# Midterm Test 2 — Key

Stat 150 — Fall 2018

1. (a) **[2 marks]**
   If $X_n \geq 0$ is a martingale then $X_\infty = \lim_{n\to\infty} X_n$ exists and $\mathbb{E}X_\infty \leq \mathbb{E}X_0$.

   (b) **[2 marks]**
   See Example 5.15 in [D] p173.

   (c) **[3 marks]**
   This is the "critical case" in Example 5.15.

2. (a) **[3 marks]**
   See Lemma 5.18 in [D] p172.

   (b) **[4 marks]**
   Example 5.14 "Polya's urn" (setting $k = 1$) in [D] p172 shows that the proportion of white balls $X_n$ after $n$ steps is a martingale. So by (a), $\mathbb{P}(\max_{n\geq 0} X_n > 3/4) \leq (4/3)\mathbb{E}X_0 = 2/3$.

3. **[7 marks]**

   See [PK] p256 ("sum quota sampling").

4. This is similar to the example in [PK] p334 ("redundancy and the burn-in pehonomenon"), except here there are two repair facilities.

   (a) **[3 marks]**
   $$G = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$

   (b) **[2 marks]**
   Solving $\pi G = 0$, we find $(\pi_0, \pi_1, \pi_2) = (\mu^2, 2\lambda\mu, 2\lambda^2)/(\lambda^2 + (\lambda+\mu)^2)$.

   (c) **[2 marks]**
   Setting $\mu = 1$ and solving for $\lambda$ in $\pi_1 + \pi_2 = 2\lambda(1+\lambda)/(\lambda^2 + (1+\lambda)^2) = 4/5$, we find $\lambda = 1$.

5. (a) **[2 marks]**
   $\mathbb{E}(M_{n+1}|X_n, \ldots, X_0) = M_n + \mathbb{E}(X_{n+1} - \mu) = M_n$.

   (b) **[2 marks]**
   $\mathbb{E}(|M_{n+1} - M_n||X_n, \ldots, X_0) = \mathbb{E}|X_{n+1} + \mu| \leq \mathbb{E}|X_{n+1}| + |\mu| \leq 2\mathbb{E}|X| < \infty$.

   (c) **[2 marks]**
   By the OST, $\mathbb{E}M_T = \mathbb{E}(\sum_{i=1}^T X_i) - \mu\mathbb{E}T = 0 = \mathbb{E}M_0$, so $\mathbb{E}(\sum_{i=1}^T X_i) = \mathbb{E}(T)\mathbb{E}(X)$.

   (d) **[3 marks]**
   For SRW $S_n = S_0 + \sum_{i=1}^n X_i$ where $X_i$ are IID $\pm 1$ with probability $1/2$, we clearly have $\mathbb{E}|X| < \infty$. Since SRW is REC we have $\mathbb{E}T < \infty$. So we can apply Wald's Equation. By FSA and symmetry, $\mathbb{E}_1 T = 1 + \mathbb{E}_0 T$. Suppose that $\mathbb{E}_0 T < \infty$. Then by Wald's Equation, $1 = \mathbb{E}_0 S_T = \mathbb{E}_0(T)\mathbb{E}_0(X) = \mathbb{E}_0(T)\cdot 0 = 0$ (since $\mathbb{E}_0(T) < \infty$), a contradiction. Hence $\mathbb{E}_0(T) = \infty$, and so $\mathbb{E}_1(T) = \infty$. Therefore $S_n$ is NULL REC (since it is IRR and NULL REC is a class property.

6. (a) **[1 mark]**

   $i > 0$ means there are $i$ customers in line, $i = 0$ means there are no customers or taxis waiting, $i < 0$ means there are $i$ taxis waiting and no customers.

   (b) **[2 marks]**

   $q(i, i+1) = \lambda$ for $i \geq -2$, $q(i, i-1) = \mu$ for $i \geq -1$, and all other $q(i, j) = 0$.

   (c) **[3 marks]**

   We solve the DBE $w_i q_{ij} = w_j q_{ji}$ and then normalize $\pi_i = w_i / \sum_j w_j$. Setting $w_{-2} = 1$ and noting that DBE imply that $w_i = (\lambda/\mu) w_{i-1}$ for $i \geq -1$, we find that $w_i = (\lambda/\mu)^{i+2}$. Note $\sum_{i \geq -2} (\lambda/\mu)^{i+2} = \sum_{i \geq 0} (\lambda/\mu)^i = 1/(1 - \lambda/\mu)$, since $\mu > \lambda$. Therefore $\pi_i = (1 - \lambda/\mu)(\lambda/\mu)^{i+2}$.

   (d) **[2 marks]**

   The LR probability that a customer has to wait is $\sum_{i \geq 1} \pi_i = (1 - \lambda/\mu) \sum_{i \geq 3} (\lambda/\mu)^i = (\lambda/\mu)^3$. Hence the LR average wait time per customer is $(\lambda/\mu)^3/(\mu - \lambda)$.

   Alternatively, this can be calculated as $\sum_{i \geq 1} (i/\mu) \pi_i = (1/\mu)(1 - \lambda/\mu) \sum_{i \geq 1} i (\lambda/\mu)^{i+2} = (1/\mu)(\lambda/\mu)^3/(1 - \lambda/\mu) = (\lambda/\mu)^3/(\mu - \lambda)$, recalling that $\sum_{i \geq 0} i x^i = x/(1 - x)^2$ for $|x| < 1$ (found by differentiating $\sum_{i \geq 0} x^i = 1/(1 - x)$).