Tutorial 3

STAT 150

Exercise 1

A communication class C of a Markov Chain is open if it is possible to leave, i.e. $p_{ij} > 0$ for some i in C and j not in C. Otherwise, a class is called closed. Argue that for any Markov Chain on a finite state space, there is at least one closed class.

Exercise 2

Show that periodicity is a class property.

We use the notation $d(i) = gcd\{n \geq 1 : p_{ii}^{(n)} > 0\}$. Show that if $i \leftrightarrow j$ then $d(i) = d(j)$.

Exercise 3 *Characterization of recurrence/transcience*

Suppose (X_n) is a Markov chain on a state space Ω with transition matrix $P = (P_{i,j})$. We fix $i \in \Omega$, we introduce N_i as the number of times the Markov chain visits the state i and $T_i = \min\{n \geq 1 : X_n = i\}$ the first time we revisit i. Finally let $f_i = \mathbb{P}[T_i < \infty | X_0 = i]$. Show that :

(a) $\mathbb{P}[N_i = k] = f_i^{k-1}(1 - f_i)$

(b) i is recurrent if and only if $\mathbb{E}[N_i] = \infty$

(c) Deduce that i is recurrent if and only if : $\sum_{n=0}^{\infty} P_{i,i}^n = \infty$

Exercise 4

(a) Show that recurrence is a class property.

(b) Show that a finite closed class is always recurrent (all the states are recurrent).