Homework 7 : Solutions
STAT 150

Problem 8.1.5 :

Let S_n be a simple random walk, and let $M_n = \max_{0\leq k\leq n} S_k$ and $Y_n = M_n - S_n$. We define :

$$\tau := \min\{n \geq 0 : Y_n = a\}$$

(a) We can see that :

$$\mathbb{P}(M_\tau = 0) = \mathbb{P}(S_n \text{ reaches } -a \text{ before } 1\ ) = \tfrac{1}{1+a}$$

(b) The event $\{M_\tau \geq 2\}$ means that we have reached 1 before $-a$ first, and then starting afresh from 1 we reach 2 before $-a+1$, those two event have the same probability due to the spatial homogeneity and are independents due to the Markov property. Thus :

$$\mathbb{P}(M_\tau \geq 2) = (\mathbb{P}(M_\tau \geq 1))^2 = (\tfrac{a}{a+1})^2$$

The same argument repeated may times gives us : $\mathbb{P}(M_\tau \geq k) = (\frac{a}{a+1})^k$, hence M_τ has a geometric distribution with parameter $\frac{1}{1+a}$.

(c) The brownian motion has the same distribution as a the scaled random walk $(\frac{S_{nt}}{\sqrt{n}}, 0 \leq t \leq 1)$, thus by replacing a by $\sqrt{n}a$ we get :

$\mathbb{P}(M_\tau > x) = \lim_{n\to\infty} \mathbb{P}(M_\tau^n > \sqrt{n}x) = \lim_{n\to\infty} (\frac{\sqrt{n}a}{1+\sqrt{n}a})^{\sqrt{n}x} = e^{-\frac{x}{a}}$.

So M_τ has an exponential distribution with rate $\frac{1}{a}$.

Problem 8.2.3 :

We have $\mathbb{P}(M(t) > a) = 2(1-\Phi_t(a)) = \mathbb{P}(|B_t| > a)$, so $M(t)$ has the same distribution as $|B_t|$. However as we always have $M_t \geq M_s$ but not necessarly $|B_t| \geq |B_s|$, then the joint distributions are not equal.

$$\mathbb{E}[M_t] = \mathbb{E}[\sqrt{t}|N(0,1)|] = \sqrt{\tfrac{2t}{\pi}}$$

Problem 8.2.4 :

By considering the process reflected after attaining the point τ_z with respect to the line $y = z$, and seeing that it has the same distribution as the initial Brownian motion, then for $0 < x < z$:

$$\begin{aligned}\mathbb{P}(M_t \geq z, B_t \leq x) &= \mathbb{P}(\tau_z \leq t, B_t \leq x)\\ &= \mathbb{P}(2z - B_t \leq x)\\ &= \mathbb{P}(B_t \geq 2z - x)\\ &= 1 - \Phi(\frac{2z - x}{\sqrt{t}})\end{aligned}$$

Hence by differentiating with respect to x and z respectively we get the following joint density :

$$f_{M(t),B(t)}(z, x) = \frac{2z - x}{t} \frac{2}{\sqrt{t}} \phi(\frac{2z - x}{\sqrt{t}})$$

Problem 8.2.5 :

It suffices to use the variable change $(z, x) \to (z, z-x) = (z, y)$ in the last density to get that the density of the couple $(M(t), M(t) - B(t))$ is equal to :

$$f_{M(t),B(t)-M(t)}(z, y) = \frac{z + y}{t} \frac{2}{\sqrt{t}} \phi(\frac{z + y}{\sqrt{t}})$$

Problem 8.2.6 :

It is clear that the last density is symmetric for z and y, that means that $Y(t) = M(t) - B(t)$ has the same distribution as $M(t)$ which also has the same distribution as $|B(t)|$ from the Problem 8.2.3.

Problem 8.3.3 :

Because $B(t)$ is a Gaussian process, any two linear combinations of vectors of this process are independent if and only if they are uncorrelated. Now it suffices to see that for $t \leq 1$ we have:

$$\text{Cov}(B_t - tB_1, B_1) = t - t = 0$$

(a) We have :

$$\mathbb{P}(B_t \leq x | B_1 = 0) = \mathbb{P}(B_t - tB_1 \leq x | B_1 = 0) = \mathbb{P}(B_t - tB_1 \leq x)$$

Thus $B_t^0 = B_t - tB_1$ is a Brownian bridge.

(b) We have for $0 < s < t < 1$:

$$\mathrm{Cov}(B_t^0, B_s^0) = \mathrm{Cov}(B_t - tB_1, B_s - sB_1) = s - st - st + st = s(1-t)$$

Problem 8.3.4 :

Let $0 < s < t < 1$, we have :

$$\begin{aligned} \mathrm{Cov}(W^0(s), W^0(t)) &= \mathrm{Cov}((1-s)B(\frac{s}{1-s}, (1-t)B(\frac{t}{1-t}) \\ &= (1-s)(1-t)\frac{s}{1-s} \\ &= s(1-t) \end{aligned}$$

It is the same covariance as the Brownian bridge.

Problem 8.4.1 :

The probability that the Brownian motion hits the line $a + bt$ is the same as the limit of the probability that the Brownian motion with drift $-b$ hits a before $-N$ where N goes to infinity. Using the formula (8.40) we get that this probability is equal to :

$$\mathbb{P}(\max_{t \geq 0} (B_t - bt) \geq a) = e^{-2ab}$$

Problem 8.4.2 :

It suffices to see that using the problem 4.1 we have :

$$\mathbb{P}(\max_{t \geq 0} (\tfrac{b+B_t}{1+t}) \geq a) = \mathbb{P}((\max_{t \geq 0} (B_t - at) \geq a - b) = e^{-2a(a-b)}$$

Problem 8.4.3 :

We have :

$$\begin{aligned}\mathbb{P}(\max_{0\le u\le 1} B^0(u) > a) &= \mathbb{P}(\max_{0\le u\le 1} (1-u)B(\frac{u}{1-u}) > a)\\ &= \mathbb{P}(\max_{u>0} (\frac{1}{u+1})B(u) > a)\\ &= \mathbb{P}(\max_{u>0} (B(u) - au) > a)\\ &= e^{-2a^2}\end{aligned}$$

Problem 2 :

It suffices to see that $B_t + B_s = (B_t - B_s) + 2B_s$, as both $B_t - B_s$ and $2B_s$ are both normal random variables with mean 0 and variance ($t - s$ and $4s$ respectively), then their sum is also a normal random variable with mean 0 and variance $t + 3s$.

Problem 3 :

(a) It is clear that $X(0) = 0$ and that the increments of $(X(t))_{t\ge 0}$ are independent and we have $X(t) - X(s) = \frac{1}{\sqrt{2}}(B_1(t) - B_1(s)) + \frac{1}{\sqrt{2}}(B_2(t) - B_2(s))$, this is the sum of two independent normal random variables with both mean 0 and variance $\frac{1}{2}(t-s)$, so the sum is a normal random variable with variance $t - s$, this proves that the process X is a Brownian motion.

(b) For any stopping time τ with respect to a Brownain motion B, we introduce the process B^* reflected at time τ, that is $B^*_t = B_t$ for $t \le \tau$, and $B^*_t = 2B_\tau - B_t$ for $t \ge \tau$. The process B^* is then itself also a Brownian motion. This is especially true when $\tau = \tau_x$ that is the first time the brownian motion B reaches a certain point x, and then $B_\tau = x$. This is due to the fact, that the process $(B_{u+\tau} - B_\tau)_{u\ge 0}$ is itself a BM independent from what happened at the past up time to τ, and then use the fact that the reflection of a BM with respect to zero is also a BM. Now, we have :

$$
\begin{aligned}
\mathbb{P}[\sup_{0\le s\le t} B_s > x] = \mathbb{P}[\tau_x \le t] \\
&= \mathbb{P}[\tau_x \le t, B_t > x] + \mathbb{P}[\tau_x \le t, B_t^* > x] \\
&= 2\mathbb{P}[B_t > x]
\end{aligned}
$$

(c) Suppose $\epsilon > 0$, as $M(\epsilon)$ has the same distribution as $|B_\epsilon|$, then $\mathbb{P}[M_\epsilon = 0] = \mathbb{P}[B_\epsilon = 0] = 0$.

(d) We take $\epsilon = \frac{1}{n}$ for any $n \in \mathbb{N}$, then from (c) we know that $\mathbb{P}[\forall n \in \mathbb{N} : M(\frac{1}{n}) > 0] = 1$, also by symmetry it is easy to see that we have also the fact that $\mathbb{P}[\forall n \in \mathbb{N} : m(\frac{1}{n}) < 0] = 1$, for $m_\epsilon = \min\limits_{0\le t\le \epsilon} X_t$, that means almost surely there is two strictly decreasing sequences (t_n) and (s_n) such that $X(t_n) > 0$ and $X(s_n) < 0$ for all $n \in \mathbb{N}$, and thus from the continuity of the paths of X, there is infinitely many times (r_n) such that $X(r_n) = 0$, thus infinitely many times t where $B_1(t) = B_2(t)$.

Problem 4 :

(a) By exchanging the order of the integral and the expectation it is clear that $\mathbb{E}[Z(t)] = \displaystyle\int_0^t \mathbb{E}[B_s]ds = 0$.

(b) We have :

$$
\begin{aligned}
\mathbb{E}[Z(s)Z(t)] &= \mathbb{E}[\int_0^t \int_0^s B_u B_v du dv] \\
&= \int_0^t \int_0^s \mathbb{E}[B_u B_v] du dv \\
&= \int_0^t \int_0^s \min\,(u,v) du dv \\
&= \int_0^s \int_0^s \min\,(u,v) du dv + \int_s^t \int_0^s u du dv \\
&= \int_0^s (\frac{v^2}{2} + v(s-v)) dv + (t-s)\frac{s^2}{2} \\
&= \frac{s^3}{3} + (t-s)\frac{s^2}{2}
\end{aligned}
$$

(c) We have $Z(t) = Z(s) + \int_s^t B_u du = Z(s) + (t-s)B_s + \int_s^t (B_u - B_s)du$, if we suppose that $Z(t)$ is Markov then : $\mathbb{E}[Z(t)|\mathcal{F}_s] = \mathbb{E}[Z(t)|Z(s)]$, which means that $B_s = \mathbb{E}[B_s|Z(s)]$, which is clearly not true, as $B(s)$ cannot be determined only from $Z(s)$.

Problem 5 :

(a) As $(B_{t\wedge T})_{t\geq 0}$ is a bounded MG, then by OST we have :

$$\mathbb{P}_a[B_T] = a$$

which gives $b\mathbb{P}_a[B_T = b] = a$.

(b) In tutorial 10, we showed that for every θ we have $\exp(\theta B_t - \frac{\theta^2}{2}t)$ is a MG, and it suffices to see that :

$$\mathbb{E}[e^{\theta B_t}] = \mathbb{E}[e^{\theta\sqrt{t}\mathcal{N}(0,1)}] = e^{\frac{\theta^2 t}{2}}$$

hence $M_t^\theta = \frac{e^{\theta B_t}}{\mathbb{E}[e^{\theta B_t}]} = \exp(\theta B_t - \frac{\theta^2 t}{2})$ is a MG. Now take $\theta = -2\mu$ we have then :

$$M_t^{-2\mu} = \exp(-2\mu(B_t + \mu t)) = \exp(-2\mu X_t)$$

and now using the fact that $\mathbb{E}_a[M_T] = e^{-2\mu a}$, then we have :

$$\mathbb{P}_a[X_T = 0] + e^{-2\mu b}\mathbb{P}_a[X_T = b] = e^{-2\mu a}$$

and thus :

$$\mathbb{P}_a[X_T = b] = \frac{1 - e^{-2\mu a}}{1 - e^{-2\mu b}}$$