7 Renewal Phenomena

7.1 Definition of a Renewal Process and Related Concepts

Renewal theory began with the study of stochastic systems whose evolution through time was interspersed with renewals or regeneration times when, in a statistical sense, the process began anew. Today, the subject is viewed as the study of general functions of independent, identically distributed, nonnegative random variables representing the successive intervals between renewals. The results are applicable in a wide variety of both theoretical and practical probability models.

A *renewal (counting) process* $\{N(t), t \geq 0\}$ is a nonnegative integer-valued stochastic process that registers the successive occurrences of an event during the time interval $(0, t]$, where the times between consecutive events are *positive, independent, identically distributed* random variables. Let the successive durations between events be $\{X_k\}_{k=1}^{\infty}$ (often representing the lifetimes of some units successively placed into service) such that X_i is the elapsed time from the $(i-1)$st event until the occurrence of the ith event. We write

$$F(x) = \Pr\{X_k \leq x\}, \quad k = 1, 2, 3, \ldots,$$

for the common probability distribution of $X_1, X_2, \ldots$. A basic stipulation for renewal processes is $F(0) = 0$, signifying that $X_1, X_2, \ldots$ are positive random variables. We refer to

$$W_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1 \qquad (7.1)$$
$$(W_0 = 0 \text{ by convention}),$$

as the *waiting time* until the occurrence of the nth event.

The relation between the interoccurrence times $\{X_k\}$ and the renewal counting process $\{N(t), t \geq 0\}$ is depicted in Figure 7.1. Note formally that

$$N(t) = \text{number of indices } n \text{ for which } 0 < W_n \leq t. \qquad (7.2)$$

In common practice, the counting process $\{N(t), t \geq 0\}$ and the partial sum process $\{W_n, n \geq 0\}$ are interchangeably called the "renewal process." The prototypical renewal model involves successive replacements of lightbulbs. A bulb is installed for service at time $W_0 = 0$, fails at time $W_1 = X_1$, and is then exchanged for a fresh bulb. The second bulb fails at time $W_2 = X_1 + X_2$ and is replaced by a third bulb. In general,

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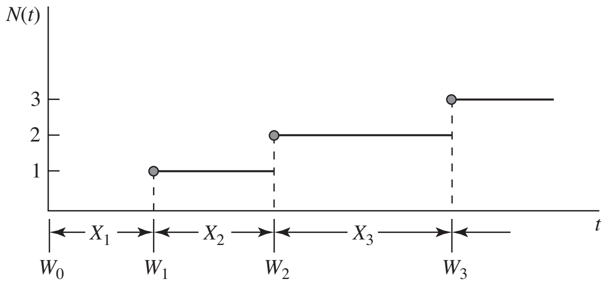


Figure 7.1 The relation between the interoccurrence times X_k and the renewal counting process $N(t)$.

the nth bulb burns out at time $W_n = X_1 + \cdots + X_n$ and is immediately replaced, and the process continues. It is natural to assume that the successive lifetimes are statistically independent, with probabilistically identical characteristics in that

$$\Pr\{X_k \leq x\} = F(x) \quad \text{for } k = 1, 2, \ldots.$$

In this process, $N(t)$ records the number of lightbulb replacements up to time t.

The principal objective of renewal theory is to derive properties of certain random variables associated with $\{N(t)\}$ and $\{W_n\}$ from knowledge of the interoccurrence distribution F. For example, it is of significance and relevance to compute the expected number of renewals for the time duration $(0, t]$:

$$E[N(t)] = M(t)$$

is called the *renewal function*. To this end, several pertinent relationships and formulas are worth recording. In principle, the probability law of $W_n = X_1 + \cdots + X_n$ can be calculated in accordance with the convolution formula

$$\Pr\{W_n \leq x\} = F_n(x),$$

where $F_1(x) = F(x)$ is assumed known or prescribed, and then

$$F_n(x) = \int_0^\infty F_{n-1}(x-y)\mathrm{d}F(y) = \int_0^x F_{n-1}(x-y)\mathrm{d}F(y).$$

Such convolution formulas were reviewed in Chapter 1, Section 1.2.5.

The fundamental connecting link between the waiting time process $\{W_n\}$ and the renewal counting process $\{N(t)\}$ is the observation that

$$N(t) \geq k \quad \text{if and only if} \quad W_k \leq t. \tag{7.3}$$

In words, equation (7.3) asserts that the number of renewals up to time t is at least k if and only if the kth renewal occurred on or before time t. Since this equivalence is the basis for much that follows, the reader should verify instances of it by referring to Figure 7.1.

It follows from (7.3) that

$$\begin{aligned}\Pr\{N(t) \geq k\} &= \Pr\{W_k \leq t\} \\ &= F_k(t), \quad t \geq 0, k = 1, 2, \ldots,\end{aligned} \tag{7.4}$$

and consequently,

$$\begin{aligned}\Pr\{N(t) = k\} &= \Pr\{N(t) \geq k\} - \Pr\{N(t) \geq k+1\} \\ &= F_k(t) - F_{k+1}(t), \quad t \geq 0, k = 1, 2, \ldots.\end{aligned} \tag{7.5}$$

For the renewal function $M(t) = E[N(t)]$, we sum the tail probabilities in the manner $E[N(t)] = \Sigma_{k=1}^{\infty} \Pr\{N(t) \geq k\}$, as derived in Chapter 1, equation (1.49), and then use (7.4) to obtain

$$\begin{aligned}M(t) = E[N(t)] &= \sum_{k=1}^{\infty} \Pr\{N(t) \geq k\} \\ &= \sum_{k=1}^{\infty} \Pr\{W_k \leq t\} = \sum_{k=1}^{\infty} F_k(t).\end{aligned} \tag{7.6}$$

There are a number of other random variables of interest in renewal theory. Three of these are the *excess life* (also called the excess random variable), the *current life* (also called the age random variable), and the *total life*, defined, respectively, by

$$\begin{aligned}&\gamma_t = W_{N(t)+1} - t && \text{(excess or residual lifetime)}, \\ &\delta_t = t - W_{N(t)} && \text{(current life or age random variable)}, \\ &\beta_t = \gamma_t + \delta_t && \text{(total life)}.\end{aligned}$$

A pictorial description of these random variables is given in Figure 7.2.

An important identity enables us to evaluate the mean of $W_{N(t)+1}$ in terms of the mean lifetime $\mu = E[X_1]$ of each unit and the renewal function $M(t)$. Namely, it is true for every renewal process that

$$\begin{aligned}E[W_{N(t)+1}] &= E[X_1 + \cdots + X_{N(t)+1}] \\ &= E[X_1]\{E[N(t) + 1]\},\end{aligned}$$

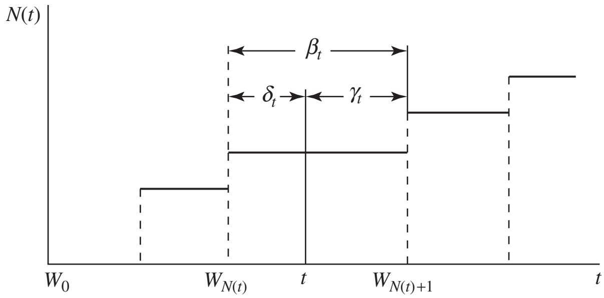


Figure 7.2 The excess life γ_t, the current life δ_t, and the total life β_t.

or

$$E[W_{N(t)+1}] = \mu\{M(t)+1\}. \tag{7.7}$$

At first glance, this identity resembles the formula given in Chapter 2, equation (2.30) for the mean of a random sum, which asserts that $E[X_1 + \cdots + X_N] = E[X_1]E[N]$ when N is an integer-valued random variable *that is independent of* $X_1, X_2, \ldots$. The random sum approach does not apply in the current context, however, the crucial difference being that the random number of summands $N(t)+1$ is *not* independent of the summands themselves. Indeed, in Section 7.3, on the Poisson process viewed as a renewal process, we will show that the last summand $X_{N(t)+1}$ has a mean that approaches twice the unconditional mean $\mu = E[X_1]$ for t large. For this reason, it is *not* correct, in particular, that $E[W_{N(t)}]$ can be evaluated as the product of $E[X_1]$ and $E[N(t)]$. In view of these comments, the identity expressed in equation (7.7) becomes more intriguing and remarkable.

To derive (7.7), we will use the fundamental equivalence (7.3) in the form

$$N(t) \geq j-1 \quad \text{if and only if} \quad X_1 + \cdots + X_{j-1} \leq t,$$

which expressed in terms of indicator random variables becomes

$$\mathbf{1}\{N(t) \geq j-1\} = \mathbf{1}\{X_1 + \cdots + X_{j-1} \leq t\}.$$

Since this indicator random variable is a function only of the random variables $X_1, \ldots, X_{j-1}$, it is independent of X_j, and thus we may evaluate

$$\begin{aligned} E[X_j \mathbf{1}\{X_1 + \cdots + X_{j-1} \leq t\}] &= E[X_j]E[\mathbf{1}\{X_1 + \cdots + X_{j-1} \leq t\}] \\ &= E[X_j]\Pr\{X_1 + \cdots + X_{j-1} \leq t\} \\ &= \mu F_{j-1}(t). \end{aligned} \tag{7.8}$$

With (7.8) in hand, the evaluation of the equivalence expressed in (7.7) becomes straightforward. We have

$$
\begin{aligned}
E[W_{N(t)+1}] &= E[X_1 + \cdots + X_{N(t)+1}] \\
&= E[X_1] + E\left[\sum_{j=2}^{N(t)+1} X_j\right] \\
&= \mu + E\left[\sum_{j=2}^{\infty} X_j \mathbf{1}\{N(t)+1 \geq j\}\right] \\
&= \mu + \sum_{j=2}^{\infty} E[X_j \mathbf{1}\{X_1 + \cdots + X_{j-1} \leq t\}] \\
&= \mu + \mu \sum_{j=2}^{\infty} F_{j-1}(t) \qquad \text{(using (7.8))} \\
&= \mu[1 + M(t)] \qquad \text{(using (7.6))}.
\end{aligned}
$$

Some examples of the use of the identity (7.7) will appear in the exercises, and an alternative proof in the case of a discrete renewal process can be found in Section 7.6.

Exercises

7.1.1 Verify the following equivalences for the age and the excess life in a renewal process $N(t)$:

$$\gamma_t > x \quad \text{if and only if} \quad N(t+x) - N(t) = 0;$$

and for $0 < x < t$,

$$\delta_t > x \quad \text{if and only if} \quad N(t) - N(t-x) = 0.$$

Why is the condition $x < t$ important in the second case but not the first?

7.1.2 Consider a renewal process in which the interoccurrence times have an exponential distribution with parameter λ:

$$f(x) = \lambda \mathrm{e}^{-\lambda x}, \text{ and } F(x) = 1 - \mathrm{e}^{-\lambda x} \quad \text{for } x > 0.$$

Calculate $F_2(t)$ by carrying out the appropriate convolution [see the equation just prior to (7.3)], and then determine $\Pr\{N(t) = 1\}$ from equation (7.5).

7.1.3 Which of the following are true statements?

(a) $N(t) < k$ if and only if $W_k > t$.

(b) $N(t) \leq k$ if and only if $W_k \geq t$.

(c) $N(t) > k$ if and only if $W_k < t$.

7.1.4 Consider a renewal process for which the lifetimes $X_1, X_2, \ldots$ are discrete random variables having the Poisson distribution with mean λ. That is,

$$\Pr\{X_k = n\} = \frac{e^{-\lambda}\lambda^n}{n!} \quad \text{for } n = 0, 1, \ldots.$$

(a) What is the distribution of the waiting time W_k?
(b) Determine $\Pr\{N(t) = k\}$.

Problems

7.1.1 Verify the following equivalences for the age and the excess life in a renewal process $N(t)$: (Assume $t > x$.)

$$\begin{aligned}
\Pr\{\delta_t \geq x, \gamma_t > y\} &= \Pr\{N(t-x) = N(t+y)\} \\
&= \sum_{k=0}^{\infty} \Pr\{W_k < t-x, W_{k+1} > t+y\} \\
&= [1 - F(t+y)] \\
&\quad + \sum_{k=1}^{\infty} \int_0^{t-x} [1 - F(t+y-z)] \mathrm{d}F_k(z).
\end{aligned}$$

Carry out the evaluation when the interoccurrence times are exponentially distributed with parameter λ, so that $\mathrm{d}F_k$ is the gamma density

$$\mathrm{d}F_k(z) = \frac{\lambda^k z^{k-1}}{(k-1)!} e^{-\lambda z} \mathrm{d}z \quad \text{for } z > 0.$$

7.1.2 From equation (7.5), and for $k \geq 1$, verify that

$$\begin{aligned}
\Pr\{N(t) = k\} &= \Pr\{W_k \leq t < W_{k+1}\} \\
&= \int_0^t [1 - F(t-x)] \mathrm{d}F_k(x),
\end{aligned}$$

and carry out the evaluation when the interoccurrence times are exponentially distributed with parameter λ, so that $\mathrm{d}F_k$ is the gamma density

$$\mathrm{d}F_k(z) = \frac{\lambda^k z^{k-1}}{(k-1)!} e^{-\lambda z} \mathrm{d}z \quad \text{for } z > 0.$$

7.1.3 A fundamental identity involving the renewal function, valid for all renewal processes, is

$$E[W_{N(t)+1}] = E[X_1](M(t) + 1).$$

See equation (7.7). Using this identity, show that the mean excess life can be evaluated in terms of the renewal function via the relation

$$E[\gamma_t] = E[X_1](1 + M(t)) - t.$$

7.1.4 Let γ_t be the excess life and δ_t the age in a renewal process having interoccurrence distribution function $F(x)$. Determine the conditional probability $\Pr\{\gamma_t > y|\delta_t = x\}$ and the conditional mean $E[\gamma_t|\delta_t = x]$.

7.2 Some Examples of Renewal Processes

Stochastic models often contain random times at which they, or some part of them, begin afresh in a statistical sense. These renewal instants form natural embedded renewal processes, and they are found in many diverse fields of applied probability including branching processes, insurance risk models, phenomena of population growth, evolutionary genetic mechanisms, engineering systems, and econometric structures. When a renewal process is discovered embedded within a model, the powerful results of renewal theory become available for deducing implications.

7.2.1 Brief Sketches of Renewal Situations

The synopses that follow suggest the wide scope and diverse contexts in which renewal processes arise. Several of the examples will be studied in more detail in later sections.

(a) *Poisson Processes* A Poisson process $\{N(t), t \geq 0\}$ with parameter λ is a renewal counting process having the exponential interoccurrence distribution

$$F(x) = 1 - \mathrm{e}^{-\lambda x}, \quad x \geq 0,$$

as established in Chapter 5, Theorem 5.5. This particular renewal process possesses a host of special features, highlighted later in Section 7.3.

(b) *Counter Processes* The times between successive electrical impulses or signals impinging on a recording device (counter) are often assumed to form a renewal process. Most physically realizable counters lock for some duration immediately upon registering an impulse and will not record impulses arriving during this dead period. Impulses are recorded only when the counter is free (i.e., unlocked). Under quite reasonable assumptions, the sequence of events of the times of recorded impulses forms a renewal process, but it should be emphasized that the renewal process of recorded impulses is a secondary renewal process derived from the original renewal process comprising the totality of all arriving impulses.

(c) *Traffic Flow* The distances between successive cars on an indefinitely long single-lane highway are often assumed to form a renewal process. So also are the time durations between consecutive cars passing a fixed location.

(d) *Renewal Processes Associated with Queues* In a single-server queueing process, there are embedded many natural renewal processes. We cite two examples:
  (i) If customer arrival times form a renewal process, then the times of the starts of successive busy periods generate a second renewal process.
  (ii) For the situation in which the input process (the arrival pattern of customers) is Poisson, the successive moments in which the server passes from a busy to a free state determine a renewal process.

(e) *Inventory Systems* In the analysis of most inventory processes, it is customary to assume that the pattern of demands forms a renewal process. Most of the standard inventory policies induce renewal sequences, e.g., the times of replenishment of stock.

(f) *Renewal Processes in Markov Chains* Let $Z_0, Z_1, \ldots$ be a recurrent Markov chain. Suppose $Z_0 = i$, and consider the durations (elapsed number of steps) between successive visits to state i. Specifically, let $W_0 = 0$,

$$W_1 = \min\{n > 0; Z_n = i\},$$

and

$$W_{k+1} = \min\{n > W_k; Z_n = i\}, \quad k = 1, 2, \ldots.$$

Since each of these times is computed from the same starting state i, the Markov property guarantees that $X_k = W_k - W_{k-1}$ are independent and identically distributed, and thus $\{X_k\}$ generates a renewal process.

7.2.2 Block Replacement

Consider a lightbulb whose life, measured in discrete units, is a random variable X, where $\Pr\{X = k\} = p_k$ for $k = 1, 2, \ldots$. Assuming that one starts with a fresh bulb and that each bulb is replaced by a new one when it burns out, let $M(n) = E[N(n)]$ be the expected number of replacements up to time n.

Because of economies of scale, in a large building such as a factory or office it is often cheaper, on a per bulb basis, to replace all the bulbs, failed or not, than it is to replace a single bulb. A *block replacement policy* attempts to take advantage of this reduced cost by fixing a block period K and then replacing bulbs as they fail during periods $1, 2, \ldots, K-1$, and replacing all bulbs, failed or not, in period K. This strategy is also known as "group relamping." If c_1 is the per bulb block replacement cost and c_2 is the per bulb failure replacement cost ($c_1 < c_2$), then the mean total cost during the block replacement cycle is $c_1 + c_2 M(K-1)$, where $M(K-1) = E[N(K-1)]$ is the mean number of failure replacements. Since the block replacement cycle consists of K periods, the mean total cost per bulb per unit time is

$$\theta(K) = \frac{c_1 + c_2 M(K-1)}{K}.$$

If we can determine the renewal function $M(n)$ from the life distribution $\{p_k\}$, then we can choose the block period $K = K^*$ so as to minimize the cost rate $\theta(K)$. Of course, this cost must be compared to the cost of replacing only upon failure.

The renewal function $M(n)$, or expected number of replacements up to time n, solves the equation

$$M(n) = F_X(n) + \sum_{k=1}^{n-1} p_k M(n-k) \quad \text{for } n = 1, 2, \ldots.$$

To derive this equation, condition on the life X_1 of the first bulb. If it fails after time n, there are no replacements during periods $[1, 2, \ldots, n]$. On the other hand, if it fails at time $k < n$, then we have its failure plus, on the average, $M(n-k)$ additional replacements during the interval $[k+1,\ k+2, \ldots, n]$. Using the law of total probability to sum these contributions, we obtain

$$\begin{aligned} M(n) &= \sum_{k=n+1}^{\infty} p_k(0) + \sum_{k=1}^{n} p_k[1 + M(n-k)] \\ &= F_X(n) + \sum_{k=1}^{n-1} p_k M(n-k) \quad [\text{because } M(0) = 0], \end{aligned}$$

as asserted.

Thus, we determine

$$\begin{aligned} M(1) &= F_X(1), \\ M(2) &= F_X(2) + p_1 M(1), \\ M(3) &= F_X(3) + p_1 M(2) + p_2 M(1), \end{aligned}$$

and so on.

To consider a numerical example, suppose that

$$p_1 = 0.1, \quad p_2 = 0.4, \quad p_3 = 0.3, \quad \text{and} \quad p_4 = 0.2,$$

and

$$c_1 = 2 \quad \text{and} \quad c_2 = 3.$$

Then,

$$\begin{aligned} M(1) &= p_1 = 0.1, \\ M(2) &= (p_1 + p_2) + p_1 M(1) = (0.1 + 0.4) + 0.1(0.1) = 0.51, \\ M(3) &= (p_1 + p_2 + p_3) + p_1 M(2) + p_2 M(1) \\ &= (0.1 + 0.4 + 0.3) + 0.1(0.51) + 0.4(0.1) = 0.891, \\ M(4) &= (p_1 + p_2 + p_3 + p_4) + p_1 M(3) + p_2 M(2) + p_3 M(1) \\ &= 1 + 0.1(0.891) + 0.4(0.51) + 0.3(0.1) = 1.3231. \end{aligned}$$

The average costs are shown in the following table:

Block Period K	**Cost** $= \dfrac{c_1 + c_2 M(K-1)}{K} = \theta(K)$
1	2.00000
2	1.15000
3	1.17667
4	1.16825
5	1.19386

The minimum cost block period is $K^* = 2$.

We wish to elicit one more insight from this example. Forgetting about block replacement, we continue to calculate

$$
\begin{aligned}
M(5) &= 1.6617, \\
M(6) &= 2.0647, \\
M(7) &= 2.4463, \\
M(8) &= 2.8336, \\
M(9) &= 3.2136, \\
M(10) &= 3.6016.
\end{aligned}
$$

Let u_n be the probability that a replacement occurs in period n. Then, $M(n) = M(n-1) + u_n$ asserts that the mean replacements up to time n is the mean replacements up to time $n-1$ plus the probability that a replacement occurs in period n. The calculations are shown in the following table:

n	$u_n = M(n) - M(n-1)$
1	0.1000
2	0.4100
3	0.3810
4	0.4321
5	0.3386
6	0.4030
7	0.3816
8	0.3873
9	0.3800
10	0.3880

The probability of a replacement in period n seems to be converging. This is indeed the case, and the limit is the reciprocal of the mean bulb lifetime:

$$\frac{1}{E[X_1]} = \frac{1}{0.1(1)+0.4(2)+0.3(3)+0.2(4)}$$
$$= 0.3846\cdots.$$

This calculation makes sense. If a lightbulb lasts, on the average, $E[X_1]$ time units, then the probability that it will need to be replaced in any period should approximate $1/E[X_1]$. Actually, the relationship is not as simple as just stated. Further discussion takes place in Sections 7.4 and 7.6.

Exercises

7.2.1 Let $\{X_n; n = 0, 1, \ldots\}$ be a two-state Markov chain with the transition probability matrix

$$\mathbf{P} = \begin{array}{c|cc|} & 0 & 1 \\ \hline 0 & 1-a & a \\ 1 & b & 1-b \end{array}.$$

State 0 represents an *operating* state of some system, while state 1 represents a *repair* state. We assume that the process begins in state $X_0 = 0$, and then the successive returns to state 0 from the repair state form a renewal process. Determine the mean duration of one of these renewal intervals.

7.2.2 A certain type component has two states: 0 = OFF and 1 = OPERATING. In state 0, the process remains there a random length of time, which is exponentially distributed with parameter α, and then moves to state 1. The time in state 1 is exponentially distributed with parameter β, after which the process returns to state 0.

The *system* has two of these components, A and B, with distinct parameters:

Component	Operating Failure Rate	Repair Rate
A	β_A	α_A
B	β_B	α_B

In order for the *system* to operate, at least one of components A and B must be operating (a parallel system). Assume that the component stochastic processes are independent of one another. Consider the successive instants that the *system* enters the failed state from an operating state. Use the memoryless property of the exponential distribution to argue that these instants form a renewal process.

7.2.3 Calculate the mean number of renewals $M(n) = E[N(n)]$ for the renewal process having interoccurrence distribution

$$p_1 = 0.4, \quad p_2 = 0.1, \quad p_3 = 0.3, \quad p_4 = 0.2$$

for $n = 1, 2, \ldots, 10$. Also calculate $u_n = M(n) - M(n-1)$.

Problems

7.2.1 For the block replacement example of this section for which $p_1 = 0.1, p_2 = 0.4, p_3 = 0.3$, and $p_4 = 0.2$, suppose the costs are $c_1 = 4$ and $c_2 = 5$. Determine the minimal cost block period K^* and the cost of replacing upon failure alone.

7.2.2 Let $X_1, X_2, \ldots$ be the interoccurrence times in a renewal process. Suppose $\Pr\{X_k = 1\} = p$ and $\Pr\{X_k = 2\} = q = 1 - p$. Verify that

$$M(n) = E[N(n)] = \frac{n}{1+q} - \frac{q^2}{(1+q)^2}\left[1 - (-q)^n\right]$$

for $n = 2, 4, 6, \ldots$.

7.2.3 Determine $M(n)$ when the interoccurrence times have the geometric distribution

$$\Pr\{X_1 = k\} = p_k = \beta(1-\beta)^{k-1} \quad \text{for } k = 1, 2, \ldots,$$

where $0 < \beta < 1$.

7.3 The Poisson Process Viewed as a Renewal Process

As mentioned earlier, the Poisson process with parameter λ is a renewal process whose interoccurrence times have the exponential distribution $F(x) = 1 - e^{-\lambda x}, x \geq 0$. The memoryless property of the exponential distribution (see Sections 1.4.2, 1.5.2 of Chapter 1, and Chapter 5) serves decisively in yielding the explicit computation of a number of functionals of the Poisson renewal process.

The Renewal Function

Since $N(t)$ has a Poisson distribution, then

$$\Pr\{N(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, \ldots,$$

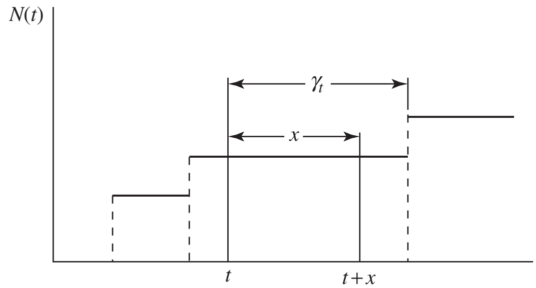


Figure 7.3 The excess life γ_t exceeds x if and only if there are no renewals in the interval $(t, t+x]$.

and

$$M(t) = E[N(t)] = \lambda t.$$

Excess Life

Observe that the excess life at time t exceeds x if and only if there are no renewals in the interval $(t, t+x]$ (Figure 7.3). This event has the same probability as that of no renewals in the interval $(0, x]$, since a Poisson process has stationary independent increments. In formal terms, we have

$$\begin{aligned}\Pr\{\gamma_t > x\} &= \Pr\{N(t+x) - N(t) = 0\} \\ &= \Pr\{N(x) = 0\} = e^{-\lambda x}.\end{aligned} \tag{7.9}$$

Thus, in a Poisson process, the excess life possesses the same exponential distribution

$$\Pr\{\gamma_t \leq x\} = 1 - e^{-\lambda x}, \quad x \geq 0, \tag{7.10}$$

as every life, another manifestation of the memoryless property of the exponential distribution.

Current Life

The current life δ_t, of course, cannot exceed t, while for $x < t$ the current life exceeds x if and only if there are no renewals in $(t-x, t]$, which again has probability $e^{-\lambda x}$. Thus, the current life follows the truncated exponential distribution

$$\Pr\{\delta_t \leq x\} = \begin{cases} 1 - e^{-\lambda x} & \text{for } 0 \leq x < t, \\ 1 & \text{for } t \leq x. \end{cases} \tag{7.11}$$

Mean Total Life

Using the evaluation of equation (1.50) in Chapter 1 for the mean of a nonnegative random variable, we have

$$\begin{aligned}E[\beta_t] &= E[\gamma_t] + E[\delta_t]\\ &= \frac{1}{\lambda} + \int_0^t \Pr\{\delta_t > x\}\mathrm{d}x\\ &= \frac{1}{\lambda} + \int_0^t \mathrm{e}^{-\lambda x}\mathrm{d}x\\ &= \frac{1}{\lambda} + \frac{1}{\lambda}\left(1-\mathrm{e}^{-\lambda t}\right).\end{aligned}$$

Observe that the mean total life is significantly larger than the mean life $1/\lambda = E[X_k]$ of any particular renewal interval. A more striking expression of this phenomenon is revealed when t is large, where the process has been in operation for a long duration. Then, the mean total life $E[\beta_t]$ is approximately twice the mean life. These facts appear at first paradoxical.

Let us reexamine the definition of the total life β_t with a view to explaining on an intuitive basis the seeming discrepancy. First, an arbitrary time point t is fixed. Then, β_t measures the length of the renewal interval containing the point t. Such a procedure will tend with higher likelihood to favor a lengthy renewal interval rather than one of short duration. The phenomenon is known as length-biased sampling and occurs, well disguised, in a number of sampling situations.

Joint Distribution of γ_t and δ_t

The joint distribution of γ_t and δ_t is determined in the same manner as the marginals. In fact, for any $x > 0$ and $0 < y < t$, the event $\{\gamma_t > x, \delta_t > y\}$ occurs if and only if there are no renewals in the interval $(t-y, t+x]$, which has probability $\mathrm{e}^{-\lambda(x+y)}$. Thus,

$$\Pr\{\gamma_t > x, \delta_t > y\} = \begin{cases}\mathrm{e}^{-\lambda(x+y)} & \text{if } x > 0, 0 < y < t,\\ 0 & \text{if } y \geq t.\end{cases} \tag{7.12}$$

For the Poisson process, observe that γ_t and δ_t are independent, since their joint distribution factors as the product of their marginal distributions.

Exercises

7.3.1 Let $W_1, W_2, \ldots$ be the event times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ. Evaluate

$$\Pr\{W_{N(t)+1} > t+s\} \quad \text{and} \quad E[W_{N(t)+1}].$$

7.3.2 Particles arrive at a counter according to a Poisson process of rate λ. An arriving particle is recorded with probability p and lost with probability $1-p$ independently of the other particles. Show that the sequence of recorded particles is a Poisson process of rate λp.

7.3.3 Let $W_1, W_2, \ldots$ be the event times in a Poisson process $\{N(t); t \geq 0\}$ of rate λ. Show that

$$N(t) \quad \text{and} \quad W_{N(t)+1}$$

are independent random variables by evaluating

$$\Pr\{N(t) = n \quad \text{and} \quad W_{N(t)+1} > t + s\}.$$

Problems

7.3.1 In another form of *sum quota sampling* (see Chapter 5, Section 5.4.2), a sequence of nonnegative independent and identically distributed random variables $X_1, X_2, \ldots$ is observed, the sampling continuing until the first time that the sum of the observations *exceeds* the quota t. In renewal process terminology, the sample size is $N(t) + 1$. The sample mean is

$$\frac{W_{N(t)+1}}{N(t)+1} = \frac{X_1 + \cdots + X_{N(t)+1}}{N(t)+1}.$$

An important question in statistical theory is whether or not this sample mean is unbiased. That is, how does the expected value of this sample mean relate to the expected value of, say, X_1? Assume that the individual X summands are exponentially distributed with parameter λ, so that $N(t)$ is a Poisson process, and evaluate the expected value of the foregoing sample mean and show that

$$E\left[\frac{W_{N(t)+1}}{N(t)+1}\right] = \frac{1}{\lambda}\left[1 - \mathrm{e}^{-\lambda t}\right]\left(1 + \frac{1}{\lambda t}\right).$$

Hint: Use the result of the previous exercise, that

$$W_{N(t)+1} \quad \text{and} \quad N(t)$$

are independent, and then evaluate separately

$$E[W_{N(t)+1}] \quad \text{and} \quad E\left[\frac{1}{N(t)+1}\right].$$

7.3.2 A fundamental identity involving the renewal function, valid for all renewal processes, is

$$E\left[W_{N(t)+1}\right] = E[X_1](M(t)+1).$$

See equation (7.7). Evaluate the left side and verify the identity when the renewal counting process is a Poisson process.

7.3.3 Pulses arrive at a counter according to a Poisson process of rate λ. All physically realizable counters are imperfect, incapable of detecting all signals that enter their detection chambers. After a particle or signal arrives, a counter must recuperate, or renew itself, in preparation for the next arrival. Signals arriving during the readjustment period, called dead time or locked time, are lost. We must distinguish between the arriving particles and the recorded particles. The experimenter observes only the particles recorded; from this observation he desires to infer the properties of the arrival process.

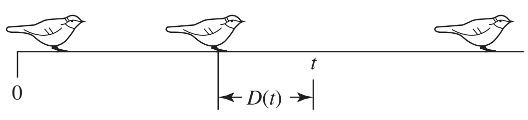
Suppose that each arriving pulse locks the counter for a fixed time τ. Determine the probability $p(t)$ that the counter is free at time t.

7.3.4 This problem is designed to aid in the understanding of length-biased sampling. Let X be a uniformly distributed random variable on $[0, 1]$. Then, X divides $[0, 1]$ into the subintervals $[0, X]$ and $(X, 1]$. By symmetry, each subinterval has mean length $\frac{1}{2}$. Now pick one of these subintervals at random in the following way: Let Y be independent of X and uniformly distributed on $[0, 1]$, and pick the subinterval $[0, X]$ or $(X, 1]$ that Y falls in. Let L be the length of the subinterval so chosen. Formally,

$$L = \begin{cases} X & \text{if } Y \le X, \\ 1 - X & \text{if } Y > X. \end{cases}$$

Determine the mean of L.

7.3.5 Birds are perched along a wire as shown according to a Poisson process of rate λ per unit distance:



At a fixed point t along the wire, let $D(t)$ be the random distance to the nearest bird. What is the mean value of $D(t)$? What is the probability density function $f_t(x)$ for $D(t)$?

7.4 The Asymptotic Behavior of Renewal Processes

A large number of the functionals that have explicit expressions for Poisson renewal processes are far more difficult to compute for other renewal processes. There are, however, many simple formulas that describe the asymptotic behavior, for large values of t, of a general renewal process. We summarize some of these asymptotic results in this section.

7.4.1 The Elementary Renewal Theorem

The Poisson process is the only renewal process (in continuous time) whose renewal function $M(t) = E[N(t)]$ is exactly linear. All renewal functions are asymptotically linear, however, in the sense that

$$\lim_{t\to\infty} \frac{M(t)}{t} = \lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{\mu}, \tag{7.13}$$

where $\mu = E[X_k]$ is the mean interoccurrence time. This fundamental result, known as the *elementary renewal theorem*, is invoked repeatedly to compute functionals describing the long run behavior of stochastic models having renewal processes associated with them.

The elementary renewal theorem (7.4.1) holds even when the interoccurrence times have infinite mean, and then $\lim_{t\to\infty} M(t)/t = 1/\infty = 0$.

The elementary renewal theorem is so intuitively plausible that it has often been viewed as obvious. The left side, $\lim_{t\to\infty} M(t)/t$, describes the long run mean number of renewals or replacements per unit time. The right side, $1/\mu$, is the reciprocal of the mean life of a component. Isn't it obvious that if a component lasts, on the average, μ time units, then in the long run these components will be replaced at the rate of $1/\mu$ per unit time? However plausible and convincing this argument may be, it is not obvious, and to establish the elementary renewal theorem requires several steps of mathematical analysis, beginning with the law of large numbers. As our main concern is stochastic modeling, we omit this derivation, as well as the derivations of the other asymptotic results summarized in this section, in order to give more space to their application.

Example *Age Replacement Policies* Let $X_1, X_2, \ldots$ represent the lifetimes of items (lightbulbs, transistor cards, machines, etc.) that are successively placed in service, the next item commencing service immediately following the failure of the previous one. We stipulate that $\{X_k\}$ are independent and identically distributed positive random variables with finite mean $\mu = E[X_k]$. The elementary renewal theorem tells us to expect to replace items over the long run at a mean rate of $1/\mu$ per unit time.

In the long run, any replacement strategy that substitutes items prior to their failure will use more than $1/\mu$ items per unit time. Nonetheless, where there is some benefit in avoiding failure in service, and where units deteriorate, in some sense, with age, there may be an economic or reliability advantage in considering alternative replacement strategies. Telephone or utility poles serve as good illustrations of this concept. Clearly, it is disadvantageous to allow these poles to fail in service because of the damage to the wires they carry, the damage to adjoining property, overtime wages paid for emergency replacements, and revenue lost while service is down. Therefore, an attempt is usually made to replace older utility poles before they fail. Other instances of planned replacement occur in preventative maintenance strategies for aircraft, where “time” is now measured by operating hours.

An age replacement policy calls for replacing an item upon its failure or upon its reaching age T, whichever occurs first. Arguing intuitively, we would expect that the long run fraction of failure replacements, items that fail before age T, will be $F(T)$, and the corresponding fraction of (conceivably less expensive) planned replacements will be $1-F(T)$. A renewal interval for this modified age replacement policy obviously follows a distribution law

$$F_T(x)=\begin{cases}F(x) & \text{for } x<T,\\ 1 & \text{for } x\geq T,\end{cases}$$

and the mean renewal duration is

$$\mu_T=\int_0^{\infty}\{1-F_T(x)\}\,\mathrm{d}x=\int_{\mathrm{O}}^{T}\{1-F(x)\}\mathrm{d}x<\mu.$$

The elementary renewal theorem indicates that the long run mean replacement rate under age replacement is increased to $1/\mu_T$.

Now, let $Y_1, Y_2, \ldots$ denote the times between actual successive failures. The random variable Y_1 is composed of a random number of time periods of length T (corresponding to replacements not associated with failures), plus a last time period in which the distribution is that of a failure conditioned on failure before age T; i.e., Y_1 has the distribution of $NT+Z$, where

$$\Pr\{N\geq k\}=\{1-F(T)\}^k,\quad k=0,1,\ldots,$$

and

$$\Pr\{Z\leq z\}=\frac{F(z)}{F(T)},\quad 0\leq z\leq T.$$

Hence,

$$\begin{aligned}E[Y_1]&=\frac{1}{F(T)}\left\{T[1-F(T)]+\int_0^T(F(T)-F(x))\mathrm{d}x\right\}\\&=\frac{1}{F(T)}\int_0^T\{1-F(x)\}\mathrm{d}x=\frac{\mu_T}{F(T)}.\end{aligned}$$

The sequence of random variables for interoccurrence times of the bona fide failure $\{Y_i\}$ generates a renewal process whose mean rate of failures per unit time in the long run is $1/E[Y_1]$. This inference again relies on the elementary renewal theorem.

Depending on F, the modified failure rate $1/E[Y_1]$ may possibly yield a lower failure rate than $1/\mu$, the rate when replacements are made only upon failure.

Let us suppose that each replacement, whether planned or not, costs K dollars, and that each failure incurs an additional penalty of c dollars. Multiplying these costs by the appropriate rates gives the long run mean cost per unit time as a function of the replacement age T:

$$\begin{aligned} C(T) &= \frac{K}{\mu_T} + \frac{c}{E[Y_1]} \\ &= \frac{K + cF(T)}{\int_0^T [1-F(x)]\mathrm{d}x}. \end{aligned}$$

In any particular situation, a routine calculus exercise or recourse to numerical computation produces the value of T that minimizes the long run cost rate. For example, if $K = 1, c = 4$, and lifetimes are uniformly distributed on $[0, 1]$, then $F(x) = x$ for $0 \le x \le 1$, and

$$\int_0^T [1 - F(x)]\mathrm{d}x = T\left(1 - \frac{1}{2}T\right)$$

and

$$C(T) = \frac{1+4T}{T(1-T/2)}.$$

To obtain the cost minimizing T, we differentiate $C(T)$ with respect to T and equate to zero, thereby obtaining

$$\begin{aligned} \frac{\mathrm{d}C(T)}{\mathrm{d}T} = 0 &= \frac{4T(1-T/2) - (1+4T)(1-T)}{[T(1-T/2)]^2}, \\ 0 &= 4T - 2T^2 - 1 + T - 4T + 4T^2, \\ 0 &= 2T^2 + T - 1, \\ T &= \frac{-1 \pm \sqrt{1+8}}{4} = \left(\frac{1}{2}, -1\right), \end{aligned}$$

and the optimal choice is $T^* = \frac{1}{2}$. Routine calculus will verify that this choice leads to a minimum cost, and not a maximum or inflection point.

7.4.2 The Renewal Theorem for Continuous Lifetimes

The elementary renewal theorem asserts that

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}.$$

It is tempting to conclude from this that $M(t)$ behaves like t/μ as t grows large, but the precise meaning of the phrase "behaves like" is rather subtle. For example, suppose that all of the lifetimes are deterministic, say $X_k = 1$ for $k = 1, 2, \ldots$. Then, it is straightforward to calculate

$$\begin{aligned} M(t) = N(t) &= 0 \qquad \text{for } 0 \le t < 1, \\ &= 1 \qquad \text{for } 1 \le t < 2, \\ &= k \qquad \text{for } k \le t < k+1. \end{aligned}$$

That is, $M(t) = [t]$, where $[t]$ denotes the greatest integer not exceeding t. Since $\mu = 1$ in this example, then $M(t) - t/\mu = [t] - t$, a function that oscillates indefinitely between 0 and -1. While it remains true in this illustration that $M(t)/t = [t]/t \to 1 = 1/\mu$, it is not clear in what sense $M(t)$ "behaves like" t/μ. If we rule out the periodic behavior that is exemplified in the extreme by this deterministic example, then $M(t)$ behaves like t/μ in the sense described by the *renewal theorem*, which we now explain. Let $M(t, t+h] = M(t+h) - M(t)$ denote the mean number of renewals in the interval $(t, t+h]$. The renewal theorem asserts that when periodic behavior is precluded, then

$$\lim_{t\to\infty} M(t, \mathrm{t}+h] = h/\mu \quad \text{for any fixed } h > 0. \tag{7.14}$$

In words, asymptotically, the mean number of renewals in an interval is proportional to the interval's length, with proportionality constant $1/\mu$.

A simple and prevalent situation in which the renewal theorem (7.4.2) is valid occurs when the lifetimes $X_1, X_2, \ldots$ are continuous random variables having the probability density function $f(x)$. In this circumstance, the renewal function is differentiable, and

$$m(t) = \frac{\mathrm{d}M(t)}{\mathrm{d}t} = \sum_{n=1}^{\infty} f_n(t), \tag{7.15}$$

where $f_n(t)$ is the probability density function for $W_n = X_1 + \cdots + X_n$. Now (7.14) may be written in the form

$$\frac{M(t+h) - M(t)}{h} \to \frac{1}{\mu} \quad \text{as } t \to \infty,$$

which, when h is small, suggests that

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} \frac{\mathrm{d}M(t)}{\mathrm{d}t} = \frac{1}{\mu}, \tag{7.16}$$

and indeed, this is the case in all but the most pathological of circumstances when $X_1, X_2, \ldots$ are continuous random variables.

If in addition to being continuous, the lifetimes $X_1, X_2, \ldots$ have a finite mean μ and finite variance σ^2, then the renewal theorem can be refined to include a second term.

Under the stated conditions, we have

$$\lim_{t\to\infty}\left[M(t)-\frac{t}{\mu}\right]=\frac{\sigma^2-\mu^2}{2\mu^2}. \tag{7.17}$$

Example When the lifetimes $X_1, X_2, \dots$ have the gamma density function

$$f(x) = x\mathrm{e}^{-x} \quad \text{for } x > 0, \tag{7.18}$$

then the waiting times $W_n = X_1 + \cdots + X_n$ have the gamma density

$$f_n(x) = \frac{x^{2n-1}}{(2n-1)!}\mathrm{e}^{-x} \quad \text{for } x > 0,$$

as may be verified by performing the appropriate convolutions. (See Chapter 1, Section 1.2.5.) Substitution into (7.15) yields

$$\begin{aligned} m(x) &= \sum_{n=1}^{\infty} f_n(x) = \mathrm{e}^{-x}\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!} \\ &= \mathrm{e}^{-x}\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2} = \frac{1}{2}\left(1-\mathrm{e}^{-2x}\right), \end{aligned}$$

and

$$M(t) = \int_0^t m(x)\mathrm{d}x = \frac{1}{2}t - \frac{1}{4}\left[1-\mathrm{e}^{-2t}\right].$$

Since the gamma density in (7.18) has moments $\mu = 2$ and $\sigma^2 = 2$, we verify that $m(t) \to 1/\mu$ as $t \to \infty$ and $M(t) - t/\mu \to -\frac{1}{4} = \left(\sigma^2 - \mu^2\right)/2\mu^2$, in agreement with (7.16) and (7.17).

7.4.3 The Asymptotic Distribution of $N(t)$

The elementary renewal theorem

$$\lim_{t\to x}\frac{E[N(t)]}{t} = \frac{1}{\mu} \tag{7.19}$$

implies that the asymptotic mean of $N(t)$ is approximately t/μ. When $\mu = E[X_k]$ and $\sigma^2 = \mathrm{Var}[X_k] = E\left[(X_k-\mu)^2\right]$ are finite, then the asymptotic variance of $N(t)$ behaves according to

$$\lim_{t\to\infty}\frac{\mathrm{Var}[N(t)]}{t} = \frac{\sigma^2}{\mu^3}. \tag{7.20}$$

That is, the asymptotic variance of $N(t)$ is approximately $t\sigma^2/\mu^3$. If we standardize $N(t)$ by subtracting its asymptotic mean and dividing by its asymptotic standard deviation, we get the following convergence to the normal distribution:

$$\lim_{t\to\infty} \Pr\left\{\frac{N(t)-t/\mu}{\sqrt{t\sigma^2/\mu^3}} \le x\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\mathrm{d}y.$$

In words, for large values of t, the number of renewals $N(t)$ is approximately normally distributed with mean and variance given by (7.19) and (7.20), respectively.

7.4.4 The Limiting Distribution of Age and Excess Life

Again we assume that the lifetimes $X_1, X_2, \ldots$ are continuous random variables with finite mean μ. Let $\gamma_t = W_{N(t)+1} - t$ be the excess life at time t. The excess life has the limiting distribution

$$\lim_{t\to\infty} \Pr\{\gamma_t \le x\} = \frac{1}{\mu}\int_0^x [1-F(y)]\mathrm{d}y. \tag{7.21}$$

The reader should verify that the right side of (7.21) defines a valid distribution function, which we denote by $H(x)$. The corresponding probability density function is $h(y) = \mu^{-1}[1-F(y)]$. The mean of this limiting distribution is determined according to

$$\begin{aligned}
\int_0^\infty yh(y)\mathrm{d}y &= \frac{1}{\mu}\int_0^\infty y[1-F(y)]\mathrm{d}y \\
&= \frac{1}{\mu}\int_0^\infty y\left\{\int_y^\infty f(t)\mathrm{d}t\right\}\mathrm{d}y \\
&= \frac{1}{\mu}\int_0^\infty f(t)\left\{\int_0^t y\backslash \mathrm{d}y\right\}\mathrm{d}t \\
&= \frac{1}{2\mu}\int_0^\infty t^2 f(t)\mathrm{d}t \\
&= \frac{\sigma^2+\mu^2}{2\mu},
\end{aligned}$$

where σ^2 is the common variance of the lifetimes $X_1, X_2, \ldots$.

$$\{\gamma_t \ge x \text{ and } \delta_t \ge y\} \quad \text{if and only if} \quad \{\gamma_{t-y} \ge x+y\}. \tag{7.22}$$

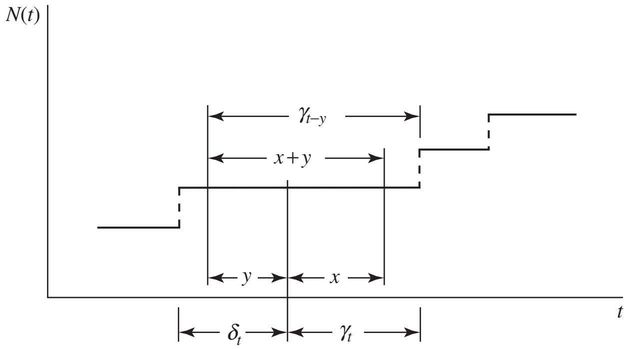


Figure 7.4 $\{\delta_t \geq y \text{ and } \gamma_t \geq x\}$ if and only if $\{\gamma_{t-y} \geq x+y\}$.

It follows that

$$\begin{aligned}\lim_{t\to\infty} \Pr\{\gamma_t \geq x, \delta_t \geq y\} &= \lim_{t\to\infty} \Pr\{\gamma_{t-y} \geq x+y\}\\ &= \mu^{-1} \int_{x+y}^{\infty} [1-F(z)]\mathrm{d}z,\end{aligned}$$

exhibiting the joint limiting distribution of (γ_t, δ_t). In particular,

$$\begin{aligned}\lim_{t\to\infty} \Pr\{\delta_t \geq y\} &= \lim_{t\to\infty} \Pr\{\gamma_t \geq 0, \delta_t \geq y\}\\ &= \mu^{-1} \int_{y}^{\infty} [1-F(z)]\mathrm{d}z\\ &= 1-H(y).\end{aligned}$$

The limiting distribution for the current life, or age, $\delta_t = t - W_{N(t)}$ can be deduced from the corresponding result (7.21) for the excess life. With the aid of Figure 7.4, corroborate the equivalence.

Exercises

7.4.1 Consider the triangular lifetime density $f(x) = 2x$ for $0 < x < 1$. Determine an asymptotic expression for the expected number of renewals up to time t.

Hint: Use equation (7.17).

7.4.2 Consider the triangular lifetime density $f(x) = 2x$ for $0 < x < 1$. Determine an asymptotic expression for the probability distribution of excess life. Using this distribution, determine the limiting mean excess life and compare with the general result following equation (7.21).

7.4.3 Consider the triangular lifetime density function $f(x) = 2x$, for $0 < x < 1$. Determine the optimal replacement age in an age replacement model with replacement cost $K = 1$ and failure penalty $c = 4$ (cf. the example in Section 7.4.1).

7.4.4 Show that the optimal age replacement policy is to replace upon failure alone when lifetimes are exponentially distributed with parameter λ. Can you provide an intuitive explanation?

7.4.5 What is the limiting distribution of excess life when renewal lifetimes have the uniform density $f(x) = 1$, for $0 < x < 1$?

7.4.6 A machine can be in either of two states: "up" or "down." It is up at time zero and thereafter alternates between being up and down. The lengths $X_1, X_2, \ldots$ of successive up times are independent and identically distributed random variables with mean α, and the lengths $Y_1, Y_2, \ldots$ of successive down times are independent and identically distributed with mean β.

(a) In the long run, what fraction of time is the machine up?

(b) If the machine earns income at a rate of \$13 per unit time while up, what is the long run total rate of income earned by the machine?

(c) If each down time costs \$7, regardless of how long the machine is down, what is the long run total down time cost per unit time?

Problems

7.4.1 Suppose that a renewal function has the form $M(t) = t + [1 - \exp(-at)]$. Determine the mean and variance of the interoccurrence distribution.

7.4.2 A system is subject to failures. Each failure requires a repair time that is exponentially distributed with rate parameter α. The operating time of the system until the next failure is exponentially distributed with rate parameter β. The repair times and the operating times are all statistically independent. Suppose that the system is operating at time 0. Using equation (7.17), determine an approximate expression for the mean number of failures up to time t, the approximation holding for $t \gg 0$.

7.4.3 Suppose that the life of a lightbulb is a random variable X with hazard rate $h(x) = \theta x$ for $x > 0$. Each failed lightbulb is immediately replaced with a new one. Determine an asymptotic expression for the mean age of the lightbulb in service at time t, valid for $t \gg 0$.

7.4.4 A developing country is attempting to control its population growth by placing restrictions on the number of children each family can have. This society places a high premium on female children, and it is felt that any policy that ignores the desire to have female children will fail. The proposed policy is to allow any married couple to have children up to the first female baby, at which point they must cease having children. Assume that male and female children are equally likely. The number of children in any family is a random variable N. In the population as a whole, what fraction of children are female? Use the elementary renewal theorem to justify your answer.

7.4.5 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.7 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0.5 & 0.5 \end{array} \right\| \end{array}.$$

A *sojourn* in a state is an uninterrupted sequence of consecutive visits to that state.

(a) Determine the mean duration of a typical sojourn in state 0.

(b) Using renewal theory, determine the long run fraction of time that the process is in state 1.

7.5 Generalizations and Variations on Renewal Processes

7.5.1 Delayed Renewal Processes

We continue to assume that $\{X_k\}$ are all independent positive random variables, but only $X_2, X_3, \ldots$ (from the second on) are identically distributed with distribution function F, while X_1 has possibly a different distribution function G. Such a process is called a *delayed renewal process*. We have all the ingredients for an ordinary renewal process except that the initial time to the first renewal has a distribution different from that of the other interoccurrence times.

A delayed renewal process will arise when the component in operation at time $t = 0$ is not new, but all subsequent replacements are new. For example, suppose that the time origin is taken y time units after the start of an ordinary renewal process. Then, the time to the first renewal after the origin in the delayed process will have the distribution of the excess life at time y of an ordinary renewal process.

As before, let $W_0 = 0$ and $W_n = X_1 + \cdots + X_n$, and let $N(t)$ count the number of renewals up to time t. But now it is essential to distinguish between the mean number of renewals in the delayed process

$$M_D(t) = E[N(t)], \tag{7.23}$$

and the renewal function associated with the distribution F,

$$M(t) = \sum_{k=1}^{\infty} F_k(t). \tag{7.24}$$

For the delayed process, the elementary renewal theorem is

$$\lim_{t\to\infty} \frac{M_D(t)}{t} = \frac{1}{\mu}, \quad \text{where } \mu = E[X_2], \tag{7.25}$$

and the renewal theorem states that

$$\lim_{t\to\infty}[M_D(t)-M_D(t-h)]=\frac{h}{\mu},$$

provided $X_2, X_3, \ldots$ are continuous random variables.

7.5.2 Stationary Renewal Processes

A delayed renewal process for which the first life has the distribution function

$$G(x)=\mu^{-1}\int_0^x\{1-F(y)\}\mathrm{d}y$$

is called a stationary renewal process. We are attempting to model a renewal process that began indefinitely far in the past, so that the remaining life of the item in service at the origin has the limiting distribution of the excess life in an ordinary renewal process. We recognize G as this limiting distribution.

It is anticipated that such a process exhibits a number of stationary, or time-invariant, properties. For a stationary renewal process,

$$M_D(t)=E[N(t)]=\frac{t}{\mu} \tag{7.26}$$

and

$$\Pr\left\{\gamma_t^D\le x\right\}=G(x),$$

for all t. Thus, what is in general only an asymptotic renewal relation becomes an identity, holding for all t, in a stationary renewal process.

7.5.3 Cumulative and Related Processes

Suppose associated with the ith unit, or lifetime interval, is a second random variable Y_i ($\{Y_i\}$ identically distributed) in addition to the lifetime X_i. We allow X_i and Y_i to be dependent but assume that the pairs (X_1, Y_1), $(X_2, Y_2), \ldots$ are independent. We use the notation $F(x)=\Pr\{X_i\le x\}$, $G(y)=\Pr\{Y_i\le y\}$, $\mu=E[X_i]$, and $\nu=E[Y_i]$.

A number of problems of practical and theoretical interest have a natural formulation in those terms.

Renewal Processes Involving Two Components to Each Renewal interval

Suppose that Y_i represents a portion of the duration X_i. Figure 7.5 illustrates the model. There we have depicted the Y portion occurring at the beginning of the interval, but this assumption is not essential for the results that follow.

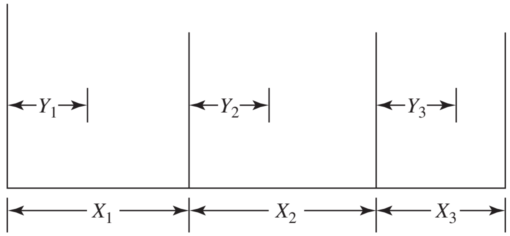


Figure 7.5 A renewal process in which an associated random variable Y_i, represents a portion of the ith renewal interval.

Let $p(t)$ be the probability that t falls in a Y portion of some renewal interval. When $X_1, X_2, \dots$ are continuous random variables, the renewal theorem implies the following important asymptotic evaluation:

$$\lim_{t\to\infty} p(t) = \frac{E[Y_1]}{E[X_1]}. \tag{7.27}$$

Here are some concrete examples.

A Replacement Model

Consider a replacement model in which replacement is not instantaneous. Let Y_i be the operating time and Z_i the lag period preceding installment of the $(i+1)$st operating unit. (The delay in replacement can be conceived as a period of repair of the service unit.) We assume that the sequence of times between successive replacements $X_k = Y_k + Z_k, k = 1, 2, \dots,$ constitutes a renewal process. Then $p(t)$, the probability that the system is in operation at time t, converges to $E[Y_1]/E[X_1]$.

A Queueing Model

A queueing process is a process in which customers arrive at some designated place where a service of some kind is being rendered, e.g., at the teller's window in a bank or beside the cashier at a supermarket. It is assumed that the time between arrivals, or interarrival time, and the time that is spent in providing service for a given customer are governed by probabilistic laws.

If arrivals to a queue follow a Poisson process of intensity λ, then the successive times X_k from the commencement of the kth busy period to the start of the next busy period form a renewal process. (A busy period is an uninterrupted duration when the queue is not empty.) Each X_k is composed of a busy portion Z_k and an idle portion Y_k. Then $p(t)$, the probability that the queue is empty at time t, converges to $E[Y_1]/E[X_1]$. This example is treated more fully in Chapter 9, which is devoted to queueing systems.

The Peter Principle

The "Peter Principle" asserts that a worker will be promoted until finally reaching a position in which he or she is incompetent. When this happens, the person stays in that job until retirement. Consider the following single job model of the Peter Principle: A person is selected at random from the population and placed in the job. If the person is competent, he or she remains in the job for a random time having cumulative distribution function F and mean μ and is promoted. If incompetent, the person remains for a random time having cumulative distribution function G and mean $\nu > \mu$ and retires. Once the job is vacated, another person is selected at random and the process repeats. Assume that the infinite population contains the fraction p of competent people and $q = 1 - p$ incompetent ones.

In the long run, what fraction of time is the position held by an incompetent person?

A renewal occurs every time the position is filled, and therefore the mean duration of a renewal cycle is

$$E[X_k] = p\mu + (1-p)\nu.$$

To answer the question, we let $Y_k = X_k$ if the kth person is incompetent, and $Y_k = 0$ if the kth person is competent. Then, the long run fraction of time that the position is held by an incompetent person is

$$\frac{E[Y_1]}{E[X_1]} = \frac{(1-p)\nu}{p\mu + (1-p)\nu}.$$

Suppose that $p = \frac{1}{2}$ of the people are competent, and that $\nu = 10$, while $\mu = 1$. Then,

$$\frac{E[Y_1]}{E[X_1]} = \frac{(1/2)(10)}{(1/2)(10) + (1/2)(1)} = \frac{10}{11} = 0.91.$$

Thus, while half of the people in the population are competent, the job is filled by a competent person only 9% of the time!

Cumulative Processes

Interpret Y_i as a cost or value associated with the ith renewal cycle. A class of problems with a natural setting in this general context of pairs (X_i, Y_i), where X_i generates a renewal process, will now be considered. Interest here focuses on the so-called cumulative process

$$W(t) = \sum_{k=1}^{N(t)+1} Y_k,$$

the accumulated costs or value up to time t (assuming that transactions are made at the beginning of a renewal cycle).

The elementary renewal theorem asserts in this case that

$$\lim_{t\to\infty} \frac{1}{t} E[W(t)] = \frac{E[Y_1]}{\mu}. \tag{7.28}$$

This equation justifies the interpretation of $E[Y_1]/\mu$ as a long run mean cost or value per unit time, an interpretation that was used repeatedly in the examples of Section 7.2.

Here are some examples of cumulative processes.

Replacement Models

Suppose Y_i is the cost of the ith replacement. Let us suppose that under an age replacement strategy (see Section 7.3 and the example entitled "Age Replacement Policies" in Section 7.4) a planned replacement at age T costs c_1 dollars, while a failure replaced at time $x < T$ costs c_2 dollars. If Y_k is the cost incurred at the kth replacement cycle, then

$$Y_k = \begin{cases} c_1 & \text{with probability } 1 - F(T), \\ c_2 & \text{with probability } F(T), \end{cases}$$

and $E[Y_k] = c_1[1 - F(T)] + c_2F(T)$. Since the expected length of a replacement cycle is

$$E[\min\{X_k, T\}] = \int_0^T [1 - F(x)]\mathrm{d}x,$$

we have that the long run cost per unit time is

$$\frac{c_1[1 - F(T)] + c_2F(T)}{\int_0^T [1 - F(x)]\mathrm{d}x},$$

and in any particular situation a routine calculus exercise or recourse to numerical computation produces the value of T that minimizes the long run cost per unit time.

Under a block replacement policy, there is one planned replacement every T units of time and, on the average, $M(T)$ failure replacements, so the expected cost is $E[Y_k] = c_1 + c_2M(T)$, and the long run mean cost per unit time is $\{c_1 + c_2M(T)\}/T$.

Risk Theory

Suppose claims arrive at an insurance company according to a renewal process with interoccurrence times $X_1, X_2, \ldots$. Let Y_k be the magnitude of the kth claim. Then, $W(t) = \Sigma_{k=1}^{N(t)+1} Y_k$ represents the cumulative amount claimed up to time t, and the long run mean claim rate is

$$\lim_{t\to\infty} \frac{1}{t}E[W(t)] = \frac{E[Y_1]}{E[X_1]}.$$

Maintaining Current Control of a Process

A production process produces items one by one. At any instance, the process is in one of two possible states, which we label *in-control* and *out-of-control*. These states are not directly observable. Production begins with the process in-control, and it remains in-control for a random and unobservable length of time before a breakdown occurs, after which the process is out-of-control. A control chart is to be used to help detect when the out-of-control state occurs, so that corrective action may be taken.

To be more specific, we assume that the quality of an individual item is a normally distributed random variable having an unknown mean and a known variance σ^2. If the process is in-control, the mean equals a standard target, or design value, μ_0. Process breakdown takes the form of shift in mean away from standard to $\mu_1 = \mu_0 \pm \delta\sigma$, where δ is the amount of the shift in standard deviation units.

The Shewhart control chart method for maintaining process control calls for measuring the qualities of the items as they are produced, and then plotting these qualities versus time on a chart that has lines drawn at the target value μ_0 and above and below this target value at $\mu_0 \pm k\sigma$, where k is a parameter of the control scheme being used. As long as the plotted qualities fall inside these so-called *action lines* at $\mu_0 \pm k\sigma$, the process is assumed to be operating in-control, but if ever a point falls outside these lines, the process is assumed to have left the in-control state, and investigation and repair are instituted. There are obviously two possible types of errors that can be made while, thus, controlling the process: (1) needless investigation and repair when the process is in-control yet an observed quality purely by chance falls outside the action lines and (2) continued operation with the process out-of-control because the observed qualities are falling inside the action lines, again by chance.

Our concern is the rational choice of the parameter k, i.e., the rational spacing of the action lines, so as to balance, in some sense, these two possible errors.

The probability that a single quality will fall outside the action lines when the process is in-control is given by an appropriate area under the normal density curve. Denoting this probability by α, we have

$$\alpha = \Phi(-k) + 1 - \Phi(k) = 2\Phi(-k),$$

where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp\left(-y^2/2\right) \mathrm{d}y$ is the standard cumulative normal distribution function. Representative values are given in the following table:

k	α
1.645	0.10
1.96	0.05

Similarly, the probability, denoted by p, that a single point will fall outside the action lines when the process is out-of-control is given by

$$p = \Phi(-\delta - k) + 1 - \Phi(-\delta + k).$$

Let S denote the number of items inspected before an out-of-control signal arises assuming that the process is out-of-control. Then, $\Pr\{S=1\}=p$, $\Pr\{S=2\}=(1-p)p$, and in general, $\Pr\{S=n\} = (1-p)^{n-1}p$. Thus, S has a geometric distribution, and

$$E[S] = \frac{1}{p}.$$

Let T be the number of items produced while the process is in-control. We suppose that the mean operating time in-control $E[T]$ is known from past records.

The sequence of durations between detected and repaired out-of-control conditions forms a renewal process because each such duration begins with a newly repaired process and is a probabilistic replica of all other such intervals. It follows from the general elementary renewal theorem that the long run fraction of time spent out-of-control (O.C.) is

$$\text{O.C.} = \frac{E[S]}{E[S]+E[T]} = \frac{1}{1+pE[T]}.$$

The long run number of repairs per unit time is

$$R = \frac{1}{E[S]+E[T]} = \frac{p}{1+pE[T]}.$$

Let N be the random number of "false alarms" while the process is in-control, i.e., during the time up to T, the first out-of-control. Then, conditioned on T, the random variable N has a binomial distribution with probability parameter α, and thus $E[N|T\}$ $= \alpha T$ and $E[N] = \alpha E[T]$. Again, it follows from the general elementary renewal theorem that the long run false alarms per unit time (F.A.) is

$$\text{F.A.} = \frac{E[N]}{E[S]+E[T]} = \frac{\alpha p E[T]}{1+pE[T]}.$$

If each false alarm costs c dollars, each repair cost K dollars, and the cost rate while operating out-of-control is C dollars, then we have the long run average cost per unit time of

$$\begin{aligned} \text{A.C.} &= C(\text{O.C.}) + K(R) + c(\text{F.A.}) \\ &= \frac{C + Kp + c\alpha p E[T]}{1+pE[T]}. \end{aligned}$$

By trial and error one may now choose k, which determines α and p, so as to minimize this average cost expression.

Exercises

7.5.1 Jobs arrive at a certain service system according to a Poisson process of rate λ. The server will accept an arriving customer only if it is idle at the time of arrival. Potential customers arriving when the system is busy are lost. Suppose that the service times are independent random variables with mean service time μ. Show that the long run fraction of time that the server is idle is $1/(1+\lambda\mu)$. What is the long run fraction of potential customers that are lost?

7.5.2 The weather in a certain locale consists of alternating wet and dry spells. Suppose that the number of days in each rainy spell is Poisson distributed with parameter 2, and that a dry spell follows a geometric distribution with a mean of 7 days. Assume that the successive durations of rainy and dry spells are statistically independent random variables. In the long run, what is the probability on a given day that it will be raining?

7.5.3 Consider a lightbulb whose life is a continuous random variable X with probability density function $f(x)$, for $x > 0$. Assuming that one starts with a fresh bulb and that each failed bulb is immediately replaced by a new one, let $M(t) = E[N(t)]$ be the expected number of renewals up to time t. Consider a *block replacement policy* (see Section 7.2.1) that replaces each failed bulb immediately at a cost of c per bulb and replaces all bulbs at the fixed times $T, 2T, 3T, \ldots$. Let the block replacement cost per bulb be $b < c$. Show that the long run total mean cost per bulb per unit time is

$$\Theta(T) = \frac{b + cM(T)}{T}.$$

Investigate the choice of a cost minimizing value T^* when $M(t) = t + 1 - \exp(-at)$.

Problems

7.5.1 A certain type component has two states: $0 = \text{OFF}$ and $1 = \text{OPERATING}$. In state 0, the process remains there a random length of time, which is exponentially distributed with parameter α, and then moves to state 1. The time in state 1 is exponentially distributed with parameter β, after which the process returns to state 0.

The *system* has two of these components, A and B, with distinct parameters:

Component	Operating Failure Rate	Repair Rate
A	β_A	α_A
B	β_B	α_B

In order for the *system* to operate, at least one of components A and B must be operating (a parallel system). Assume that the component stochastic processes are independent of one another.

(a) In the long run, what fraction of time is the system inoperational (not operating)?

(b) Once the system enters the failed state, what is the mean duration there prior to returning to operation?

(c) Define a cycle as the time between the instant that the system first enters the failed state and the next such instant. Using renewal theory, find the mean duration of a cycle.

(d) What is the mean system operating duration between successive system failures?

7.5.2 The random lifetime X of an item has a distribution function $F(x)$. What is the mean total life $E[X|X > x]$ of an item of age x?

7.5.3 At the beginning of each period, customers arrive at a taxi stand at times of a renewal process with distribution law $F(x)$. Assume an unlimited supply of cabs, such as might occur at an airport. Suppose that each customer pays a random fee at the stand following the distribution law $G(x)$, for $x > 0$. Write an expression for the sum $W(t)$ of money collected at the stand by time t, and then determine the limit expectation

$$\lim_{t\to\infty} \frac{E[W(t)]}{t}.$$

7.5.4 A lazy professor has a ceiling fixture in his office that contains two lightbulbs. To replace a bulb, the professor must fetch a ladder, and being lazy, when a single bulb fails, he waits until the second bulb fails before replacing them both. Assume that the length of life of the bulbs are independent random variables.

(a) If the lifetimes of the bulbs are exponentially distributed, with the same parameter, what fraction of time, in the long run, is our professor's office half lit?

(b) What fraction of time, in the long run, is our professor's office half lit if the bulbs that he buys have the same uniform (0, 1) lifetime distribution?

7.6 Discrete Renewal Theory*

In this section, we outline the renewal theory that pertains to nonnegative integer-valued lifetimes. We emphasize renewal equations, the renewal argument, and the renewal theorem (Theorem 7.1).

Consider a lightbulb whose life, measured in discrete units, is a random variable X where $\Pr\{X = k\} = p_k$ for $k = 0, 1, \ldots$. If one starts with a fresh bulb and if each bulb when it burns out is replaced by a new one, then $M(n)$, the expected number of renewals (not including the initial bulb) up to time n, solves the equation

$$M(n) = F_X(n) + \sum_{k=0}^{n} p_k M(n-k), \tag{7.29}$$

where $F_X(n) = p_0 + \cdots + p_n$ is the cumulative distribution function of the random variable X. A vector or functional equation of the form (7.29) in the unknowns $M(0), M(1), \ldots$ is termed a *renewal equation*. The equation is established by a *renewal argument*, a first step analysis that proceeds by conditioning on the life of the first bulb and then invoking the law of total probability. In the case of (7.29), e.g., if the first bulb fails at time $k \leq n$, then we have its failure plus, on the average, $M(n-k)$ additional failures in the interval $[k, k+1, \ldots, n]$. We weight this conditional mean by

* The discrete renewal model is a special case in the general renewal theory presented in Sections 7.1–7.5 and does not arise in later chapters.

the probability $p_k = \Pr\{X_1 = k\}$ and sum according to the law of total probability to obtain

$$M(n) = \sum_{k=0}^{n} [1 + M(n-k)] p_k$$

$$= F_X(n) = \sum_{k=0}^{n} p_k M(n-k).$$

Equation (7.29) is only a particular instance of what is called a renewal equation. In general, a renewal equation is prescribed by a given bounded sequence $\{b_k\}$ and takes the form

$$v_n = b_n + \sum_{k=0}^{n} p_k v_{n-k} \qquad \text{for } n = 0, 1, \ldots. \tag{7.30}$$

The unknown variables are $v_0, v_1, \ldots,$ and $p_0, p_1, \ldots$ is a probability distribution for which, to avoid trivialities, we always assume $p_0 < 1$.

Let us first note that there is one and only one sequence $v_0, v_1, \ldots$ satisfying a renewal equation, because we may solve (7.30) successively to get

$$\begin{aligned} v_0 &= \frac{b_0}{1 - p_0}, \\ v_1 &= \frac{b_1 + p_1 v_0}{1 - p_0}, \end{aligned} \tag{7.31}$$

and so on.

Let u_n be the mean number of renewals that take place exactly in period n. When $p_0 = 0$, so that the lifetimes are strictly positive and at most one renewal can occur in any period, then u_n is the probability that a single renewal occurs in period n. The sequence $u_0, u_1, \ldots$ satisfies a renewal equation that is of fundamental importance in the general theory. Let

$$\delta_n = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases} \tag{7.32}$$

Then, $\{u_n\}$ satisfies the renewal equation

$$u_n = \delta_n + \sum_{k=0}^{n} p_k u_{n-k} \qquad \text{for } n = 0, 1, \ldots. \tag{7.33}$$

Again, equation (7.33) is established via a renewal argument. First, observe that δ_n counts the initial bulb, the renewal at time 0. Next, condition on the lifetime of this first bulb. If it fails in period $k \le n$, which occurs with probability p_k, then the process begins afresh and the conditional probability of a renewal in period n becomes u_{n-k}. Weighting the contingency represented by u_{n-k} by its respective probability p_k and summing according to the law of total probability then yields (7.33).

The next lemma shows how the solution $\{v_n\}$ to the general renewal equation (7.30) can be expressed in terms of the solution $\{u_n\}$ to the particular equation (7.33).

Lemma 7.1. *If $\{v_n\}$ satisfies (7.30) and $\{u_n\}$ satisfies (7.33), then*

$$v_n = \sum_{k=0}^{n} b_{n-k}u_k \qquad \textit{for } n = 0, 1, \ldots.$$

Proof. In view of our remarks on the existence and uniqueness of solutions to equation (7.30), we need only verify that $v_n = \Sigma_{k=0}^{n} b_{n-k}u_k$ satisfies (7.30). We have

$$\begin{aligned}
v_n &= \sum_{k=0}^{n} b_{n-k}u_k \\
&= \sum_{k=0}^{n} b_{n-k}\left\{\delta_k + \sum_{l=0}^{k} p_{k-l}u_l\right\} \\
&= b_n + \sum_{k=0}^{n}\sum_{l=0}^{k} b_{n-k}p_{k-l}u_l \\
&= b_n + \sum_{l=0}^{n}\sum_{k=l}^{n} b_{n-k}p_{k-l}u_l \\
&= b_n + \sum_{l=0}^{n}\sum_{j=0}^{n-l} p_j b_{n-l-j}u_l \\
&= b_n + \sum_{j=0}^{n}\sum_{l=0}^{n-j} p_j b_{n-j-l}u_l \\
&= b_n + \sum_{j=0}^{n} p_j v_{n-j}.
\end{aligned}$$

■

Example Let $X_1, X_2, \ldots$ be the successive lifetimes of the bulbs and let $W_0 = 0$ and $W_n = X_1 + \cdots + X_n$ be the replacement times. We assume that $p_0 = \Pr\{X_1 = 0\} = 0$. The number of replacements (not including the initial bulb) up to time n is given by

$$N(n) = k \quad \text{for } W_k \le n < W_{k+1}.$$

The $M(n) = E[N(n)]$ satisfies the renewal equation (7.29)

$$M(n) = p_0 + \cdots + p_n + \sum_{k=0}^{n} p_k M(n-k),$$

and elementary algebra shows that $m_n = E[N(n)+1] = M(n)+1$ satisfies

$$m_n = 1 + \sum_{k=0}^{n} p_k m_{n-k} \quad \text{for } n = 0, 1, \ldots. \tag{7.34}$$

Then, (7.34) is a renewal equation for which $b_n \equiv 1$ for all n. In view of Lemma 7.1, we conclude that

$$m_n = \sum_{k=0}^{n} 1 u_k = u_0 + \cdots + u_n.$$

Conversely, $u_n = m_n - m_{n-1} = M(n) - M(n-1)$.

To continue with the example, let $g_n = E[W_{N(n)+1}]$. The definition is illustrated in Figure 7.6. We will argue that g_n satisfies a certain renewal equation. As shown in Figure 7.6, $W_{N(n)+1}$ always includes the first renewal duration X_1. In addition, if $X_1 = k \le n$, which occurs with probability p_k, then the conditional mean of the added lives constituting $W_{N(n)+1}$ is g_{n-k}. Weighting these conditional means by their respective probabilities and summing according to the law of total probability then gives

$$g_n = E[X_1] + \sum_{k=0}^{n} g_{n-k} p_k.$$

Hence, by Lemma 7.1,

$$g_n = \sum_{k=0}^{n} E[X_1] u_k = E[X_1] m_n.$$

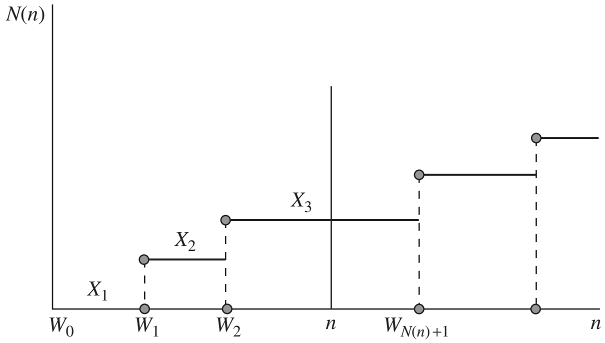


Figure 7.6 $W_{N(n)+1}$ always contains X_1 and contains additional durations when $X_1 = k \le n$.

We get the interesting formula [see (7.7)]

$$E[X_1 + \cdots + X_{N(n)+1}] = E[X_1] \times E[N(n) + 1]. \tag{7.35}$$

Note that $N(n)$ is not independent of $\{X_k\}$, and yet (7.35) still prevails.

7.6.1 The Discrete Renewal Theorem

The renewal theorem provides conditions under which the solution $\{v_n\}$ to a renewal equation will converge as n grows large. Certain periodic behavior, such as failures occurring only at even ages, must be precluded, and the simplest assumption assuring this preclusion is that $p_1 > 0$.

Theorem 7.1. *Suppose that* $0 < p_1 < 1$ *and that* $\{u_n\}$ *and* $\{v_n\}$ *are the solutions to the renewal equations (7.33) and (7.30), respectively. Then (a)* $\lim_{n\to\infty} u_n = 1/\Sigma_{k=0}^{\infty} kp_k$*; and (b) if* $\Sigma_{k=0}^{\infty}|b_k| < \infty$*, then* $\lim_{n\to\infty} v_n = \left\{\Sigma_{k=0}^{\infty} b_k\right\}/\left\{\Sigma_{k=0}^{\infty} kp_k\right\}$.

We recognize that $\Sigma_{k=0}^{\infty} kp_k = E[X_1]$ is the mean lifetime of a unit. Thus, (a) in Theorem 7.1 asserts that in the long run, the probability of a renewal occurring in a given interval is one divided by the mean life of a unit.

Remark Theorem 7.1 holds in certain circumstances when $p_1 = 0$. It suffices to assume that the greatest common divisor of the integers k for which $p_k > 0$ is one.

Example Let $\gamma_n = W_{N(n)+1} - n$ be the excess life at time n. For a fixed integer m, let $f_n(m) = \Pr\{\gamma_n = m\}$. We will establish a renewal equation for $f_n(m)$ by conditioning on the first life X_1. For $m \geq 1$,

$$\Pr\{\gamma_n = m | X_1 = k\} = \begin{cases} f_{n-k}(m) & \text{if } 0 \leq k \leq n, \\ 1 & \text{if } k = n + m, \\ 0 & \text{otherwise.} \end{cases}$$

(The student is urged to diagram the alternatives arising in $\Pr\{\gamma_n = m | X_1 = k\}$.) Then, by the law of total probability,

$$\begin{aligned} f_n(m) = \Pr\{\gamma_n = m\} &= \sum_{k=0}^{\infty} \Pr\{\gamma_n = m | X_1 = k\} p_k \\ &= p_{m+n} + \sum_{k=0}^{n} f_{n-k}(m) p_k. \end{aligned} \tag{7.36}$$

We apply Theorem 7.1 with $b_n = p_{m+n}$ to conclude that

$$\begin{aligned} \lim_{n\to\infty} \Pr\{\gamma_n = m\} &= \frac{\Sigma_{k=0}^{\infty} p_{m+k}}{\Sigma_{k=0}^{\infty} kp_k}. \\ &= \frac{\Pr\{X_1 \geq m\}}{E[X_1]}, \quad m = 1, 2, \ldots. \end{aligned}$$

The limit is a bona fide probability mass function, since its terms sum to one:

$$\frac{\Sigma_{m=1}^{\infty}\Pr\{X_1 \geq m\}}{E[X_1]} = \frac{E[X_1]}{E[X_1]} = 1.$$

7.6.2 Deterministic Population Growth with Age Distribution

In this section, we will discuss a simple deterministic model of population growth that takes into account the age structure of the population. Surprisingly, the discrete renewal theorem (Theorem 7.1) will play a role in the analysis. As the language will suggest, the deterministic model that we treat may be viewed as describing the mean population size in a more elaborate stochastic model that is beyond our scope to develop fully.

A Simple Growth Model

Let us set the stage by reviewing a simple model that has no age structure. We consider a single species evolving in discrete time $t = 0, 1, 2, \ldots,$ and we let N_t be the population size at time t. We assume that each individual present in the population at time t gives rise to a constant number λ of offspring that form the population at time $t+1$. (If death does not occur in the model, then we include the parent as one of the offspring, and then necessarily $\lambda \geq 1$.) If N_0 is the initial population size, and each individual gives rise to λ offspring, then

$$N_1 = \lambda N_0,$$
$$N_2 = \lambda N_1 = \lambda^2 N_0,$$

and in general,

$$N_t = \lambda^t N_0. \tag{7.37}$$

If $\lambda > 1$, then the population grows indefinitely in time; if $\lambda < 1$, then the population dies out; while if $\lambda = 1$, then the population size remains constant at $N_t = N_0$ for all $t = 0, 1, \ldots.$

The Model with Age Structure

We shall now introduce an age structure in the population. We need the following notation:

$n_{u,t}$ = the number of individuals of age u in the population at time t;

$N_t = \Sigma_{u=0}^{\infty} n_{u,t}$ = the total number of individuals in the population at time t;

b_t = the number of new individuals created in the population at time t, the number of births;

β_u = the expected number of progeny of a single individual of age u in one time period;

l_u = the probability that an individual will survive, from birth, at least to age u.

The conditional probability that an individual survives at least to age u, given that he has survived to age $u-1$, is simply the ration l_u/l_{u-1}. The *net maternity function* is the product

$$m_u = l_u \beta_u$$

and is the birth rate adjusted for the death of some fraction of the population. That is, m_u is the expected number of offspring at age u of an individual now of age 0.

Let us derive the total progeny of a single individual during its lifespan. An individual survives at least to age u with probability l_u, and then during the next unit of time gives rise to β_u offspring. Summing $l_u \beta_u = m_u$ over all ages u then gives the total progeny of a single individual:

$$M = \sum_{u=0}^{\infty} l_u \beta_u = \sum_{u=0}^{\infty} m_u. \tag{7.38}$$

If $M > 1$, then we would expect the population to increase over time; if $M < 1$, then we would expect the population to decrease; while if $M = 1$, then the population size should neither increase nor decrease in the long run. This is indeed the case, but the exact description of the population evolution is more complex, as we will now determine.

In considering the effect of age structure on a growing population, our interest will center on b_t, the number of new individuals created in the population at time t. We regard β_u, l_u, and $n_{u,0}$ as known, and the problem is to determine b_t for $t \geq 0$. Once b_t is known, then $n_{u,t}$ and N_t may be determined according to, e.g.,

$$n_{0,1} = b_1, \tag{7.39}$$

$$n_{u,1} = n_{u-1,0} \left[\frac{l_u}{l_{u-1}} \right] \quad \text{for } u \geq 1, \tag{7.40}$$

and

$$N_1 = \sum_{u=0}^{\infty} n_{u,1}. \tag{7.41}$$

In the first of these simple relations, $n_{0,1}$, is the number in the population at time 1 of age 0, which obviously is the same as b_1, those born in the population at time 1. For the second equation, $n_{u,1}$ is the number in the population at time 1 of age u. These individuals must have survived from the $n_{u-1,0}$ individuals in the population at time 0 of age $u-1$; the conditional probability of survivorship is $[l_u/l_{u-1}]$, which explains the second equation. The last relation simply asserts that the total population size results by summing the numbers of individuals of all ages. The generalizations of (7.39) through (7.41) are

$$n_{0,t} = b_t, \tag{7.42}$$

$$n_{u,t} = n_{u-1,t-1}\left[\frac{l_u}{l_{u-1}}\right] \quad \text{for } u \geq 1, \tag{7.43}$$

and

$$N_t = \sum_{u=0}^{\infty} n_{u,t} \quad \text{for } t \geq 1. \tag{7.44}$$

Having explained how $n_{u,t}$ and N_t are found once b_t is known, we turn to determining b_t. The number of individuals created at time t has two components. One component, a_t, say, counts the offspring of those individuals in the population at time t who already existed at time 0. In the simplest case, the population begins at time $t = 0$ with a single ancestor of age $u = 0$, and then the number of offspring of this individual at time t is $a_t = m_t$, the net maternity function. More generally, assume that there were $n_{u,0}$ individuals of age u at time 0. The probability that an individual of age u at time 0 will survive to time t (at which time it will be of age $t+u$) is l_{t+u}/l_u. Hence the number of individuals of age u at time 0 that survive to time t is $n_{u,0}(l_{t+u}/l_u)$, and each of these individuals, now of age $t+u$, will produce β_{t+u} new offspring. Adding over all ages we obtain

$$\begin{aligned} a_t &= \sum_{u=0}^{\infty} \beta_{t+u} n_{u,0} \frac{l_{t+u}}{l_u} \\ &= \sum_{u=0}^{\infty} \frac{m_{t+u} n_{u,0}}{l_u}. \end{aligned} \tag{7.45}$$

The second component of b_t counts those individuals created at time t whose parents were not initially in the population but were born after time 0. Now, the number of individuals created at time τ is b_τ. The probability that one of these individuals survives to time t, at which time he will be of age $t-\tau$, is $l_{t-\tau}$. The rate of births for individuals of age $t-\tau$ is $\beta_{t-\tau}$. The second component results from summing over τ and gives

$$\begin{aligned} b_t &= a_t + \sum_{\tau=0}^{t} \beta_{t-\tau} l_{t-\tau} b_\tau \\ &= a_t + \sum_{\tau=0}^{t} m_{t-\tau} b_\tau. \end{aligned} \tag{7.46}$$

Example Consider an organism that produces two offspring at age 1, and two more at age 2, and then dies. The population begins with a single organism of age 0 at time 0.

We have the data

$$
\begin{aligned}
&n_{0,0} = 1,\ n_{u,0} = 0 \quad \text{for } u \geq 1,\\
&b_1 = b_2 = 2,\\
&l_0 = l_1 = l_2 = 1 \quad \text{and} \quad l_u = 0 \quad \text{for} \quad u > 2.
\end{aligned}
$$

We calculate from (7.45) that

$$
a_0 = 0, \quad a_1 = 2, \quad a_2 = 2, \quad \text{and} \quad a_t = 0, \quad \text{for } t > 2.
$$

Finally, (7.46) is solved recursively as

$$
\begin{aligned}
b_0 &= 0,\\
b_1 &= a_1 + m_0 b_1 + m_1 b_0\\
&= 2 + 0 + 0 = 2,\\
b_2 &= a_2 + m_0 b_2 + m_1 b_1 + m_2 b_0\\
&= 2 + 0 + (2)(2) + 0 = 6,\\
b_3 &= a_3 + m_0 b_3 + m_1 b_2 + m_2 b_1 + m_3 b_0\\
&= 0 + 0 + (2)(6) + (2)(2) + 0 = 16.
\end{aligned}
$$

Thus, e.g., an individual of age 0 at time 0 gives rise to 16 new individuals entering the population at time 3.

The Long Run Behavior

Somewhat surprisingly, since no "renewals" are readily apparent, the discrete renewal theorem (Theorem 7.1) will be invoked to deduce the long run behavior of this age-structured population model. Observe that (7.46)

$$
\begin{aligned}
b_t &= a_t + \sum_{\tau=0}^{t} m_{t-\tau} b_\tau\\
&= a_t + \sum_{v=0}^{t} m_v b_{t-v}
\end{aligned}
\tag{7.47}
$$

has the form of a renewal equation except that $\{m_v\}$ is not necessarily a bona fide probability distribution in that, typically, $\{m_v\}$ will not sum to one. Fortunately, there is a trick that overcomes this difficulty. We introduce a variable s, whose value will be chosen later, and let

$$
m_v^{\#} = m_v s^v, \quad b_v^{\#} = b_v s^v, \quad \text{and} \quad a_v^{\#} = a_v s^v.
$$

Now multiply (7.47) by s^t and observe that $s^t m_v b_{t-v} = (m_v s^v)(b_{t-v} s^{t-v}) = m_v^\# b_{t-v}^\#$ to get

$$b_t^\# = a_t^\# + \sum_{v=0}^{t} m_v^\# b_{t-v}^\#. \tag{7.48}$$

This renewal equation holds no matter what value we choose for s. We, therefore, choose s such that $\{m_v^\#\}$ is a bona fide probability distribution. That is, we fix the value of s such that

$$\sum_{v=0}^{\infty} m_v^\# = \sum_{v=0}^{\infty} m_v s^v = 1.$$

There is always a unique such s whenever $1 < \Sigma_{v=0}^{\infty} m_v < \infty$. We may now apply the renewal theorem to (7.48), provided that its hypothesis concerning nonperiodic behavior is satisfied. For this it suffices, e.g., that $m_1 > 0$. Then, we conclude that

$$\lim_{t\to\infty} b_t^\# = \lim_{t\to\infty} b_t s^t = \frac{\Sigma_{v=0}^{\infty} a_v^\#}{\Sigma_{v=0}^{\infty} v m_v^\#}. \tag{7.49}$$

We set $\lambda = 1/s$ and $K = \Sigma_{v=0}^{\infty} a_v^\# / \Sigma_{v=0}^{\infty} v m_v^\#$ to write (7.49) in the form

$$b_t \sim K\lambda^t \quad \text{for } t \text{ large.}$$

In words, asymptotically, the population grows at rate λ where $\lambda = 1/s$ is the solution to

$$\sum_{v=0}^{\infty} m_v \lambda^{-v} = 1.$$

When t is large ($t > u$), then (7.43) may be iterated in the manner

$$\begin{aligned}
n_{u,t} &= n_{u-1,t-1}\left[\frac{l_u}{l_{u-1}}\right] \\
&= n_{u-2,t-2}\left[\frac{l_{u-1}}{l_{u-2}}\right]\left[\frac{l_u}{l_{u-1}}\right] \\
&= n_{u-2,t-2}\left[\frac{l_u}{l_{u-2}}\right] \\
&\ \vdots \\
&= n_{0,t-u}\left[\frac{l_u}{l_0}\right] = b_{t-u} l_u.
\end{aligned}$$

This simply expresses that those of age u at time t were born $t-u$ time units ago and survived. Since for large t we have $b_{t-u} \sim K\lambda^{t-u}$, then

$$n_{u,t} \sim Kl_u\lambda^{t-u} = K(l_u\lambda^{-u})\lambda^t,$$
$$N_t = \sum_{u=0}^{\infty} n_{u,t} \sim K\sum_{u=0}^{\infty}(l_u\lambda^{-u})\lambda^t,$$

and

$$\lim_{t\to\infty}\frac{n_{u,t}}{N_t} = \frac{l_u\lambda^{-u}}{\Sigma_{v=0}^{\infty} l_v\lambda^{-v}}.$$

This last expression furnishes the asymptotic, or *stable, age distribution* in the population.

Example Continuing the example in which $m_1 = m_2 = 2$ and $m_k = 0$ otherwise, then we have

$$\sum_{v=0}^{\infty} m_v s^v = 2s + 2s^2 = 1,$$

which we solve to obtain

$$s = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$
$$= (0.366, -1.366).$$

The relevant root is $s = 0.366$, whence $\lambda = 1/s = 2.732$. Thus asymptotically, the population grows geometrically at rate $\lambda = 2.732\cdots$, and the stable age distribution is as shown in the following table:

Age	**Fraction of Population**
0	$1/\left(1+s+s^2\right) = 0.6667$
1	$s/\left(1+s+s^2\right) = 0.2440$
2	$s^2/\left(1+s+s^2\right) = 0.0893$

Exercises

7.6.1 Solve for v_n for $n = 0, 1, \ldots, 10$ in the renewal equation

$$v_n = b_n + \sum_{k=0}^{n} p_k v_{n-k} \quad \text{for } n = 0, 1, \ldots,$$

where $b_0 = b_1 = \frac{1}{2}, b_2 = b_3 = \cdots = 0$, and $p_0 = \frac{1}{4}, p_1 = \frac{1}{2}$, and $p_2 = \frac{1}{4}$.

7.6.2 (Continuation of Exercise 7.6.1)

(a) Solve for u_n for $n = 0, 1, \ldots, 10$ in the renewal equation

$$u_n = \delta_n + \sum_{k=0}^{n} p_k u_{n-k} \quad \text{for } n = 0, 1, \ldots,$$

where $\delta_0 = 1, \delta_1 = \delta_2 = \cdots = 0$, and $\{p_k\}$ is as defined in Exercise 7.6.1.

(b) Verify that the solution v_n in Exercise 7.6.1 and u_n are related according to $v_n = \Sigma_{k=0}^{n} b_k u_{n-k}$.

7.6.3 Using the data of Exercises 7.6.1 and 7.6.2, determine

(a) $\lim_{n\to\infty} u_n$.

(b) $\lim_{n\to\infty} v_n$.

Problems

7.6.1 Suppose the lifetimes $X_1, X_2, \ldots$ have the geometric distribution

$$\Pr\{X_1 = k\} = \alpha(1-\alpha)^{k-1} \quad \text{for } k = 1, 2, \ldots,$$

where $0 < \alpha < 1$.

(a) Determine u_n for $n = 1, 2, \ldots$.

(b) Determine the distribution of excess life γ_n by using Lemma 7.1 and (7.36).

7.6.2 Marlene has a fair die with the usual six sides. She throws the die and records the number. She throws the die again and adds the second number to the first. She repeats this until the cumulative sum of all the tosses first exceeds a prescribed number n. (a) When $n = 10$, what is the probability that she stops at a cumulative sum of 13? (b) When n is large, what is the approximate probability that she stops at a sum of $n + 3$?

7.6.3 Determine the long run population growth rate for a population whose individual net maternity function is $m_2 = m_3 = 2$, and $m_k = 0$ otherwise. Why does delaying the age at which offspring are first produced cause a reduction in the population growth rate? (The population growth rate when $m_1 = m_2 = 2$, and $m_k = 0$ otherwise was determined in the last example of this section.)

7.6.4 Determine the long run population growth rate for a population whose individual net maternity function is $m_0 = m_1 = 0$ and $m_2 = m_3 = \cdots = a > 0$. Compare this with the population growth rate when $m_2 = a$, and $m_k = 0$ for $k \neq 2$.