Stat 200A, Fall 2018
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WEEK 7 EXERCISES

You are expected to do all these problems, but for **Homework 7** please turn in **only Problems 2, 3, 5, and 6** on **Thursday October 11 at the start of lecture**.

1. Poisson MGF

Let X have Poisson (μ) distribution, and let Y independent of X have Poisson (λ) distribution.

a) Find the mgf of X.

$$M_X(t) = E\left[e^{tX}\right] = \sum_{k=0}^{\infty} e^{tk}\frac{\mu^k}{k!}e^{-\mu} = e^{-\mu}\sum_{k=0}^{\infty}\frac{(e^t\mu)^k}{k!} = e^{-\mu}e^{e^t\mu} = e^{\mu(e^t-1)}$$

b) Use the result of Part **a** to show that the distribution of $X+Y$ is Poisson.

$$M_{X+Y}(t) = E\left[e^{t(X+Y)}\right] = E\left[e^{tX}e^{tY}\right] = E\left[e^{tX}\right]E\left[e^{tY}\right] = e^{\mu(e^t-1)}\cdot e^{\lambda(e^t-1)} = e^{(\mu+\lambda)(e^t-1)},$$

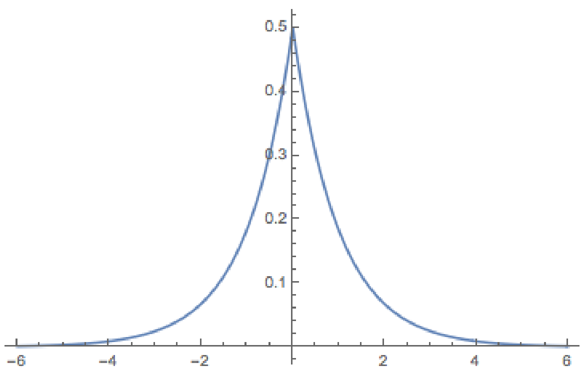
which is the mgf of the $Poisson(\mu+\lambda)$ distribution, hence, $X+Y \sim Poisson(\mu+\lambda)$.

2. Bilateral Exponential Moments

Let X have density

$$f_X(x) \quad = \quad \frac{1}{2}e^{-|x|}, \qquad -\infty < x < \infty$$

a) Sketch a graph of f_X.



b) Find the mgf of X (careful about where it is defined).

$$
\begin{aligned}
M_X(t) = E\left[e^{tX}\right] &= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{2} e^{-|x|} dx \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{tx} e^{x} dx + \frac{1}{2}\int_{0}^{\infty} e^{tx} e^{-x} dx \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{(t+1)x} dx + \frac{1}{2}\int_{0}^{\infty} e^{(t-1)x} dx \\
&= \frac{1}{2}\left[\frac{e^{(t+1)x}}{t+1}\right]_{x=-\infty}^{0} + \frac{1}{2}\left[\frac{e^{(t-1)x}}{t-1}\right]_{x=0}^{\infty} \\
&= \frac{1}{2}\cdot\frac{1}{t+1} - \frac{1}{2}\cdot\frac{1}{t-1} = \frac{1}{(1+t)(1-t)}
\end{aligned}
$$

if both $1+t$ and $1-t$ are positive, that is, $-1 < t < 1$. (Otherwise, one of the two integrals will be infinite.)

c) Use the mgf to find the even moments of X. The odd moments are of course all 0.

The moments are given by the derivatives of the mgf evaluated at 0. In the calculations below $t \in (-1, 1)$.

$$
\begin{aligned}
M_X(t) &= \frac{1}{2}\left(\frac{1}{1+1} + \frac{1}{1-t}\right) \\
M_X'(t) &= \frac{1}{2}\left(\frac{-1}{(1+t)^2} + \frac{1}{(1-t)^2}\right) \\
M_X''(t) &= \frac{1}{2}\left(\frac{2}{(1+t)^3} + \frac{2}{(1-t)^3}\right) \\
M_X'''(t) &= \frac{1}{2}\left(\frac{-3!}{(1+t)^4} + \frac{3!}{(1-t)^4}\right)
\end{aligned}
$$

We can recognize the pattern (and prove by induction) that

$$
\frac{d^k M_X}{dt^k}(t) = \frac{1}{2}\left(\frac{(-1)^k k!}{(1+t)^{k+1}} + \frac{k!}{(1-t)^{k+1}}\right).
$$

Hence, the even moments are given by the formula

$$
E\left[X^k\right] = \frac{d^k M_X}{dt^k}(0) = k!
$$

for $k = 0, 2, 4, 6, \ldots$

d) Check your answer to Part **c** by finding the even moments by direct integration.

$$
E\left[X^k\right] = \int_{-\infty}^{\infty} x^k \cdot \frac{1}{2} e^{-|x|} dx = \frac{1}{2}\underbrace{\int_{-\infty}^{0} x^k e^x dx}_{I_1(k)} + \frac{1}{2}\underbrace{\int_{0}^{\infty} x^k e^{-x} dx}_{I_2(k)}
$$

For $k = 0$, $I_1(0) = I_2(0) = 1$, so we get $E\left[X^0\right] = 1$ as we expected. If $k > 0$ even, then we can express $I_1(k)$ and $I_2(k)$ in terms of $I_1(k-2)$ and $I_2(k-2)$ using integration by parts twice:

$$
\begin{aligned}
I_1(k) = \int_{-\infty}^{0} x^k e^x dx &= \left[x^k e^x\right]_{x=-\infty}^{0} - \int_{-\infty}^{0} k x^{k-1} e^x dx = -\int_{-\infty}^{0} k x^{k-1} e^x dx \\
&= -\left[k x^{k-1} e^x\right]_{x=-\infty}^{0} + \int_{-\infty}^{0} k(k-1) x^{k-2} e^x dx = \int_{-\infty}^{0} k(k-1) x^{k-2} e^x dx \\
&= k(k-1) I_1(k-2)
\end{aligned}
$$

Very similarly, $I_2(k) = k(k-1)I_2(k-2)$. Hence, $E\left[X^k\right] = k(k-1)E\left[X^{k-2}\right]$. This, along with the base case $k = 0$, inductively proves that $E\left[X^k\right] = k!$ for even k.

3. Gamma Tail Bound

Let X have the gamma (r, λ) distribution. Show that

$$P\left(X \geq \frac{2r}{\lambda}\right) \leq \left(\frac{2}{e}\right)^r$$

First, notice that $P\left(X \geq \frac{2r}{\lambda}\right) = P\left(\lambda X \geq 2r\right)$, and $Y = \lambda X$ has the Gamma $(r, 1)$ distribution. Now, using Markov's inequality for the exponentials $(t > 0)$:

$$P\left(Y \geq 2r\right) = P\left(e^{tY} \geq e^{2rt}\right) \leq \frac{E\left[e^{tY}\right]}{e^{2rt}} = \left(\frac{1}{1-t}\right)^r \cdot e^{-2rt},$$

and this equals $\left(\frac{2}{e}\right)^r$ for $t = 1/2$.

4. Poisson Phone Calls

This is Pitman Exercise 4.2.5.

Suppose calls are arriving at a telephone exchange at an average rate of 2 per second, according to a Poisson arrival process. Find

a) the probability that the fourth call after time $t = 0$ arrives within 2 seconds of the third call;

Let us denote the arrival times of the calls in order by $T_1, T_2, \ldots$ Then $T_4 - T_3 \sim Exp(2)$ (the parameter being the rate of the process), so $\mathbb{P}(T_4 - T_3 < 2) = 1 - e^{-2\cdot 2} = 1 - e^{-4}$.

b) the probability that the fourth call arrives by time $t = 5$ seconds;

Let N_t be the number of calls up to time t. Then $N_t \sim Poisson(2t)$, so

$$\mathbb{P}(T_4 < 5) = \mathbb{P}(N_5 \geq 4) = 1 - \mathbb{P}(N_5 \leq 3) = 1 - \sum_{k=0}^{3} \frac{10^k}{k!} e^{-10}.$$

c) the expected time at which the fourth call arrives

Using that the waiting times are Exp(2) variables,

$$E[T_4] = E[T_1] + E[T_2 - T_1] + E[T_3 - T_2] + E[T_4 - T_3] = 4 \cdot \frac{1}{2} = 2.$$

5. More Poisson Phone Calls

This is Pitman Exercise 4.Review.13.

Local calls are coming into a telephone exchange according to a Poisson process with rate λ_{loc} calls per minute. Indpendently of this, long distance calls are coming in at a rate of λ_{dis} calls per minute. Write down expressions for the probabilities of the following events:

a) exactly 5 local calls and 3 long distance calls arrive in a given minute

If $N^{loc}_{[0,t]}$ is the number of local calls between times 0 and t, and $N^{dis}_{[0,t]}$ is the number of long distance calls in the same period, then $N^{loc}_{[0,t]} \sim Poisson(t\lambda_{loc})$, $N^{dis}_{[0,t]} \sim Poisson(t\lambda_{dis})$, and they are independent. Thus,

$$\mathbb{P}(N^{loc}_{[0,1]} = 5, N^{dis}_{[0,1]} = 3) = e^{-\lambda_{loc}} \frac{\lambda^5_{loc}}{5!} \cdot e^{-\lambda_{dis}} \frac{\lambda^3_{dis}}{3!}.$$

b) exactly 50 calls (counting both local and long distance) arrive in a given three-minute period

The total number of calls in the time interval $[0,t]$, $N_{[0,t]} = N^{loc}_{[0,t]} + N^{dis}_{[0,t]}$ is the sum of two independent Poisson variables, so it has the Poisson$(t(\lambda_{loc} + \lambda_{dis}))$ distribution. Hence,

$$\mathbb{P}(N_{[0,3]} = 50) = e^{-3(\lambda_{loc}+\lambda_{dis})} \frac{(3(\lambda_{loc} + \lambda_{dis}))^{50}}{50!}.$$

c) starting from a fixed time, the first ten calls to arrive are local

This can be computed in several ways. For an intuitive way, we first figure out the probability that a given call is local. The arrival times of all the calls form a Poisson point process with rate $\lambda_{loc} + \lambda_{dis}$ (as in the approach of part b), and each of them is independently local with some probability p. If we pick each call with probability p, we get a PPP with rate $p(\lambda_{loc}+\lambda_{dis})$. But this has to be the PPP given by the local calls, which has rate λ_{loc}. So $p = \frac{\lambda_{loc}}{\lambda_{loc}+\lambda_{dis}}$. Thus, the probability that the first ten calls are local is $\left(\frac{\lambda_{loc}}{\lambda_{loc}+\lambda_{dis}}\right)^{10}$.

For another solution, if the arrival of the first local call is T^{loc}_1, and that of the first long distance one is T^{dis}_1, then $T^{loc}_1 \sim Exp(\lambda_{loc})$, and $T^{dis}_1 \sim Exp(\lambda_{dis})$ independently of each other. The probability that the first call is local then is $\mathbb{P}(T^{loc}_1 < T^{dis}_1)$, which is $\frac{\lambda_{loc}}{\lambda_{loc}+\lambda_{dis}}$ by an earlier exercise. The memoryless property then implies that the chance that the second call is local will be the same, and so on, for all the calls.

6. Poisson Particles

This is Pitman Exercise 4.Review.14.

Particles arrive at a Geiger counter according to a Poisson process with rate 3 per minute.

a) Find the chance that fewer than 4 particles arrive in the time interval 0 to 2 minutes.

The number of particles arriving in the time interval $[0,t]$, $N_{[0,t]}$ follows the Poisson$(3t)$ distribution. Hence,

$$\mathbb{P}(N_{[0,2]} < 4) = \sum_{k=0}^{3} e^{-6} \frac{6^k}{k!}.$$

b) Let T_n minutes denote the arrival time of the nth particle. Find

$$P(T_1 < 1, T_2 - T_1 < 1, T_3 - T_2 < 1)$$

Since the waiting times are i.i.d Exp(3) variables, this probability is

$$\mathbb{P}(T_1 < 1)\mathbb{P}(T_2 - T_1 < 1)\mathbb{P}(T_3 - T_2 < 1) = (1 - e^{-3})^3.$$

c) Find the conditional distribution of the number of arrivals in 0 to 2 minutes given that there were 10 arrivals in 0 to 4 minutes. Recognize this as a named distribution, and state the parameters.

$$
\begin{aligned}
\mathbb{P}(N_{[0,2]} = k \mid N_{[0,4]} = 10) &= \frac{\mathbb{P}(N_{[0,2]} = k, N_{[0,4]} = 10)}{\mathbb{P}(N_{[0,4]} = 10)} = \frac{\mathbb{P}(N_{[0,2]} = k, N_{[2,4]} = 10-k)}{\mathbb{P}(N_{[0,4]} = 10)} \\
&= \frac{\mathbb{P}(N_{[0,2]} = k)\mathbb{P}(N_{[2,4]} = 10-k)}{\mathbb{P}(N_{[0,4]} = 10)} \\
&= \frac{e^{-6} \cdot \frac{6^k}{k!} \cdot e^{-6} \cdot \frac{6^{10-k}}{(10-k)!}}{e^{-12} \cdot \frac{12^{10}}{10!}} = \frac{10!}{k!(10-k)!} \left(\frac{6}{12}\right)^k \left(\frac{6}{12}\right)^{10-k}.
\end{aligned}
$$

Therefore, the distribution of $N_{[0,2]}$ given $N_{[0,4]} = 10$ is Binomial(10,0.5).

7. Poisson Toll Booth

This is a version of Pitman Exercise 4.Review.16

Cars arrive at a toll booth according to a Poisson process at rate of 3 arrivals per minute. Of the cars arriving at the booth, it is known that over the long run 60% are Japanese imports. What is the probability that the cars that arrive in a given 10-minute period consist of 10 that are Japanese imports and 5 that are not? State your assumptions clearly.

If N is the number of cars arriving in the first 10 minutes, then $N \sim Poisson(30)$. Of these N cars each one is Japanese with probability 0.6, independently of each other. Hence, the number of Japanese cars among them, N_J, and the number of non-Japanese ones, N_N are independent, and both follow a Poisson distribution: $N_J \sim Poisson(30 \cdot 0.6)$, and $N_N \sim Poisson(30 \cdot 0.4)$. Thus,

$$
\mathbb{P}(N_J = 10, N_N = 5) = \mathbb{P}(N_J = 10) \cdot \mathbb{P}(N_N = 5) = \frac{18^{10}}{10!} e^{-18} \cdot \frac{12^5}{5!} e^{-12}.
$$