Homework 1 Solutions
Statistics 200B
Due Jan. 31, 2019

1. Let $X \sim N(0,1)$ and let $Y = e^X$. Find the PDF for Y.

   Solution:

   Method 1: The CDF of Y is

   $$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \log y) = F_X(\log y) \overset{X \sim N(0,1)}{=} \Phi(\log y),$$

   where Φ is the CDF of $N(0,1)$.

   The PDF of Y can be calculated by taking the first derivative of its CDF.

   $$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{y} f_X(\log y) = \frac{1}{y}\phi(\log y) = \frac{1}{y\sqrt{2\pi}} e^{-(\log y)^2/2},$$

   where ϕ is the CDF of $N(0,1)$. $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

   Method 2: Since $Y = e^X$ and e^x is a monotone function, with $\log x$ as its inverse, we have the PDF of Y as

   $$f_Y(y) = f_X(\log y) \left| \frac{d \log y}{dy} \right| \overset{X \sim N(0,1)}{=} \phi(\log y) \frac{1}{y} = \frac{1}{y\sqrt{2\pi}} e^{-(\log y)^2/2}.$$

   Aside: This is the PDF of the log-normal distribution with $\mu = 0$ and $\sigma^2 = 1$.

2. Let $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\mu)$ and assume that X and Y are independent. Show that the distribution of X given that $X + Y = n$ is $Binomial(n, \pi)$ where $\pi = \lambda/(\lambda + \mu)$.

   Solution:

   If $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\mu)$, and X and Y are independent, then $X + Y \sim Poisson(\lambda + \mu)$.
   Note that $\{X = x, X + Y = n\} = \{X = x, Y = n - x\}$.

   $$p_{X|X+Y=n}(k) = \frac{P(X = k, X + Y = n)}{P(X + Y = n)} = \frac{P(X = k, Y = n - k)}{P(X + Y = n)}$$

$$= \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)} = \frac{e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{(n-k)}}{(n-k)!}}{e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!}}.$$

$$= \binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k}$$

The conditional distribution of X given $X+Y=n$ is a binomial distribution with parameters n and $\pi = \frac{\lambda}{\lambda+\mu}$

3. Let X have PDF $f_X(x) = \begin{cases} 1/4 & 0 < x < 1 \\ 3/8 & 3 < x < 5 \\ 0 & \text{otherwise} \end{cases}$

   (a) Find the CDF of X.

   (b) Let $Y = 1/X$. Find the probability density function $f_Y(y)$ for Y. Hint: Consider three cases: $1/5 \le y \le 1/3$, $1/3 \le y \le 1$, and $y > 1$.

   Solution:

   (a) $F_X(x) = \begin{cases} 1/4x & 0 < x < 1 \\ 1/4 & 1 < x < 3 \\ 3/8x - 7/8 & 3 < x < 5 \\ 0 & \text{otherwise} \end{cases}$

   (b) $F_Y(y) = P(Y \le y) = P(1/X \le y) = P(X \ge 1/y) = 1 - P(X \le 1/y)$

   $F_Y(y) = \begin{cases} 1 - 1/(4y) & y > 1 \\ 15/8 - 3/(8y) & 1/5 < y < 1/3 \\ 3/4 & 1/3 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

   $f_Y(y) = \begin{cases} 1/(4y^2) & y > 1 \\ 3/(8y^2) & 1/5 < y < 1/3 \\ 0 & \text{otherwise} \end{cases}$

4. Let X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} c(x+y) & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find c.

   (b) Find $f_{Y|X}(y|x)$.

(c) Find $P(Y > 1/2|X = 1)$.

(d) Find $P(Y > 1/2|X < 1/2)$

Solution:

(a) Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$, given the joint density, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = \int_0^1\int_0^1 c(x+y)dxdy = c.$$

Hence, $c = 1$.

(b) Find f_X first.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^1 (x+y)dy = x + \frac{1}{2}.$$

Then

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{2(x+y)}{2x+1} & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

(c)

$$P(Y > 1/2|X = 1) = \int_{1/2}^1 f_{Y|X}(y|X = 1)dy = \int_{1/2}^1 \frac{2}{3}(1+y)dy = \frac{7}{12}.$$

(d)

$$P(Y > 1/2|X < 1/2) = \frac{P(Y > 1/2, X < 1/2)}{P(X < 1/2)} = \frac{\int_{1/2}^1\int_0^{1/2}(x+y)dxdy}{\int_0^{1/2}(x+1/2)dx} = \frac{2}{3}.$$

5. Let X be a continuous random variable with CDF F. Suppose that $P(X > 0) = 1$ and that $E[X]$ exists. Show that $E[X] = \int_0^{\infty} P(X > x)dx$. Hint: Consider integrating by parts. The following fact is helpful: if $E[X]$ exists then $\lim_{x\to\infty} x[1 - F(x)] = 0$.

Proof:

$$
\begin{aligned}
\int_0^\infty P(X > x)dx &= \int_0^\infty [1 - F(x)]dx \\
\text{(Let } u = 1 - F(x), v = x) \quad &= \int_0^\infty udv = uv\Big|_0^\infty - \int_0^\infty vdu \\
&= [1 - F(x)]x\Big|_0^\infty - \int_0^\infty x[-f(x)]dx \\
\text{(Use the fact that } \lim_{x\to\infty} x[1 - F(x)] = 0) \quad &= 0 - 0 + \int_0^\infty xf(x)dx \\
&= E[X].
\end{aligned}
$$

Note: The fact that if $E[X]$ exists then $\lim_{x\to\infty} x[1-F(x)] = 0$ can be proved as follows.

Since $E[X] = \int_{-\infty}^\infty xf(x)dx = \int_0^\infty xf(x)dx$ exists,

$$\lim_{x\to\infty} \int_x^\infty yf(y)dy = 0.$$

Also, since

$$\int_x^\infty yf(y)dy \geq \int_x^\infty xf(y)dy = x[1 - F(x)] \geq 0,$$

we have

$$\lim_{x\to\infty} x[1 - F(x)] = 0.$$

6. Let $X \sim \textit{Exponential}(\beta)$. (See page 29 of the textbook for the definition.) Find $P(|X - \mu_X| \geq k\sigma_X)$ for $k > 1$, where μ_X and σ_X denote the mean and standard deviation of the distribution, both equal to β in this case. Calculate an upper bound for this probability using Chebyshev's inequality. Make a plot (a rough sketch is ok) comparing the exact probability to the bound, both as a function of k.

   Solution:

   Since $X \sim$ Exponential(β), $f(x) = \frac{1}{\beta}e^{-x/\beta}, x > 0$, and $\mu_X = \beta$, $\sigma_X = \beta$.

$$
\begin{aligned}
P(|X - \mu_X| \geq k\sigma_X) &= P(|X - \beta| \geq k\beta) \\
&= P(X \geq (k+1)\beta) + P(X \leq (1-k)\beta) \\
\text{(Since } k > 1,\ (1-k)\beta < 0) \quad &= \int_{(k+1)\beta}^\infty f(x)dx + 0 \\
&= e^{-(k+1)}.
\end{aligned}
$$

By using Chebyshev's inequality,

$$P(|X - \mu_X| \geq k\sigma_X) \leq \frac{\sigma_X^2}{(k\sigma_X)^2} = \frac{1}{k^2}.$$

The exact probability and Chebyshev's bound are plotted both as a function of k below.

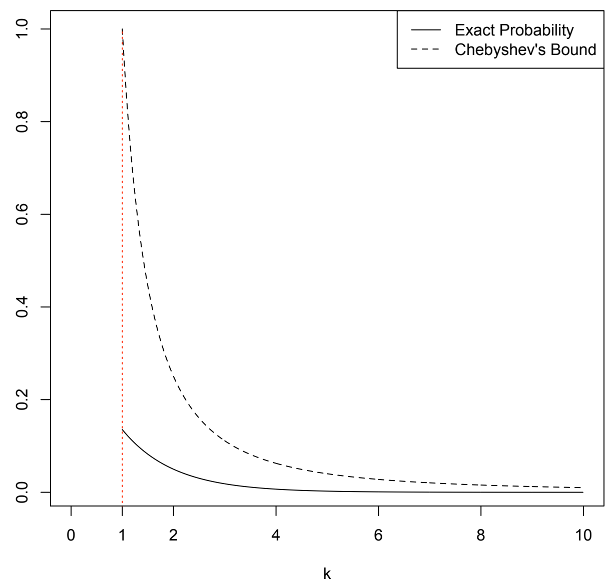

Figure 1: Problem 6

7. Suppose that $P(X = 1) = P(X = -1) = 1/2$. Define

$$X_n = \begin{cases} X & \text{with probability } 1 - 1/n \\ e^n & \text{with probability } 1/n \end{cases}$$

Show why X_n does or does not converge to X

(a) in probability.

(b) in distribution.

(c) in quadratic mean.

Solution:

(a) For every $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) = P(X_n = e^n) = \frac{1}{n} \to 0,$$

as $n \to \infty$. So yes, $X_n \xrightarrow{P} X$.

(b) From 5.4 Theorem (Page 73), $X_n \xrightarrow{P} X$ implies that $X_n \xrightarrow{D} X$. So yes.

(c)

$$\begin{aligned} E[(X_n - X)^2] &= E[E[(X_n - X)^2|X]] = E[0^2(1 - \frac{1}{n}) + (e^n - X)^2 \frac{1}{n}] \\ &= \frac{1}{n} E[(e^n - X)^2] = \frac{1}{n}\left[(e^n - 1)^2 \frac{1}{2} + (e^n + 1)^2 \frac{1}{2}\right] \\ &\to \infty. \end{aligned}$$

So $X_n \overset{qm}{\nrightarrow} X$.

8. Given a sequence of random variables such that X_n converges to μ (μ is a constant) in probability, give one example where:

   (a) $E(X_n)$ does not converge to μ.

   (b) $E(X_n - \mu)^2$ does not converge to 0.

   Solution: Define

$$X_n = \begin{cases} n & \text{with probability } 1/n \\ 0 & \text{with probability } 1 - 1/n \end{cases}$$

Then, X_n converges to 0 in probability but $E(X_n) = 1$ and $E(X_n^2) = n$ does not coverges to 0.