Midterm Test 2

Stat 150 — Fall 2018

Instructions: Show all of your work. Fully explain your reasoning. Cite any results from the textbook, lectures, etc. which you use.

Attempt as many question as you can. There are 45 possible marks. The test will be marked out of 35 marks. The maximum grade is 35/35.

First name: ______________________________

Last name: ______________________________

Signature: ______________________________

Student ID: ______________________________

Question	1	2	3	4	5	6	Total
Marks	7	7	7	7	9	8	45
Score							/35

1. (a) **[2 marks]** State the *Martingale Convergence Theorem*.

(b) **[2 marks]** Let $(Z_n, n \geq 0)$ be a branching process with offspring distribution ξ with mean $\mu = \mathbb{E}\xi \in (0, \infty)$ and $Z_0 = 1$. Recall that Z_n denotes the number of particles in the nth generation. Show that $(Z_n/\mu^n, n \geq 0)$ is a martingale.

(c) **[3 marks]** Suppose that $\mu = 1$ and $\mathbb{P}(\xi = 1) < 1$. Using the Martingale Convergence Theorem, argue that $\mathbb{P}(Z_n > 0) \to 0$ as $n \to \infty$.

2. (a) [**3 marks**] Prove the *Martingale Maximal Inequality:*

If $(X_n, n \geq 0)$ is a martingale and $\lambda > 0$, then $\mathbb{P}(\max_{n\geq 0} X_n > \lambda) \leq \mathbb{E}(X_0)/\lambda$.

(b) [**4 marks**] An urn initially contains one white and one black ball. At each step, a uniformly random ball is drawn from the urn, and then put back into the urn along with an additional ball of the same color. Show that the proportion of white balls in the urn is ever as large as 3/4 with probability at most 2/3. Fully justify your answer.

3. **[7 marks]** Customers arrive at a store according to a Poisson process $(N_t, t \geq 0)$ with an unknown rate λ, which the store manager wants to estimate. Recall that N_t denotes the number of customers by time t, and W_n the time of the nth customer. In order to estimate the average inter-arrival time $1/\lambda$, the quantity W_{N_t}/N_t is calculated at time t. Note that this is the average of the N_t complete inter-arrival times $X_i = W_i - W_{i-1}$, $1 \leq i \leq N_t$, observed by time t. To obtain an indication of the accuracy of this approximation method, show that

$$\mathbb{E}(W_{N_t}/N_t | N_t > 0) = \frac{1}{\lambda}\left(1 - \frac{\lambda t}{e^{\lambda t} - 1}\right).$$

4. A airline owns two computers which it uses to operate its online reservation system. When in operation, a computer fails at rate μ. The system requires only one computer, so if both computers are in working order, one of the computers is kept in reserve (during which time it cannot fail), and immediately replaces the other computer once it fails. Each computer has its own repair technician, who begins to fix the computer at rate λ once its fails. All operating and repair times are independent. Let $X(t)$ be the number of computers in working order at time t.

   (a) **[3 marks]** Find the generator matrix G for the Markov chain $(X(t) : t \geq 0)$.

   (b) **[2 marks]** Find its stationary distribution π.

(c) **[2 marks]** Suppose that the computers fail at rate $\mu = 1$. Find the repair rate λ required of the technicians so that in the long run the reservation system is functioning 80% of the time.

5. One version of the *Optional Stopping Theorem* states that

   If $(M_n, n \geq 0)$ is a martingale and T is a stopping time with respect to $(X_n, n \geq 0)$ and
   (1) $\mathbb{E}T < \infty$
   (2) for some $K < \infty$, $\mathbb{E}(|M_{n+1} - M_n||X_n, \ldots, X_0) < K$ for all n
   then $\mathbb{E}M_T = \mathbb{E}M_0$.

   In this question, you will use this to prove *Wald's Equation:*

   For IID random variables $(X_n, n \geq 0)$ and a stopping time T with respect to $(X_n, n \geq 0)$ such that $\mathbb{E}|X| < \infty$ and $\mathbb{E}T < \infty$, we have that $\mathbb{E}(\sum_{i=1}^{T} X_i) = \mathbb{E}(T)\mathbb{E}(X)$.

   (a) **[2 marks]** Let $\mu = \mathbb{E}X$ and $M_n = \sum_{i=1}^{n}(X_i - \mu)$. Show that $(M_n, n \geq 0)$ is a martingale with respect to $(X_n, n \geq 0)$.

   (b) **[2 marks]** Show that condition (2) of the Optional Stopping Theorem holds for $(M_n, n \geq 0)$.

(c) **[2 marks]** Deduce Wald's Equation.

(d) **[3 marks]** Use Wald's Equation to show that symmetric Simple Random Walk $(S_n, n \geq 0)$ on $\mathbb{Z}$ is null recurrent. *Hint:* Define $T = \min\{n \geq 1 : S_n = 1\}$. Show that $\mathbb{E}_1 T = 1 + \mathbb{E}_0 T$. Then use Wald's equation to argue by contradiction that $\mathbb{E}_0 T = \infty$, and so $\mathbb{E}_1 T = \infty$.

6. Customers looking for a taxi arrive at a small airport according to a Poisson process with rate λ. If a taxi is waiting they take it immediately, and otherwise wait in line for a taxi. Taxis arrive according to an independent Poisson process with rate $\mu > \lambda$, however if at least 2 taxis are waiting for a customer when a taxi arrives, it leaves the airport immediately without picking up a customer. Model this process as a continuous-time Markov chain $(X_t : t \geq 0)$ on state space $S = \{-2, -1, 0, 1, \ldots\}$.

   (a) **[1 mark]** Describe (in terms of customers and taxis) what it means for $X(t) = i$, for $i \in S$.

   (b) **[2 marks]** Specify the transition rates q_{ij} of the Markov chain.

(c) **[3 marks]** Find its stationary distribution π.

(d) **[2 marks]** Calculate the long run average wait time per customer. *Hint:* Recall that is $1/(\mu - \lambda)$ is the average wait time in a single server queue with arrival rate λ and service rate $\mu > \lambda$.

Scrap paper 1 of 3

Scrap paper 2 of 3

Scrap paper 3 of 3