Stat 200A, Fall 2018
A. Adhikari and Z. Bartha

WEEK 9 EXERCISES

You are expected to do all these problems, but for **Homework 9** please turn in **only Problems 1, 6, 7, and 8** on **Thursday October 25 at the start of lecture**.

Note that there is no Homework 8. I'm keeping the number of the homework the same as the week number.

Also please note that it's important that you do the problems you missed on the midterm. Ask Zsolt or Prof. A. if you're not sure if you're doing them correctly.

1. Maximum of a Random Number of Uniforms

Let $U_1, U_2, \ldots$ be i.i.d. uniform on $(0, 1)$, and let N have the Poisson (μ) distribution independent of $U_1, U_2, \ldots$. Let $M = \max\{U_1, U_2, \ldots, U_N\}$, and define M to be 0 if $N = 0$.

Find the distribution of M.

For $z \in (0, 1)$:

$$\begin{aligned}
\mathbb{P}(M \le z) &= \sum_{n=0}^{\infty} \mathbb{P}(M \le z, N = n) \\
&= \mathbb{P}(N = 0) + \sum_{n=1}^{\infty} \mathbb{P}(N = n)\mathbb{P}(\max\{U_1, \ldots, U_n\} \le z) \\
&= e^{-\mu} + \sum_{n=1}^{\infty} e^{-\mu}\frac{\mu^n}{n!}z^n \\
&= e^{-\mu} + e^{-\mu}\sum_{n=1}^{\infty} \frac{(\mu z)^n}{n!} \\
&= e^{-\mu} + e^{-\mu}(e^{\mu z} - 1) = e^{-\mu(1-z)}
\end{aligned}$$

Note that while this is a continuous distribution on $(0, 1)$, it has mass $e^{-\mu}$ at 0, which corresponds to the case $N = 0$.

2. Moments of the Beta

Let X have the beta (r, s) distribution, where $r > 0$ and $s > 0$ are not necessarily integers. For each $k \ge 1$, find $E(X^k)$.

$$\begin{aligned}
E(X^k) &= \int_0^1 x^k \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}\,dx \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r+k-1}(1-x)^{s-1}\,dx \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \cdot \frac{\Gamma(r+k)\Gamma(s)}{\Gamma(r+k+s)} = \frac{\Gamma(r+s)}{\Gamma(r)} \cdot \frac{\Gamma(r+k)}{\Gamma(r+k+s)} \\
&= \frac{r(r+1)\cdot \ldots \cdot (r+k-1)}{(r+s)(r+s+1)\cdot \ldots \cdot (r+s+k-1)}
\end{aligned}$$

3. General Order Statistics

Let $X_1, X_2, \ldots, X_n$ be i.i.d., each with density f and cdf F. For $1 \leq k \leq n$ let $X_{(k)}$ be the kth order statistic of $X_1, X_2, \ldots, X_n$.

a) Find the density of $X_{(k)}$ by using the method we used in class in the case where each X_i had the uniform $(0, 1)$ distribution.

If f_k is the density of $X_{(k)}$, then for any x:

$$f_k(x)dx = \mathbb{P}(X_{(k)} \in dx) = n\binom{n-1}{k-1}f(x)dx(F(x))^{k-1}(1-F(x))^{n-k},$$

so

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!}f(x)(F(x))^{k-1}(1-F(x))^{n-k}.$$

b) Find the cdf of $X_{(k)}$ by using the method we used in class in the case where each X_i had the uniform $(0, 1)$ distribution.

If F_k is the cdf of $X_{(k)}$, then for any x:

$$F_k(x) = \mathbb{P}(X_{(k)} \leq x) = \sum_{l=k}^{n}\binom{n}{l}(F(x))^l(1-F(x))^{n-l},$$

where l is the number of variables among $X_1, \ldots, X_n$ that are less than or equal to x.

4. Which One is the Max?

This is 4.6.4 of Pitman's text.

Let $X = \min(S, T)$ and $Y = \max(S, T)$ for independent random variables S and T with a common density f. Let I denote the indicator of the event $S < T$.

a) What is the distribution of I?

Since S and T are independent, and have the same distribution, the pair (S, T) has the same joint distribution as the pair (T, S). This notion of symmetry is what we use throughout this exercise.

Now this symmetry implies $P(S < T) = \mathbb{P}(T < S)$, while $\mathbb{P}(S < T) + \mathbb{P}(T < S) = 1$ (because $\mathbb{P}(S = T) = 0$, since S and T have continuous distributions). Thus, $\mathbb{P}(S < T) = 1/2$, so $I \sim \mathit{Bernoulli}(1/2)$.

b) Are X and I independent? Are Y and I independent? Are (X, Y) and I independent?

For any x:

$$\mathbb{P}(X \in dx, I = 1) = \mathbb{P}(\min\{S, T\} \in dx, S < T) = \mathbb{P}(S \in dx, T > x).$$

On the other hand

$$\mathbb{P}(X \in dx, I = 0) = \mathbb{P}(\min\{S, T\} \in dx, T < S) = \mathbb{P}(T \in dx, S > x).$$

These two probabilities (on the right-hand sides) are the same by symmetry and they sum up to $\mathbb{P}(X \in dx)$ (by looking at the left-hand sides), so both have to be equal to $\mathbb{P}(X \in dx)/2$, which is $\mathbb{P}(X \in dx)\mathbb{P}(I = 1)$. Hence, $\mathbb{P}(X \in dx, I = 1) = \mathbb{P}(X \in dx)\mathbb{P}(I = 1)$, which proves the independence of X and I.

Similarly, Y and I are independent.

As for (X, Y) and I, we have for $y > x$

$$\mathbb{P}(X \in dx, Y \in dy, I = 1) = \mathbb{P}(\min\{S, T\} \in dx, \max\{S, T\} \in dy, S < T) = \mathbb{P}(S \in dx, T \in dy),$$

and proceeding similarly as above we get the independence of (X, Y) and I.

c) How can these conclusions be extended to the order statistics of three or more independent random variables all with the same density?

A similar reasoning yields that the joint distribution of all the order statistics of these variables is independent of order of the variables, and the order of them is uniform over all the their permutations.

5. Beta-Geometric

Let X have the beta (r, s) distribution. Given $X = x$, let Y be the number of tosses till an x-coin lands heads.

a) What is the posterior density of X given $Y = k$? Identify it as one of the famous ones and state its name and parameters.

For $x \in (0, 1)$

$$\begin{aligned}
\mathbb{P}(X \in dx \mid Y = k) &= \frac{\mathbb{P}(X \in dx, Y = k)}{\mathbb{P}(Y = k)} = \frac{\mathbb{P}(X \in dx)\mathbb{P}(Y = k \mid X \in dx)}{\mathbb{P}(Y = k)} \\
&= \frac{\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}(1-x)^{k-1}x}{\mathbb{P}(Y = k)} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)\mathbb{P}(Y = k)} x^{r}(1-x)^{s+k-2},
\end{aligned}$$

so $(X \mid Y = k) \sim \beta(r + 1, s + k - 1)$.

b) What is the distribution of Y?

For $k \in \{1, 2, \ldots\}$

$$\begin{aligned}
\mathbb{P}(Y = k) &= \int_0^1 \mathbb{P}(X \in dx)\mathbb{P}(Y = k \mid X \in dx)\, dx \\
&= \int_0^1 \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}(1-x)^{k-1}x\, dx \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r}(1-x)^{s+k-2}\, dx \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \cdot \frac{\Gamma(r+1)\Gamma(s+k-1)}{\Gamma(r+s+k)}.
\end{aligned}$$

($\mathbb{P}(Y = k)$ is the normalizing constant (in x) that makes the density that we got in part (a) integrate to 1.)

6. Gamma-Poisson

Let V have the gamma (r, λ) distribution. Given $V = v$, let the conditional distribution of W be Poisson with parameter v.

a) What is the posterior density of V given $W = k$? Identify it as one of the famous ones and state its name and parameters.

For $v > 0$

$$\begin{aligned}
\mathbb{P}(V \in dv \mid W = k) &= \frac{\mathbb{P}(V \in dv)\mathbb{P}(W = k \mid V \in dv)}{\mathbb{P}(W = k)} \\
&= \frac{\frac{1}{\Gamma(r)}\lambda^r v^{r-1} e^{-\lambda v} e^{-v} \frac{v^k}{k!}}{\mathbb{P}(W = k)} \\
&= \frac{\lambda^r}{\Gamma(r) k! \mathbb{P}(W = k)} v^{r+k-1} e^{-(\lambda+1)v},
\end{aligned}$$

so $(V \mid W = k) \sim \Gamma(r + k, \lambda + 1)$.

b) What is the distribution of W?

For $k \in \{0, 1, \ldots\}$, $\mathbb{P}(W = k)$ is the normalizing constant in the density above, hence

$$\begin{aligned}
\mathbb{P}(W = k) &= \int_0^\infty \frac{\lambda^r}{\Gamma(r) k!} v^{r+k-1} e^{-(\lambda+1)v} \, dv \\
&= \frac{\lambda^r}{\Gamma(r) k!} \cdot \frac{\Gamma(r + k)}{(\lambda + 1)^{r+k}} \\
&= \frac{\Gamma(r + k)}{\Gamma(r) k!} \left(\frac{\lambda}{\lambda + 1}\right)^r \left(\frac{1}{\lambda + 1}\right)^k.
\end{aligned}$$

Thus, W has the negative binomial distribution with parameters $(r, \lambda/(\lambda + 1))$. (We didn't learn about it when r is not an integer, but this is how it is defined.)

7. Range of an IID Uniform Sample

Let $U_1, U_2, \ldots, U_n$ be an i.i.d. uniform $(0, 1)$ random sample. As usual let $U_{(1)}$ be the minimum and $U_{(n)}$ the maximum of the sample.

a) Find the joint density of $U_{(1)}$ and $U_{(n)}$.

For $0 < x < y < 1$

$$f_{U_{(1)}, U_{(n)}}(x, y) = n(n - 1)(y - x)^{n-2}.$$

b) Let $R_n = U_{(n)} - U_{(1)}$ be the *range* of the sample. Find $E(R_n)$ and $\lim_{n\to\infty} E(R_n)$. Explain why the limit makes intuitive sense.

We know that $U_{(1)} \sim \beta(1, n)$ and $U_{(n)} \sim \beta(n, 1)$, so

$$E(R_n) = E(U_{(n)}) - E(U_{(1)}) = \frac{n}{n + 1} - \frac{1}{n + 1} = \frac{n - 1}{n + 1} \to 1$$

as $n \to \infty$. Intuitively, as $n \to \infty$, $U_{(1)}$ becomes more and more concentrated around 0, whereas $U_{(n)}$ becomes more and more concentrated around 1.

c) Find the distribution of R_n.

Implementing a change of variables $(U_{(1)}, U_{(n)}) \to (U_{(1)}, R_n)$ with the arguments (x, y) with $0 < x < y < 1$ and (x, z) with $z = y - x$ (having the inverse $y = x + z$) we have

$$\begin{aligned}
f_{U_{(1)}, R_n}(x, z) &= f_{U_{(1)}, U_{(n)}}(x, y) \left| \det \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial z} \end{pmatrix} \right| \\
&= n(n - 1)(y - x)^{n-2} \left| \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right| \\
&= n(n - 1) z^{n-2},
\end{aligned}$$

where the range of (x, z) is $0 < x < 1$, $0 < z < 1 - x$. Now the marginal density of R_n is

$$f_{R_n}(z) = \int_0^{1-z} f_{U_{(1)}, R_n}(x, z)\, dx = \int_0^{1-z} n(n-1)z^{n-2}\, dx = n(n-1)z^{n-2}(1-z),$$

so $R_n \sim \beta(n-1, 2)$.

8. Random Segments

This problem is about order statistics of uniforms, but we will change notation as it will help us keep track of sample size.

Let $0 = U_{n,0} < U_{n,1} < U_{n,2} < \cdots < U_{n,n} < U_{n,n+1} = 1$ be the order statistics of n i.i.d. uniform $(0, 1)$ random variables. Previously, we had called these $0 < U_{(1)} < U_{(2)} < \cdots < U_{(n)} < 1$, suppressing the sample size in the notation for each order statistic.

The order statistics give rise to $n + 1$ *spacings* $U_{n,1} - 0, U_{n,2} - U_{n,1}, \ldots, 1 - U_{n,n}$. Just like the spacings formed by the aces in a shuffled deck, these spacings are *exchangeable*, that is, every permutation of them has the same distribution. In particular, this implies that they all have the same distribution.

For this problem, just assume the exchangeability. The next problem (which is optional) outlines a proof.

a) Explain why for $1 \le k \le n + 1$, the distribution of the spacing $U_{n,k} - U_{n,k-1}$ is beta, and identify the parameters.

By exchangeability, $U_{n,k} - U_{n,k-1}$ has the same distribution as $U_{n,1} - 0 = U_{n,1}$, which has the $\beta(1, n)$ distribution.

b) Suppose the unit interval is cut at the $U_{n,k}$'s to form $n+1$ segments of lengths $U_{n,k} - U_{n,k-1}$, $1 \le k \le n+1$. Independent of the $U_{n,k}$'s a number U is picked uniformly in $(0, 1)$. Let V_n be the length of the segment containing U.

With only a minimum of calculation, explain why V_n has a beta distribution, and identify its parameters. Is it the same beta distribution as in Part (a)?

If we add U to our sample $(U_1, \ldots, U_n)$, we are dealing with $n + 1$ i.i.d. uniform(0,1) variables, their order statistics being $U_{n+1,1}, \ldots, U_{n+1,n+1}$. The value of U is one of them, $U_{n+1,k}$ say, and V_n is then $U_{n+1,k+1} - U_{n+1,k-1}$. Using exchangeability for this sample of $n + 1$ variables, we get that V_n has the same distribution as $U_{n+1,2} - 0$, which has the $\beta(2, n)$ distribution. This is not the same as the one in part (a), it shows that conditioning on the fact that U falls in a particular segment creates a bias, and makes the conditional length of that segment bigger on average.

9. Optional

For $U_{n,k}$ as in Exercise 8, use beta-gamma algebra (or any method you prefer) to show that

$$(U_{n,k} - U_{n,k-1} : 1 \le k \le n + 1) \overset{d}{=} (W_k : 1 \le k \le n + 1)/S_{n+1}$$

where $W_1, W_2, \ldots, W_{n+1}$ are i.i.d. exponential (1) variables with sum S_{n+1}. Hence show that all the $n + 1$ spacings are exchangeable.

We will show that

$$(U_{n,1}, U_{n,2}, \ldots, U_{n,n}) \overset{d}{=} \left(\frac{W_1}{S_{n+1}}, \frac{W_1 + W_2}{S_{n+1}}, \ldots, \frac{W_1 + \ldots W_n}{S_{n+1}} \right). \tag{1}$$

By a change of variables formula, we get the density of the left-hand side of (1) for $0 < x_1 < x_2 < \ldots < x_n < 1$

as

$$f_{U_{n,1},U_{n,2},\ldots,U_{n,n}}(x_1,\ldots,x_n) = \sum_{\substack{(y_1,\ldots,y_n) \\ \text{a permutation of } (x_1,\ldots,x_n)}} f_{U_1,\ldots,U_n}(y_1,\ldots,y_n) \cdot 1 = n!,$$

since there are $n!$ permutations, the uniform joint density is constant 1, and the Jacobian is a permutation matrix that has determinant ± 1.

To get the joint density of the right-hand side of (1) (denote it by f_*), we first notice that the joint density of $W_1,\ldots,W_{n+1}$ is

$$f_{W_1,\ldots,W_{n+1}}(z_1,\ldots,z_{n+1}) = e^{-z_1}\ldots e^{-z_n} = e^{-(z_1+\ldots+z_n)}$$

for $z_1,\ldots,z_n > 0$. We can implement the change of variables

$$(W_1,\ldots,W_{n+1}) \to \left(\frac{W_1}{S_{n+1}}, \frac{W_1+W_2}{S_{n+1}}, \ldots, \frac{W_1+\ldots W_n}{S_{n+1}}, S_{n+1}\right)$$

with arguments $(z_1,\ldots,z_{n+1})$ for the left-hand side and $(x_1,\ldots,x_n,s)$ for the right-hand side with the inverse transformation being

$$\begin{aligned} z_1 &= x_1 s \\ z_2 &= (x_2 - x_1)s \\ &\vdots \\ z_n &= (x_n - x_{n-1})s \\ z_{n+1} &= (1 - x_n)s. \end{aligned}$$

It is not hard to compute that the Jacobian has determinant s^n, so the density $f_{*,S_{n+1}}$ that we get is

$$f_{*,S_{n+1}}(x_1,\ldots,x_n,s) = e^{-s}s^n.$$

Integrating out s (a gamma integral) yields that $f_*(x_1,\ldots,x_n) = n!$, what we wanted to prove.