Tutorial 5
STAT 150

Exercise 1

Let (X_n) be a Markov chain with state space Ω. Show that positive recurrence is a class property.

Hint : Use the fact that for every state i if we denote R_i the first return time to i and $m_i = \mathbb{E}[R_i]$, then we have :

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{i,i}^{(k)} = \frac{1}{m_i}$$

Exercise 2

Let (X_n) be an irreducible and recurrent Markov chain, then show that for every two states x and y we have :

$$\mathbb{P}_x(T_y < \infty) = \mathbb{P}(\exists n \text{ s.t } X_n = y | X_0 = x] = 1$$

i.e : we visit any given state eventually with probability 1 whenever we start from.

Exercise 3

Let $\{X_n : n \geq 0\}$ be an irreducible, ergodic Markov chain on a state space S. Let π denote its stationary distribution. Suppose that the Markov chain is started in equilibrium, $X_0 \sim \pi$. Let $R = \inf\{n \geq 1 : X_n = X_0\}$ be the first time it returns to its starting position. Find $\mathbb{E}[R]$.

Exercise 4

Consider a Markov chain $\{X_n : n \geq 0\}$ on a finite state space S, where each state $i \in S$ is given a positive weight $w_i > 0$. The Markov chain starts from a uniformly random state X_0 in S. For each subsequent time $n \geq 1$, if $X_{n-1} = i$ the Markov chain transitions at time n according to the following two-step procedure:

- First, a state $j \neq i$ is selected uniformly at random.
- Then, given the state j, with probability $\frac{w_j}{w_i+w_j}$ the Markov chain transitions to j (that is, $X_n = j$), and with the remaining probability $\frac{w_i}{w_i+w_j}$ it stays put (that is, $X_n = i$).

Find $\lim_{n\to\infty} \mathbb{P}[X_n = i]$, for a state $i \in S$.

Exercise 5

Let Z_n be a branching process with offspring distribution Geom($\frac{1}{2}$). We know that the branching process becomes extinct with probability 1.

(a) Show that $G_{Z_n}(s) = \frac{n-(n-1)s}{n+1-ns}$.

(b) Calculate the expected number of steps to extinction.