Tutorial 6

STAT 150

Exercise 1 (*Question 4 of the midterm*)

Let (X_n) be a branching process with $X_0 = 1$ and offspring distribution ξ such that $\mu = \mathbb{E}[\xi] < \infty$ and $\mathbb{P}[\xi = 1] \neq 1$.

(a) Find $\mathbb{E}[X_n]$ and then use it to calculate the mean of the total population $\sum_{n=0}^{\infty} X_n$.

(b) Let $p \in (0, 1)$, and denote $q = 1 - p$. Consider the Markov chain $(Y_n : n \geq 0)$ on state space $S = \{0, 1, 2, \dots\}$ with transition probabilities

$$p_{i,j} = \begin{cases} qp^j & \text{if } i = 0, j \geq 0 \\ qp^{j-i+1} & \text{if } i \geq 1, j \geq i - 1 \end{cases}$$

and all other $p_{i,j} = 0$. Using a "slowed down" version of a branching process, determine the values of p for which (Y_n) is transient, null recurrent or positive recurrent.

(c) For p such that (Y_n) is positive recurrent, find the long run proportion of time spent in state 1.

Exercise 2

Customers arrive at a store according to a Poisson process with rate λ. Each customer remains in the store for an amount of time that is distributed as an exponential random variable with rate μ, independently of all other customers. Let $t = 0$ be the time when the store opens (at which point there are not yet any customers in the store). Let N_t be the number of customers that have arrived by time t and S_t the number of customers in the store at time t.

(a) Given that the first customer arrives at time $t = 1$, find the expected number of customers which arrive by $t = 2$.

(b) Find $\mathbb{E}[S_t | N_t]$ and then use this to compute $\mathbb{E}[S_t]$, the expected number of customers in the store at time t.

Exercise 3

Let $X(t)$ be a Poisson process of rate λ, and W_n the time of the nth event.

Find the joint distribution of W_1 and W_2.