Tutorial 8
STAT 150

Exercise 1 :

Let $S_n = X_1 + ... + X_n$ where the X_i are independent with $\mathbb{E}[X_i] = 0$ and $\text{Var}(X_i) = \sigma^2$.
(a) Show that $S_n^2 - n\sigma^2$ is a martingale.
(b) Let $\tau = \min\{n : |S_n| > a\}$. Show that $\mathbb{E}[\tau] \geq \frac{a^2}{\sigma^2}$. (Hint: use Fatou's Lemma: for any sequence of positive r.v's X_n we have $\mathbb{E}[\lim_{n\to\infty} X_n] \leq \lim_{n\to\infty} \mathbb{E}[X_n]$)

Exercise 2 :

Let $Y_1, .., Y_n$ a sequence of i.i.d random variables such that $\mathbb{P}[Y_i = 0] = 1 - p$ and $\mathbb{P}[Y_i = 1] = p$. We define $M_n = \dfrac{1}{p^n} \displaystyle\prod_{i=1}^{n} Y_i$
(a) Prove that M_n is a martingale with respect to Y_n.
(b) Show that M_n converges to a limit M_∞ and prove that this limit is almost surely equal to 0.

Exercise 3 :

Consider a Markov chain $(X_n)_n$ with state space S. Let $A \subset S$ and suppose that $C = S - A$ is a finite set. Let $V_A = \min\{n \geq 0 : X_n \in A\}$ be the time of the first visit to A. Suppose that $g(x) = 0$ for $x \in A$, while for $x \in C$ we have $g(x) = 1 + \displaystyle\sum_y p(x, y)g(y)$

(a) Show that $g(X_{V_A \wedge n}) + (V_A \wedge n)$ is a martingale.
(b) Prove that for all $x \in C$ we have $g(x) = \mathbb{E}_x[V_A]$.

Exercise 4 :

Let $\xi_1, \xi_2, ..$ be independent with $\mathbb{P}(\xi_i = 1) = p$ and $\mathbb{P}(\xi_i = -1) = q = 1 - p$ where $p < \frac{1}{2}$.

Let $S_n = S_0 + \xi_1 + .. + \xi_n$ and $V_0 = \min\{n \geq 0 : S_n = 0\}$.

(a) Let ϕ be the Laplace transform of ξ_1 (i.e : $\phi(\theta) = \mathbb{E}[e^{\theta \xi_1}]$). Show that $\frac{e^{\theta S_n}}{\phi(\theta)^n}$ is a martingale.

(b) Using the optimal stopping theorem show that if $\theta \leq 0$ and $x > 0$ then $e^{\theta x} = \mathbb{E}_x[\phi(\theta)^{-V_0}]$.

(c) Let $0 < s < 1$. Solve the equation $\phi(\theta) = \frac{1}{s}$, then use the previous question to conclude that the generating function of V_0 has the following form :

$$\mathbb{E}_x[s^{V_0}] = \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2ps}\right)^x$$