Homework 3 : Solutions
STAT 150

Problem 4.1.5:

The transition matrix P of the Markov Chain is equal to :

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

The probability to find in the long run the train in town D is simply equal to the value of the stationary distribution at this state. It is easy to see after calculation that we find the following values :

$$\pi_A = \frac{2}{8}, \pi_B = \frac{3}{8}, \pi_C = \frac{1}{8}, \pi_D = \frac{2}{8}$$

The value we are looking for is thus : $p = \frac{2}{8}$.

Problem 4.4.1:

(a) $\pi_0 = \frac{\beta}{\alpha+\beta}, \pi_1 = \frac{\alpha}{\alpha+\beta}$.

(b) A first return to 0 at time n entails leaving 0 on the first step to 1, staying in 1 for $n-2$ transitions, and then returning to 0, hence : $f_{00}^{(n)} = \alpha\beta(1-\beta)^{n-2}$ for $n \geq 2$, and $f_{00}^{(1)} = 1-\alpha$

(c) We have :
$m_0 = \displaystyle\sum_{n=1}^{\infty} n f_{00}^{(n)} = 1-\alpha+\sum_{n=2}^{\infty} n\alpha\beta(1-\beta)^{n-2} = 1-\alpha+\alpha\beta(\frac{1}{\beta}+\frac{1}{\beta^2}) = \frac{\alpha+\beta}{\beta} = \frac{1}{\pi_0}$

Problem 4.4.6:

It is easy to see from the transition graph that it is possible to get from 0 to 0 in both 4 and 5 steps (either by going first to 1 or -1), hence as those two integers are relatively prime, then the period of 0 is equal to 1.

Problem 4.4.8:

We have that : $\pi(0) = \sum_{k=0}^{\infty} \frac{\pi(k)}{k+2}$ and for every $i \geq 1$:

$$\pi(i) = \sum_{k=i-1}^{\infty} \frac{\pi(k)}{k+2}$$

Hence it is easy to see that : $\pi(0) = \pi(1)$ and for every $k \geq 1$ we have : $\pi(k+1) = \pi(k) - \frac{\pi(k-1)}{k+1}$, hence by induction we get that $\pi(k) = \frac{\pi(0)}{k!}$ for every k, and thus $\pi(0) = e^{-1}$, π is thus the Poisson distribution of parameter 1.

Problem 4.4.4 :

We have to distinguish two cases in our approach. If either only a finite number of the α_i's is non zero or not. In the first case, where for i large enough $\alpha_i = 0$, let $n = \min\{i \geq 1 : \alpha_i = 0\}$, then all the states $i \geq n+1$ are transient, and thus the limiting distribution is equal to zero on those states. Now, if we restrict our Markov chain on the set $\{0, 1, .., n-1\}$, this is a irreducible, aperiodic and positive recurrent (because it is a finite state) Markov chain and thus admits a limiting distribution. On the other hand, if there is infinitely many non zeros in the sequence (α_i), then the Markov chain is irreducible, aperiodic. Suppose there exists a limiting distribution in this case, then this one is necesarly a stationary distribution, thus :

Let π be an eventual stationary distribution of the Markov chain, we have then :

$$\pi_0 = (\sum_{k=1}^{\infty} \alpha_k)\pi_0$$

$\pi_0 = \alpha_1\pi_0 + \pi_1$ hence, $\pi_1 = (1 - \alpha_1)\pi_0 = (\sum_{k=2}^{\infty} \alpha_k)\pi_0$ $\pi_1 = \alpha_2\pi_0 + \pi_2$

$$\text{then } \pi_2 = (1-\alpha_1-\alpha_2)\pi_0 = (\sum_{k=3}^{\infty}\alpha_k)\pi_0$$

Inductively we get that for every n : $\pi_n = (\sum_{k=n+1}^{\infty}\alpha_k)\pi_0$, so in order to ensure that π exists we must have : $\sum_{n=0}^{\infty}\pi_n = 1$ i.e : $\pi_0(\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\alpha_k) = 1$, i.e : $\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\alpha_k < \infty$, which translates to $\sum_{k=1}^{\infty}k\alpha_k < \infty$.

On the other hand, if we suppose that $\sum_{k=1}^{\infty}k\alpha_k < \infty$ then as we know that $m_0 = \mathbb{E}[R_0|X_0=0] = \sum_{k=1}^{\infty}k\alpha_k$, so in this case $m_0 < \infty$, so 0 is recurrent and thus as the chain is irreducible it is recurrent and verifies the conditions of Theorem 4.3 and hence the limiting distribution exists.

We see then that the only condition on the probability distribution $\{\alpha_i, i=1,..\}$ to guarantee that the Markov chain has a limiting distribution is the fact that $\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\alpha_k < \infty$, which is the same as $\mathbb{E}[X] < \infty$ for X a random variable such that $\mathbb{P}[X=k] = \alpha_k$. Also this same condition guarantee the existence of a stationary distribution too.

Problem 4.5.1 :

It is clear that the states $0, 1, 2$ are transient, and each of the two classes $\{3, 4\}$ and $\{5, 6, 7\}$ are recurrent and closed. Hence if the chain starts in one of those two classes it stays in it, it is easy to check that the stationary distributions of P_B and P_C are respectively : $\pi_B = (\pi_3, \pi_4) = (\frac{1}{2}, \frac{1}{2})$ and $\pi_C = (\pi_5, \pi_6, \pi_7) = (0.4227, 0.2371, 0.3402)$.

The two classes B and C can be seen as absorbant states from the matrix P_A so if we starts from $0, 1$ or 2, the probability to hit each one of those is equal to :

$$\begin{array}{cc} & \begin{array}{cc} 3-4 & 5-7 \end{array} \\ U = & \begin{pmatrix} 0.4569 & 0.5431 \\ 0.1638 & 0.8362 \\ 0.4741 & 0.5259 \end{pmatrix} \end{array}$$

so thus we can get the fraction of time we will be at one point from either B or C if we start from 0 as the product between the probability of reaching the class and the fraction of time it spends at this point. Hence :

- If we start from 0 : $P_0^\infty = (0, 0, 0, 0.2284, 0.2284, 0.2296, 0.1288, 0.1848)$.

- If we start from 3 : $P_3^\infty = (0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$.

-If we start from 5 : $P_5^\infty = (0, 0, 0, 0, 0, 0.4227, 0.2371, 0.3402)$.

Problem 3:

This Markov chain has the following probability transition matrix :

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

It is clearly irreducible and aperiodic. Hence, it admits a limiting distribution and it is a stationary distribution we will call π, we have :

$\pi_0 = \pi_1 + \frac{\pi_2}{2} + \frac{\pi}{3}$ and $\pi_1 = \frac{\pi_2}{2} + \frac{\pi_3}{3}$ and $\pi_2 = \frac{\pi_3}{3}$ and finally $\pi_3 = \pi_0$, thus we get that :
$\pi_3 = \pi_0$, $\pi_2 = \frac{\pi_0}{3}$ and $\pi_1 = \frac{\pi_0}{2}$ and as $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$, then $\pi_0 = \frac{1}{2+\frac{1}{2}+\frac{1}{3}} = \frac{6}{17}$, which is the long run proportion of time the

Markov chain spend at 0.

Problem 4:

(a) We have :

$\mathbb{P}[Y_{n+1}=j|Y_n=i,Y_{n-1}=x_{n-1},..,Y_0=x_0]=\dfrac{\mathbb{P}[Y_{n+1}=j,Y_n=i,Y_{n-1}=x_{n-1},..,Y_0=x_0]}{\mathbb{P}[Y_n=i,Y_{n-1}=x_{n-1},..,Y_0=x_0]}$

$=\dfrac{\mathbb{P}[X_{N-n-1}=j,X_{N-n}=i,..,X_N=x_0]}{\mathbb{P}[X_{N-n}=i,X_{N-n+1}=x_{n-1},.,X_N=x_0]}$

$=\dfrac{\mathbb{P}[X_{N-n-1}=j,X_{N-n}=i]\mathbb{P}[X_N=x_0,..,X_{N-n+1}=x_{n-1}|X_{N-n}=i]}{\mathbb{P}[X_{N-n}=i]\mathbb{P}[X_N=x_0,.,X_{N-n+1}=x_{n-1}|X_{N-n}=i]}$

$=\dfrac{\mathbb{P}[X_{N-n-1}=j,X_{N-n}=i]}{\mathbb{P}[X_{N-n}=i]}=\dfrac{\pi_j P_{j,i}}{\pi_i}$

So (Y_n) is a Markov chain with transition matrix Q such that $Q_{i,j}=\frac{\pi_j P_{j,i}}{\pi_i}$, and as $\displaystyle\sum_i \pi_i Q_{i,j}=\sum_i \pi_j P_{j,i}=\pi_j$, then π is also the stationary distribution of Q.

Problem 5:

Let C be a finite, closed class. As positive recurrence is a class property, then it suffices to see that as :

$$1=\sum_{j\in C}\frac{\#\{k\le n : X_k=j|X_0=i\}}{n}\underset{n\to\infty}{\rightarrow}\sum_{j\in C}\frac{1}{m_j}$$

Thus the m_j's cannot be all equal to zero, thus at least one is non zero, i.e that state is positive recurrent, and so are all the states in C.

Problem 6:

We consider the branching process where we start with one particle at time 0, at time 1 we either have two children or 0, so $Z_1=2$ or $Z_1=0$. Now if $Z_1=2$, we check the birth of every particle (from the two present), that means Z_2 is either equal to 3 or 1 with equal probability,etc ... . We see that it follows the same distribution as the simple random walk. As for every step $n\to n+1$, one particle disappear and either is replaced by two particles or not replaced at all, that translates to adding $+1$ or -1 to our total with probability $\frac{1}{2}$ for each. Hence, as this branching process becomes extinct with probability 1 (as $\mu = \mathbb{E}[\xi] = 1$), then the random walk started from 1 attains 0 with probability 1. Hence using first step analysis, the simple random walk is recurrent if and only if it returns to 0 after moving to 1 or -1 (who are symmetrical), and this event happens with probability 1, thus the simple random walk is recurrent.