Homework 5 : Solutions
STAT 150

Problem 6.1.1 :

It is forward by conditioning on the value of U that :

$$\begin{aligned}\mathbb{P}(X(U)=k) &= \int_0^1 e^{-\beta u}(1-e^{-\beta u})^{k-1}du \\ &= \int_0^{1-e^{-\beta}} \frac{x^{k-1}}{\beta}dx \\ &= \frac{1}{\beta k}(1-e^{-\beta})^k\end{aligned}$$

Problem 6.1.3 :

The probability that the number of infected people increase by 1 between t and $t+h$ given that $X(t)=k$, is equal to the probability that one of the k infected people transmit the disease to the $n-k$ non infected people, thus the probability is equal to :

$$\mathbb{P}(X(t+h)-X(t)=1|X(t)=k) = k(N-k)\alpha h + o(h)$$

Thus the birth rate is equal to $\lambda_k = \alpha k(N-k)$.

Problem 6.1.7 :

(a) Each S_k has an exponential distribution with rate λ_k, and they all are independent, thus by immediate computation (by using the law of total expectation), we get :

$$\mathbb{P}(S_0 > t) = e^{-\lambda_0 t}$$

$$\mathbb{P}(S_0+S_1>t) = \tfrac{\lambda_0}{\lambda_1-\lambda_0}e^{-\lambda_1 t} - \tfrac{\lambda_1}{\lambda_1-\lambda_0}e^{-\lambda_0 t}$$

$$\mathbb{P}(S_0+S_1+S_2>t) = \tfrac{\lambda_1\lambda_2}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 t} + \tfrac{\lambda_2\lambda_0}{(\lambda_2-\lambda_1)(\lambda_0-\lambda_1)}e^{-\lambda_1 t} + \tfrac{\lambda_1\lambda_0}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}e^{-\lambda_2 t}$$

(b) It is easy to see that $P_2(t) = \mathbb{P}(X(t) = 2) = \mathbb{P}(\{S_0 + S_1 + S_2 > t\}, \{S_0 + S_1 \leq t\}) = \mathbb{P}(S_0 + S_1 + S_2 > t) - \mathbb{P}(S_0 + S_1 > t)$.

Thus, $P_2(t) = \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 t} + \frac{\lambda_2\lambda_0}{(\lambda_2-\lambda_1)(\lambda_0-\lambda_1)}e^{-\lambda_1 t} + \frac{\lambda_1\lambda_0}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}e^{-\lambda_2 t} - \frac{\lambda_0}{\lambda_1-\lambda_0}e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_1-\lambda_0}e^{-\lambda_0 t}$.
Which is also equal to :

$$P_2(t) = \lambda_0\left[\frac{1}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 t} + \frac{1}{(\lambda_2-\lambda_1)(\lambda_0-\lambda_1)}e^{-\lambda_1 t} + \frac{1}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}e^{-\lambda_2 t}\right]$$

Problem 6.2.1 :

By using the memoryless property of the exponential distribution of T we get that :

$$\begin{aligned}
\mathbb{P}(X(T) = 0) &= \mathbb{P}(T > W_n) \\
&= \prod_{i=1}^{n} \mathbb{P}(T > W_i | T > W_{i-1}) \\
&= \prod_{i=1}^{n} \mathbb{P}(T > W_i - W_{i-1} = S_i) \\
&= \prod_{i=1}^{n} \frac{\mu_i}{\mu_i + \theta}
\end{aligned}$$

Problem 6.3.1 :

We have : $\mathbb{P}[X(t+h) = 1 \,|X(t) = 0] = \mathbb{P}[N_{t+h} - N_t = 1]p(0,1) = \lambda h + o(h)$ so $\lambda_0 = \lambda$. On the other hand we have :

$\mathbb{P}[X(t+h) = 0 \,|X(t) = 1] = \mathbb{P}[N_{t+h} - N_t = 1]p(0,1) = \lambda(1-\alpha)h + o(h)$ so $\mu_1 = \lambda(1-\alpha)$.

Problem 4.3 :

The Markov chain represents the states at which the two machines are at. The state 0 is when the two machines are both operational. The state 1 is when only the machine 1 is being repaired and 2 is operational, the state 2 is when only the machine 2 is being repaired and 1 is operational, and finally the state 12 is when the machine 1 is being repaired and the machine 2 is also broken, similarly for the state 21. The generator matrix has the following form then :

$$G = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 & 0 \\ \mu_2 & 0 & -(\mu_2+\lambda_1) & 0 & \lambda_1 \\ 0 & 0 & \mu_1 & -\mu_1 & 0 \\ 0 & \mu_2 & 0 & 0 & -\mu_2 \end{bmatrix}$$

and the stationary distribution verifies $\pi G = 0$, thus $\pi = (\frac{44}{129}, \frac{16}{129}, \frac{36}{129}, \frac{24}{129}, \frac{9}{129})$.

Problem 4.8 :

The Markov chain represents the state of the two machines (wether they are busy or free). The state 0 is when both are free, the state 1 is when only 1 is busy and 2 is free (the same for state 2), and finally state 12 is when both are busy. The generator matrix is equal to :

$$G = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 2 & 0 & -4 & 2 \\ 0 & 2 & 4 & -6 \end{bmatrix}$$

The stationary distribution solves the equation $\pi G = 0$, thus $\pi = (\frac{1}{3}, \frac{2}{9}, \frac{1}{3}, \frac{1}{9})$. In the long run the proportion of customers that enters the system is equal to the probability that the first server is free so it is equal to $\pi_0 + \pi_2 = \frac{2}{3}$. The proportion of customers that visit server 2 is the proportion of customers that enter the system times the probability that the second server is free, so it is equal to $(\pi_0 + \pi_2)(\pi_0 + \pi_1) = \frac{2}{9}$.

Problem 4.10 :

The Markov chain obviously concerns the state of failure of the machine, with the state 0 being that the machine is operational.

The generator matrix is equal to :

$$G = \begin{bmatrix} -(\lambda_1+\lambda_2+\lambda_3) & \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & -\mu_1 & 0 & 0 \\ \mu_2 & 0 & -\mu_2 & 0 \\ \mu_3 & 0 & 0 & -\mu_3 \end{bmatrix}$$

Again $\pi G = 0$ gives : $\pi = (\frac{1}{1+\sum_{i=1}^{3}\frac{\lambda_i}{\mu_i}}, \frac{\frac{\lambda_1}{\mu_1}}{1+\sum_{i=1}^{3}\frac{\lambda_i}{\mu_i}}, \frac{\frac{\lambda_2}{\mu_2}}{1+\sum_{i=1}^{3}\frac{\lambda_i}{\mu_i}}, \frac{\frac{\lambda_3}{\mu_3}}{1+\sum_{i=1}^{3}\frac{\lambda_i}{\mu_i}})$

Problem 4.22 :

The Markov chain represents the number of cabs available. The state space is $\{0,1,2,3\}$, the rate at which a cab is liberated is 3 and the rate of arrivals of the customers is 2, hence the generator matrix is :

$$G = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 3 & -5 & 2 & 0 \\ 0 & 6 & -8 & 2 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

thus the stationary distribution is equal to $\pi = (\frac{81}{157}, \frac{54}{157}, \frac{18}{157}, \frac{4}{157})$. Hence the probability that all the cabs are busy when a call comes in is $\frac{4}{157}$. Customers are served when there is at least one cab available which happens with probability $\frac{153}{157}$, and as customers arrive with a rate of 2 per hour, then on average $\frac{306}{157} \approx 1.95$ customers are served per hour.

Problem 4.24 :

(a) If π is the stationary distribution that satisfies the detailed balance then we have $\pi_{x_{i-1}} q(x_{i-1}, x_i) = \pi_{x_i} q(x_i, x_{i-1})$, then because $(x_i)_{i=0,..,n}$ is a cycle, then by multiplying the last equality for $i = 1, .., n$ we get the desired result.

(b) Fix the state a, for every b, we fix a path from b to a that we denote $x_{k+1} = b, .., x_l = a$ (which exists because the MC is irreducible), thus we have $x_0, .., x_k, x_{k+1}, ., x_l$ is a cycle no matters how we chose the first path from a to b and using the cycle condition we get that :

$$\pi(b) = \prod_{i=1} k \frac{q(x_{i-1}, x_i)}{q(x_i, x_{i-1})} = \prod_{i=k+1}^{l} \frac{q(x_i, x_{i-1})}{q(x_{i-1}, x_i)}$$

and we see that the left hand side doesn't depend on how we chose the initial path $x_0, ..x_k$, so $\pi(b)$ is well defined. It is obvious that $q(b,c) = 0$ if and only if $q(c,b) = 0$ from the cycle condition, so we may as well suppose that $q(b,c) > 0$ and $q(c,b) > 0$ for two states c, b, let $x_0 = a, .., x_k = b$ be a path to go from a to b, and then we construct the path from a to c by adding $x_{k+1} = c$. Hence :

$$\pi(c) = \prod_{i=1}^{k+1} \frac{q(x_{i-1}, x_i)}{q(x_i, x_{i-1})} = \prod_{i=1}^{k} \frac{q(x_{i-1}, x_i)}{q(x_i, x_{i-1})} \frac{q(b,c)}{q(c,b)} = \pi(b)\frac{q(b,c)}{q(c,b)}.$$

Hence, the detailed balance condition verified for π.