Stat 150, Fall 2018, HW #2

Due Thurs Sept 13 at the start of class 9:30 AM in Evans 10.
Late assignments will not be accepted.

1. Pinsky and Karlin [PK], Problems (*not* exercises):

   3.5.1 (read Section 3.5.2 for background), 3.6.7 (read Section 3.6.1 for background), 3.9.5, 3.9.6

2. Before doing this question, read Section 3.6.1 in [PK]. This section studies a *Birth and Death Markov chain* (a Markov chain where for all n, X_{n+1} can only take one of three values: X_n, $X_n - 1$ or $X_n + 1$) on $S = \{0, 1, \ldots, N\}$ with absorbing boundaries. This generalizes the gambler's ruin problem from class.

   Random walk around the clock: Consider a Markov chain on $S = \{1, 2, \ldots, 12\}$ such that $p_{ij} = 1/2$ if i, j are adjacent hours on a clock, and $p_{ij} = 0$ otherwise.

   (a) Find the expected number of steps until the Markov chain returns to its starting position.

   (b) Calculate the probability that all other hours are visited before returning to the starting position.

   (c) Suppose that instead, for some $p \neq 1/2$, we have that $p_{ij} = p$ (resp. $p_{ij} = 1 - p$) if j is clockwise (resp. counter-clockwise) adjacent to i, and $p_{ij} = 0$ otherwise. Find the expected number of visits to six o'clock, starting from six o'clock, before visiting twelve o'clock.

3. Using first step analysis, show that the expected number of steps until symmetric Simple Random Walk on the integer line $\mathbb{Z}$ revisits its starting position is infinite. *Hint:* For $x \in \mathbb{Z}$, let $f(x) = \mathbb{E}(T_0 | X_0 = x)$, where $T_0 = \min\{n \geq 1 : X_n = 0\}$. Using a proof by contradiction, show that $f(0) = \infty$.

4. A jar contains infinitely many coins which come up Tails with probability p. A person selects a coin from the jar and flips it on a table. In the following way, they continue flipping coins until all coins on the table show Tails: If the last flip came up Heads, they select a coin from the jar and flip it on the table. On the other hand, if the last flip came up Tails, they select a coin from the table which shows Heads and flip it on the table. Find the probability that this process eventually ends.

5. Let $\{X_n : n \geq 1\}$ be a branching process with $X_0 = 1$ whose offspring ξ distribution has mean $\mu = \mathbb{E}(\xi) \in (0, \infty)$. Recall that X_n denotes the number of particles in the nth generation of the process. Let $N_n = \sum_{i=0}^{n} X_i$ denote the number of particles born by time n, and $N = \sum_{i=0}^{\infty} X_i$ the number of particles ever born. Let ϕ, ψ_n and ψ be the probability generating functions for the random variables ξ, N_n and N, respectively.

   (a) Show that $\psi_n(s) = s\phi(\psi_{n-1}(s))$, and then take $n \to \infty$ to conclude that $\psi(s) = s\phi(\psi(s))$.

   (b) Use this formula to show that if $\mu < 1$ then $\mathbb{E}(N) = 1/(1 - \mu)$.