$

In terms of the first return distribution

$$f_{00}^{(n)} = \Pr\{R_0 = n|X_0 = 0\},$$

we have

$$\Pr\{R_0 > k|X_0 = 0\} = 1 - \sum_{n=1}^{k} f_{00}^{(n)},$$

or

$$\sum_{n=1}^{k} f_{00}^{(n)} = 1 - \Pr\{R_0 > k|X_0 = 0\} = 1 - \prod_{i=0}^{k-1}(1-p_i).$$

By definition, state 0 is recurrent provided $\sum_{n=1}^{\infty} f_{00}^{(n)} = 1$. In terms of $p_0, p_1, \ldots$ then, state 0 is recurrent whenever $\lim_{k\to\infty} \Pi_{i=0}^{k-1}(1-p_i) = \Pi_{i=0}^{\infty}(1-p_i) = 0$. Lemma 4.1 shows that $\Pi_{i=0}^{\infty}(1-p_i) = 0$ is equivalent, in this case, to the condition $\sum_{i=0}^{\infty} p_i = \infty$.

Lemma 4.1. *If* $0 < p_i < 1, i = 0, 1, 2, \ldots,$ *then* $u_m = \Pi_{i=0}^{m}(1-p_l) \to 0$ *as* $m \to \infty$ *if and only if* $\sum_{i=0}^{\infty} p_i = \infty$.

Proof. Assume $\sum_{i=0}^{\infty} p_i = \infty$. Since the series expansion for $\exp(-p_i)$ is an alternating series with terms decreasing in absolute value, we can write

$$1 - p_i < 1 - p_i + \frac{p_i^2}{2!} - \frac{p_i^3}{3!} + \cdots = \exp(-p_i), \quad i = 0, 1, 2, \ldots. \tag{4.29}$$

Since (4.29) holds for all i, we obtain $\Pi_{i=0}^{m}(1 - p_i) < \exp\left(-\sum_{i=0}^{m} p_i\right)$. But by assumption,

$$\lim_{m\to\infty} \sum_{i=0}^{m} p_i = \infty;$$

hence,

$$\lim_{m\to\infty} \prod_{i=0}^{m} (1 - p_i) = 0.$$

To prove necessity, observe that from a straightforward induction,

$$\prod_{i=j}^{m} (1 - p_i) > (1 - p_j - p_{j+1} - \cdots - p_m)$$

for any j and all $m = j+1, j+2, \ldots$. Assume now that $\sum_{i=1}^{\infty} p_i < \infty$; then, $0 < \sum_{i=j}^{\infty} p_i < 1$ for some $j > 1$. Thus,

$$\lim_{m\to\infty} \prod_{i=j}^{m} (1 - p_i) > \lim_{m\to\infty} \left(1 - \sum_{i=j}^{m} p_i\right) > 0,$$

which contradicts $u_m \to 0$. ■

State 0 is recurrent when $\Pi_{i=0}^{\infty}(1 - p_i) = 0$, or equivalently, when $\sum_{i=0}^{\infty} p_i = \infty$. The state is positive recurrent when $m_0 = E[R_0|X_0 = 0] < \infty$. But

$$\begin{aligned} m_0 &= \sum_{k=0}^{\infty} \Pr\{R_0 > k|X_0 = 0\} \\ &= 1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} (1 - p_i). \end{aligned}$$

Thus, positive recurrence requires the stronger condition that $\sum_{k=1}^{\infty} \Pi_{i=0}^{k-1}(1 - p_i) < \infty$, and in this case, the stationary probability π_0 is given by

$$\pi_0 = \frac{1}{m_0} = \frac{1}{1 + \sum_{k=1}^{\infty} \Pi_{i=0}^{k-1}(1 - p_i)}.$$

From the equations for the stationary distribution, we have

$$
\begin{aligned}
(1-p_0)\pi_0 &= \pi_1, \\
(1-p_1)\pi_1 &= \pi_2, \\
(1-p_2)\pi_2 &= \pi_3, \\
&\vdots
\end{aligned}
$$

or

$$
\begin{aligned}
\pi_1 &= (1-p_0)\pi_0, \\
\pi_2 &= (1-p_1)\pi_1 = (1-p_1)(1-p_0)\pi_0, \\
\pi_3 &= (1-p_2)\pi_2 = (1-p_2)(1-p_1)(1-p_0)\pi_0,
\end{aligned}
$$

and, in general,

$$\pi_k = \pi_0 \prod_{i=0}^{k-1}(1-p_i) \quad \text{for } k \geq 1.$$

In the special case where $p_i = p = 1-q$ for $i = 0, 1, \ldots,$ then $\Pi_{i=0}^{k-1}(1-p_i) = q^k$,

$$m_0 = 1 + \sum_{k=1}^{\infty} q^k = \frac{1}{p}$$

so that $\pi_k = pq^k$ for $k = 0, 1, \ldots$.

Remark Suppose $a_0, a_1, a_2, \ldots$ is a convergent sequence of real numbers where $a_n \to a$ as $n \to \infty$. Then, it can be proved by elementary methods that the partial averages of the sequence also converge in the form

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = a. \tag{4.30}$$

Applying (4.30) with $a_n = P_{ii}^{(n)}$, where i is a member of a positive recurrent aperiodic class, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} P_{ii}^{(m)} = \pi_i = \frac{1}{m_i} > 0, \tag{4.31}$$

where $\pi = (\pi_0, \pi_1, \ldots)$ is the stationary distribution and where m_i is the mean return time for state i. Let $M_i^{(n)}$ be the random variable that counts the total number of visits to state i during time periods $0, 1, \ldots, n-1$. We may write

$$M_i^{(n)} = \sum_{k=0}^{n-1} \mathbf{1}\{X_k = i\}, \tag{4.32}$$

where

$$\mathbf{1}\{X_k = i\} = \begin{cases} 1 & \text{if } X_k = i \\ 0 & \text{if } X_k \neq i \end{cases} \tag{4.33}$$

and then see that

$$E\left[M_i^{(n)}|X_0 = i\right] = \sum_{k=0}^{n-1} E[\mathbf{1}\{X_k = i\}|X_0 = i] = \sum_{k=0}^{n-1} P_{ii}^{(k)}. \tag{4.34}$$

Then, referring to (4.31), we have

$$\lim_{n\to\infty} \frac{1}{n} E\left[M_i^{(n)}|X_0 = i\right] = \frac{1}{m_i}. \tag{4.35}$$

In words, the long run ($n \to \infty$) mean visits to state i per unit time equals π_i, the probability of state i under the stationary distribution.

Next, let $r(i)$ define a cost or rate to be accumulated upon each visit to state i. The total cost accumulated during the first n stages is

$$\begin{aligned} R^{(n-1)} &= \sum_{k=0}^{n-1} r(X_k) = \sum_{k=0}^{n-1} \sum_{i} \mathbf{1}\{X_k = i\} r(i) \\ &= \sum_{i=0}^{\infty} M_i^{(n)} r(i). \end{aligned} \tag{4.36}$$

This leads to the following derivation showing that the long run mean cost per unit time equals the mean cost evaluated over the stationary distribution:

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} E[R^{(n-1)}|X_0 = i] &= \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{1}{n} E\left[M_i^{(n)}|X_0 = i\right] r(i) \\ &= \sum_{i=0}^{\infty} \pi_i r(i). \end{aligned} \tag{4.37}$$

(When the Markov chain has an infinite number of states, then the derivation requires that a limit and infinite sum be interchanged. A sufficient condition to justify this interchange is that $r(i)$ be a bounded function of i.)

Remark *The Periodic Case* If i is a member of a recurrent periodic irreducible Markov chain with period d, one can show that $P_{ii}^m = 0$ if m is not a multiple of d (i.e., if $m \neq nd$ for any n), and that

$$\lim_{n\to\infty} P_{ii}^{nd} = \frac{d}{m_i}.$$

These last two results are easily combined with (4.30) to show that (4.31) also holds in the periodic case. If $m_i < \infty$, then the chain is positive recurrent and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} P_{ii}^{(m)} = \pi_i = \frac{1}{m_i}, \tag{4.38}$$

where $\pi = (\pi_0, \pi_1, \ldots)$ is given as the unique nonnegative solution to

$$\pi_j = \sum_{k=0} \pi_k P_{kj}, \quad j = 0, 1, \ldots,$$

and

$$\sum_{j=0}^{\infty} \pi_j = 1.$$

That is, a unique stationary distribution $\pi = (\pi_0, \pi_1, \ldots)$ exists for a positive recurrent periodic irreducible Markov chain, and the mean fraction of time in state i converges to π_i as the number of stages n grows to infinity.

The convergence of (4.38) does not require the chain to start in state i. Under the same conditions,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} P_{ki}^{(m)} = \pi_i = \frac{1}{m_i}$$

holds for all states $k = 0, 1, \ldots$ as well.

Exercises

4.4.1 Determine the limiting distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{ccccc} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 \end{array} \right\| \end{array},$$

where $p > 0, q > 0$, and $p + q = 1$.

4.4.2 Consider the Markov chain whose transition probability *matrix* is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0.1 & 0.4 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 & 0 \end{array} \right\| \end{array}$$

(a) Determine the limiting probability π_0 that the process is in state 0.
(b) By pretending that state 0 is absorbing, use a first step analysis (Chapter 3, Section 3.4) and calculate the mean time m_{10} for the process to go from state 1 to state 0.
(c) Because the process always goes directly to state 1 from state 0, the mean *return* time to state 0 is $m_0 = 1 + m_{10}$. Verify equation (4.26), $\pi_0 = 1/m_0$.

4.4.3 Determine the stationary distribution for the periodic Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array} \right\| \end{array}.$$

Problems

4.4.1 Consider the Markov chain on {0, 1} whose transition probability matrix is

$$\begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 1-\alpha & \alpha \\ \beta & 1-\beta \end{array} \right\| \end{array}, \quad 0 < \alpha, \beta < 1.$$

(a) Verify that $(\pi_0, \pi_1) = (\beta/(\alpha+\beta), \alpha/(\alpha+\beta))$ is a stationary distribution.
(b) Show that the first return distribution to state 0 is given by $f_{00}^{(1)} = (1-\alpha)$ and $f_{00}^{(n)} = \alpha\beta(1-\beta)^{n-2}$ for $n = 2, 3, \ldots$.
(c) Calculate the mean return time $m_0 = \sum_{n=1}^{\infty} n f_{00}^{(n)}$ and verify that $\pi_0 = 1/m_0$.

4.4.2 Determine the stationary distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 2 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 3 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \end{array}.$$

4.4.3 Consider a random walk Markov chain on state $0, 1, \ldots, N$ with transition probability matrix

$$\begin{array}{c|cccccccccc|} & 0 & 1 & 2 & 3 & 4 & 5 & & N-1 & N \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & q_1 & 0 & p_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & q_2 & 0 & p_2 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 0 & 0 & q_3 & 0 & p_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ N-1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & p_{N-1} \\ N & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{array},$$

where $p_i + q_i = 1, p_i > 0, q_i > 0$ for all i.

The transition probabilities from state 0 and N "reflect" the process back into state $1, 2, \ldots, N-1$. Determine the limiting distribution.

4.4.4 Let $\{\alpha_i : i = 1, 2, \ldots\}$ be a probability distribution, and consider the Markov chain whose transition probability matrix is

$$\begin{array}{c|ccccccc|} & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}.$$

What condition on the probability distribution $\{\alpha_i : i = 1, 2, \ldots\}$ is necessary and sufficient in order that a limiting distribution exist, and what is this limiting distribution? Assume $\alpha_1 > 0$ and $\alpha_2 > 0$ so that the chain is aperiodic.

4.4.5 Let P be the transition probability matrix of a finite-state regular Markov chain. Let $\mathbf{M} = \|m_{ij}\|$ be the matrix of mean *return* times.

(a) Use a first step argument to establish that

$$m_{ij} = 1 + \sum_{k \neq j} P_{ik} m_{kj}.$$

(b) Multiply both sides of the preceding by π_i and sum to obtain

$$\sum_i \pi_i m_{ij} = \sum_i \pi_i + \sum_{k \neq j} \sum_i \pi_i P_{ik} m_{kj}.$$

Simplify this to show (see equation (4.26))

$$\pi_j m_{jj} = 1, \quad \text{or } \pi_j = 1/m_{jj}.$$

4.4.6 Determine the period of state 0 in the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{r} \\ 3 \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ -3 \\ -4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \end{array} \\ \left\| \begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right\| \end{array}.$$

4.4.7 An individual either drives his car or walks in going from his home to his office in the morning, and from his office to his home in the afternoon. He uses the following strategy: If it is raining in the morning, then he drives the car, provided it is at home to be taken. Similarly, if it is raining in the afternoon and his car is at the office, then he drives the car home. He walks on any morning or afternoon that it is not raining or the car is not where he is. Assume that, independent of the past, it rains during successive mornings and afternoons with constant probability p. In the long run, on what fraction of *days* does our man walk in the rain? What if he owns two cars?

4.4.8 A Markov chain on states $0, 1, \ldots$ has transition probabilities

$$P_{ij} = \frac{1}{i+2} \quad \text{for } j = 0, 1, \ldots, i, i+1.$$

Find the stationary distribution.

4.5 Reducible Markov Chains*

Recall that states i and j *communicate* if it is possible to reach state j starting from state i, and vice versa, and a Markov chain is *irreducible* if all pairs of states communicate. In this section, we show, mostly by example, how to analyze more general Markov chains.

Consider first the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{Vmatrix}.$$

Which we write in the form

$$\mathbf{P} = \begin{Vmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{Vmatrix},$$

where

$$\mathbf{P} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{Vmatrix} \quad \text{and} \quad \mathbf{P} = \begin{Vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{Vmatrix}$$

The chain has two communicating classes, the first two states forming one class and the last two states forming the other. Then,

$$\mathbf{P}^2 = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{Vmatrix} \times \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{Vmatrix}$$

* The overwhelming majority of Markov chains encountered in stochastic modeling are irreducible. Reducible Markov chains form a specialized topic.

$$= \begin{Vmatrix} \frac{3}{8} & \frac{5}{8} & 0 & 0 \\ \frac{5}{16} & \frac{11}{16} & 0 & 0 \\ 0 & 0 & \frac{5}{9} & \frac{4}{9} \\ 0 & 0 & \frac{4}{9} & \frac{5}{9} \end{Vmatrix} = \begin{Vmatrix} \mathbf{P}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2^2 \end{Vmatrix},$$

and, in general,

$$\mathbf{P}^n = \begin{Vmatrix} \mathbf{P}_1^n & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2^n \end{Vmatrix}, \quad n \geq 1. \tag{4.39}$$

Equation (4.39) is the mathematical expression of the property that it is not possible to communicate back and forth between distinct communicating classes; once in the first class, the process remains there thereafter; and similarly, once in the second class, the process remains there. In effect, two completely unrelated processes have been labeled together. The transition probability matrix $\mathbf{P}$ is reducible to the irreducible matrices $\mathbf{P}_1$ and $\mathbf{P}_2$. It follows from (4.39) that

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{Vmatrix} \pi_0^{(1)} & \pi_1^{(1)} & 0 & 0 \\ \pi_0^{(1)} & \pi_1^{(1)} & 0 & 0 \\ 0 & 0 & \pi_0^{(2)} & \pi_1^{(2)} \\ 0 & 0 & \pi_0^{(2)} & \pi_1^{(2)} \end{Vmatrix},$$

where

$$\lim_{n\to\infty} \mathbf{P}_1^n = \begin{Vmatrix} \pi_0^{(1)} & \pi_1^{(1)} \\ \pi_0^{(1)} & \pi_1^{(1)} \end{Vmatrix} \quad \text{and} \quad \lim_{n\to\infty} \mathbf{P}_2^n = \begin{Vmatrix} \pi_0^{(2)} & \pi_1^{(2)} \\ \pi_0^{(2)} & \pi_1^{(2)} \end{Vmatrix}.$$

We solve for $\pi^{(1)} = \left(\pi_0^{(1)}, \pi_1^{(1)}\right)$ and $\pi^{(2)} = \left(\pi_0^{(2)}, \pi_1^{(2)}\right)$ in the usual way:

$$\frac{1}{2}\pi_0^{(1)} + \frac{1}{4}\pi_1^{(1)} = \pi_0^{(1)},$$

$$\frac{1}{2}\pi_0^{(1)} + \frac{3}{4}\pi_1^{(1)} = \pi_1^{(1)},$$

$$\pi_0^{(1)} + \pi_1^{(1)} = 1,$$

or

$$\pi_0^{(1)} = \frac{1}{3}, \quad \pi_1^{(1)} = \frac{2}{3}, \tag{4.40}$$

and because $\mathbf{P}_2$ is doubly stochastic (see Section 4.1.1), it follows that $\pi_0^{(2)} = \frac{1}{2}$, $\pi_1^{(2)} = \frac{1}{2}$.

The basic limit theorem of Markov chains, Theorem 4.3, referred to an irreducible Markov chain. The limit theorem applies verbatim to any aperiodic recurrent class in a reducible Markov chain. If i,j are in the same aperiodic recurrent class, then $P_{ij}^{(n)} \to 1/m_j \geq 0$ as $n \to \infty$. If i,j are in the same periodic recurrent class, then $n^{-1}\sum_{m=0}^{n-1} P_{ij}^{(m)} \to 1/m_j \geq 0$ as $n \to \infty$.

If j is a transient state, then $P_{jj}^{(n)} \to 0$ as $n \to \infty$, and, more generally, $P_{ij}^{(n)} \to 0$ as $n \to \infty$ for all initial states i.

In order to complete the discussion of the limiting behavior of $P_{ij}^{(n)}$, we still must consider the case where i is transient and j is recurrent. Consider the transition probability matrix

$$\mathbf{P} = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 2 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 3 & 0 & 0 & 0 & 1 \end{array}.$$

There are three classes: $\{0, 1\}, \{2\}$, and $\{3\}$, and of these, $\{0, 1\}$ and $\{3\}$ are recurrent, while $\{2\}$ is transient. Starting from state 2, the process ultimately gets absorbed in one of the other classes. The question is, Which one? or more precisely, What are the probabilities of absorption in the two recurrent classes starting from state 2?

A first step analysis answers the question. Let u denote the probability of absorption in class $\{0, 1\}$ starting from state 2. Then, $1-u$ is the probability of absorption in class $\{3\}$. Conditioning on the first step, we have

$$u = \left(\frac{1}{4} + \frac{1}{4}\right) 1 + \frac{1}{4}u + \frac{1}{4}(0) = \frac{1}{2} + \frac{1}{4}u,$$

or $u = \frac{2}{3}$. With probability $\frac{2}{3}$, the process enters $\{0, 1\}$ and remains there ever after. The stationary distribution for the recurrent class $\{0, 1\}$, computed in (5.2), is $\pi_0 = \frac{1}{3}, \pi_1 = \frac{2}{3}$. Therefore, $\lim_{n\to\infty} P_{20}^{(n)} = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}, \lim_{n\to\infty} P_{21}^{(n)} = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$. That is, we multiply the probability of entering the class $\{0, 1\}$ by the appropriate probabilities under the stationary distribution for the various states in the class. In matrix form, the limiting behavior of $\mathbf{P}^n$ is given by

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{Vmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{9} & \frac{4}{9} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

To firm up the principles, consider one last example:

$$\mathbf{P}^n = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\Vert \begin{array}{cc|cc|cc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \hline \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{6} \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right\Vert \end{array}.$$

There are three classes: $C_1 = \{0,1\}, C_2 = \{2,3\}$, and $C_3 = \{4,5\}$. The stationary distribution in C_1 is (π_0, π_1), where

$$\begin{aligned} \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 &= \pi_0, \\ \frac{1}{2}\pi_0 + \frac{2}{3}\pi_1 &= \pi_1, \\ \pi_0 + \pi_1 &= 1. \end{aligned}$$

Then, $\pi_0 = \frac{2}{5}$ and $\pi_1 = \frac{3}{5}$.

Class C_3 is periodic, and $P_{ij}^{(n)}$ does not converge for i,j in $C_3 = \{4,5\}$. The time averages do converge, however; and $\lim_{n\to\infty} n^{-1}\sum_{m=0}^{n-1} P_{ij}^{(m)} = \frac{1}{2}$ for $i = 3,4$ and $j = 3,4$.

For the transient class $C_2 = \{2, 3\}$, let u_i be the probability of ultimate absorption in class $C_1 = \{0, 1\}$ starting from state i for $i = 2, 3$. From a first step analysis, then

$$u_2 = \frac{1}{3}(1) + 0(1) + 0u_2 + \frac{1}{3}u_3 + \frac{1}{6}(0) + \frac{1}{6}(0),$$
$$u_3 = \frac{1}{6}(1) + \frac{1}{6}(1) + \frac{1}{6}u_2 + 0u_3 + \frac{1}{3}(0) + \frac{1}{6}(0),$$

or

$$u_2 = \frac{1}{3}\frac{1}{3}u_3; \quad u_3 = \frac{1}{3} + \frac{1}{6}u_2.$$

The solution is $u_2 = \frac{8}{17}$ and $u_3 = \frac{7}{17}$. Combining these partial answers in matrix form, we have

$$\lim_{n\to\infty}\mathbf{P} = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left\| \begin{array}{cccccc} \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ \left(\frac{8}{17}\right)\left(\frac{2}{5}\right) & \left(\frac{8}{17}\right)\left(\frac{3}{5}\right) & 0 & 0 & \mathrm{X} & \mathrm{X} \\ \left(\frac{7}{17}\right)\left(\frac{2}{5}\right) & \left(\frac{7}{17}\right)\left(\frac{3}{5}\right) & 0 & 0 & \mathrm{X} & \mathrm{X} \\ 0 & 0 & 0 & 0 & \mathrm{X} & \mathrm{X} \\ 0 & 0 & 0 & 0 & \mathrm{X} & \mathrm{X} \end{array} \right\| \end{array},$$

where X denotes that the limit does not exist. For the time average, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}\mathbf{P}^m = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left\| \begin{array}{cccccc} \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ \left(\frac{8}{17}\right)\left(\frac{2}{5}\right) & \left(\frac{8}{17}\right)\left(\frac{3}{5}\right) & 0 & 0 & \left(\frac{9}{17}\right)\left(\frac{1}{2}\right) & \left(\frac{9}{17}\right)\left(\frac{1}{2}\right) \\ \left(\frac{7}{17}\right)\left(\frac{2}{5}\right) & \left(\frac{7}{17}\right)\left(\frac{3}{5}\right) & 0 & 0 & \left(\frac{10}{17}\right)\left(\frac{1}{2}\right) & \left(\frac{10}{17}\right)\left(\frac{1}{2}\right) \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \right\| \end{array}.$$

One possible behavior remains to be illustrated. It can occur only when there are an infinite number of states. In this case, it is possible that all states are transient or null recurrent and $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ for all states i, j. For example, consider the deterministic Markov chain described by $X_n = X_0 + n$. The transition probability matrix is

$$\mathbf{P}^n = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 1 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 1 & \cdots \\ 3 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}.$$

Then, all states are transient, and $\lim_{n\to\infty} P_{ij}^{(n)} = \lim_{n\to\infty} \Pr\{X_n = j|X_0 = i\} = 0$ for all states i, j.

If there is only a finite number M of states, then there are no null recurrent states and not all states can be transient. In fact, since $\sum_{j=0}^{M-1} P_{ij}^{(n)} = 1$ for all n, it cannot happen that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ for all j.

Exercises

4.5.1 Given the transition matrix

$$\mathbf{P} = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \end{array},$$

determine the limits, as $n \to \infty$, of $P_{i0}^{(n)}$ for $i = 0, 1, \ldots, 4$.

4.5.2 Given the transition matrix

$$\mathbf{P} = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 2 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{4} & \frac{1}{4} \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array},$$

derive the following limits, where they exist:

(a) $\lim_{n\to\infty} P_{11}^{(n)}$ (e) $\lim_{n\to\infty} P_{21}^{(n)}$

(b) $\lim_{n\to\infty} P_{31}^{(n)}$ (f) $\lim_{n\to\infty} P_{33}^{(n)}$

(c) $\lim_{n\to\infty} P_{61}^{(n)}$ (g) $\lim_{n\to\infty} P_{67}^{(n)}$

(d) $\lim_{n\to\infty} P_{63}^{(n)}$ (h) $\lim_{n\to\infty} P_{64}^{(n)}$

Problems

4.5.1 Describe the limiting behavior of the Markov chain whose transition probability matrix is

$$\begin{array}{c|cccccccc|} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 0.1 & 0.1 & 0.2 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 1 & 0 & 0.1 & 0.1 & 0.1 & 0 & 0.3 & 0.2 & 0.2 \\ 2 & 0.6 & 0 & 0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 \\ 3 & 0 & 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0.7 & 0.3 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.4 & 0.3 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0.9 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0.2 & 0 \end{array}.$$

Hint: First consider the matrices

$$
\mathbf{P}_A = \begin{array}{c|ccccc|}
 & 0 & 1 & 2 & 3-4 & 5-7 \\
\hline
0 & 0.1 & 0.1 & 0.2 & 0.3 & 0.3 \\
1 & 0 & 0.1 & 0.1 & 0.1 & 0.7 \\
2 & 0.6 & 0 & 0 & 0.2 & 0.2 \\
3-4 & 0 & 0 & 0 & 1 & 0 \\
5-7 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

and

$$
\mathbf{P}_B = \begin{array}{c|cc|}
 & 3 & 4 \\
\hline
3 & 0.3 & 0.7 \\
4 & 0.7 & 0.3
\end{array}, \qquad
\mathbf{P}_C = \left\|\begin{array}{ccc}
5 & 6 & 7 \\
0.3 & 0.4 & 0.3 \\
0.1 & 0 & 0.9 \\
0.8 & 0.2 & 0
\end{array}\right\|.
$$

4.5.2 Determine the limiting behavior of the Markov chain whose transition probability matrix is

$$
\mathbf{P} = \begin{array}{c|cccccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 \\
1 & 0 & 0.1 & 0.2 & 0.1 & 0 & 0.3 & 0.1 & 0.2 \\
2 & 0.5 & 0 & 0 & 0.2 & 0.1 & 0.1 & 0.1 & 0 \\
3 & 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0.6 & 0.4 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.4 & 0.3 \\
6 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.6 \\
7 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0
\end{array}.
$$