A New Approach to Finite Element Simulation of General Relativity

Vincent Quenneville-Bélair with Douglas N. Arnold
University of Minnesota

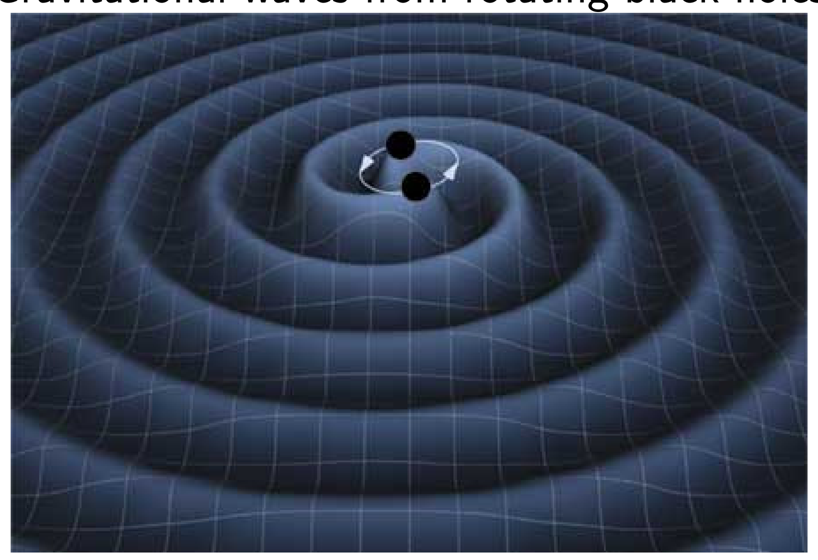
Objective

Design and implementation of new mixed finite element method for propagation of gravitational waves, by adapting *Finite Element Exterior Calculus* (FEEC) framework.

Gravitational Waves

As gravitational waves are very weak, they have never been observed before. To detect, new gravitational wave observatories are being built. Computer simulations are essential for determining expected signals and interpreting the data.

Gravitational waves from rotating black holes



Curvature

Curvature is given by

$$\text{Riemann} = \underset{\text{Shape change}}{\text{Weyl}} + \underset{\text{Volume change}}{\text{Ricci}}$$

In vacuum, Einstein's equations are

$$\text{Ricci} = 0$$

This geometric problem is converted to PDE in order to recover Weyl.

Evolution with EB

Weyl divides into $5 + 5 = 10$ as

$\boldsymbol{E}$ and $\boldsymbol{B}$ symmetric traceless matrices

Then, second Bianchi identity

$$\operatorname{div}_4 \text{Weyl} = 0$$

translates to linearized EB system in vacuum

$$\dot{\boldsymbol{E}} + \operatorname{curl} \boldsymbol{B} = 0$$
$$\dot{\boldsymbol{B}} - \operatorname{curl} \boldsymbol{E} = 0$$

with $\operatorname{div} \boldsymbol{E} = \operatorname{div} \boldsymbol{B} = 0$

… like Maxwell's equations!

Maxwell

Maxwell's equations can be fit in the FEEC framework as a *Hodge wave equation*

$$\dot{\sigma} - \operatorname{div} \boldsymbol{E} = 0$$
$$\dot{\boldsymbol{E}} + \operatorname{curl} \boldsymbol{B} + \operatorname{grad} \sigma = 0$$
$$\dot{\boldsymbol{B}} - \operatorname{curl} \boldsymbol{E} = 0$$

for de Rham complex

$$\begin{tikzcd} \underset{\sigma}{L^2(\mathbb{R})} \arrow[r, shift left, "\operatorname{grad}"] & \underset{\boldsymbol{E}}{L^2(\mathbb{V})} \arrow[l, shift left, "\operatorname{div}"] \arrow[r, shift left, "\operatorname{curl}"] & \underset{\boldsymbol{B}}{L^2(\mathbb{V})} \arrow[l, shift left, "\operatorname{curl}"] \end{tikzcd}$$

where $\mathbb{V}$ are vectors.

EB with strong symmetries

Using strong symmetries complex

$$\begin{tikzcd} \underset{\sigma}{L^2(\mathbb{R})} \arrow[r, shift left, "\operatorname{grad}\operatorname{grad}"] & \underset{\boldsymbol{E}}{L^2(\mathbb{S})} \arrow[l, shift left, "\operatorname{div}\operatorname{div}"] \arrow[r, shift left, "\operatorname{curl}"] & \underset{\boldsymbol{B}}{L^2(\mathbb{T})} \arrow[l, shift left, "\operatorname{sym}\operatorname{curl}"] \end{tikzcd}$$

where $\mathbb{S}$ and $\mathbb{T}$ are symmetric and traceless matrices.

EB fits FEEC framework

$$\dot{\sigma} - \operatorname{div}\operatorname{div} \boldsymbol{E} = 0$$
$$\dot{\boldsymbol{E}} + \operatorname{sym}\operatorname{curl} \boldsymbol{B} + \operatorname{grad}\operatorname{grad} \sigma = 0$$
$$\dot{\boldsymbol{B}} - \operatorname{curl} \boldsymbol{E} = 0$$

To discretize each spaces, inspiration from
plate bending for $\operatorname{grad}\operatorname{grad}$,
elasticity for symmetries.

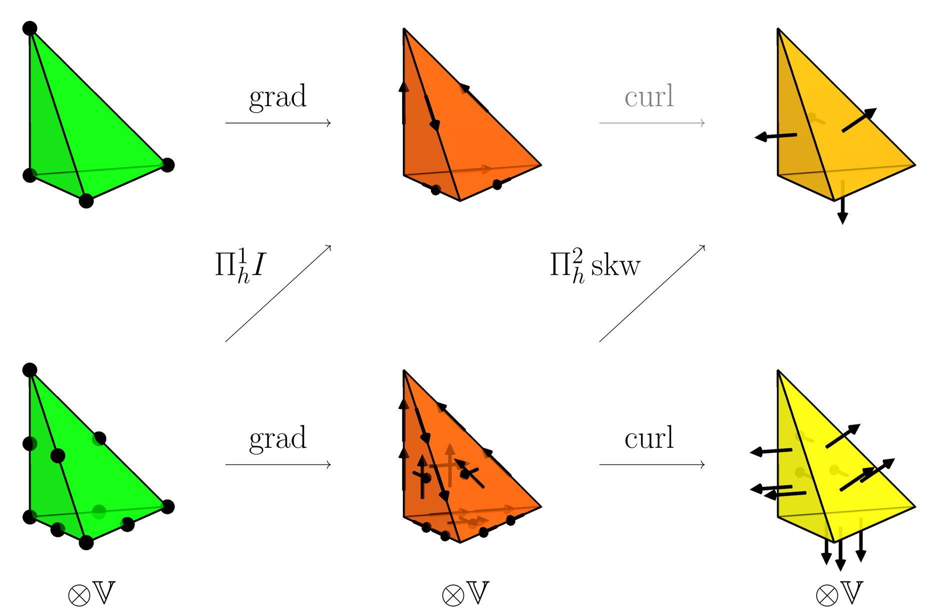
Finding compatible spaces

Compatible spaces are found through the Bernstein-Gelfand-Gelfand resolution: a way of combining two de Rham complexes to generate new complexes as above.

At the continuous level …

$$\begin{tikzcd} L^2(\mathbb{R}) \arrow[r, "\operatorname{grad}"] & L^2(\mathbb{V}) \arrow[r, "\operatorname{curl}"] \arrow[dl, "I"'] & L^2(\mathbb{V}) \arrow[dl, "\operatorname{skw}"'] \\ L^2(\mathbb{V}) \arrow[r, "\operatorname{grad}"] & L^2(\mathbb{M}) \arrow[r, "\operatorname{curl}"] & L^2(\mathbb{M}) \end{tikzcd}$$

At the discrete level …



where Π_h^1 and Π_h^2 are interpolations in the target space.

EB with weak symmetries

The formulation with weak symmetries is to find

$$\begin{tikzcd} \underset{\text{scalar}}{\sigma} \arrow[r] & \underset{\text{vector}}{\lambda} \arrow[r] & \underset{\text{vector}}{\boldsymbol{M}} \\ \underset{\text{vector}}{\phi} \arrow[r] \arrow[ur, dash] & \underset{\text{matrix}}{\boldsymbol{E}} \arrow[r] \arrow[ur, dash] & \underset{\text{matrix}}{\boldsymbol{B}} \end{tikzcd}$$

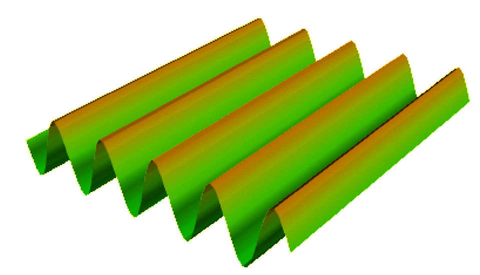
satisfying

$$\dot{\sigma} + \operatorname{div} \dot{\lambda} = 0$$
$$\dot{\lambda} + \operatorname{div} \boldsymbol{E} = 0$$
$$\dot{\phi} - \operatorname{grad} \dot{\sigma} = 0$$
$$\dot{\boldsymbol{E}} + \operatorname{inc} \boldsymbol{M} + \operatorname{curl} \boldsymbol{B} + \operatorname{grad} \phi = 0$$
$$\dot{\boldsymbol{M}} - \operatorname{skw} \boldsymbol{E} = 0$$
$$\dot{\boldsymbol{B}} - \operatorname{curl} \boldsymbol{E} = 0$$

No constraints are imposed during the evolution. Less smoothness is needed from the finite elements, since $\operatorname{grad}\operatorname{grad}$ and $\operatorname{div}\operatorname{div}$ are avoided.

Implementation

When discretizing, I and skw become $\Pi_h^1 I$ and $\Pi_h^2 \operatorname{skw}$. Operators implemented through hybridization: integrals over edges and faces. Requires manual assembly. Implementation using FEniCS, open source project with focus on solving PDE by finite element methods.



Main Directions

A priori error analysis with Finite Element Exterior Calculus. Linear implementation with new mixed finite elements with weak symmetries. Full nonlinear implementation.

References

A. Anderson, Y. Choquet-Bruhat, and J. W. York, Jr., *Einstein-Bianchi hyperbolic system for general relativity*, Topol. Methods Nonlinear Anal., 1997.

D.N. Arnold, R.S. Falk, and R. Winther, *Mixed finite element methods for linear elasticity with weakly imposed symmetry*, Math. Comp., 2007.

______, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bull. Amer. Math. Soc., 2010.

Ricardo Durán and Elsa Liberman, *On mixed finite element methods for the Reissner-Mindlin plate model*, Math. Comp., 1992.

Figure from LIGO, www.ligo.org/science/GW-Overview/

Figures from FEniCS book, dx.doi.org/10.1007/978-3-642-23099-8

vqb@umn.edu umn.edu/~vqb