Tutorial 7
STAT 150

Exercise 1 :

For a continuous time Markov chain $(X_t)_{t\geq 0}$ with transition matrix $(p_t(i,j))$, we let for every two states i,j , $q(i,j) = \lim_{h\to 0} \frac{p_h(i,j)}{h}$. If we let $\lambda_i = \sum_{j\neq i} q(i,j)$, we define the generator G of $(X_t)_{t\geq 0}$ as the matrix that has the following entries :

$$G(i,j) = q(i,j) \text{ if } i \neq j \text{ and } G(i,i) = -\lambda_i.$$

Hence, we have the transition matrix has the following form : $p_h(i,j) = G(i,j)h + o(h)$ if $i \neq j$ and $p_h(i,i) = 1 + G(i,i)h + o(h)$.

π is said to be a stationary distribution of $(X_t)_{t\geq 0}$ if $\pi p_t = \pi$ for every $t \geq 0$.

(a)- Find the forward equations in function of $p_t(i,j)$ and G.

(b)- Prove that π verifies $\pi G = 0$

Exercise 2 :
Let $X(t)$ be a continuous time Markov chain with states $\{1,2\}$ and :

$$P_{i,j}(t) = \tfrac{1}{2} - \tfrac{1}{2}e^{-t} \text{ if } i \neq j \quad \text{and}$$
$$P_{i,i}(t) = \tfrac{1}{2} + \tfrac{1}{2}e^{-t}.$$

(a) What is the generator of $X(t)$.

(b) What is its stationary distribution.

Exercise 3 : Let X be a Markov chain with states $\{1,2\}$ with generator

$$G = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

(a) Write down the forward equations for the transition probabilities $P_{i,j}(t)$.

(b) Using the forward equations or otherwise find $P_{1,1}(t)$.

(c) Find the stationary distribution.

(d) Compute $\mathbb{P}[X(1) = 2 | X(0) = 1, X(2) = 1]$

Exercise 4 :

Red cars arrive according to a rate α Poisson process and blue cars according to an independent rate β Poisson process. We are told that exactly one blue car has arrived by time 1.

(a) Conditional on this event find the distribution of the time of the first blue car.

(b) Find the conditional distribution of the first car of either colour.