Practice Midterm II
Statistics 200B

1. Suppose that $X_1, \ldots, X_n$ form a random sample from a uniform distribution on the interval $(0, \theta)$, and that the following hypotheses are to be tested:

$$H_0: \quad \theta \geq 2$$
$$H_1: \quad \theta < 2$$

Let $Y_n = max\{X_1, \ldots, X_n\}$, and consider a test whose rejection region contains all the outcomes for which $Y_n \leq 1.5$.

   (a) Determine the power function of the test.

   (b) Determine the size of the test.

2. Suppose we observe m *iid Bernoulli*(θ) random variables, denoted by $Y_1, \ldots, Y_m$. Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

   (a) Calculate the MLE for θ under the restriction $\theta \leq \theta_0$.

   (b) Show that the likelihood ratio test will reject H_0 if $\sum_{i=1}^{m} Y_i > b$ for some constant b. (You do not need to determine what b is; it will depend on the size of the test.)

   (c) How would you compute the (exact) p-value in this case?

3. Suppose $X|\theta \sim Bin(n, \theta)$.

   (a) What is the Jeffreys prior distribution for θ?

   (b) Is the Jeffreys prior proper? Why or why not?

   (c) What is the posterior distribution for θ given X when using the Jeffreys prior?

4. Assume that Θ consists of finitely many values, say $\Theta = \{\theta_1, \theta_2, \ldots, \theta_k\}$. Suppose that the prior f assigns positive probability to each $\theta_i \in \Theta$. Let $\hat{\theta}^f$ be a Bayes rule with respect to f. Prove that $\hat{\theta}^f$ is admissible. *Hint: Try proof by contradiction.*

5. Suppose we observe $X = (X_1, X_2, \ldots, X_k)$, a sample from a multinomial distribution with parameter $p = (p_1, p_2, \ldots, p_k)$. The PDF for X is

$$f(x|p) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

Suppose furthermore that we we take p to have prior PDF

$$f(p) = \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^k \alpha_i)} \prod_{i=1}^k p_i^{\alpha_i - 1}.$$

This is the PDF of a Dirichlet distribution with parameter $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. In answering the following, you may refer to the facts below about the $Dirichlet(\alpha)$ distribution.

(a) What is the posterior distribution for p?

(b) What is the Bayes rule under squared error loss?

Facts about the Dirichlet distribution

Suppose $\theta \sim Dirichlet(\alpha)$, where both θ and α have length k. Define $\alpha_0 = \sum_{i=1}^k \alpha_i$. Then

- The marginal distributions are $\theta_i \sim Beta(\alpha_i, \alpha_0 - \alpha_i)$.
- $E[\theta_i] = \alpha_i / \alpha_0$.
- $V[\theta_i] = \frac{\alpha_i(\alpha_0 - \alpha_i)}{\alpha_0^2(\alpha_0 + 1)}$.
- The mode of the distribution is a vector whose i^{th} element is $\frac{\alpha_i - 1}{\alpha_0 - k}$.