OUT OF THE CODE – A FENICS RISES

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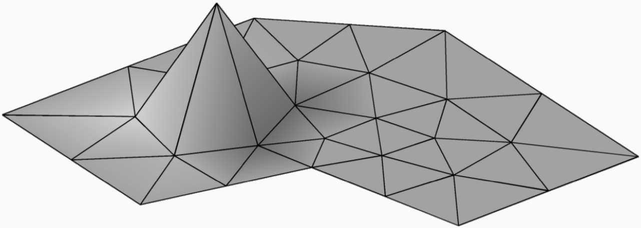
MAIN REFERENCES

Anders Logg, Kent-Andre Mardal, and Garth N. Wells.
Automated solution of differential equations by the finite element method, 2012.

Douglas N. Arnold, Richard S. Falk, and Ragnar Winther.
Finite element exterior calculus: from Hodge theory to numerical stability, 2010.

FENICS

POLYNOMIAL BASIS



SOLVING PHYSICAL PROBLEM IN FENICS

1. Identify PDE along with IC and BC.
2. Reformulate PDE in variational form.
3. Make code for mesh, input data, and variational form.
4. Add statements to solve system, and visualize.

POISSON'S PROBLEM

First Step

Find u such that $-\Delta u = \operatorname{div}\operatorname{grad} u = f$ in square Ω,
with $u = 0$ on $\partial\Omega$.

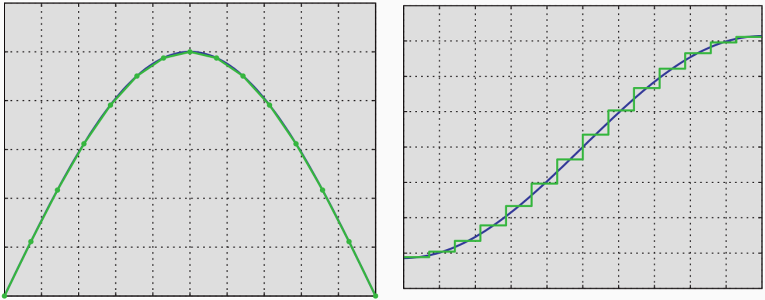
Second Step

Find $u \in V = \mathring{H}^1(\Omega)$ such that

$$\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds = \int_\Omega f u \, dx$$

for any $v \in V$.

Python!



u on left. $-u'$ on right.

PRIMAL VS MIXED

In primal form...

$$\begin{aligned} -\operatorname{div}\operatorname{grad} u &= f \quad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

In mixed form...

$$\begin{aligned} -\operatorname{div}\sigma &= f \quad && \text{in } \Omega \\ \sigma &= \operatorname{grad} u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

First Step

Find (σ, u) such that $-\operatorname{div} \sigma = f$ and $\sigma = \operatorname{grad} u$ in square Ω, with $u = 0$ on $\partial\Omega$.

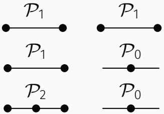
Second Step

Find $(\sigma, u) \in V = H(\operatorname{div}, \Omega) \times L^2(\Omega)$ such that

$$\int_\Omega \operatorname{div} \sigma \, v \, dx = -\int_\Omega f v \, dx,$$
$$\int_\Omega \sigma \cdot \tau \, dx + \int_\Omega u \operatorname{div} \tau \, dx - \int_{\partial\Omega} u \, (\tau \cdot \boldsymbol{n}) \, ds = 0,$$

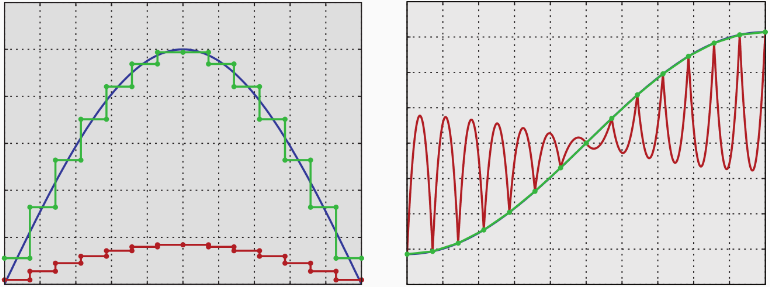
for any $(\tau, v) \in V$.

In 1D, pick (σ_h, u_h) in ...



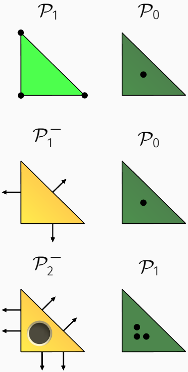
singular? stable? unstable?

Python!



(σ, u) in $\mathcal{P}_1 \times \mathcal{P}_0$ or $\mathcal{P}_2 \times \mathcal{P}_0$

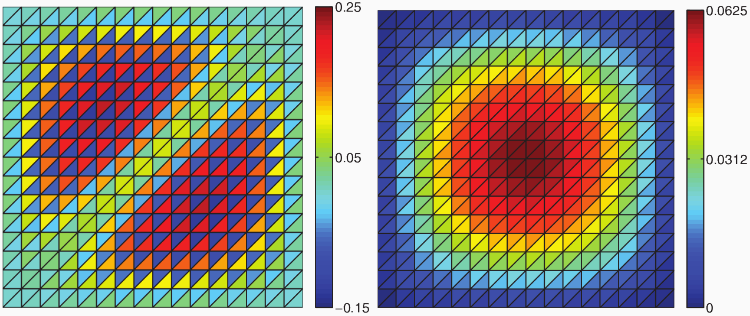
In 2D, pick (σ_h, u_h) in ...



unstable? stable?

Python!

NUMERICAL RESULT



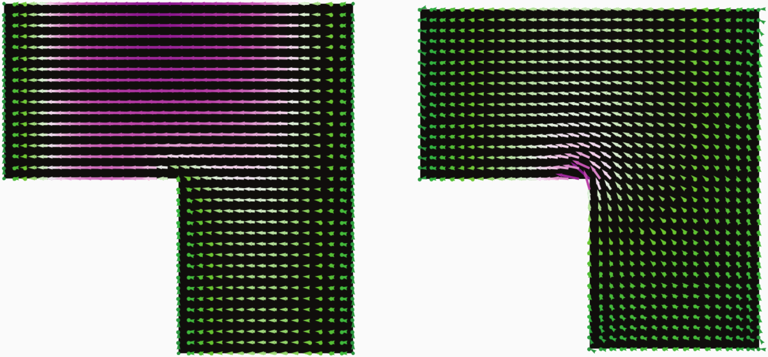
(σ, u) in $\mathcal{P}_1 \times \mathcal{P}_0$ or $\mathcal{P}_1^- \times \mathcal{P}_0$

CONVERGENCE

HOW TO GET CONVERGENCE?

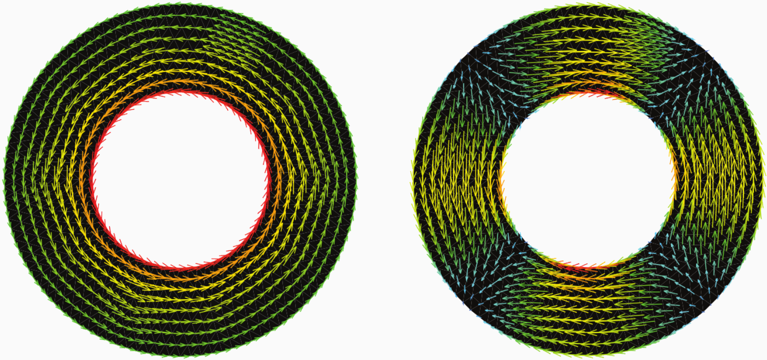
Consistency – is discrete problem close to continuous one?
Stability – is discrete problem well-posed?

…then convergence!



Consistency – is discrete problem close to continuous one?

$H^1 \subsetneq H(\mathrm{curl}) \cap H(\mathrm{div})$



Stability – is discrete problem well-posed?

Harmonic functions

HODGE LAPLACIAN

POISSON'S EQUATION

The complex

$$
\begin{tikzcd}
\underset{\sigma}{L^2(\mathbb{V})} \arrow[r, "\mathrm{div}"] & \underset{u}{L^2(\mathbb{R})} \arrow[l, bend left, "-\,\mathrm{grad}"] \arrow[r] & 0 \arrow[l, bend left]
\end{tikzcd}
$$

...associated 2nd order operator

$$-\operatorname{div} \underbrace{\operatorname{grad} u}_{\sigma} = 0$$

...leads to the mixed formulation

$$\sigma - \operatorname{grad} u = 0$$

$$-\operatorname{div} \sigma = 0$$

The *de Rham complex* in 3D

$$
\begin{tikzcd}
0 \arrow[r] & L^2(\mathbb{R}) \arrow[r, "\mathrm{grad}"] & L^2(\mathbb{V}) \arrow[r, "\mathrm{curl}"] \arrow[l, bend left, "-\mathrm{div}"] & L^2(\mathbb{V}) \arrow[r, "\mathrm{div}"] \arrow[l, bend left, "\mathrm{curl}"] & L^2(\mathbb{R}) \arrow[r] \arrow[l, bend left, "-\mathrm{grad}"] & 0
\end{tikzcd}
$$

...associated *Hodge Laplacian*

$$d^*\underbrace{du}_{\sigma} + \underbrace{dd^*u}_{\rho} = 0$$

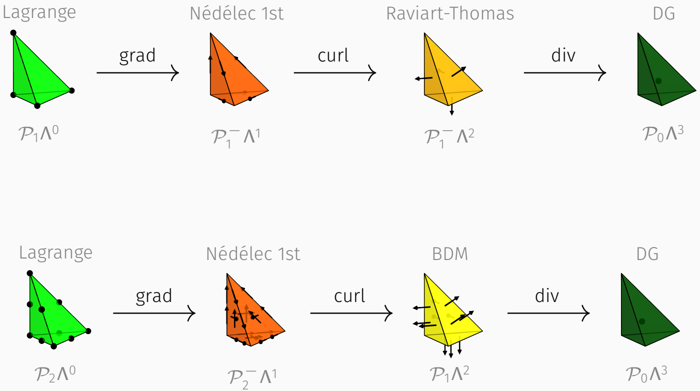
...leads to the mixed formulation

$$\sigma - d^*u = 0$$

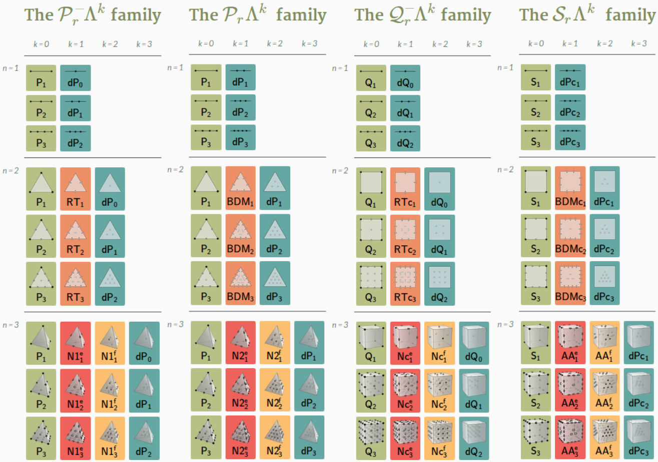
$$d^*\rho + d\sigma = 0$$

$$\rho - du = 0$$

POLYNOMIAL DE RHAM COMPLEXES



PERIODIC TABLE OF FINITE ELEMENTS



FEMtable.org

Questions?