Basic Discrete Probability

1. What is the probability that a five-card poker hand contains a royal flush, that is, the 10, jack, queen, king, and ace of one suit?

Solution Here we can just enumerate possibilities by hand. There are four five-card hands that are royal flushes, one for each suit. The sample space has size $\binom{52}{5}$. Thus the probability is $\frac{4}{\binom{52}{5}}$.

2. What is the probability that Abby, Barry, and Sylvia win the first, second, and third prizes, respectively, in a drawing if 200 people enter a contest and

   (a) no one can win more than one prize.

   Solution The probability is simply $\frac{1}{200} \cdot \frac{1}{199} \cdot \frac{1}{198}$, as we draw without replace independently.

   (b) winning more than one prize is allowed.

   Solution This time it is $\frac{1}{200}^3$, since we draw with replacement.

3. What is the probability of these events when we randomly select a permutation of $\{1, 2, 3\}$?

   (a) 1 precedes 3.

   Solution There are three permutations in which 1 precedes 3, 132, 123, and 231. Thus it is $\frac{3}{3!}$.

   (b) 3 precedes 1.

   Solution Same as above.

   (c) 3 precedes 1 and 3 precedes 2.

   Solution Here there are two permutations 312 and 321, giving $\frac{2}{3!}$.

4. Assume that the probability a child is a boy is 0.51 and that sexes of children borin into a family are independent. What is the probability a family of five children has

   (a) exactly three boys?

   Solution This is $\binom{5}{3}0.51^3 \cdot 0.49^2$.

   (b) at least one boy?

   Solution Here we can consider the complement (having all girls), yielding $1 - 0.49^5$.

   (c) at least one girl?

   Solution Again we consider the complement to get $1 - 0.51^5$.

   (d) two boys, conditional on there being at least two girls?

   Solution The probability of at least two girls $1 - 0.51^5 - \binom{5}{1}0.49 \cdot 0.51^4$. For there two be exactly two boys and at least two girls forces there to be two boys and three girls. This event has probability $\binom{5}{3}0.49^3 \cdot 0.51^2$. Thus the probability is

$$\frac{\binom{5}{3}0.49^3 \cdot 0.51^2}{1 - 0.51^5 - \binom{5}{1}0.49 \cdot 0.51^4}$$

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5. Assume that the probability of a 0 is 0.8 and a 1 is 0.2 for a randomly generated bit string of length six. What is the probability that there

(a) are at least 3 zeros?

Solution Here it is $\binom{6}{3}0.8^3 \cdot 0.2^3 + \binom{6}{4}0.8^4 \cdot 0.2^2 + \binom{6}{5}0.8^5 \cdot 0.2 + 0.8^6$. Notice that the complement is not much less work.

(b) are two ones, conditional on the first digit being a zero?

Solution The first digit is a zero with probability 0.8. The intersection is the probability the first digit is a zero and there are two ones, which is $\binom{5}{2}0.8 \cdot 0.2^2$. Thus the probability is $\frac{\binom{5}{2}0.8\cdot 0.2^2}{0.8}$.

(c) is a run of exactly two zeros in a row?

Solution The probability of two zeros in the first two positions followed by a one is $0.8^2 \cdot 0.2$. Note that after the first one, we don't care what is there. If our two zeros start at positions one through four, we need a one to the left *and* the right. This has probability $0.2 \cdot 0.8^2 \cdot 0.2$. Lastly, if our two zeros are the end, this has probability $0.2 \cdot 0.8^2$. Thus the final answer is $2 \cdot 0.8^2 \cdot 0.2 + 3 \cdot 0.8^2 \cdot 0.2^2$.

(d) is a run of exactly two zeros in a row, conditional on the last digit being a one?

Solution The last digit is a one with probability 0.2. The event that there are exactly two zeros *and* the last digit is a one eliminates the last case above. Thus the final answer is

$$\frac{0.8^2 \cdot 0.2 + 3 \cdot 0.8^2 \cdot 0.2^2}{0.8}$$

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Source: Rosen's *Discrete Mathematics and its Applications.*