8 Brownian Motion and Related Processes

8.1 Brownian Motion and Gaussian Processes

The Brownian motion stochastic process arose early in this century as an attempt to explain the ceaseless irregular motions of tiny particles suspended in a fluid, such as dust motes floating in air. Today, the Brownian motion process and its many generalizations and extensions occur in numerous and diverse areas of pure and applied science such as economics, communication theory, biology, management science, and mathematical statistics.

8.1.1 A Little History

The story begins in the summer of 1827, when the English botanist Robert Brown observed that microscopic pollen grains suspended in a drop of water moved constantly in haphazard zigzag trajectories. Following the reporting of his findings, other scientists verified the strange phenomenon. Similar *Brownian motion* was apparent whenever very small particles were suspended in a fluid medium, e.g., smoke particles in air. Over time, it was established that finer particles move more rapidly, that the motion is stimulated by heat, and that the movement becomes more active with a decrease in fluid viscosity.

A satisfactory explanation had to wait until the next century, when in 1905, Einstein would assert that the Brownian motion originates in the continual bombardment of the pollen grains by the molecules of the surrounding water, with successive molecular impacts coming from different directions and contributing different impulses to the particles. Einstein argued that as a result of the continual collisions, the particles themselves had the same average kinetic energy as the molecules. Belief in molecules and atoms was far from universal in 1905, and the success of Einstein's explanation of the well-documented existence of Brownian motion did much to convince a number of distinguished scientists that such things as atoms actually exist. Incidentally, 1905 is the same year in which Einstein set forth his theory of relativity and his quantum explanation for the photoelectric effect. Any single one of his 1905 contributions would have brought him recognition by his fellow physicists. Today, a search in a university library under the subject heading "Brownian motion" is likely to turn up dozens of books on the stochastic process called Brownian motion and few, if any, on the irregular movements observed by Robert Brown. The literature on the model has far surpassed and overwhelmed the literature on the phenomenon itself!

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Brownian motion is complicated because the molecular bombardment of the pollen grain is itself a complicated process, so it is not surprising that it took more than another decade to get a clear picture of the Brownian motion stochastic process. It was not until 1923 that Norbert Wiener set forth the modern mathematical foundation. The reader may also encounter "Wiener process" or "Wiener–Einstein process" as names for the stochastic process that we will henceforth simply call "Brownian motion."

Predating Einstein by several years, in 1900 in Paris, Louis Bachelier proposed what we would now call a "Brownian motion model" for the movement of prices in the French bond market. While Bachelier's paper was largely ignored by academics for many decades, his work now stands as the innovative first step in a mathematical theory of stock markets that has greatly altered the financial world of today. Later in this chapter, we will have much to say about Brownian motion and related models in finance.

8.1.2 The Brownian Motion Stochastic Process

In terms of our general framework of stochastic processes (cf. Chapter 1, Section 1.1.1), the Brownian motion process is an example of a continuous-time, continuous-state-space Markov process. Let $B(t)$ be the y component (as a function of time) of a particle in Brownian motion. Let x be the position of the particle at time t_0; i.e., $B(t_0) = x$. Let $p(y,t|x)$ be the probability density function, in y, of $B(t_0+t)$, given that $B(t_0) = x$. We postulate that the probability law governing the transitions is stationary in time, and therefore $p(y,t|x)$ does not depend on the initial time t_0.

Since $p(y,t|x)$ is a probability density function in y, we have the properties

$$p(y,t|x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} p(y,t|x)\mathrm{d}y = 1. \tag{8.1}$$

Further, we stipulate that $B(t_0+t)$ is likely to be near $B(t_0) = x$ for small values of t. This is done formally by requiring that

$$\lim_{t\to 0} p(y,t|x) = 0 \quad \text{for} \quad y \neq x. \tag{8.2}$$

From physical principles, Einstein showed that $p(y,t|x)$ must satisfy the partial differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2}. \tag{8.3}$$

This is called the *diffusion equation*, and σ^2 is the *diffusion coefficient*, which Einstein showed to be given by $\sigma^2 = RT/Nf$, where R is the gas constant, T is the temperature, N is Avogadro's number, and f is a coefficient of friction. By choosing the proper scale, we may take $\sigma^2 = 1$. With this choice, we can verify directly (see Exercise 8.1.3) that

$$p(y,t|x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(y-x)^2\right) \tag{8.4}$$

is a solution of (8.3). In fact, it is the only solution under the conditions (8.1) and (8.2). We recognize (8.4) as a normal probability density function whose mean is x and whose variance is t. That is, the position of the particle t time units after observations begin is normally distributed. The mean position is the initial location x, and the variance is the time of observation t.

Because the normal distribution will appear over and over in this chapter, we are amply justified in standardizing some notation to deal with it.
Let

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z2}, \quad -\infty < z < \infty, \tag{8.5}$$

be the standard normal probability density function, and let

$$\Phi(z) = \int_{-\infty}^{z} \phi(x) \mathrm{d}x \tag{8.6}$$

be the corresponding cumulative distribution function. A small table (Table 8.1) of the cumulative normal distribution appears at the end of this section. Let

$$\phi_t(z) = \frac{1}{\sqrt{t}} \phi(z/\sqrt{t}), \tag{8.7}$$

Table 8.1 The Cumulative Normal Distribution

$$\boldsymbol{\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du}$$

x	$\Phi(x)$
−3	0.00135
−2	0.02275
−1	0.1587
0	0.5000
1	0.8413
2	0.97725
3	0.99865
−2.326	0.01
−1.96	0.025
−1.645	0.05
−1.282	0.10
1.282	0.90
1.645	0.95
1.96	0.975
2.236	0.99

and

$$\Phi_t(z) = \int_{-\infty}^{z} \phi_t(x)\mathrm{d}x = \Phi(z/\sqrt{t}) \tag{8.8}$$

be the probability density function and cumulative distribution function, respectively, for the normal distribution with mean zero and variance t. In this notation, the transition density in (8.4) is given by

$$p(y,t|x) = \phi_t(y-x), \tag{8.9}$$

and

$$\Pr\{B(t) \le y | B(0) = x\} = \Phi\left(\frac{y-x}{\sqrt{t}}\right).$$

The transition probability density function in (8.4) or (8.9) gives only the probability distribution of $B(t) - B(0)$. The complete description of the Brownian motion process with diffusion coefficient σ^2 is given by the following definition.

Definition Brownian motion with diffusion coefficient σ^2 is a stochastic process $\{B(t); t \ge 0\}$ with the properties:

(a) Every increment $B(s+t) - B(s)$ is normally distributed with mean zero and variance $\sigma^2 t$; $\sigma^2 > 0$ is a fixed parameter.
(b) For every pair of disjoint time intervals $(t_1, t_2]$, $(t_3, t_4]$, with $0 \le t_1 < t_2 \le t_3 < t_4$, the increments $B(t_4) - B(t_3)$ and $B(t_2) - B(t_1)$ are independent random variables, and similarly for n disjoint time intervals, where n is an arbitrary positive integer.
(c) $B(0) = 0$, and $B(t)$ is continuous as a function of t.

The definition says that a displacement $B(s+t) - B(s)$ is independent of the past, or alternatively, if we know $B(s) = x$, then no further knowledge of the values of $B(\tau)$ for past times $\tau < s$ has any effect on our knowledge of the probability law governing the future movement $B(s+t) - B(s)$. This is a statement of the Markov character of the process. We emphasize, however, that the independent increment assumption (b) is actually more restrictive than the Markov property. A typical Brownian motion path is illustrated in Figure 8.1.

The choice $B(0) = 0$ is arbitrary, and we often consider *Brownian motion starting at* x, for which $B(0) = x$ for some fixed point x. For Brownian motion starting at x, the variance of $B(t)$ is $\sigma^2 t$, and σ^2 is termed the *variance parameter* in the stochastic process literature. The process $\tilde{B}(t) = B(t)/\sigma$ is a Brownian motion process whose variance parameter is one, the so-called *standard Brownian motion*. By this device, we may always reduce an arbitrary Brownian motion to a standard Brownian motion; for the most part, we derive results only for the latter. By part (a) of the definition, for

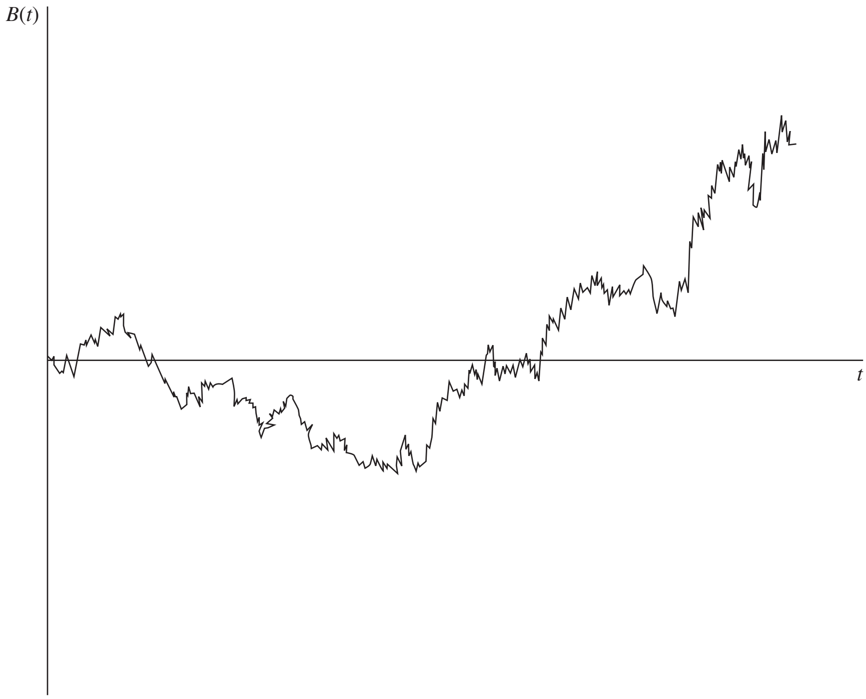

Figure 8.1 A typical Brownian motion path.

a standard Brownian motion ($\sigma^2 = 1$), we have

$$\begin{aligned}\Pr\{B(s+t) \le y | B(s) = x\} &= \Pr\{B(s+t) - B(s) \le y - x\} \\ &= \Phi\left(\frac{y-x}{\sqrt{t}}\right).\end{aligned}$$

Remark Let us look, for the moment, at a Brownian displacement $B(\Delta t)$ after a small elapsed time Δt. The mean displacement is zero, and the variance of the displacement is Δt itself. It is much more common in practical work to use a standard deviation, the square root of the variance, to measure variability. For the normal distribution, for instance, the probability of an observation more than 2 standard deviations away from the mean is about 5%, and the standard deviation is in the same units as the original measurement, and not (units)^2. The standard deviation of the Brownian displacement is $\sqrt{\Delta t}$, which is much larger than Δt itself when Δt is small. Indeed, $\mathrm{StdDev}[B(\Delta t)]/\Delta t = \sqrt{\Delta t}/\Delta t = 1/\sqrt{\Delta t} \to \infty$ as $\Delta t \to 0$. This is simply another manifestation of the erratic movements of the Brownian particle, yet it is a point that Bachelier and others had difficulty in handling. *But the variance, being linear in time, and not the standard deviation, is the only possibility if displacements over disjoint time intervals are to be stationary and independent.* Write a total displacement $B(s+t) - B(0)$ as the sum of two incremental steps in the form

$B(s+t)-B(0)=\{B(s)-B(0)\}+\{B(s+t)-B(s)\}$. The incremental steps being statistically independent, their variances must add. The stationary assumption is that the statistics of the second step $B(t+s)-B(t)$ do not depend on the time t when the step began, but only on the duration s of the movement. We must have, then, $\operatorname{Var}[B(t+s)]=\operatorname{Var}[B(t)]+\operatorname{Var}[B(s)]$, and the only nonnegative solution to such an equation is to have the variance of the displacement a linear function of time.

The Covariance Function

Using the independent increments assumption (b), we will determine the covariance of the Brownian motion. Recall that $E[B(t)]=0$ and that $E\left[B(t)^2\right]=\sigma^2 t$. Then, for $0\le s<t$,

$$\begin{aligned}
\operatorname{Cov}[B(s),B(t)] &= E[B(s)B(t)]\\
&= E[B(s)\{B(t)-B(s)+B(s)\}]\\
&= E\left[B(s)^2\right]+E[B(s)\{B(t)-B(s)\}]\\
&= \sigma^2 s+E[B(s)]E[B(t)-B(s)] \quad \text{(by (b))}\\
&= \sigma^2 s \quad (\text{since } E[B(s)]=0).
\end{aligned}$$

Similarly, if $0\le t<s$, we obtain $\operatorname{Cov}[B(s),B(t)]=\sigma^2 t$. Both cases may be treated in a single expression via

$$\operatorname{Cov}[B(s),B(t)]=\sigma^2\min\{s,t\}, \quad \text{for } s,t\ge 0. \tag{8.10}$$

8.1.3 The Central Limit Theorem and the Invariance Principle

Let $S_n=\xi_1+\cdots+\xi_n$ be the sum of n independent and identically distributed random variables $\xi_1,\ldots,\xi_n$ having zero means and unit variances. In this case, the central limit theorem asserts that

$$\lim_{n\to\infty}\Pr\left\{\frac{S_n}{\sqrt{n}}\le x\right\}=\Phi(x) \quad \text{for all } x.$$

The central limit theorem is stated as a limit. In stochastic modeling, it is used to justify the normal distribution as appropriate for a random quantity whose value results from numerous small random effects, all acting independently and additively. It also justifies the approximate calculation of probabilities for the sum of independent and identically distributed summands in the form $\Pr\{S_n\le x\}\approx\Phi(x/\sqrt{n})$, the approximation known to be excellent even for moderate values of n in most cases in which the distribution of the summands is not too skewed.

In a similar manner, functionals computed for a Brownian motion can often serve as excellent approximations for analogous functionals of a partial sum *process*, as we now explain. As a function of the continuous variable t, define

$$B_n(t)=\frac{S_{[nt]}}{\sqrt{n}}, \quad t\ge 0, \tag{8.11}$$

where $[x]$ is the greatest integer less than or equal to x. Observe that

$$B_n(t) = \frac{S_k}{\sqrt{n}} = \frac{\sqrt{[nt]}}{\sqrt{n}} \frac{S_k}{\sqrt{k}}, \qquad \text{for } \frac{k}{n} \leq t < \frac{k}{n} + \frac{1}{n}.$$

Because $S_k/\sqrt{k}$ has unit variance, the variance of $B_n(t)$ is $[nt]/n$, which converges to t as $n \to \infty$. When n is large, then $k = [nt]$ is large, and $S_k/\sqrt{k}$ is approximately normally distributed by the central limit theorem, and, finally, $B_n(t)$ inherits the independent increments property (b) from the postulated independence of the summands. It is reasonable, then, to believe that $B_n(t)$ should behave much like a standard Brownian motion process, at least when n is large. This is indeed true, and while we cannot explain the precise way in which it holds in an introductory text such as this, the reader should leave with some intuitive feeling for the usefulness of the result and, we hope, a motivation to learn more about stochastic processes. The convergence of the *sequence of stochastic processes* defined in (8.11) to a standard Brownian motion is termed the *invariance principle*. It asserts that some functionals of a partial sum process of independent and identically distributed zero mean and unit variance random variables should not depend too heavily on (should be invariant of) the actual distribution of the summands, but be approximately given by the analogous functional of a standard Brownian motion, provided only that the summands are not too badly behaved.

Example Suppose that the summands have the distribution in which $\xi = \pm 1$, each with probability $\frac{1}{2}$. Then, the partial sum process S_n is a simple random walk for which we calculated in Chapter 3, Section 3.5.3 (using a different notation)

$$\begin{aligned} &\Pr\{S_n \text{ reaches } -a < 0 \text{ before } b > 0 | S_0 = 0\} \\ &\qquad = \frac{b}{a+b}. \end{aligned} \tag{8.12}$$

Upon changing the scale in accordance with (8.11), we have

$$\begin{aligned} &\Pr\{B_n(t) \text{ reaches } -a < 0 \text{ before } b > 0 | S_0 = 0\} \\ &\qquad = \frac{b\sqrt{n}}{a\sqrt{n} + b\sqrt{n}} = \frac{b}{a+b}, \end{aligned}$$

and invoking the invariance principle, it should be, and is, the case that for a standard Brownian motion we have

$$\begin{aligned} &\Pr\{B(t) \text{ reaches } -a < 0 \text{ before } b > 0 | B(0) = 0\} \\ &\qquad = \frac{b}{a+b}. \end{aligned} \tag{8.13}$$

Finally, invoking the invariance principle for a second time, the evaluation in (8.12) should hold approximately for an arbitrary partial sum process, provided only that the independent and identically distributed summands have zero means and unit variances.

8.1.4 Gaussian Processes

A random vector $X_1, \ldots, X_n$ is said to have a *multivariate normal distribution*, or a *joint normal distribution*, if every linear combination $\alpha_1 X_1 + \cdots + \alpha_n X_n, \alpha_i$ real, has a univariate normal distribution. Obviously, if $X_1, \ldots, X_n$ has a joint normal distribution, then so does the random vector $Y_1, \ldots, Y_m$, defined by the linear transformation in which

$$Y_j = \alpha_{j1} X_1 + \cdots + \alpha_{jn} X_n, \quad \text{for } j = 1, \ldots, m,$$

for arbitrary constants α_{ji}.

The multivariate normal distribution is specified by two parameters, the mean values $\mu_i = E[X_i]$ and the covariance matrix whose entries are $\Gamma_{ij} = \operatorname{Cov}\left[X_i, X_j\right]$. In the joint normal distribution, $\Gamma_{ij} = 0$ is sufficient to imply that X_i and X_j are independent random variables.

Let T be an abstract set and $\{X(t); t \text{ in } T\}$ a stochastic process. We call $\{X(t); t \text{ in } T\}$ a *Gaussian process* if for every $n = 1, 2, \ldots$ and every finite subset $\{t_1, \ldots, t_n\}$ of T, the random vector $(X(t_1), \ldots, X(t_n))$ has a multivariate normal distribution. Equivalently, the process is Gaussian if every linear combination

$$\alpha_1 X(t_1) + \cdots + \alpha_n X(t_n), \quad \alpha_i \text{ real},$$

has a univariate normal distribution. Every Gaussian process is described uniquely by its two parameters, the mean and covariance functions, given respectively by

$$\mu(t) = E[X(t)], \quad t \text{ in } T,$$

and

$$\Gamma(s,t) = E[\{X(s) - \mu(s)\}\{X(t) - \mu(t)\}], \quad s, t \text{ in } T.$$

The covariance function is positive definite in the sense that for every $n = 1, 2, \ldots$, real numbers $\alpha_1, \ldots, \alpha_n$, and elements $t_1, \ldots, t_1$ in T,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \Gamma\left(t_i, t_j\right) \geq 0.$$

One need only evaluate the expected value of $\left(\Sigma_{i=1}^{n} \alpha_i \{X(t_i) - \mu(t_i)\}\right)^2 \geq 0$ in terms of the covariance function in order to verify this.

Conversely, given an arbitrary mean value function $\mu(t)$ and a positive definite covariance function $\Gamma(s,t)$, there, then, exists a corresponding Gaussian process. Brownian motion is the unique Gaussian process having continuous trajectories, zero mean, and covariance function (8.10). We shall use this feature, that the mean value and covariance functions define a Gaussian process, several times in what follows.

We have seen how the invariance principle leads to the Gaussian process called Brownian motion. Gaussian processes also arise as the limits of normalized sums of independent and identically distributed random *functions*. To sketch out this idea, let $\xi_1(t), \xi_2(t), \ldots$ be independent and identically distributed random functions, or stochastic processes. Let $\mu(t) = E[\xi(t)]$ and $\Gamma(s,t) = \mathrm{Cov}[\xi(s), \xi(t)]$ be the mean value and covariance functions, respectively. Motivated by the central limit theorem, we define

$$X_N(t) = \frac{\Sigma_{i=1}^N \{\xi_i(t) - \mu(t)\}}{\sqrt{N}}.$$

The central limit theorem tells us that the distribution of $X_N(t)$ converges to the normal distribution for each fixed time point t. A multivariate extension of the central limit theorem asserts that for any finite set of time points $(t_1, \ldots, t_n)$, the random vector

$$(X_N(t_1), \ldots, X_N(t_n))$$

has, in the limit for large N, a multivariate normal distribution. It is not difficult to believe, then, that under ordinary circumstances, the stochastic processes $\{X_N(t); t \geq 0\}$ would converge, in an appropriate sense, to a Gaussian process $\{X(t); t \geq 0\}$ whose mean is zero and whose covariance function is $\Gamma(s,t)$. We call this the *central limit principle for random functions*. Several instances of its application appear in this chapter, the first of which is next.

Example *Cable Strength Under Equal Load Sharing* Consider a cable constructed from N wires in parallel. Suspension bridge cables are usually built this way. A section of the cable is clamped at each end and elongated by increasing the distance between the clamps. The problem is to determine the maximum tensile load that the cable will sustain in terms of the probabilistic and mechanical characteristics of the individual wires.

Let L_0 be the reference length of an unstretched unloaded strand of wire, and let L be the length after elongation. The *nominal strain* is defined to be $t = (L - L_0)/L_0$. Steadily increasing t causes the strand to stretch and exert a force $\xi(t)$ on the clamps, up to some random failure strain ζ, at which point the wire breaks. Hooke's law of elasticity asserts that the wire force is proportional to wire strain, with Young's modulus K as the proportionality constant. Taken all together, we write the force on the wire as a function of the nominal strain as

$$\xi(t) = \begin{cases} Kt, & \text{for } 0 \leq t < \zeta, \\ 0 & \text{for } \zeta \leq t. \end{cases} \tag{8.14}$$

A typical load function is depicted in Figure 8.2.

We will let $F(x) = \Pr\{\zeta \leq x\}$ be the cumulative distribution function of the failure strain. We easily determine the mean load on the wire to be

$$\mu(t) = E[\xi(t)] = E[Kt\mathbf{1}\{t < \zeta\}] = Kt[1 - F(t)]. \tag{8.15}$$

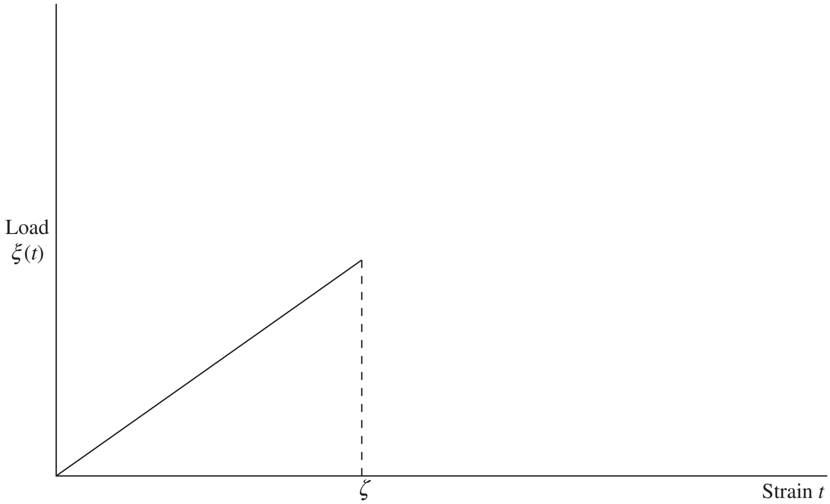

Figure 8.2 The load on an elastic wire as a function of nominal strain. At a strain of ζ the wire fails, and the load carried drops to zero.

The higher moments are, for $0 < s < t$,

$$\begin{aligned} E[\xi(s)\xi(t)] &= K^2 st E[\mathbf{1}\{s < \zeta\}\mathbf{1}\{t < \zeta\}] \\ &= K^2 st[1 - F(t)] \end{aligned}$$

and

$$\begin{aligned} \Gamma(s,t) &= E[\xi(s)\xi(t)] - E[\xi(s)]E[\xi(t)] \\ &= K^2 st F(s)[1 - F(t)], \quad \text{for } 0 < s < t. \end{aligned} \tag{8.16}$$

Turning to the cable, if it is clamped at the ends and elongated, then each wire within it is elongated the same amount. The total force $S_N(t)$ on the cable is the sum of the forces exerted by the individual wires. If we assume that the wires are independent and *a priori* identical, then these wire forces $\xi_1(t), \xi_2(t), \ldots$ are independent and identically distributed random functions, and

$$S_N(t) = \sum_{i=1}^{N} \xi_i(t)$$

is the random load experienced by the cable as a function of the cable strain. An illustration when $N = 5$ is given in Figure 8.3.

We are interested in the maximum load that the cable could carry without failing. This is

$$Q_N = \max\{S_N(t); t \geq 0\}.$$

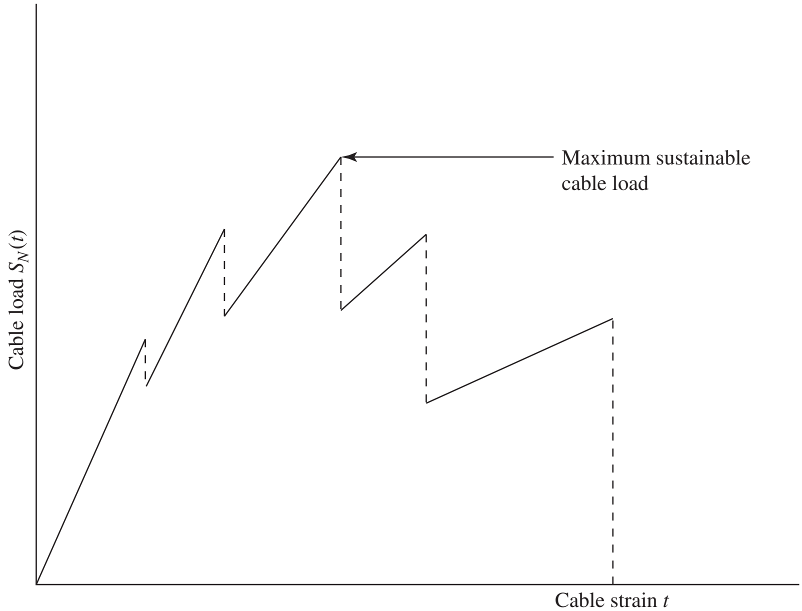

Figure 8.3 The load experienced by a cable composed of five elastic wire strands as a function of the cable strain t.

To obtain an approximation to the distribution of Q_N, we apply the central limit principle for random functions. This leads us to believe that

$$X_N(t) = \frac{S_N(t) - N\mu(t)}{\sqrt{N}}$$

should, for large N, be approximately a Gaussian process $X(t)$ with mean zero and covariance function given by (8.16). We write this approximation in the form

$$S_N(t) \approx N\mu(t) + \sqrt{N}X(t). \tag{8.17}$$

When N is large, the dominant term on the right of (8.17) is $N\mu(t)$. Let t^* be the value of t that maximizes $\mu(t)$. We assume that t^* is unique and that the second derivative of $\mu(t)$ is strictly negative at $t = t^*$. We would then expect that

$$Q_N = \max S_N(t) \approx N\mu\left(t^*\right) + \sqrt{N}X\left(t^*\right). \tag{8.18}$$

That is, we would expect that the cable strength would be approximately normally distributed with mean $N\mu(t^*)$ and variance $N\Gamma(t^*, t^*)$. To carry out a numerical example, suppose that $F(x) = 1 - \exp\left\{-x^5\right\}$, a Weibull distribution with shape parameter 5. It is easily checked that $t^* = 1/5^{0.2} = 0.7248$, that $\mu(t^*) = 0.5934K$, and $\Gamma(t^*, t^*) = (0.2792)^2K^2$. A cable composed of $N = 30$ wires would have a strength that is approximately normally distributed with mean $30(0.5934)K = 17.8K$ and standard deviation $0.2792\sqrt{30}K = 1.5292K$.

The above heuristics can be justified, and, indeed, significant refinements in the approximation have been made. We have referred to the approach as the central limit principle for random functions because we have not supplied sufficient details to label it a theorem. Nevertheless, we will see several more applications of the principle in subsequent sections of this chapter.

Exercises

8.1.1 Let $\{B(t); t \geq 0\}$ be a standard Brownian motion.
- **(a)** Evaluate $\Pr\{B(4) \leq 3 | B(0) = 1\}$.
- **(b)** Find the number c for which $\Pr\{B(9) > c | B(0) = 1\} = 0.10$.

8.1.2 Let $\{B(t); t \geq 0\}$ be a standard Brownian motion and $c > 0$ a constant. Show that the process defined by $W(t) = cB\left(t/c^2\right)$ is a standard Brownian motion.

8.1.3 **(a)** Show that

$$\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} = \phi'(x) = -x\phi(x),$$

where $\phi(x)$ is given in (8.5).

(b) Use the result in (a) together with the chain rule of differentiation to show that

$$p(y,t|x) = \phi_t(y-x) = \frac{1}{\sqrt{t}}\phi\left(\frac{y-x}{\sqrt{t}}\right)$$

satisfies the diffusion equation (8.3).

8.1.4 Consider a standard Brownian motion $\{B(t); t \geq 0\}$ at times $0 < u < u+v < u+v+w$, where $u, v, w > 0$.
- **(a)** Evaluate the product moment $E[B(u)B(u+v)B(u+v+w)]$.
- **(b)** Evaluate the product moment

$$E[B(u)B(u+v)B(u+v+w)B(u+v+w+x)]$$

where $x > 0$.

8.1.5 Determine the covariance functions for the stochastic processes
- **(a)** $U(t) = \mathrm{e}^{-t}B\left(\mathrm{e}^{2t}\right)$, for $t \geq 0$.
- **(b)** $V(t) = (1-t)B(t/(1-t))$, for $0 < t < 1$.
- **(c)** $W(t) = tB(1/t)$, with $W(0) = 0$.

$B(t)$ is standard Brownian motion.

8.1.6 Consider a standard Brownian motion $\{B(t); t \geq 0\}$ at times $0 < u < u+v < u+v+w$, where $u, v, w > 0$.
- **(a)** What is the probability distribution of $B(u) + B(u+v)$?
- **(b)** What is the probability distribution of $B(u) + B(u+v) + B(u+v+w)$?

8.1.7 Suppose that in the absence of intervention, the cash on hand for a certain corporation fluctuates according to a standard Brownian motion $\{B(t); t \geq 0\}$. The company manages its cash using an (s, S) policy: If the cash level ever drops to zero, it is instantaneously replenished up to level s; If the cash level ever rises up to S, sufficient cash is invested in long-term securities to bring the cash-on-hand down to level s. In the long run, what fraction of cash interventions are investments of excess cash?

Hint: Use equation (8.13).

Problems

8.1.1 Consider the simple random walk

$$S_n = \xi_1 + \cdots + \xi_n, \quad S_0 = 0,$$

in which the summands are independent with $\Pr\{\xi = \pm 1\} = \frac{1}{2}$. In Chapter 3, Section 3.5.3, we showed that the mean time for the random walk to first reach $-a < 0$ or $b > 0$ is ab. Use this together with the invariance principle to show that $E[T] = ab$, where

$$T = T_{a,b} = \min\{t \geq 0; B(t) = -a \text{ or } B(t) = b\},$$

and $B(t)$ is standard Brownian motion.

Hint: The approximate Brownian motion (8.11) rescales the random walk in both time and space.

8.1.2 Evaluate $E\left[e^{\lambda B(t)}\right]$ for an arbitrary constant λ and standard Brownian motion $B(t)$.

8.1.3 For a positive constant ϵ, show that

$$\Pr\left\{\frac{|B(t)|}{t} > \epsilon\right\} = 2\{1 - \Phi(\epsilon\sqrt{t})\}.$$

How does this behave when t is large ($t \to \infty$)? How does it behave when t is small ($t \approx 0$)?

8.1.4 Let $\alpha_1, \ldots, \alpha_n$ be real constants. Argue that

$$\sum_{i=1}^{n} \alpha_i B(t_i)$$

is normally distributed with mean zero and variance

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \min\{t_i, t_j\}.$$

8.1.5 Consider the simple random walk

$$S_n = \xi_1 + \cdots + \xi_n, \quad S_0 = 0,$$

in which the summands are independent with $\Pr\{\xi = \pm 1\} = \frac{1}{2}$. We are going to stop this random walk when it first drops a units below its maximum to date. Accordingly, let

$$M_n = \max_{0 \le k \le n} S_k, \quad Y_n = M_n - S_n, \quad \text{and}$$
$$\tau = \tau_a = \min\{n \ge 0; Y_n = a\}.$$

(a) Use a first step analysis to show that

$$\Pr\{M_\tau = 0\} = \frac{1}{1+a}.$$

(b) Why is $\Pr\{M_\tau \ge 2\} = \Pr\{M_\tau \ge 1\}^2$, and

$$\Pr\{M_\tau \ge k\} = \left(\frac{a}{1+a}\right)^k?$$

Identify the distribution of M_τ.

(c) Let $B(t)$ be standard Brownian motion, $M(t) = \max\{B(u); 0 \le u \le t\}$, $Y(t) = M(t) - B(t)$, and $\tau = \min\{t \ge 0; Y(t) = a\}$. Use the invariance principle to argue that $M(\tau)$ has an exponential distribution with mean a.

Note: τ is a popular strategy for timing the sale of a stock. It calls for keeping the stock as long as it is going up, but to sell it the first time that it drops a units from its best price to date. We have shown that $E[M(\tau)] = a$, whence $E[B(\tau)] = E[M(\tau)] - a = 0$, so that the strategy does not gain a profit, on average, in the Brownian motion model for stock prices.

8.1.6 Manufacturers of crunchy munchies such as cheese crisps use compression testing machines to gauge product quality. The crisp, or whatever, is placed between opposing plates, which then move together. As the crisp is crunched, the force is measured as a function of the distance that the plates have moved. The output of the compression testing machine is a graph of force versus distance that is much like Figure 8.3. What aspects of the graph might be measures of product quality? Model the test as a row of tiny balloons between parallel plates. Each single balloon might follow a force–distance behavior of the form $\sigma = Ke(1 - q(e))$, where σ is the force, K is Young's modulus, e is strain or distance, and $q(e)$ is a function that measures departures from Hooke's law, to allow for soggy crisps. Each balloon obeys this relationship up until the random strain ζ at which it bursts. Determine the mean force as a function of strain. Use $F(x)$ for the cumulative distribution function of failure strain.

8.1.7 For $n = 0, 1, \ldots,$ show that (a) $B(n)$ and (b) $B(n)^2 - n$ are martingales (see Chapter 2, Section 2.5).

8.1.8 *Computer Challenge* A problem of considerable contemporary importance is how to simulate a Brownian motion stochastic process. The invariance principle provides one possible approach. An infinite series expression that N. Wiener introduced may provide another approach. Let $Z_0,\ Z_1, \ldots$ be a series of independent standard normal random variables. The infinite series

$$B(t) = \frac{t}{\sqrt{\pi}} Z_0 + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{\sin mt}{m} Z_m, \quad 0 \le t \le 1,$$

is a standard Brownian motion for $0 \le t \le 1$. Try to simulate a Brownian motion stochastic process, at least approximately, by using finite sums of the form

$$B_N(t) = \frac{t}{\sqrt{\pi}} Z_0 + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{N} \frac{\sin mt}{m} Z_m, \quad 0 \le t \le 1.$$

If $B(t), 0 \le t \le 1$, is a standard Brownian motion on the interval $[0, 1]$, then $B'(t) = (1+t)B(1/(1+t)), 0 \le t < \infty$, is a standard Brownian motion on the interval $[0, \infty)$. This suggests

$$B'_N(t) = (1+t)B_N\left(\frac{1}{1+t}\right), \quad 0 \le t < \infty,$$

as an approximate standard Brownian motion. In what ways do these finite approximations behave like Brownian motion? Clearly, they are zero mean Gaussian processes. What is the covariance function, and how does it compare to that of Brownian motion? Do the gambler's ruin probabilities of (8.13) accurately describe their behavior? It is known that the squared variation of a Brownian motion stochastic process is not random, but constant:

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left| B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right|^2 = 1.$$

This is a further consequence of the variance relation $E\left[(\Delta B)^2\right] = \Delta t$ (see the remark in Section 8.1.2). To what degree do the finite approximations meet this criterion?

8.2 The Maximum Variable and the Reflection Principle

Using the continuity of the trajectories of Brownian motion and the symmetry of the normal distribution, we will determine a variety of interesting probability expressions for the Brownian motion process. The starting point is the *reflection principle*.

8.2.1 The Reflection Principle

Let $B(t)$ be a standard Brownian motion. Fix a value $x > 0$ and a time $t > 0$. Bearing in mind the continuity of the Brownian motion, property (c) of the definition, consider the collection of sample paths $B(u)$ for $u \geq 0$ with $B(0) = 0$ and for which $B(t) > x$. Since $B(u)$ is continuous and $B(0) = 0$, there exists a time τ, itself a random variable depending on the particular sample trajectory, at which the Brownian motion $B(u)$ first attains the value x.

We next describe a new path $B^*(u)$ obtained from $B(u)$ by reflection. For $u > \tau$, we reflect $B(u)$ about the horizontal line at height $x > 0$ to obtain

$$B^*(u) = \begin{cases} B(u), & \text{for } u \leq \tau, \\ x - [B(u) - x], & \text{for } u > \tau. \end{cases}$$

Figure 8.4 illustrates the construction. Note that $B^*(t) < x$ because $B(t) > x$.

Because the conditional probability law of the path for $u > \tau$, given that $B(\tau) = x$, is symmetric with respect to the values $y > x$ and $y < x$, and independent of the history prior to τ,* the reflection argument displays for every sample path with $B(t) > x$ two

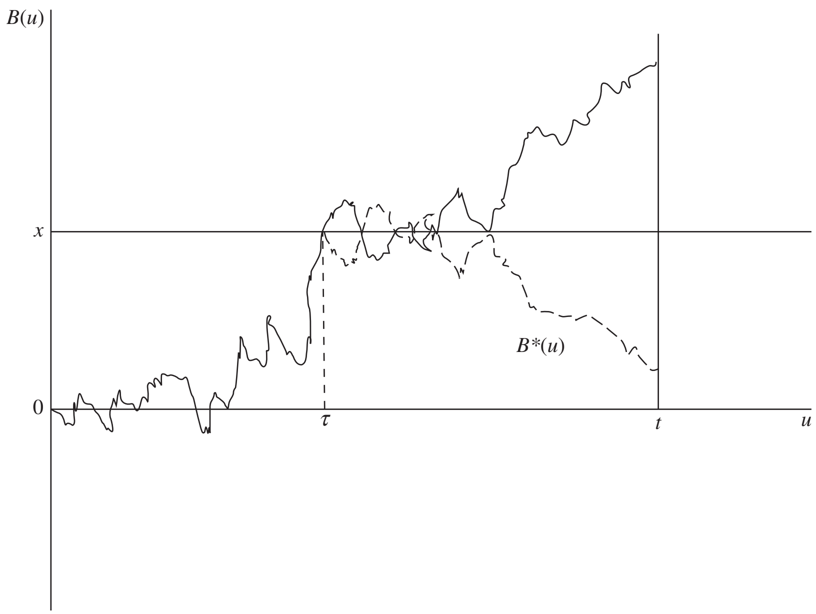

Figure 8.4 The path $B(u)$ is reflected about the horizontal line at x, showing that for every path ending at $B(t) > x$, there are two paths, $B(u)$ and $B^*(u)$, that attain the value x somewhere in the interval $0 \leq u \leq t$.

* The argument is not quite complete because the definition asserts that an increment in a Brownian motion after a fixed time t is independent of the past, whereas here we are restarting from the random time τ. While the argument is incomplete, the assertion is true: A Brownian path begins afresh from *hitting times* such as τ.

equally likely sample paths, $B(u)$ and $B^*(u)$, for which both

$$\max_{0\le u\le t} B(u) > x \quad \text{and} \quad \max_{0\le u\le t} B^*(u) > x.$$

Conversely, by the nature of this correspondence, every sample path $B(u)$ for which $\max_{0\le u\le t} B(u) > x$ results from either of two equally likely sample paths, exactly one of which is such that $B(t) > x$. The two-to-one correspondence fails only if $B(t) = x$, but because $B(t)$ is a continuous random variable (normal distribution), we have $\Pr\{B(t) = x\} = 0$, and this case can be safely ignored. Thus, we conclude that

$$\Pr\left\{\max_{0\le u\le t} B(u) > x\right\} = 2\Pr\{B(t) > x\}.$$

In terms of the maximum process defined by

$$M(t) = \max_{0\le u\le t} B(u), \tag{8.19}$$

and using the notation set forth in (8.8), we have

$$\Pr\{M(t) > x\} = 2[1 - \Phi_t(x)]. \tag{8.20}$$

8.2.2 The Time to First Reach a Level

With the help of (8.20), we may determine the probability distribution of the random time τ_x at which the Brownian motion first attains a prescribed value $x > 0$ starting from $B(0) = 0$. Formally, define the *hitting time*

$$\tau_x = \min\{u \ge 0; B(u) = x\}. \tag{8.21}$$

Clearly, $\tau_x \le t$ if and only if $M(t) \ge x$. In words, the Brownian motion attains the level $x > 0$ before time t if and only if at time t the maximum of the process is at least x. If the two events are equivalent, then their probabilities must be the same. That is,

$$\begin{aligned} \Pr\{\tau_x \le t\} &= \Pr\{M(t) \ge x\} = 2\,[1 - \Phi_t(x)] \\ &= \frac{2}{\sqrt{2\pi t}} \int_x^\infty \mathrm{e}^{-\xi^2/(2t)} \mathrm{d}\xi. \end{aligned} \tag{8.22}$$

The change of variable $\xi = \eta\sqrt{t}$ leads to

$$\Pr\{\tau_x \le t\} = \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^\infty \mathrm{e}^{-\eta^2/2} \mathrm{d}\eta. \tag{8.23}$$

The probability density function of the random time τ is obtained by differentiating (8.23) with respect to t, giving

$$f_{\tau_x}(t) = \frac{xt^{-3/2}}{\sqrt{2\pi}} e^{-x^2/(2t)} \quad \text{for} \quad 0 < t < \infty. \tag{8.24}$$

8.2.3 The Zeros of Brownian Motion

As a final illustration of the far-reaching consequences of the reflection principle and equation (8.24), we will determine the probability that a standard Brownian motion $B(t)$, with $B(0) = 0$, will cross the t axis at least once in the time interval $(t, t+s]$ for $t, s > 0$. Let us denote this quantity by $\vartheta(t, t+s)$. The result is

$$\begin{aligned} \vartheta(t, t+s) &= \Pr\{B(u) = 0 \text{ for some } u \text{ in } (t, t+s]\} \\ &= \frac{2}{\pi} \arctan \sqrt{s/t} \\ &= \frac{2}{\pi} \arccos \sqrt{t/(t+s)}. \end{aligned} \tag{8.25}$$

First, let us define some notation concerning the hitting time τ_x defined in (8.21). Let

$$H_t(z, x) = \Pr\{\tau_x \le t | B(0) = z\}$$

be the probability that a standard Brownian motion starting from $B(0) = z$ will reach the level x before time t. In equation (8.22), we gave an integral that evaluated

$$H_t(0, x) = \Pr\{\tau_x \le t | B(0) = 0\}, \qquad \text{for } x > 0.$$

The symmetry and spatial homogeneity of the Brownian motion make it clear that $H_t(0, x) = H_t(x, 0)$. That is, the probability of reaching $x > 0$ starting from $B(0) = 0$ before time t is the same as the probability of reaching 0 starting from $B(0) = x$. Consequently, from (8.24) we have

$$\begin{aligned} H_t(0, x) &= H_t(x, 0) = \Pr\{\tau_0 \le t | B(0) = x\} \\ &= \int_0^t \frac{x}{\sqrt{2\pi}} \xi^{-3/2} e^{-x^2/(2\xi)} d\xi. \end{aligned} \tag{8.26}$$

We will condition on the value of the Brownian motion at time t and use the law of total probability to derive (8.25). Accordingly, we have

$$\vartheta(t, t+s) = \int_{-\infty}^{\infty} \Pr\{B(u) = 0 \text{ for some } u \text{ in } (t, t+s] | B(t) = x\} \phi_t(x) dx,$$

where $\phi_t(x)$ is the probability density function for $B(t)$ as given in (8.7). Then, using (8.26),

$$\begin{aligned}\vartheta(t,t+s) &= 2\int_0^\infty H_s(x,0)\phi_t(x)\mathrm{d}x \\ &= 2\int_0^\infty \left\{\int_0^s \frac{x}{\sqrt{2\pi}}\xi^{-3/2}\mathrm{e}^{-x^2/(2\xi)}\mathrm{d}\xi\right\}\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-x^2/2t}\mathrm{d}x \\ &= \frac{1}{\pi\sqrt{t}}\int_0^s \left\{\int_0^\infty x\mathrm{e}^{-x^2/(2\xi)-x^2/(2t)}\mathrm{d}x\right\}\xi^{-3/2}\mathrm{d}\xi.\end{aligned}$$

To evaluate the inner integral, we let

$$v = \frac{x^2}{2}\left(\frac{1}{\xi}+\frac{1}{t}\right),$$

whence

$$\mathrm{d}v = x\left(\frac{1}{\xi}+\frac{1}{t}\right)\mathrm{d}x,$$

and so

$$\begin{aligned}\vartheta(t,t+s) &= \frac{1}{\pi\sqrt{t}}\int_0^s \left(\frac{1}{\xi}+\frac{1}{t}\right)^{-1}\left\{\int_0^\infty v\mathrm{e}^{-v}\mathrm{d}v\right\}\xi^{-3/2}\mathrm{d}\xi \\ &= \frac{\sqrt{t}}{\pi}\int_0^s \frac{\mathrm{d}\xi}{(t+\xi)\sqrt{\xi}}.\end{aligned}$$

The change of variable $\eta = \sqrt{\xi/t}$ gives

$$\vartheta(t,t+s) = \frac{2}{\pi}\int_0^{\sqrt{s/t}} \frac{\mathrm{d}\eta}{1+\eta^2} = \frac{2}{\pi}\arccos\sqrt{t/(t+s)}.$$

Finally, Exercise 8.2.2 asks the student to use standard trigonometric identities to show the equivalence $\arctan\sqrt{s/t} = \arccos\sqrt{t/(t+s)}$.

Exercises

8.2.1 Let $\{B(t);\, t\geq 0\}$ be a standard Brownian motion, with $B(0)=0$, and let $M(t) = \max\{B(u);\, 0\leq u\leq t\}$.

(a) Evaluate $\Pr\{M(4) \le 2\}$.
(b) Find the number c for which $\Pr\{M(9) > c\} = 0.10$.

8.2.2 Show that

$$\arctan\sqrt{s/t} = \arccos\sqrt{t/(s+t)}.$$

8.2.3 Suppose that net inflows to a reservoir are described by a standard Brownian motion. If at time 0, the reservoir has $x = 3.29$ units of water on hand, what is the probability that the reservoir never becomes empty in the first $t = 4$ units of time?

8.2.4 Consider the simple random walk

$$S_n = \xi_1 + \cdots + \xi_n, \quad S_0 = 0,$$

in which the summands are independent with $\Pr\{\xi = \pm 1\} = \frac{1}{2}$. Let $M_n = \max_{0 \le k \le n} S_k$. Use a reflection argument to show that

$$\Pr\{M_n \ge a\} = 2\Pr\{S_n > a\} + \Pr\{S_n = a\}, \quad a > 0.$$

8.2.5 Let τ_0 be the largest zero of a standard Brownian motion not exceeding $a > 0$. That is, $\tau_0 = \max\{u \ge 0; B(u) = 0 \text{ and } u \le a\}$. Show that

$$\Pr\{\tau_0 < t\} = \frac{2}{\pi}\arcsin\sqrt{t/a}.$$

8.2.6 Let τ_1 be the smallest zero of a standard Brownian motion that exceeds $b > 0$. Show that

$$\Pr\{\tau_1 < t\} = \frac{2}{\pi}\arccos\sqrt{b/t}.$$

Problems

8.2.1 Find the conditional probability that a standard Brownian motion is not zero in the interval $(t, t+b]$ given that it is not zero in the interval $(t, t+a]$, where $0 < a < b$ and $t > 0$.

8.2.2 Find the conditional probability that a standard Brownian motion is not zero in the interval $(0, b]$ given that it is not zero in the interval $(0, a]$, where $0 < a < b$.

Hint: Let $t \to 0$ in the result of Problem 8.2.1.

8.2.3 For a fixed $t > 0$, show that $M(t)$ and $|B(t)|$ have the same marginal probability distribution, whence

$$f_{M(t)}(z) = \frac{2}{\sqrt{t}}\phi\left(\frac{z}{\sqrt{t}}\right) \quad \text{for} \quad z > 0.$$

(Here $M(t) = \max_{0 \le u \le t} B(u)$.) Show that

$$E[M(t)] = \sqrt{2t/\pi}.$$

For $0 < s < t$, do $(M(s), M(t))$ have the same joint distribution as $(|B(s)|, |B(t)|)$?

8.2.4 Use the reflection principle to obtain

$$\begin{aligned}\Pr\{M(t) \geq z, B(t) \leq x\} &= \Pr\{B(t) \geq 2z - x\}\\ &= 1 - \Phi\left(\frac{2z - x}{\sqrt{t}}\right) \quad \text{for} \quad 0 < x < m.\end{aligned}$$

($M(t)$ is the maximum defined in (8.19).) Differentiate with respect to x, and then with respect to z, to obtain the joint density function for $M(t)$ and $B(t)$:

$$f_{M(t),B(t)}(z, x) = \frac{2z - x}{t} \frac{2}{\sqrt{t}} \phi\left(\frac{2z - x}{\sqrt{t}}\right).$$

8.2.5 Show that the joint density function for $M(t)$ and $Y(t) = M(t) - B(t)$ is given by

$$f_{M(t),Y(t)}(z, y) = \frac{z + y}{t} \frac{2}{\sqrt{t}} \phi\left(\frac{z + y}{\sqrt{t}}\right).$$

8.2.6 Use the result of Problem 8.2.5 to show that $Y(t) = M(t) - B(t)$ has the same distribution as $|B(t)|$.

8.3 Variations and Extensions

A variety of processes derived from Brownian motion find relevance and application in stochastic modeling. We briefly describe a few of these.

8.3.1 Reflected Brownian Motion

Let $\{B(t); t \geq 0\}$ be a standard Brownian motion process. The stochastic process

$$R(t) = |B(t)| = \begin{cases} B(t), & \text{if } B(t) \geq 0,\\ -B(t), & \text{if } B(t) < 0,\end{cases}$$

is called *Brownian motion reflected at the origin*, or, more briefly, *reflected Brownian motion*. Reflected Brownian motion reverberates back to positive values whenever it reaches the zero level and, thus, might be used to model the movement of a pollen grain in the vicinity of a container boundary that the grain cannot cross.

Since the moments of $R(t)$ are the same as those of $|B(t)|$, the mean and variance of reflected Brownian motion are easily determined. Under the condition that $R(0) = 0$, e.g., we have

$$
\begin{aligned}
E[R(t)] &= \int_{-\infty}^{\infty} |x|\phi_t(x)\mathrm{d}x \\
&= 2\int_0^{\infty} \frac{x}{\sqrt{2\pi t}} \exp\left(-x^2/2t\right)\mathrm{d}x \\
&= \sqrt{2t/\pi}.
\end{aligned}
\tag{8.27}
$$

The integral was evaluated through the change of variable $y = x/\sqrt{t}$. Also,

$$
\begin{aligned}
\operatorname{Var}[R(t)] &= E\left[R(t)^2\right] - \{E[R(t)]\}^2 \\
&= E\left[B(t)^2\right] - 2t/\pi \\
&= \left(1 - \frac{2}{\pi}\right)t.
\end{aligned}
\tag{8.28}
$$

Reflected Brownian motion is a second example of a continuous-time, continuous-state-space Markov process. Its transition density $p(y,t|x)$ is derived from that of Brownian motion by differentiating

$$
\begin{aligned}
\Pr\{R(t) \le y | R(0) = x\} &= \Pr\{-y \le B(t) \le y | B(0) = x\} \\
&= \int_{-y}^{y} \phi_t(z - x)\mathrm{d}z
\end{aligned}
$$

with respect to y to get

$$
\begin{aligned}
p(y,t|x) &= \phi_t(y - x) + \phi_t(-y - x) \\
&= \phi_t(y - x) + \phi_t(y + x).
\end{aligned}
$$

8.3.2 Absorbed Brownian Motion

Suppose that the initial value $B(0) = x$ of a standard Brownian motion process is positive, and let τ be the first time that the process reaches zero. The stochastic process

$$
A(t) = \begin{cases} B(t) & \text{for } t \le \tau, \\ 0 & \text{for } t > \tau \end{cases}
$$

is called *Brownian motion absorbed at the origin*, which we will shorten to *absorbed Brownian motion*. Absorbed Brownian motion might be used to model the price of a share of stock in a company that becomes bankrupt at some future instant. We can evaluate the transition probabilities for absorbed Brownian motion by another use of the reflection principle introduced in Section 8.2. For $x > 0$ and $y > 0$, let

$$\begin{aligned} G_t(x,y) &= \Pr\{A(t) > y | A(0) = x\} \\ &= \Pr\left\{B(t) > y, \min_{0 \le u \le t} B(u) > 0 | B(0) = x\right\}. \end{aligned} \tag{8.29}$$

To determine (8.29), we start with the obvious relation

$$\Pr\{B(t) > y | B(0) = x\} = G_t(x,y) + \Pr\left\{B(t) > y, \min_{0 \le u \le t} B(u) \le 0 | B(0) = x\right\}.$$

The reflection principle is applied to the last term; Figure 8.5 is the appropriate picture to guide the analysis. We will argue that

$$\begin{aligned} &\Pr\left\{B(t) > y, \min_{0 \le u \le t} B(u) \le 0 | B(0) = x\right\} \\ &\quad = \Pr\left\{B(t) < -y, \min_{0 \le u \le t} B(u) \le 0 | B(0) = x\right\} \\ &\quad = \Pr\{B(t) < -y | B(0) = x\} = \Phi_t(-y - x). \end{aligned} \tag{8.30}$$

The reasoning behind (8.30) goes as follows: Consider a path starting at $x > 0$, satisfying $B(t) > y$, and which reaches zero at some intermediate time τ. By reflecting such

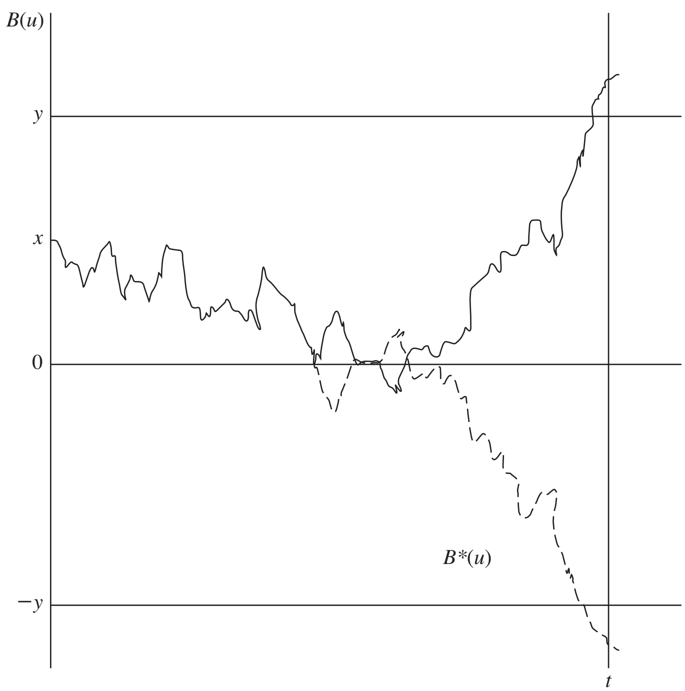

Figure 8.5 For every path $B(u)$ starting at x, ending at $B(t) > y$, and reaching zero in the interval, there is another path $B^*(u)$ starting at x and ending at $B^*(t) < -y$.

a path about zero after time τ, we obtain an equally likely path starting from x and assuming a value below $-y$ at time t. This implies the equality of the first two terms of (8.30). The equality of the last terms is clear from their meaning, since the condition that the minimum be below zero is superfluous in view of the requirement that the path end below $-y(y > 0)$. Inserting (8.30) into (8.29) yields

$$\begin{aligned} G_t(x,y) &= \Pr\{B(t) > y|B(0) = x\} - \Pr\{B(t) < -y|B(0) = x\} \\ &= 1 - \Phi_t(y-x) - \Phi_t(-(y+x)) \\ &= \Phi_t(y+x) - \Phi_t(y-x) \\ &= \int_{y-x}^{y+x} \phi_t(z)\mathrm{d}z = \Phi\left(\frac{y+x}{\sqrt{t}}\right) - \Phi\left(\frac{y-x}{\sqrt{t}}\right). \end{aligned} \tag{8.31}$$

From (8.29) and (8.31), we obtain the transition distribution for absorbed Brownian motion:

$$\Pr\{A(t) > y|A(0) = x\} = \Phi\left(\frac{y+x}{\sqrt{t}}\right) - \Phi\left(\frac{y-x}{\sqrt{t}}\right). \tag{8.32}$$

Under the condition that $A(0) = x > 0, A(t)$ is a random variable that has both discrete and continuous parts. The discrete part is

$$\begin{aligned} \Pr\{A(t) = 0|A(0) = x\} &= 1 - G_t(x,0) \\ &= 1 - \int_{-x}^{x} \phi_t(z)\mathrm{d}z \\ &= 2[1 - \Phi_t(x)]. \end{aligned}$$

In the region $y > 0, A(t)$ is a continuous random variable whose transition density $p(y,t|x)$ is obtained by differentiating with respect to y in (8.32) and suitably changing the sign:

$$p(y,t|x) = \phi_t(y-x) - \phi_t(y+x).$$

8.3.3 The Brownian Bridge

The *Brownian bridge* $\left\{B^0(t); t \geq 0\right\}$ is constructed from a standard Brownian motion $\{B(t); t \geq 0\}$ by conditioning on the event $\{B(0) = B(1) = 0\}$. The Brownian bridge is used to describe certain random functionals arising in nonparametric statistics, and as a model for the publicly traded prices of bonds having a specified redemption value on a fixed expiration date.

We will determine the probability distribution for $B^0(t)$ by using the conditional density formula for jointly normally distributed random variables derived in Chapter 2, Problem 2.4.8. First, for $0 < t < 1$, the random variables $B(t)$ and $B(1) - B(t)$ are independent and normally distributed according to the definition of Brownian motion. It follows that $X = B(t)$ and $Y = B(1) = B(t) + \{B(1) - B(t)\}$ have a joint normal distribution (see Chapter 1, Section 1.4.6) for which we have determined $\mu_X = \mu_Y = 0, \sigma_X = \sqrt{t}, \sigma_Y = 1$, and $\rho = \mathrm{Cov}[X, Y]/\sigma_X\sigma_Y = \sqrt{t}$. Using the results of Chapter 2, Problem 2.4.8, it then follows that given $Y = B(1) = y$, the conditional distribution of $X = B(t)$ is normal, with

$$\mu_{X|Y} = \mu_X + \frac{\rho\sigma_X}{\sigma_Y}(y - \mu_Y) = y\sqrt{t} = 0 \quad \text{when} \quad y = 0,$$

and

$$\sigma_{X|Y} = \sigma_X\sqrt{1 - \rho^2} = \sqrt{t(1 - t)}.$$

For the Brownian bridge, $B^0(t)$ is normally distributed with $E\left[B^0(t)\right] = 0$ and $\mathrm{Var}\left[B^0(t)\right] = t(1 - t)$. Notice how the condition $B(0) = B(1) = 0$ causes the variance of $B^0(t)$ to vanish at $t = 0$ and $t = 1$.

The foregoing calculation of the variance can be extended to determine the covariance function. Consider times s, t with $0 < s < t < 1$. By first obtaining the joint distribution of $(B(s), B(t), B(1))$, and then the conditional joint distribution of $(B(s), B(t))$, given that $B(1) = 0$, one can verify that the Brownian bridge is a normally distributed stochastic process with mean zero and covariance function $\Gamma(s, t) = \mathrm{Cov}\left[B^0(s), B^0(t)\right] = s(1 - t)$, for $0 < s < t < 1$. (See Problem 8.3.3 for an alternative approach.)

Example *The Empirical Distribution Function* Let $X_1, X_2, \ldots$ be independent and identically distributed random variables. The *empirical cumulative distribution function* corresponding to a sample of size N is defined by

$$\begin{aligned} F_N(t) &= \frac{1}{N}\#\{X_i \le t \quad \text{for } i = 1, \ldots, N\} \\ &= \frac{1}{N}\sum_{i=1}^{N}\xi_i(t), \end{aligned} \tag{8.33}$$

where

$$\xi_i(t) = \begin{cases} 1, & \text{if } X_i \le t, \\ 0, & \text{if } X_i > t. \end{cases}$$

The empirical distribution function is an estimate, based on the observed sample, of the true distribution function $F(t) = \Pr\{X \le t\}$. We will use the central limit principle for random functions (Section 8.1.4) to approximate the empirical distribution function by a Brownian bridge, assuming that the observations are uniformly distributed over the interval $(0, 1)$. (Problem 8.3.9 calls for the student to explore the case of a general distribution.) In the uniform case, $F(t) = t$ for $0 < t < 1$, and $\mu(t) = E[\xi(t)] = F(t) = t$. For the higher moments, when $0 < s < t < 1$, $E[\xi(s)\xi(t)] = F(s) = s$, and $\Gamma(s,t) = \mathrm{Cov}[\xi(s), \xi(t)] = E[\xi(s)\xi(t)] - E[\xi(s)]E[\xi(t)] = s - st = s(1-t)$.

In view of equation (8.33), which expresses the empirical distribution function in terms of a sum of independent and identically distributed random functions, we might expect the central limit principle for random functions to yield an approximation in terms of a Gaussian limit. Following the guidelines in Section 8.1.4, we would expect that

$$\begin{aligned} X_N(t) &= \frac{\Sigma_{i=1}^N \{\xi_i(t) - \mu(t)\}}{\sqrt{N}} \\ &= \frac{NF_N(t) - Nt}{\sqrt{N}} \\ &= \sqrt{N}\{F_N(t) - t\} \end{aligned}$$

would converge, in an appropriate sense, to a Gaussian process with zero mean and covariance $\Gamma(s,t) = s(1-t)$, for $0 < s < t < 1$. As we have just seen, this process is a Brownian bridge. Therefore, we would expect the approximation

$$F_N(t) \approx t + \frac{1}{\sqrt{N}} B^0(t), \quad 0 < t < 1.$$

Such approximations are heavily used in the theory of nonparametric statistics.

8.3.4 Brownian Meander

Brownian meander $\left\{B^+(t);\, t \geq 0\right\}$ is Brownian motion conditioned to be positive. Recall (8.29) and (8.31):

$$\begin{aligned} G_t(x,y) &= \Pr\left\{B(t) > y, \min{}_{0\leq u\leq t} B(u) > 0 | B(0) = x\right\} \\ &= \Phi\left(\frac{y+x}{\sqrt{t}}\right) - \Phi\left(\frac{y-x}{\sqrt{t}}\right), \end{aligned}$$

so that

$$\begin{aligned} G_t(x,0) &= \Pr\left\{\min{}_{0\leq u\leq t} B(u) > 0 | B(0) = x\right\} \\ &= \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{-x}{\sqrt{t}}\right). \end{aligned}$$

The transition law for Brownian meander is

$$\Pr\left\{B^+(t) > y | B^+(0) = x\right\} = \Pr\left\{B(t) > y | \min{}_{0\leq u\leq t} B(u) > 0, B(0) = x\right\},$$

whence

$$\Pr\left\{B^+(t) > y | B^+(0) = x\right\} = \frac{G_t(x,y)}{G_t(x,0)}$$

$$= \frac{\Phi\left(\frac{y+x}{\sqrt{t}}\right) - \Phi\left(\frac{y-x}{\sqrt{t}}\right)}{\Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{-x}{\sqrt{t}}\right)}.$$

Of most interest is the limiting case as $x \to 0$,

$$\Pr\left\{B^+(t) > y | B^+(0) = 0\right\} = \lim_{x \to 0} \frac{\Phi\left(\frac{y+x}{\sqrt{t}}\right) - \Phi\left(\frac{y-x}{\sqrt{t}}\right)}{\Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{-x}{\sqrt{t}}\right)}$$

$$= \frac{\phi\left(\frac{y}{\sqrt{t}}\right)}{\phi(0)}$$

$$= \mathrm{e}^{-\frac{1}{2}y^2/t}.$$

A simple integration yields the mean

$$E\left[B^+(t) | B^+(0) = 0\right] = \int_0^\infty \Pr\left\{B^+(t) > y | B^+(0) = 0\right\} \mathrm{d}y$$

$$= \int_0^\infty \mathrm{e}^{-\frac{1}{2}y^2/t} \mathrm{d}y$$

$$= \frac{1}{2}\sqrt{2\pi t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(2\pi t)}} \mathrm{e}^{-\frac{1}{2}y^2/t} \mathrm{d}y$$

$$= \sqrt{\pi t/2}.$$

Exercises

8.3.1 Show that the cumulative distribution function for reflected Brownian motion is

$$\Pr\{R(t) < y | R(0) = x\} = \Phi\left(\frac{y-x}{\sqrt{t}}\right) - \Phi\left(\frac{-y-x}{\sqrt{t}}\right)$$

$$= \Phi\left(\frac{y-x}{\sqrt{t}}\right) + \Phi\left(\frac{y+x}{\sqrt{t}}\right) - 1$$

$$= \Phi\left(\frac{x+y}{\sqrt{t}}\right) - \Phi\left(\frac{x-y}{\sqrt{t}}\right).$$

Evaluate this probability when $x = 1, y = 3$, and $t = 4$.

8.3.2 The price fluctuations of a share of stock of a certain company are well described by a Brownian motion process. Suppose that the company is bankrupt if ever the share price drops to zero. If the starting share price is $A(0) = 5$, what is the probability that the company is bankrupt at time $t = 25$? What is the probability that the share price is above 10 at time $t = 25$?

8.3.3 The net inflow to a reservoir is well described by a Brownian motion. Because a reservoir cannot contain a negative amount of water, we suppose that the water level $R(t)$ at time t is a reflected Brownian motion. What is the probability that the reservoir contains more than 10 units of water at time $t = 25$? Assume that the reservoir has unlimited capacity and that $R(0) = 5$.

8.3.4 Suppose that the net inflows to a reservoir follow a Brownian motion. Suppose that the reservoir was known to be empty 25 time units ago but has never been empty since. Use a Brownian meander process to evaluate the probability that there is more than 10 units of water in the reservoir today.

8.3.5 Is reflected Brownian motion a Gaussian process? Is absorbed Brownian motion (cf. Section 8.1.4)?

Problems

8.3.1 Let $B_1(t)$ and $B_2(t)$ be independent standard Brownian motion processes. Define

$$R(t) = \sqrt{B_1(t)^2 + B_2(t)^2}, \quad t \geq 0.$$

$R(t)$ is the radial distance to the origin in a *two-dimensional Brownian motion*. Determine the mean of $R(t)$.

8.3.2 Let $B(t)$ be a standard Brownian motion process. Determine the conditional mean and variance of $B(t), 0 < t < 1$, given that $B(1) = b$.

8.3.3 Let $B(t)$ be a standard Brownian motion. Show that $B(u) - uB(1), 0 < u < 1$, is independent of $B(1)$.

(a) Use this to show that $B^0(t) = B(t) - tB(1), 0 \leq t \leq 1$, is a Brownian bridge.

(b) Use the representation in (a) to evaluate the covariance function for a Brownian bridge.

8.3.4 Let $B(t)$ be a standard Brownian motion. Determine the covariance function for

$$W^0(s) = (1-s)B\left(\frac{s}{1-s}\right), \quad 0 < s < 1,$$

and compare it to that for a Brownian bridge.

8.3.5 Determine the expected value for absorbed Brownian motion $A(t)$ at time $t = 1$ by integrating the transition density (8.32) according to

$$\begin{aligned} E[A(1)|A(0) = x] &= \int_0^\infty yp(y, 1|x)\mathrm{d}y \\ &= \int_0^\infty y[\phi(y-x) - \phi(y+x)]\mathrm{d}y. \end{aligned}$$

The answer is $E[A(1)|A(0)=x]=x$. Show that $E[A(t)|A(0)=x]=x$ for all $t>0$.

8.3.6 Let $M=\max\{A(t); t\geq 0\}$ be the largest value assumed by an absorbed Brownian motion $A(t)$. Show that $\Pr\{M>z|A(0)=x\}=x/z$ for $0<x<z$.

8.3.7 Let $t_0=0<t_1<t_2<\cdots$ be time points, and define $X_n=A(t_n)$, where $A(t)$ is absorbed Brownian motion starting from $A(0)=x$. Show that $\{X_n\}$ is a nonnegative martingale. Compare the maximal inequality (2.53) in Chapter 2 with the result in Problem 8.3.6.

8.3.8 Show that the transition densities for both reflected Brownian motion and absorbed Brownian motion satisfy the diffusion equation (8.3) in the region $0<x<\infty$.

8.3.9 Let $F(t)$ be a cumulative distribution function and $B^0(u)$ a Brownian bridge.

(a) Determine the covariance function for $B^0(F(t))$.

(b) Use the central limit principle for random functions to argue that the empirical distribution functions for random variables obeying $F(t)$ might be approximated by the process in (a).

8.4 Brownian Motion with Drift

Let $\{B(t); t\geq 0\}$ be a standard Brownian motion process, and let μ and $\sigma>0$ be fixed. The *Brownian motion with drift parameter μ and variance parameter σ^2* is the process

$$X(t)=\mu t+\sigma B(t) \quad \text{for} \quad t\geq 0. \tag{8.34}$$

Alternatively, Brownian motion with drift parameter μ and variance parameter σ^2 is the process whose increments over disjoint time intervals are independent (property (b) of the definition of standard Brownian motion) and whose increments $X(t+s)-X(t)$, $t,s>0$, are normally distributed with mean μs and variance $\sigma^2 s$. When $X(0)=x$, we have

$$\begin{aligned}\Pr\{X(t)\leq y|X(0)=x\}&=\Pr\{\mu t+\sigma B(t)\leq y|\sigma B(0)=x\}\\&=\Pr\left\{B(t)\leq\frac{y-\mu t}{\sigma}\,\middle|\,B(0)=\frac{x}{\sigma}\right\}\\&=\Phi_t\left(\frac{y-x-\mu t}{\sigma}\right)=\Phi\left(\frac{y-x-\mu t}{\sigma\sqrt{t}}\right).\end{aligned}$$

Brownian motion with drift is not symmetric when $\mu\neq 0$, and the reflection principle cannot be used to compute the distribution of the maximum of the process. We will use an infinitesimal first step analysis to determine some properties of Brownian motion with drift. To set this up, let us introduce some notation to describe changes in the Brownian motion with drift over small time intervals of length Δt. We let $\Delta X=X(t+\Delta t)-X(t)$ and $\Delta B=B(t+\Delta t)-B(t)$. Then, $\Delta X=\mu\Delta t+\sigma\Delta B$, and

$$X(t+\Delta t)=X(t)+\Delta X=X(t)+\mu\Delta t+\sigma\Delta B. \tag{8.35}$$

We observe that the conditional moments of $\triangle$t, given $X(t) = x$, are

$$E[\Delta X|X(t) = x] = \mu\Delta t + \sigma E[\Delta B] = \mu\Delta t, \tag{8.36}$$

$$\mathrm{Var}[\Delta X|X(t) = x] = \sigma^2 E\left[(\Delta B)^2\right] = \sigma^2\Delta t, \tag{8.37}$$

and

$$E\left[(\Delta X)^2|X(t) = x\right] = \sigma^2\Delta t + (\mu\Delta t)^2 = \sigma^2\Delta t + o(\Delta t), \tag{8.38}$$

while

$$E\left[(\Delta X)^c\right] = o(\Delta t) \quad \text{for} \quad c > 2. \tag{8.39}$$

8.4.1 The Gambler's Ruin Problem

Let us suppose that $X(0) = x$, and that $a < x$ and $b > x$ are fixed quantities. We will be interested in some properties of the random time T at which the process first assumes one of the values a or b. This so-called *hitting time* is formally defined by

$$T = T_{ab} = \min\{t \geq 0; X(t) = a \text{ or } X(t) = b\}.$$

Analogous to the gambler's ruin problem in a random walk (Chapter 3, Section 3.5.3), we will determine the probability that when the Brownian motion exits the interval (a, b), it does so at the point b. The solution for a standard Brownian motion was obtained in Section 8.1 by using the invariance principle. Here we solve the problem for Brownian motion with drift by instituting an infinitesimal first step analysis.

Theorem 8.1. *For a Brownian motion with drift parameter μ and variance parameter σ^2, and $a < x < b$,*

$$u(x) = \Pr\{X(T_{ab}) = b|X(0) = x\} = \frac{e^{-2\mu x/\sigma^2} - e^{-2\mu a/\sigma^2}}{e^{-2\mu b/\sigma^2} - e^{-2\mu a/\sigma^2}}. \tag{8.40}$$

Proof. Our proof is not entirely complete in that we will assume (1) that $u(x)$ is twice continuously differentiable, and (2) that we can choose a time increment Δt so small that exiting the interval (a, b) prior to time Δt can be neglected. With these provisos, at time Δt the Brownian motion will be at the position $X(0) + \Delta X = x + \Delta X$, and the conditional probability of exiting at the upper point b is now $u(x + \Delta X)$. Invoking the law of total probability, it must be that $u(x) = \Pr\{X(T) = b|X(0) = x\} = E[\Pr\{X(T) = b|X(0) = x, X(\Delta t) = x + \Delta X\}|X(0) = x] = E[u(x + \Delta X)]$, where $a < x < b$. ∎

The next step is to expand $u(x + \Delta X)$ in a Taylor series, whereby $u(x + \Delta X) = u(x) + u'(x)\Delta X + \frac{1}{2}u''(x)(\Delta X)^2 + o(\Delta X)^2$. Then,

$$\begin{aligned} u(x) &= E[u(x + \Delta X)] \\ &= u(x) + u'(x)E[\Delta X] + \frac{1}{2}u''(x)E\left[(\Delta X)^2\right] + E[o(\Delta t)]. \end{aligned}$$

We use (8.36), (8.38), and (8.39) to evaluate the moments of ΔX, obtaining

$$u(x) = u(x) + u'(x)\mu\Delta t + \frac{1}{2}u''(x)\sigma^2\Delta t + o(\Delta t),$$

which, after subtracting $u(x)$, dividing by Δt, and letting $\Delta t \to 0$, becomes the differential equation

$$0 = \mu u'(x) + \frac{1}{2}\sigma^2 u''(x) \quad \text{for} \quad a < x < b. \tag{8.41}$$

The solution to (8.41) is

$$u(x) = A\mathrm{e}^{-2\mu x/\sigma^2} + B,$$

where A and B are constants of integration. These constants are determined by the conditions $u(a) = 0$ and $u(b) = 1$. In words, the probability of exiting at b if the process starts at a is zero, while the probability of exiting at b if the process starts at b is one. When these conditions are used to determine A and B, then (8.40) results.

Example Suppose that the fluctuations in the price of a share of stock in a certain company are well described by a Brownian motion with drift $\mu = 1/10$ and variance $\sigma^2 = 4$. A speculator buys a share of this stock at a price of \$100 and will sell if ever the price rises to \$110 (a profit) or drops to \$95 (a loss). What is the probability that the speculator sells at a profit? We apply (8.40) with $a = 95$, $x = 100$, $b = 110$, and $2\mu/\sigma^2 = 2(0.1)/4 = 1/20$. Then,

$$\Pr\{\text{Sell at profit}\} = \frac{\mathrm{e}^{-100/20} - \mathrm{e}^{-95/20}}{\mathrm{e}^{-110/20} - \mathrm{e}^{-95/20}} = 0.419.$$

The Mean Time to Exit an Interval

Using another infinitesimal first step analysis, the mean time to exit an interval may be determined for Brownian motion with drift.

Theorem 8.2. *For a Brownian motion with drift parameter μ and variance parameter σ^2, and $a < x < b$,*

$$E[T_{ab}|X(0) = x] = \frac{1}{\mu}[u(x)(b-a) - (x-a)], \tag{8.42}$$

where $u(x)$ is given in (8.40).

Proof. Let $v(x) = E[T_{ab}|X(0) = x]$. As in the proof of Theorem 8.1, we will assume (1) that $v(x)$ is twice continuously differentiable, and (2) that we can choose a time increment Δt so small that exiting the interval (a, b) prior to time Δt can be neglected. With these provisos, after time Δt the Brownian motion will be at the position $X(0) + \Delta X = x + \Delta X$, and the conditional mean time to exit the interval is now $\Delta t + v(x + \Delta X)$. Invoking the law of total probability, it must be that $v(x) = E[T|X(0) = x] = E[\Delta t + E\{T - \Delta t|X(0) = x, X(\Delta t) = x + \Delta X\}|X(0) = x] = \Delta t + E[v(x + \Delta X)]$, where $a < x < b$. ■

The next step is to expand $v(x + \Delta X)$ in a Taylor series, whereby $v(x + \Delta X) = v(x) + v'(x)\Delta X + \frac{1}{2}v''(x)(\Delta X)^2 + o(\Delta X)^2$. Then,

$$\begin{aligned} v(\mathrm{x}) &= \Delta t + E[v(x + \Delta X)] \\ &= \Delta t + v(x) + v'(x)E[\Delta X] + \frac{1}{2}v''(x)E\left[(\Delta X)^2\right] + E\left[o(\Delta X)^2\right]. \end{aligned}$$

We use (8.36), (8.38), and (8.39) to evaluate the moments of ΔX, obtaining

$$v(x) = \Delta t + v(x) + v'(x)\mu\Delta t + \frac{1}{2}v''(x)\sigma^2\Delta t + o(\Delta t),$$

which, after subtracting $v(x)$, dividing by Δt, and letting $\Delta t \to 0$, becomes the differential equation

$$-1 = \mu v'(x) + \frac{1}{2}\sigma^2 v''(x) \quad \text{for} \quad a < x < b. \tag{8.43}$$

Since it takes no time to reach the boundary if the process starts at the boundary, the conditions are $v(a) = v(b) = 0$. Subject to these conditions, the solution to (8.43) is uniquely given by (8.42), as is easily verified (Problem 8.4.1).

Example *A Sequential Decision Procedure* A Brownian motion $X(t)$ either (1) has drift $\mu = +\frac{1}{2}\delta > 0$, or (2) has drift $\mu = -\frac{1}{2}\delta < 0$, and it is desired to determine which is the case by observing the process. The process will be monitored until it first reaches the level $b > 0$, in which case we will decide that the drift is $\mu = +\frac{1}{2}\delta$, or until it first drops to the level $a < 0$, which occurrence will cause us to decide in favor of $\mu = -\frac{1}{2}\delta$. This decision procedure is, of course, open to error, but we can evaluate these error probabilities and choose a and b so as to keep the error probabilities acceptably small. We have

$$\begin{aligned} \alpha &= \Pr\left\{\text{Decide } \mu = +\frac{1}{2}\delta|\mu = -\frac{1}{2}\delta\right\} \\ &= \Pr\left\{X(T) = b|E[X(t)] = -\frac{1}{2}\delta t\right\} \\ &= \frac{1 - \mathrm{e}^{+\delta a/\sigma^2}}{\mathrm{e}^{+\delta b/\sigma^2} - \mathrm{e}^{+\delta a/\sigma^2}}, \quad \text{(using (8.40))} \end{aligned} \tag{8.44}$$

and

$$\begin{aligned}1-\beta &= \Pr\left\{\text{Decide } \mu = -\frac{1}{2}\delta | \mu = +\frac{1}{2}\delta\right\}\\ &= \Pr\left\{X(T) = b | E[X(t)] = +\frac{1}{2}\delta t\right\} \\ &= \frac{1-\mathrm{e}^{-\delta a/\sigma^2}}{\mathrm{e}^{-\delta b/\sigma^2} - \mathrm{e}^{-\delta a/\sigma^2}}.\end{aligned} \tag{8.45}$$

If acceptable levels of the error probabilities α and β are prescribed, then we can solve in (8.44) and (8.45) to determine the boundaries to be used in the decision procedure. The reader should verify that these boundaries are

$$a = -\frac{\sigma^2}{\delta}\log\left(\frac{1-\alpha}{\beta}\right), \quad \text{and} \quad b = \frac{\sigma^2}{\delta}\log\left(\frac{1-\beta}{\alpha}\right). \tag{8.46}$$

For a numerical example, if $\sigma^2 = 4$ and we are attempting to decide between $\mu = -\frac{1}{2}$ and $\mu = +\frac{1}{2}$, and the acceptable error probabilities are chosen to be $\alpha = 0.05$ and $\beta = 0.10$, then the decision boundaries that should be used are $a = -4\log(0.95/0.10) = -9.01$, and $b = 4\log(0.90/0.05) = 11.56$.

In the above procedure for deciding the drift of a Brownian motion, the observation duration until a decision is reached will be a random variable whose mean will depend upon the true value of the drift. Using (8.42) with $x = 0$ and μ replaced by $\pm\frac{1}{2}\delta$ gives us the mean observation interval, as a function of the true mean μ:

$$E\left[T|\mu = -\frac{1}{2}\delta\right] = 2\left(\frac{\sigma}{\delta}\right)^2\left[(1-\alpha)\log\left(\frac{1-\alpha}{\beta}\right) - \alpha\log\left(\frac{1-\beta}{\alpha}\right)\right]$$

and

$$E\left[T|\mu = +\frac{1}{2}\delta\right] = 2\left(\frac{\sigma}{\delta}\right)^2\left[(1-\beta)\log\left(\frac{1-\beta}{\alpha}\right) - \beta\log\left(\frac{1-\alpha}{\beta}\right)\right].$$

We have developed a sequential decision procedure for evaluating the drift of a Brownian motion. However, invoking the invariance principle leads us to believe that similar results should maintain, at least approximately, in analogous situations in which the Brownian motion is replaced by a partial sum process of independent and identically distributed summands. The result is known as *Wald's approximation* for his celebrated sequential probability ratio test of a statistical hypothesis.

The Maximum of a Brownian Motion with Negative Drift

Consider a Brownian motion with drift $\{X(t)\}$, where the drift parameter μ is negative. Over time, such a process will tend toward ever lower values, and its maximum $M = \max\{X(t) - X(0); t \geq 0\}$ will be a well-defined and finite random variable.

Theorem 8.1 will enable us to show that M has an exponential distribution with parameter $2|\mu|/\sigma^2$. To see this, let us suppose that $X(0) = 0$ and that $a < 0 < b$ are constants. Then, Theorem 8.1 states that

$$\Pr\{X(T_{ab}) = b|X(0) = x\} = \frac{1 - e^{-2\mu a/\sigma^2}}{e^{-2\mu b/\sigma^2} - e^{-2\mu a/\sigma^2}}, \tag{8.47}$$

where T_{ab} is the random time at which the process first reaches $a < 0$ or $b > 0$. That is, the probability that the Brownian motion reaches $b > 0$ before it ever drops to $a < 0$ is given by the right side of (8.47). Because both $\mu < 0$ and $a < 0$, then $a\mu > 0$ and

$$e^{-2\mu a/\sigma^2} = e^{-2|\mu a|/\sigma^2} \to 0 \quad \text{as} \quad a \to -\infty, \quad \text{and then}$$

$$\lim_{a\to-\infty} \Pr\{X(T_{ab}) = b\} = \frac{1-0}{e^{-2\mu b/\sigma^2} - 0} = e^{-2|\mu|b/\sigma^2}.$$

But as $a \to -\infty$ the left side of (8.47) becomes the probability that the process ever reaches the point b, i.e., the probability that the maximum M of the process ever exceeds b. We have deduced, then, the desired exponential distribution

$$\Pr\{M > b\} = e^{-2|\mu b/\sigma^2}, \quad b > 0. \tag{8.48}$$

8.4.2 Geometric Brownian Motion

A stochastic process $\{Z(t); t \geq 0\}$ is called a *geometric Brownian motion* with drift parameter α if $X(t) = \log Z(t)$ is a Brownian motion with drift $\mu = \alpha - \frac{1}{2}\sigma^2$ and variance parameter σ^2. Equivalently, $Z(t)$ is geometric Brownian motion starting from $Z(0) = z$ if

$$Z(t) = ze^{X(t)} = ze^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B(t)}, \tag{8.49}$$

where $B(t)$ is a standard Brownian motion starting from $B(0) = 0$.

Modern mathematical economists usually prefer geometric Brownian motion over Brownian motion as a model for prices of assets, say shares of stock, that are traded in a perfect market. Such prices are nonnegative and exhibit random fluctuations about a long-term exponential decay or growth curve. Both of these properties are possessed by geometric Brownian motion, but not by Brownian motion itself. More importantly, if $t_0 < t_1 < \cdots < t_n$ are time points, then the successive ratios

$$\frac{Z(t_1)}{Z(t_0)}, \frac{Z(t_2)}{Z(t_1)}, \ldots, \frac{Z(t_n)}{Z_{(t_{n-1}})}$$

are independent random variables, so that crudely speaking, the percentage changes over nonoverlapping time intervals are independent.

We turn to determining the mean and variance of geometric Brownian motion. Let ξ be a normally distributed random variable with mean zero and variance one. We begin by establishing the formula

$$E\left[e^{\lambda\xi}\right] = e^{\frac{1}{2}\lambda^2}, \quad -\infty < \lambda < \infty,$$

which results immediately from

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-\lambda)^2} du \quad \text{(area under normal density)} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(u^2-2\lambda u+\lambda^2\right)} du \\
&= e^{-\frac{1}{2}\lambda^2} \int_{-\infty}^{\infty} e^{\lambda u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \\
&= e^{-\frac{1}{2}\lambda^2} E\left[e^{\lambda\xi}\right].
\end{aligned}$$

To obtain the mean of geometric Brownian motion $Z(t) = z e^{X(t)} = z e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B(t)}$, we use the fact that $\xi = B(t)/\sqrt{t}$ is normally distributed with mean zero and variance one, whence

$$\begin{aligned}
E[Z(t)|Z(0) = z] &= zE\left[e^{\left(\alpha-\frac{1}{2}\sigma^2\right)t+\sigma B(t)}\right] \\
&= z e^{\left(\alpha-\frac{1}{2}\sigma^2\right)^t} E\left[e^{\sigma\sqrt{t}\xi}\right] \quad (\xi = B(t)/\sqrt{t}) \qquad (8.50) \\
&= z e^{\left(\alpha-\frac{1}{2}\sigma^2\right)t} e^{\frac{1}{2}\sigma^2 t} = z e^{\alpha t}.
\end{aligned}$$

Equation (8.50) has interesting economic implications in the case where α is positive but small relative to the variance parameter σ^2. On the one hand, if α is positive, then the mean $E[Z(t)] = z\exp(\alpha t) \to \infty$ as $t \to \infty$. On the other hand, if α is positive but $\alpha < \frac{1}{2}\sigma^2$, then $\alpha - \frac{1}{2}\sigma^2 < 0$, and $X(t) = \left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B(t)$ is drifting in the negative direction. As a consequence of the law of large numbers, it can be shown that $X(t) \to -\infty$ as $t \to \infty$ under these circumstances, so that $Z(t) = z\exp[X(t)] \to \exp(-\infty) = 0$. The geometric Brownian motion process is drifting ever closer to zero, while simultaneously, its mean or expected value is continually increasing! Here is yet another stochastic model in which the mean value function is entirely misleading as a sole description of the process.

The variance of the geometric Brownian motion is derived in much the same manner as the mean. First

$$
\begin{aligned}
E\left[Z(t)^2|Z(0)=z\right] &= z^2E\left[e^{2X(t)}\right] = z^2E\left[e^{2\left(\alpha-\frac{1}{2}\sigma^2\right)t+2\sigma B(t)}\right] \\
&= z^2e^{2\left(\alpha+\frac{1}{2}\sigma^2\right)t} \quad \text{(as in (8.50))},
\end{aligned}
$$

and then

$$
\begin{aligned}
\mathrm{Var}[Z(t)] &= E\left[Z(t)^2\right] - \{E[Z(t)]\}^2 \\
&= z^2e^{2\left(\alpha+\frac{1}{2}\sigma^2\right)t} - z^2e^{2\alpha t} \qquad (8.51) \\
&= z^2e^{2\alpha t}\left(e^{\sigma^2 t}-1\right).
\end{aligned}
$$

Because of their close relation as expressed in the definition (8.49), many results for Brownian motion can be directly translated into analogous results for geometric Brownian motion. For example, let us translate the gambler's ruin probability in Theorem 8.1. For $A < 1$ and $B > 1$, define

$$
T = T_{A,B} = \min\left\{t \geq 0; \frac{Z(t)}{Z(0)} = A \text{ or } \frac{Z(t)}{Z(0)} = B\right\}.
$$

Theorem 8.3. *For a geometric Brownian motion with drift parameter α and variance parameter σ^2, and $A < 1 < B$,*

$$
\Pr\left\{\frac{Z(T)}{Z(0)} = B\right\} = \frac{1-A^{1-2\alpha/\sigma^2}}{B^{1-2\alpha/\sigma^2}-A^{1-2\alpha/\sigma^2}}. \qquad (8.52)
$$

Example Suppose that the fluctuations in the price of a share of stock in a certain company are well described by a geometric Brownian motion with drift $\alpha = 1/10$ and variance $\sigma^2 = 4$. A speculator buys a share of this stock at a price of \$100 and will sell if ever the price rises to \$110 (a profit) or drops to \$95 (a loss). What is the probability that the speculator sells at a profit? We apply (8.52) with $\mathrm{A} = 0.95$, $\mathrm{B} = 1.10$, and $1 - 2\alpha/\sigma^2 = 1 - 2(0.1)/4 = 0.95$. Then,

$$
\Pr\{\text{Sell at profit}\} = \frac{1-0.95^{0.95}}{1.10^{0.95}-0.95^{0.95}} = 0.3342.
$$

Example *The Black-Scholes Option Pricing Formula* A *call*, or *warrant*, is an option entitling the holder to buy a block of shares in a given company at a specified price at any time during a stated interval. Thus, the call listed in the financial section of the newspaper as

Hewlett Aug \$60 \$6

means that for a price of \$6 per share, one may purchase the privilege (option) of buying the stock of Hewlett-Packard at a price of \$60 per share at any time between now and August (by convention, always the third Friday of the month). The \$60 figure is called the *striking price*. Since the most recent closing price of Hewlett was \$59, the option of choosing when to buy, or not to buy at all, carries a *premium* of $\$7 = \$60 + \$6 - \$59$ over a direct purchase of the stock today.

Should the price of Hewlett rise to, say, \$70 between now and the third Friday of August, the owner of such an option could exercise it, buying at the striking price of \$60 and immediately selling at the then current market price of \$70 for a \$10 profit, less, of course, the \$6 cost of the option itself. On the other hand, should the price of Hewlett fall, the option owner's loss is limited to his \$6 cost of the option. Note that the seller (technically called the "writer") of the option has a profit limited to the \$6 that he receives for the option but could experience a huge loss should the price of Hewlett soar, say to \$100. The writer would then either have to give up his own Hewlett shares or buy them at \$100 on the open market in order to fulfill his obligation to sell them to the option holder at \$60.

What should such an option be worth? Is \$6 for this privilege a fair price? While early researchers had studied these questions using a geometric Brownian motion model for the price fluctuations of the stock, they all assumed that the option should yield a higher mean return than the mean return from the stock itself because of the unlimited potential risk to the option writer. This assumption of a higher return was shown to be false in 1973 when Fisher Black, a financial consultant with a Ph.D. in applied mathematics, and Myron Scholes, an assistant professor in finance at MIT, published an entirely new and innovative analysis. In an idealized setting that included no transaction costs and an ability to borrow or lend limitless amounts of capital at the same fixed interest rate, they showed that an owner, or a writer, of a call option could simultaneously buy or sell the underlying stock ("program trading") in such a way as to exactly match the returns of the option. Having available two investment opportunities with exactly the same return effectively eliminates all risk, or randomness, by allowing an investor to buy one while selling the other. The implications of their result are many. First, since writing an option potentially carries no risk, its return must be the same as that for other riskless investments in the economy. Otherwise, limitless profit opportunities bearing no risk would arise. Second, since owning an option carries no risk, one should not exercise it early, but hold it until its expiration date, when, if the market price exceeds the striking price, it should be exercised, and otherwise not. These two implications then lead to a third, a formula that established the worth, or value, of the option.

The Black-Scholes paper spawned hundreds, if not thousands, of further academic studies. At the same time, their valuation formula quickly invaded the financial world, where soon virtually all option trades were taking place at or near their Black-Scholes value. It is remarkable that the valuation formula was adopted so quickly in the real world in spite of the esoteric nature of its derivation and the ideal world of its assumptions.

In order to present the Black-Scholes formula, we need some notation. Let $S(t)$ be the price at time t of a share of the stock under study. We assume that $S(t)$ is described by a geometric Brownian motion with drift parameter α and variance parameter σ^2. Let $F(z, \tau)$ be the value of an option, where z is the current price of the stock and τ is the time remaining until expiration. Let a be the striking price. When $\tau = 0$, and there is no time remaining, one exercises the option for a profit of $z - a$ if $z > a$ (market price greater than striking price) and does not exercise the option, but lets it lapse, or expire, if $z \le a$. This leads to the condition

$$F(z, 0) = (z - a)^+ = \max\{z - a, 0\}.$$

The Black-Scholes analysis resulted in the valuation

$$F(z, \tau) = \mathrm{e}^{-r\tau} E\left[(Z(\tau) - a)^+ | Z(0) = z\right], \tag{8.53}$$

where r is the return rate for secure, or riskless, investments in the economy, and where $Z(t)$ is a second geometric Brownian motion having drift parameter r and variance parameter σ^2. Looking at (8.53), the careful reader will wonder whether we have made a mistake. No, the worth of the option does *not* depend on the drift parameter α of the underlying stock.

In order to put the valuation formula into a useful form, we write

$$Z(\tau) = z\mathrm{e}^{\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}\xi}, \quad \xi = B(\tau)/\sqrt{\tau}, \tag{8.54}$$

and observe that

$$z\mathrm{e}^{\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}\xi} > a$$

is the same as

$$\xi > v_0 = \frac{\log(a/z) - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}. \tag{8.55}$$

Then,

$$\begin{aligned}
\mathrm{e}^{r\tau} F(z, \tau) &= E\left[(Z(\tau) - a)^+ | Z(0) = z\right] \\
&= E\left[\left(z\mathrm{e}^{\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}\xi} - a\right)^+\right] \\
&= \int_{v_0}^{\infty} \left[z\mathrm{e}^{\left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}v} - a\right] \phi(v)\mathrm{d}v. \\
&= z\mathrm{e}^{\left(r - \frac{1}{2}\sigma^2\right)\tau} \int_{v_0}^{\infty} \mathrm{e}^{\sigma\sqrt{\tau}v} \phi(\mathrm{v})\mathrm{d}v - a \int_{v_0}^{\infty} \phi(v)\mathrm{d}v.
\end{aligned}$$

Completing the square in the form

$$-\frac{1}{2}v^2 + \sigma\sqrt{\tau}v = -\frac{1}{2}\left[\left(v - \sigma\sqrt{\tau}\right)^2 - \sigma^2\tau\right]$$

shows that

$$e^{\sigma\sqrt{\tau}v}\phi(v) = e^{\frac{1}{2}\sigma^2\tau}\phi(v-\sigma\sqrt{\tau}),$$

whence

$$\begin{aligned} e^{r\tau}F(z,\tau) &= ze^{\left(r-\frac{1}{2}\sigma^2\right)\tau}e^{+\frac{1}{2}\sigma^2\tau}\int_{v_0}^{\infty}\phi(v-\sigma\sqrt{\tau})\mathrm{d}v - a[1-\Phi(v_0)] \\ &= ze^{r\tau}[1-\Phi(v_0-\sigma\sqrt{\tau})] - a[1-\Phi(v_0)]. \end{aligned}$$

Finally, note that

$$v_0 - \sigma\sqrt{\tau} = \frac{\log(a/z) - \left(r+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

and that

$$1-\Phi(x) = \Phi(-x) \quad \text{and} \quad \log(a/z) = -\log(z/a)$$

to get, after multiplying by $e^{-r\tau}$, the end result

$$\begin{aligned} F(z,\tau) = z\Phi&\left(\frac{\log(z/a)+\left(r+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\ &- ae^{-r\tau}\Phi\left(\frac{\log(z/a)+\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right). \end{aligned} \tag{8.56}$$

Equation (8.56) is the Black-Scholes valuation formula. Four of the five factors that go into it are easily and objectively evaluated: The current market prize z, the striking price a, the time τ until the option expires, and the rate r of return from secure investments such as short-term government securities. It is the fifth factor, σ, sometimes called the *volatility*, that presents problems. It is, of course, possible to estimate this parameter based on past records of price movements. However, it should be emphasized that it is the volatility in the future that will affect the profitability of the option, and when economic conditions are changing, past history may not accurately indicate the future. One way around this difficulty is to work backwards and use the Black-Scholes formula to impute a volatility from an existing market price of the option. For example, the Hewlett-Packard call option that expires in August, six months or, $\tau = \frac{1}{2}$ year, in the future, with a current price of Hewlett-Packard stock of \$59, a striking price of \$60, and secure investments returning about $r = 0.05$, a volatility of $\sigma = 0.35$ is consistent with the listed option price of \$6. (When $\sigma = 0.35$ is used in the Black-Scholes formula, the resulting valuation is \$6.03.) A volatility derived in this manner is called an *imputed* or *implied* volatility. Someone who believes that the future will be more variable might regard the option at \$6 as a good buy. Someone who believes the future to be less variable than the imputed volatility of $\sigma = 0.35$ might be inclined to offer a Hewlett-Packard option at \$6.

Striking Price a	Time to Expiration (Years) τ	Offered Price	Black-Scholes Valuation $F(z,\tau)$
130	1/12	\$17.00	\$17.45
130	2/12	19.25	18.87
135	1/12	13.50	13.09
135	2/12	15.13	14.92
140	1/12	8.50	9.26
140	2/12	12.00	11.46
145	1/12	5.50	6.14
145	2/12	9.13	8.52
145	5/12	13.63	13.51
150	1/12	3.13	3.80
150	2/12	6.38	6.14
155	1/12	1.63	2.18
155	2/12	4.00	4.28
155	5/12	9.75	9.05

The above table compares actual offering prices on February 26, 1997, for options in IBM stock with their Black-Scholes valuation using (8.56). The current market price of IBM stock is \$146.50, and, in all cases, the same volatility $\sigma = 0.30$ was used.

The agreement between the actual option prices and their Black-Scholes valuations seems quite good.

Exercises

8.4.1 A Brownian motion $\{X(t)\}$ has parameters $\mu = -0.1$ and $\sigma = 2$. What is the probability that the process is above $y = 9$ at time $t = 4$, given that it starts at $x = 2.82$?

8.4.2 A Brownian motion $\{X(t)\}$ has parameters $\mu = 0.1$ and $\sigma = 2$. Evaluate the probability of exiting the interval $(a, b]$ at the point b starting from $X(0) = 0$ for $b = 1, 10,$ and 100 and $a = -b$. Why do the probabilities change when a/b is the same in all cases?

8.4.3 A Brownian motion $\{X(t)\}$ has parameters $\mu = 0.1$ and $\sigma = 2$. Evaluate the mean time to exit the interval $(a, b]$ from X(0) $= 0$ for $b = 1, 10,$ and 100 and $a = -b$. Can you guess how this mean time varies with b for b large?

8.4.4 A Brownian motion $X(t)$ either (1) has drift $\mu = +\frac{1}{2}\delta > 0$, or (2) has drift $\mu = -\frac{1}{2}\delta < 0$, and it is desired to determine which is the case by observing the process for a *fixed* duration τ. If $X(\tau) > 0$, then the decision will be that $\mu = +\frac{1}{2}\delta$; If $X(\tau) \leq 0$, then $\mu = -\frac{1}{2}\delta$ will be stated. What should be the length τ of the observation period if the design error probabilities are set at $\alpha = \beta = 0.05$? Use $\delta = 1$ and $\sigma = 2$. Compare this fixed duration with the average duration of the sequential decision plan in the example of Section 8.4.1.

8.4.5 Suppose that the fluctuations in the price of a share of stock in a certain company are well described by a geometric Brownian motion with drift $\alpha = -0.1$ and variance $\sigma^2 = 4$. A speculator buys a share of this stock at a price of \$100 and will sell if ever the price rises to \$110 (a profit) or drops to \$95 (a loss). What is the probability that the speculator sells at a profit?

8.4.6 Let ξ be a standard normal random variable.

(a) For an arbitrary constant a, show that

$$E\left[(\xi - a)^+\right] = \phi(a) - a[1 - \Phi(a)].$$

(b) Let X be normally distributed with mean μ and variance σ^2. Show that

$$E\left[(X - b)^+\right] = \sigma\left\{\phi\left(\frac{b-\mu}{\sigma}\right) - \left(\frac{b-\mu}{\sigma}\right)\left[1 - \Phi\left(\frac{b-\mu}{\sigma}\right)\right]\right\}.$$

Problems

8.4.1 What is the probability that a standard Brownian motion $\{B(t)\}$ ever crosses the line $a + bt (a > 0, b > 0)$?

8.4.2 Show that

$$\Pr\left\{\max_{t\geq 0} \frac{b + B(t)}{1+t} > a\right\} = \mathrm{e}^{-2a(a-b)}, \quad a > 0, b < a.$$

8.4.3 If $B^0(s), 0 < s < 1$, is a Brownian bridge process, then

$$B(t) = (1+t)B^0\left(\frac{t}{1+t}\right)$$

is a standard Brownian motion. Use this representation and the result of Problem 8.4.2 to show that for a Brownian bridge $B^0(t)$,

$$\Pr\left\{\max_{0\leq u\leq 1} B^0(u) > a\right\} = \mathrm{e}^{-2a^2}.$$

8.4.4 A Brownian motion $X(t)$ either (1) has drift $\mu = \mu_0$, or (2) has drift $\mu = \mu_1$, where $\mu_0 < \mu_1$ are known constants. It is desired to determine which is the case by observing the process. Derive a sequential decision procedure that meets prespecified error probabilities α and β.

Hint: Base your decision on the process $X'(t) = X(t) - \frac{1}{2}(\mu_0 + \mu_1)$.

8.4.5 Change a Brownian motion with drift $X(t)$ into an absorbed Brownian motion with drift $X^A(t)$ by defining

$$X^A(t) = \begin{cases} X(t), & \text{for } t < \tau, \\ 0, & \text{for } t \geq \tau, \end{cases}$$

where

$$\tau = \min\{t \geq 0; X(t) = 0\}.$$

(We suppose that $X(0) = x > 0$ and that $\mu < 0$, so that absorption is sure to occur eventually.) What is the probability that the absorbed Brownian motion ever reaches the height $b > x$?

8.4.6 What is the probability that a geometric Brownian motion with drift parameter $\alpha = 0$ ever rises to more than twice its initial value? (You buy stock whose fluctuations are described by a geometric Brownian motion with $\alpha = 0$. What are your chances to double your money?)

8.4.7 A call option is said to be "in the money" if the market price of the stock is higher than the striking price. Suppose that the stock follows a geometric Brownian motion with drift α, variance σ^2, and has a current market price of z What is the probability that the option is in the money at the expiration time τ? The striking price is a.

8.4.8 Verify the Hewlett-Packard option valuation of \$6.03 stated in the text when $\tau = \frac{1}{2}, z = \$59, a = 60, r = 0.05,$ and $\sigma = 0.35$. What is the Black-Scholes valuation if $\sigma = 0.30$?

8.4.9 Let τ be the first time that a standard Brownian motion $B(t)$ starting from $B(0) = x > 0$ reaches zero. Let λ be a positive constant. Show that

$$w(x) = E\left[e^{-\lambda\tau} | B(0) = x\right] = e^{-\sqrt{2\lambda}x}.$$

Hint: Develop an appropriate differential equation by instituting an infinitesimal first step analysis according to

$$w(x) = E\left[E\left\{e^{-\lambda\tau} | B(\Delta t)\right\} | B(0) = x\right] = E\left[e^{-\lambda\Delta t} w(x + \Delta B)\right].$$

8.4.10 Let $t_0 = 0 < t_1 < t_2 < \cdots$ be time points, and define $X_n = Z(t_n)\exp(-rt_n)$, where $Z(t)$ is geometric Brownian motion with drift parameters r and variance parameter σ^2 (see the geometric Brownian motion in the Black-Scholes formula (8.53)). Show that $\{X_n\}$ is a martingale.

8.5 The Ornstein–Uhlenbeck Process*

The Ornstein–Uhlenbeck process $\{V(t); t \geq 0\}$ has two parameters, a drift coefficient $\beta > 0$ and a diffusion parameter σ^2. The process, starting from $V(0) = v$, is defined in

* This section contains material of a more specialized nature.

terms of a standard Brownian motion $\{B(t)\}$ by scale changes in both space and time:

$$V(t) = v\mathrm{e}^{-\beta t} + \frac{\sigma \mathrm{e}^{-\beta t}}{\sqrt{2\beta}} B\left(\mathrm{e}^{2\beta t} - 1\right), \quad \text{for } t \geq 0. \tag{8.57}$$

The first term on the right of (8.57) describes an exponentially decreasing trend towards the origin. The second term represents the fluctuations about this trend in terms of a rescaled Brownian motion. The Ornstein–Uhlenbeck process is another example of a continuous-state-space, continuous-time Markov process having continuous paths, inheriting these properties from the Brownian motion in the representation (8.57). It is a Gaussian process (see the discussion in Section 8.1.4), and (8.57) easily shows its mean and variance to be

$$E[V(t)|V(0) = v] = v\mathrm{e}^{-\beta t}, \tag{8.58}$$

and

$$\begin{aligned}\operatorname{Var}[V(t)|V(0) = x] &= \mathrm{e}^{-2\beta t}\frac{\sigma^2}{2\beta}\operatorname{Var}\left[B\left(\mathrm{e}^{2\beta t} - 1\right)\right] \\ &= \sigma^2\left(\frac{1 - \mathrm{e}^{-2\beta t}}{2\beta}\right).\end{aligned} \tag{8.59}$$

Knowledge of the mean and variance of a normally distributed random variable allows its cumulative distribution function to be written in terms of the standard normal distribution (8.6), and by this means we can immediately express the transition distribution for the Ornstein–Uhlenbeck process as

$$\Pr\{V(t) \leq y|V(0) = x\} = \Phi\left(\frac{\sqrt{2\beta}\left(y - x\mathrm{e}^{-\beta t}\right)}{\sigma\sqrt{1 - \mathrm{e}^{-2\beta t}}}\right). \tag{8.60}$$

The Covariance Function

Suppose that $0 < u < s$, and that $V(0) = x$. Upon subtracting the mean as given by (8.58), we obtain

$$\begin{aligned}\operatorname{Cov}[V(u), V(s)] &= E\left[\left\{V(u) - x\mathrm{e}^{-\beta u}\right\}\left\{V(s) - x\mathrm{e}^{-\beta s}\right\}\right] \\ &= \frac{\sigma^2}{2\beta}\mathrm{e}^{-\beta(u+s)}E\left[\left\{B\left(\mathrm{e}^{2\beta u} - 1\right)\right\}\left\{B\left(\mathrm{e}^{2\beta s} - 1\right)\right\}\right] \\ &= \frac{\sigma^2}{2\beta}\mathrm{e}^{-\beta(u+s)}\left(\mathrm{e}^{2\beta u} - 1\right) \\ &= \frac{\sigma^2}{2\beta}\left(\mathrm{e}^{-\beta(s-u)} - \mathrm{e}^{-\beta(s+u)}\right).\end{aligned} \tag{8.61}$$

8.5.1 A Second Approach to Physical Brownian Motion

The path that we have taken to introduce the Ornstein–Uhlenbeck process is not faithful to the way in which the process came about. To begin an explanation, let us recognize that all models of physical phenomena have deficiencies, and the Brownian motion stochastic process as a model for the Brownian motion of a particle is no exception. If $B(t)$ is the position of a pollen grain at time t and if this position is changing over time, then the pollen grain must have a velocity. Velocity is the infinitesimal change in position over infinitesimal time, and where $B(t)$ is the position of the pollen grain at time t, the velocity of the grain would be the derivative $\mathrm{d}B(t)/\mathrm{d}t$. *But while the paths of the Brownian motion stochastic process are continuous, they are not differentiable*. This remarkable statement is difficult to comprehend. Indeed, many elementary calculus explanations implicitly tend to assume that all continuous functions are differentiable, and if we were to be asked to find an example of one that was not, we might consider it quite a challenge. Yet each path of a continuous Brownian motion stochastic process is (with probability one) differentiable at no point. We have encountered yet another intriguing facet of stochastic processes that we cannot treat in full detail but must leave for future study. We will attempt some motivation, however. Recall that the variance of the Brownian increment ΔB is Δt. But variations in the normal distribution are not scaled in terms of the variance, but in terms of its square root, the standard deviation, so that the Brownian increment ΔB is roughly on the order of $\sqrt{\Delta t}$, and the approximate derivative

$$\frac{\Delta B}{\Delta t} = \frac{\Delta B}{\sqrt{\Delta t}} \cdot \frac{1}{\sqrt{\Delta t}}$$

is roughly on the order of $1/\sqrt{\Delta t}$. This, of course, becomes infinite as $\Delta t \to 0$, which suggests that a derivative of Brownian motion, were it to exist, could only take the values $\pm\infty$. As a consequence, the Brownian path cannot have a derivative. The reader can see from our attempt at explanation that the topic is well beyond the scope of an introductory text.

Although its movements may be erratic, a pollen grain, being a physical object of positive mass, must have a velocity, and the Ornstein–Uhlenbeck process arose as an attempt to model this velocity directly. Two factors are postulated to affect the particle's velocity over a small time interval. First, the frictional resistance or viscosity of the surrounding medium is assumed to reduce the magnitude of the velocity by a deterministic proportional amount, the constant of proportionality being $\beta > 0$. Second, there are random changes in velocity caused by collisions with neighboring molecules, the magnitude of these random changes being measured by a variance coefficient σ^2. That is, if $V(t)$ is the velocity at time t, and ΔV is the change in velocity over $(t, t+\Delta t]$, we might express the viscosity factor as

$$E[\Delta V|V(t) = v] = -\beta v \Delta t + o(\Delta t) \tag{8.62}$$

and the random factor by

$$\mathrm{Var}[\Delta V|V(t) = v] = \sigma^2 \Delta t + o(\Delta t). \tag{8.63}$$

The Ornstein–Uhlenbeck process was developed by taking (8.62) and (8.63) together with the Markov property as the postulates, and from them deriving the transition probabilities (8.60). While we have chosen not to follow this path, we will verify that the mean and variance given in (8.58) and (8.59) do satisfy (8.62) and (8.63) over small time increments. Beginning with (8.58) and the Markov property, the first step is

$$E[V(t+\Delta t)|V(t)=v] = v\mathrm{e}^{-\beta\Delta t} = v[1-\beta\Delta t + o(\Delta t)],$$

and then,

$$\begin{aligned} E[\Delta V|V(t)=v] &= E[V(t+\Delta t)|V(t)=v] - v \\ &= -\beta v\Delta t + o(\Delta t), \end{aligned}$$

and over small time intervals the mean change in velocity is the proportional decrease desired in (8.62). For the variance, we have

$$\begin{aligned} \mathrm{Var}[\Delta V|V(t)=v] &= \mathrm{Var}[V(t+\Delta t)|V(t)=v] \\ &= \sigma^2\left(\frac{1-\mathrm{e}^{-2\beta\Delta t}}{2\beta}\right) \\ &= \sigma^2\Delta t + o(\Delta t), \end{aligned}$$

and the variance of the velocity increment behaves as desired in (8.63). In fact, (8.62) and (8.63) together with the Markov property can be taken as the definition of the Ornstein–Uhlenbeck process in much the same way, but involving far deeper analysis, that the infinitesimal postulates of Chapter 5, Section 5.2.1, serve to define the Poisson process.

Example *Tracking Error* Let $V(t)$ be the measurement error of a radar system that is attempting to track a randomly moving target. We assume $V(t)$ to be an Ornstein–Uhlenbeck process. The mean increment $E[\Delta V|V(t)=v] = -\beta v\Delta t + o(\Delta t)$ represents the controller's effort to reduce the current error, while the variance term reflects the unpredictable motion of the target. If $\beta = 0.1, \sigma = 2$, and the system starts on target ($v=0$), the probability that the error is less than one at time $t=1$ is, using (8.60),

$$\begin{aligned} \Pr\{|V(t)|<1\} &= \Phi\left(\frac{\sqrt{2\beta}}{\sigma\sqrt{1-\mathrm{e}^{-2\beta t}}}\right) - \Phi\left(\frac{\sqrt{-2\beta}}{\sigma\sqrt{1-\mathrm{e}^{-2\beta t}}}\right) \\ &= \Phi\left(\frac{1}{\sqrt{20\left(1-\mathrm{e}^{-0.2}\right)}}\right) - \Phi\left(\frac{-1}{\sqrt{20\left(1-\mathrm{e}^{-0.2}\right)}}\right) \\ &= \Phi(0.53) - \Phi(-0.53) = 0.4038. \end{aligned}$$

As time passes, this near-target probability drops to $\Phi(1/\sqrt{20}) - \Phi(-1/\sqrt{20}) = \Phi(0.22) - \Phi(-0.22) = 0.1742$.

Example *Genetic Fluctuations Under Mutation* In Chapter 6, Section 6.4, we introduced a model describing fluctuations in gene frequency in a population of N individuals, each either of gene type **a** or gene type **A**. With $X(t)$ being the number of type **a** individuals at time t, we reasoned that $X(t)$ would be a birth and death process with parameters

$$\lambda_j = \lambda N\left(1 - \frac{j}{N}\right)\left[\frac{j}{N}(1-\gamma_1) + \left(1 - \frac{j}{N}\right)\gamma_2\right]$$

and

$$\mu_j = \lambda N \frac{j}{N}\left[\frac{j}{N}\gamma_1 + \left(1 - \frac{j}{N}\right)(1-\gamma_2)\right].$$

The parameters γ_1 and γ_2 measured the rate of mutation from **a**-type to **A**-type, and **A**-type to **a**-type, respectively. Here we attempt a simplified description of the model when the population size N is large. The steady state fluctuations in the relative gene frequency $X(t)/N$ are centered on the mean

$$\pi = \frac{\gamma_2}{\gamma_1 + \gamma_2}.$$

Accordingly, we define the rescaled and centered process

$$V_N(t) = \sqrt{N}\left(\frac{X(t)}{N} - \pi\right).$$

With

$$\Delta V = V_N(t + \Delta t) - V_N(t), \quad \text{and} \quad \Delta X = X(t + \Delta t) - X(t),$$

we have

$$E[\Delta X | X(t) = j] = \left(\lambda_j - \mu_j\right)\Delta t + o(\Delta t),$$

which becomes, after substitution and simplification,

$$E[\Delta X | X(t) = j] = N\lambda\left[\left(1 - \frac{j}{N}\right)\gamma_2 - \frac{j}{N}\gamma_1\right]\Delta t + o(\Delta t).$$

More tedious calculations show that

$$E\left[\Delta X^2 | X(t) = j\right] = N\lambda\left[\frac{2\gamma_1\gamma_2}{(\gamma_1 + \gamma_2)^2} + o\left(\frac{1}{N}\right)\right]\Delta t.$$

Our next step is to rescale these in terms of v, using

$$\frac{j}{N} = \pi + \frac{v}{\sqrt{N}}.$$

In the rescaled variables,

$$\begin{aligned} E[\Delta V|V_N(t) = v] &= \frac{1}{\sqrt{N}} E\left[\Delta X \left| \frac{X(t)}{N} = \frac{j}{N} = \pi + \frac{v}{\sqrt{N}} \right.\right] \\ &= \lambda\sqrt{N}\left[\left(1 - \pi - \frac{v}{\sqrt{N}}\right)\gamma_2 - \left(\pi + \frac{v}{\sqrt{N}}\right)\gamma_1\right]\Delta t + o(\Delta t) \\ &= -\lambda(\gamma_1 + \gamma_2)v\Delta t + o(\Delta t). \end{aligned}$$

A similar substitution shows that

$$E\left[\Delta V^2|V_N(t) = v\right] = \frac{2\lambda\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2}\Delta t + o(\Delta t).$$

Similar computations show that the higher moments of ΔV are negligible. Since the relations (8.62) and (8.63) serve to characterize the Ornstein–Uhlenbeck process, the evidence is compelling that the rescaled gene processes $\{V_N(t)\}$ will converge in some appropriate sense to an Ornstein–Uhlenbeck process $V(t)$ with

$$\beta = \lambda(\gamma_1 + \gamma_2) \quad \text{and} \quad \sigma^2 = \frac{2\lambda\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2},$$

and

$$X(t) \approx N\pi + \sqrt{N}V(t) \qquad \text{for large } N.$$

This is indeed the case, but it represents another topic that we must leave for future study.

8.5.2 The Position Process

If $V(t)$ is the velocity of the pollen grain, then its position at time t would be

$$S(t) = S(0) + \int_0^t V(u)\mathrm{d}u. \tag{8.64}$$

Because the Ornstein–Uhlenbeck process is continuous, the integral in (8.64) is well defined. The Ornstein–Uhlenbeck process is normally distributed, and so is each approximating sum to the integral in (8.64). It must be, then, that the position process $S(t)$ is normally distributed, and, to describe it, we need only evaluate its mean and covariance functions. To simplify the mathematics without losing any essentials, let us assume that $S(0) = V(0) = 0$. Then,

$$E[S(t)] = E\left[\int_0^t V(s)\mathrm{d}s\right] = \int_0^t E[V(s)]\mathrm{d}s = 0.$$

(The interchange of integral and expectation needs justification. Since the expected value of a sum is always the sum of the expected values, the interchange of expectation with Riemann approximating sums is clearly valid. What is needed is justification in the limit as the approximating sums converge to the integrals.)

$$\begin{aligned}
\mathrm{Var}[S(t)] = E[S(t)^2] &= E\left[\left\{\int_0^t V(s)\mathrm{d}s\right\}^2\right] \\
&= E\left[\left\{\int_0^t V(u)\mathrm{d}u\right\}\left\{\int_0^t V(s)\mathrm{d}s\right\}\right] \\
&= \int_0^t\int_0^t E[V(s)V(u)]\mathrm{d}u\,\mathrm{d}s \\
&= 2\int_0^t\int_0^s E[V(s)V(u)]\,\mathrm{d}u\,\mathrm{d}s \qquad (8.65) \\
&= \frac{\sigma^2}{\beta}\int_0^t\int_0^s \left(\mathrm{e}^{-\beta(s-u)} - \mathrm{e}^{-\beta(s+u)}\right)\mathrm{d}u\,\mathrm{d}s \qquad \text{(Using (8.61))} \\
&= \frac{\sigma^2}{\beta^2}\int_0^t \mathrm{e}^{-\beta s}\left(\mathrm{e}^{\beta s} - 1 - 1 + \mathrm{e}^{-\beta s}\right)\mathrm{d}s \\
&= \frac{\sigma^2}{\beta^2}\left[t - \frac{2}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right) + \frac{1}{2\beta}\left(1 - \mathrm{e}^{-2\beta t}\right)\right].
\end{aligned}$$

This variance behaves like that of a Brownian motion when t is large in the sense that

$$\frac{\mathrm{Var}[S(t)]}{t} \to \frac{\sigma^2}{\beta^2} \quad \text{as} \quad t \to \infty.$$

That is, observed over a long time span, the particle's position as modeled by an Ornstein–Uhlenbeck velocity behaves much like a Brownian motion with variance parameter σ^2/β^2. In this sense, the Ornstein–Uhlenbeck model agrees with the Brownian motion model over long time spans and improves upon it for short durations. Section 8.5.4 offers another approach.

Example *Stock Prices* It is sometimes assumed that the market price of a share of stock follows the position process under an Ornstein–Uhlenbeck velocity. The model is consistent with the Brownian motion model over long time spans. In the short term, the price changes are not independent but have an exponentially decreasing correlation meant to capture some notion of a market momentum. Suppose a call option is to be exercised, if profitable at a striking price of a, at some fixed time t in the future. If $V(0) = 0$ and $S(0) = z$ is the current stock price, then the expected value of the option is

$$E\left[(S(t) - a)^+\right] = \tau\left\{\phi\left(\frac{a-\mu}{\tau}\right) - \left(\frac{a-\mu}{\tau}\right)\left[1 - \Phi\left(\frac{a-\mu}{\tau}\right)\right]\right\}, \tag{8.66}$$

where

$$\mu = z,$$

and

$$\tau^2 = \frac{\sigma^2}{\beta^2}\left[t - \frac{2}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right) + \frac{1}{2\beta}\left(1 - \mathrm{e}^{-2\beta t}\right)\right].$$

Note that μ and τ^2 are the mean and variance of $S(t)$. The derivation is left for Problem 8.5.4.

8.5.3 The Long Run Behavior

It is easily seen from (8.58) and (8.59) that for large values of t, the mean of the Ornstein–Uhlenbeck process converges to zero and the variance to $\sigma^2/2\beta$. This leads to a limiting distribution for the process in which

$$\lim_{t\to\infty} \Pr\{V(t) < y | V(0) = x\} = \Phi\left(\frac{\sqrt{2\beta} y}{\sigma}\right). \tag{8.67}$$

That is, the limiting distribution of the process is normal with mean zero and variance $\sigma^2/(2\beta)$. We now set forth a representation of a *stationary Ornstein–Uhlenbeck process*, a process for which the limiting distribution in (8.67) holds for all finite times as well as in the limit. The stationary Ornstein–Uhlenbeck process $\{V^s(t); -\infty < t < \infty\}$ is represented in terms of a Brownian motion by

$$V^s(t) = \frac{\sigma}{\sqrt{2\beta}}\mathrm{e}^{-\beta t}B\left(\mathrm{e}^{2\beta t}\right), \quad -\infty < t < \infty. \tag{8.68}$$

The stationary Ornstein–Uhlenbeck process is Gaussian (see Section 8.1.4) and has mean zero. The covariance calculation is

$$\begin{aligned}
\Gamma(s,t) &= \mathrm{Cov}\left[V^s(s), V^s(t)\right] \\
&= \frac{\sigma^2}{2\beta}\mathrm{e}^{-\beta(s+t)}\mathrm{Cov}\left[B\left(\mathrm{e}^{2\beta s}\right), B\left(\mathrm{e}^{2\beta t}\right)\right] \\
&= \frac{\sigma^2}{2\beta}\mathrm{e}^{-\beta(s+t)}\min\left\{\mathrm{e}^{2\beta s}, \mathrm{e}^{2\beta t}\right\} \\
&= \frac{\sigma^2}{2\beta}\mathrm{e}^{-\beta|t-s|}.
\end{aligned} \tag{8.69}$$

The stationary Ornstein–Uhlenbeck process is the unique Gaussian process with mean zero and covariance (8.69).

The independence of the Brownian increments implies that the stationary Ornstein–Uhlenbeck process is a Markov process, and it is straightforward to verify that the transition probabilities are given by (8.60).

Example *An Ehrenfest Urn Model in Continuous Time* A single particle switches repeatedly between urn A and urn B. Suppose that the duration it spends in an urn before moving is an exponentially distributed random variable with parameter β, and that all durations are independent. Let $\xi(t) = 1$ if the particle is in urn A at time t, and $\xi(t) = -1$ if in urn B. Then, $\{\xi(t); t \geq 0\}$ is a two-state Markov process in continuous time for which (see Chapter 6, (6.30))

$$\Pr\{\xi(t+s) = 1 | \xi(t) = 1\} = \frac{1}{2} + \frac{1}{2}\mathrm{e}^{-2\beta s}. \tag{8.70}$$

Let us further stipulate that the particle is equally likely to be in either urn at time zero. It follows, then, that it is equally likely to be in either urn at all times, and that therefore, $E[\xi(t)] = 0$ for all t. Using (8.70) and the symmetry of the process, we may derive the covariance. We have

$$\begin{aligned} E[\xi(t)\xi(t+s)] &= \frac{1}{2}\Pr\{\xi(t+s) = 1|\xi(t) = 1\} \\ &\quad + \frac{1}{2}\Pr\{\xi(t+s) = -1|\xi(t) = -1\} \\ &\quad - \frac{1}{2}\Pr\{\xi(t+s) = -1|\xi(t) = 1\} \\ &\quad - \frac{1}{2}\Pr\{\xi(t+s) = 1|\xi(t) = -1\} \\ &= \mathrm{e}^{-2\beta s}. \end{aligned} \tag{8.71}$$

Now consider N of these particles, each alternating between the urns independently of the others, and let $\xi_i(t)$ track the position of the ith particle at time t. The disparity between the numbers of particles in the two urns is measured by

$$S_N(t) = \sum_{i=1}^{N} \xi_i(t).$$

If $S_N(t) = 0$, then the urns contain equal numbers of particles. If $S_N(t) = k$, then there are $(N+k)/2$ particles in urn A. The central limit principle for random functions suggests that

$$V_N(t) = \frac{1}{\sqrt{N}} S_N(t)$$

should, for large N, behave similarly to a Gaussian process with mean zero and covariance $\Gamma(s,t) = \exp\{-2\beta|t-s|\}$. This limiting process is a stationary Ornstein–Uhlenbeck process with $\sigma^2 = 2\beta$. Thus, we have derived the approximation

$$S_N(t) \approx \sqrt{N} V^s(t), \qquad t > 0,$$

for the behavior of this continuous-time urn model when the number of particles is large.

8.5.4 Brownian Measure and Integration*

We state, in the form of a theorem without proof, an exceedingly useful formula for computing certain functionals of Gaussian processes. This theorem provides a tiny glimpse into a vast and rich area of stochastic process theory, and we included it in this elementary text in a blatant attempt to entice the student towards further study.

Theorem 8.4. *Let $g(x)$ be a continuous function and let $\{B(t); t \geq 0\}$ be a standard Brownian motion. For each fixed value of $t > 0$, there exists a random variable*

$$\mathcal{I}(g) = \int_0^t g(x)\mathrm{d}B(x) \tag{8.72}$$

that is the limit of the approximating sums

$$\mathcal{I}_n(g) = \sum_{k=1}^{2^n} g\left(\frac{k}{2^n}t\right)\left[B\left(\frac{k}{2^n}t\right) - B\left(\frac{k-1}{2^n}t\right)\right] \tag{8.73}$$

as $n \to \infty$. The random variable $\mathcal{I}(g)$ is normally distributed with mean zero and variance

$$\mathrm{Var}[\mathcal{I}(g)] = \int_0^t g^2(u)\mathrm{d}u. \tag{8.74}$$

If $f(x)$ is another continuous function of x, then $\mathcal{I}(f)$ and $\mathcal{I}(g)$ have a joint normal distribution with covariance

$$E[\mathcal{I}(f)\mathcal{I}(g)] = \int_0^t f(x)g(x)\mathrm{d}x. \tag{8.75}$$

The proof of the theorem is more tedious than difficult, but it does require knowledge of facts that are not included among our prerequisites. The theorem asserts that a

* This subsection is both more advanced and more abstract than those that have preceded it.

sequence of random variables, the Riemann approximations, converge to another random variable. The usual proof begins with showing that the expected mean square of the difference between distinct Riemann approximations satisfies the Cauchy criterion for convergence, and then goes on from there.

When $g(x)$ is differentiable, then an integration by parts may be validated, which shows that

$$\int_0^t g(x)\mathrm{d}B(x) = g(t)B(t) - \int_0^t B(x)g'(x)\mathrm{d}x, \tag{8.76}$$

and this approach may yield a concrete representation of the integral in certain circumstances. For example, if $g(x) = 1$, then $g'(x) = 0$, and

$$\int_0^t 1\mathrm{d}B(x) = g(t)B(t) - 0 = B(t),$$

as one would hope. When $g(x) = t - x$, then $g'(x) = -1$, and

$$\int_0^t (t-x)\mathrm{d}B(x) = \int_0^t B(x)\mathrm{d}x. \tag{8.77}$$

The process on the right side of (8.77) is called *integrated Brownian motion*. Theorem 8.4, then, asserts that integrated Brownian motion is normally distributed with mean zero and variance

$$\mathrm{Var}\left[\int_0^t B(x)\mathrm{d}x\right] = \int_0^t (t-x)^2\mathrm{d}x = \frac{t^3}{3}.$$

The calculus of the *Brownian integral* of Theorem 8.4 offers a fresh and convenient way to determine functionals of some Gaussian processes. For example, in the case of the Ornstein–Uhlenbeck process, we have the integral representation

$$V(t) = v\mathrm{e}^{-\beta t} + \sigma\int_0^t \mathrm{e}^{-\beta(t-u)}\mathrm{d}B(u). \tag{8.78}$$

The second term on the right of (8.78) expresses the random component of the Ornstein–Uhlenbeck process as an exponentially weighted moving average of infinitesimal Brownian increments. According to Theorem 8.4, this random component has mean zero and is normally distributed. We use Theorem 8.4 to determine the covariance. For $0 < s < t$,

$$
\begin{aligned}
\mathrm{Cov}[V(s), V(t)] &= E\left[\left\{V(s) - v\mathrm{e}^{-\beta s}\right\}\left\{V(t) - v\mathrm{e}^{-\beta t}\right\}\right] \\
&= \sigma^2 E\left[\int_0^s \mathrm{e}^{-\beta(s-u)}\mathrm{d}B(u) \int_0^t \mathrm{e}^{-\beta(t-w)}\mathrm{d}B(w)\right] \\
&= \sigma^2 \int_0^t 1(u < s)\mathrm{e}^{-\beta(s-u)}\mathrm{e}^{-\beta(t-u)}\mathrm{d}u \\
&= \sigma^2 \mathrm{e}^{-\beta(s+t)} \int_0^s \mathrm{e}^{2\beta u}\mathrm{d}u \\
&= \frac{\sigma^2}{2\beta}\mathrm{e}^{-\beta(s+t)}\left(\mathrm{e}^{2\beta s} - 1\right) \\
&= \frac{\sigma^2}{2\beta}\left(\mathrm{e}^{-\beta(t-s)} - \mathrm{e}^{-\beta(t+s)}\right),
\end{aligned}
$$

in agreement with (8.61).

Example *The Position Process Revisited* Let us assume that $V(0) = v = 0$. The integral of the Ornstein–Uhlenbeck velocity process gives the particle's position $S(t)$ at time t. If we replace the integrand by its representation (8.78) ($v = 0$), we obtain

$$
\begin{aligned}
S(t) = \int_0^t V(s)\mathrm{d}s &= \sigma \int_0^t \int_0^s \mathrm{e}^{-\beta(s-u)}\mathrm{d}B(u)\mathrm{d}s \\
&= \sigma \int_0^t \int_u^t \mathrm{e}^{-\beta(s-u)}\mathrm{d}s\,\mathrm{d}B(u) \\
&= \sigma \int_0^t \mathrm{e}^{\beta u} \int_u^t \mathrm{e}^{-\beta s}\mathrm{d}s\,\mathrm{d}B(u) \qquad (8.79) \\
&= \frac{\sigma}{\beta} \int_0^t \mathrm{e}^{\beta u}\left(\mathrm{e}^{-\beta u} - \mathrm{e}^{-\beta t}\right)\mathrm{d}B(u) \\
&= \frac{\sigma}{\beta} \int_0^t \left(1 - \mathrm{e}^{-\beta(t-u)}\right)\mathrm{d}B(u).
\end{aligned}
$$

Theorem 8.4 applied to (8.79) tells us that the position $S(t)$ at time t is normally distributed with mean zero and variance

$$\begin{aligned}\mathrm{Var}[S(t)] &= \frac{\sigma^2}{\beta^2}\int_0^t \left[1-\mathrm{e}^{-\beta(t-u)}\right]^2 \mathrm{d}u \\ &= \frac{\sigma^2}{\beta^2}\int_0^t \left(1-\mathrm{e}^{-\beta w}\right)^2 \mathrm{d}w \\ &= \frac{\sigma^2}{\beta^2}\left[t-\frac{2}{\beta}\left(1-\mathrm{e}^{-\beta t}\right)+\frac{1}{2\beta}\left(1-\mathrm{e}^{-2\beta t}\right)\right],\end{aligned}$$

in agreement with (8.65). Problem 8.5.4 calls for using Theorem 8.4 to determine the covariance between the velocity $V(t)$ and position $S(t)$.

The position process under an Ornstein–Uhlenbeck velocity behaves like a Brownian motion over large time spans, and we can see this more clearly from the Brownian integral representation in (8.79). If we carry out the first term in the integral (8.79) and recognize that the second part is $V(t)$ itself, we see that

$$\begin{aligned}S(t) &= \frac{\sigma}{\beta}\int_0^t \left(1-\mathrm{e}^{-\beta(t-u)}\right)\mathrm{d}B(u) \\ &= \frac{\sigma}{\beta}\left[B(t)-\int_0^t \mathrm{e}^{-\beta(t-u)}\mathrm{d}B(u)\right] \\ &= \frac{1}{\beta}[\sigma B(t)-V(t)].\end{aligned} \tag{8.80}$$

Let us introduce a rescaled position process that will allow us to better see changes in position over large time spans. Accordingly, for $N>0$, let

$$\begin{aligned}S_N(t) &= \frac{1}{\sqrt{N}}S(Nt) \\ &= \frac{1}{\beta}\left[\frac{\sigma}{\sqrt{N}}B(Nt)+\frac{1}{\sqrt{N}}V(t)\right] \\ &= \frac{1}{\beta}\left[\sigma\tilde{B}(t)+\frac{1}{\sqrt{N}}V(t)\right],\end{aligned} \tag{8.81}$$

where $\tilde{B}(t)=B(Nt)/\sqrt{N}$ remains a standard Brownian motion. (See Exercise 8.1.2.) Because the variance of $V(t)$ is always less than or equal to $\sigma^2/(2\beta)$, the variance of $V(t)/\sqrt{N}$ becomes negligible for large N. Equation (8.81), then, shows more clearly in what manner the position process becomes like a Brownian motion: For large N,

$$S_N(t) \approx \frac{\sigma}{\beta}\tilde{B}(t).$$

Exercises

8.5.1 An Ornstein–Uhlenbeck process $V(t)$ has $\sigma^2 = 1$ and $\beta = 0.2$. What is the probability that $V(t) \leq 1$ for $t = 1, 10$, and 100? Assume that $V(0) = 0$.

8.5.2 The velocity of a certain particle follows an Ornstein–Uhlenbeck process with $\sigma^2 = 1$ and $\beta = 0.2$. The particle starts at rest ($v = 0$) from position $S(0) = 0$. What is the probability that it is more than one unit away from its origin at time $t = 1$. What is the probability at times $t = 10$ and $t = 100$?

8.5.3 Let $\xi_1, \xi_2, \ldots$ be independent standard normal random variables and β a constant, $0 < \beta < 1$. A discrete analog to the Ornstein–Uhlenbeck process may be constructed by setting

$$V_0 = v \quad \text{and} \quad V_n = (1-\beta)V_{n-1} + \xi_n \quad \text{for} \quad n \geq 1.$$

(a) Determine the mean value function and covariance function for $\{V_n\}$.

(b) Let $\Delta V = V_{n+1} - V_n$. Determine the conditional mean and variance of ΔV, given that $V_n = v$.

Problems

8.5.1 Let $\xi_1, \xi_2, \ldots$ be independent standard normal random variables and β a constant, $0 < \beta < 1$. A discrete analog to the Ornstein–Uhlenbeck process may be constructed by setting

$$V_0 = v \quad \text{and} \quad V_n = (1-\beta)V_{n-1} + \xi_n \quad \text{for} \quad n \geq 1.$$

(a) Show that

$$V_n = (1-\beta)^n v + \sum_{k=1}^{n} (1-\beta)^{n-k}\xi_k.$$

Comment on the comparison with (8.78).

(b) Let $\Delta V_n = V_n - V_{n-1}, S_1 = v + V_1 + \cdots + V_n,$ and $B_n = \xi_1 + \cdots + \xi_n$. Show that

$$V_n = v - \beta S_{n-1} + B_n.$$

Compare and contrast with (8.80).

8.5.2 Let $S(t)$ be the position process corresponding to an Ornstein–Uhlenbeck velocity $V(t)$. Assume that $S(0) = V(0) = 0$. Obtain the covariance between $S(t)$ and $V(t)$.

8.5.3 Verify the option valuation formulation (8.66).

Hint: Use the result of Exercise 8.4.6.

8.5.4 In the Ehrenfest urn model (see Chapter 3, Section 3.3.2) for molecular diffusion through a membrane, if there are i particles in urn A, the probability that there will be $i+1$ after one time unit is $1-i/(2N)$, and the probability of $i-1$ is $i/(2N)$, where $2N$ is the aggregate number of particles in both urns. Following Chapter 3, Section 3.3.2, let Y_n be the number of particles in urn A after the nth transition, and let $X_n = Y_n - N$. Let $\Delta X = X_{n+1} - X_n$ be the change in urn composition. The probability law is

$$\Pr\{\Delta X = \pm 1 | X_n = x\} = \frac{1}{2} \mp \frac{x}{2N}.$$

We anticipate a limiting process in which the time between transitions becomes small and the number of particles becomes large. Accordingly, let $\Delta t = 1/N$ and measure fluctuations of a rescaled process in units of order $1/\sqrt{N}$. The definition of the rescaled process is

$$V_N(t) = \frac{X_{[Nt]}}{\sqrt{N}}.$$

Note that in the duration $t=0$ to $t=1$ in the rescaled process, there are N transitions in the urns, and a unit change in the rescaled process corresponds to a fluctuation of order $\sqrt{N}$ in the urn composition. Let $\Delta V = V_N(t+1/N) - V_N(t)$ be the displacement in the rescaled process over the time interval $\Delta t = 1/N$. Show that

$$\begin{aligned} E[\Delta V | V_N(t) = v] &= \frac{1}{\sqrt{N}}\left(\frac{1}{2} - \frac{v}{2\sqrt{N}}\right) - \frac{1}{\sqrt{N}}\left(\frac{1}{2} + \frac{v}{2\sqrt{N}}\right) \\ &= -v\left(\frac{1}{N}\right) = -v\Delta t, \end{aligned}$$

and that $(\Delta V)^2 = 1/N$, whence

$$\begin{aligned} \operatorname{Var}[\Delta V | V_N(t) = v] &= E\left[(\Delta V)^2\right] - \{E[\Delta V]\}^2 \\ &= \frac{1}{N} + o\left(\frac{1}{N}\right) = \Delta t + o(\Delta t). \end{aligned}$$