5 Poisson Processes

5.1 The Poisson Distribution and the Poisson Process

Poisson behavior is so pervasive in natural phenomena and the Poisson distribution is so amenable to extensive and elaborate analysis as to make the Poisson process a cornerstone of stochastic modeling.

5.1.1 The Poisson Distribution

The Poisson distribution with parameter $\mu > 0$ is given by

$$p_k = \frac{e^{-\mu}\mu^k}{k!} \quad \text{for } k = 0, 1, \ldots. \tag{5.1}$$

Let X be a random variable having the Poisson distribution in (5.1). We evaluate the mean, or first moment, via

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} k p_k = \sum_{k=1}^{\infty} \frac{k e^{-\mu}\mu^k}{k!} \\ &= \mu e^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^{(k-1)}}{(k-1)!} \\ &= \mu. \end{aligned}$$

To evaluate the variance, it is easier first to determine

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^{\infty} k(k-1)p_k \\ &= \mu^2 e^{-\mu} \sum_{k=2}^{\infty} \frac{\mu^{(k-2)}}{(k-2)!} \\ &= \mu^2. \end{aligned}$$

Then

$$\begin{aligned} E\left[X^2\right] &= E[X(X-1)] + E[X] \\ &= \mu^2 + \mu, \end{aligned}$$

An Introduction to Stochastic Modeling. DOI: 10.1016/B978-0-12-381416-6.00005-8

while

$$\sigma_X^2 = \mathrm{Var}[X] = E\left[X^2\right] - \{E[X]\}^2$$
$$= \mu^2 + \mu - \mu^2 = \mu.$$

Thus, the Poisson distribution has the unusual characteristic that both the mean and the variance are given by the same value μ.

Two fundamental properties of the Poisson distribution, which will arise later in a variety of forms, concern the sum of independent Poisson random variables and certain random decompositions of Poisson phenomena. We state these properties formally as Theorems 5.1 and 5.2.

Theorem 5.1. *Let X and Y be independent random variables having Poisson distributions with parameters μ and ν, respectively. Then the sum $X+Y$ has a Poisson distribution with parameter $\mu+\nu$.*

Proof. By the law of total probability,

$$\begin{aligned}\Pr\{X+Y=n\} &= \sum_{k=0}^{n} \Pr\{X=k, Y=n-k\} \\ &= \sum_{k=0}^{n} \Pr\{X=k\}\ \Pr\{Y=n-k\} \\ &\qquad\qquad (X \text{ and } Y \text{ are independent}) \\ &= \sum_{k=0}^{n} \left\{\frac{\mu^k \mathrm{e}^{-\mu}}{k!}\right\}\left\{\frac{\nu^{n-k}\mathrm{e}^{-\nu}}{(n-k)!}\right\} \\ &= \frac{\mathrm{e}^{-(\mu+\nu)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\mu^k\nu^{n-k}.\end{aligned} \tag{5.2}$$

The binomial expansion of $(\mu+\nu)^n$ is, of course,

$$(\mu+\nu)^n = \sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\mu^k\nu^{n-k},$$

and so (5.2) simplifies to

$$\Pr\{X+Y=n\} = \frac{\mathrm{e}^{-(\mu+\nu)}(\mu+\nu)^n}{n!}, \quad n=0,1,\ldots,$$

the desired Poisson distribution.

To describe the second result, we consider first a Poisson random variable N where the parameter is $\mu > 0$. Write N as a sum of ones in the form

$$N = \underbrace{1 + 1 + \cdots + 1}_{N \text{ ones}},$$

and next, considering each one separately and independently, erase it with probability $1 - p$ and keep it with probability p. What is the distribution of the resulting sum M, of the form $M = 1 + 0 + 0 + 1 + \cdots + 1$?

The next theorem states and answers the question in a more precise wording. ■

Theorem 5.2. *Let N be a Poisson random variable with parameter μ, and conditional on N, let M have a binomial distribution with parameters N and p. Then the unconditional distribution of M is Poisson with parameter μp.*

Proof. The verification proceeds via a direct application of the law of total probability. Then

$$\begin{aligned}
\Pr\{M = k\} &= \sum_{n=0}^{\infty} \Pr\{M = k|N = n\} \Pr\{N = n\} \\
&= \sum_{n=k}^{\infty} \left\{ \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \right\} \left\{ \frac{\mu^n \mathrm{e}^{-\mu}}{n!} \right\} \\
&= \frac{\mathrm{e}^{-\mu} (\mu p)^k}{k!} \sum_{n=k}^{\infty} \frac{[\mu(1-p)]^{n-k}}{(n-k)!} \\
&= \frac{\mathrm{e}^{-\mu} (\mu p)^k}{k!} \mathrm{e}^{\mu(1-p)} \\
&= \frac{\mathrm{e}^{-\mu p} (\mu p)^k}{k!} \quad \text{for } k = 0, 1, \ldots,
\end{aligned}$$

which is the claimed Poisson distribution. ■

5.1.2 The Poisson Process

The Poisson process entails notions of both independence and the Poisson distribution.

Definition A Poisson process of intensity, or rate, $\lambda > 0$ is an integer-valued stochastic process $\{X(t);\, t \geq 0\}$ for which

1. for any time points $t_0 = 0 < t_1 < t_2 < \cdots < t_n$, the process increments

$$X(t_1) - X(t_0),\ X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent random variables;

2. for $s \geq 0$ and $t > 0$, the random variable $X(s+t) - X(s)$ has the Poisson distribution

$$\Pr\{X(s+t) - X(s) = k\} = \frac{(\lambda t)^k \mathrm{e}^{-\lambda t}}{k!} \quad \text{for } k = 0, 1, \ldots;$$

3. $X(0) = 0$.

In particular, observe that if $X(t)$ is a Poisson process of rate $\lambda > 0$, then the moments are

$$E[X(t)] = \lambda t \quad \text{and} \quad \mathrm{Var}[X(t)] = \sigma^2_{X(t)} = \lambda t.$$

Example Defects occur along an undersea cable according to a Poisson process of rate $\lambda = 0.1$ per mile. (a) What is the probability that no defects appear in the first two miles of cable? (b) Given that there are no defects in the first two miles of cable, what is the conditional probability of no defects between mile points two and three? To answer (a) we observe that $X(2)$ has a Poisson distribution whose parameter is $(0.1)(2) = 0.2$. Thus, $\Pr\{X(2) = 0\} = \mathrm{e}^{-0.2} = 0.8187$. In part (b), we use the independence of $X(3) - X(2)$ and $X(2) - X(0) = X(2)$. Thus, the conditional probability is the same as the unconditional probability, and

$$\Pr\{X(3) - X(2) = 0\} = \Pr\{X(1) = 0\} = \mathrm{e}^{-0.1} = 0.9048.$$

Example Customers arrive in a certain store according to a Poisson process of rate $\lambda = 4$ per hour. Given that the store opens at 9:00 A.M., what is the probability that exactly one customer has arrived by 9:30 and a total of five have arrived by 11:30 A.M.?

Measuring time t in hours from 9:00 A.M., we are asked to determine $\Pr\left\{X\left(\frac{1}{2}\right) = 1, X\left(\frac{5}{2}\right) = 5\right\}$. We use the independence of $X\left(\frac{5}{2}\right) - X\left(\frac{1}{2}\right)$ and $X\left(\frac{1}{2}\right)$ to reformulate the question thus:

$$\begin{aligned}
\Pr\left\{X\left(\frac{1}{2}\right) = 1, X\left(\frac{5}{2}\right) = 5\right\} &= \Pr\left\{X\left(\frac{1}{2}\right) = 1, X\left(\frac{5}{2}\right) - X\left(\frac{1}{2}\right) = 4\right\} \\
&= \left\{\frac{\mathrm{e}^{-4(1/2)} 4\left(\frac{1}{2}\right)}{1!}\right\} \left\{\frac{\mathrm{e}^{-4(2)}[4(2)]^4}{4!}\right\} \\
&= \left(2\mathrm{e}^{-2}\right)\left(\frac{512}{3}\mathrm{e}^{-8}\right) = 0.0154965.
\end{aligned}$$

5.1.3 Nonhomogeneous Processes

The rate λ in a Poisson process $X(t)$ is the proportionality constant in the probability of an event occurring during an arbitrarily small interval. To explain this more precisely,

$$\begin{aligned}
\Pr\{X(t+h) - X(t) = 1\} &= \frac{(\lambda h)\mathrm{e}^{-\lambda h}}{1!} \\
&= (\lambda h)\left(1 - \lambda h + \frac{1}{2}\lambda^2 h^2 - \cdots\right) \\
&= \lambda h + o(h),
\end{aligned}$$

where $o(h)$ denotes a general and unspecified remainder term of smaller order than h.

It is pertinent in many applications to consider rates $\lambda = \lambda(t)$ that vary with time. Such a process is termed a *nonhomogeneous* or *nonstationary* Poisson process to distinguish it from the stationary, or homogeneous, process that we primarily consider. If $X(t)$ is a nonhomogeneous Poisson process with rate $\lambda(t)$, then an increment $X(t) - X(s)$, giving the number of events in an interval $(s, t]$, has a Poisson distribution with parameter $\int_s^t \lambda(u)\,\mathrm{d}u$, and increments over disjoint intervals are independent random variables.

Example Demands on a first aid facility in a certain location occur according to a nonhomogeneous Poisson process having the rate function

$$\lambda(t) = \begin{cases} 2t & \text{for } 0 \le t < 1, \\ 2 & \text{for } 1 \le t < 2, \\ 4 - t & \text{for } 2 \le t \le 4, \end{cases}$$

where t is measured in hours from the opening time of the facility. What is the probability that two demands occur in the first 2 h of operation and two in the second 2 h? Since demands during disjoint intervals are independent random variables, we can answer the two questions separately. The mean for the first 2 h is $\mu = \int_0^1 2t\,\mathrm{d}t + \int_1^2 2\mathrm{d}t = 3$, and thus

$$\Pr\{X(2) = 2\} = \frac{\mathrm{e}^{-3}(3)^2}{2!} = 0.2240.$$

For the second 2 h, $\mu = \int_2^4 (4 - t)\,\mathrm{d}t = 2$, and

$$\Pr\{X(4) - X(2) = 2\} = \frac{\mathrm{e}^{-2}(2)^2}{2!} = 0.2707.$$

Let $X(t)$ be a nonhomogeneous Poisson process of rate $\lambda(t) > 0$ and define $\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u$. Make a deterministic change in the time scale and define a new process $Y(s) = X(t)$, where $s = \Lambda(t)$. Observe that $\Delta s = \lambda(t)\Delta t + o(\Delta t)$. Then

$$\begin{aligned} \Pr\{Y(s + \Delta s) - Y(s) = 1\} &= \Pr\{X(t + \Delta t) - X(t) = 1\} \\ &= \lambda(t)\Delta t + o(\Delta t) \\ &= \Delta s + o(\Delta s), \end{aligned}$$

so that $Y(s)$ is a homogeneous Poisson process of unit rate. By this means, questions about nonhomogeneous Poisson processes can be transformed into corresponding questions about homogeneous processes. For this reason, we concentrate our exposition on the latter.

5.1.4 Cox Processes

Suppose that $X(t)$ is a nonhomogeneous Poisson process, but where the rate function $\{\lambda(t), t \ge 0\}$ is itself a stochastic process. Such processes were introduced in 1955 as models for fibrous threads by Sir David Cox, who called them *doubly stochastic Poisson processes*. Now they are most often referred to as *Cox processes* in honor of their discoverer. Since their introduction, Cox processes have been used to model a myriad of phenomena, e.g., bursts of rainfall, where the likelihood of rain may vary with the season; inputs to a queueing system, where the rate of input varies over time, depending on changing and unmeasured factors; and defects along a fiber, where the rate and type of defect may change due to variations in material or manufacture. As these applications suggest, the process increments over disjoint intervals are, in general, statistically dependent in a Cox process, as contrasted with their postulated independence in a Poisson process.

Let $\{X(t);\, t \geq 0\}$ be a Poisson process of constant rate $\lambda = 1$. The very simplest Cox process, sometimes called a *mixed Poisson process*, involves choosing a single random variable Θ, and then observing the process $X'(t) = X(\Theta t)$. Given Θ, then X' is, conditionally, a Poisson process of constant rate $\lambda = \Theta$, but Θ is random, and typically, unobservable. If Θ is a continuous random variable with probability density function $f(\theta)$, then, upon removing the condition via the law of total probability, we obtain the marginal distribution

$$\Pr\{X'(t) = k\} = \int_0^\infty \frac{(\theta t)^k e^{-\theta t}}{k!} f(\theta)\mathrm{d}\theta. \tag{5.3}$$

Problem 5.1.12 calls for carrying out the integration in (5.3) in the particular instance in which Θ has an exponential density.

Chapter 6, Section 6.7 develops a model for defects along a fiber in which a Markov chain in continuous time is the random intensity function for a Poisson process. A variety of functionals are evaluated for the resulting Cox process.

Exercises

5.1.1 Defects occur along the length of a filament at a rate of $\lambda = 2$ per foot.

- **(a)** Calculate the probability that there are no defects in the first foot of the filament.
- **(b)** Calculate the conditional probability that there are no defects in the second foot of the filament, given that the first foot contained a single defect.

5.1.2 Let $p_k = \Pr\{X = k\}$ be the probability mass function corresponding to a Poisson distribution with parameter λ. Verify that $p_0 = \exp\{-\lambda\}$, and that p_k may be computed recursively by $p_k = (\lambda/k)p_{k-1}$.

5.1.3 Let X and Y be independent Poisson distributed random variables with parameters α and β, respectively. Determine the conditional distribution of X, given that $N = X + Y = n$.

5.1.4 Customers arrive at a service facility according to a Poisson process of rate λ customer/hour. Let $X(t)$ be the number of customers that have arrived up to time t.

- **(a)** What is $\Pr\{X(t) = k\}$ for $k = 0, 1, \ldots$?
- **(b)** Consider fixed times $0 < s < t$. Determine the conditional probability $\Pr\{X(t) = n + k | X(s) = n\}$ and the expected value $E[X(t)X(s)]$.

5.1.5 Suppose that a random variable X is distributed according to a Poisson distribution with parameter λ. The parameter λ is itself a random variable, exponentially distributed with density $f(x) = \theta e^{-\theta x}$ for $x \geq 0$. Find the probability mass function for X.

5.1.6 Messages arrive at a telegraph office as a Poisson process with mean rate of 3 messages per hour.

(a) What is the probability that no messages arrive during the morning hours 8:00 A.M. to noon?

(b) What is the distribution of the time at which the first afternoon message arrives?

5.1.7 Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities and conditional probabilities:

(a) $\Pr\{X(1) = 2\}$.

(b) $\Pr\{X(1) = 2 \text{ and } X(3) = 6\}$.

(c) $\Pr\{X(1) = 2 | X(3) = 6\}$.

(d) $\Pr\{X(3) = 6 | X(1) = 2\}$.

5.1.8 Let $\{X(t);\, t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the numerical values to two decimal places for the following probabilities:

(a) $\Pr\{X(1) \leq 2\}$.

(b) $\Pr\{X(1) = 1 \text{ and } X(2) = 3\}$.

(c) $\Pr\{X(1) \geq 2 | X(1) \geq 1\}$.

5.1.9 Let $\{X(t);\, t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the following expectations:

(a) $E[X(2)]$.

(b) $E\left[\{X(1)\}^2\right]$.

(c) $E[X(1)X(2)]$.

Problems

5.1.1 Let $\xi_1, \xi_2, \ldots$ be independent random variables, each having an exponential distribution with parameter λ. Define a new random variable X as follows: If $\xi_1 > 1$, then $X = 0$; if $\xi_1 \leq 1$ but $\xi_1 + \xi_2 > 1$, then set $X = 1$; in general, set $X = k$ if

$$\xi_1 + \cdots + \xi_k \leq 1 < \xi_1 + \cdots + \xi_k + \xi_{k+1}.$$

Show that X has a Poisson distribution with parameter λ. (Thus, the method outlined can be used to simulate a Poisson distribution.)

Hint: $\xi_1 + \cdots + \xi_k$ has a gamma density

$$f_k(x) = \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x} \quad \text{for } x > 0.$$

Condition on $\xi_1 + \cdots + \xi_k$ and use the law of total probability to show

$$\Pr\{X = k\} = \int_0^1 [1 - F(1 - x)] f_k(x)\,\mathrm{d}x,$$

where $F(x)$ is the exponential distribution function.

5.1.2 Suppose that minor defects are distributed over the length of a cable as a Poisson process with rate α, and that, independently, major defects are distributed over the cable according to a Poisson process of rate β. Let $X(t)$ be the number of defects, either major or minor, in the cable up to length t. Argue that $X(t)$ must be a Poisson process of rate $\alpha + \beta$.

5.1.3 The *generating function* of a probability mass function $p_k = \Pr\{X = k\}$, for $k = 0, 1, \ldots$, is defined by

$$g_X(s) = E\left[s^X\right] = \sum_{k=0}^{\infty} p_k s^k \quad \text{for } |s| < 1.$$

Show that the generating function for a Poisson random variable X with mean μ is given by

$$g_X(s) = \mathrm{e}^{-\mu(1-s)}.$$

5.1.4 (Continuation) Let X and Y be independent random variables, Poisson distributed with parameters α and β, respectively. Show that the generating function of their sum $N = X + Y$ is given by

$$g_N(s) = \mathrm{e}^{-(\alpha+\beta)(1-s)}.$$

Hint: Verify and use the fact that the generating function of a sum of independent random variables is the product of their respective generating functions. See Chapter 3, Section 3.9.2.

5.1.5 For each value of $h > 0$, let $X(h)$ have a Poisson distribution with parameter λh. Let $p_k(h) = \Pr\{X(h) = k\}$ for $k = 0, 1, \ldots$. Verify that

$$\lim_{h\to 0} \frac{1 - p_0(h)}{h} = \lambda, \quad \text{or } p_0(h) = 1 - \lambda h + o(h);$$

$$\lim_{h\to 0} \frac{p_1(h)}{h} = \lambda, \quad \text{or } p_1(h) = \lambda h + o(h);$$

$$\lim_{h\to 0} \frac{p_2(h)}{h} = 0, \quad \text{or } p_2(h) = o(h).$$

Here $o(h)$ stands for any remainder term of order less than h as $h \to 0$.

5.1.6 Let $\{X(t); t \geq 0\}$ be a Poisson process of rate λ. For $s, t > 0$, determine the conditional distribution of $X(t)$, given that $X(t+s) = n$.

5.1.7 Shocks occur to a system according to a Poisson process of rate λ. Suppose that the system survives each shock with probability α, independently of other shocks, so that its probability of surviving k shocks is α^k. What is the probability that the system is surviving at time t?

5.1.8 Find the probability $\Pr\{X(t) = 1, 3, 5, \ldots\}$ that a Poisson process having rate λ is odd.

5.1.9 Arrivals of passengers at a bus stop form a Poisson process $X(t)$ with rate $\lambda = 2$ per unit time. Assume that a bus departed at time $t = 0$ leaving no customers behind. Let T denote the arrival time of the next bus. Then, the number of passengers present when it arrives is $X(T)$. Suppose that the bus arrival time T is independent of the Poisson process and that T has the uniform probability density function

$$f_T(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Determine the conditional moments $E[X(T)|T = t]$ and $E\left[\{X(T)\}^2|T = t\right]$.

(b) Determine the mean $E[X(T)]$ and variance $\mathrm{Var}[X(T)]$.

5.1.10 Customers arrive at a facility at random according to a Poisson process of rate λ. There is a waiting time cost of c per customer per unit time. The customers gather at the facility and are processed or dispatched in groups at fixed times $T, 2T, 3T, \ldots$. There is a dispatch cost of K. The process is depicted in the following graph.

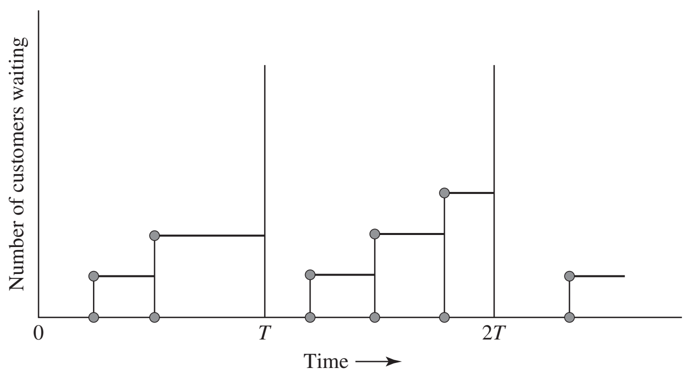

Figure 5.1 The number of customers in a dispatching system as a function of time.

(a) What is the total dispatch cost during the first cycle from time 0 to time T?

(b) What is the mean total customer waiting cost during the first cycle?

(c) What is the mean total customer waiting + dispatch cost per unit time during the first cycle?

(d) What value of T minimizes this mean cost per unit time?

5.1.11 Assume that a device fails when a cumulative effect of k shocks occurs. If the shocks happen according to a Poisson process of parameter λ, what is the density function for the life T of the device?

5.1.12 Consider the mixed Poisson process of Section 5.1.4, and suppose that the mixing parameter Θ has the exponential density $f(\theta) = e^{-\theta}$ for $\theta > 0$.

(a) Show that equation (5.3) becomes

$$\Pr\{X'(t) = j\} = \left(\frac{t}{1+t}\right)^j \left(\frac{1}{1+t}\right), \quad \text{for } j = 0, 1, \ldots.$$

(b) Show that

$$\Pr\{X'(t) = j, X'(t+s) = j+k\} = \binom{j+k}{j} t^j s^k \left(\frac{1}{1+s+t}\right)^{j+k+1},$$

so that $X'(t)$ and the increment $X'(t+s) - X'(t)$ are not independent random variables, in contrast to the simple Poisson process as defined in Section 5.1.2.

5.2 The Law of Rare Events

The common occurrence of the Poisson distribution in nature is explained by the *law of rare events*. Informally, this law asserts that where a certain event may occur in any of a large number of possibilities, but where the probability that the event does occur in any given possibility is small, then the total number of events that do happen should follow, approximately, the Poisson distribution.

A more formal statement in a particular instance follows. Consider a large number N of independent Bernoulli trials where the probability p of success on each trial is small and constant from trial to trial. Let $X_{N,p}$ denote the total number of successes in the N trials, where $X_{N,p}$ follows the binomial distribution

$$\Pr\left\{X_{N,p} = k\right\} = \frac{N!}{k!\,(N-k)!} p^k (1-p)^{N-k} \quad \text{for } k = 0, \ldots, N. \tag{5.4}$$

Now let us consider the limiting case in which $N \to \infty$ and $p \to 0$ in such a way that $Np = \mu > 0$ where μ is constant. It is a familiar fact (see Chapter 1, Section 1.3) that the distribution for $X_{N,p}$ becomes, in the limit, the Poisson distribution

$$\Pr\left\{X_\mu = k\right\} = \frac{e^{-\mu}\mu^k}{k!} \quad \text{for } k = 0, 1, \ldots. \tag{5.5}$$

This form of the law of rare events is stated as a limit. In stochastic modeling, the law is used to suggest circumstances under which one might expect the Poisson distribution to prevail, at least approximately. For example, a large number of cars may pass through a given stretch of highway on any particular day. The probability that any specified car is in an accident is, we hope, small. Therefore, one might expect that the actual number of accidents on a given day along that stretch of highway would be, at least approximately, Poisson distributed.

While we have formally stated this form of the law of rare events as a mathematical limit, in older texts, (5.5) is often called "the Poisson approximation" to the binomial, the idea being that when N is large and p is small, the binomial probability (5.4) may be approximately evaluated by the Poisson probability (5.5) with $\mu = Np$. With today's computing power, exact binomial probabilities are not difficult to obtain, so there is little need to approximate them with Poisson probabilities. Such is not the case if the problem is altered slightly by allowing the probability of success to vary from trial to trial. To examine this proposal in detail, let $\epsilon_1, \epsilon_2, \ldots$ be independent Bernoulli random variables, where

$$\Pr\{\epsilon_i = 1\} = p_i \quad \text{and } \Pr\{\epsilon_i = 0\} = 1 - p_i,$$

and let $S_n = \epsilon_1 + \cdots + \epsilon_n$. When $p_1 = p_2 = \cdots = p$, then S_n has a binomial distribution, and the probability $\Pr\{S_n = k\}$ for some $k = 0, 1, \ldots$ is easily computed. It is not so easily evaluated when the p's are unequal, with the binomial formula generalizing to

$$\Pr\{S_n = k\} = \Sigma^{(k)} \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}, \tag{5.6}$$

where $\Sigma^{(k)}$ denotes the sum over all 0, 1 valued x_i's such that $x_1 + \cdots + x_n = k$.

Fortunately, Poisson approximation may still prove accurate and allow the computational challenges presented by equation (5.6) to be avoided.

Theorem 5.3. *Let $\epsilon_1, \epsilon_2, \ldots$ be independent Bernoulli random variables, where*

$$\Pr\{\epsilon_i = 1\} = p_i \quad \textit{and } \Pr\{\epsilon_i = 0\} = 1 - p_i,$$

and let $S_n = \epsilon_1 + \cdots + \epsilon_n$. The exact probabilities for S_n, determined using (5.6), and Poisson probabilities with $\mu = p_l + \cdots + p_n$ differ by at most

$$\left| \Pr\{S_n = k\} - \frac{\mu^k e^{-\mu}}{k!} \right| \leq \sum_{i=1}^{n} p_i^2. \tag{5.7}$$

Not only does the inequality of Theorem 5.3 extend the law of rare events to the case of unequal p's, it also directly confronts the approximation issue by providing a numerical measure of the approximation error. Thus, the Poisson distribution provides a good approximation to the exact probabilities whenever the p_i's are uniformly small as measured by the right side of (5.7). For instance, when $p_1 = p_2 = \cdots = \mu/n$, then the right side of (5.7) reduces to μ^2/n, which is small when n is large, and thus (5.7) provides another means of obtaining the Poisson distribution (5.5) as a limit of the binomial probabilities (5.4).

We defer the proof of Theorem 5.3 to the end of this section, choosing to concentrate now on its implications. As an immediate consequence, e.g., in the context of the earlier car accident vignette, we see that the individual cars need not all have the same accident probabilities in order for the Poisson approximation to apply.

5.2.1 The Law of Rare Events and the Poisson Process

Consider events occurring along the positive axis $[0,\infty)$ in the manner shown in Figure 5.2. Concrete examples of such processes are the time points of the X-ray emissions of a substance undergoing radioactive decay, the instances of telephone calls originating in a given locality, the occurrence of accidents at a certain intersection, the location of faults or defects along the length of a fiber or filament, and the successive arrival times of customers for service.

Let $N((a,b])$ denote the number of events that occur during the interval $(a,b]$. That is, if $t_1 < t_2 < t_3 < \cdots$ denote the times (or locations, etc.) of successive events, then $N((a,b])$ is the number of values t_i for which $a < t_i \le b$.

We make the following postulates:

1. The numbers of events happening in disjoint intervals are independent random variables. That is, for every integer $m = 2,3,\ldots$ and time points $t_0 = 0 < t_1 < t_2 < \cdots < t_m$, the random variables

$$N((t_0,t_1]), N((t_1,t_2]), \ldots, N((t_{m-1},t_m])$$

are independent.
2. For any time t and positive number h, the probability distribution of $N((t,t+h])$, the number of events occurring between time t and $t+h$, depends only on the interval length h and not on the time t.
3. There is a positive constant λ for which the probability of at least one event happening in a time interval of length h is

$$\Pr\{N((t,t+h]) \ge 1\} = \lambda h + o(h) \quad \text{as } h \downarrow 0.$$

(Conforming to a common notation, here $o(h)$ as $h \downarrow 0$ stands for a general and unspecified remainder term for which $o(h)/h \to 0$ as $h \downarrow 0$. That is, a remainder term of smaller order than h as h vanishes.)

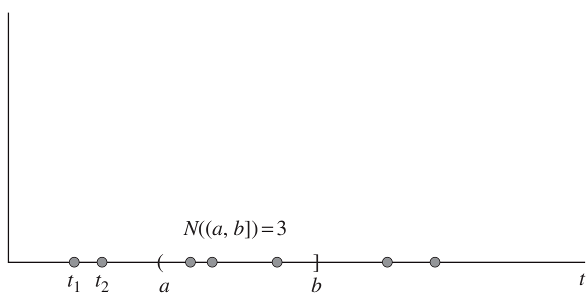

Figure 5.2 A Poisson point process.

4. The probability of two or more events occurring in an interval of length h is $o(h)$, or

$$\Pr\{N((t,t+h]) \geq 2\} = o(h), \quad h \downarrow 0.$$

Postulate 3 is a specific formulation of the notion that events are rare. Postulate 4 is tantamount to excluding the possibility of the simultaneous occurrence of two or more events. In the presence of Postulates 1 and 2, Postulates 3 and 4 are equivalent to the apparently weaker assumption that events occur singly and discretely, with only a finite number in any finite interval. In the concrete illustrations cited earlier, this requirement is usually satisfied.

Disjoint intervals are independent by 1, and 2 asserts that the distribution of $N((s,t])$ is the same as that of $N((0,t-s])$. Therefore, to describe the probability law of the system, it suffices to determine the probability distribution of $N((0,t])$ for an arbitrary value of t. Let

$$P_k(t) = \Pr\{N((0,t]) = k\}.$$

We will show that Postulates 1 through 4 require that $P_k(t)$ be the Poisson distribution

$$P_k(t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad \text{for } k = 0, 1, \ldots. \tag{5.8}$$

To establish (5.8), divide the interval $(0,t]$ into n subintervals of equal length $h = t/n$, and let

$$\epsilon_i = \begin{cases} 1 & \text{if there is at least one event in the interval } ((i-1)t/n, it/n], \\ 0 & \text{otherwise.} \end{cases}$$

Then, $S_n = \epsilon_1 + \cdots + \epsilon_n$ counts the total number of subintervals that contain at least one event, and

$$p_i = \Pr\{\epsilon_i = 1\} = \lambda t/n + o(t/n) \tag{5.9}$$

according to Postulate 3. Upon applying (5.7), we see that

$$\begin{aligned} \left| \Pr\{S_n = k\} - \frac{\mu^k e^{-\mu}}{k!} \right| &\leq n[\lambda t/n + o(t/n)]^2 \\ &= \frac{(\lambda t)^2}{n} + 2\lambda t o\left(\frac{t}{n}\right) + no\left(\frac{t}{n}\right)^2, \end{aligned}$$

where

$$\mu = \sum_{i=1}^{n} p_i = \lambda t + no(t/n). \tag{5.10}$$

Because $o(h) = o(t/n)$ is a term of order smaller than $h = t/n$ for large n, it follows that

$$no(t/n) = t\frac{o(t/n)}{t/n} = t\frac{o(h)}{h}$$

vanishes for arbitrarily large n. Passing to the limit as $n \to \infty$, then, we deduce that

$$\lim_{n\to\infty} \Pr\{S_n = k\} = \frac{\mu^k e^{-\mu}}{k!}, \quad \text{with } \mu = \lambda t.$$

To complete the demonstration, we need only show that

$$\lim_{n\to\infty} \Pr\{S_n = k\} = \Pr\{N((0,t]) = k\} = P_k(t).$$

But S_n differs from $N((0,t])$ only if at least one of the subintervals contains two or more events, and Postulate 4 precludes this because

$$\begin{aligned}
|P_k(t) - \Pr\{S_n = k\}| &\le \Pr\{N((0,t]) \ne S_n\} \\
&\le \sum_{i=1}^{n} \Pr\left\{N\left(\left(\frac{(i-1)t}{n}, \frac{it}{n}\right]\right) \ge 2\right\} \\
&\le no(t/n) \quad \text{(by Postulate 4)} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

By making n arbitrarily large, or equivalently, by dividing the interval $(0,t]$ into arbitrarily small subintervals, we see that it must be the case that

$$\Pr\{N((0,t]) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad \text{for } k \ge 0,$$

and Postulates 1 through 4 imply the Poisson distribution.

Postulates 1 through 4 arise as physically plausible assumptions in many circumstances of stochastic modeling. The postulates seem rather weak. Surprisingly, they are sufficiently strong to force the Poisson behavior just derived. This motivates the following definition.

Definition Let $N((s,t])$ be a random variable counting the number of events occurring in an interval $(s,t]$. Then, $N((s,t])$ is a Poisson point process of intensity $\lambda > 0$ if

1. for every $m = 2, 3, \ldots$ and distinct time points $t_0 = 0 < t_1 < t_2 < \cdots < t_m$, the random variables

$$N((t_0,t_1]), N((t_1,t_2]), \ldots, N((t_{m-1},t_m])$$

are independent; and

2. for any times $s < t$ the random variable $N((s,t])$ has the Poisson distribution

$$\Pr\{N((s,t]) = k\} = \frac{[\lambda(t-s)]^k \mathrm{e}^{-\lambda(t-s)}}{k!}, \quad k = 0, 1, \ldots.$$

Poisson point processes often arise in a form where the time parameter is replaced by a suitable spatial parameter. The following formal example illustrates this vein of ideas. Consider an array of points distributed in a space E (E is a Euclidean space of dimension $d \geq 1$). Let $N(A)$ denote the number of points (finite or infinite) contained in the region A of E. We postulate that $N(A)$ is a random variable. The collection $\{N(A)\}$ of random variables, where A varies over all possible subsets of E, is said to be a homogeneous Poisson process if the following assumptions are fulfilled:

1. The numbers of points in nonoverlapping regions are independent random variables.
2. For any region A of finite volume, $N(A)$ is Poisson distributed with mean $\lambda|A|$, where $|A|$ is the volume of A. The parameter λ is fixed and measures in a sense the intensity component of the distribution, which is independent of the size or shape. Spatial Poisson processes arise in considering such phenomena as the distribution of stars or galaxies in space, the spatial distribution of plants and animals, and the spatial distribution of bacteria on a slide. These ideas and concepts will be further studied in Section 5.5.

5.2.2 Proof of Theorem 5.3

First, some notation. Let $\epsilon(p)$ denote a Bernoulli random variable with success probability p, and let $X(\theta)$ be a Poisson distributed random variable with parameter θ. We are given probabilities $p_1, \ldots, p_n$ and let $\mu = p_1 + \cdots + p_n$. With $\epsilon(p_1), \ldots, \epsilon(p_n)$ assumed to be independent, we have $S_n = \epsilon(p_1) + \cdots + \epsilon(p_n)$, and according to Theorem 5.1, we may write $X(\mu)$ as the sum of independent Poisson distributed random variables in the form $X(\mu) = X(p_1) + \cdots + X(p_n)$. We are asked to compare $\Pr\{S_n = k\}$ with $\Pr\{X(\mu) = k\}$, and, as a first step, we observe that if S_n and $X(\mu)$ are unequal, then at least one of the pairs $\epsilon(p_k)$ and $X(p_k)$ must differ, whence

$$|\Pr\{S_n = k\} - \Pr\{X(\mu) = k\}| \leq \sum_{k=1}^{n} \Pr\{\epsilon(p_k) \neq X(p_k)\}. \tag{5.11}$$

As the second step, observe that the quantities that are compared on the left of (5.11) are the marginal distributions of S_n and $X(\mu)$, while the bound on the right is a joint probability. This leaves us free to choose the joint distribution that makes our task the easiest. That is, we are free to specify the joint distribution of each $\epsilon(p_k)$ and $X(p_k)$, as we please, provided only that the marginal distributions are Bernoulli and Poisson, respectively.

To complete the proof, we need to show that $\Pr\{\epsilon(p) \neq X(p)\} \leq p^2$ for some Bernoulli random variable $\epsilon(p)$ and Poisson random variable $X(p)$, since this reduces the right side of (5.11) to that of (5.7). Equivalently, we want to show that $1 - p^2 \leq \Pr\{\epsilon(p) = X(p)\} = \Pr\{\epsilon(p) = X(p) = 0\} + \Pr\{\epsilon(p) = X(p) = 1\}$, and we are free to choose the joint distribution, provided that the marginal distributions are correct.

Let U be a random variable that is uniformly distributed over the interval $(0, 1]$. Define

$$\epsilon(p) = \begin{cases} 1 & \text{if } 0 < U \le p, \\ 0 & \text{if } p < U \le 1, \end{cases}$$

and for $k = 0, 1, \ldots,$ set

$$X(p) = k \quad \text{when} \quad \sum_{i=0}^{k-1} \frac{p^i e^{-p}}{i!} < U \le \sum_{i=0}^{k} \frac{p^i e^{-p}}{i!}.$$

It is elementary to verify that $\epsilon(p)$ and $X(p)$ have the correct marginal distributions. Furthermore, because $1 - p \le e^{-p}$, we have $\epsilon(p) = X(p) = 0$ only for $U \le 1 - p$, whence $\Pr\{\epsilon(p) = X(p) = 0\} = 1 - p$. Similarly, $\epsilon(p) = X(p) = 1$ only when $e^{-p} < U \le (1+p)e^{-p}$, whence $\Pr\{\epsilon(p) = X(p) = 1\} = pe^{-p}$. Upon summing these two evaluations, we obtain

$$\Pr\{\epsilon(p) = X(p)\} = 1 - p + pe^{-p} = 1 - p^2 + p^3/2 \cdots \ge 1 - p^2$$

as was to be shown. This completes the proof of (5.7).

Problem 2.10 calls for the reader to review the proof and to discover the single line that needs to be changed in order to establish the stronger result

$$|\Pr\{S_n \text{ in } I\} - \Pr\{X(\mu) \text{ in } I\}| \le \sum_{k=1}^{n} p_i^2$$

for any set of nonnegative integers I.

Exercises

5.2.1 Determine numerical values to three decimal places for $\Pr\{X = k\}$, $k = 0, 1, 2,$ when

(a) X has a binomial distribution with parameters $n = 20$ and $p = 0.06$.

(b) X has a binomial distribution with parameters $n = 40$ and $p = 0.03$.

(c) X has a Poisson distribution with parameter $\lambda = 1.2$.

5.2.2 Explain in general terms why it might be plausible to assume that the following random variables follow a Poisson distribution:

(a) The number of customers that enter a store in a fixed time period.

(b) The number of customers that enter a store and buy something in a fixed time period.

(c) The number of atomic particles in a radioactive mass that disintegrate in a fixed time period.

5.2.3 A large number of distinct pairs of socks are in a drawer, all mixed up. A small number of individual socks are removed. Explain in general terms why it might be plausible to assume that the number of pairs among the socks removed might follow a Poisson distribution.

5.2.4 Suppose that a book of 600 pages contains a total of 240 typographical errors. Develop a Poisson approximation for the probability that three particular successive pages are error-free.

Problems

5.2.1 Let $X(n,p)$ have a binomial distribution with parameters n and p. Let $n \to \infty$ and $p \to 0$ in such a way that $np = \lambda$. Show that

$$\lim_{n\to\infty} \Pr\{X(n,p) = 0\} = \mathrm{e}^{-\lambda}$$

and

$$\lim_{n\to\infty} \frac{\Pr\{X(n,p) = k+1\}}{\Pr\{X(n,p) = k\}} = \frac{\lambda}{k+1} \quad \text{for } k = 0, 1, \ldots.$$

5.2.2 Suppose that 100 tags, numbered $1, 2, \ldots, 100$, are placed into an urn, and 10 tags are drawn successively, *with replacement*. Let A be the event that no tag is drawn twice. Show that

$$\Pr\{A\} = \left(1 - \frac{1}{100}\right)\left(1 - \frac{2}{100}\right) \cdots \left(1 - \frac{9}{100}\right) = 0.6282.$$

Use the approximation

$$1 - x \approx \mathrm{e}^{-x} \quad \text{for } x \approx 0$$

to get

$$\Pr\{A\} \approx \exp\left\{-\frac{1}{100}(1 + 2 + \cdots + 9)\right\} = \mathrm{e}^{-0.45} = 0.6376.$$

Interpret this in terms of the law of rare events.

5.2.3 Suppose that N pairs of socks are sent to a laundry, where they are washed and thoroughly mixed up to create a mass of unmatched socks. Then, n socks are drawn at random without replacement from the pile. Let A be the event that no pair is among the n socks so selected. Show that

$$\Pr\{A\} = \frac{2^n \binom{N}{n}}{\binom{2N}{n}} = \prod_{i=1}^{n-1} \left(1 - \frac{i}{2N - i}\right).$$

Use the approximation

$$1 - x \approx e^{-x} \quad \text{for } x \approx 0$$

to get

$$\Pr\{A\} \approx \exp\left\{-\sum_{i=1}^{n-1} \frac{i}{2N-i}\right\} \approx \exp\left\{-\frac{n(n-1)}{4N}\right\},$$

the approximations holding when n is small relative to N, which is large. Evaluate the exact expression and each approximation when $N = 100$ and $n = 10$. Is the approximation here consistent with the actual number of pairs of socks among the n socks drawn having a Poisson distribution?

Answer: Exact 0.7895; Approximate 0.7985.

5.2.4 Suppose that N points are uniformly distributed over the interval $[0, N)$. Determine the probability distribution for the number of points in the interval $[0, 1)$ as $N \to \infty$.

5.2.5 Suppose that N points are uniformly distributed over the surface of a circular disk of radius r. Determine the probability distribution for the number of points within a distance of one of the origin as $N \to \infty$, $r \to \infty$, $N/\left(\pi r^2\right) = \lambda$.

5.2.6 Certain computer coding systems use randomization to assign memory storage locations to account numbers. Suppose that $N = M\lambda$ different accounts are to be randomly located among M storage locations. Let X_i be the number of accounts assigned to the ith location. If the accounts are distributed independently and each location is equally likely to be chosen, show that $\Pr\{X_i = k\} \to e^{-\lambda}\lambda^k/k!$ as $N \to \infty$. Show that X_i and X_j are independent random variables in the limit, for distinct locations $i \neq j$. In the limit, what fraction of storage locations have two or more accounts assigned to them?

5.2.7 N bacteria are spread independently with uniform distribution on a microscope slide of area A. An arbitrary region having area a is selected for observation. Determine the probability of k bacteria within the region of area a. Show that as $N \to \infty$ and $a \to 0$ such that $(a/A)N \to c (0 < c < \infty)$, then $p(k) \to e^{-c}c^k/k!$.

5.2.8 Using (5.6), evaluate the exact probabilities for S_n and the Poisson approximation and error bound in (5.7) when $n = 4$ and $p_1 = 0.1, p_2 = 0.2, p_3 = 0.3$, and $p_4 = 0.4$.

5.2.9 Using (5.6), evaluate the exact probabilities for S_n and the Poisson approximation and error bound in (5.7) when $n = 4$ and $p_1 = 0.1, p_2 = 0.1, p_3 = 0.1$, and $p_4 = 0.2$.

5.2.10 Review the proof of Theorem 5.3 in Section 5.2.2 and establish the stronger result

$$|\Pr\{S_n \text{ in } I\} - \Pr\{X(\mu) \text{ in } I\}| \leq \sum_{k=1}^{n} p_i^2$$

for any set of nonnegative integers I.

5.2.11 Let X and Y be jointly distributed random variables and B an arbitrary set. Fill in the details that justify the inequality $|\Pr\{X \text{ in } B\} - \Pr\{Y \text{ in } B\}| \le \Pr\{X \neq Y\}$.

Hint: Begin with

$$\begin{aligned}\{X \text{ in } B\} &= \{X \text{ in } B \text{ and } Y \text{ in } B\} \quad \text{or} \quad \{X \text{ in } B \text{ and } Y \text{ not in B}\}\\ &\subset \{Y \text{ in } B\} \quad \text{or} \quad \{X \neq Y\}.\end{aligned}$$

5.2.12 *Computer Challenge* Most computers have available a routine for simulating a sequence $U_0, U_1, \ldots$ of independent random variables, each uniformly distributed on the interval $(0, 1)$. Plot, say, 10,000 pairs (U_{2n}, U_{2n+1}) on the unit square. Does the plot look like what you would expect? Repeat the experiment several times. Do the points in a fixed number of disjoint squares of area 1/10,000 look like independent unit Poisson random variables?

5.3 Distributions Associated with the Poisson Process

A *Poisson point process* $N((s,t])$ counts the number of events occurring in an interval $(s,t]$. A *Poisson counting process*, or more simply a *Poisson process* $X(t)$, counts the number of events occurring up to time t. Formally, $X(t) = N((0,t])$.

Poisson events occurring in space can best be modeled as a point process. For Poisson events occurring on the positive time axis, whether we view them as a Poisson point process or Poisson counting process is largely a matter of convenience, and we will freely do both. The two descriptions are equivalent for Poisson events occurring along a line. The Poisson process is the more common and traditional description in this case because it allows a pictorial representation as an increasing integer-valued random function taking unit steps.

Figure 5.3 shows a typical sample path of a Poisson process where W_n is the time of occurrence of the nth event, the so-called *waiting time*. It is often convenient to set $W_0 = 0$. The differences $S_n = W_{n+1} - W_n$ are called *sojourn times*; S_n measures the duration that the Poisson process sojourns in state n.

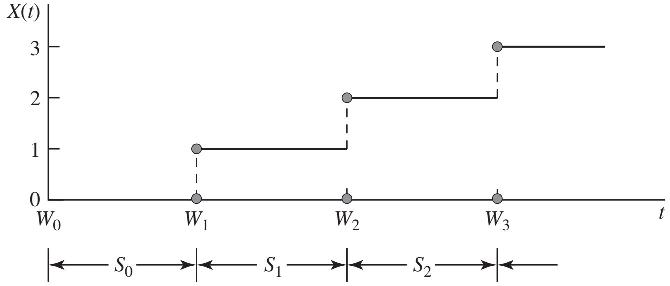

Figure 5.3 A typical sample path of a Poisson process showing the waiting times W_n and the sojourn times S_n.

In this section, we will determine a number of probability distributions associated with the Poisson process $X(t)$, the waiting times W_n, and the sojourn times S_n.

Theorem 5.4. *The waiting time W_n has the gamma distribution whose probability density function is*

$$f_{W_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \quad n = 1, 2, \ldots, t \geq 0. \tag{5.12}$$

In particular, W_1, the time to the first event, is exponentially distributed:

$$f_{W_1}(t) = \lambda e^{-\lambda t}, \quad t \geq 0. \tag{5.13}$$

Proof. The event $W_n \leq t$ occurs if and only if there are at least n events in the interval $(0, t]$, and since the number of events in $(0, t]$ has a Poisson distribution with mean λt we obtain the cumulative distribution function of W_n via

$$\begin{aligned}
F_{W_n}(t) &= \Pr\{W_n \leq t\} = \Pr\{X(t) \geq n\} \\
&= \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\
&= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad n = 1, 2, \ldots, t \geq 0.
\end{aligned}$$

We obtain the probability density function $f_{W_n}(t)$ by differentiating the cumulative distribution function. Then

$$\begin{aligned}
f_{W_n}(t) &= \frac{d}{dt} F_{W_n}(t) \\
&= \frac{d}{dt} \left\{ 1 - e^{-\lambda t} \left[1 - \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{n-1}}{(n-1)!} \right] \right\} \\
&= -e^{-\lambda t} \left[\lambda + \frac{\lambda(\lambda t)}{1!} + \lambda \frac{(\lambda t)^2}{2!} + \cdots + \lambda \frac{(\lambda t)^{n-2}}{(n-2)!} \right] \\
&\quad + \lambda e^{-\lambda t} \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{n-1}}{(n-1)!} \right] \\
&= \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \quad n = 1, 2, \ldots, t \geq 0.
\end{aligned}$$

There is an alternative derivation of the density in (5.12) that uses the Poisson point process $N((s, t])$ and proceeds directly without differentiation. The event $t < W_n \leq t + \Delta t$ corresponds exactly to $n - 1$ occurrences in $(0, t]$ and one in $(t, t + \Delta t]$, as depicted in Figure 5.4.

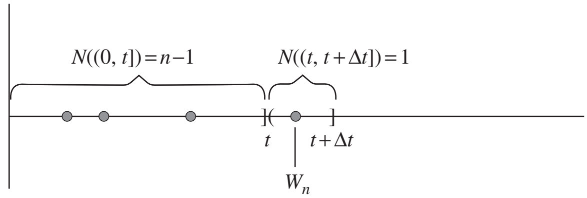

Figure 5.4

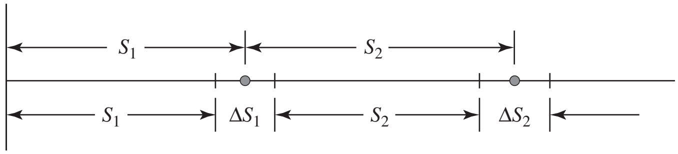

Figure 5.5

Then

$$\begin{aligned} f_{W_n}(t)\Delta t &\approx \Pr\{t < W_n \le t+\Delta t\} + o(\Delta t) \quad \text{[see Chapter 1, equation (1.5)]} \\ &= \Pr\{N((0,t]) = n-1\}\Pr\{N((t,t+\Delta t]) = 1\} + o(\Delta t) \\ &= \frac{(\lambda t)^{n-1}e^{-\lambda t}}{(n-1)!}\lambda(\Delta t) + o(\Delta t). \end{aligned}$$

Dividing by Δt and passing to the limit as $\Delta t \to 0$ we obtain (5.13).

Observe that $\Pr\{N((t,t+\Delta t]) \ge 1\} = \Pr\{N((t,t+\Delta t]) = 1\} + o(\Delta t) = \lambda(\Delta t) + o(\Delta t)$. ■

Theorem 5.5. *The sojourn times $S_0, S_1, \ldots, S_{n-1}$ are independent random variables, each having the exponential probability density function*

$$f_{S_k}(s) = \lambda e^{-\lambda s}, \quad s \ge 0. \tag{5.14}$$

Proof. We are being asked to show that the joint probability density function of $S_0, S_1, \ldots, S_{n-1}$ is the product of the exponential densities given by

$$f_{S_0,S_1,\ldots,S_{n-1}}(s_0,s_1,\ldots,s_{n-1}) = \left(\lambda e^{-\lambda s_0}\right)\left(\lambda e^{-\lambda s_1}\right)\cdots\left(\lambda e^{-\lambda s_{n-1}}\right). \tag{5.15}$$

We give the proof only in the case $n = 2$, the general case being entirely similar. Referring to Figure 5.5 we see that the joint occurrence of

$$s_1 < S_1 < s_1 + \Delta s_1 \quad \text{and} \quad s_2 < S_2 < s_2 + \Delta s_2$$

corresponds to no events in the intervals $(0, s_1]$ and $(s_1 + \Delta s_1, s_1 + \Delta s_1 + s_2]$ and exactly one event in each of the intervals $(s_1, s_1 + \Delta s_1]$ and $(s_1 + \Delta s_1 + s_2, s_1 + \Delta s_1 + s_2 + \Delta s_2]$. Thus

$$
\begin{aligned}
f_{s_1,s_2}(s_1, s_2)\Delta s_1 \Delta s_2 &= \Pr\{s_1 < S_1 < s_1 + \Delta s_1, s_2 < S_2 < s_2 + \Delta s_2\} \\
&\quad + o(\Delta s_1 \Delta s_2) \\
&= \Pr\{N((0, s_1]) = 0\} \\
&\quad \times \Pr\{N((s_1 + \Delta s_1, s_1 + \Delta s_1 + s_2]) = 0\} \\
&\quad \times \Pr\{N((s_1, s_1 + \Delta s_1]) = 1\} \\
&\quad \times \Pr\{N((s_1 + \Delta s_1 + s_2, s_1 + \Delta s_1 + s_2 + \Delta s_2]) = 1\} \\
&\quad + o(\Delta s_1 \Delta s_2) \\
&= \mathrm{e}^{-\lambda s_1}\mathrm{e}^{-\lambda s_2}\mathrm{e}^{-\lambda \Delta s_1}\mathrm{e}^{-\lambda \Delta s_2}\lambda(\Delta s_1)\lambda(\Delta s_2) + o(\Delta s_1 \Delta s_2) \\
&= (\lambda \mathrm{e}^{-\lambda s_1})(\lambda \mathrm{e}^{-\lambda s_2})(\Delta s_1)(\Delta s_2) + o(\Delta s_1 \Delta s_2).
\end{aligned}
$$

Upon dividing both sides by $(\Delta s_1)(\Delta s_2)$ and passing to the limit as $\Delta s_1 \to 0$ and $\Delta s_2 \to 0$, we obtain (5.15) in the case $n = 2$. ■

The binomial distribution also arises in the context of Poisson processes.

Theorem 5.6. *Let $\{X(t)\}$ be a Poisson process of rate $\lambda > 0$. Then for $0 < u < t$ and $0 \le k \le n$,*

$$
\Pr\{X(u) = k | X(t) = n\} = \frac{n!}{k!\,(n-k)!}\left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}. \tag{5.16}
$$

Proof. Straightforward computations give

$$
\begin{aligned}
\Pr\{X(u) = k | X(t) = n\} &= \frac{\Pr\{X(u) = k \text{ and } X(t) = n\}}{\Pr\{X(t) = n\}} \\
&= \frac{\Pr\{X(u) = k \text{ and } X(t) - X(u) = n - k\}}{\Pr\{X(t) = n\}} \\
&= \frac{\left\{\mathrm{e}^{-\lambda u}(\lambda u)^k/k!\right\}\left\{\mathrm{e}^{-\lambda(t-u)}[\lambda(t-u)]^{n-k}/(n-k)!\right\}}{\mathrm{e}^{-\lambda t}(\lambda t)^n/n!} \\
&= \frac{n!}{k!\,(n-k)!}\frac{u^k(t-u)^{n-k}}{t^n},
\end{aligned}
$$

which establishes (5.16). ■

Exercises

5.3.1 A radioactive source emits particles according to a Poisson process of rate $\lambda = 2$ particles per minute. What is the probability that the first particle appears after 3 min?

5.3.2 A radioactive source emits particles according to a Poisson process of rate $\lambda = 2$ particles per minute.
- **(a)** What is the probability that the first particle appears some time after 3 min but before 5 min?
- **(b)** What is the probability that exactly one particle is emitted in the interval from 3 to 5 min?

5.3.3 Customers enter a store according to a Poisson process of rate $\lambda = 6$ per hour. Suppose it is known that only a single customer entered during the first hour. What is the conditional probability that this person entered during the first 15 min?

5.3.4 Let $X(t)$ be a Poisson process of rate $\xi = 3$ per hour. Find the conditional probability that there were two events in the first hour, given that there were five events in the first 3 h.

5.3.5 Let $X(t)$ be a Poisson process of rate θ per hour. Find the conditional probability that there were m events in the first t hours, given that there were n events in the first T hours. Assume $0 \leq m \leq n$ and $0 < t < T$.

5.3.6 For $i = 1, \ldots, n$, let $\{X_i(t); t \geq 0\}$ be independent Poisson processes, each with the same parameter λ. Find the distribution of the first time that at least one event has occurred in every process.

5.3.7 Customers arrive at a service facility according to a Poisson process of rate λ customers/hour. Let $X(t)$ be the number of customers that have arrived up to time t. Let $W_1, W_2, \ldots$ be the successive arrival times of the customers. Determine the conditional mean $E[W_5 | X(t) = 3]$.

5.3.8 Customers arrive at a service facility according to a Poisson process of rate $\lambda = 5$ per hour. Given that 12 customers arrived during the first two hours of service, what is the conditional probability that 5 customers arrived during the first hour?

5.3.9 Let $X(t)$ be a Poisson process of rate λ. Determine the cumulative distribution function of the gamma density as a sum of Poisson probabilities by first verifying and then using the identity $W_r \leq t$ if and only if $X(t) \geq r$.

Problems

5.3.1 Let $X(t)$ be a Poisson process of rate λ. Validate the identity

$$\{W_1 > w_1, W_2 > w_2\}$$

if and only if

$$\{X(w_1) = 0, X(w_2) - X(w_1) = 0 \text{ or } 1\}.$$

Use this to determine the joint upper tail probability

$$\begin{aligned}\Pr\{W_1 > w_1, W_2 > w_2\} &= \Pr\{X(w_1) = 0, X(w_2) - X(w_1) = 0 \text{ or } 1\} \\ &= e^{-\lambda w_1}[1 + \lambda(w_2 - w_1)]e^{-\lambda(w_2 - w_1)}.\end{aligned}$$

Finally, differentiate twice to obtain the joint density function

$$f(w_1, w_2) = \lambda^2 \exp\{-\lambda w_2\} \quad \text{for } 0 < w_1 < w_2.$$

5.3.2 The joint probability density function for the waiting times W_1 and W_2 is given by

$$f(w_1, w_2) = \lambda^2 \exp\{-\lambda w_2\} \quad \text{for } 0 < w_1 < w_2.$$

Determine the conditional probability density function for W_1, given that $W_2 = w_2$. How does this result differ from that in Theorem 5.6 when $n = 2$ and $k = 1$?

5.3.3 The joint probability density function for the waiting times W_1 and W_2 is given by

$$f(w_1, w_2) = \lambda^2 \exp\{-\lambda w_2\} \quad \text{for } 0 < w_1 < w_2.$$

Change variables according to

$$S_0 = W_1 \quad \text{and} \quad S_1 = W_2 - W_1$$

and determine the joint distribution of the first two sojourn times. Compare with Theorem 5.5.

5.3.4 The joint probability density function for the waiting times W_1 and W_2 is given by

$$f(w_1, w_2) = \lambda^2 \exp\{-\lambda w_2\} \quad \text{for } 0 < w_1 < w_2.$$

Determine the marginal density functions for W_1 and W_2, and check your work by comparison with Theorem 5.4.

5.3.5 Let $X(t)$ be a Poisson process with parameter λ. Independently, let T be a random variable with the exponential density

$$f_T(t) = \theta e^{-\theta t} \quad \text{for } t > 0.$$

Determine the probability mass function for $X(T)$.

Hint: Use the law of total probability and Chapter 1, (1.54). Alternatively, use the results of Chapter 1, Section 1.5.2.

5.3.6 Customers arrive at a holding facility at random according to a Poisson process having rate λ. The facility processes in batches of size Q. That is, the first $Q - 1$ customers wait until the arrival of the Qth customer. Then, all are passed simultaneously, and the process repeats. Service times are instantaneous. Let $N(t)$ be the number of customers in the holding facility at time t. Assume that $N(0) = 0$ and let $T = \min\{t \geq 0 : N(t) = Q\}$ be the first dispatch time. Show that $E[T] = Q/\lambda$ and $E\left[\int_0^T N(t)\mathrm{d}t\right] = [1 + 2 + \cdots + (Q - 1)]/\lambda = Q(Q - 1)/2\lambda$.

5.3.7 A critical component on a submarine has an operating lifetime that is exponentially distributed with mean 0.50 years. As soon as a component fails, it is replaced by a new one having statistically identical properties. What is the smallest number of *spare* components that the submarine should stock if it is leaving for a one-year tour and wishes the probability of having an inoperable unit caused by failures exceeding the spare inventory to be less than 0.02?

5.3.8 Consider a Poisson process with parameter λ. Given that $X(t) = n$ events occur in time t, find the density function for W_r, the time of occurrence of the rth event. Assume that $r \le n$.

5.3.9 The following calculations arise in certain highly simplified models of learning processes. Let $X_1(t)$ and $X_2(t)$ be independent Poisson processes having parameters λ_1 and λ_2, respectively.

(a) What is the probability that $X_1(t) = 1$ before $X_2(t) = 1$?

(b) What is the probability that $X_1(t) = 2$ before $X_2(t) = 2$?

5.3.10 Let $\{W_n\}$ be the sequence of waiting times in a Poisson process of intensity $\lambda = 1$. Show that $X_n = 2^n \exp\{-W_n\}$ defines a nonnegative martingale.

5.4 The Uniform Distribution and Poisson Processes

The major result of this section, Theorem 5.7, provides an important tool for computing certain functionals on a Poisson process. It asserts that, conditioned on a fixed total number of events in an interval, the locations of those events are uniformly distributed in a certain way.

After a complete discussion of the theorem and its proof, its application in a wide range of problems will be given.

In order to completely understand the theorem, consider first the following experiment. We begin with a line segment t units long and a fixed number n of darts and throw darts at the line segment in such a way that each dart's position upon landing is uniformly distributed along the segment, independent of the location of the other darts. Let U_1 be the position of the first dart thrown, U_2 the position of the second, and so on up to U_n. The probability density function is the uniform density

$$f_U(u) = \begin{cases} \dfrac{1}{t} & \text{for } 0 \le u \le t, \\ 0 & \text{elsewhere.} \end{cases}$$

Now let $W_1 \le W_2 \le \cdots \le W_n$ denote these same positions, not in the order in which the darts were thrown, but instead in the order in which they appear along the line. Figure 5.6 depicts a typical relation between $U_1, U_2, \ldots, U_n$ and $W_1, W_2, \ldots, W_n$.

The joint probability density function for $W_1, W_2, \ldots, W_n$ is

$$f_{W_1,\ldots,W_n}(w_1, \ldots, w_n) = n!\,t^{-n} \quad \text{for } 0 < w_1 < w_2 < \cdots < w_n \le t. \tag{5.17}$$

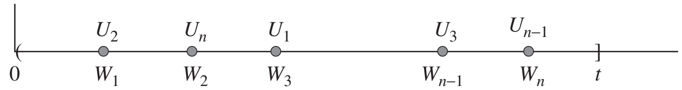

Figure 5.6 $W_1, W_2, \ldots, W_n$ are the values $U_1, U_2, \ldots, U_n$ arranged in increasing order.

For example, to establish (5.17) in the case $n = 2$ we have

$$\begin{aligned}
&f_{W_1,W_2}(w_1, w_2)\Delta w_1 \Delta w_2\\
&\qquad = \Pr\{w_1 < W_1 \le w_1 + \Delta w_1, w_2 < W_2 \le w_2 + \Delta w_2\}\\
&\qquad = \Pr\{w_1 < U_1 \le w_1 + \Delta w_1, w_2 < U_2 < w_2 + \Delta w_2\}\\
&\qquad\quad + \Pr\{w_1 < U_2 \le w_1 + \Delta w_1, w_2 < U_1 \le w_2 + \Delta w_2\}\\
&\qquad = 2\left(\frac{\Delta w_1}{t}\right)\left(\frac{\Delta w_2}{t}\right) = 2t^{-2}\Delta w_1 \Delta w_2.
\end{aligned}$$

Dividing by $\Delta w_1 \Delta w_2$ and passing to the limit gives (5.17). When $n = 2$, there are two ways that U_1 and U_2 can be ordered; either U_1 is less than U_2, or U_2 is less than U_1. In general, there are $n!$ arrangements of $U_1, \ldots, U_n$ that lead to the same ordered values $W_1 \le \cdots \le W_n$, thus giving (5.17).

Theorem 5.7. *Let $W_1, W_2, \ldots$ be the occurrence times in a Poisson process of rate $\lambda > 0$. Conditioned on $N(t) = n$, the random variables $W_1, W_2, \ldots, W_n$ have the joint probability density function*

$$f_{W_1,\ldots,W_n|X(t)=n}(w_1, \ldots, w_n) = n!\,t^{-n} \quad \text{for } 0 < w_1 < \cdots < w_n \le t. \tag{5.18}$$

Proof. The event $w_i < W_i \le w_i + \Delta w_i$ for $i = 1, \ldots, n$ and $N(t) = n$ corresponds to no events occurring in any of the intervals $(0, w_1], (w_1 + \Delta w_1, w_2], \ldots, (w_{n-1} + \Delta w_{n-1}, w_n], (w_n + \Delta w_n, t]$, and exactly one event in each of the intervals $(w_1, w_1 + \Delta w_1], (w_2, w_2 + \Delta w_2], \ldots, (w_n, w_n + \Delta w_n]$. These intervals are disjoint, and

$$\begin{aligned}
&\Pr\{N((0, w_1]) = 0, \ldots, N((w_n + \Delta w_n, t]) = 0\}\\
&\quad = e^{-\lambda w_1} e^{-\lambda(w_2 - w_1 - \Delta w_1)} \cdots e^{-\lambda(w_n - w_{n-1} - \Delta w_{n-1})} e^{-\lambda(t - w_n - \Delta w_n)}\\
&\quad = e^{-\lambda t}\left[e^{\lambda(\Delta w_1 + \cdots + \Delta w_n)}\right]\\
&\quad = e^{-\lambda t}[1 + o(\max\{\Delta w_i\})],
\end{aligned}$$

while

$$\begin{aligned}
&\Pr\{N((w_1, w_1 + \Delta w_1]) = 1, \ldots, N((w_n, w_n + \Delta w_n]) = 1\}\\
&\quad = \lambda(\Delta w_1) \cdots \lambda(\Delta w_n)[1 + o(\max\{\Delta w_i\})].
\end{aligned}$$

Thus

$$
\begin{aligned}
&f_{W_1,\ldots,W_n|X(t)=n}(w_1,\ldots,w_n)\Delta w_1\cdots\Delta w_n \\
&\quad=\Pr\{w_1<W_1\le w_1+\Delta w_1,\ldots,w_n<W_n\le w_n+\Delta w_n|N(t)=n\} \\
&\qquad+o(\Delta w_1\cdots\Delta w_n) \\
&\quad=\frac{\Pr\{w_i<W_i\le w_i+\Delta w_i, i=1,\ldots,n, N(t)=n\}}{\Pr\{N(t)=n\}} \\
&\qquad+o(\Delta w_1\cdots\Delta w_1) \\
&\quad=\frac{\mathrm{e}^{-\lambda t}\lambda(\Delta w_1)\cdots\lambda(\Delta w_n)}{\mathrm{e}^{-\lambda t}(\lambda t)^n/n!}[1+o(\max\{\Delta w_i\})] \\
&\quad=n!\,t^{-n}(\Delta w_1)\cdots(\Delta w_n)[1+o(\max\{\Delta w_i\})].
\end{aligned}
$$

Dividing both sides by $(\Delta w_1)\cdots(\Delta w_n)$ and letting $\Delta w_1\to 0,\ldots,\Delta w_n\to 0$ establishes (5.18). ■

Theorem 5.7 has important applications in evaluating certain symmetric functionals on Poisson processes. Some sample instances follow.

Example Customers arrive at a facility according to a Poisson process of rate λ. Each customer pays \$1 on arrival, and it is desired to evaluate the expected value of the total sum collected during the interval $(0,t]$ discounted back to time 0. This quantity is given by

$$
M=E\left[\sum_{k=1}^{X(t)}\mathrm{e}^{-\beta W_k}\right],
$$

where β is the discount rate, $W_1, W_2,\ldots$ are the arrival times, and $X(t)$ is the total number of arrivals in $(0,t]$. The process is shown in Figure 5.7.

We evaluate the mean total discounted sum M by conditioning on $X(t)=n$. Then

$$
M=\sum_{n=1}^{\infty}E\left[\sum_{k=1}^{n}\mathrm{e}^{-\beta W_k}\middle|X(t)=n\right]\Pr\{X(t)=n\}. \tag{5.19}
$$

Let $U_1,\ldots,U_n$ denote independent random variables that are uniformly distributed in $(0,t]$. Because of the symmetry of the functional $\Sigma_{k=1}^{n}\exp\{-\beta W_k\}$ and Theorem 5.7,

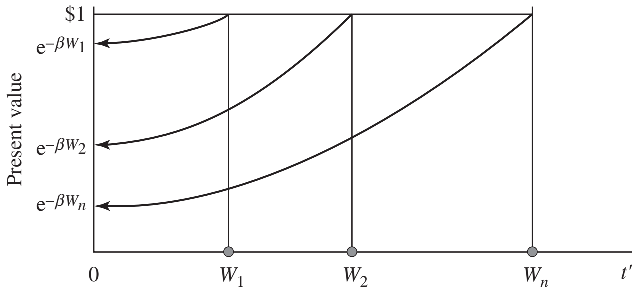

Figure 5.7 A dollar received at time W_k is discounted to a present value at time 0 of $\exp\{-\beta W_k\}$.

we have

$$\begin{aligned}E\left[\sum_{k=1}^{n} e^{-\beta W_k}\middle|X(t)=n\right] &= E\left[\sum_{k=1}^{n} e^{-\beta U_k}\right]\\ &= nE\left[e^{-\beta U_1}\right]\\ &= nt^{-1}\int_0^t e^{-\beta u}\,du\\ &= \frac{n}{\beta t}\left[1-e^{-\beta t}\right].\end{aligned}$$

Substitution into (5.19) then gives

$$\begin{aligned}M &= \frac{1}{\beta t}\left[1-e^{-\beta t}\right]\sum_{n=1}^{\infty} n\Pr\{X(t)=n\}\\ &= \frac{1}{\beta t}\left[1-e^{-\beta t}\right]E[X(t)]\\ &= \frac{\lambda}{\beta}\left[1-e^{-\beta t}\right].\end{aligned}$$

Example Viewing a fixed mass of a certain radioactive material, suppose that *alpha* particles appear in time according to a Poisson process of intensity λ. Each particle exists for a random duration and is then annihilated. Suppose that the successive lifetimes $Y_1, Y_2, \ldots$ of distinct particles are independent random variables having the common distribution function $G(y) = \Pr\{Y_k \le y\}$. Let $M(t)$ count the number of alpha particles existing at time t. The process is depicted in Figure 5.8.

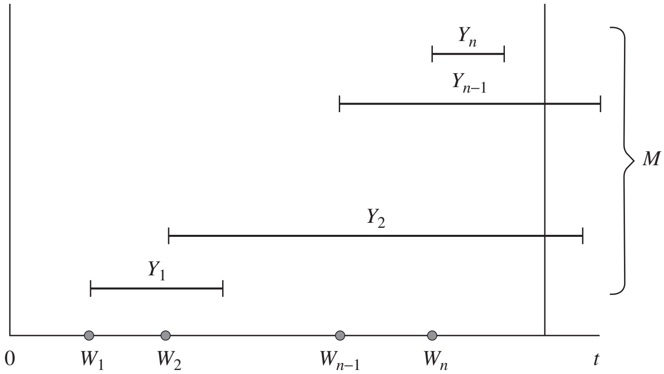

Figure 5.8 A particle created at time $W_k \le t$ still exists at time t if $W_k + Y_k \ge t$.

We will use Theorem 5.7 to evaluate the probability distribution of $M(t)$ under the condition that $M(0) = 0$.

Let $X(t)$ be the number of particles created up to time t, by assumption a Poisson process of intensity λ. Observe that $M(t) \le X(t)$; the number of existing particles cannot exceed the number of particles created. Condition on $X(t) = n$ and let $W_1, \ldots, W_n \le t$ be the times of particle creation. Then, particle k exists at time t if and only if $W_k + Y_k \ge t$. Let

$$\mathbf{1}\{W_k + Y_k \ge t\} = \begin{cases} 1 & \text{if } W_k + Y_k \ge t, \\ 0 & \text{if } W_k + Y_k < t. \end{cases}$$

Then, $\mathbf{1}\ \{W_k + Y_k \ge t\} = 1$ if and only if the kth particle is alive at time t. Thus

$$\Pr\{M(t) = m | X(t) = n\} = \Pr\left\{\sum_{k=1}^{n} \mathbf{1}\{W_k + Y_k \ge t\} = m | X(t) = n\right\}.$$

Invoking Theorem 5.7 and the symmetry among particles, we have

$$\begin{aligned} &\Pr\left\{\sum_{k=1}^{n} \mathbf{1}\{W_k + Y_k \ge t\} = m | X(t) = n\right\} \\ &\quad = \Pr\left\{\sum_{k=1}^{n} \mathbf{1}\{U_k + Y_k \ge t\} = m\right\}, \end{aligned} \tag{5.20}$$

where $U_1, U_2, \ldots, U_n$ are independent and uniformly distributed on $(0, t]$. The right-hand side of (5.20) is readily recognized as the binomial distribution in which

$$\begin{aligned} p = \Pr\{U_k + Y_k \geq t\} &= \frac{1}{t}\int_0^t \Pr\{Y_k \geq t-u\}\mathrm{d}u \\ &= \frac{1}{t}\int_0^t [1 - G(t-u)]\mathrm{d}u \\ &= \frac{1}{t}\int_0^t [1 - G(z)]\mathrm{d}z. \end{aligned} \tag{5.21}$$

Thus, explicitly writing the binomial distribution, we have

$$\Pr\{M(t) = m | X(t) = n\} = \frac{n!}{m!\,(n-m)!}p^m(1-p)^{n-m},$$

with p^n given by (5.21). Finally,

$$\begin{aligned} \Pr\{M(t) = m\} &= \sum_{n=m}^{\infty} \Pr\{M(t) = m | X(t) = n\}\Pr\{X(t) = n\} \\ &= \sum_{n=m}^{\infty} \frac{n!}{m!\,(n-m)!}p^m(1-p)^{n-m}\frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!} \\ &= \mathrm{e}^{-\lambda t}\frac{(\lambda p t)^m}{m!}\sum_{n=m}^{\infty}\frac{(1-p)^{n-m}(\lambda t)^{n-m}}{(n-m)!}. \end{aligned} \tag{5.22}$$

The infinite sum is an exponential series and reduces according to

$$\sum_{n=m}^{\infty}\frac{(1-p)^{n-m}(\lambda t)^{n-m}}{(n-m)!} = \sum_{j=0}^{\infty}\frac{[\lambda t(1-p)]^j}{j!} = \mathrm{e}^{\lambda t(1-p)},$$

and this simplifies (5.22) to

$$\Pr\{M(t) = m\} = \frac{\mathrm{e}^{-\lambda p t}(\lambda p t)^m}{m!} \quad \text{for } m = 0, 1, \ldots.$$

In words, the number of particles existing at time t has a Poisson distribution with mean

$$\lambda p t = \lambda \int_0^t [1 - G(y)]\mathrm{d}y. \tag{5.23}$$

It is often relevant to let $t \to \infty$ in (5.23) and determine the corresponding long run distribution. Let $\mu = E[Y_k] = \int_0^\infty [1 - G(y)]\mathrm{d}y$ be the mean lifetime of an alpha particle. It is immediate from (5.23) that as $t \to \infty$, the distribution of $M(t)$ converges to the Poisson distribution with parameter $\lambda\mu$. A great simplification has taken place. In the long run, the probability distribution for existing particles depends only on the mean lifetime μ, and not otherwise on the lifetime distribution $G(y)$. In practical terms, this statement implies that in order to apply this model, only the mean lifetime μ need be known.

5.4.1 Shot Noise

The shot noise process is a model for fluctuations in electrical currents that are due to chance arrivals of electrons to an anode. Variants of the phenomenon arise in physics and communication engineering. Assume:

1. Electrons arrive at an anode according to a Poisson process $\{X(t); t \geq 0\}$ of constant rate λ;
2. An arriving electron produces a current whose intensity x time units after arrival is given by the *impulse response function* $h(x)$.

The intensity of the current at time t is, then, the shot noise

$$I(t) = \sum_{k=1}^{X(t)} h(t - W_k), \tag{5.24}$$

where W_1, W_2 are the arrival times of the electrons.

Common impulse response functions include triangles, rectangles, decaying exponentials of the form

$$h(x) = \mathrm{e}^{-\theta x}, \quad x > 0,$$

where $\theta > 0$ is a parameter, and *power law* shot noise for which

$$h(x) = x^{-\theta}, \quad \text{for } x > 0.$$

We will show that *for a fixed time point t*, the shot noise $I(t)$ has the same probability distribution as a certain random sum that we now describe. Independent of the Poisson process $X(t)$, let $U_1, U_2, \ldots$ be independent random variables, each uniformly distributed over the interval $(0, t]$, and define $\epsilon_k = h(U_k)$ for $k = 1, 2, \ldots$. The claim is that $I(t)$ has the same probability distribution as the random sum

$$S(t) = \epsilon_1 + \cdots + \epsilon_{X(t)}. \tag{5.25}$$

With this result in hand, the mean, variance, and distribution of the shot noise $I(t)$ may be readily obtained using the results on random sums developed in Chapter 2, Section 2.3. For example, Chapter 2, equation (2.30) immediately gives us

$$E[I(t)] = E[S(t)] = \lambda t E[h(U_1)] = \lambda \int_0^t h(u)\mathrm{d}u$$

and

$$\begin{aligned}\mathrm{Var}[I(t)] &= \lambda t \left\{\mathrm{Var}[h(U_1)] + E[h(U_1)]^2\right\} \\ &= \lambda t E\left[h(U_1)^2\right] = \lambda \int_0^t h(u)^2 \mathrm{d}u.\end{aligned}$$

In order to establish that the shot noise $I(t)$ and the random sum $S(t)$ share the same probability distribution, we need to show that $\Pr\{I(t) \le x\} = \Pr\{S(t) \le x\}$ for a fixed $t > 0$. Begin with

$$\begin{aligned}\Pr\{I(t) \le x\} &= \Pr\left\{\sum_{k=1}^{X(t)} h(t - W_k) \le x\right\} \\ &= \sum_{n=0}^{\infty} \Pr\left\{\sum_{k=1}^{X(t)} h(t - W_k) \le x | X(t) = n\right\} \Pr\{X(t) = n\} \\ &= \sum_{n=0}^{\infty} \Pr\left\{\sum_{k=1}^{n} h(t - W_k) \le x | X(t) = n\right\} \Pr\{X(t) = n\},\end{aligned}$$

and now invoking Theorem 5.7,

$$\begin{aligned}&= \sum_{n=0}^{\infty} \Pr\left\{\sum_{k=1}^{n} h(t - U_k) \le x\right\} \Pr\{X(t) = n\} \\ &= \sum_{n=0}^{\infty} \Pr\left\{\sum_{k=1}^{n} h(U_k) \le x\right\} \Pr\{X(t) = n\}\end{aligned}$$

(because U_k and $t - U_k$ have the same distribution)

$$\begin{aligned}&= \sum_{n=0}^{\infty} \Pr\{\epsilon_1 + \cdots + \epsilon_n \le x\} \Pr\{X(t) = n\} \\ &= \Pr\left\{\epsilon_1 + \cdots + \epsilon_{X(t)} \le x\right\} \\ &= \Pr\{S(t) \le x\},\end{aligned}$$

which completes the claim.

5.4.2 Sum Quota Sampling

A common procedure in statistical inference is to observe a fixed number n of independent and identically distributed random variables $X_1, \ldots, X_n$ and use their sample mean

$$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n}$$

as an estimate of the population mean or expected value $E[X_1]$. But suppose we are asked to advise an airline that wishes to estimate the failure rate *in service* of a particular component, or, what is nearly the same thing, to estimate the mean service life of the part. The airline monitors a new plane for two years and observes that the original component lasted 7 months before failing. Its replacement lasted 5 months, and the third component lasted 9 months. No further failures were observed during the remaining 3 months of the observation period. Is it correct to estimate the mean life in service as the observed average $(7+5+9)/3 = 7$ months?

This airline scenario provides a realistic example of a situation in which the sample size is not fixed in advance but is determined by a preassigned quota $t > 0$. In *sum quota sampling*, a sequence of independent and identically distributed nonnegative random variables $X_1, X_2, \ldots$ is observed sequentially, with the sampling continuing as long as the sum of the observations is less than the quota t. Let this random sample size be denoted by $N(t)$. Formally,

$$N(t) = \max\{n \geq 0; X_1 + \cdots + X_n < t\}.$$

The sample mean is

$$\overline{X}_{N(t)} = \frac{W_{N(t)}}{N(t)} = \frac{X_1 + \cdots + X_{N(t)}}{N(t)}.$$

Of course it is possible that $X_1 \geq t$, and then $N(t) = 0$, and the sample mean is undefined. Thus, we must assume, or condition on, the event that $N(t) \geq 1$. An important question in statistical theory is whether or not this sample mean is unbiased. That is, how does the expected value of this sample mean relate to the expected value of, say, X_1?

In general, the determination of the expected value of the sample mean under sum quota sampling is very difficult. It can be carried out, however, in the special case in which the individual X summands are exponentially distributed with common parameter λ, so that $N(t)$ is a Poisson process. One hopes that the results in the special case will shed some light on the behavior of the sample mean under other distributions.

The key is the use of Theorem 5.7 to evaluate the conditional expectation

$$\begin{aligned} E[W_{N(t)}|N(t)=n] &= E[\max\{U_1, \ldots, U_n\}] \\ &= t\left(\frac{n}{n+1}\right), \end{aligned}$$

where $U_1, \ldots, U_n$ are independent and uniformly distributed over the interval $(0, t]$. Note also that

$$\Pr\{N(t) = n | N(t) > 0\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!\,(1 - e^{-\lambda t})}.$$

Then

$$\begin{aligned}
E\left[\frac{W_{N(t)}}{N(t)}\middle| N(t) > 0\right] &= \sum_{n=1}^{\infty} E\left[\frac{W_n}{n}\middle| N(t) = n\right] \Pr\{N(t) = n | N(t) > 0\} \\
&= \sum_{n=1}^{\infty} t\left(\frac{n}{n+1}\right)\left(\frac{1}{n}\right)\left\{\frac{(\lambda t)^n e^{-\lambda t}}{n!\,(1 - e^{-\lambda t})}\right\} \\
&= \frac{1}{\lambda}\left(\frac{1}{e^{\lambda t} - 1}\right)\sum_{n=1}^{\infty} \frac{(\lambda t)^{n+1}}{(n+1)!} \\
&= \frac{1}{\lambda}\left(\frac{1}{e^{\lambda t} - 1}\right)\left(e^{\lambda t} - 1 - \lambda t\right) \\
&= \frac{1}{\lambda}\left(1 - \frac{\lambda t}{e^{\lambda t} - 1}\right).
\end{aligned}$$

We can perhaps more clearly see the effect of the sum quota sampling if we express the preceding calculation in terms of the ratio of the bias to the true mean $E[X_1] = 1/\lambda$. We then have

$$\frac{E[X_1] - E[\overline{X}_{N(t)}]}{E[X_1]} = \frac{\lambda t}{e^{\lambda t} - 1} = \frac{E[N(t)]}{e^{E[N(t)]} - 1}.$$

The left side is the fraction of bias, and the right side expresses this fraction bias as a function of the expected sample size under sum quota sampling. The following table relates some values:

Fraction Bias	$\boldsymbol{E[N(t)]}$
0.58	1
0.31	2
0.16	3
0.17	4
0.03	5
0.015	6
0.0005	10

In the airline example, we observed $N(t) = 3$ failures in the two-year period, and upon consulting the above table, we might estimate the fraction bias to be something on the order of -16%. Since we observed $\overline{X}_{N(t)} = 7$, a more accurate estimate of the mean time between failures (MTBF $= E[X_1]$) might be $7/.84 = 8.33$, an estimate that attempts to correct, at least on average, for the bias due to the sampling method.

Looking once again at the table, we may conclude in general, that the bias due to sum quota sampling can be made acceptably small by choosing the quota t sufficiently large so that, on average, the sample size so selected is reasonably large. If the individual observations are exponentially distributed, the bias can be kept within 0.05% of the true value, provided that the quota t is large enough to give an average sample size of 10 or more.

Exercises

5.4.1 Let $\{X(t); t \geq 0\}$ be a Poisson process of rate λ. Suppose it is known that $X(1) = n$. For $n = 1, 2, \ldots$, determine the mean of the first arrival time W_1.

5.4.2 Let $\{X(t); t \geq 0\}$ be a Poisson process of rate λ. Suppose it is known that $X(1) = 2$. Determine the mean of $W_1 W_2$, the product of the first two arrival times.

5.4.3 Customers arrive at a certain facility according to a Poisson process of rate λ. Suppose that it is known that five customers arrived in the first hour. Determine the mean total waiting time $E[W_1 + W_2 + \cdots + W_5]$.

5.4.4 Customers arrive at a service facility according to a Poisson process of intensity λ. The service times $Y_1, Y_2, \ldots$ of the arriving customers are independent random variables having the common probability distribution function $G(y) = \Pr\{Y_k \leq y\}$. Assume that there is no limit to the number of customers that can be serviced simultaneously; i.e., there is an infinite number of servers available. Let $M(t)$ count the number of customers in the system at time t. Argue that $M(t)$ has a Poisson distribution with mean λpt, where

$$p = t^{-1} \int_0^t [1 - G(y)] \mathrm{d}y.$$

5.4.5 Customers arrive at a certain facility according to a Poisson process of rate λ. Suppose that it is known that five customers arrived in the first hour. Each customer spends a time in the store that is a random variable, exponentially distributed with parameter α and independent of the other customer times, and then departs. What is the probability that the store is empty at the end of this first hour?

Problems

5.4.1 Let $W_1, W_2, \ldots$ be the event times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ. Suppose it is known that $X(1) = n$. For $k < n$, what is the conditional density function of $W_1, \ldots, W_{k-1}, W_{k+1}, \ldots, W_n$, given that $W_k = w$?

5.4.2 Let $\{N(t); t \geq 0\}$ be a Poisson process of rate λ, representing the arrival process of customers entering a store. Each customer spends a duration in the store that is a random variable with cumulative distribution function G. The customer durations are independent of each other and of the arrival process. Let $X(t)$ denote the number of customers remaining in the store at time t, and let $Y(t)$ be the number of customers who have arrived and departed by time t. Determine the joint distribution of $X(t)$ and $Y(t)$.

5.4.3 Let $W_1, W_2, \ldots$ be the waiting times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ. Under the condition that $X(1) = 3$, determine the joint distribution of $U = W_1/W_2$ and $V = (1 - W_3)/(1 - W_2)$.

5.4.4 Let $W_1, W_2, \ldots$ be the waiting times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ. Independent of the process, let $Z_1, Z_2, \ldots$ be independent and identically distributed random variables with common probability density function $f(x), 0 < x < \infty$. Determine $\Pr\{Z > z\}$, where

$$Z = \min\{W_1 + Z_1, W_2 + Z_2, \ldots\}.$$

5.4.5 Let $W_1, W_2, \ldots$ be the waiting times in a Poisson process $\{N(t); t \geq 0\}$ of rate λ. Determine the limiting distribution of W_1, under the condition that $N(t) = n$ as $n \to \infty$ and $t \to \infty$ in such a way that $n/t = \beta > 0$.

5.4.6 Customers arrive at a service facility according to a Poisson process of rate λ customers/hour. Let $X(t)$ be the number of customers that have arrived up to time t. Let $W_1, W_2, \ldots$ be the successive arrival times of the customers.

(a) Determine the conditional mean $E[W_1|X(t) = 2]$.

(b) Determine the conditional mean $E[W_3|X(t) = 5]$.

(c) Determine the conditional probability density function for W_2, given that $X(t) = 5$.

5.4.7 Let $W_1, W_2, \ldots$ be the event times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ, and let $f(w)$ be an arbitrary function. Verify that

$$E\left[\sum_{i=1}^{X(t)} f(W_i)\right] = \lambda \int_0^t f(w)\mathrm{d}w.$$

5.4.8 Electrical pulses with independent and identically distributed random amplitudes $\xi_1, \xi_2, \ldots$ arrive at a detector at random times $W_1, W_2, \ldots$ according to a Poisson process of rate λ. The detector output $\theta_k(t)$ for the kth pulse at time t is

$$\theta_k(t) = \begin{cases} 0 & \text{for } t < W_k, \\ \xi_k \exp\{-\alpha(t - W_k)\} & \text{for } t \geq W_k. \end{cases}$$

That is, the amplitude impressed on the detector when the pulse arrives is ξ_k, and its effect thereafter decays exponentially at rate α. Assume that the detector is additive, so that if $N(t)$ pulses arrive during the time interval $[0, t]$, then the output at time t is

$$Z(t) = \sum_{k=1}^{N(t)} \theta_k(t).$$

Determine the mean output $E[Z(t)]$ assuming $N(0) = 0$. Assume that the amplitudes $\xi_1, \xi_2, \ldots$ are independent of the arrival times $W_1, W_2, \ldots$.

5.4.9 Customers arrive at a service facility according to a Poisson process of rate λ customers per hour. Let $N(t)$ be the number of customers that have arrived up to time t, and let $W_1, W_2, \ldots$ be the successive arrival times of the customers. Determine the expected value of the product of the waiting times up to time t. (Assume that $W_1 W_2 \cdots W_{N(t)} = 1$ when $N(t) = 0$.)

5.4.10 Compare and contrast the example immediately following Theorem 5.7, the shot noise process of Section 5.4.1, and the model of Problem 4.8. Can you formulate a general process of which these three examples are special cases?

5.4.11 *Computer Challenge* Let $U_0, U_1, \ldots$ be independent random variables, each uniformly distributed on the interval $(0, 1)$. Define a stochastic process $\{S_n\}$ recursively by setting

$$S_0 = 0 \quad \text{and} \quad S_{n+1} = U_n(1 + S_n) \quad \text{for } n > 0.$$

(This is an example of a discrete-time, continuous-state, Markov process.) When n becomes large, the distribution of S_n approaches that of a random variable $S = S_\infty$, and S must have the same probability distribution as $U(1 + S)$, where U and S are independent. We write this in the form

$$S \overset{\mathscr{D}}{=} U(1 + S),$$

from which it is easy to determine that $E[S] = 1, \mathrm{Var}[S] = \frac{1}{2}$, and even (the Laplace transform)

$$E\left[\mathrm{e}^{-\theta S}\right] = \exp\left\{-\int_{0+}^{\theta} \frac{1 - \mathrm{e}^{-u}}{u} \mathrm{d}u\right\}, \quad \theta > 0.$$

The probability density function $f(s)$ satisfies

$$f(s) = 0 \quad \text{for} \quad s \leq 0, \quad \text{and}$$
$$\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{1}{s} f(s - 1), \quad \text{for } s > 0.$$

What is the 99th percentile of the distribution of S? (Note: Consider the shot noise process of Section 5.4.1. When the Poisson process has rate $\lambda = 1$ and the impulse response function is the exponential $h(x) = \exp\{-x\}$, then the shot noise $I(t)$ has, in the limit for large t, the same distribution as S.)

5.5 Spatial Poisson Processes

In this section, we define some versions of multidimensional Poisson processes and describe some examples and applications.

Let S be a set in n-dimensional space and let $\mathscr{A}$ be a family of subsets of S. A *point process* in S is a stochastic process $N(A)$ indexed by the sets A in $\mathscr{A}$ and having the set of nonnegative integers $\{0, 1, 2, \ldots\}$ as its possible values. We think of "points" being scattered over S in some random manner and of $N(A)$ as counting the number of points in the set A. Because $N(A)$ is a counting function, there are certain obvious requirements that it must satisfy. For example, if A and B are disjoint sets in $\mathscr{A}$ whose union $A \cup B$ is also in $\mathscr{A}$, then it must be that $N(A \cup B) = N(A) + N(B)$. In words, the number of points in A or B equals the number of points in A plus the number of points in B when A and B are disjoint.

The one-dimensional case, in which S is the positive half line and $\mathscr{A}$ comprises all intervals of the form $A = (s, t]$, for $0 \leq s < t$, was introduced in Section 5.3. The straightforward generalization to the plane and three-dimensional space that is now being discussed has relevance when we consider the spatial distribution of stars or galaxies in astronomy, of plants or animals in ecology, of bacteria on a slide in medicine, and of defects on a surface or in a volume in reliability engineering.

Let S be a subset of the real line, two-dimensional plane, or three-dimensional space; let $\mathscr{A}$ be the family of subsets of S and for any set A in $\mathscr{A}$; let $|A|$ denote the size (length, area, or volume, respectively) of A. Then, $\{N(A); A \text{ in } \mathscr{A}\}$ is a *homogeneous Poisson point process* of intensity $\lambda > 0$ if

1. for each A in $\mathscr{A}$, the random variable $N(A)$ has a Poisson distribution with parameter $\lambda|A|$;
2. for every finite collection $\{A_1, \ldots, A_n\}$ of disjoint subsets of S, the random variables $N(A_1)$, $\ldots, N(A_n)$ are independent.

In Section 5.2, the law of rare events was invoked to derive the Poisson process as a consequence of certain physically plausible postulates. This implication serves to justify the Poisson process as a model in those situations where the postulates may be expected to hold. An analogous result is available in the multidimensional case at hand. Given an arbitrary point process $\{N(A); A \text{ in } \mathscr{A}\}$, the required postulates are as follows:

1. The possible values for $N(A)$ are the nonnegative integers $\{0, 1, 2, \ldots\}$ and $0 < \Pr\{N(A) = 0\} < 1$ if $0 < |A| < \infty$.
2. The probability distribution of $N(A)$ depends on the set A only through its size (length, area, or volume) $|A|$, with the further property that $\Pr\{N(A) \geq 1\} = \lambda|A| + o(|A|)$ as $|A| \downarrow 0$.
3. For $m = 2, 3, \ldots$, if $A_1, A_2, \ldots, A_m$ are disjoint regions, then $N(A_1)$, $N(A_2), \ldots, N(A_m)$ are independent random variables and $N(A_1 \cup A_2 \cup \cdots \cup A_m) = N(A_1) + N(A_2) + \cdots + N(A_m)$.
4. $\displaystyle \lim_{|A| \to 0} \frac{\Pr\{N(A) \geq 1\}}{\Pr\{N(A) = 1\}} = 1.$

The motivation and interpretation of these postulates is quite evident. Postulate 2 asserts that the probability distribution of $N(A)$ does not depend on the shape or location of A, but only on its size. Postulate 3 requires that the outcome in one region not influence or be influenced by the outcome in a second region that does not overlap the first. Postulate 4 precludes the possibility of two points occupying the same location.

If a random point process $N(A)$ defined with respect to subsets A of Euclidean n-space satisfies Postulates 1 through 4, then $N(A)$ is a homogeneous Poisson point process of intensity $\lambda > 0$, and

$$\Pr\{N(A) = k\} = \frac{e^{-\lambda|A|}(\lambda|A|)^k}{k!} \quad \text{for } k = 0, 1, \ldots. \tag{5.26}$$

As in the one-dimensional case, homogeneous Poisson point processes in n-dimensions are highly amenable to analysis, and many results are known for them. We elaborate a few of these consequences next, beginning with the uniform distribution of a single point. Consider a region A of positive size $|A| > 0$, and suppose it is known that A contains exactly one point; i.e., $N(A) = 1$. Where in A is this point located? We claim that the point is uniformly distributed in the sense that

$$\Pr\{N(B) = 1|N(A) = 1\} = \frac{|B|}{|A|} \quad \text{for any set } B \subset A. \tag{5.27}$$

In words, the probability of the point being in any subset B of A is proportional to the size of B; i.e., the point is uniformly distributed in A. The uniform distribution expressed in (5.27) is an immediate consequence of elementary conditional probability manipulations. We write $A = B \cup C$, where B is an arbitrary subset of A and C is the portion of A not included in B. Then, B and C are disjoint, so that $N(B)$ and $N(C)$ are independent Poisson random variables with respective means $\lambda|B|$ and $\lambda|C|$. Then

$$\begin{aligned}\Pr\{N(B) = 1|N(A) = 1\} &= \frac{\Pr\{N(B) = 1, N(C) = 0\}}{\Pr\{N(A) = 1\}}\\ &= \frac{\lambda|B|e^{-\lambda|B|}e^{-\lambda|c|}}{\lambda|A|e^{-\lambda|A|}}\\ &= \frac{|B|}{|A|} \quad (\text{because } |B| + |C| = |A|),\end{aligned}$$

and the proof is complete.

The generalization to n points in a region A is stated as follows. Consider a set A of positive size $|A| > 0$ and containing $N(A) = n \geq 1$ points. Then, these n points are independent and uniformly distributed in A in the sense that for any disjoint partition $A_1, \ldots, A_m$ of A, where $A_1 \cup \cdots \cup A_m = A$, and any positive integers $k_1, \ldots, k_m$, where $k_1 + \cdots + k_m = n$, we have

$$\begin{aligned}&\Pr\{N(A_1) = k_1, \ldots, N(A_m) = k_m|N(A) = n\}\\ &\quad = \frac{n!}{k_1! \cdots k_m!}\left(\frac{|A_1|}{|A|}\right)^{k_1} \cdots \left(\frac{|A_m|}{|A|}\right)^{k_m}.\end{aligned} \tag{5.28}$$

Equation (5.28) expresses the multinomial distribution for the conditional distribution of $N(A_1), \ldots, N(A_m)$ given that $N(A) = n$.

Example *An Application in Astronomy* Consider stars distributed in space in accordance with a three-dimensional Poisson point process of intensity $\lambda > 0$.

Let $\mathbf{x}$ and $\mathbf{y}$ designate general three-dimensional vectors, and assume that the light intensity exerted at $\mathbf{x}$ by a star located at $\mathbf{y}$ is $f(\mathbf{x},\mathbf{y},\alpha) = \alpha/\|\mathbf{x}-\mathbf{y}\|^2 = \alpha/\left[(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right]$, where α is a random parameter depending on the intensity of the star at $\mathbf{y}$. We assume that the intensities α associated with different stars are independent, identically distributed random variables possessing a common mean μ_α and variance σ_α^2. We also assume that the combined intensity exerted at the point $\mathbf{x}$ due to light created by different stars accumulates additively. Let $Z(\mathbf{x},A)$ denote the total light intensity at the point $\mathbf{x}$ due to signals emanating from all sources located in region A. Then

$$\begin{aligned} Z(\mathbf{x},A) &= \sum_{r=1}^{N(A)} f(\mathbf{x},\mathbf{y}_r,\alpha_r) \\ &= \sum_{r=1}^{N(A)} \frac{\alpha_r}{\|\mathbf{x}-\mathbf{y}_r\|^2}, \end{aligned} \tag{5.29}$$

where $\mathbf{y}_r$ is the location of the rth star in A. We recognize (5.29) as a random sum, as discussed in Chapter 2, Section 2.3.2. Accordingly, we have the mean intensity at $\mathbf{x}$ given by

$$E[Z(\mathbf{x},A)] = (E[N(A)])(E[f(\mathbf{x},\mathbf{y},\alpha)]). \tag{5.30}$$

Note that $E[N(A)] = \lambda|A|$, while because we have assumed α and $\mathbf{y}$ to be independent,

$$E[f(\mathbf{x},\mathbf{y},\alpha)] = E[\alpha]E\left[\|\mathbf{x}-\mathbf{y}\|^{-2}\right].$$

But as a consequence of the Poisson distribution of stars in space, we may take $\mathbf{y}$ to be uniformly distributed in A. Thus

$$E\left[\|\mathbf{x}-\mathbf{y}\|^{-2}\right] = \frac{1}{|A|}\int_A \frac{\mathrm{d}\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^2}.$$

With $\mu_\alpha = E[\alpha]$, then (5.30) reduces to

$$E[Z(\mathbf{x},A)] = \lambda\mu_\alpha \int_A \frac{\mathrm{d}y}{\|\mathbf{x}-\mathbf{y}\|2}.$$

Exercises

5.5.1 Bacteria are distributed throughout a volume of liquid according to a Poisson process of intensity $\theta = 0.6$ organisms per mm^3. A measuring device counts the number of bacteria in a 10 mm^3 volume of the liquid. What is the probability that more than two bacteria are in this measured volume?

5.5.2 Customer arrivals at a certain service facility follow a Poisson process of unknown rate. Suppose it is known that 12 customers have arrived during the first 3 h. Let N_i be the number of customers who arrive during the ith hour, $i = 1, 2, 3$. Determine the probability that $N_1 = 3, N_2 = 4$, and $N_3 = 5$.

5.5.3 Defects (air bubbles, contaminants, chips) occur over the surface of a varnished tabletop according to a Poisson process at a mean rate of one defect per top. If two inspectors each check separate halves of a given table, what is the probability that both inspectors find defects?

Problems

5.5.1 A piece of a fibrous composite material is sliced across its circular cross section of radius R, revealing fiber ends distributed across the circular area according to a Poisson process of rate 100 fibers per cross section. The locations of the fibers are measured, and the radial distance of each fiber from the center of the circle is computed. What is the probability density function of this radial distance X for a randomly chosen fiber?

5.5.2 Points are placed on the surface of a circular disk of radius one according to the following scheme. First, a Poisson distributed random variable N is observed. If $N = n$, then n random variables $\theta_1, \ldots, \theta_n$ are independently generated, each uniformly distributed over the interval $[0, 2\pi)$, and n random variables $R_1, \ldots, R_n$ are independently generated, each with the triangular density $f(r) = 2r, 0 < r < 1$. Finally, the points are located at the positions with polar coordinates $(R_i, \theta_i), i = 1, \ldots, n$. What is the distribution of the resulting point process on the disk?

5.5.3 Let $\left\{N(A); A \in \mathbf{R}^2\right\}$ be a homogeneous Poisson point process in the plane, where the intensity is λ. Divide the $(0, t] \times (0, t]$ square into n^2 boxes of side length $d = t/n$. Suppose there is a reaction between two or more points whenever they are located within the same box. Determine the distribution for the number of reactions, valid in the limit as $t \to \infty$ and $d \to 0$ in such a way that $td \to \mu > 0$.

5.5.4 Consider spheres in three-dimensional space with centers distributed according to a Poisson distribution with parameter $\lambda|A|$, where $|A|$ now represents the volume of the set A. If the radii of all spheres are distributed according to $F(r)$ with density $f(r)$ and finite third moment, show that the number of spheres that cover a point $\mathbf{t}$ is a Poisson random variable with parameter $\frac{4}{3}\lambda\pi \int_0^\infty r^3 f(r)\mathrm{d}r$.

5.5.5 Consider a two-dimensional Poisson process of particles in the plane with intensity parameter ν. Determine the distribution $F_D(x)$ of the distance between a particle and its nearest neighbor. Compute the mean distance.

5.5.6 Suppose that stars are distributed in space following a Poisson point process of intensity λ. Fix a star *alpha* and let R be the distance from alpha to its nearest neighbor. Show that R has the probability density function

$$f_R(x) = (4\lambda\pi x^2)\exp\left\{\frac{-4\lambda\pi x^3}{3}\right\}, \quad x > 0.$$

5.5.7 Consider a collection of circles in the plane whose centers are distributed according to a spatial Poisson process with parameter $\lambda|A|$, where $|A|$ denotes the area of the set A. (In particular, the number of centers $\xi(A)$ in the set A follows the distribution law $\Pr\{\xi(A)=k\}=\mathrm{e}^{-\lambda|A}\left[(\lambda|A|)^k/k!\right]$.) The radius of each circle is assumed to be a random variable independent of the location of the center of the circle, with density function $f(r)$ and finite second moment.

(a) Show that $C(r)$, defined to be the number of circles that cover the origin and have centers at a distance less than r from the origin, determines a variable-time Poisson process, where the time variable is now taken to be the distance r.

Hint: Prove that an event occurring between r and $r+\mathrm{d}r$ (i.e., there is a circle that covers the origin and whose center is in the ring of radius r to $r+\mathrm{d}r$) has probability $\lambda 2\pi r\,\mathrm{d}r\int_r^\infty f(\rho)\,\mathrm{d}\rho + o(\mathrm{d}r)$, and events occurring over disjoint intervals constitute independent random variables. Show that $C(r)$ is a variable-time (nonhomogeneous) Poisson process with parameter

$$\lambda(r) = 2\pi\lambda r\int_r^\infty f(\rho)\mathrm{d}\rho.$$

(b) Show that the number of circles that cover the origin is a Poisson random variable with parameter $\lambda\int_0^\infty \pi r^2 f(r)\mathrm{d}r$.

5.6 Compound and Marked Poisson Processes

Given a Poisson process $X(t)$ of rate $\lambda > 0$, suppose that each event has associated with it a random variable, possibly representing a value or a cost. Examples will appear shortly. The successive values $Y_1, Y_2, \ldots$ are assumed to be independent, independent of the Poisson process, and random variables sharing the common distribution function

$$G(y) = \Pr\{Y_k \le y\}.$$

A *compound Poisson process* is the cumulative value process defined by

$$Z(t) = \sum_{k=1}^{X(t)} Y_k \quad \text{for } t \ge 0. \tag{5.31}$$

A *marked Poisson process* is the sequence of pairs $(W_i, Y_1), (W_2, Y_2), \ldots,$ where $W_1, W_2, \ldots$ are the waiting times or event times in the Poisson process $X(t)$.

Both compound Poisson and marked Poisson processes appear often as models of physical phenomena.

5.6.1 Compound Poisson Processes

Consider the compound Poisson process $Z(t) = \Sigma_{k=1}^{x(t)} Y_k$. If $\lambda > 0$ is the rate for the process $X(t)$ and $\mu = E[Y_1]$ and $\nu^2 = \mathrm{Var}[Y_1]$ are the common mean and variance for $Y_1, Y_2, \ldots$ then the moments of $Z(t)$ can be determined from the random sums formulas of Chapter 2, Section 2.3.2 and are

$$E[Z(t)] = \lambda\mu t; \quad \mathrm{Var}[Z(t)] = \lambda\left(\nu^2 + \mu^2\right)t. \tag{5.32}$$

Examples

(a) *Risk Theory* Suppose claims arrive at an insurance company in accordance with a Poisson process having rate λ. Let Y_k be the magnitude of the kth claim. Then, $Z(t) = \Sigma_{k=1}^{X(t)} Y_k$ represents the cumulative amount claimed up to time t.

(b) *Stock Prices* Suppose that transactions in a certain stock take place according to a Poisson process of rate λ. Let Y_k denote the change in market price of the stock between the kth and $(k-1)$th transaction.

The *random walk hypothesis* asserts that $Y_1, Y_2, \ldots$ are independent random variables. The random walk hypothesis, which has a history dating back to 1900, can be deduced formally from certain assumptions describing a "perfect market."

Then, $Z(t) = \Sigma_{k=1}^{X(t)} Y_k$ represents the total price change up to time t.

This stock price model has been proposed as an explanation for why stock price changes do not follow a Gaussian (normal) distribution.

The distribution function for the compound Poisson process $Z(t) = \Sigma_{k=1}^{X(t)} Y_k$ can be represented explicitly after conditioning on the values of $X(t)$. Recall the convolution notation

$$\begin{aligned} G^{(n)}(y) &= \Pr\{Y_1 + \cdots + Y_n \le y\} \\ &= \int_{-\infty}^{+\infty} G^{(n-1)}(y-z)\mathrm{d}G(z) \end{aligned} \tag{5.33}$$

with

$$G^{(0)}(y) = \begin{cases} 1 & \text{for } y \ge 0, \\ 0 & \text{for } y < 0. \end{cases}$$

Then

$$\begin{aligned} \Pr\{Z(t) \le z\} &= \Pr\left\{\sum_{k=1}^{X(t)} Y_k \le z\right\} \\ &= \sum_{n=0}^{\infty} \Pr\left\{\sum_{k=1}^{X(t)} Y_k \le z | X(t) = n\right\} \frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!} G^{(n)}(z) \quad \text{(since } X(t) \text{ is independent of } Y_1, Y_2, \ldots). \end{aligned} \tag{5.34}$$

Example *A Shock Model* Let $X(t)$ be the number of shocks to a system up to time t and let Y_k be the damage or wear incurred by the kth shock. We assume that damage is positive, i.e., that $\Pr\{Y_k \ge 0\} = 1$, and that the damage accumulates additively, so that $Z(t) = \Sigma_{k=1}^{X(t)} Y_k$ represents the total damage sustained up to time t. Suppose that the system continues to operate as long as this total damage is less than some critical value a and fails in the contrary circumstance. Let T be the time of system failure.

Then

$$\{T > t\} \quad \text{if and only if} \quad \{Z(t) < a\}. \quad \text{(Why?)} \tag{5.35}$$

In view of (5.34) and (5.35), we have

$$\Pr\{T > t\} = \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} G^{(n)}(a).$$

All summands are nonnegative, so we may interchange integration and summation to get the mean system failure time

$$\begin{aligned} E[T] &= \int_0^{\infty} \Pr\{T > t\}\, dt \\ &= \sum_{n=0}^{\infty} \left(\int_0^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} dt \right) G^{(n)}(a) \\ &= \lambda^{-1} \sum_{n=0}^{\infty} G^{(n)}(a). \end{aligned}$$

This expression simplifies greatly in the special case in which $Y_1, Y_2, \ldots$ are each exponentially distributed according to the density $g_Y(y) = \mu e^{-\mu y}$ for $y \ge 0$. Then, the sum $Y_1 + \cdots + Y_n$ has the gamma distribution

$$G^{(n)}(z) = 1 - \sum_{k=0}^{n-1} \frac{(\mu z)^k e^{-\mu z}}{k!} = \sum_{k=n}^{\infty} \frac{(\mu z)^k e^{-\mu z}}{k!},$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} G^{(n)}(a) &= \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{(\mu a)^k e^{-\mu a}}{k!} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{(\mu a)^k e^{-\mu a}}{k!} \\ &= \sum_{k=0}^{\infty} (1 + k) \frac{(\mu a)^k e^{-\mu a}}{k!} \\ &= 1 + \mu a. \end{aligned}$$

When $Y_1, Y_2, \ldots$ are exponentially distributed, then

$$E[T] = \frac{1+\mu a}{\lambda}.$$

5.6.2 Marked Poisson Processes

Again suppose that a random variable Y_k is associated with the kth event in a Poisson process of rate λ. We stipulate that $Y_1, Y_2, \ldots$ are independent, independent of the Poisson process, and share the common distribution function

$$G(y) = \Pr\{Y_k \leq y\}.$$

The sequence of pairs $(W_1, Y_1), (W_2, Y_2), \ldots$ is called a *marked Poisson process*.

We begin the analysis of marked Poisson processes with one of the simplest cases. For a fixed value $p (0 < p < 1)$, suppose

$$\Pr\{Y_k = 1\} = p, \quad \Pr\{Y_k = 0\} = q = 1 - p.$$

Now consider separately the processes of points marked with ones and of points marked with zeros. In this case, we can define the relevant Poisson processes explicitly by

$$X_1(t) = \sum_{k=1}^{X(t)} Y_k \quad \text{and} \quad X_0(t) = X(t) - X_1(t).$$

Then, nonoverlapping increments in $X_1(t)$ are independent random variables, $X_1(0) = 0$, and finally, Theorem 5.2 applies to assert that $X_1(t)$ has a Poisson distribution with mean λpt. In summary, $X_1(t)$ is a Poisson process with rate λp, and the parallel argument shows that $X_0(t)$ is a Poisson process with rate $\lambda(1-p)$. *What is even more interesting and surprising is that $X_0(t)$ and $X_1(t)$ are independent processes*! The relevant property to check is that $\Pr\{X_0(t) = j \text{ and } X_1(t) = k\} = \Pr\{X_0(t) = j\} \times \Pr\{X_1(t) = k\}$ for $j, k = 0, 1, \ldots$. We establish this independence by writing

$$\begin{aligned}
\Pr\{X_0(t) = j, X_1(t) = k\} &= \Pr\{X_0(t) = j + k, X_1(t) = k\} \\
&= \Pr\{X_1(t) = k | X(t) = j + k\} \Pr\{X(t) = j + k\} \\
&= \frac{(j+k)!}{j!k!} p^k (1-p)^j \frac{(\lambda t)^{j+k} e^{-\lambda t}}{(j+k)!} \\
&= \left[\frac{e^{-\lambda pt}(\lambda pt)^k}{k!}\right] \left[\frac{e^{-\lambda(1-p)t}(\lambda(1-p)t)^j}{j!}\right] \\
&= \Pr\{X_1(t) = k\} \Pr\{X_0(t) = j\} \\
&\qquad \text{for } j, k = 0, 1, \ldots.
\end{aligned}$$

Example Customers enter a store according to a Poisson process of rate $\lambda = 10$ per hour. Independently, each customer buys something with probability $p = 0.3$ and leaves without making a purchase with probability $q = 1 - p = 0.7$. What is the probability that during the first hour 9 people enter the store and that 3 of these people make a purchase and 6 do not?

Let $X_1 = X_1(1)$ be the number of customers who make a purchase during the first hour and $X_0 = X_0(1)$ be the number of people who do not. Then, X_1 and X_0 are independent Poisson random variables having respective rates $0.3(10) = 3$ and $0.7(10) = 7$. According to the Poisson distribution,

$$\Pr\{X_1 = 3\} = \frac{3^3 e^{-3}}{3!} = 0.2240,$$

$$\Pr\{X_0 = 6\} = \frac{7^6 e^{-7}}{6!} = 0.1490,$$

and

$$\Pr\{X_1 = 3, X_0 = 6\} = \Pr\{X_1 = 3\}\Pr\{X_0 = 6\} = (0.2240)(0.1490) = 0.0334.$$

In our study of marked Poisson processes, let us next consider the case where the value random variables $Y_1, Y_2, \ldots$ are discrete, with possible values $0, 1, 2, \ldots$ and

$$\Pr\{Y_n = k\} = a_k > 0 \quad \text{for } k = 0, 1, \ldots, \text{ with } \sum_k a_k = 1.$$

In Figure 5.9, the original Poisson event times $W_1, W_2, \ldots$ are shown on the bottom axis. Then, a point is placed in the (t, y) plane at (W_n, Y_n) for every n. For every integer $k = 0, 1, 2, \ldots$, one obtains a point process that corresponds to the times W_n for which $Y_n = k$. The same reasoning as in the zero–one case applies to imply that each of these

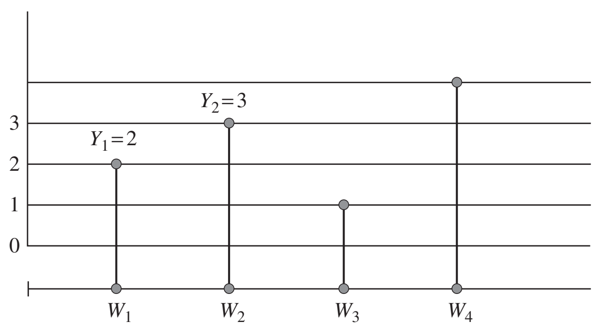

Figure 5.9 A marked Poisson process. $W_1, W_2, \ldots$ are the event times in a Poisson process of rate λ. The random variables $Y_1, Y_2, \ldots$ are the markings, assumed to be independent and identically distributed, and independent of the Poisson process.

processes is Poisson, the rate for the kth process being λa_k, and that processes for distinct values of k are independent.

To state the corresponding decomposition result when the values $Y_1, Y_2, \ldots$ are continuous random variables requires a higher level of sophistication, although the underlying ideas are basically the same. To set the stage for the formal statement, we first define what we mean by a nonhomogeneous Poisson point process in the plane, thus extending the homogeneous processes of the previous section. Let $\theta = \theta(x, y)$ be a nonnegative function defined on a region S in the (x, y) plane. For each subset A of S, let $\mu(A) = \iint_A \theta(x, y)\mathrm{d}x\,\mathrm{d}y$ be the volume under $\theta(x, y)$ enclosed by A. A nonhomogeneous Poisson point process of intensity function $\theta(x, y)$ is a point process $\{N(A); A \subset S\}$ for which

1. for each subset A of S, the random variable $N(A)$ has a Poisson distribution with mean $\mu(A)$;
2. for disjoint subsets $A_1, \ldots, A_m$ of S, the random variables $N(A_1), \ldots, N(A_m)$ are independent.

It is easily seen that the homogeneous Poisson point process of intensity λ corresponds to the function $\theta(x, y)$ being constant, and $\theta(x, y) = \lambda$ for all x, y.

With this definition in hand, we state the appropriate decomposition result for general marked Poisson processes.

Theorem 5.8. *Let $(W_1, Y_1), (W_2, Y_2), \ldots$ be a marked Poisson process where $W_1, W_2, \ldots$ are the waiting times in a Poisson process of rate λ and $Y_1, Y_2, \ldots$ are independent identically distributed continuous random variables having probability density function $g(y)$. Then $(W_1, Y_1), (W_2, Y_2), \ldots$ form a two-dimensional nonhomogeneous Poisson point process in the (t, y) plane, where the mean number of points in a region A is given by*

$$\mu(A) = \iint_A \lambda g(y)\mathrm{d}y\,\mathrm{d}t. \tag{5.36}$$

Figure 5.10 diagrams the scene.

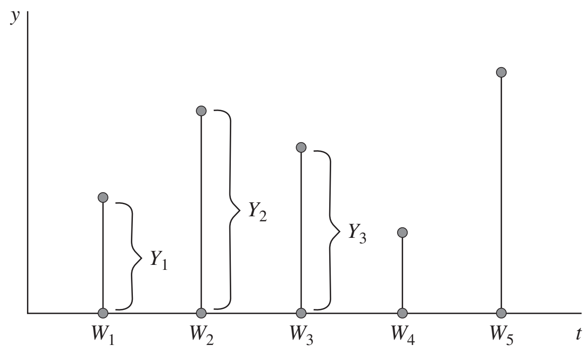

Figure 5.10 A marked Poisson process.

Theorem 5.8 asserts that the numbers of points in disjoint intervals are independent random variables. For example, the waiting times corresponding to positive values $Y_1, Y_2, \ldots$ form a Poisson process, as do the times associated with negative values, and these two processes are independent.

Example *Crack Failure* The following model is proposed to describe the failure time of a sheet or volume of material subjected to a constant stress σ. The failure time is viewed in two parts, crack initiation and crack propagation.

Crack initiation occurs according to a Poisson process whose rate per unit time and unit volume is a constant $\lambda_\sigma > 0$ depending on the stress level σ. Then, crack initiation per unit time is a Poisson process of rate $\lambda_\sigma|V|$, where $|V|$ is the volume of material under consideration. We let $W_1, W_2, \ldots$ be the times of crack initiation.

Once begun, a crack grows at a random rate until it reaches a critical size, at which instant structural failure occurs. Let Y_k be the time to reach critical size for the kth crack. The cumulative distribution function $G_\sigma(y) = \Pr\{Y_k \le y\}$ depends on the constant stress level σ.

We assume that crack initiations are sufficiently sparse as to make $Y_1, Y_2, \ldots$ independent random variables. That is, we do not allow two small cracks to join and form a larger one.

The structural failure time Z is the smallest of $W_1 + Y_1, W_2 + Y_2, \ldots$. It is not necessarily the case that the first crack to appear will cause system failure. A later crack may grow to critical size faster.

In the (t, y) plane, the event $\{\min\{W_k + Y_k\} > z\}$ corresponds to no points falling in the triangle $\triangle = \{(t, y) : t + y \le z, t \ge 0, y \ge 0\}$, as shown in Figure 5.11.

The number of points $N(\triangle)$ falling in the triangle $\triangle$ has a Poisson distribution with mean $\mu(\triangle)$ given, according to (5.36), by

$$\mu(\triangle) = \iint_{\triangle} \lambda_\sigma |V| \mathrm{d}s g_\sigma(u) \mathrm{d}u$$

$$= \int_0^z \lambda_\sigma |V| \left\{ \int_0^{z-s} g_\sigma(u) \mathrm{d}u \right\} \mathrm{d}s$$

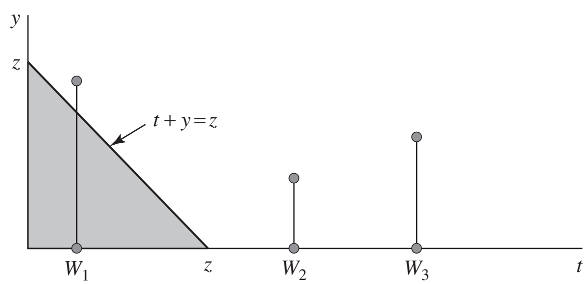

Figure 5.11 A crack failure model.

$$= \lambda_\sigma |V| \int_0^z G_\sigma(z-s)\mathrm{d}s$$

$$= \lambda_\sigma |V| \int_0^z G_\sigma(\nu)\mathrm{d}\nu.$$

From this we obtain the cumulative distribution function for structural failure time,

$$\begin{aligned}\Pr\{Z \le z\} &= 1 - \Pr\{Z > z\} = 1 - \Pr\{N(\triangle) = 0\}\\ &= 1 - \exp\left\{ -\lambda_\sigma |V| \int_0^z G_\sigma(\nu)\mathrm{d}\nu \right\}.\end{aligned}$$

Observe the appearance of the so-called *size effect* in the model, wherein the structure volume $|V|$ affects the structural failure time even at constant stress level σ. The parameter λ_σ and distribution function $G_\sigma(y)$ would require experimental determination.

Example *The Strength Versus Length of Filaments* It was noted that the logarithm of mean tensile strength of brittle fibers, such as boron filaments, in general varies linearly with the logarithm of the filament length, but that this relation did not hold for short filaments. It was suspected that the breakdown in the log linear relation might be due to testing or measurement problems, rather than being an inherent property of short filaments. Evidence supporting this idea was the observation that short filaments would break in the test clamps, rather than between them as desired, more often than would long filaments. Some means of correcting observed mean strengths to account for filaments breaking in, rather than between, the clamps was desired. It was decided to compute the ratio between the actual mean strength and an ideal mean strength, obtained under the assumption that there was no stress in the clamps, as a correction factor.

Since the molecular bonding strength is several orders of magnitude higher than generally observed strengths, it was felt that failure typically was caused by flaws. There are a number of different types of flaws, both internal flaws such as voids, inclusions, and weak grain boundaries, and external, or surface, flaws such as notches and cracks that cause stress concentrations. Let us suppose that flaws occur independently in a Poisson manner along the length of the filament. We let Y_k be the strength of the filament at the kth flaw and suppose Y_k has the cumulative distribution function $G(y), y > 0$. We have plotted this information in Figure 5.12. The flaws reduce the strength. Opposing the strength is the stress in the filament. Ideally, the stress should be constant along the filament between the clamp faces and zero within the clamp. In practice, the stress tapers off to zero over some positive length in the clamp. As a first approximation it is reasonable to assume that the stress decreases linearly. Let l be the length of the clamp and t the distance between the clamps, called the *gauge length*, as illustrated in Figure 5.12 on the next page.

The filament holds as long as the stress has not exceeded the strength as determined by the weakest flaw. That is, the filament will support a stress of y as long as no flaw points fall in the stress trapezoid of Figure 5.12. The number of points in this trapezoid has a Poisson distribution with mean $\mu(B) + 2\mu(A)$. In particular, no points fall there with probability $e^{-[\mu(B)+2\mu(A)]}$. If we let S be the strength of the filament, then

$$\Pr\{S > y\} = e^{-2\mu(A)-\mu(B)}.$$

We compute

$$\mu(A) = \int_0^l G\left(\frac{xy}{l}\right)\lambda\, dx$$

and

$$\mu(B) = \lambda t G(y).$$

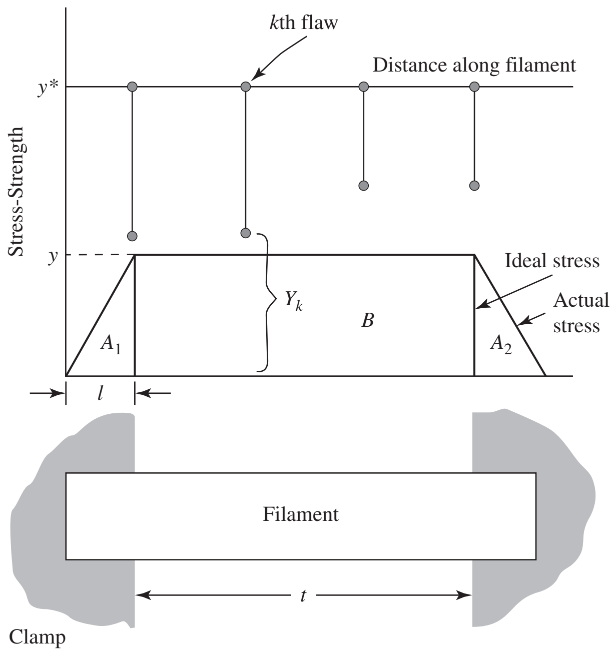

Figure 5.12 The stress versus strength of a filament under tension. Flaws reducing the strength of a filament below its theoretical maximum y^* are distributed randomly along its length. The stress in the filament is constant at the level y between clamps and tapers off to zero within the clamps. The filament fails if at any point along its length a flaw reduces its strength below its stress.

Finally, the mean strength of the filament is

$$E[S] = \int_0^\infty \Pr\{S > y\}\,\mathrm{d}y = \int_0^\infty \exp\left\{-\lambda\left[tG(y) + 2\int_0^l c\left(\frac{xy}{l}\right)\mathrm{d}x\right]\right\}\mathrm{d}y.$$

For an ideal filament, we use the same expression but with $l = 0$.

Exercises

5.6.1 Customers demanding service at a central processing facility arrive according to a Poisson process of intensity $\theta = 8$ per unit time. Independently, each customer is classified as *high priority* with probability $\alpha = 0.2$, or *low priority* with probability $1 - \alpha = 0.8$. What is the probability that three high priority and five low priority customers arrive during the first unit of time?

5.6.2 Shocks occur to a system according to a Poisson process of intensity λ. Each shock causes some damage to the system, and these damages accumulate. Let $N(t)$ be the number of shocks up to time t, and let Y_i be the damage caused by the ith shock. Then

$$X(t) = Y_1 + \cdots + Y_{N(t)}$$

is the total damage up to time t. Determine the mean and variance of the total damage at time t when the individual shock damages are exponentially distributed with parameter θ.

5.6.3 Let $\{N(t); t \geq 0\}$ be a Poisson process of intensity λ, and let $Y_1, Y_2, \ldots$ be independent and identically distributed nonnegative random variables with cumulative distribution function $G(y) = \Pr\{Y \leq y\}$. Determine $\Pr\{Z(t) > z | N(t) > 0\}$, where

$$Z(t) = \min\left\{Y_1, Y_2, \ldots, Y_{N(t)}\right\}.$$

5.6.4 Men and women enter a supermarket according to independent Poisson processes having respective rates of two and four per minute.

(a) Starting at an arbitrary time, what is the probability that at least two men arrive before the first woman arrives?

(b) What is the probability that at least two men arrive before the third woman arrives?

5.6.5 Alpha particles are emitted from a fixed mass of material according to a Poisson process of rate λ. Each particle exists for a random duration and is then annihilated. Suppose that the successive lifetimes $Y_1, Y_2, \ldots$ of distinct particles are independent random variables having the common distribution function $G(y) = \Pr\{Y_k \leq y\}$. Let $M(t)$ be the number of particles existing at time t. By considering the lifetimes as markings, identify the region in the lifetime, arrival-time space that corresponds to $M(t)$, and thereby deduce the probability distribution of $M(t)$.

Problems

5.6.1 Suppose that points are distributed over the half line $[0, \infty)$ according to a Poisson process of rate λ. A sequence of independent and identically distributed nonnegative random variables $Y_1, Y_2, \ldots$ is used to reposition the points so that a point formerly at location W_k is moved to the location $W_k + Y_k$. Completely describe the distribution of the relocated points.

5.6.2 Suppose that particles are distributed on the surface of a circular region according to a spatial Poisson process of intensity λ particles per unit area. The polar coordinates of each point are determined, and each angular coordinate is shifted by a random amount, with the amounts shifted for distinct points being independent random variables following a fixed probability distribution. Show that at the end of the point movement process, the points are still Poisson distributed over the region.

5.6.3 Shocks occur to a system according to a Poisson process of intensity λ. Each shock causes some damage to the system, and these damages accumulate. Let $N(t)$ be the number of shocks up to time t, and let Y_i be the damage caused by the ith shock. Then

$$X(t) = Y_1 + \cdots + Y_{N(t)}$$

is the total damage up to time t. Suppose that the system continues to operate as long as the total damage is strictly less than some critical value a, and fails in the contrary circumstance. Determine the mean time to system failure when the individual damages Y_i have a geometric distribution with $\Pr\{Y = k\} = p(1-p)^k, k = 0, 1, \ldots$.

5.6.4 Let $\{X(t); t \geq 0\}$ and $\{Y(t); t \geq 0\}$ be independent Poisson processes with respective parameters λ and μ. For a fixed integer a, let $T_a = \min\{t \geq 0; Y(t) = a\}$ be the random time that the Y process first reaches the value a. Determine $\Pr\{X(T_a) = k\}$ for $k = 0, 1, \ldots$.

Hint: First consider $\xi = X(T_1)$ in the case in which $a = 1$. Then, ξ has a geometric distribution. Then, argue that $X(T_a)$ for general a has the same distribution as the sum of a independent ξs and hence has a negative binomial distribution.

5.6.5. Let $\{X(t); t \geq 0\}$ and $\{Y(t); t \geq 0\}$ be independent Poisson processes with respective parameters λ and μ. Let $T = \min\{t \geq 0; Y(t) = 1\}$ be the random time of the first event in the Y process. Determine $\Pr\{X(T/2) = k\}$ for $k = 0, 1, \ldots$.

5.6.6 Let $W_1, W_2, \ldots$ be the event times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ. A new point process is created as follows: Each point W_k is replaced by two new points located at $W_k + X_k$ and $W_k + Y_k$, where $X_1, Y_1, X_2, Y_2, \ldots$ are independent and identically distributed nonnegative random variables, independent of the Poisson process. Describe the distribution of the resulting point process.

5.6.7 Let $\{N(t); t \geq 0\}$ be a Poisson process of intensity λ, and let $Y_1, Y_2, \ldots$ be independent and identically distributed nonnegative random variables with cumulative distribution function

$$G(y) = y^{\alpha} \quad \text{for } 0 < y < 1.$$

Determine $\Pr\{Z(t) > z | N(t) > 0\}$, where

$$Z(t) = \min\left\{Y_1, Y_2, \ldots, Y_{N(t)}\right\}.$$

Describe the behavior for large t.

5.6.8 Let $\{N(t); t \geq 0\}$ be a nonhomogeneous Poisson process of intensity $\lambda(t)$, $t > 0$, and let $Y_1, Y_2, \ldots$ be independent and identically distributed nonnegative random variables with cumulative distribution function

$$G(y) = y^{\alpha} \quad \text{for } 0 < y < 1.$$

Suppose that the intensity process averages out in the sense that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \lambda(u) \mathrm{d}u = \theta.$$

Let

$$Z(t) = \min\left\{Y_1, Y_2, \ldots, Y_{N(t)}\right\}.$$

Determine

$$\lim_{t \to \infty} \Pr\left\{t^{1/\alpha} Z(t) > z\right\}.$$

5.6.9 Let $W_1, W_2, \ldots$ be the event times in a Poisson process of rate λ, and let $N(t) = N((0, t])$ be the number of points in the interval $(0, t]$. Evaluate

$$E\left[\sum_{k=1}^{N(t)} (W_k)^2\right].$$

Note: $\Sigma_{k=1}^{0} (W_k)^2 = 0$.

5.6.10 *A Bidding Model* Let $U_1, U_2, \ldots$ be independent random variables, each uniformly distributed over the interval $(0, 1]$. These random variables represent successive bids on an asset that you are trying to sell, and that you must sell by time $t = 1$, when the asset becomes worthless. As a strategy, you adopt a secret number θ, and you will accept the first offer that is greater than θ. For example, you accept the second offer if $U_1 \leq \theta$ while $U_2 > \theta$. Suppose that the offers arrive according to a unit rate Poisson process ($\lambda = 1$).

(a) What is the probability that you sell the asset by time $t = 1$?

(b) What is the value for θ that maximizes your expected return? (You get nothing if you don't sell the asset by time $t = 1$.)

(c) To improve your return, you adopt a new strategy, which is to accept an offer at time t if it exceeds $\theta(t) = (1 - t)/(3 - t)$. What are your new chances of selling the asset, and what is your new expected return?