Stat 150, Fall 2018, HW #7

Due Thurs Nov 29 at the start of class 9:30 AM in Evans 10.
Late assignments will not be accepted.

1. Pinsky and Karlin [PK], Problems (*not* exercises):

   Bonus: 8.1.5

   8.2.3–8.2.6

   8.3.3

   8.3.4

   Bonus: 8.4.1–8.4.3

2. Let $B(t)$ be a standard Brownian motion. Find the distribution of $B(s) + B(t)$ for $0 < s < t$.

3. Let $B_1(t)$ and $B_2(t)$ be independent standard Brownian motions.

   (a) Show that $X(t) = (B_1(t) - B_2(t))/\sqrt{2}$ is a standard Brownian motion.

   (b) State the Reflection Principle for standard Brownian motion, and briefly describe its proof.

   (c) Let $\varepsilon > 0$. Show that $\mathbb{P}(M(\varepsilon) > 0) = 1$, where $M(\varepsilon) = \max\{X(t) : 0 \le t \le \varepsilon\}$.

   (d) Argue that $B_1(t) = B_2(t)$ at infinitely many times t (with probability 1).

4. Let $B(t)$ be a standard Brownian motion, and $Z(t) = \int_0^t B(s)ds$.

   (a) Calculate $\mathbb{E}Z(t)$.

   (b) Find $\mathbb{E}[Z(s)Z(t)]$ for $s < t$.

   (c) In words, explain why or why not $Z(t)$ is a Markov process.

Bonus: For the following question, you may use the following analogue of Theorem 5.13 in [D]:

Theorem. *If $(X_t, t \ge 0)$ is a continuous-time martingale with continuous paths and T is a stopping time, then $(X_{t\wedge T}, t \ge 0)$ is a martingale.*

(a) Recall that standard Brownian motion $(B_t, t \ge 0)$ is a martingale. For $0 < a < b$ let $T = \min\{t \ge 0 : B(t) \in \{0, b\}\}$. You may assume that $\mathbb{P}_a(T < \infty) = 1$. Show that $\mathbb{P}_a(B_T = b) = a/b$.

(b) Consider Brownian motion with drift, $X(t) = B(t) + \mu t$. Show that $M(t) = e^{\theta B(t)}/\mathbb{E}e^{\theta B(t)}$ is a martingale, for any θ. Then use this to find $\mathbb{P}_a(X_T = b)$, where here $T = \min\{t \ge 0 : X(t) \in \{0, b\}\}$.