Stat 200A, Fall 2018
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WEEK 2 EXERCISES

You are expected to do all these problems, but for **Homework 2** please turn in **only Problems 2, 4, 6, and 7** on **Thursday September 6 at the start of lecture**.

1. Coin Tossing Distributions

A coin that lands heads with probability p is tossed repeatedly.

(a) What is the chance of the sequence HHTTT? How does it compare with the chance of the sequence THTHT?

(b) Let X be the number of heads in the first n tosses. Use the observation in part (a) to find the distribution of X. This is called the *binomial* (n, p) distribution.

(c) In class we found the distribution of the waiting time till the first head; that is, the number of trials needed to get the first head. Fix an integer $r \geq 1$. Find the distribution of the waiting time till the rth head. Check that your answer agrees with what we got in class in the case $r = 1$.

(d) Find the distribution of the number of tails before the rth head. This is called the *negative binomial* (r, p) distribution.

(e) A gambler bets repeatedly. On each bet, the chance that she wins is p. You can assume that the bets are independent of each other. The gambler decides to stop betting once she wins k bets. What is the chance that she has to make no more than n bets?

2. More Coin Tossing Distributions

A coin that lands heads with probability p is tossed repeatedly.

(a) Let $0 < m < n$ be integers. Let X be the number of heads in the first m tosses and Y the number of heads in the first n tosses. Find the joint distribution of X and Y.

(b) Continuing Part (a): Let $k \leq m$. Find the conditional distribution of Y given $X = k$.

(c) Continuing Part (a): Let $k \leq n$. Find the conditional distribution of X given $Y = k$.

(d) Find the distribution of the number of tosses needed till both of the faces of the coin have appeared. For example, if the sequence is TTTTH then 5 tosses were needed.

(e) Fix an integer $k \geq 1$. Find the distribution of the number of tosses needed to get at least k heads or at least k tails, whichever happens sooner.

3. Simple Random Sampling

A population consists of N elements of which G are "good" and the remaining $B = N - G$ are "bad". A simple random sample (SRS) is a sample drawn at random without replacement from the population. Suppose a simple random sample of size n is drawn. As always, a sample is just a subset of the population and hence unordered.

(a) How many samples of size n are possible?

(b) For a fixed g, how many samples contain exactly g good elements?

(c) Find the distribution of the number of good elements in the sample. This is called the *hypergeometric* distribution with parameters N, G, and n. Compare your formula with the answer to 2c.

(d) For a description of a standard deck of cards, please refer to Exercise 4 of the Week 1 Exercises. A poker hand is a simple random sample of five cards dealt from a standard deck. Find the distribution of the number of aces in a poker hand.

(e) Find the joint distribution of the number of aces and the number of kings in a poker hand.

4. Using Discrete Joint Distributions

(a) A move in the game Monopoly is determined by S, the total number of spots in two rolls of a die. Find the distribution of S and hence find $P(S > 9)$.

(b) U_1 and U_2 are independent, and each is uniformly distributed on $\{1, 2, \ldots, n\}$. Let $S = U_1 + U_2$. Find the distribution of S. Please prove your answer; don't just infer from (a).

(c) You roll n dice, and so do I. What is the chance that we both get the same number of sixes? Yes, zero is a number.

(d) A coin lands heads with chance p. I toss it until I get H. Then you toss it until you get H. What is the chance that we both make the same number of tosses?

(e) Let X have the binomial (n, p) distribution, and let Y independent of X have the binomial (m, p) distribution. What is the distribution of $X + Y$, and why?

5. Radial Distance of Random Point

A point is selected uniformly from the unit disc, that is, the disc with radius 1 centered at the origin $(0, 0)$. Let R be the distance of the point from the origin.

(a) Find the cdf and the density of R.

(b) Let (X, Y) be the coordinates of the point. Are X and Y independent? Explain.

(c) Find the density of X.

6. Functions of Uniform Random Variables

Let X and Y have joint density

$$f(x, y) = \begin{cases} 90(y - x)^8, & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

In what follows, please do the calculus yourself and show your work.

(a) Find $P(Y > 2X)$.

(b) Find the marginal density of X.

(c) Fill in the blanks (explain briefly): The joint density f above is the joint density of the ___________ and ___________ of ten independent uniform $(0, 1)$ random variables.

7. Functions of Exponential Random Variables

Let X and Y be independent exponential random variables with rates λ and μ respectively.

(a) Let $W = \min(X, Y)$. Find the distribution of W.

(b) Let c be a positive constant. Find the distribution of cY.

(c) Let c be a positive constant. Use part (b) and a useful result from lecture to find $P(X > cY)$ without integration.

(d) Use part (c) to find the cdf of $\frac{X}{Y}$.

8. The Exponential and the Geometric

This is from Exercise 10 in Section 4.2 of Pitman's book. You can assume $0 < p < 1$. Also recall that the geometric (p) distribution on $\{0, 1, 2, \ldots\}$ is the distribution of the number of tails before the first head in tosses of a coin that lands heads with probability p.

(a) Let T have exponential distribution with rate λ. Let $Y = int(T)$ be the "integer part" of T, that is, the greatest integer less than or equal to T. Show that Y has a geometric (p) distribution on $\{0, 1, 2, \ldots\}$, and find p in terms of λ.

(b) For positive integer m, let $T_m = \frac{int(mT)}{m}$ be the greatest multiple of $1/m$ that is less than or equal to T. Show that T has exponential distribution with parameter λ for some $\lambda > 0$ if and only if for every m there is some p_m such that mT_m has geometric (p_m) distribution on $\{0, 1, 2, \ldots\}$. Find p_m in terms of λ.