Stat 200A, Fall 2018
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WEEK 14 EXERCISES

1. MGF of a Random Sum

Let N be a non-negative integer valued random variable, and let $X, X_1, X_2, \ldots$ be i.i.d. and independent of N. Define the "random sum" S by

$$\begin{aligned} S &= 0 \text{ if } N = 0 \\ &= X_1 + X_2 + \cdots + X_n \text{ if } N = n > 0 \end{aligned}$$

(a) Let M be our usual notation for moment generating functions. By conditioning on N, show that

$$M_S(t) = M_N(\log M_X(t))$$

assuming that all the quantities above are well defined. [The identity $w = e^{\log(w)}$ might be handy.]

(b) Let N have the geometric (p) distribution on $\{1, 2, 3, \ldots\}$. Find the mgf of N. This doesn't use Part (a).

(c) Let $X_1, X_2, \ldots$ be i.i.d. exponential (λ) variables and let N be geometric as in Part (b). Use the results of Parts (a) and (b) to identify the distribution of S.

(d) Find the density of S by conditioning on N, and hence confirm the result of Part (c).

[Find $P(S \in ds)$ by conditioning on N.]

(e) Use a Poisson (λ) process and Poissonization (also known as "thinning" in the context of Poisson processes) to find yet another way of confirming the result of Part (c).

2.

Let X and Y be independent random variables. Let X have moment generating function

$$M_X(t) = e^{5t+2t^2}, \quad -\infty < t < \infty$$

and let Y have moment generating function

$$M_Y(t) = e^{8t^2}, \quad -\infty < t < \infty$$

(a) Find the moment generating function of $X - 2Y - 3$.

(b) Find $P(X > 2Y + 3)$.

3.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. with the normal (μ, σ^2) distribution. Define the sample mean M as

$$M = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and for each i in the range 1 through n let the ith deviation from the mean be D_i defined by

$$D_i = X_i - M.$$

(a) Find the joint distribution of D_1 and D_2.

(b) Pick one option and justify your choice: the random variables M and D_1 are

1. neither uncorrelated nor independent
2. uncorrelated but not independent
3. independent but not uncorrelated
4. uncorrelated and independent

4.

Let M have the Gamma(r, λ) density. Given $M = m$, let N have the Poisson distribution with parameter m. Compute the following:

(a) $E(N \mid M)$

(b) $Var(N \mid M)$

(c) $E(N)$

(d) $Var(N)$

(e) For $m > 0$ and non-negative integer n, $P(M \in dm, N = n)$

(f) The posterior distribution of M given $N = n$

5.

Let Z have the standard normal density. Then $E(Z^k)$ is well-defined and finite for every positive integer k. In this question you will find the numerical value of $E(Z^k)$ for each positive k.

(a) Let n be a positive integer and consider the odd integer $k = 2n - 1$. What is the value of $E(Z^{2n-1})$ and why?

(b) Write the formula for the density of Z^2. You don't have to derive the formula if you remember it or can work it out from the formula sheets.

(c) Let n be a positive integer and consider the even integer $k = 2n$. Use part (b) to find $E(Z^{2n})$ in terms of the Gamma function.

(d) For each positive integer n, find an integer c_n such that $E(Z^{2n}) = c_n E(Z^{2n-2})$. Then use induction to derive a formula for $E(Z^{2n})$ that does not involve the Gamma function.