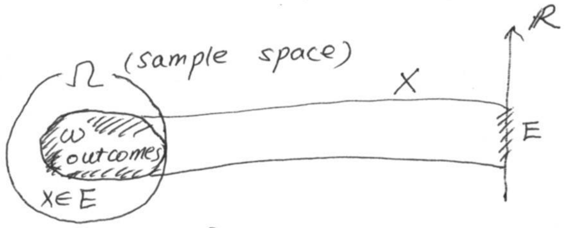
Week 1.

r.v. $X : \Omega \to \mathbb{R}$

random variable is a mapping that assigns a real number $X(\omega)$ to each outcome ω

quantifying outcomes, the sample space (Ω)



$\{\omega : X(\omega) \in E\}$

CDF $\quad F_X(x) := \mathbb{P}(X \le x), \quad F_X : \mathbb{R} \to [0, 1]$

cumulative distribution function

i) Non decreasing $\quad x_1 < x_2 \Rightarrow F(x_1) \le F(x_2)$

ii) Normalised $\quad \lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to +\infty} F(x) = 1$

iii) Right-continuous $\quad \lim_{\substack{y \to x \\ y > x}} F(y) = F(x)$

$X \sim F$ r.v X has distribution F

$X \overset{D}{=} Y$ X and Y are equal in distribution; $F_X(x) = F_Y(x) \quad \forall x$.

density

X discrete

pmf   probability mass function

$p(x) = \mathbb{P}(X = x)$

continuous   pdf   probability density function

$f_X(x)$

i) $f_X(x) \ge 0 \quad \forall x$

ii) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

iii) $\mathbb{P}(a < X < b) = \int_a^b f_X(x)\, dx$

$f_X(x) = F_X'(x) \quad \forall x$ at which F_X is differentiable

$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$

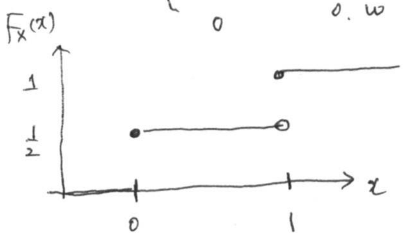
Example : Bernoulli distribution

$$X = \begin{cases} 1 & \text{if Head (Success)} \\ 0 & \text{Tail (failure)} \end{cases}$$

$$p_X(x) = \begin{cases} \frac{1}{2} & \text{if } X = 1 \\ \frac{1}{2} & \text{if } X = 0 \end{cases}$$

$*$ $p_X(x) = p^x(1-p)^{1-x}$ for $x = \{0, 1\}$

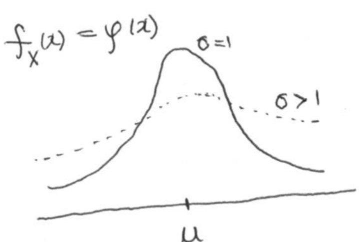
$$F_X(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, 1) \\ 1 & x \geq 1 \\ 0 & \text{o.w} \end{cases}$$

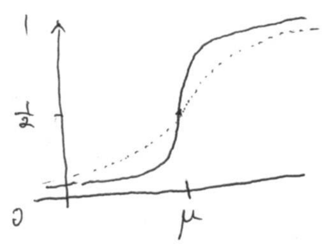


Example : Normal distribution

$X \sim N(\mu, \sigma^2)$

$F_X(x) = \Phi(x)$





$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

No a simple closed form.

$*$ parameter? characteristics of population
numerical summary of population

- location parameter
- disperson parameter
- shape parameter

Example : Indicatior function

$X \sim F$, pdf : f

A : a subset of the real line

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

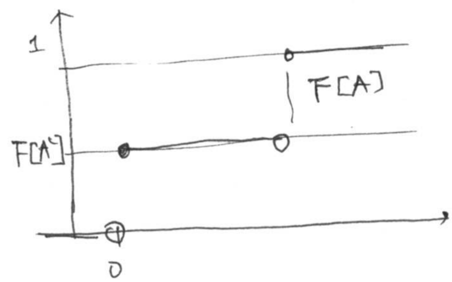
during class,
I assume $A = [a, b]$
since f is continuous.

$$Y = \begin{cases} 1 & \text{if } X \in A \\ 0 & \text{o.w} \end{cases}$$

pmf $\mathbb{P}(Y=1) = \mathbb{P}(X \in A) = \int_{x \in A} f(x)\,dx$

$\mathbb{P}(Y=0) = \mathbb{P}(X \notin A) = \int_{x \in A^c} f(x)\,dx$

CDF



where $F[A^c] = \int_{x \in A^c} f(x)\,dx$

$F[A] = \int_{x \in A} f(x)\,dx$

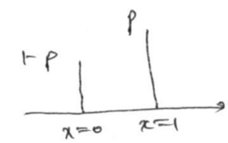
<u>Example</u> $\quad X \neq Y \quad$ but $\quad X \overset{D}{=} Y$

$$X = \begin{cases} 1 & \text{if } H \\ -1 & \text{if } T \end{cases} \qquad Y = \begin{cases} -1 & \text{if } H \\ 1 & T \end{cases}$$

$\mathbb{P}(X=1) = \mathbb{P}(X=-1) = \frac{1}{2} \qquad \mathbb{P}(Y=1) = \mathbb{P}(Y=-1) = \frac{1}{2}$

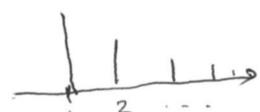
<u>Example</u>

Bernoulli (p) $\quad p^x(1-p)^{1-x} \quad x = \{0, 1\}$

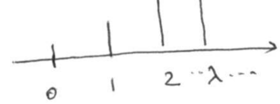


Binomial (n, p) $\quad \binom{n}{x} p^x (1-p)^{n-x} \quad x = \{0, \ldots, n\}$

Geometric (p) $\quad (1-p)^{x-1} p \quad x = \{1, 2, 3, \ldots\}$

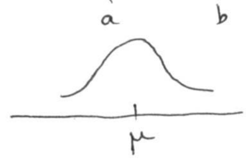


Poisson (λ) $\quad \dfrac{\lambda^x e^{-\lambda}}{x!} \quad x = \{0, 1, \ldots\}$



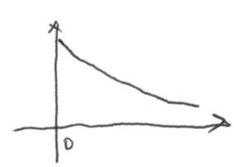
Uniform (a, b) $\quad I_{[a,b]}(x) \dfrac{1}{b-a} \quad x \in [a, b]$

$N(\mu, \sigma^2) \quad \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right) \quad x \in (-\infty, \infty)$

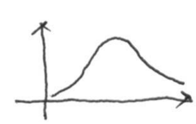


Exponential (β) $\quad \dfrac{e^{-x/\beta}}{\beta}$ (β: scale parameter)

Exponential (λ) $\quad$ or $\lambda \cdot e^{-\lambda x} \quad x \in [0, \infty)$ (λ: rate parameter)



Gamma(α, β) $\quad \dfrac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad x \in (0, \infty)$



Joint CDF

$F(x, y) = \mathbb{P}(X \leq x, Y \leq y)$

X, Y discrete — joint pmf $P(X=x, Y=y)$

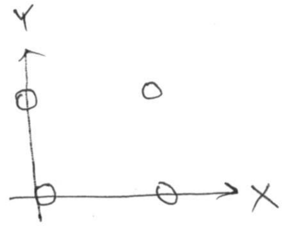
continuous — joint pdf $f(x,y)$

$f(x,y) \geq 0 \quad \forall (x,y)$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy$

example



$$(X, Y) = \begin{cases} (0,0) & TT \\ (0,1) & TH \\ (1,0) & HT \\ (1,1) & HH \end{cases}$$

$P(X=x, Y=y) = \frac{1}{4}$

Marginal density

X discrete $\quad f_X(x) = P_X(x) = \sum_y f(x,y)$

X continuous $\quad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

X, Y independent? $\quad f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)$

$X_1, \ldots, X_n$ iid $\sim F$ $\quad f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_1}(x_i)$ [indepent and identitally distributed]

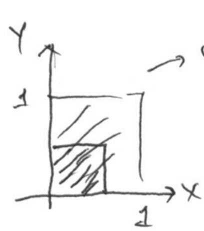
example $P(X=) = \begin{cases} \frac{1}{2} & (H) \\ \frac{1}{2} & (T) \end{cases}$ $\quad P(Y=Y) = \begin{cases} \frac{1}{2} & (H) \\ \frac{1}{2} & (T) \end{cases}$

$P(X=0, Y=0) = P(X=0)\,P(Y=0)$

$\vdots$

X and Y are independent.

example $\quad f(X, Y) = \begin{cases} 1 & \text{if } \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{array} \\ 0 \end{cases}$



uniform on the unit square

$P(X < \frac{1}{2}, Y < \frac{1}{2}) = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} 1\,dx\,dy = \frac{1}{4}$

$P(X < \frac{1}{2}) \times P(Y < \frac{1}{2}) = \frac{1}{4}$

Conditional distribution

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

X, Y discrete $\quad P(X=x \mid Y=y) = f_{X|Y}(x|y)$

X, Y continuous $\quad P(X\in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx$

Transformation

$$Y = r(X)$$

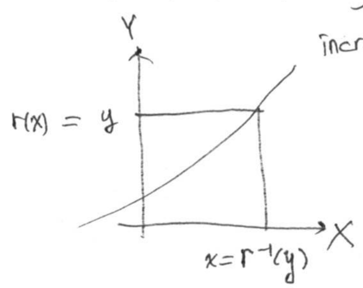
Want pdf $f_Y(y)$ from F_X

discrete $\quad f_Y(y) = \mathbb{P}(Y=y) = \mathbb{P}(r(X)=y)$
$= \mathbb{P}(\{x : r(x) = y\})$
$= \mathbb{P}(X \in r^{-1}(y))$
$= \sum_{x\in r^{-1}(y)} f(x)$

continuous $\quad$ CDF $\quad F_Y(y) = \mathbb{P}(Y\leq y) = p(r(X)<y) = \mathbb{P}(\{x : r(x)\leq y\})$
$= \int_{A_y} f_X(x)\,dx$, where $A_y = \{x : r(x) \leq y\}$

calculate pdf from CDF

* r : strictly monotone : $\quad f_Y(y) = f_X(r^{-1}(y)) \left| \frac{d r^{-1}(y)}{dy} \right|$



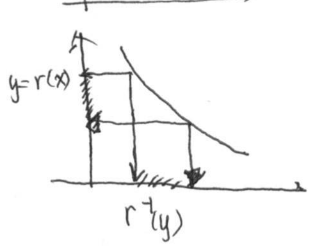
increasing

cdf: $\mathbb{P}(Y\leq y) = \mathbb{P}(X \leq r^{-1}(y)) = F_X(r^{-1}(y))$

pdf: $\quad f_Y(y) = \frac{d}{dy} F_X(r^{-1}(y)) = f_X(r^{-1}(y)) \frac{\partial}{\partial y} r^{-1}(y)$

decreasing

cdf: $\quad \mathbb{P}(Y\leq y) = \mathbb{P}(X \geq r^{-1}(y)) = 1 - \mathbb{P}(X<r^{-1}(y))$
$= 1 - \mathbb{P}(X\leq r^{-1}(y))$
$= 1 - F_X(r^{-1}(y))$



pdf: $\quad f_Y(y) = -f_X(r^{-1}(y)) \underbrace{\frac{\partial}{\partial y} r^{-1}(y)}_{\text{sign } (-)}$

<u>example</u> $X \sim F$, $f_X(x) = e^{-x}$ (exponential (1))

$Y = \log X$

<u>Method 1</u>

$f_X(x) = e^{-x}$

$r(x) = \log x$

$r^{-1}(y) = e^y$

$f_Y(y) = f_X(r^{-1}(y)) \left|\frac{\partial}{\partial y} r^{-1}(y)\right| = e^{-e^y} \cdot e^y$

<u>Method2</u> find a $A_y = \{x : x \le e^y\}$

$= \{x : \log x \le y\} = \{x : x \le e^y\}$

$F_X(x) = \int_0^x f_X(x)\,dx = \int_0^x e^{-x}dx = -[e^{-x}]_0^x = 1 - e^{-x}$

$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\log X \le y) = \mathbb{P}(X \le e^y) = F_X(e^y) = 1 - e^{-e^y}$

$f_Y(y) = e^{-e^y} \cdot e^y$

<u>example</u> $X \sim U(0,1)$ ind recall $f_X(x) = 1_{[0,1]}(x)$, $F_X(x) = \begin{cases} x\, 1_{[0,1]}(x) \\ 0 \quad \text{o.w} \end{cases}$

$Y \sim U(0,1)$

$Z = \max(X, Y)$

$F_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}(\max(X, Y) \le z)$

$= \mathbb{P}(X \le z, Y \le z)$

$\overset{ind}{=} \mathbb{P}(X \le z)\ \mathbb{P}(Y \le z)$

$\overset{identical}{=} [F_X(z)]^2$

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ z^2 & 0 \le z < 1 \\ 1 & z \ge 1 \end{cases}$$

$$f_Z(z) = \begin{cases} 0 & z < 0 \\ 2z & 0 \le z < 1 \\ 1 & z \ge 1 \end{cases}$$

* In general, $X_1, \ldots, X_n \sim U(0, \theta)$

$Z = \max(X_1 \ldots X_n)$ pdf $f_Z(z) = n \cdot \frac{z^{n-1}}{\theta^n}$