10 Random Evolutions

In the previous chapters, we have examined stochastic models whose path functions are either of the jump variety or related to Brownian motion—which is continuous but has infinite velocity. The aim of this chapter is to introduce a class of continuous parameter processes, which move in a piecewise linear fashion and whose slopes jump at the times of a Poisson process. The transition probabilities satisfy a system of linear partial differential equations. In the simplest case, the components of the system satisfy the one-dimensional telegraph equation, which was studied by Mark Kac[1] and Sidney Goldstein[2] in the 1950s.

10.1 Two-State Velocity Model

We begin with the simplest case of random evolution, based on a set of two real numbers $v_0 = 1, v_1 = -1$, which are interpreted as velocities. Meanwhile, we introduce a probability space $(\Omega, \mathcal{F}, \Pr)$ on which is defined a sequence of independent random variables with the common exponential distribution

$$\Pr[e_n > t] = \mathrm{e}^{-\lambda t}, \quad 0 < t < \infty, \quad n = 1, 2, \ldots \tag{10.1}$$

and $\lambda > 0$ is a parameter, interpreted as the *rate*.

An increasing sequence of times is defined by forming the sums

$$\tau_n := e_1 + \cdots + e_n, \quad n = 1, 2, \ldots \tau_0 := 0. \tag{10.2}$$

As we showed in Theorem 5.4, p. 242, τ_n has a gamma distribution with parameters (n, λ) so that

$$\Pr[\tau_n \in \mathrm{d}t] = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda \, \mathrm{e}^{-\lambda t} \, \mathrm{d}t. \tag{10.3}$$

Closely associated to τ_n is the *counting process*, defined as

$$N(t) := \#\{k : \tau_k \le t\}. \tag{10.4}$$

[1] Kac, M. (1974). *Rocky Mountain Journal of Mathematics*, 4, 497–520.
[2] Goldstein, S. (1951). *Quarterly Journal of Mechanics and Applied Mathematics*, 4, 129–156.

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Proposition 10.1. $N(t)$ has a Poisson distribution with parameter λ.

Proof.

$$\begin{aligned}
\Pr[N(t)=k] &= \Pr[\tau_k \le t < \tau_{k+1}] \\
&= \Pr[\tau_{k+1} > t] - \Pr[\tau_k > t] \\
&= \int_t^\infty \frac{(\lambda s)^k}{k!}\lambda e^{-\lambda s}\,ds - \int_t^\infty \frac{(\lambda s)^{k-1}}{(k-1)!}\lambda e^{-\lambda s}\,ds \\
&= -\int_t^\infty \frac{d}{ds}\left[\frac{(\lambda s)^k}{k!}e^{-\lambda s}\right] ds \\
&= \frac{(\lambda t)^k}{k!}e^{-\lambda t},
\end{aligned}$$

which was to be proved. ■

The two-state velocity process is defined by

$$V(t) = \begin{cases} 1 & \text{for } \tau_{2k} < t \le \tau_{2k+1} \\ -1 & \text{otherwise.} \end{cases}$$

Equivalently, we can write $V(t) = (-1)^{N(t)}$. This process is a special case of the two-state continuous Markov chain. The transition probabilities $P_{ij}(t)$ of a two-state continuous Markov chain have been computed in Chapter 7. Alternatively, we can exploit the symmetry of the problem and the initial conditions $V(0) = +1$. Then, $P_{00}(t) = \Pr[N(t)\text{even}]$; the values of the process are ± 1 so that the transition matrix can be computed in terms of the oddness/evenness of the state variable. We have

$$\begin{aligned}
P_{1,1}(t) = \Pr[V(t)=1|V(0)=1] = \Pr[N(t)\text{ even}] &= \sum_{k=0}^{\infty}\frac{(\lambda t)^{2k}}{(2k)!}e^{-\lambda t} \\
&= e^{-\lambda t}\cosh \lambda t \\
&= \frac{1}{2}\left(1+e^{-2\lambda t}\right).
\end{aligned}$$

To compute $\Pr[V(t)=-1|V(0)=1]$, use the fact that the sum of each row of the matrix $P_{ij}(t)$ is equal to one. Thus, $P_{1,-1}(t) = \frac{1}{2}\left(1-e^{-2\lambda t}\right)$. Symmetry considerations suggest that $P_{1,-1}(t) = P_{-1,1}(t), P_{-1,-1}(t) = P_{1,1}(t)$.

From these considerations, we conclude that the transition matrix is

$$P(t) = \begin{pmatrix} \frac{1}{2}\left(1+e^{-2\lambda t}\right) & \frac{1}{2}\left(1-e^{-2\lambda t}\right) \\ \frac{1}{2}\left(1-e^{-2\lambda t}\right) & \frac{1}{2}\left(1+e^{-2\lambda t}\right) \end{pmatrix}.$$

The properties of the matrices $P_{ij}(t)$ are listed as follows:

Proposition 10.2

1. $P(t+s) = P(t)P(s) \quad t, s > 0$
2. $\lim_{t\to 0} P(t) = I, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
3. $P'(t) = QP(t) = P(t)Q \quad Q := \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$
4. $\lim_{t\to\infty} P(t) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Exercise 10.1

Suppose that $P(t)$ is the family of transition matrices that correspond to a two-state Markov chain with the values v_1, v_2 and

$$P[V(t) = v_1 | V(0) = v_1] = 1 - tb + o(t),$$
$$P[V(t) = v_2 | V(0) = v_2] = 1 - ta + o(t), t \downarrow 0,$$

where $a > 0, b > 0$.

(a) Show that the transition matrix is given by the explicit formula

$$P(t) = \frac{1}{a+b} \begin{pmatrix} (a + b\mathrm{e}^{-\mu t}) & (b - b\mathrm{e}^{-\mu t}) \\ (a - a\mathrm{e}^{-\mu t}) & (b + a\mathrm{e}^{-\mu t}) \end{pmatrix},$$

where $\mu = a + b$.

(b) Show that

$$E[V(t)|V(0) = v_1] = \frac{av_1 + bv_2}{a+b} + \mathrm{e}^{-\mu t} \frac{bv_1 - bv_2}{a+b}.$$

(c) Show that

$$E[V(t)|V(0) = v_2] = \frac{av_1 + bv_2}{a+b} + \mathrm{e}^{-\mu t} \frac{av_2 - av_1}{a+b}.$$

(d) Show that $E[V(t)^2 | V(0 = v_1] = \frac{av_1^2 + bv_2^2}{a+b} + b\mathrm{e}^{-\mu t} \frac{v_1^2 - v_2^2}{a+b}$.

(e) Show that $E[V(t)^2 | V(0) = v_2] = \frac{av_1^2 + bv_2^2}{a+b} + a\mathrm{e}^{-\mu t} \frac{v_2^2 - v_1^2}{a+b}$.

(f) Re-work Problem 6.3.3 of Chapter 6, p. 304.

Many of the above properties of two-state Markov chains extend to the case of several states. The *transition matrix* consists of a set of nonnegative functions $P_{ij}(t)$

so that

$$\sum_{k=1}^{N} P_{ik}(t) = 1, \quad \lim_{t\to 0} P_{ij}(t) = 0 \quad (i \neq j), \quad \lim_{t\to 0} P_{ij}(t) = 1 \quad (i = j).$$

$$P_{ij}(s+t) = \sum_{k=1}^{N} P_{ik}(t)P_{kj}(s) \quad \text{or} \quad P(s+t) = P(t)P(s).$$

We use these to conclude continuity of $t \to P(t)$. $P(t+h) = P(t)P(h)$ proves that the right-hand limit $P^{+}(t)$ exists and is equal to $P(t)$. Furthermore, $P(h)$ has an inverse for small h so that we can write $P(t) = P(t-h)P(h), P(t-h) = P(t)P(h)^{-1}$, which implies that the left-hand limit exists with $P(t^{-}) = P(t)$.

We can also use these ideas to prove the differentiability of $t \to P(t)$. Writing $P(t+s) = P(t)P(s)$, we have for a $\delta > 0$

$$\int_{t}^{t+b} P(u)\,\mathrm{d}u = RP(t), \quad R := \int_{0}^{\delta} P(u)\,\mathrm{d}u.$$

If δ is sufficiently small, then the matrix R has an inverse and we can write for $0 < h < \delta$

$$P(t) = R^{-1} \int_{t}^{t+h} P(u)\,\mathrm{d}u.$$

Any function of this form has a derivative, given by

$$P'(t) = R^{-1}[P(t+h) - P(t)].$$

In particular, we can set $t = 0$ to express $P'(0) = R^{-1}[P(h) - I]$; this proves the existence of the rates $P'_{ij}(0)$ for all i,j—assuming only the continuity of $P_{ij}(t)$ at $t = 0$. The set of numbers $Q_{ij} = P'_{ij}(0)$ is the *infinitesimal matrix*.

10.1.1 Two-State Random Evolution

Beginning with $V(t)$, a Markov chain with two states, ± 1, we define the associated random evolution by

$$X(t) = x + \int_{0}^{t} V(s)\,\mathrm{d}s, \tag{10.5}$$

where we assume unit rates: $q_1 = 1 = q_2$. A major step is to determine a set of partial differential equations, corresponding to the backward equations studied in Chapter 6, pp. 295–296. In order to determine these equations, we let $\mathbf{f} = (f_1, f_{-1})$ be a pair of bounded and differentiable functions on $\mathbf{R}$ and define

$$\mathbf{u}(x,t) = E\left(\mathbf{f}\left(x + \int_0^t V(s)\mathrm{d}s\right)\right).$$

Fix t and consider the integrand separately on the sets $\tau_1 \le t$ and $\tau_1 > t$. Thus,

$$\begin{aligned}\mathbf{u}(x,t) &= E\left(\mathbf{f}(x + \int_0^t V(s)\mathrm{d}s\right) I_{\tau_1 \le t} + E\left(\mathbf{f}(x + \int_0^t V(s)\mathrm{d}s\right) I_{\tau_1 > t} \\ &= \int_0^t \mathbf{u}(x+s)\lambda \mathrm{e}^{-\lambda s}\mathrm{d}s + \mathrm{e}^{-\lambda t}\mathbf{f}(x+t).\end{aligned}$$

The event $N(t) = 0$ has probability $\mathrm{e}^{-\lambda t} = 1 - \lambda t + o(t), t \to 0$. On this set, we have $V(t) = 1, X(t) = x + t$.

The event $N(t) = 1$ has probability $\lambda t \mathrm{e}^{-\lambda t} = \lambda t + o(t), t \to 0$. On this event, we have $V(t) = -1, X(t) = x - t$.

The event $N(t) \ge 1$ has probability $= o(t), t \to 0$ and can be ignored. Combining these yields the result

$$T_t f(x,v) = (1-\lambda t) f_i(x+t) + \lambda t f_{-i}(x + O(t)) + o(t).$$

When we subtract $f_i(x)$ from both sides and take the limit when $t \to 0$, we obtain

$$\lim_{t\downarrow 0} t^{-1}[T_t f(x,i) - f(x,i)] = v_i f' - v_i + \lambda[f_{-i} - f_i]. \tag{10.6}$$

This information can be efficiently summarized in terms of a matrix-valued partial differential equation. We are implicitly using the Markov property $T_{t+s}f(x) = T_t(T_s f(x))$ to extend (10.6) from $s = 0$ to all $s > 0$.

Corollary 10.1. Let $\mathbf{f} = (f_1, f_{-1})$ be bounded and differentiable. Then, we have the system of backward equations

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_{-1} \end{pmatrix} = \begin{pmatrix} u_1' + \lambda(u_{-1} - u_1) \\ -u_{-1}' + \lambda(u_1 - u_{-1}) \end{pmatrix}, \tag{10.7}$$

where $' = \mathrm{d}/\mathrm{d}x$.

10.1.2 The Telegraph Equation

The telegraph equation with rate λ is, by definition, the following partial differential equation for a function u of two variables (t, x):

$$\frac{\partial^2 u}{\partial t^2} + 2\lambda \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{10.8}$$

where $\lambda > 0$ is a positive constant. The telegraph equation is related to random evolution in much the same way that the heat equation is related to Brownian motion.

The history of the telegraph equation in probability theory reveals a different source: in 1951, Goldstein demonstrated that a two-step random walk, when properly scaled, has a distributional limit that satisfies the telegraph equation. Our approach will derive the telegraph equation directly from a continuous-time model, without taking limits. To see this in detail, recall the backward equations that govern the evolution of the distributions:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial u_1}{\partial x} + \lambda(u_{-1} - u_1) \\ \frac{\partial u_{-1}}{\partial t} &= -\frac{\partial u_{-1}}{\partial x} + \lambda(u_1 - u_{-1}). \end{aligned} \tag{10.9}$$

On the one hand, the unique solution of the system (10.9) can be written as the operator $\mathbf{f} \to \mathbf{u} = E[\mathbf{f}(\cdot + \int_0^t V)]$. Choosing $\mathbf{f}$ to be the indicator of an interval, then $\mathbf{u}$ is the probability that $X(t)$ falls in that interval. This gives a justification for studying the solutions of the system (10.9).

We now proceed to demonstrate that the all solutions of (10.9) satisfy the telegraph equation.

Set $U = u_1 + u_{-1}$, $V = u_1 - u_{-1}$. With this notation, we have

$$\frac{\partial U}{\partial t} = \frac{\partial V}{\partial x}, \tag{10.10}$$

$$\frac{\partial V}{\partial t} = \frac{\partial U}{\partial x} - 2\lambda V. \tag{10.11}$$

We differentiate (10.10) with respect to t to obtain

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2} &= \frac{\partial^2 V}{\partial x \partial t} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} - 2\lambda V\right) \\ &= \frac{\partial^2 U}{\partial x^2} - 2\lambda \frac{\partial V}{\partial x} \\ &= \frac{\partial^2 U}{\partial x^2} - 2\lambda \frac{\partial U}{\partial t}, \end{aligned}$$

which proves that U satisfies the telegraph equation. On the other hand, from (10.11), we have

$$\begin{aligned}\frac{\partial^2 V}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial U}{\partial x} - 2\lambda V\right)\\ &= \frac{\partial^2 U}{\partial t \partial x} - 2\lambda \frac{\partial V}{\partial t}\\ &= \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) - 2\lambda \frac{\partial V}{\partial t}\\ &= \frac{\partial^2 V}{\partial x^2} - 2\lambda \frac{\partial V}{\partial t},\end{aligned}$$

which proves that V also satisfies the telegraph equation. But $u_1 = (U+V)/2, u_{-1} = (U-V)/2$ so that both components u_1, u_2 satisfy the telegraph equation $u_{tt} + 2\lambda u_t = u_{xx}$. We summarize these computations as follows:

Proposition 10.3. Let (u_1, u_{-1}) be a solution of the system (10.9). Then, each component u_j satisfies the telegraph equation (10.8) for $j = \pm 1$.

The converse is false: if we are given two solutions v_1, v_{-1} of the telegraph equation, it is not necessary that they be the components of a solution of a first-order system of the form (10.9). For example, if v_1 is a solution of the telegraph equation, then the first equation of (10.9) implies that $v_{-1} = v_1 + (v_1)_t - (v_1)_x$, proving that v_{-1} cannot be arbitrary.

10.1.3 Distribution Functions and Densities in the Two-State Model

In order to discuss the explicit form of the distribution function of the two-state process, we appeal to some elementary analysis. We have shown that the distribution of the two-state process is a solution of the telegraph equation. On the other hand, if we have a solution of the telegraph equation with the same initial conditions (Cauchy data), then we can be assured that we have found the explicit distribution function. We also present a computation of the distribution functions, which avoids the Fourier transform. For notational simplicity, we take $\lambda = 1$.

Solution Using Fourier Analysis

In order to solve the telegraph equation, we use the method of characteristic functions, otherwise known as the Fourier transform and defined by

$$\hat{P}_{i,j}(t,\mu) := \int_{-\infty}^{\infty} e^{\sqrt{-1}\mu y} P_{ij}(t,0,dy). \tag{10.12}$$

For simplicity, we take $\lambda = 1$. Here, μ is a real parameter that measures the spatial frequencies which are present in P. The measure P has a density inside the interval of length $2t$ and two weights at the ends of this interval.

The Fourier transform of the telegraph equation is

$$\hat{P}'' + 2\hat{P}' + \mu^2 \hat{P} = 0. \tag{10.13}$$

Exercise 10.2

Define the Fourier transform of an integrable function f by $\hat{f}(y) = \int_R f(x)e^{-i\mu x}\,dx$. Show that if $f(x,t)$ is a soluton of the telegraph equation, then $\hat{f} = g$ is a solution of the ordinary differential equation $g_{tt} + 2g_t + \mu^2 g = 0$. Assume as much smoothness and decay as you need.

The general solution of (10.13) is obtained by first finding the characteristic exponents, solutions of the algebraic equation

$$r^2 + 2r + \mu^2 = 0, \quad r = -1 \pm \sqrt{1-\mu^2}$$

in case $|\mu| < 1$ and with a corresponding formula if $|\mu| > 1$.

The general solution of (10.13) for $|\mu| < 1$ is written in terms of hyperbolic functions:

$$\hat{P}_{ij}(\mu) = A_{ij}(\mu)e^{-t}\cosh t\sqrt{1-\mu^2} + B_{ij}(\mu)e^{-t}\sinh t\sqrt{1-\mu^2}.$$

For $|\mu| > 1$, the hyperbolic functions can be replaced by suitable trigonometric functions. In this way, we can obtain the representation of the distribution function in terms of its Fourier transform.

Exercise 10.3

Let $g(x)$ be a solution of the differential equation $g'' + 2g' + \mu^2 g = 0$, where μ is a real parameter. Find the general solution in case $(1)|\mu| < 1, (2)|\mu| = 1,$ and $(3)|\mu| > 1$.

In detail, we have the following results:

$$\hat{P}_{1,1}(t,\mu) = e^{-t}\left[\cosh t\sqrt{1-\mu^2} + \frac{i\mu}{\sqrt{1-\mu^2}}\sinh t\sqrt{1-\mu^2}\right]$$

$$\hat{P}_{1,-1}(t,\mu) = e^{-t}\frac{\sinh t\sqrt{1-\mu^2}}{\sqrt{1-\mu^2}}$$

$$\hat{P}_{-1,1}(t,\mu) = e^{-t}\frac{\sinh t\sqrt{1-\mu^2}}{\sqrt{1-\mu^2}}$$

$$\hat{P}_{-1,-1}(t,\mu) = e^{-t}\left[\cosh t\sqrt{1-\mu^2} - \frac{i\mu}{\sqrt{1-\mu^2}}\sinh t\sqrt{1-\mu^2}\right].$$

These formulas may be inverted to obtain the densities $\mathrm{d}P/\mathrm{d}y$ if we apply a classical formula for the modified Bessel function[3]

$$\frac{\sinh t\sqrt{1-\mu^2}}{\sqrt{1-\mu^2}} = \frac{1}{2}\int_{-t}^{t} I_0(\sqrt{t^2-x^2})\mathrm{e}^{i\mu x}\,\mathrm{d}x, \tag{10.14}$$

where I_0 is the modified Bessel function. Formula (10.14) makes it clear that $\hat{P}(t,\cdot)$ is the Fourier transform of a measure of compact support. Hence, the distribution $P_{ij}(t,x,y) = 0$ for $|y-x| > t$, whereas $|y-x| \le t$ gives the values

$$\begin{aligned}
\frac{\mathrm{d}P_{1,1}}{\mathrm{d}y} &= \mathrm{e}^{-t}\left[\frac{t+y-x}{2}\frac{I_1(\sqrt{t^2-(x-y)^2}}{\sqrt{t^2-(y-x)^2}} + \delta(y-(x+t))\right] \\
\frac{\mathrm{d}P_{1,-1}}{\mathrm{d}y} &= \frac{\mathrm{e}^{-t}}{2} I_0\left(\sqrt{t^2-(x-y)^2}\right) \\
\frac{\mathrm{d}P_{-1,1}}{\mathrm{d}y} &= \frac{\mathrm{e}^{-t}}{2} I_0\left(\sqrt{t^2-(x-y)^2}\right) \\
\frac{\mathrm{d}P_{-1,-1}}{\mathrm{d}y} &= \mathrm{e}^{-t}\left[\frac{t-(y-x)}{2}\frac{{}_1(\sqrt{t^2-(x-y)^2}}{\sqrt{t^2-(y-x)^2}} + \delta(y-(x-t))\right].
\end{aligned}$$

Probabilistic Approach

For a more probabilistic approach, begin with the entire function

$$I_0(2\sqrt{z}) = \sum_{n=0}^{\infty}\frac{z^n}{(n!)^2}.$$

We will group the exponential random variables e_n into odd and even indices, thus

$$U_n = e_1 + e_3 + \cdots + e_{2n-1}, \qquad V_n = e_2 + e_4 + \cdots + e_{2n}.$$

For each $n \ge 1$, the random variables $U_n, V_n, e_{2n+1}, e_{2n+2}$ are independent, and the densities are given by

$$\Pr[U_n \in \mathrm{d}u] = \frac{u^{n-1}}{(n-1)!}\mathrm{e}^{u}\,\mathrm{d}u = \Pr[V_n \in \mathrm{d}u], \qquad n \ge 1$$

The random velocity process $V(t)$ satisfies

$$\begin{aligned}
V(t) &= U_n - V_n + (t - U_n - V_n) = t - 2V_n \\
&\qquad \text{on the set } U_n + V_n \le t < U_n + V_n + e_{2n+1}
\end{aligned}$$

[3] Bateman, Tables of Integral Transforms, 1, 57.

and

$$V(t) = U_{n+1} - V_n + (t - U_{n+1} - V_n) = 2U_{n+1} - t$$
$$\text{on the set } U_{n+1} + V_n \le t < U_{n+1} + V_n + e_{2n+2}.$$

These events are disjoint and exhaust the sample space. Hence, the distribution functions can be obtained by summation of the respective probabilities. Thus,

$$\Pr[X(t) \le y, V(t) = 1] = \sum_{n=0}^{\infty} \Pr[e_{2n+1} > t - U_n - V_n, t - 2V_n \le y, U_n + V_n \le t]$$
$$\Pr[X(t) \le y, V(t) = -1] = \sum_{n=0}^{\infty} \Pr[e_{2n+2} > t - U_{n+1} - V_n, t - 2V_n \le y, U_{n+1} + V_n \le t].$$

The term with $n = 0$ in the first sum is $\Pr[e_1 > t] = \mathrm{e}^{-t}$. For the terms with $n \ge 1$, note that the conditional probability of $e_{2n+1} > t - U_n - V_n$ given U_n and V_n is equal to $\mathrm{e}^{-(t-U_n-V_n)}$. Therefore, the required probability is given by the series of double integrals

$$\sum_{n=1}^{\infty} \int \int_{0 \le u+v<t, v \ge \frac{1}{2}(t-y)} \mathrm{e}^{-(t-u-v)} \frac{u^{n-1}\mathrm{e}^{-u}}{(n-1)!} \frac{v^{n-1}\mathrm{e}^{-v}}{(n-1)!}$$
$$= \mathrm{e}^{-t} \int \int_{0 \le u+v<t, v \ge \frac{1}{2}(t-y)} \sum_{n=1}^{\infty} \left(\frac{u^{n-1}v^{n-1}}{(n-1)!^2} \right) \mathrm{d}u\,\mathrm{d}v$$
$$= \mathrm{e}^{-t} \int \int_{0 \le u+v<t, v \ge \frac{1}{2}(t-y)} \sum_{n=1}^{\infty} \int \int_{0 \le u+v<t, v \ge \frac{1}{2}(t-y)} I_0(2\sqrt{uv})\mathrm{d}u\,\mathrm{d}v.$$

The second sum is handled in a similar manner. For $n = 0$, we have

$$\Pr[e_2 > t - U_1, 2U_1 \le t + y, U_1 \le t] = \mathrm{e}^{-t} \frac{t+y}{2}.$$

For $n \ge 1$, we have

$$\sum_{n=1}^{\infty} \int \int_{0 \le u+v<t, v \ge \frac{1}{2}(t-y)} \mathrm{e}^{-(t-u-v)} \frac{u^n \mathrm{e}^{-u}}{n!} \frac{v^{n-1}\mathrm{e}^{-v}}{(n-1)!} \mathrm{d}u\,\mathrm{d}v$$
$$= \mathrm{e}^{-t} \int \int_{0 \le u+v<t, v \ge \frac{1}{2}(t-y)} \sum_{n=1}^{\infty} \left(\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{u^n v^{n-1}}{(n-1)!^2} \right) \mathrm{d}u\,\mathrm{d}v$$

$$= \mathrm{e}^{-t} \int \int_{0 \le u+v<t, v \ge \frac{1}{2}(t-y)} \sum_{n=1}^{\infty} \int_0^{\frac{1}{2}(t+y)} \frac{(t-u)^n}{n!} \frac{u^n}{n!}.$$

The term $n = 0$ is also given correctly by this formula so that we can perform the sum for $n \ge 0$ in the form

$$\mathrm{e}^{-t} \int_0^{\frac{1}{2}(t+y)} \sum_{n=0}^{\infty} \frac{u^n (t-u)^n}{n!^2} \,\mathrm{d}u = \mathrm{e}^{-t} \int_0^{\frac{1}{2}(t+y)} I_0(2\sqrt{u(t-u)}\,\mathrm{d}u.$$

The above calculations show that the distribution functions of $X(t)$ are given by

$$\Pr[X(t) \le y, V(t) = 1] = \mathrm{e}^{-t} \left[1 + \int \int_{0 \le u+v<t, v \ge \frac{1}{2}(t-y)} I_0(2\sqrt{uv})\mathrm{d}u\,\mathrm{d}v \right]$$

$$\Pr[X(t) \le y, V(t) = -1] = \int_{0 \le \frac{1}{2}(t+y)} I_0(2\sqrt{u(t-u)}\,\mathrm{d}u.$$

10.1.4 Passage Time Distributions

In Chapter 8, Section 8.2.2, p. 407, we established the distributional properties of the first hitting time of Brownian motion $B(t), t > 0$ defined as

$$\tau_y := \inf\{t > 0 : B(t) = y\}. \tag{10.15}$$

An explicit formula was derived for the density of this random variable. This formula shows, in particular, that the moments of order less than 1/2 are finite but higher moments are infinite. We will now discuss the corresponding properties for two-state random evolution.

In order to discuss the hitting time of the two-state random evolution process, we use a discrete variable $j \in \{1, -1\}$ as well as the continuous parameter $y \in \mathbf{R}$, which defines the position on the real line. We define

$$T_{y,j} = \inf\{t > 0 : (X(t), V(t)) = (y, j)\}. \tag{10.16}$$

In order to obtain some perspective, think of a two-state process where the velocities do not sum to zero; in that case, the process tends to infinity and cannot be expected to hit all points.

To generate a computational algorithm, consider the *Laplace transform*, defined as

$$\phi_i(\alpha, x, y) = E\left(\mathrm{e}^{-\alpha T_{y,1}} | X(0) = x, V(0) = i\right), \quad i = \pm 1. \tag{10.17}$$

For $x < y$, they satisfy the following system of integral equations

$$\phi_1(\alpha, x, y) = \mathrm{e}^{-(1+\alpha)(y-x)} + \int_0^{y-x} \mathrm{e}^{-(1+\alpha)s} \phi_{-1}(\alpha, x+s, y)\,\mathrm{d}s \tag{10.18}$$

$$\phi_{-1}(\alpha, x, y) = \int_0^{\infty} \mathrm{e}^{-(1+\alpha)s} \phi_1(\alpha, x-s, y)\,\mathrm{d}s. \tag{10.19}$$

When we differentiate (10.18) with respect to x and simplify, we obtain the following first-order system:

$$\frac{\partial \phi_1}{\partial x} + (\phi_{-1} - \phi_1) = \alpha\,\phi_1 \tag{10.20}$$

$$-\frac{\partial \phi_{-1}}{\partial x} + (\phi_1 - \phi_{-1}) = \alpha\,\phi_{-1}. \tag{10.21}$$

Both components ϕ_1, ϕ_{-1} are solutions of the single second-order equation $\partial^2\phi/\partial x^2 = \alpha(\alpha+2)\phi$ and are bounded functions when $x < y$. Hence, one cannot use the exponent that is positive. Substituting into (10.20), this can be solved to obtain

$$\phi_1(\alpha, x, y) = \mathrm{e}^{-(y-x)\sqrt{\alpha(\alpha+2)}} \quad x < y \tag{10.22}$$

$$\phi_{-1}(\alpha, x, y) = \mathrm{e}^{-(y-x)\sqrt{\alpha(\alpha+2)}}\,[1+\alpha+\sqrt{\alpha(\alpha+2)}]^{-1}. \tag{10.23}$$

These Laplace transforms can be inverted in terms of the modified Bessel functions if we apply the formula of (Bateman, *Higher Transcendental Functions*, vol. 2, p. 200, no. 18):

$$\Pr[T_{y,1} \in \mathrm{d}s | V(0) = 1, X(0) = x] = (y-x)\mathrm{e}^{-s} \frac{I_1(\sqrt{s^2-(y-x)^2}}{\sqrt{s^2-(y-x)^2}}. \quad s < y-x \tag{10.24}$$

For $s > y-x$, the density function is zero, whereas there is a point mass of weight e^{y-x} at the point $s = y-x$, corresponding to those polygonal paths that have suffered no changes of direction.

The above formulas can be applied to study the *recurrence* of the Kac–Goldstein process. In general, we say that a point is recurrent if and only if $P[\tau_y < \infty | X(0) = y, V(0) = i] = 1$. In this case, we have

$$P[\tau_y < \infty | X(0) = x, V(0) = i] = \lim_{\alpha \to 0} \phi(\alpha, x, y) = 1$$

for all $y < x$. Taking $y \to x$ yields the conclusion.

Proposition 10.4. The passage time distributions of the two-state random evolution are given by (10.24). Every point is recurrent.

10.2 N-State Random Evolution

We now generalize the notion of random evolution to systems based on a Markov chain in continuous-time with an arbitrary finite number of states. We first discuss the random velocity model—a Markov chain with finitely many states. The spectral properties of the Markov chain are used to prove the law of large numbers and the central limit theorem.

10.2.1 Finite Markov Chains and Random Velocity Models

A continuous-time finite-state Markov chain is associated with a one-parameter family of matrices $P(t) = P_{ij}(t), 1 \le i, j \le N$, which has the properties

$$P_{ij}(t) \ge 0, \quad \sum_{j=1}^{N} P_{ij}(t) = 1, 1 \le i \le N$$

$$P_{ij}(t+s) = \sum_{k=1}^{N} P_{ik}(t) P_{kj}(s), \ \lim_{t\to 0} P(t) = I.$$

From the results in Chapter 6, Section 6.6, we recall that $t \to P(t)$ is continuous at every $t > 0$ and the derivative $P'(t)$ exists, especially at $t = 0$. This defines the *infinitesimal rates* of the Markov chain:

$$q_{ij} := \lim_{t\to 0} \frac{P_{ij}(t) - \delta_{ij}}{t},$$

which satisfies $\sum_{j=1}^{N} q_{ij} = 0, q_{ii} \le 0$. The row sums of Q are zero, and the diagonal elements of Q are negative or zero, so we can define $q_i := -q_{ii}, 1 \le i \le N$ where $q_i \ge 0$. The case $q_i = 0$ is trivial and is excluded, so we can divide by q_j to obtain a stochastic matrix

$$p_{ij} = \frac{q_{ij}}{q_j}. \quad 1 \le i \ne j \le N, \quad p_{jj} = 0$$

The matrix p_{ij} is a stochastic matrix, since

$$p_{ij} \ge 0, \quad \sum_{j=1}^{N} p_{ij} = 1.$$

10.2.2 Constructive Approach of Random Velocity Models

Given a Q matrix, as above, and a finite set of real numbers

$$\Lambda = \{v_1 < v_2 < \cdots < v_N\},$$

let W be the set of all piecewise constant, right continuous functions $t \to V(t,\omega)$.

We construct a family of random process $\Pr_x, x \in \Lambda$ on W as follows:

A pair of stochastic sequences $(e_n, Z_n)_{n\geq 1} \in R \times \Lambda$ is defined by:

$$\Pr_x(e_1 > t) = \mathrm{e}^{-q_x}, \quad \Pr_x[Z_1 = z|e_1] = q_{xz}/q_x.$$

Assuming that $e_1, Z_1, \ldots e_N, Z_N$ have been defined, the conditional distributions of e_{N+1}, Z_{N+1} are postulated as follows:

$$\Pr[e_{N+1} > t|e_1, \ldots, e_N, Z_N] = \mathrm{e}^{-tq_{Z_N}}, \Pr[\mathrm{Z}_{N+1} = z|e_1, \ldots, e_N, Z_N] = q_{Z_N,z}/q_{Z_n}.$$

It is directly verified that

- The random variables $Z_n, n \geq 1$ form a discrete-time Markov chain.
- If q_x is constant for $x \in \Lambda$, then the random variables $e_n, n \geq 1$ are independent and identically distributed: $\Pr_x[e_n > t] = \mathrm{e}^{-q_x t}$.

10.2.3 Random Evolution Processes

Having constructed the random velocity model, we can define the random evolution process on $\mathbf{R} \times \Lambda$ by setting

$$Z_x(t,\omega) = \left(x + \int_0^t V(s,\omega)\,\mathrm{d}s, V(t,\omega) \right). \tag{10.25}$$

Although the second component $(V(t))$ has the Markov property, this is not true for the first component alone. However, the joint process $Z(t)$ enjoys the Markov property, written in the form

$$\Pr[Z. \in A|Z_r, r \leq s)$$

that depends only upon Z_s.

The next theorem is the *backward equation* for the general random evolution process. It gives the time evolution of the probabilities of general sets, through the equation $P(Z(t) \in A|Z(0) = x) = E(1_A(Z(t))|Z(0) = x)$, where 1_A is the indicator function of the set A: $1_A = 1$ on A, $1_A = 0$ on A^c.

Theorem 10.1. *Let* $\mathbf{f} = (f_1, f_2, \ldots, f_N)$ *be an n-tuple of differentiable functions.* $u(t,x,v) := E[\mathbf{f}(Z(t))|Z(0) = (y,v)]$. *Then, u satisfies the system of partial differential equations*

$$\frac{\partial u}{\partial t}(t,x,v_i) = v\frac{\partial u}{\partial x}(t,x,v_i) + \sum_{j=1}^{N} q_{ij} u(t,x,v_j), \quad v = v_i, x \in \mathbf{R}, t > 0, i \leq i \leq N. \tag{10.26}$$

This is proved using a corresponding system of integral equations.

Proposition 10.5. If $\mathbf{f} = (f_1, \ldots, f_N)$ is an n-tuple of differentiable functions and $v = v_i, x \in \mathbf{R}, t > 0, 1 \le i \le N$, then

$$u(t,x,v) = \mathrm{e}^{-tq_x} f_v(x+vt) + \sum_{j \neq i} q_{ij} \int_0^t \mathrm{e}^{-sq_i} u(t-s, x+v_i s)\mathrm{d}s. \tag{10.27}$$

This is proved in the same manner as in the case $N = 2$ studied in Section 10.1. At a fixed moment $t > 0$, either the first jump has not occurred and this event has probability q_x or the first jump occurs at some time τ, which is distributed on the interval $[0,t]$ according to the exponential distribution with density $q_x \mathrm{e}^{-sq_x}$.

10.2.4 Existence-Uniqueness of the First-Order System (10.26)

Quite apart from the probabilistic model, it is important to know the properties of the system (10.26).

Proposition 10.6. If $\mathbf{f} = (f_1, \ldots, f_N)$ is an n-tuple of differentiable functions, the equation (10.26) has a solution $u(t,\cdot)$ that is unique within the class of bounded functions and satisfies $u(0,\cdot) = f$.

Proof of Existence. Let

$$\begin{aligned} u_0(t,x,v_i) &= \mathrm{e}^{-tq_i} f_i(x+v_i t) \\ u_{n+1}(t,x,v_i) &= \mathrm{e}^{-tq_i} f_i(x+v_i t) + \sum_{j \neq i} q_{ij} \int_0^t \mathrm{e}^{-sq_i} u_n(t-s, x+v_i s, v_j) \quad n \ge 0. \end{aligned} \tag{10.28}$$

Since u_0 is in $C(\Lambda)$, mathematical induction shows that $u_n(t,\cdot) \in C(\Lambda)$ for all $n \ge 0$. Now,

$$\begin{aligned} & u_{n+1}(t,x,v_i) - u_n(t,x,v_i) \\ &\quad = \sum_{j \neq i} q_{ij} \int_0^t \mathrm{e}^{-sq_i} \big(u_n(t-s, x+v_i s, v_j) - u_{n-1}(t-s, x+v_i s, v_j)\big)\, \mathrm{d}s. \end{aligned}$$

Let

$$\phi_n(t) = \sup_{a,v,s \le t} |u_{n+1}(s,a,v) - u_n(s,a,v)|.$$

Upon iteration, this becomes the factorial estimate

$$\phi_n(t) \le \frac{(Qt)^n}{n!} \phi_0(t),$$

which shows that the series $u_0 + \sum_0^\infty (u_{n+1} - u_n)$ converges uniformly in a to a continuous limit u_∞. Using this uniform convergence, it follows that u_∞ is a solution of (10.26). ■

Proof of Uniqueness. Let u_1, u_2 be two solutions and set $U := u_1 - u_2$. It satisfies

$$U(t,x,v_i) = \sum_{j\neq i} q_{ij} \int_0^t \mathrm{e}^{-q_i s} U(t-s, x+v_i s, v_j)\,\mathrm{d}s.$$

Letting $\Phi(t) = \sup_{a,t} U(a,t)$, we find that $\Phi(t) \le Q\int_0^t \Phi(s)\,\mathrm{d}s$ whose only solution is $\Phi(t) = 0$. ■

10.2.5 Single Hyperbolic Equation

The system of first-order partial differential equations

$$\frac{\partial u_i}{\partial t} = v_i \frac{\partial u_i}{\partial x} + \sum_{j=1}^N q_{ij} u_j \quad 1 \le i \le N \tag{10.29}$$

can be related to a single partial differential equation of the Nth order, satisfied by each of the component functions u_i, $1 \le i \le N$. In case $N = 2$ and $v_1 = 1 = -v_2, q_1 = q = q_2$, this PDE includes the telegraph equation with rate q, which has been shown in the previous section.

Lemma 10.1. *Let Q, V be N-dimensional real matrices. A polynomial P is defined by*

$$P(\lambda,\mu) = \det(Q + \lambda V - \mu) = \sum_{k+l\le N} a_{kl}\lambda^k \mu^l$$

for suitable constants a_{kl}. A differential operator on functions is defined by

$$\mathcal{H} := \det(Q + V\partial_x - \partial_t) = \sum_{k+l\le N} a_{kl}\partial_t^k \partial_x^l, \quad \partial_t := \frac{\partial}{\partial t}, \partial_x := \frac{\partial}{\partial x} \tag{10.30}$$

meaning that we compute $P(\lambda,\mu)$ and make the substitution $\lambda \to \partial_x, \mu \to \partial_t$. In full detail, $\mathcal{H} = \sum_{k+l\le N} a_{kl}\partial_x^k \partial_t^l$.

Let $\boldsymbol{u} = (u_i)$ be a solution of the first-order linear system

$$\partial_t u = V\partial_x u + Qu.$$

Then, for each i, $1 \le i \le N$, we have $\mathcal{H}u_i = 0$.

Proof. For any matrix A, we have the identity $\det A = (\text{adj } A) \times A$; apply this to the case $A = Q + \lambda V - \mu$. A is a linear function of (λ, μ) and the classical adjoint adj A is also a polynomial in λ, μ, of degree $N-1$. Applying this to $A = Q + \lambda V - \mu$, we have

$$P(\lambda, \mu) = (\text{adj } A)(Q + \lambda V - \mu).$$

Making the substitutions for λ, μ, we have for each i, $1 \le i \le N$

$$\mathcal{H} u_i = (\text{adj } A)(Q + V\partial_x - \partial_t)\mathbf{u} = 0.$$

Hence for each i, we have $\mathcal{H} u_i = 0$, as required. ■

Returning to equation (10.29), it follows that each component $u = u_i$ satisfies the (scalar) PDE

$$\det \begin{pmatrix} q_{11} + v_1\partial_x - \partial_t & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} + v_2\partial_x - \partial_t & \cdots & \cdots \\ \vdots & \vdots & \vdots\vdots & \\ q_{n1} & q_{n2} & \cdots & q_{nn} + v_n\partial_x - \partial_t \end{pmatrix} u_i = 0. \qquad (10.31)$$

In case $N = 2$, $q_1 = q = q_2, v_1 = 1 = -v_2$, this is the statement that both components satisfy the telegraph equation with rate q:

$$\det \begin{pmatrix} -1 - \partial/\partial t + \partial/\partial x & 1 \\ 1 & -1 - \partial/\partial t - \partial/\partial x \end{pmatrix} u = 0.$$

Example The most general two-state random evolution has $v_1 \neq v_2$, $q_1 \neq q_2$. In this case, the second-order PDE (10.31) is

$$\left(\frac{\partial^2}{\partial t^2} - (v_1 + v_2)\frac{\partial^2}{\partial t \partial x} + v_1 v_2 \frac{\partial^2}{\partial x^2} + (q_1 + q_2)\frac{\partial}{\partial t} - (q_1 v_2 + q_2 v_1)\frac{\partial}{\partial x}\right) u = 0. \qquad (10.32)$$

In the special case $q_1 = q_2 = 1, v_1 = 1 = -v_2$, we obtain the telegraph equation with rate 1: $u_{tt} + 2u_t - u_{xx} = 0$.

Exercise 10.4

Show that (10.32) follows from (10.31), when we take $N = 2$.

If we write the PDE (10.31) in the form

$$\mathcal{H} u = \sum_{k+l \le 2} a_{kl} D_t^k D_x^l u = 0, \qquad (10.33)$$

then $a_{20} = 1$, $a_{11} = (v_1 + v_2)$, $a_{02} = v_1 v_2$, $a_{10} = q_1 + q_2$, $a_{01} = v_1 q_2 + v_2 q_1$, $a_0 = 0$. Then, a short calculation shows that

(i) The polynomial $\lambda^2 + a_{11}\lambda + a_{02} = 0$ has two distinct real roots $v_1 < v_2$.
(ii) $a_{10} > 0$.
(iii) $v_1 < a_{01}/a_{10} < v_2$.

Conversely, one may characterize the coefficients as follows: suppose that we are given a second-order constant coefficient linear PDE written in the form (10.33) and satisfying the conditions (i)-(ii)-(iii). Then, there exist constants $v_1 < v_2, q_1 > 0, q_2 > 0$ such that (10.32) holds.

Exercise 10.5
Prove that any PDE of the form (10.33) satisfies conditions (i)-(ii)-(iii), when one writes it in the form (10.33).

Exercise 10.6
Suppose that we are given constants $v_1 < v_2, q_1 > 0, q_2 > 0$ satisfying the conditions (i)-(ii)-(iii). Then, we can define a_{10}, a_{01} so that (10.33) holds.

Example Let $N = 3$ and (v_i, q_{ij}) be otherwise arbitrary. Then, the system (10.29) becomes the single third-order equation

$$\begin{gathered}
L := D_t^3 - (v_1 + v_2 + v_3)D_t^2 D_x + (v_1 v_2 + v_1 v_3 + v_2 v_3)D_t D_x^2 - v_1 v_2 v_3 D_x^3 \\
-(q_{11} + q_{22} + q_{33})D_t^2 + q_{11}(v_2 + v_3) + q_{22}(v_1 + v_3) + q_{33}(v_1 + v_2)D_t D_x \\
-(q_{11} v_2 v_3 + q_{22} v_1 v_3 + q_{33} v_1 v_2)D_x^2 \\
+(q_{11}q_{33} - q_{13}q_{31} + q_{11}q_{22} + q_{12}q_{21} + q_{22}q_{33} - q_{23}q_{32})D_t \\
+(v_1(q_{23}q_{32} - q_{22}q_{33}) + v_2(q_{13}q_{31} - q_{11}q_{33}) + v_3(q_{12}q_{21} - q_{11}q_{22})u = 0.
\end{gathered}$$

Writing $\mathcal{H}$ in the form (10.33), it is easy to see that the coefficients obey the following necessary conditions:

(i') The polynomial $\lambda^3 + a_{21}\lambda^2 + a_{12}\lambda + a_{03}$ has distinct real roots.
(ii') $a_{20} > 0, v_1 + v_2 < a_{11}/a_{20} < v_2 + v_3, ; a_{02}/a_{20} \in [\min_{i \neq j} v_i v_j, \max_{i \neq j} v_i v_j]$.
(iii') There exists $\delta > 0$ such that $\delta < a_{10} < q_1 q_2 + q_1 q_3 + q_2 q_3$. In case q_i is constant q, then $\delta = (9q^2/4)$.

Exercise 10.7
Prove the necessary conditions (i'), (ii'), and (iii').

10.2.6 Spectral Properties of the Transition Matrix

In this section, we return briefly to study N-state Markov chains. The results will be used to obtain the central limit theorem and the law of large numbers for the random evolution process.

To study the asymptotic properties of the transition matrix $P(t)$, we need to obtain information about its eigenvalues. Clearly, the complex number $\gamma = 0$ is an eigenvalue with eigenvector $(1, 1, \ldots, 1)^T$. The next lemma gives further information.

Lemma 10.2. *If γ is any eigenvalue of the matrix Q, then* $\mathrm{Re}\gamma \le 0$.
If γ is a purely imaginary eigenvalue of the matrix Q, then $\gamma = 0$.

Proof. If γ is any eigenvalue, then there exists a set of complex numbers (c_k), not all zero, so that

$$\sum_{k=1}^{N} q_{ik}c_k = \gamma c_i. \quad 1 \le i \le N$$

Moving the term $q_{ii}c_i$ to the other side and taking the modulus, we have

$$\begin{aligned}|\gamma + q_i||c_i| &= \left|\sum_{k\neq i} q_{ik}c_k\right| \quad 1 \le i\ leN\\ &\le \max_k \left|\sum_{k\neq i} q_{ik}c_k\right|\\ &= \max_k |c_k|\, q_i.\end{aligned}$$

Now, choose i to maximize $|c_i|$. This allows one to cancel a common factor, resulting in the inequality $|\gamma + q_i| \le q_i$ which states that γ lies in a circle of radius q_i, centered at $-q_i$. In particular $\mathrm{Re}\gamma \le 0$, with equality if and only if $\gamma = 0$.

In order to treat the law of large numbers and the central limit theorem, we need to develop the properties of the matrix $Q + i\xi V$, where ξ is real, V is a real diagonal matrix, and Q is the infinitesimal matrix of an irreducible continuous-time Markov chain. In particular, the matrix Q has a simple eigenvalue $\gamma = 0$, and all other eigenvalues lie in the strict left half-plane $\mathrm{Re}(\gamma) < 0$. The detailed behavior is in the next proposition. ∎

Proposition 10.7. With the above assumptions, we have the following behavior: there exist eigenvalues $\gamma_1(\xi), \ldots, \gamma_N(\xi)$ of $Q + i\xi - V$ and $\delta > 0$ so that

$$\mathrm{Re}\,\gamma_k(\xi) \le -\delta < 0 \quad \text{if} - \infty < \xi < \infty,\ 2 \le k \le N, \tag{10.34}$$

$$\mathrm{Re}\,\gamma_1(\xi) \le -\delta < 0 \quad \text{if} |\xi| > \delta, \tag{10.35}$$

$$\gamma_1(\xi) = \bar{\phi}\xi + \frac{\sigma^2\xi^2}{2} + O(\xi^3), \quad \xi \to 0 \quad \bar{\phi} := \sum_j v_j\pi_j. \tag{10.36}$$

Proof. To prove the first two statements, we recall that $P(\gamma,\lambda) = \sum_{k+l\le N} a_{kl}\lambda^k\gamma^l$ and that $\gamma(\lambda)$ is obtained by solving $P(\gamma,\lambda) = 0$. Let $\nu(\lambda) := \lambda\gamma(1/\lambda)$. Then, $B(\nu,\lambda) := \sum a_{k+l\le N}\nu^k\lambda^{n-k-l} = 0$. Setting $\lambda = 0$, it follows that $\sum a_{k+l=N}\nu^k = 0$. This shows that $\lim_{\nu\to 0}\nu(\lambda) = iv_j$ for some j. Since (v_j) are distinct, it follows that

$\nu(\lambda) = iv_j + \lambda\phi(1/\lambda)$ for some smooth function $\phi(\lambda)$. Translating back into the λ language, this is written as

$$\gamma(\lambda) = i\lambda v_j + \phi(1/\lambda),$$

where $\lambda \to \infty$. If we substitute this expansion into the original equation $\det(Q + \lambda V - \gamma) = 0$, it follows that $\phi(0) = q_{jj}$, which was to be proved. ■

The eigenvalue equation for $\gamma(\xi)$ is

$$(Q + i\xi V)c = \gamma c$$
$$\sum_{k=1}^{N} q_{jk}c_k + i\xi v_j c_j = \gamma c_j \quad 1 \le j \le N$$
$$\sum_{k \ne j} q_{jk}c_k = (\gamma - i\xi v_j - q_j)c_j$$
$$|\gamma - i\xi v_j + q_j|c_j \le \max_j |c_j| \sum_{k \ne j} q_{jk} = |c_j|q_j.$$

Choose j so that $|c_j| = \max_k |c_k|$, which leads to the inequality

$$|\gamma + q_j - i\xi v_j| \le q_j.$$

This is the equation of a disk centered at $(-q_j, v_j\xi)$ of radius q_j. A glance at the γ plane shows that every point in the disk satisfies the inequality $\mathrm{Re}(\gamma) \ge q_j - |v_j| > 0$. But $\mathrm{Re}\gamma(0) < 0$ for $2 \le k \le N$ from which (10.34) and (10.35) follow.

To prove (10.36), recall that the eigenvectors and eigenvalues have asymptotic expansions about a simple eigenvalue, e.g., $\gamma = 0$. Thus,

$$\mathbf{e}(\xi) = \mathbf{e}_0 + \mathbf{e}_1\xi + O\big(|\xi|^2\big) \tag{10.37}$$
$$\gamma(\xi) = \gamma_0 + \gamma_1\xi + \gamma_2\xi^2 + O\big(|\xi^3\big). \tag{10.38}$$

The coefficients need to be chosen so that $(Q + i\xi V)\mathbf{e} = \gamma(\xi)\mathbf{e}$. This requires that

$$Q\mathbf{e}_0 = \gamma_0\mathbf{e}_0 \tag{10.39}$$
$$Q\mathbf{e}_1 + iV\mathbf{e}_0 = \gamma_1\mathbf{e}_0 + \gamma_0\mathbf{e}_1 \tag{10.40}$$
$$Q\mathbf{e}_2 + iV\mathbf{e}_1 = \gamma_2\mathbf{e}_0 + \gamma_1\mathbf{e}_1 + \gamma_0\mathbf{e}_2 \tag{10.41}$$

and so forth. Equation (10.39) is solved by taking $\gamma_0 = 0, \mathbf{e}_0 = 1$. To solve the second equation, take the inner product of each side with the stationary distribution π_j, solution of $\sum_j \pi_j q_{jk} = 0$. The final term is already zero from the choice of γ_0. We are left with two terms involving Ve_0 and γ_1, which yield the first nonzero term in the expansion (10.38) of the eigenvalue. The next term is obtained by computing the inner product of e_0 and Qe_2.

10.2.7 Recurrence Properties of Random Evolution

It is well known (see Section 4.3.3) that a random walk in one dimension is recurrent if and only if the common distribution has mean value zero. The same holds true for one-dimensional Brownian motion: the process is recurrent if and only if the drift (mean value) is zero.

When we come to random evolution a similar criterion is valid: the process is recurrent if and only if the overall process has mean zero. As in the case of irreducible Markov chains, returning once is equivalent to returning infinitely often, which we make precise below.

Let $\Lambda = \mathbf{R} \times \{1, 2, \ldots N\}$ be the state space of a random evolution process $Z(t) = (X(t), Y(t))$. If $z = (a, i) \in \Lambda$, $w = (b, j) \in \Lambda$, we write the *hitting time* as the extended random variable

$$T_w = \inf\{t > 0 : Z(t) = w\}, \quad T_w = +\infty \quad \text{otherwise}$$
$$\pi(z, w) := P_z[T_w < \infty]$$

is the hitting probability of w starting at z. In these terms, recurrence means that $\pi(z, z) = 1$, for all pairs $z \in \Lambda$.

The proof hinges on a system of integral equations satisfied by the Laplace transform of the hitting time distribution, defined by

$$u_\alpha(z, w) = E_z\left[\mathrm{e}^{-\alpha T_w}\right] \quad z, w \in \Lambda, \alpha > 0. \tag{10.42}$$

Lemma 10.3. *If $I(z) = I(w)$ and $(b-a)/v_i > 0$, then*

$$u_\alpha(z, w) = \mathrm{e}^{-(\alpha+q_i)(b-a)/v_i} + \sum_{k \neq i} q_{ik} \int_0^{(b-a)/v_i} \mathrm{e}^{-(\alpha+q_i)s} u_\alpha(z_s, w)\mathrm{d}s, \tag{10.43}$$

where $z_s := (a + v_i s)$, $I(z) = i$.

If $I(z) \neq I(w)$ or $(b-a)/v_i \leq 0$, then

$$u_\alpha(z, w) = \sum_{k \neq i} q_{ik} \int_0^\infty \mathrm{e}^{-(\alpha+q_i)s} u_\alpha(z_s, w)\mathrm{d}s. \tag{10.44}$$

In the first case, the random process hits w (with a positive probability) before changing directions. In the second case, the process changes direction before hitting w, with probability one.

Letting $\alpha \to 0$, we have the system of integral equations for the hitting probabilities.

Lemma 10.4. *If* $I(z) = I(w)$ *and* $(b-a)v_i > 0$, *then* $\pi(z,w) = \mathrm{e}^{-(q_i(b-a)/v_i} + \sum_{k\neq i}\int_0^{(b-a)/v_i} \mathrm{e}^{-q_i s}\pi(z_s, w)\mathrm{d}s$.

If $I(z) \neq I(w)$ *or* $(b-a)/v_i \leq 0$, *then* $\pi(z,w) = \sum_{k\neq i}\int_0^{\infty} \mathrm{e}^{-q_I s}\pi(z_s, w)\mathrm{d}s$.

These integral equations allow us to deduce the smoothness properties of the hitting probabilities.

Lemma 10.5. *For fixed* $w = (b,j)$, *the mapping* $z \to \pi(z,w)$ *is continuous everywhere, with the possible exception of the place* $z = w$. *The mapping is infinitely differentiable for* $z \neq w$ *provided that* $v_i \neq 0$.

This is proved by changing variables in the integrals, which represent π. *The proof is left to the reader.*

A simple example shows that the statement of the lemma cannot be improved in general. To see this, take $N = 2$, $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, and $v_1 > 0 > v_2, v_1 + v_2 > 0$. Direct computation using lemma (10.4) yields the formulas

$$\pi(x,1;0,2) = \frac{v_2}{v_1+v_2}(1-\mathrm{e}^{\mu x})1_{(-\infty,0)}(x)$$
$$\pi(x,2;0,2) = \frac{1}{v_1+v_2}(v_2+v_1\mathrm{e}^{\mu x})1_{(-\infty,0)}(x),$$

where $\mu = (v_1+v_2)/v_1v_2 > 0$. This reflects the fact that if we start at the left of zero and are moving to the right, then we almost certainly hit zero, moving to the right; if we start at the left of zero and are moving to the left, the probability of hitting zero from the left is nearly zero.

The corresponding local statements follow immediately.

Lemma 10.6. *If* (b,j) *is fixed, then the hitting probabilities are harmonic functions of* (x,i):

$$v_i\pi_i'(x) + \sum_{k=1}^{N} q_{ik}\pi_k(x) = 0, \quad 1 \leq i \leq N \tag{10.45}$$

10.3 Weak Law and Central Limit Theorem

The normal distribution plays a fundamental role in the theory of probability and stochastic processes. Brownian motion furnishes a family of normally distributed random variables with mean zero and variance proportional to the time t. Many other nonnormal distributions are well approximated by the normal distribution if one takes sufficiently many independent components. This leads to the central limit theorem—the subject of deep analysis on the one hand and numerical confidence levels on the other hand.

In the case of random evolutions, there is a counterpart of these limit theorems. Consider a finite-state Markov chain $V(t)$ with real numbers $v_1 < v_2 < \cdots < v_N$ and form

$$M_t := \int_0^t (V(s) - m)\,\mathrm{d}s. \tag{10.46}$$

With the proper choice of m, we will have $E_x[M_t] = 0$ $\mathrm{Var}[M_t^2] \sim \sigma^2 t$. The weak law of large numbers asserts the weak convergence of M_t/t to the constant m. The central limit theorem asserts the weak convergence of the distributions of $(M_t - mt)/\sqrt{t}$ to a normal distribution whose variance will be computed.

Based on the analogy with Brownian motion, we can formulate and prove the analogs of the classical weak law of large numbers and the central limit theorem. The setup is based on a continuous-parameter finite-state Markov chain $V(t), t \geq 0$ with one ergodic class and no transient states. A real-valued function ϕ is written $\phi(v_i), 1 \leq i \leq N$. For maximum flexibility, we consider a *continuous additive functional* defined by

$$X(t) = \int_0^t \phi(V(s))\,\mathrm{d}s,$$

where ϕ is a real-valued function. The weak law of large numbers states that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(V(s))\,\mathrm{d}s = \sum_1^N \pi(k)\phi(v_k), \tag{10.47}$$

where the convergence is in probability, i.e., for every $\delta > 0$

$$P\left[\omega : \left|\frac{1}{T}\int_0^T \phi(V(s)\,\mathrm{d}s - \sum_1^N \pi(k)\phi(v_k)\right| > \delta\right] \to 0 \quad N \to \infty.$$

(10.47) is equivalent to the condition that for every bounded and continuous f

$$\lim_{T\to\infty} Ef\left(\frac{1}{T}\int_0^T \phi(s)\,\mathrm{d}s\right) = f\left(\sum_1^N \pi(k)\phi(v_k)\right) = 0.$$

For example, suppose that $\phi(v) = 1$ for $v = v_1$ and $\phi(v) = 0$ otherwise. Then, the left side of (10.47) is the limiting fraction of time in the interval $[0, T]$ that the Markov chain spends at v_1. The right side of (10.47) is the stationary measure of the point v_1. In other words,

THE TIME AVERAGE EQUALS THE SPACE AVERAGE.

We can obtain an intuitive idea of the validity of (10.47) by computing expectations. Then,

$$\begin{aligned} E[X(t)|V(0)=v_i] &= \int_0^t E[\phi(V(s))|V(0)=v_i]\,\mathrm{d}s \\ &= \int_0^t \sum_{j=1}^N \phi(v_j)P_{ij}(s)\,\mathrm{d}s. \end{aligned}$$

Recalling that $\lim_{t\to\infty} P_{ij}(t) = \pi_j$, we obtain

$$\begin{aligned} \lim_{t\to\infty} t^{-1} E[X(t)|V(0)=v_i] &= \sum_{j=1}^N \phi(v_j) \lim_{t\to\infty} t^{-1} \int_0^t P_{ij}(s)\,\mathrm{d}s \\ &= \sum_{j=1}^N \phi(v_j)\pi_j := \bar{\phi}. \end{aligned}$$

This calculation shows that the mean value of the time average tends to the space average, as expressed through the stationary distribution. The weak law of large numbers shows that the *mean value* can be omitted in the previous sentence.

The central limit theorem is a refinement of the weak law of large numbers. Using the same notation as above, it states that

$$P\left[\frac{X(t)-\bar{\phi}t}{\sqrt{t}} < x\right] \to \Phi\left(\frac{x}{\sigma}\right), \quad t\to\infty$$

where $\sigma > 0$ and Φ is the standard normal distribution function, defined by the integral

$$\Phi(x) = \int_{-\infty}^x \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y, \quad -\infty < x < \infty.$$

The proof will be organized in a non-probabilistic fashion, using the Fourier transform of the various random variables. In what follows, we will use the indices (j,k) in place of (i,j), since we want to save the letter i for $\sqrt{-1}$.

$$WLLN: \; E\left[\mathrm{e}^{i\xi X(t)/t} 1_k(V(t))|V(0)=j\right] \to \mathrm{e}^{i\xi\bar{\phi}}\pi_k, \quad \xi\in R, \quad 1\le j,k\le N$$

$$CLT: \; E\left[\mathrm{e}^{i\xi\frac{X(t)-\bar{\phi}t}{\sqrt{t}}} 1_k(V(t))|V(0)=j\right] \to \mathrm{e}^{-\gamma_2\xi^2}\pi_k, \quad \xi\in R, \quad 1\le j,k\le N$$

The constant $\gamma_2 > 0$ indicates the amount of randomness in the original Markov chain.

The proof will be broken into several stages, each of which involves elementary calculations. The common hypothesis is that the matrix Q has one single ergodic class and no transient states. In particular, zero is a simple eigenvalue and all other eigenvalues of Q are strictly in the left half-plane.

Step 1: (Solution of Poisson's equation) The matrix equation $QH = HQ = -I + \Pi$ has the solution $H_{ij} = \int_0^\infty (P_{ij}(t) - \pi_j)\,\mathrm{d}t$ where the convergent of the integral is exponentially fast, and π_j is the stable probability distribution, solution of $\pi Q = 0$.

Step 2: (Quadratic forms in the Q matrix) The matrix Q satisfies the identity

$$< Qv, v >_\pi := \sum_{i,j=1}^{N} \pi_i q_{ij} v_i v_j = -\frac{1}{2} \sum_{i,j=1}^{N} \pi_i q_{ij} (v_i - v_j)^2 \leq 0$$

with equality iff $v = c(1, 1, \ldots, 1)$ for some constant c.

Let $\hat{P}_{jk}(t, \xi)$ be the Fourier transform, defined by

$$\hat{P}_{jk}(t, \xi) = E\left[\mathrm{e}^{i\xi X(t)}\, 1_{V(t)=k} | V(0) = j\right].$$

Step 3: The Fourier transform can be written as the matrix exponential

$$\hat{P}_{jk}(t, \xi) = E\left[\mathrm{e}^{t(Q+i\xi\phi)}\right]_{jk}. \tag{10.48}$$

From the elementary theory of matrices, it is known that the solutions of the equation $(Q + i\xi\phi)\mathbf{e}(\xi) = \gamma(\xi)\mathbf{e}(\xi)$ have expansions as analytic functions of ξ. The solution $\gamma 1$ tends to zero while the other branches satisfy $\mathrm{Re}(\gamma_j) \leq -\delta < 0$ for some $\delta > 0$ and all ξ, $-\infty < \xi < \infty$.

Step 4: There exists a solution of the eigenvalue problem $(Q + i\xi\phi)\mathbf{e} = \gamma(\xi)\mathbf{e}(\xi)$ with the expansions about $\xi = 0$

$$\gamma(\xi) = \bar{\phi}\xi + \sigma_2\xi^2 + O\big(\xi^3\big), \quad \xi \to 0 \tag{10.49}$$

$$\mathbf{e}(\xi) = 1 + \mathbf{e}_1(\xi) + O\big(\xi^2\big), \quad \xi \to 0. \tag{10.50}$$

Proof of the WLLN. We have the matrix exponential representation

$$E(\exp(iX(t)/t))_{jk} = \exp t(Q + i\xi/t) = \mathrm{e}^{t\gamma(\xi/t)} + O\big(\mathrm{e}^{-t\delta}\big), \quad t \to \infty.$$

When $t \to \infty$, the right-hand side converges to $\mathrm{e}^{it\bar{\phi}\xi}$. The continuity theorem proves that the distribution functions converge to a distribution concentrated at the point $x = \bar{\phi} := \sum_j \pi_j \phi_j$. ■

Proof of the CLT. We have the matrix exponential representation

$$E\Big(\exp(i(X(t) - mt)/\sqrt{t})]_{jk}\Big) = \exp t(Q + i\xi/t)_{j,k} = \sum_{l=1}^{N} \mathrm{e}^{t\gamma_l(\xi/t)}.$$

Applying Step 4, we have

$$\lim_{t\to\infty} E\left[\exp\left[i\xi\frac{X(t)-t\bar{\phi}}{\sqrt{t}}\right]1_{v(t)=j}\right] = \pi_j e^{\gamma_2\xi^2}.$$

We outline the main steps to prove Step 1. The other proofs are left to the reader. ■

Proposition 10.8. If zero is a simple eigenvalue of Q, then $\pi_k = \lim_{t\to\infty} p_{jk}(t)$ exists, is independent of j, and satisfies $\pi Q = 0$. The convergence is exponentially fast and the integral

$$H_{ij} = \int_0^\infty (p_{ij}(t) - \pi_j)\,\mathrm{d}t \tag{10.51}$$

is convergent. It satisfies the equations $H1 = 0$, $< Hv, 1 >= 0$, for any vector v and satisfies the Poisson equation

$$QH = HQ = -I + \Pi,$$

where Π is the projection onto the constant vectors, defined by $\Pi f = \sum_1^N f_k\pi_k$.

Proof. The exponential convergence is guaranteed by the location of the eigenvalues in the left half-plane, coupled with the Jordan canonical form. From the differential equation $P'(t) = QP(t) = P(t)Q$, we take $t\to\infty$ to obtain that the limiting matrix R satisfies $QR = RQ = 0$. In particular, the columns of R are eigenvectors with eigenvalue zero, hence each column of R is a constant: $R_{ij} = \pi_j$. Furthermore, the equation $RQ = 0$ shows that the row vector π_i is a left eigenvector, solution of $\pi Q = 0$. Finally, we come to the H matrix:

$$\begin{aligned}\sum_{k=1}^N p_{ik}(s)H_{kj} &= \sum_{k=1}^N p_{ik}(s)\int_0^\infty (p_{kj}(t)-\pi_j)\,\mathrm{d}t\\ &= \int_0^\infty (p_{ij}(s+t)-\pi_j)\,\mathrm{d}t\\ &= \int_s^\infty (p_{ij}(u)-\pi_j)\,\mathrm{d}u.\end{aligned}$$

Therefore,

$$\sum_{k=1}^n q_{ik}H_{kj} = \frac{\mathrm{d}}{\mathrm{d}s}\int_s^\infty (p_{ij}(u)-\pi_j)\mathrm{d}u|_{s=0} = \pi_j - \delta_{ij}.$$

■

10.4 Isotropic Transport in Higher Dimensions

In this section, we consider models of random motion in several dimensions. The finite Markov chain models of the previous sections are not well suited for this purpose, since one needs a continuum of directions as soon as the dimension is two or greater.

10.4.1 The Rayleigh Problem of Random Flights

The classical theory of probability provides a starting point for the higher-dimensional case, by means of the *Rayleigh problem of random flights*. This was first posed by Karl Pearson in 1905 in the following terms: "A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. It is required to find the probability that after n stretches he is at a distance between r and $r+\mathrm{d}r$ from his starting point O."

To solve Pearson's problem, we assume that each step is a random variable and that the collection of steps forms a sequence of R^2-valued independent and isotropic random variables with the same distribution. This is written

$$S_n = X_1 + \cdots + X_n, \quad \Pr[x_1 \in \mathrm{d}r \times \mathrm{d}\theta] = f(r)\mathrm{d}r\,\mathrm{d}\theta, \tag{10.52}$$

where $f(r)$ is a nonnegative f on $[0,\infty)$ normalized so that $\int_0^\infty f(r)\,\mathrm{d}r = 1/2\pi$.

The behavior of the sum is most effectively studied by means of the Fourier transform

$$\begin{aligned}\Phi_n(\lambda) :&= E\big[\mathrm{e}^{i\lambda\dot{S}_n}\big]\\ &= E\big[\mathrm{e}^{i\lambda\cdot X_1}\big]^n\\ &= \left[\int_0^\infty\int_0^{2\pi} \mathrm{e}^{i|\lambda| r\cos\theta} f(r)\mathrm{d}r\,\mathrm{d}\theta\right]^n\\ &= \left[2\pi\int_0^\infty J_0(|\lambda| r f(r)\,\mathrm{d}r\right]^n 2.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\mathrm{Prob}[S_n \in (r, r+\mathrm{d}r)]}{\mathrm{d}x} &= \frac{1}{(2\pi)^2}\int_{R^2} \Phi_n(\lambda)\mathrm{e}^{-i\lambda\cdot x}\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^\infty \Phi(\lambda)\mathrm{e}^{-i\lambda r\cos\phi}\lambda\,\mathrm{d}\lambda\mathrm{d}\phi\\ &= \int_0^\infty J_0(r\lambda)\Phi_n(r\lambda)\lambda\mathrm{d}\lambda.\end{aligned}$$

The solution of Pearson's problem is given formally by

$$\frac{P[S_n \in r, r + \mathrm{d}r)]}{\mathrm{d}r} = \int_0^\infty J_0(r\lambda)\Phi_n(\lambda)r\lambda \mathrm{d}\lambda. \tag{10.53}$$

The exact computation of the integral defining $\Phi_n(\lambda)$ may be difficult, depending on the form of the density function $f(r)$. Moreover, the inversion of the resultant nth power may be formidable. Nevertheless, it is straightforward to obtain the analog of the law of large numbers and the central limit theorem for this model. We recall the asymptotic behavior of the Bessel function J_0:

$$\begin{aligned} J_0(x) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{iz\cos\theta} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left[1 + ix\cos\theta - \frac{x^2}{2}\cos^2\theta + O\big(|x|^3\big)\right]\mathrm{d}\theta \\ &= 1 - \frac{x^2}{4} + O\big(|x|^3\big), \quad x \to 0 \end{aligned}$$

When we replace λ by λ/n, the integral $= 1 + O(1/n^2)$, which, when taken to the nth power, tends to 1. This gives the law of large numbers in the form

$$\lim_{n\to\infty} E\big[\mathrm{e}^{i\lambda\cdot S_n/n}\big] = 1 \quad \lambda \in R^2. \tag{10.54}$$

To obtain the appropriate form of the central limit theorem, we replace λ by $\lambda/\sqrt{n}$, which yields an integral of the form $1 - \big(\pi|\lambda|^2/2n\big)\int_0^\infty r^2 f(r)\mathrm{d}r + O\big(1/n^{3/2}\big)$. When we take this to the nth power, we obtain the central limit theorem in the form

$$\lim_{n\to\infty} E\left[\mathrm{e}^{i\lambda S_n/\sqrt{n}}\right] = \mathrm{e}^{-\lambda^2\sigma^2/2}, \tag{10.55}$$

where $\sigma^2 = \int_0^\infty \pi r^2 f(r)\,\mathrm{d}r$.

These results are the counterparts of the asymptotic results obtained for the two-state velocity model, studied in Section 10.1, which is the one-dimensional continuous-time analog of the Rayleigh problem of random flights. The variance parameter σ^2 depends on the form of the radial density function $f(r)$.

Exercise 10.8

Suppose that we have an exponential distribution with radial density function $f(r) = (2\pi a)^{-1}\mathrm{e}^{-r/a}$. Then, $\sigma^2\mathrm{e}^{-r/a}\,\mathrm{d}r = a^2$.

10.4.2 Three-Dimensional Rayleigh Model

The three-dimensional counterpart of Pearson's problem can be solved by reduction to a well-studied one-dimensional problem—as is often the case in three-dimensional spherically symmetric models.

Let $\{X_n\}$ be a sequence of independent and uniformly distributed random variables on the unit sphere in three-dimensional space; $S_n := X_1 + \cdots + X_n$. The radial density function $f(r)$ is defined as the quotient

$$f_n(r) = \frac{P[|S_n| \in (r, r + \mathrm{d}r)]}{\mathrm{d}r}. \tag{10.56}$$

This can be expressed in terms of the density function of a sum of independent and uniformly distributed random variables on the interval $[-1, 1]$, as follows.

Proposition 10.9. The density function is expressed in the form

$$f_n(r) = -rg_n'(r),$$

where

$$g_n(r) = \frac{P[[Y_1 + \cdots + Y_n] \in (r, r + \mathrm{d}r)]}{\mathrm{d}r}$$

and where $\{Y_n, n \geq 1\}$ is a sequence of real-valued and independent random variables with the uniform distribution on $[-1/2, 1/2]$.

Proof. We have

$$\begin{aligned} E\big[\mathrm{e}^{i\lambda\cdot S_n}\big] &= E\big[\mathrm{e}^{i\lambda\cdot X_1}\big]^n \\ &= \left[\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \mathrm{e}^{i\lambda\cos\theta}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\right] \\ &= \left(\frac{\sin\lambda}{\lambda}\right)^n \end{aligned}$$

$$\begin{aligned} \frac{P[S_n \in \mathrm{d}x]}{\mathrm{d}x} &= \left(\frac{1}{2\pi}\right)^3 \int_{R^3} \left(\frac{\sin\lambda}{\lambda}\right)^n \mathrm{e}^{-i\lambda\cdot x}\mathrm{d}^3x \\ &= \left(\frac{1}{2\pi}\right)^3 \int_0^{\infty} \left(\frac{\sin\lambda}{\lambda}\right)^n \lambda^2\mathrm{d}\lambda \int_0^{2\pi}\int_0^{\pi} \mathrm{e}^{-i\lambda r\cos\theta}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{1}{2\pi^2}\int_0^{\infty} \left(\frac{\sin\lambda}{\lambda}\right)^n \left(\frac{\sin\lambda r}{\lambda r}\right)\lambda^2\mathrm{d}\lambda \end{aligned}$$

$$
\begin{aligned}
f_n(r) &= \frac{P[S_n \in (r, r + \mathrm{d}r)]}{\mathrm{d}r} \\
&= 4\pi^2 \frac{1}{2\pi^2} \int_0^\infty \left(\frac{\sin\lambda}{\lambda}\right)^n \left(\frac{\sin\lambda r}{\lambda r}\right) \lambda^2 \, \mathrm{d}\lambda \\
&= \frac{2}{\pi} \int_0^\infty \left(\frac{\sin\lambda}{\lambda}\right)^n r\lambda \sin(r\lambda) \, \mathrm{d}\lambda.
\end{aligned}
$$

On the other hand, the density of the sum $\bar{S}_n := Y_1 + \cdots + Y_n$ is computed as

$$
\begin{aligned}
E\left[\mathrm{e}^{it\bar{S}_n}\right] &= E\left[\mathrm{e}^{itX_1}\right]^n \\
&= \left(\frac{1}{2} \int_{-1}^{1} \mathrm{e}^{ity} \, \mathrm{d}y\right)^n \\
&= \left(\frac{\sin t}{t}\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
g_n(x) &= \frac{P[S_n \in \mathrm{d}x]}{\mathrm{d}x} \\
&= \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^n \cos tx \, \mathrm{d}t \\
g_n'(x) &= -\frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^n t \sin tx \, \mathrm{d}t,
\end{aligned}
$$

which completes the proof that $f_n(r) = -rg_n'(r)$. The functions $g_n(r)$ are polynomials on the interval $0 \le r \le n$ and can be directly computed. ■

The Rayleigh model can be generalized to an arbitrary number of dimensions, where the Bessel function J_0 is replaced by $J_{(p-2)/2}$ in dimension p. It can also be carried out in the case of a continuous parameter t: a skater moves along a straight line at constant velocity for an exponentially distributed amount of time, at the end of which he or she chooses a new direction at random according to a uniform distribution on the unit sphere. In fact, the displacement after n changes of direction will be given by a discrete-time model with an exponential distribution of radial displacement. These considerations can be carried out on a surface or higher dimensional manifold.