Homework 6 : Solutions
STAT 150

Problem 4.26 :

(a) Let T be the first return time to 0, the busy period is the time we spent before reaching 0 starting from 1, hence we are looking for the quantity $g(1) = \mathbb{E}_1[T]$, also we define $g(i) = \mathbb{E}_i[T]$ for every state i, by the first step analysis we get the following equalities :

$$g(1) = \frac{1}{4} + g(2)$$
$$g(2) = \frac{1}{4} + \frac{1}{2}g(3)$$
$$g(3) = \frac{1}{6} + \frac{1}{3}g(1) + \frac{2}{3}g(2)$$

Hence we get that $g(1) = 1$

(b) We know by first step analysis that $g(0) = \frac{1}{2} + g(1)$, on the other hand we have $g(0) = \frac{1}{\lambda_0 \pi_0} = \frac{3}{2}$, hence $g(1) = 1$.

Problem 4.33 :

The transition rates are :

$$q(i, i+1) = \lambda$$
$$q(i, i-1) = \mu + \delta(i-1)$$

and q zero otherwise. This is a birth-death chain so we look for a distribution that verifies the detailed balance, hence the equation $\pi_i q(i, i+1) = \pi_{i+1} q(i+1, i)$ gives us :

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \frac{\lambda}{\mu + \delta j} = \pi_0 \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + \delta j)} \quad (1)$$

Hence a stationary distribution exists if $\displaystyle\sum_{i=0}^{\infty} \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + \delta j)} < \infty$, however as $\delta > 0$ then there exists a $K > 0$ such that for $j > K$ we have $\mu + \delta j \geq 2\lambda$, hence :

$$\frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + \delta j)} \leq C(\frac{1}{2})^{i-K} \text{ for } i \geq K$$

where C is a constant, which proves our claim. The stationary distribution is given by (1) for :

$$\pi_0 = \frac{1}{\sum_{i=0}^{\infty} \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu+\delta j)}}$$

Problem 4.36 :

The transition probabilities are :

$$q(0,a) = q(0,b) = \tfrac{\lambda}{2} \text{ and } q(a,2) = q(b,2) = \lambda$$
$$q(a,0) = \mu_a,\ q(b,0) = \mu_b,\ q(2,a) = \mu_b \text{ and } q(2,b) = \mu_a$$
$$q(i,i+1) = \lambda \text{ for } i \geq 2 \text{ and } q(i,i-1) = \mu_a + \mu_b \text{ for } i \geq 3$$

By writing the system of equations of time reversibility we get that : $\pi_0 = \dfrac{2c\mu_a\mu_b}{\lambda^2}$ $\quad \pi_a = \dfrac{c\mu_b}{\lambda}$ $\quad \pi_b = \dfrac{c\mu_a}{\lambda}$

and for the remaining of the Markov chain it behaves like a death-birth chain starting from 2 so we get that :

$$\pi_n = c(\frac{\lambda}{\mu_a + \mu_b})^{n-2}$$

Now using the fact that $\displaystyle\sum_i \pi_i = 1$ we get that :

$$c = \frac{1}{\frac{2\mu_a\mu_b}{\lambda^2} + \frac{\mu_a+\mu_b}{\lambda} + \frac{\mu_a+\mu_b}{\mu_a+\mu_b-\lambda}}$$

Now, for all the martingales that are used in the problems, we denote $\mathscr{F}_n := \{X_0, .., X_n\}$ all the information up to time n.

Problem 5.2 :

(a) From the Example 1.9, we know that the distribution of X_{n+1} given X_n is binomial with parameters $(N, \frac{X_n}{N})$, hence using the fact that X_n is a Markov Chain we have :

$$\mathbb{E}[X_{n+1}|\mathscr{F}_n] = \mathbb{E}[X_{n+1}|X_n] = \mathbb{E}[Bin(N, \tfrac{X_n}{N})|X_n] = N.\tfrac{X_n}{N} = X_n$$

(b)

$$
\begin{aligned}
\mathbb{E}[Y_{n+1}|\mathscr{F}_n] &= \mathbb{E}[\frac{X_{n+1}(N-X_{n+1})}{(1-\frac{1}{N})^{n+1}}|X_n] \\
&= \frac{1}{(1-\frac{1}{N})^{n+1}}(N\mathbb{E}[Bin(N,\frac{X_n}{N})] - \mathbb{E}[Bin(N,\frac{X_n}{N})^2]) \\
&= \frac{1}{(1-\frac{1}{N})^{n+1}}(NX_n - (\mathrm{Var}(Bin(N,\frac{X_n}{N})) + X_n^2)) \\
&= \frac{1}{(1-\frac{1}{N})^{n+1}}(NX_n - (X_n(1-\frac{X_n}{N}) + X_n^2)) \\
&= \frac{X_n(N-X_n)}{(1-\frac{1}{N})^{n}} = Y_n
\end{aligned}
$$

(c) Using the fact that for $x \in \{1,..,N-1\}$ we have :

$$N-1 \leq x(N-x) \leq \tfrac{N^2}{4} \ (1)$$

Also, Y_n has a constant expectation as it is martingale, thus we have $\mathbb{E}_x[X_n(N-X_n)] = \mathbb{E}_x[X_n(N-X_n)1_{0<X_n<N}] = x(N-x)(1-\frac{1}{N})^n$, thus by using the inequality (1) on X_n we get :

$$(N-1) \leq \frac{x(N-x)(1-\frac{1}{N})^n}{\mathbb{P}_x(0<X_n<N)} \leq \frac{N^2}{4}$$

Problem 5.4 :

Let $T = \inf\{n : X_n \geq 0.9\}$, then as $X_{n\wedge T}$ is a martingale then :

$$\tfrac{1}{2} = \mathbb{E}[X_0] = \mathbb{E}[X_{n\wedge T}] \geq \mathbb{E}[X_{n\wedge T}1_{T<\infty}] \rightarrow_{n\infty} 0.9\mathbb{P}(T<\infty)$$

Hence : $\mathbb{P}(X_n \geq 0.9 \text{ for some } n) = \mathbb{P}(T<\infty) \leq \frac{5}{9}$

Problem 5.5 :

We start with g green balls and r red balls at time 0, thus at time n, X_n take values in the set $\{\frac{r}{g+r+n},..,\frac{r+n}{g+r+n}\}$. Now for $0 \leq j \leq n$ let's consider the probability to pick the j red balls in the first j draws and then only pick green balls in the remaining $n-j$ draws.

We have then :

$$\mathbb{P}(\text{first j draws are red balls}) = \frac{r}{g+r}\frac{r+1}{g+r+1}\cdots\frac{r+j-1}{g+r+j-1}\frac{g}{g+r+j}\cdots\frac{g+n-j-1}{g+r+n-1} = \frac{\Gamma(g+r)\Gamma(r+j)\Gamma(g+n-j)}{\Gamma(r)\Gamma(g)\Gamma(g+r+n)}$$

We see that the probability remains the same no matter how we choose the order of the draws of the red balls, thus :

$$\begin{aligned}
\mathbb{P}(X_n = \frac{r+j}{g+r+n}) &= \binom{n}{j}\frac{\Gamma(g+r)\Gamma(r+j)\Gamma(g+n-j)}{\Gamma(r)\Gamma(g)\Gamma(g+r+n)} \\
&= \frac{\Gamma(n+1)\Gamma(g+r)\Gamma(r+j)\Gamma(g+n-j)}{\Gamma(j+1)\Gamma(n-j+1)\Gamma(r)\Gamma(g)\Gamma(g+r+n)} \\
&= [\frac{\Gamma(g+r)}{\Gamma(r)\Gamma(g)}][\frac{\Gamma(r+j)}{\Gamma(j+1)}][\frac{\Gamma(n-j+g)}{\Gamma(n-j+1)}][\frac{\Gamma(n+1)}{\Gamma(g+r+n)}] \qquad (1)
\end{aligned}$$

Using the fact that for every a and b we have $\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim_{x\to\infty} x^{a-b}$, then we get from (1) by setting $j = \lfloor nx \rfloor$ for $x \in (0,1)$:

$$\mathbb{P}(X_n = \tfrac{r+\lfloor nx \rfloor}{g+r+n}) \sim_{n\to\infty} \tfrac{\Gamma(g+r)}{\Gamma(r)\Gamma(g)}(nx)^{r-1}(n(1-x))^{g-1}n^{1-g-r}$$

Hence:

$$f_n(x) = n\mathbb{P}(X_n = \tfrac{r+\lfloor nx \rfloor}{g+r+n}) \sim_{n\to\infty} \tfrac{\Gamma(g+r)}{\Gamma(r)\Gamma(g)}x^{r-1}(1-x)^{g-1} = f(x)$$

It is also easy to see that we have the uniform convergence of the sequence of functions f_n to f (i.e $\sup_{x\in(0,1)} |f_n(x) - f(x)| \underset{n\to\infty}{\to} 0$)

Thus :

$$\mathbb{P}(X_n \le t) \sim_{n\to\infty} \int_0^t f_n(x)dx \underset{n\to\infty}{\to} \int_0^t f(x)dx$$

So, X_n converges in distribution to the random variable with density f which is Beta(r, g).

Problem 5.6 :

(a) $\mathbb{E}[Y_{n+1}|\mathscr{F}_n] = \mathbb{E}[U_n]Y_n = \frac{Y_n}{2}$, hence $M_n = 2^n Y_n$ is a martingale.

(b) The random variables U_i are independent, then by the law of the large numbers :

$$\tfrac{1}{n}\log(Y_n) = \tfrac{1}{n}\sum_{k=1}^{n}\log(U_k) \underset{n\to\infty}{\to} \mathbb{E}[\log(U)] = -1$$

(c) M_n is a nonnegative martingale so it converges almost surely, now we have $\log(M_n) = n\log(2) + \log(Y_n) \to -\infty$, because $\log(2) < 1$, hence $M_n \to 0$.

Problem 5.10 :

It is easy to see that $S_n - n(p-q)$ is a martingale and thus so is $S_{n\wedge V_0} - (n\wedge V_0)(p-q)$, hence : $x = \mathbb{E}_x[S_{n\wedge V_0} - (n\wedge V_0)(p-q)]$.

Now from Example 5.10 we can see that max S_m is an integrable random variable, and so because $0 \le S_{n\wedge V_0} \le \max S_m$, hence we get that : $|p-q|\mathbb{E}(n\wedge V_0) \le |x| + \mathbb{E}_x(\max S_m) < \infty$, thus $\mathbb{E}_x(V_0) < \infty$. We can use the Wald's equality and we get that :

$$\mathbb{E}_x(V_0) = \frac{x}{q-p} = \frac{x}{1-2p}$$

Problem 5.15 :

(a) $\mathbb{E}[\theta^{Z_{n+1}}|Z_n] = \mathbb{E}[\theta^{\sum_{k=1}^{Z_n}\xi_i}|Z_n] = \prod_{i=1}^{Z_n}\mathbb{E}[\theta^{\xi_i}] = (\phi(\theta))^{Z_n}$.

(b) Let ρ the unique solution < 1 of the equation $\phi(\rho) = \rho$, hence by (a) ρ^{Z_n} is clearly a martingale. Now let T_0 be the time of extinction, T_0 is a stopping time and thus $\rho^{Z_{n\wedge T_0}}$ is a martingale, hence $\rho^k = \mathbb{E}_k(\rho^{Z_{n\wedge T_0}}) = \mathbb{E}(\rho^{Z_{n\wedge T_0}}1_{T_0<\infty} + \rho^{Z_n}1_{T_0=\infty}]$.

However, since $p_0 > 0$, we can prove that given $\{T_0 = \infty\}$ we have $Z_n \to \infty$. Indeed, as $p_0 > 0$ then if Z_n doesn't diverge, then there is a $k \in \mathbb{N}$ such that $Z_n = k$ infinitely often, however everytime we hit k we can go back to 0 with probability p_0^k and thus never come back again to k. Rigorously speaking, if we put N as the number of passages to the state k, then by the Markov property we get $\mathbb{P}(N \ge m) \le \mathbb{P}(N \ge m-1)(1-p_0^k)$, thus N is finite almost surely.

That gives us that $\mathbb{E}_k[\rho^{Z_n} 1_{T_0=\infty}] \to 0$ and $\mathbb{E}(\rho^{Z_{n\wedge T_0}} 1_{T_0<\infty}] \to \mathbb{P}_k(T_0 < \infty)$, hence $\mathbb{P}_k(T_0 < \infty) = \rho^k$

Problem 5.19 :

(a) We have :

$$\begin{aligned}\mathbb{E}_x[X_1 - x] &= \mathbb{E}_x[(x-\eta_1)_+ - x] \\ &= -\mathbb{E}[\eta_1 1_{\{\eta_1 \le x\}}] - x\mathbb{P}\{\eta_1 > x\} \\ &\le -\mathbb{E}[\eta_1 1_{\{\eta_1 \le x\}}]\end{aligned}$$

and we conclude using the fact that $\mathbb{E}[\eta_1 1_{\{\eta_1 \le x\}}] \underset{x\to\infty}{\to} \mathbb{E}[\eta_1] \ge 2\epsilon$

(b) We know that :

$\mathbb{E}[X_{U_k\wedge n+1} + \epsilon(U_k \wedge n+1) - X_{U_k\wedge n} - \epsilon(U_k \wedge n)|\mathscr{F}_n] = 1_{\{U_k \ge n+1\}}(\mathbb{E}[X_{n+1} - X_n|\mathscr{F}_n] + \epsilon) \le 1_{\{X_n > K\}}(-\epsilon + \epsilon) = 0$ so $(X_{U_k\wedge n} + \epsilon(U_k \wedge n))$ is a supermartingale, and hence :

$$\epsilon\mathbb{E}_x[U_k] \le \mathbb{E}_x[X_{U_k} + \epsilon U_k] \le x$$