Stat 200A, Fall 2018
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WEEK 9 EXERCISES

You are expected to do all these problems, but for **Homework 9** please turn in **only Problems 1, 6, 7, and 8** on **Thursday October 25 at the start of lecture**.

Note that there is no Homework 8. I'm keeping the number of the homework the same as the week number.

Also please note that it's important that you do the problems you missed on the midterm. Ask Zsolt or Prof. A. if you're not sure if you're doing them correctly.

1. Maximum of a Random Number of Uniforms

Let $U_1, U_2, \ldots$ be i.i.d. uniform on $(0,1)$, and let N have the Poisson (μ) distribution independent of $U_1, U_2, \ldots$. Let $M = \max\{U_1, U_2, \ldots, U_N\}$, and define M to be 0 if $N = 0$.

Find the distribution of M.

2. Moments of the Beta

Let X have the beta (r, s) distribution, where $r > 0$ and $s > 0$ are not necessarily integers. For each $k \geq 1$, find $E(X^k)$.

3. General Order Statistics

Let $X_1, X_2, \ldots, X_n$ be i.i.d., each with density f and cdf F. For $1 \leq k \leq n$ let $X_{(k)}$ be the kth order statistic of $X_1, X_2, \ldots, X_n$.

a) Find the density of $X_{(k)}$ by using the method we used in class in the case where each X_i has the uniform $(0,1)$ distribution.

b) Find the cdf of $X_{(k)}$ by using the method we used in class in the case where each X_i has the uniform $(0,1)$ distribution.

4. Which One is the Max?

This is 4.6.4 of Pitman's text.

Let $X = \min(S,T)$ and $Y = \max(S,T)$ for independent random variables S and T with a common density f. Let I denote the indicator of the event $S < T$.

a) What is the distribution of I?

b) Are X and I independent? Are Y and I independent? Are (X, Y) and I independent?

c) How can these conclusions be extended to the order statistics of three or more independent random variables all with the same density?

5. Beta-Geometric

Let X have the beta (r, s) distribution. Given $X = x$, let Y be the number of tosses till an x-coin lands heads.

a) What is the posterior density of X given $Y = k$? Identify it as one of the famous ones and state its name and parameters.

b) What is the distribution of Y?

6. Gamma-Poisson

Let V have the gamma (r, λ) distribution. Given $V = v$, let the conditional distribution of W be Poisson with parameter v.

a) What is the posterior density of V given $W = k$? Identify it as one of the famous ones and state its name and parameters.

b) What is the distribution of W?

7. Range of an IID Uniform Sample

Let $U_1, U_2, \ldots, U_n$ be an i.i.d. uniform $(0, 1)$ random sample. As usual let $U_{(1)}$ be the minimum and $U_{(n)}$ the maximum of the sample.

a) Find the joint density of $U_{(1)}$ and $U_{(n)}$.

b) Let $R_n = U_{(n)} - U_{(1)}$ be the *range* of the sample. Find $E(R_n)$ and $\lim_{n\to\infty} E(R_n)$. Explain why the limit makes intuitive sense.

c) Find the distribution of R_n.

8. Random Segments

This problem is about order statistics of uniforms, but we will change notation as it will help us keep track of sample size.

Let $0 = U_{n,0} < U_{n,1} < U_{n,2} < \cdots < U_{n,n} < U_{n,n+1} = 1$ be the order statistics of n i.i.d. uniform $(0, 1)$ random variables. Previously, we had called these $0 < U_{(1)} < U_{(2)} < \cdots < U_{(n)} < 1$, suppressing the sample size in the notation for each order statistic.

The order statistics give rise to $n + 1$ *spacings* $U_{n,1} - 0, U_{n,2} - U_{n,1}, \ldots, 1 - U_{n,n}$. Just like the spacings formed by the aces in a shuffled deck, these spacings are *exchangeable*, that is, every permutation of them has the same distribution. In particular, this implies that they all have the same distribution.

For this problem, just assume the exchangeability. The next problem (which is optional) outlines a proof.

a) Explain why for $1 \le k \le n + 1$, the distribution of the spacing $U_{n,k} - U_{n,k-1}$ is beta, and identify the parameters.

b) Suppose the unit interval is cut at the $U_{n,k}$'s to form $n+1$ segments of lengths $U_{n,k} - U_{n,k-1}$, $1 \le k \le n+1$. Independent of the $U_{n,k}$'s a number U is picked uniformly in $(0, 1)$. Let V_n be the length of the segment containing U.

With only a minimum of calculation, explain why V_n has a beta distribution, and identify its parameters. Is it the same beta distribution as in Part (a)?

9. Optional

For $U_{n,k}$ as in Exercise 8, use beta-gamma algebra (or any method you prefer) to show that

$$(U_{n,k} - U_{n,k-1} : 1 \le k \le n + 1) \stackrel{d}{=} (W_k : 1 \le k \le n + 1)/S_{n+1}$$

where $W_1, W_2, \ldots, W_{n+1}$ are i.i.d. exponential (1) variables with sum S_{n+1}. Hence show that all the $n + 1$ spacings are exchangeable.