# Stat 150, Fall 2018, HW #3

Due Thurs Sept 27 at the start of class 9:30 AM in Evans 10.
Late assignments will not be accepted.

1. Pinsky and Karlin [PK], Problems (*not* exercises):

   (a) 4.1.5, 4.4.1, 4.4.6, 4.4.8

   (b) 4.4.4 + let $X$ be a random variable with $\mathbb{P}(X = k) = \alpha_k$. Describe the conditions on $\{\alpha_k, k \geq 0\}$ which guarantee a stationary distribution in terms of $\mathbb{E}X$. *Hint:* recall that $\mathbb{E}X = \sum_{k=0}^{\infty} \mathbb{P}(X > k)$.

   *Note:* This Markov chain models a *renewal process*. Components have IID lifetimes distributed as $X$. When a component breaks down (when the Markov chain is in state 0), it is immediately replaced with a new component (a renewal time).

   (c) 4.5.1: Find the limiting distribution starting in states $0, 3, 5$. That is, for each $i \in \{0, 3, 5\}$, find $\lim_{n\to\infty} \mathbb{P}(X_n = j | X_0 = i)$ for all $j \in S$.

   *Note:* This Markov chain is reducible. It has a limiting distribution, which depends on $X_0$. For each $i \in \{0, 3, 5\}$ and closed class $C$, use first step analysis to find the probability that $(X_n)$ ends up in $C$ eventually when $X_0 = i$. Once the Markov chain is in a closed class, the Main Markov Chain Theorem gives its long run behaviour from there. You can use a computer to solve the system of equations if you wish.

2. Durrett [D], Exercises:

   There are many good exercises in [D] starting on p.62–76 to do while studying for the upcoming midterm (*Thurs., Oct. 4, in class*), although none are assigned for homework this week. You are strongly encouraged to do as many as you can. HW #4 will not be due until after the midterm, so you should have plenty of time.

3. Consider a Markov chain with state space $S = \{0, 1, 2, 3\}$ which always transitions to state 3 when in state 0, and when in some state $i \in \{1, 2, 3\}$ it transitions to one of the states in $\{0, 1, \ldots, i-1\}$ uniformly at random. Find the long run proportion of time that the Markov chain is in state 0.

4. Let $(X_n)$ be a Markov chain on state space $S$ with stationary distribution $\pi$. Suppose that $X_0 \sim \pi$, that is, $\mathbb{P}(X_0 = i) = \pi_i$ for all $i \in S$. Fix some $N > 0$, and let $(Y_n = X_{N-n})_{n=0}^N$ denote its *reversal*. Show that

   (a) $(Y_n)$ is a Markov chain

   (b) $(Y_n)$ has stationary distribution $\pi$

   (c) the one-step transition probabilities for the reverse chain are $\mathbb{P}(Y_{k+1} = j | Y_k = i) = p_{ji}\pi_j/\pi_i$.

   Hence if $(X_n)$ satisfies the Detailed Balance Equations, then starting from $\pi$, it is equal in distribution to its reversal $(Y_n)$ (informally, a "movie of its dynamics" would look qualitatively similar viewed forwards or backwards).

5. Recall that Basic Markov Chain Limit Theorem states that if a Markov chain $(X_n)$ on state space $S$ is irreducible and recurrent, then the long run proportion of time spent in state $j \in S$ is equal to the inverse mean return time to $j$, regardless of $X_0$. That is, for all $i, j \in S$, we have that (with probability 1)

$$\lim_{n\to\infty} \frac{\#\{k \le n : X_k = j | X_0 = i\}}{n} = \frac{1}{m_j}.$$

   (If $(X_n)$ is aperiodic, then moreover $\lim_{n\to\infty} p_{i,j}^{(n)} = 1/m_j$.) Use this to show that a finite, closed communication class of a Markov chain is positive recurrent. Recall that a class $C$ is closed if $p_{i,j} = 0$ for all $i \in C$ and $j \notin C$, that is, the Markov chain cannot escape from $C$.

6. Argue that Simple Random Walk on $\mathbb{Z}$ is recurrent by first step analysis and considering the branching process with offspring distribution $\xi$ such that $\mathbb{P}(\xi = 0) = 1/2$ and $\mathbb{P}(\xi = 2) = 1/2$. *Hint:* Let $T_n$ be the total number of particles alive at time $n$, where we "slow down" the process so that particles give birth (and then die) one at a time.

   *Note:* This argument can be modified to show also that Simple Random Walk is transient if $p \neq 1/2$.