Tutorial 9
STAT 150

Exercise 1:

A continuous time Markov chain $X(t)$ with states $\{1, 2, 3\}$ starts at $X(0) = 1$ and has generator :

$$G = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

(a) Find the stationary distribution of $X(t)$.

(b) Let T be the first hitting time of state 3. Find $\mathbb{E}[T]$.

(c) Let S be the total amount of time spent in state 1 before the first visit to state 3. Find the distribution of S.

Exercise 2: *Left continuous random walk*

Suppose that $X_1, X_2, ..$ are independent integer-valued random variables with $\mathbb{E}[X_i] > 0$, $\mathbb{P}[X_i \geq -1] = 1$, and $\mathbb{P}[X_i = -1] > 0$.Let $\phi(\theta) = \mathbb{E}[e^{\theta X_i}]$ and define $\alpha < 0$ by the requirement that $\phi(\alpha) = 1$. Let $a < x$ and set $V_a = \min\{n : S_n = a\}$

(a) Justify the existence of such an α

(b) Prove that $\mathbb{P}_x(V_a < \infty) = e^{\alpha(x-a)}$

Exercise 3: *Polya's urn*

Consider an urn that contains initially one red ball and one green ball. At each time n we draw out a ball chosen at random. We return it to the urn and add one more of the same color. Let X_n be the fraction of red balls at time n.

(a) Prove that $(X_n)_{n\geq 0}$ is a martingale.

(b) Find the distribution of X_∞ the limit of (X_n).

Exercise 4: *Cramer's estimate of ruin*

Let S_n be the total assets of an insurance company at the end of year n. Consider $X_1, .., X_n..$ a sequence of i.i.d random variables that are normally distributed with mean $\mu > 0$ and variance σ^2. S_n has the form :

$$S_n = S_0 + \sum_{k=1}^{n} X_k$$

Let B be the event of an eventual bankruptcy, i.e that the wealth of the company will be negative at some time n. Show that :

$$\mathbb{P}\{B\} \leq \exp(-\frac{2\mu S_0}{\sigma^2})$$