# Midterm Test 1

Stat 150 — Fall 2018

**Instructions:** Show all of your work. Fully explain your reasoning. Cite any results from the textbook, lectures, etc. which you use.

**First name:** ______________________

**Last name:** ______________________

**Signature:** ______________________

**Student ID:** ______________________

| Question | 1 | 2 | 3 | 4 | 5 | Bonus | Total |
|---|---|---|---|---|---|---|---|
| Marks | 3 | 5 | 4 | 14 | 4 | 4 | 30+4 |
| Score | | | | | | | /30 |

1. **[3 marks]** Suppose we repeatedly flip a fair coin that comes up $H$ and $T$ with equal probability $1/2$. Define a Markov chain on state space $S = \{0, 1, 2, 3\}$ and use it to compute the expected number of flips until the pattern $THH$ is observed.

2. [**5 marks**] Consider a Markov chain $\{X_n : n \geq 0\}$ on a finite state space $S$, where each state $i \in S$ is given a positive weight $w_i > 0$. The Markov chain starts from a uniformly random state $X_0$ in $S$. For each subsequent time $n \geq 1$, if $X_{n-1} = i$ the Markov chain transitions at time $n$ according to the following two-step procedure:

   - First, a state $j \neq i$ is selected uniformly at random.
   - Then, given the state $j$, with probability $w_j/(w_i + w_j)$ the Markov chain transitions to $j$ (that is, $X_n = j$), and with the remaining probability $w_i/(w_i + w_j)$ it stays put (that is, $X_n = i$).

   Find $\lim_{n\to\infty} \mathbb{P}(X_n = i)$, for a state $i \in S$. Fully justify your answer.

3. **[4 marks]** A particle moves randomly between the 8 corners of a cube as follows: When at a corner, it moves to one of the 3 neighboring corners uniformly at random (with probability 1/3 each) and stays there for one unit of time until moving again. Find the expected amount of time until the particle moves to the diametrically opposite corner from its starting position. *Hint:* Use symmetry.

4. (a) [**2 marks**] State the Branching Process Theorem for a branching process $(X_n : n \geq 0)$ with $X_0 = 1$ and offspring distribution $\xi$ with mean $\mu = \mathbb{E}(\xi) < \infty$ and $\mathbb{P}(\xi = 1) \neq 1$.

(b) [**3 marks**] Find $\mathbb{E}(X_n)$, and then use this to calculate the mean of the total progeny $\sum_{n=0}^{\infty} X_n$.

(c) **[5 marks]** Let $p \in (0,1)$, and denote $q = 1-p$. Consider the Markov chain $(Y_n : n \geq 0)$ on state space $S = \{0,1,2,\ldots\}$ with transition probabilities

$$p_{ij} = \begin{cases} qp^j & i = 0 \text{ and } j \geq 0 \\ qp^{j-i+1} & i \geq 1 \text{ and } j \geq i-1. \end{cases}$$

and all other $p_{i,j} = 0$. Using parts (a) and (b), determine the values of $p$ for which $(Y_n)$ is transient, null recurrent or positive recurrent. Fully explain your answer.

(d) **[4 marks]** For $p$ such that $(Y_n)$ is positive recurrent, find the long run proportion of time spent in state 1. Fully justify your answer.

5. [**4 marks**] Let $k \geq 1$. Initially $k$ white balls and $k$ black balls are distributed in two urns so that each urn contains $k$ balls. At each time step one ball from each urn is selected uniformly at random and then their positions are interchanged. Find the long run proportion of time that all of the white balls are contained in one of the urns. Fully justify your answer.

*Bonus* [**4 marks**] Starting from a given initial ordering of a deck of 52 playing cards, a card is chosen uniformly at random and then placed on the top of the deck. This operation is performed repeatedly. Show that, in the limit, the deck is perfectly shuffled, in the sense that it is equally likely to be in any one of the 52! possible orderings. Fully justify your answer.

Scrap paper 1 of 3

Scrap paper 2 of 3

Scrap paper 3 of 3