Guitar Tuning

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1 Abstract

This project will be to create a mathematical model of a guitar string. I will then use this mathematical model in order to tune the guitar string to a certain frequency. The process of creating, discretizing, and plotting the model will be done by hand and using MatLab.

2 Introduction

The equation I will be evaluating:

$$u_{tt} = c^2 u_{xx}$$

where

$$c = \sqrt{\frac{T}{\rho A}}.[1]$$

This equation describes the relation between the vertical acceleration and concavity of a guitar string. In this model, c is a given constant where T is the tension on the string, ρ is the density of the string, and A is the cross sectional area of the string. To describe the length of the string I will use L. We also have $u(t, x)$ as a function describing the amplitude of the string at given time t(seconds) and given distance x(centimeters). Furthermore, we have the second partial derivatives u_{tt} and u_{xx} that describe the acceleration and concavity of the string, respectively. Some other variables include the time step, $\triangle t$, and displacement in x, $\triangle x$.

3 Methods

We have the partial differential equations that we must descritize, apply a numerical partial differential equation solving routine, and plot.We have initial conditions on our partial differential equation $u(0, x) = sin(\pi x/10)$ and $u_t = 0$. Our boundary conditions are $u(0, t) = u(L, t) = 0$. Now, I have set up an IVP that I will need to solve using the Runge-Kutta method of order 2, (center difference method). After applying the center difference method to both differential equations we obtain:

$$u_{tt} = \frac{u(t + \triangle t) - 2u(t) + u(t - \triangle t)}{(\triangle t)^2}$$

where u_{tt} is the vertical acceleration of the string at distance x and

$$u_{xx} = \frac{u(x + \triangle x) - 2u(x) + u(x - \triangle x)}{(\triangle x)^2}$$

where u_{xx} describes the concavity of the string at time t. For notation purposes we will indicate x of u using lower indices and t of u using upper indices, that is:

$$u_{tt} = \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\triangle t)^2}$$

and

$$u_{xx} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\triangle x)^2}.[3]$$

Focusing on $u_{xx} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\triangle x)^2}$. If, for example, we take 5 measurements for u_i we can rewrite this equation as a matrix

$$u_{xx} = \frac{1}{(\triangle t)^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}.[2]$$

We can expand this matrix as needed for any size i. For notation purposes, we will call this coefficient matrix D_2. When solving this IVP using Runge-kutta, we are trying to find the next time step of the function u, so we must isolate u_i^{j+1}. After plugging these equations, matrix, and vector into $u_{tt} = c^2 u_{xx}$ and isolating u_i^{j+1} we obtain $\vec{u}_i^{j+1} = \delta^2 D_2 \vec{u}^j + 2\vec{u}_i^j - \vec{u}_i^{j-1}$ where $\delta = \frac{c\triangle t}{\triangle x}$. Before moving on, we must address $\triangle t$ and $\triangle x$. We must choose a $\triangle t$ and $\triangle x$ such that our condition is stable. The stability condition for our equation is $\frac{c\triangle t}{\triangle x} \leq 1$[3]. For this problem, I chose $\triangle t = 0.00001$ and $\triangle x = 0.2$. The next step is to utilize the initial conditions to solve the IVP. We have the initial vector $u_i^0 = sin(\frac{\pi x}{10})$. When using the center difference method, we not only need u^j but also u^{j-1} to solve for u^{j+1}. Although, when using $u_i^0 = u_i^j$ we need to come up with an equation for u_i^{-1}. We can use the equation $u_i^{-1} = u_i^1 - 2\triangle t u_t$[3]. Since we set our initial condition $u_t = 0$, we get that $u_i^{-1} = u_i^1$. Plugging this into our equation we get $2\vec{u}_i^{j+1} = \delta^2 D_2 \vec{u}^j + 2\vec{u}_i^j$. Now, with this equation we can get the u_i^1 using u_i^0. Now that we have our first 2 time steps for $\vec{u}$ we can begin to solve the IVP. The last step to solve the IVP is to use MatLab. The code for this solution is in the Appendix.

4 Examples

For my example I will be creating two graphs from the result of my solution to the IVP. My first graph will contain a select few amplitudes of the string at corresponding time steps.

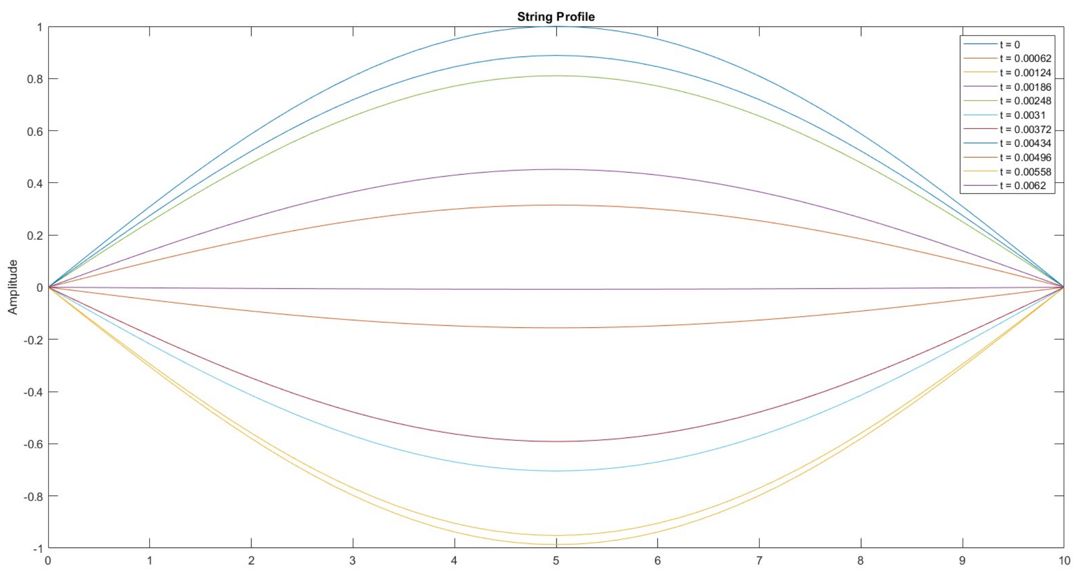
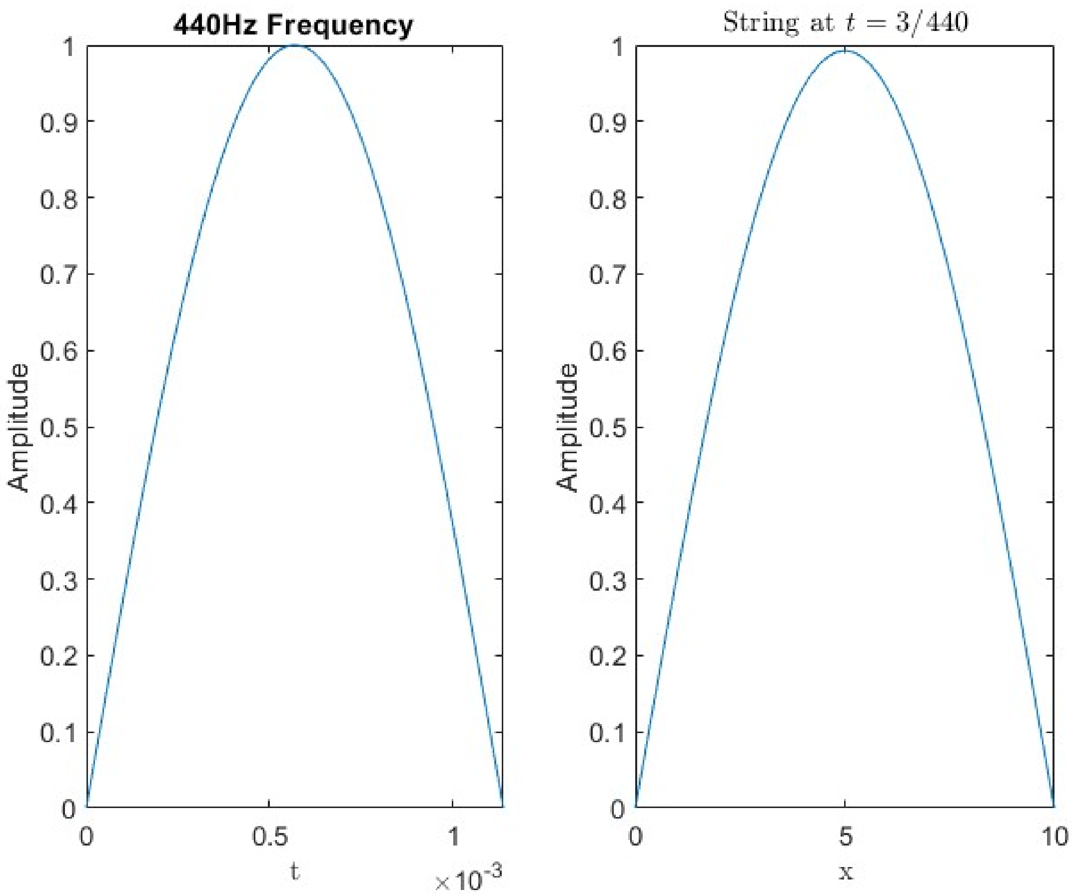


Figure 1: Resulting amplitudes at select timesteps with the interval $t \in [0, 3/440]$

My next example will be the comparison of the solution to my IVP at $t = 3/440$ to one half cycle of the 440Hz graph, $p(t) = sin(2\pi 440t)$ where $t \in [0, 3/440]$.



5 Discussion

With this project, my primary interest is to create a model of a guitar string using numerical solving methods. I have completed my primary motive for this project. I used the method Runge Kutta of order 2 in order to solve an IVP of partial differential equations. If I were to go further with this project, I would start by analyzing a more realistic string. In this project I did not account for air friction that would normally occur in the physical world. How would I go about this, you might ask, I don't know. I would assume, with the knowledge I have, that the sum of the forces acting on the string would be different and so we would have additional parameters on our equation. Something else I would try to accomplish is to change the tension, T, and analyze my results. Although, when changing the tension on the string, I must be careful to check that the system will still be stable since the stability of the system corresponds to c which is calculated using T. To take it a step further, I would go into the physical world and take measurements of a real guitar. One measurement I would like to take is the frequency of a guitar string tuned to the note c. I would then use this measurement to tune my theoretical string to the right frequency.

References

[1] .(2022, Nov 16) In *SIMIODE*. Retrieved 03:00PM, Nov 16, 2022, from *https://www.simiode.org/resources/672/download/9-12-S-PDEGuitarTuning-StudentVersion.pdf*

[2] Asaki, T., Moon, H., Snipes, M. (2015). *Heat Diffusion in Linear Algebra.*

[3] Chapter 4 the wave equation - uni-muenster.de. (n.d.). Retrieved December 6, 2022, from *https://www.uni-muenster.de/imperia/md/content/physik_tp/lectures/ws2016 − 2017/num_methods_i/wave.pdf*

6 Appendix

projectcode.m singraph.m

```
deltat = 0.00001;
deltax = 0.2;
T = 1575000;
A = pi*(0.029)^2;
rho = 7.58;
c = sqrt(T/(rho*A));
delta = (c^2)*((deltat)^2)/((deltax)^2);
t = 0:deltat:(3/440);
x = deltax:deltax:(10-deltax);
N_t = length(t);
N_x = length(x);
u = zeros(N_x,N_t);
I = eye(N_x);
D2 = zeros(N_x,N_x);
D2(1,:) = [-2,1,zeros(1,47)];
D2(2,:) = [1,-2,1,zeros(1,46)];
D2(N_x-1,:) = [zeros(1,46),1,-2,1];
D2(N_x,:) = [zeros(1,47),1,-2];
for i=3:1:(N_x)-2
    D2(i,:) = [zeros(1,i-2),1,-2,1,zeros(1,(N_x)-(1+i))];
end
for i=1:1:N_x
    u(i,1) = sin(pi*x(i)/10);
end
u(:,2) = ((delta*D2)*(u(:,1)) + 2*u(:,1))/2;
for i=2:1:N_t-1
    u(:,i+1) = (delta*D2)*u(:,i) + 2*u(:,i) - u(:,i-1);
end
```

```
%%
n = 1:62:N_t;
for i=1:62:N_t
    txt = ['t = ', num2str(t(i))];
    plot(x,u(:,i),'DisplayName', txt);
    hold on;
end
title('String Profile');
xlabel('$x$', 'interpreter', 'latex');
ylabel('Amplitude');
legend show;
```

```
%%
u = [zeros(1,N_t);u;zeros(1,N_t)];
x = [0,x,10];
subplot(1,2,1);
syms y;
f = sin(pi*2*440*y);
fplot(f,[0,3/(440*6)]);
hold on;
title('440Hz Frequency');
xlabel('t', 'interpreter', 'latex');
ylabel('Amplitude');
subplot(1,2,2);
plot(x,u(:,N_t));
title('String at $t = 3/440$', 'interpreter', 'latex');
xlabel('x','interpreter', 'latex');
ylabel('Amplitude');
```