Stat 200A, Fall 2018
A. Adhikari and Z. Bartha

WEEK 2 EXERCISES

You are expected to do all these problems, but for **Homework 2** please turn in **only Problems 2, 4, 6, and 7** on **Thursday September 6 at the start of lecture**.

1. Coin Tossing Distributions

A coin that lands heads with probability p is tossed repeatedly.

(a) What is the chance of the sequence HHTTT? How does it compare with the chance of the sequence THTHT?

Both chances are $p^2(1-p)^3$.

(b) Let X be the number of heads in the first n tosses. Use the observation in part (a) to find the distribution of X. This is called the *binomial* (n, p) distribution.

$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

(c) In class we found the distribution of the waiting time till the first head; that is, the number of trials needed to get the first head. Fix an integer $r \geq 1$. Find the distribution of the waiting time till the rth head. Check that your answer agrees with what we got in class in the case $r = 1$.

If X is the waiting time until the rth head, then for $k \geq r$

$$\mathbb{P}(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}.$$

(d) Find the distribution of the number of tails before the rth head. This is called the *negative binomial* (r, p) distribution.

If Y is the number of tails before the rth head, then $Y = X - r$ with X defined in (c), and for $k \geq 0$

$$\mathbb{P}(X = k) = \binom{k+r-1}{r-1} p^r (1-p)^k.$$

(e) A gambler bets repeatedly. On each bet, the chance that she wins is p. You can assume that the bets are independent of each other. The gambler decides to stop betting once she wins k bets. What is the chance that she has to make no more than n bets?

We can think of it in two ways. Let P be the probability that we are trying to find. First, by summing up the probabilities that she has to make exactly i bets to achieve k wins, for $k \leq i \leq n$:

$$P = \sum_{i=k}^{n} \binom{i-1}{k-1} p^k (1-p)^{i-k}.$$

Or, by summing up the probabilities that she wins on exactly i bets among n bets for $k \leq i \leq n$:

$$P = \binom{n}{i} p^i (1-p)^{n-i}.$$

2. More Coin Tossing Distributions

A coin that lands heads with probability p is tossed repeatedly.

(a) Let $0 < m < n$ be integers. Let X be the number of heads in the first m tosses and Y the number of heads in the first n tosses. Find the joint distribution of X and Y.

$\mathbb{P}(X = x, Y = y) = \binom{m}{x}\binom{n-m}{y-x}p^y(1-p)^{n-y}$

(b) Continuing Part (a): Let $k \leq m$. Find the conditional distribution of Y given $X = k$.

$\mathbb{P}(Y = y|X = k) = \frac{\mathbb{P}(X=k,Y=y)}{\mathbb{P}(X=k)} = \frac{\binom{m}{k}\binom{n-m}{y-k}p^y(1-p)^{n-y}}{\binom{m}{k}p^k(1-p)^{m-k}} = \binom{n-m}{y-k}p^{y-k}(1-p)^{(n-m)-(y-k)}$, that is, the conditional distribution of $Y - k$ given $X = k$ is Binomial$(n-m, p)$.

(c) Continuing Part (a): Let $k \leq n$. Find the conditional distribution of X given $Y = k$.

$\mathbb{P}(X = x|Y = k) = \frac{\mathbb{P}(X=x,Y=k)}{\mathbb{P}(Y=k)} = \frac{\binom{m}{x}\binom{n-m}{k-x}p^k(1-p)^{n-k}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{\binom{m}{x}\binom{n-m}{k-x}}{\binom{n}{k}}$, that is, the conditional distribution of X given $Y = k$ is HyperGeo(n, k, m).

(d) Find the distribution of the number of tosses needed till both of the faces of the coin have appeared. For example, if the sequence is TTTTH then 5 tosses were needed.

Denote the waiting time by X. Then for $k \geq 1$:

$$\mathbb{P}(X = k) = \mathbb{P}(k-1 \text{ heads first, then tails}) + \mathbb{P}(k-1 \text{ tails first, then heads}) = p^{k-1}(1-p) + (1-p)^{k-1}p.$$

(e) Fix an integer $k \geq 1$. Find the distribution of the number of tosses needed to get at least k heads or at least k tails, whichever happens sooner.

Denote the waiting time by Y. Then for $k \leq n \leq 2k-1$:

$$\begin{aligned}\mathbb{P}(Y = n) &= \mathbb{P}(k-1 \text{ heads and } n-k \text{ tails first in some order and then heads for the } n\text{th toss})\\ &\quad + \mathbb{P}(k-1 \text{ tails and } n-k \text{ heads first in some order and then tails for the } n\text{th toss})\\ &= \binom{n-1}{k-1}p^k(1-p)^{n-k} + \binom{n-1}{k-1}p^{n-k}(1-p)^k.\end{aligned}$$

3. Simple Random Sampling

A population consists of N elements of which G are "good" and the remaining $B = N - G$ are "bad". A simple random sample (SRS) is a sample drawn at random without replacement from the population. Suppose a simple random sample of size n is drawn. As always, a sample is just a subset of the population and hence unordered.

(a) How many samples of size n are possible?

$\binom{N}{n}$

(b) For a fixed g, how many samples contain exactly g good elements?

$\binom{G}{g}\binom{B}{n-g}$

(c) Find the distribution of the number of good elements in the sample. This is called the *hypergeometric* distribution with parameters N, G, and n. Compare your formula with the answer to 2c.

Denote the number of good elements in the sample by X. Then for $0 \leq g \leq G$:

$$\mathbb{P}(X = g) = \frac{\binom{G}{g}\binom{B}{n-g}}{\binom{N}{n}}.$$

(d) For a description of a standard deck of cards, please refer to Exercise 4 of the Week 1 Exercises. A poker hand is a simple random sample of five cards dealt from a standard deck. Find the distribution of the number of aces in a poker hand.

The distribution is $\text{HyperGeo}(52, 4, 5)$.

(e) Find the joint distribution of the number of aces and the number of kings in a poker hand.

Denote the number of aces by X and the number of kings by Y. Then for $0 \leq x \leq 4$ and $0 \leq y \leq 4$ such that $0 \leq x + y \leq 5$:

$$\mathbb{P}(X = x, Y = y) = \frac{\binom{4}{x}\binom{4}{y}\binom{44}{5-x-y}}{\binom{52}{5}}.$$

4. Using Discrete Joint Distributions

(a) A move in the game Monopoly is determined by S, the total number of spots in two rolls of a die. Find the distribution of S and hence find $P(S > 9)$.

The distribution table is the following:

s	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(S = s)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

(b) U_1 and U_2 are independent, and each is uniformly distributed on $\{1, 2, \ldots, n\}$. Let $S = U_1 + U_2$. Find the distribution of S. Please prove your answer; don't just infer from (a).

$$\mathbb{P}(S = s) = \begin{cases} (s-1)/n^2 & \text{if } 2 \leq s \leq n+1, \\ (2n - s + 1)/n^2 & \text{if } n+1 \leq s \leq 2n. \end{cases}$$

(c) You roll n dice, and so do I. What is the chance that we both get the same number of sixes? Yes, zero is a number.

Denote by X the number of sixes that I get, and by Y the number of sixes that you get. Then for $0 \leq k \leq n$:

$$\mathbb{P}(X = k) = \binom{n}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{n-k},$$

and the same holds for Y. Therefore,

$$\mathbb{P}(X = Y) = \sum_{k=0}^{n}\mathbb{P}(X = k)\mathbb{P}(Y = k) = \sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{n-k}\right]^2$$

(d) A coin lands heads with chance p. I toss it until I get H. Then you toss it until you get H. What is the chance that we both make the same number of tosses?

Denote by X the number of tosses that I make, and by Y the number of tosses that you make. Then for $k \geq 1$:

$$\mathbb{P}(X = k) = (1-p)^{k-1}p,$$

and the same holds for Y. Hence,

$$\mathbb{P}(X=Y)=\sum_{k=1}^{\infty}\mathbb{P}(X=k)\mathbb{P}(Y=k)=\sum_{k=1}^{\infty}\left[(1-p)^{k-1}p\right]^2=p^2\sum_{k=1}^{\infty}\left[(1-p)^2\right]^{k-1}=\frac{p^2}{1-(1-p)^2}=\frac{p}{2-p}.$$

(e) Let X have the binomial (n,p) distribution, and let Y independent of X have the binomial (m,p) distribution. What is the distribution of $X+Y$, and why?

If X is the number of successes in n independent trials, and Y is the number of successes in a disjoint set of m independent trials (all trials having success probability p), then $X+Y$ is the number of successes in $n+m$ independent trials, hence $X+Y\sim Bin(n+m,p)$.

5. Radial Distance of Random Point

A point is selected uniformly from the unit disc, that is, the disc with radius 1 centered at the origin $(0,0)$. Let R be the distance of the point from the origin.

(a) Find the cdf and the density of R.

For $0\le r\le 1$: $\mathbb{P}(R\le r)=\frac{\text{area of the disc with radius } r}{\text{area of the disc with radius } 1}=r^2$. The density f_R is the derivative of this, that is, $f_R(r)=2r$ for $0\le r\le 1$.

(b) Let (X,Y) be the coordinates of the point. Are X and Y independent? Explain.

No, since the support of (X,Y) is not a rectangle.

(c) Find the density of X.

$f_X(x)=\frac{2}{\pi}\sqrt{1-x^2}$, see Exercise 2 in Section 4.5. of Pitman Probability (page 315).

6. Functions of Uniform Random Variables

Let X and Y have joint density

$$f(x,y)=\begin{cases}90(y-x)^8, & 0<x<y<1\\ 0 & \text{otherwise}\end{cases}$$

In what follows, please do the calculus yourself and show your work.

(a) Find $P(Y>2X)$.

$$\mathbb{P}(Y>2X)=\int_0^1\int_0^{y/2}90(y-x)^8dxdy=\int_0^1[-10(y-x)^9]_{x=0}^{y/2}dy=\int_0^1 10\left(y^9-(y/2)^9\right)dy$$
$$=\left(1-(1/2)^9\right)\int_0^1 10y^9dy=\left(1-(1/2)^9\right)\left[y^{10}\right]_{y=0}^1=1-(1/2)^9$$

(b) Find the marginal density of X.

For $0\le x\le 1$:

$$f_X(x)=\int_x^1 90(y-x)^8dy=\left[10(y-x)^9\right]_{y=x}^1=10(1-x)^9.$$

(c) Fill in the blanks (explain briefly): The joint density f above is the joint density of the minimum and maximum of ten independent uniform $(0,1)$ random variables.

Denote the independent uniform(0,1) varables by $U_1, \ldots, U_{10}$ and let $0 \le x \le y \le 1$. The probability that one of them falls in a dx neighborhood of x, one of them in the dy neighborhood of y, and all the rest in between is $10 \cdot 9 \cdot dx \cdot dy \cdot (y-x)^8$, since there are 10 ways to pick the one that will be around x, given that choice, there are 9 ways to pick the one that will be around y, and the probability that a uniform(0,1) variable falls in a subinterval of $(0,1)$ equals the length of that interval. Therefore, the joint density of the minimum and the maximum is $10 \cdot 9 \cdot dx \cdot dy \cdot (y-x)^8/(dxdy) = 90(y-x)^8$.

7. Functions of Exponential Random Variables

Let X and Y be independent exponential random variables with rates λ and μ respectively.

(a) Let $W = \min(X, Y)$. Find the distribution of W.

$$\mathbb{P}(W > z) = \mathbb{P}(X > z, Y > z) = \mathbb{P}(X > z)\mathbb{P}(Y > z) = e^{-\lambda z}e^{-\mu z} = e^{-(\lambda+\mu)z}$$

for $z > 0$, hence $W \sim Exp(\lambda + \mu)$.

(b) Let c be a positive constant. Find the distribution of cY.

$$\mathbb{P}(cY < z) = \mathbb{P}(Y < z/c) = 1 - e^{-\mu(z/c)} = 1 - e^{-(\mu/c)z}$$

for $z > 0$, hence $cY \sim Exp(\mu/c)$.

(c) Let c be a positive constant. Use part (b) and a useful result from lecture to find $P(X > cY)$ without integration.

We know that if T_1 and T_2 are independent exponentially distributed variables with parameters λ_1 and λ_2 respectively, then $\mathbb{P}(T_1 < T_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$. Using this and part (b),

$$\mathbb{P}(X > cY) = \frac{\mu/c}{\lambda + \mu/c} = \frac{\mu}{c\lambda + \mu}.$$

(d) Use part (c) to find the cdf of $\frac{X}{Y}$.

For $c > 0$:

$$\mathbb{P}(X/Y < c) = \mathbb{P}(X < cY) = 1 - \frac{\mu}{c\lambda + \mu} = \frac{c\lambda}{c\lambda + \mu}.$$

8. The Exponential and the Geometric

This is from Exercise 10 in Section 4.2 of Pitman's book. You can assume $0 < p < 1$. Also recall that the geometric (p) distribution on $\{0, 1, 2, \ldots\}$ is the distribution of the number of tails before the first head in tosses of a coin that lands heads with probability p.

(a) Let T have exponential distribution with rate λ. Let $Y = int(T)$ be the "integer part" of T, that is, the greatest integer less than or equal to T. Show that Y has a geometric (p) distribution on $\{0, 1, 2, \ldots\}$, and find p in terms of λ.

Since the range of T is $(0, \infty)$, the range of $Y = int(T)$ is $\{0, 1, 2, \ldots\}$. For k in this range

$$\mathbb{P}(int(T) = k) = \mathbb{P}(k \le T < k+1) = \mathbb{P}(T \ge k) - \mathbb{P}(T \ge k+1) = e^{-\lambda k} - e^{-\lambda(k+1)} = (e^{-\lambda})^k(1 - e^{-\lambda}).$$

Therefore, $Y = int(T) \sim Geo(1 - e^{-\lambda})$.

(b) For positive integer m, let $T_m = \frac{int(mT)}{m}$ be the greatest multiple of $1/m$ that is less than or equal to T. Show that T has exponential distribution with parameter λ for some $\lambda > 0$ if and only if for every m there is some p_m such that mT_m has geometric (p_m) distribution on $\{0, 1, 2, \ldots\}$. Find p_m in terms of λ.

First, let $T \sim Exp(\lambda)$. Then the variable $mT_m = int(mT)$ takes non-negative integer values. For $k = 0, 1, 2, \ldots$:

$$\begin{aligned}\mathbb{P}(int(mT) = k) &= \mathbb{P}(k \leq mT < k+1) = \mathbb{P}(k/m \leq T < (k+1)/m)\\ &= \mathbb{P}(T \geq k/m) - \mathbb{P}(T \geq (k+1)/m) = e^{-\lambda k/m} - e^{-\lambda(k+1)/m}\\ &= (e^{-\lambda/m})^k(1 - e^{-\lambda/m}).\end{aligned}$$

Hence, $mT_m = int(mT) \sim Geo(1 - e^{-\lambda/m})$. (That is, $p_m = 1 - e^{-\lambda/m}$.)

For the other direction, we will only give an intuitive reasoning. Assume that T is the lifetime of a lightbulb. Then $mT_m = int(mT)$ is the number of consecutive time intervals of length $1/m$ that the lightbulb survived. If $mT_m \sim Geo(p_m)$ with some p_m, that means that in each of these time intervals the ligthbulb survives with probability $1 - p_m$, and burns out with probability p_m. The important thing is that the lighbulb burns out in any of these time intervals with the same probability. If this is true for any m, then, as a limit, we get that in any time interval of length dx the lightbulb burns out with the same probability, only depending on dx. This is exactly the intuition behind the exponential distribution, so this yields $T \sim Exp(\lambda)$ with some λ. (And then the calculation above will give the relationship between the p_m's and λ.)