Homework 5 Solutions
Statistics 200B
Due March 7, 2019

1. Let $X_1, \ldots, X_n \overset{iid}{\sim} Poisson(\lambda)$.

   (a) Let $\lambda_0 > 0$. Find the size α Wald test for

   $$H_0 : \lambda = \lambda_0 \text{ versus } H_1 : \lambda \neq \lambda_0.$$

   (b) (Computer experiment) Let $\lambda_0 = 1$, $n = 20$, and $\alpha = 0.05$. Simulate $X_1, \ldots, X_n \sim Poisson(\lambda_0)$ and perform the Wald test. Repeat many times and count how often you reject the null hypothesis. How close is the type I error rate to 0.05?

Solution:

(a) It can be easily shown that the MLE of λ is $\hat{\lambda} = \bar{X}_n$, and

$$se(\hat{\lambda}) = \sqrt{V(\bar{X}_n)} = \sqrt{V(\bar{X}_1)/n} = \sqrt{\lambda/n}.$$

Define the Wald statistic as

$$W = \frac{\hat{\lambda} - \lambda_0}{\hat{se}(\hat{\lambda})} = \frac{\bar{X}_n - \lambda_0}{\sqrt{\bar{X}_n/n}},$$

which is asymptotically distributed as $N(0,1)$ when $\lambda = \lambda_0$.
Because

$$P_{\lambda_0}(|W| > z_{\alpha/2}) = \alpha,$$

we have found a size α Wald test: reject H_0 when $|W| > z_{\alpha/2}$.

(b) The following `R` codes can be applied to the computer experiment:

```
B <- 1000 # do the simulation for 1000 times
n <- 20
lambda0 <- 1
```

```
alpha <- 0.05
W <- rep(0, B)
for (i in 1:B) {
X <- rpois(n, lambda0)
W[i] <- (mean(X)-lambda0) / sqrt(mean(X)/n)
}
percent_rejection <- sum(abs(W)>qnorm(1-alpha/2))/B
# 0.049
```

The percentage of rejecting H_0 for data simulated from H_0 (type I error) is 0.049, which is very close to the size $\alpha = 0.05$.

2. Let $X \sim Binomial(n, p)$. Construct the likelihood ratio test for

$$H_0 : p = p_0 \text{ versus } H_1 : p \neq p_0.$$

Compare to the Wald test.

Solution:

- LRT:
  The likelihood ratio statistic is

$$\lambda = 2 \log \left(\frac{L(\hat{p})}{L(p_0)} \right) = 2(\ell(\hat{p}) - \ell(p_0)),$$

  where $\hat{p}$ is the MLE: $\hat{p} = X/n$.
  Since

$$\ell(p) = constant + X \log p + (n - X) \log(1 - p),$$

$$\begin{aligned}\lambda &= 2[X \log \hat{p} + (n - X) \log(1 - \hat{p})] - 2[X \log p_0 + (n - X) \log(1 - p_0)] \\ &= 2X \log \left(\frac{X}{np_0} \right) + 2(n - X) \log \left(\frac{n - X}{n(1 - p_0)} \right).\end{aligned}$$

  The rejection region is $\lambda > c$, with c to be determined by the size α.
- Wald Test:
  The MLE of p is $\hat{p} = X/n$, whose standard error is $se(\hat{p}) = \sqrt{p(1-p)/n}$.
  So $\hat{se}(\hat{p}) = \sqrt{\hat{p}(1-\hat{p})/n}$. The Wald statistic is

$$W = \frac{\hat{p} - p_0}{\hat{se}(\hat{p})} = \frac{X - np_0}{\sqrt{X(n - X)/n}}.$$

  The rejection region is $|W| > z_{\alpha/2}$.

3. Suppose $X_1, \ldots, X_n$ are iid with PDF

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Find the likelihood ratio test statistic $T(X) = \frac{\sup_{\theta\in\Theta} \mathcal{L}_n(\theta)}{\sup_{\theta\in\Theta_0} \mathcal{L}_n(\theta)}$. Hint: Consider separately the cases $\min\{X_1, \ldots, X_n\} \leq \theta_0$ and $\min\{X_1, \ldots, X_n\} > \theta_0$.

Solution:

$$\mathcal{L}_n(\theta) = \prod_{i=1}^{n} f(X_i;\theta) = \prod_{i=1}^{n} e^{-(X_i-\theta)} I(X_i \geq \theta) = e^{-\sum_{i=1}^{n} X_i + n\theta} I(\min\{X_1, \ldots, X_n\} \geq \theta)$$

Since $\mathcal{L}_n(\theta)$ is an increasing function in θ on $-\infty < \theta \leq \min\{X_1, \ldots, X_n\}$ and is zero for $\theta > \min\{X_1, \ldots, X_n\}$, $\sup_{\theta\in\Theta} \mathcal{L}_n(\theta) = \mathcal{L}_n(\min\{X_1, \ldots, X_n\})$.

When $\min\{X_1, \ldots, X_n\} \leq \theta_0$, $\sup_{\theta\in\Theta_0} \mathcal{L}_n(\theta) = \mathcal{L}_n(\min\{X_1, \ldots, X_n\})$.

When $\min\{X_1, \ldots, X_n\} > \theta_0$, $\sup_{\theta\in\Theta_0} \mathcal{L}_n(\theta) = \mathcal{L}_n(\theta_0)$.

Therefore, the likelihood ratio test statistic

$$T(X) = \begin{cases} 1 & \text{if } \min\{X_1, \ldots, X_n\} \leq \theta_0 \\ \frac{\mathcal{L}_n(\min\{X_1, \ldots, X_n\})}{\mathcal{L}_n(\theta_0)} = \exp\{n(\min\{X_1, \ldots, X_n\} - \theta_0)\} & \text{if } \min\{X_1, \ldots, X_n\} > \theta_0 \end{cases}.$$

4. Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. For each of the following cases, derive the power function of the size 0.05 LRT, and plot it in `R` for $n = 1, 4, 16, 64$.

   (a) $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$

   (b) $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$

Solutions:

(a) $\Theta = \mathbb{R}$, $\Theta_0 = (-\infty, 0]$. The likelihood ratio statistic is

$$\lambda = 2\log\left(\frac{\mathcal{L}_n(\hat{\mu})}{\sup_{\mu\in\Theta_0} \mathcal{L}_n(\mu)}\right),$$

where $\hat{\mu}$ is the MLE on Θ.
From previous exercises, we know $\hat{\mu} = \bar{X}_n$.

- If $\bar{X}_n \leq 0$, then $\sup_{\mu\in\Theta_0} \mathcal{L}_n(\mu) = \mathcal{L}_n(\hat{\mu})$.

$$\lambda = 2\log\left(\frac{\mathcal{L}_n(\hat{\mu})}{\mathcal{L}_n(\hat{\mu})}\right) = 0.$$

- If $\bar{X}_n > 0$, then

$$\ell_n(\mu) = constant - n\log\sigma - \frac{\sum_{i=1}^n (X_i-\mu)^2}{2\sigma^2}.$$

$$\Rightarrow \frac{\partial \ell_n}{\partial \mu}(\mu) = \frac{\sum_{i=1}^n (X_i-\mu)}{\sigma^2} = \frac{n(\bar{X}_n-\mu)}{\sigma^2} > 0, \quad \text{for } \mu\in\Theta_0.$$

  So $\mathcal{L}_n(\mu)$ is an increasing function of μ for $\mu \in (-\infty, 0]$, and $\sup_{\mu\in\Theta_0} \mathcal{L}_n(\mu) = \mathcal{L}_n(0)$.

$$\lambda = 2\log\left(\frac{\mathcal{L}_n(\hat{\mu})}{\mathcal{L}_n(0)}\right) = 2(\ell_n(\hat{\mu}) - \ell_n(0)) = -\frac{\sum_{i=1}^n (X_i-\bar{X}_n)^2}{2\sigma^2} + \frac{\sum_{i=1}^n X_i^2}{2\sigma^2} = \frac{n(\bar{X}_n)^2}{2\sigma^2} \geq 0.$$

Since $\lambda \geq 0$, the rejection region should be $\lambda > c$, with some $c > 0$ to be determined by the size $\alpha = 0.05$. We can also see from the above that this rejection region is equivalent to $\bar{X}_n > d$, with $d > 0$ to be determined by the size $\alpha = 0.05$.
Since the test has size 0.05,

$$\begin{aligned}
0.05 &= \sup_{\mu\in\Theta_0} P_\mu(\bar{X}_n > d) \\
&\quad (\text{Since } \bar{X}_n \sim N(\mu, \sigma^2/n),) \\
&= \sup_{\mu\in\Theta_0} P_\mu\left(\frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma} > \frac{\sqrt{n}(d-\mu)}{\sigma}\right) = \sup_{\mu\in\Theta_0}\left[1-\Phi\left(\frac{\sqrt{n}(d-\mu)}{\sigma}\right)\right] \\
&= 1-\Phi(\sqrt{n}d/\sigma).
\end{aligned}$$

$$\Rightarrow d = \frac{\Phi^{-1}(0.95)\sigma}{\sqrt{n}} = \frac{z_{0.05}\sigma}{\sqrt{n}}.$$

The power function of the test is

$$\begin{aligned}
\beta(\mu) = P_\mu(\bar{X}_n > d) &= P_\mu\left(\frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma} > \frac{\sqrt{n}(d-\mu)}{\sigma}\right) = 1-\Phi\left(\frac{\sqrt{n}(d-\mu)}{\sigma}\right) \\
&= 1-\Phi\left(\Phi^{-1}(0.95) - \frac{\sqrt{n}\mu}{\sigma}\right) = 1-\Phi\left(z_{0.05} - \frac{\sqrt{n}\mu}{\sigma}\right).
\end{aligned}$$

(b) $\Theta = \mathbb{R}$, $\Theta_0 = \{0\}$. The likelihood ratio statistic is

$$\lambda = 2\log\left(\frac{\mathcal{L}_n(\hat{\mu})}{\sup_{\mu\in\Theta_0}\mathcal{L}_n(\mu)}\right) = 2\log\left(\frac{\mathcal{L}_n(\hat{\mu})}{\mathcal{L}_n(0)}\right) = 2(\ell_n(\hat{\mu}) - \ell_n(0)),$$

where $\hat{\mu}$ is the MLE on Θ.
From (a),

$$\lambda = 2(\ell_n(\hat{\mu}) - \ell_n(0)) = \frac{n(\bar{X}_n)^2}{2\sigma^2} \geq 0.$$

The rejection region is $\lambda > c$, with $c > 0$ to be determined by the level $\alpha = 0.05$. This is equivalent to $|\bar{X}_n| > d$, with $d > 0$ to be determined by $\alpha = 0.05$.

$$\begin{aligned}
0.05 &= P_{\mu=0}(|\bar{X}_n| > d) \\
&\quad \text{When } \mu = 0, \bar{X}_n \sim N(0, \sigma^2/n) \\
&= P_{\mu=0}\left(\sqrt{n}\bar{X}_n/\sigma > \sqrt{n}d/\sigma\right) + P_{\mu=0}\left(\sqrt{n}\bar{X}_n/\sigma < -\sqrt{n}d/\sigma\right) \\
&= 1 - \Phi(\sqrt{n}d/\sigma) + \Phi(-\sqrt{n}d/\sigma) = 2(1 - \Phi(\sqrt{n}d/\sigma)).
\end{aligned}$$

$$\Rightarrow d = \frac{\Phi^{-1}(0.975)\sigma}{\sqrt{n}} = \frac{z_{0.025}\sigma}{\sqrt{n}}.$$

The power function of the test is

$$\begin{aligned}
\beta(\mu) &= P_\mu\left(|\bar{X}_n| > d\right) \\
&= P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} > \frac{\sqrt{n}(d-\mu)}{\sigma}\right) + P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} < -\frac{\sqrt{n}(d-\mu)}{\sigma}\right) \\
&= 1 - \Phi\left(\frac{\sqrt{n}(d-\mu)}{\sigma}\right) + \Phi\left(-\frac{\sqrt{n}(d-\mu)}{\sigma}\right) \\
&= 1 - \Phi\left(\Phi^{-1}(0.975) - \frac{\sqrt{n}\mu}{\sigma}\right) + \Phi\left(-\Phi^{-1}(0.975) - \frac{\sqrt{n}\mu}{\sigma}\right) \\
&= 1 - \Phi\left(z_{0.025} - \frac{\sqrt{n}\mu}{\sigma}\right) + \Phi\left(-z_{0.025} - \frac{\sqrt{n}\mu}{\sigma}\right).
\end{aligned}$$

The power functions of (a) and (b) are plotted below. The `R` code is

```
beta1 <- function(n,x) 1-pnorm(qnorm(0.95) - sqrt(n)*x)
beta2 <- function(n,x) 1-pnorm(qnorm(0.975) - sqrt(n)*x) +
pnorm(qnorm(0.025) - sqrt(n)*x)
```

```
power1 <- sapply(c(1,4,16,64), FUN=function(n)
sapply(seq(-3,3,by=0.01), FUN=function(x) beta1(n,x)))
power2 <- sapply(c(1,4,16,64), FUN=function(n)
sapply(seq(-3,3,by=0.01), FUN=function(x) beta2(n,x)))
par(mfrow=c(2,1))
matplot(power1, x=seq(-3,3,by=0.01), type="l", xlab=expression(mu/sigma),
ylab=expression(beta(mu)), main="(a)")
legend("topleft", paste("n =", c(1,4,16,64)), lty=1:5, col=1:6, cex=0.7)
matplot(power2, x=seq(-3,3,by=0.01), type="l", xlab=expression(mu/sigma),
ylab=expression(beta(mu)), main="(b)")
legend("bottomleft", paste("n =", c(1,4,16,64)), lty=1:5, col=1:6, cex=0.7)
```

From the plot we can see that given the level α, when n increases, the power function of parameters in the alternative parameter space increases. This means that when the sample size becomes larger, the test becomes more powerful to reject the null hypothesis when the alternative is true.

5. There is a theory that people can postpone their death until after an important event. To test the theory, Philips and King (1988) collected data on deaths around the Jewish holiday of Passover. Of 1919 daths, 922 died the week before the holiday and 997 died the week after. Think of this as a binomial and test the null hypothesis that $\theta = 1/2$. Report and interpret the p-value.

   Solution: Define θ as the probability of death after the holiday, and X as the number of death before the holiday. The test is

   $$H_0 : \theta = 1/2 \text{ vs. } H_1 : \theta > 1/2.$$

   Use the Wald test defined in Problem 2. The test statistic is

   $$W = \frac{X - n \times (1/2)}{\sqrt{X(n-X)/n}},$$

   where $n = 1919$.

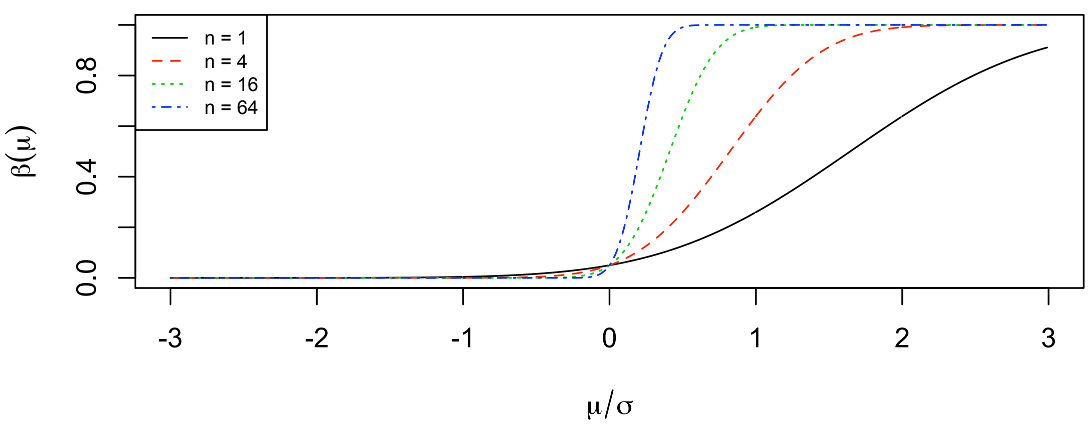
   The observed statistic value is

   $$w = \frac{x - n \times (1/2)}{\sqrt{x(n-x)/n}} = \frac{997 - 1919 \times (1/2)}{\sqrt{997(1919-997)/1919}} = 1.713.$$

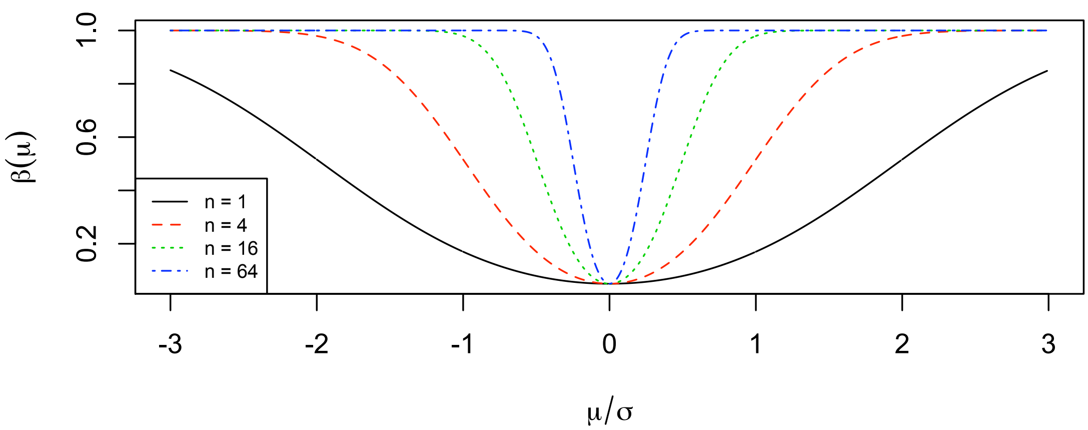
   The p-value is

   $$\text{p-value} = P_{\theta=1/2}(W > w) = 1 - \Phi(w) = 1 - \Phi(1.713) = 0.043.$$

(a)


n = 1
n = 4
n = 16
n = 64
β(μ)
0.0
0.4
0.8
-3
-2
-1
0
1
2
3
μ/σ

(b)


n = 1
n = 4
n = 16
n = 64
β(μ)
0.2
0.6
1.0
-3
-2
-1
0
1
2
3
μ/σ

This means that the null hypothesis can be rejected at level 0.05.

Note that there is another way to define the rejection region based its binomial property. The rejection region can be defined as $X > c$, with c to be determined by the size of the test. Then the p-value is

$$\text{p-value} = P_{\theta=1/2}(X > x) = \sum_{i=0}^{997} \binom{1919}{i} (1/2)^i (1-1/2)^{1919-i} = 0.041,$$

which is very close to the p-value obtained by the Wald test, and results in the same conclusion of rejecting H_0 at level 0.05.

6. In 1861, 10 essays appeared in the *New Orleans Daily Crescent.* They were signed "Quintus Curtius Snodgrass", and some people suspected they were actually written by Mark Twain. To investigate this, we will consider the proportion of three letter words found in each author's work.
From eight Twain essays we have:
.225 .262 .217 .240 .230 .229 .235 .217
From 10 Snodgrass essays we have:
.209 .205 .196 .210 .202 .207 .224 .223 .220 .201

   (a) Perform a Wald test for equality of the means. Use the nonparametric plug-in estimator. Report the p-value and a 95% confidence interval for the difference in means.

   (b) Can you see any problems with the implicit assumptions we've made in this analysis?

   Solution:

   (a) Define the sample from 8 Twain essays as $X_1, \ldots, X_8$, and its population mean as μ_1. Also define the sample from 10 Snodgrass essays as $Y_1, \ldots, Y_{10}$, and its population mean as μ_2.
   The test is

$$H_0 : \mu_1 = \mu_2 \text{ vs. } H_1 : \mu_1 \neq \mu_2$$

   Use the nonparametric plug-in estimator $\hat{\delta} = \bar{X} - \bar{Y}$ as the estimate of $\delta = \mu_1 - \mu_2$. The estimated standard error of $\hat{\delta}$ is

$$\hat{se} = \sqrt{\frac{s_1^2}{8} + \frac{s_2^2}{10}},$$

where s_1^2 and s_2^2 are the sample variances.

The Wald statistic is

$$W = \frac{\hat{\delta} - 0}{\hat{se}}.$$

The observed value of the statistic is $w = 3.7$. The p-value is

$$\text{p-value} = P_{\delta=0}(|W| > |w|) = 2\Phi(-|w|) \approx 0.0002.$$

An approximate 95% confidence interval for the difference in means is

$$\hat{\delta} \pm z_{\alpha/2}\hat{se} = 0.022 \pm 1.96 \times 0.006 = [0.010, 0.039],$$

which does not cover 0. This also shows that the null hypothesis can be rejected.

(b) This analysis assumes that the two samples are independent. However, there may be a problem with this assumption, as the two samples may be correlated if they have similar topics. If they have a negative correlation, then we have under-estimated the standard error of $\hat{\delta}$, and then under-estimated the p-value and drawn a wrong conclusion.

Also, the analysis assumes we have identically distributed observations, but in fact these are proportions constructed from essays of varying length; therefore they may have different variances.