Tutorial 2
STAT 150

Exercise 1
1. Lets consider a Markov chain (X_n) with finite state space Ω and with transition matrix $P = (p_{i,j})$, we say that a probability distribution π on Ω verifies the detailed balance with respect to (X_n) if for every $i, j \in \Omega$ we have :

$$\pi_i p_{i,j} = \pi_j p_{j,i}$$

Show that π is the stationnary distribution of (X_n).

2. Suppose we are given an undirected graph, and suppose there is a "weight" $a_{ij} = a_{ji} > 0$ on each edge (i, j). Define $a_i = \sum_j a_{ij}$. Then :

$$p_{ij} = \frac{a_{ij}}{a_i}$$

defines a transition matrix on the graph.

Find the stationnary distribution π of this Markov Chain using the first question.

Exercise 2

The probability generating function for a positive random variable X is the function G_X from $[0, 1]$ to $[0, 1]$ defined as $G_X(s) = \mathbb{E}[s^X]$. Let N be a random variable taking values in the set of integers greater than 1 and X_i a sequence of i.i.d random variables independent from N.

We denote $S = \sum_{k=1}^{N} X_k$
Prove that : $G_S(s) = G_N(G_X(s))$

Exercise 3

Let Z_n be a branching process whose offspring distribution has a mean $\mu = \mathbb{E}[\xi] < 1$. Let $Z = \sum_{k=0}^{\infty} Z_k$ be the total family size. Assuming that $Z_0 = 1$, show that $\mathbb{E}[Z] = \frac{1}{1-\mu}$.

Exercise 4

Let Z_n be a branching process with offspring distribution Geom(p). Find the extinction probability as a function of p.