Midterm Test 1 — Key

Stat 150 — Fall 2018

1. **[3 marks]**

   Consider the MC $(X_n : n \geq 0)$ on $S = \{0, 1, 2, 3\}$ such that $p_{0,0}, p_{0,1}, p_{1,1}, p_{1,2}, p_{2,1}, p_{2,3}$ are all equal to $1/2$, $p_{3,3} = 1$ and all other $p_{i,j} = 0$. The expected number of flips until THH is observed is equal to $\mathbb{E}(T_3|X_0 = 0)$, where $T_3 = \min\{n \geq 0 : T_n = 3\}$. Let $f_x = \mathbb{E}(T_3|X_0 = x)$. By FSA, $f_0 = 1 + (f_0 + f_1)/2$, $f_1 = 1 + (f_1 + f_2)/2$ and $f_2 = 1 + f_1/2$. Solving, we find that the expected number of flips is $f_0 = 8$.

2. **[5 marks]**

   This problem was done in Tutorial #5. Let $m = |S| - 1$. Since all $p_{ij} = \frac{1}{m} \frac{w_j}{w_i + w_j} > 0$ for $i \neq j$ and $p_{i,i} = \frac{1}{m} \sum_{j \neq i} \frac{w_i}{w_i + w_j} > 0$, (X_n) is IRR. In particular, since all $p_{i,i} > 0$, (X_n) is APER. As shown in the tutorial, $w = (w_i, i \in S)$ satisfies $w = wP$. Since S is finite and all $w_i > 0$, we have $\sum_j w_j < \infty$. Hence $\pi = (\pi_i = w_i / \sum_j w_j, i \in S)$ is a SD. Since (X_n) is IRR and has a SD, (X_n) is POS REC. So, by the Main MC Thm, π is its LD. Hence all $\lim_{n\to\infty} \mathbb{P}(X_n = i) = w_i / \sum_j w_j$.

3. **[4 marks]**

   From any given corner a on the cube, there are 3 corners at distance 1 from a, 3 corners at distance 2 from a, and 1 vertex at distance 3 from a (the diametrically opposite corner from a). By symmetry, it suffices to consider a MC on $S = \{0, 1, 2, 3\}$ with $p_{0,1} = 1$, $p_{1,0} = 1/3$, $p_{1,2} = 2/3$, $p_{2,1} = 2/3$, $p_{2,3} = 1/3$ and $p_{3,3} = 1$. Let $T_3 = \min\{n \geq 0 : X_n = 3\}$. Then the expected time for the particle to reach the diametrically opposite corner is equal to $\mathbb{E}(T_3|X_0 = 0)$. Let $f_x = (T_3|X_0 = x)$. By FSA, $f_0 = 1 + f_1$, $f_1 = 1 + f_0/3 + 2f_2/3$ and $f_2 = 1 + 2f_1/3$. Hence the expected time to reach the opposite corner is $f_0 = 10$.

4. (a) **[2 marks]**

   Let u be the probability that $X_n = 0$ eventually. If $\mu \leq 1$ then $u = 1$. If $\mu > 1$ then $u < 1$. Moreover u is the smallest solution in $[0, 1]$ to $\phi(u) = u$, where $\phi(s) = \mathbb{E}(s^\xi)$ is the PGF of ξ.

   (b) **[3 marks]**

   Note $\mathbb{E}(X_0) = 1$ and $\mathbb{E}(X_1) = \mu$. By induction, we claim $\mathbb{E}(X_n) = \mu^n$. Indeed, suppose that $\mathbb{E}(X_m) = \mu^m$ holds for all $m < n$. Then since $X_n|X_{n-1}$ is the sum of X_{n-1} many IID copies of ξ, and so $\mathbb{E}(X_n|X_{n-1}) = \mu X_{n-1}$, we find that $\mathbb{E}(X_n) = \mathbb{E}[\mathbb{E}(X_n|X_{n-1})] = \mu\mathbb{E}(X_{n-1}) = \mu^n$, completing the induction. Hence $\mathbb{E}(\sum_{n=0}^{\infty} X_n) = 1/(1 - \mu)$ for $\mu < 1$ and $\mathbb{E}(\sum_{n=0}^{\infty} X_n) = \infty$ for $\mu \geq 1$.

   (c) **[5 marks]**

   The reasoning here is similar to HW#3, Problem #6. We observe that between visits to state 0, this MC is equivalent to a "slowed down" BP with ξ distributed as a shifted Geometric(q), where we let particles die and give birth one a time. The total progeny of the BP gives the number of steps between visits. Note that $\mathbb{E}(\xi) = 1/q - 1 = p/q$. Hence, by (a) and (b), we find that (X_n) is TRANS, NULL REC and POS REC when $p > q$, $p = q$ and $p < q$.

(d) **[4 marks]**

First, since $p_{i,i+1} > 0$ for all $i \geq 0$ and $p_{i,i-1} > 0$ for all $i \geq 1$, (X_n) is clearly IRR. Since $p_{0,0} > 0$, (X_n) is APER. For $p < q$ it is POS REC and so by the Main MC Thm, (X_n) has a LD, $\pi = \pi P$. In particular, the LR proportion of time in state 1 is π_1. We solve for $w = wP$ with $w_0 = 1$ and then normalize $\pi = w/\sum_j w_j$. Note $w = wP$ implies that $w_0 = w_0 q + w_1 q$, $w_1 = w_0 qp + w_1 qp + w_2 q = w_0 p + w_2 q$, $w_2 = w_1 p + w_3 q$, and so on. Hence $w_0 = 1$, $w_1 = p/q$, $w_2 = (p/q - p)/q = (p/q)^2$, $w_3 = [(p/q)^2 - p(p/q)]/q = (p/q)^3$, and in general $w_i = (p/q)^i$. Since $p < q$ and so $\sum_{j=0}^{\infty}(p/q)^i = 1/(1-p/q) < \infty$, $\pi_i = (p/q)^i(1-p/q)$ is the LD, and the LR proportion of time in state 1 is $\pi_1 = (p/q)/(1-p/q)$.

5. **[4 marks]**

Let X_n be the number of white balls in the first urn after n steps. This is a B&D MC on state space $S = \{0, 1, \ldots, k\}$. Since $p_{0,0} > 0$, $p_{i,i+1} > 0$ for $0 \leq i < k$ and $p_{i,i-1} > 0$ for $0 < i \leq k$, (X_n) is IRR and APER. Since S is finite and (X_n) is IRR, (X_n) is POS REC. So by the Main MC Thm, its SD $\pi = \pi P$ is its LD. The LR proportion of time that all of the white balls are in one of the urns is $\pi_0 + \pi_k$. Since (X_n) is a B&D MC its SD π satisfies DB: $\pi_i p_{ij} = \pi_j p_{ji}$ for all i, j. By symmetry we expect that, in the limit, when $X_n = i$, the first urn is equally likely to contain any given i of the k white balls and any given $k-i$ of the k black balls. That is, we expect $\pi_i = \binom{k}{i}\binom{k}{k-i}/\binom{2k}{k} = \binom{k}{i}^2/\binom{2k}{k}$. We check that DB holds for this choice of π. Note that it suffices to check that $\pi_i p_{i,i+1} = \pi_{i+1} p_{i+1,i}$ for $0 \leq i < k$, as the equations hold trivially for all other i, j. Note that $p_{i,i+1} = (k-i)^2/k^2$ and $p_{i+1,i} = (i+1)^2/k^2$. Hence $\binom{2k}{k} k^2 \pi_i p_{i,i+1} = \binom{k}{i}^2 (k-i)^2 = \frac{k!^2}{i!^2(k-i-1)!^2} = \binom{k}{i+1}^2 (i+1)^2 = \binom{2k}{k} k^2 \pi_{i+1} p_{i+1,i}$, as required.

Bonus **[4 marks]**

Consider the MC (X_n) on all 52! orderings of the deck. The MC is clearly IRR (to get to some given ordering, select a card that is behind a card that it should not be, until the deck is in the desired order) and APER (with positive probability we select the top card, and then the ordering does not change). To show that the LD is uniform, we note that its transition matrix P is doubly stochastic, and apply the Main MC Thm. To see this note that from a given ordering $c_1, c_2, \ldots, c_{52}$ the MC transitions to any of the following 52 orderings with equal probability 1/52:

$c_1, c_2, \ldots, c_{52}$,

$c_2, c_1, c_3, c_4, \ldots, c_{52}$,

$c_3, c_1, c_2, c_4, c_5, \ldots, c_{52}$,

$\ldots$,

$c_{52}, c_1, c_2 \ldots, c_{51}$.

Likewise, if the deck is currently ordered $c_1, c_2, \ldots, c_{52}$, then it was necessarily in one of the following 52 orderings in the previous state:

$c_1, c_2, \ldots, c_{52}$,

$c_2, c_1, c_3, c_4, \ldots, c_{52}$,

$c_2, c_3, c_1, c_4, c_5, \ldots, c_{52}$,

$\ldots$,

$c_2, c_3, \ldots, c_{52}, c_1$.

Moreover, the MC transitions from each of these states to $c_1, c_2, \ldots, c_{52}$ with equal probability 1/52. Altogether, P is doubly stochastic, as claimed.