Stat 200A, Fall 2018
A. Adhikari and Z. Bartha

WEEK 14 EXERCISES

1. MGF of a Random Sum

Let N be a non-negative integer valued random variable, and let $X, X_1, X_2, \ldots$ be i.i.d. and independent of N. Define the "random sum" S by

$$\begin{aligned} S &= 0 \ \text{ if } N = 0 \\ &= X_1 + X_2 + \cdots + X_n \ \text{ if } N = n > 0 \end{aligned}$$

(a) Let M be our usual notation for moment generating functions. By conditioning on N, show that

$$M_S(t) = M_N(\log M_X(t))$$

assuming that all the quantities above are well defined. [The identity $w = e^{\log(w)}$ might be handy.]

$$M_S(t) = E[e^{tS}] = E[E[e^{tS} \mid N]] = E[E[e^{tX_1} \ldots e^{tX_N} \mid N]] = E[M_X(t)^N] = E[e^{(\log M_X(t))N}] = M_N(\log M_X(t))$$

(b) Let N have the geometric (p) distribution on $\{1, 2, 3, \ldots\}$. Find the mgf of N. This doesn't use Part (a).

$$M_N(t) = E[e^{tN}] = \sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1}p = pe^t \sum_{k=1}^{\infty}(e^t(1-p))^{k-1} = \frac{pe^t}{1-(1-p)e^t}$$

(c) Let $X_1, X_2, \ldots$ be i.i.d. exponential (λ) variables and let N be geometric as in Part (b). Use the results of Parts (a) and (b) to identify the distribution of S.

Using that $M_N(t)$ is as above, and $M_X(t) = \frac{\lambda}{\lambda - t}$, the mgf of S is

$$\begin{aligned} M_S(t) = M_N(\log M_X(t)) &= \frac{pe^{\log M_X(t)}}{1-(1-p)e^{\log M_X(t)}} = \frac{pM_X(t)}{1-(1-p)M_X(t)} = \frac{p \cdot \frac{\lambda}{\lambda - t}}{1-(1-p)\cdot \frac{\lambda}{\lambda-t}} \\ &= \frac{p\lambda}{\lambda - t - (1-p)\lambda} = \frac{p\lambda}{p\lambda - t}, \end{aligned}$$

hence, $S \sim Exp(p\lambda)$.

(d) Find the density of S by conditioning on N, and hence confirm the result of Part (c).

[Find $P(S \in ds)$ by conditioning on N.]

Given $N = n$, the distribution of S is $\Gamma(n, \lambda)$. Hence,

$$\begin{aligned} P(S \in ds) = E[P(S \in ds \mid N)] &= E\left[\frac{\lambda^N s^{N-1} e^{-\lambda s}}{(N-1)!}\right] = \lambda e^{-\lambda s} E\left[\frac{(\lambda s)^{N-1}}{(N-1)!}\right] \\ &= \lambda e^{-\lambda s} \sum_{k=1}^{\infty} p(1-p)^{k-1}\frac{(\lambda s)^{k-1}}{(k-1)!} = p\lambda e^{-\lambda s}\sum_{k=1}^{\infty}\frac{(\lambda s(1-p))^{k-1}}{(k-1)!} \\ &= p\lambda e^{-\lambda s} e^{\lambda s(1-p)} = p\lambda e^{-p\lambda s} \end{aligned}$$

for $s > 0$, so $S \sim Exp(p\lambda)$.

(e) Use a Poisson (λ) process and Poissonization (also known as "thinning" in the context of Poisson processes) to find yet another way of confirming the result of Part (c).

Let's take a Poisson point process with rate λ, waiting times $X_1, X_2, \ldots \sim Exp(\lambda)$ and arrival times $T_1, T_2 \ldots$ If we mark each arrival of this process independently with probability p, then the first marked arrival is at T_N with $N \sim Geo(p)$, and $T_N = X_1 + \ldots + X_N = S$. On the other hand, we can think of T_N as the first arrival time in the process that we get by thinning the original one with the factor p, and we know that this resulting process is another PPP with rate $p\lambda$. Hence, $S = T_N \sim Exp(p\lambda)$.

2.

Let X and Y be independent random variables. Let X have moment generating function

$$M_X(t) = e^{5t+2t^2}, \quad -\infty < t < \infty$$

and let Y have moment generating function

$$M_Y(t) = e^{8t^2}, \quad -\infty < t < \infty$$

(a) Find the moment generating function of $X - 2Y - 3$.

$$E[e^{t(X-2Y-3)}] = E[e^{tX}] \cdot E[e^{-2tY}] \cdot e^{-3t} = M_X(t)M_Y(-2t)e^{-3t} = e^{34t^2+2t}$$

(b) Find $P(X > 2Y + 3)$.

Since the mgf of the normal (μ, σ^2) distribution is $e^{\frac{\sigma^2 t^2}{2}+\mu t}$, part (a) implies that $X - 2Y - 3 \sim N(2, 68)$. Hence,

$$P(X > 2Y + 3) = P(X - 2Y - 3 > 0) = P\left(\frac{X - 2Y - 3 - 2}{\sqrt{68}} > \frac{-2}{\sqrt{68}}\right) = 1 - \Phi(-2/\sqrt{68}) = \Phi(1/\sqrt{17}).$$

3.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. with the normal (μ, σ^2) distribution. Define the sample mean M as

$$M = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and for each i in the range 1 through n let the ith deviation from the mean be D_i defined by

$$D_i = X_i - M.$$

(a) Find the joint distribution of D_1 and D_2.

Bivariate normal with $E[D_1] = E[D_2] = 0$, $Var[D_1] = Var[D_2] = (1 - 1/n)\sigma^2$, $Cov(D_1, D_2) = -\sigma^2/n$, see Exercise 5b, Week 11.

(b) Pick one option and justify your choice: the random variables M and D_1 are

1. neither uncorrelated nor independent
2. uncorrelated but not independent
3. independent but not uncorrelated
4. uncorrelated and independent

Uncorrelated and independent, see the same problem.

4.

Let M have the Gamma(r, λ) density. Given $M = m$, let N have the Poisson distribution with parameter m. Compute the following:

(a) $E(N \mid M)$

M

(b) $Var(N \mid M)$

M

(c) $E(N)$

$E[N] = E[E[N \mid M]] = E[M] = \frac{r}{\lambda}$

(d) $Var(N)$

$Var[N] = E[Var[N \mid M]] + Var[E[N \mid M]] = E[M] + Var[M] = \frac{r}{\lambda} + \frac{r}{\lambda^2}$

(e) For $m > 0$ and non-negative integer n, $P(M \in dm, N = n)$

$$P(M \in dm, N = n) = P(M \in dm)P(N = n \mid M \in dm) = \frac{\lambda^r m^{r-1} e^{-\lambda m}}{\Gamma(r)} \cdot \frac{m^n}{n!} e^{-m} = \frac{\lambda^r}{\Gamma(r) n!} m^{r+n-1} e^{-(\lambda+1)m}$$

(f) The posterior distribution of M given $N = n$

$$P(M \in dm \mid N = n) = \frac{P(M \in dm, N = n)}{P(N = n)},$$

so by looking at the shape of the distribution, given by the formula in part (e), we can see that the posterior is $\Gamma(r + n, \lambda + 1)$. (cf. Exercise 6, Week 9)

5.

Let Z have the standard normal density. Then $E(Z^k)$ is well-defined and finite for every positive integer k. In this question you will find the numerical value of $E(Z^k)$ for each positive k.

(a) Let n be a positive integer and consider the odd integer $k = 2n - 1$. What is the value of $E(Z^{2n-1})$ and why?

0, because the distribution of Z^{2n-1} is symmetric around 0.

(b) Write the formula for the density of Z^2. You don't have to derive the formula if you remember it or can work it out from the formula sheets.

The distribution of Z^2 is $\chi^2(1)$, or equivalently, $\Gamma(1/2, 1/2)$, which has density $f_{Z^2}(x) = \frac{1}{\sqrt{2}\Gamma(1/2)} x^{-1/2} e^{-x/2}$ for $x > 0$.

(c) Let n be a positive integer and consider the even integer $k = 2n$. Use part (b) to find $E(Z^{2n})$ in terms of the Gamma function.

$$E[Z^{2n}] = \int_0^\infty x^n f_{Z^2}(x)\, dx = \int_0^\infty \frac{1}{\sqrt{2}\Gamma(1/2)} x^{n-1/2} e^{-x/2}\, dx = \frac{1}{\sqrt{2}\Gamma(1/2)} \cdot \frac{\Gamma(n+1/2)}{(1/2)^{n+1/2}} = 2^n \cdot \frac{\Gamma(n+1/2)}{\Gamma(1/2)}$$

(d) For each positive integer n, find an integer c_n such that $E(Z^{2n}) = c_n E(Z^{2n-2})$. Then use induction to derive a formula for $E(Z^{2n})$ that does not involve the Gamma function.

By part (c),

$$c_n = \frac{E[Z^{2n}]}{E[Z^{2(n-1)}]} = 2 \cdot \frac{\Gamma(n+1/2)}{\Gamma(n-1/2)} = 2\left(n - \frac{1}{2}\right) = 2n - 1.$$

Hence, by induction,

$$E[Z^{2n}] = E[Z^2] \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1),$$

which is usually denoted by $(2n-1)!!$