Note on Quantum Information and Optimal Control

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Abstract

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Contents

1 Quantum sensing

In sensing, one tries to estimate an external field. The prototypical quantum sensing example involves coupling a spin-1/2 state to an unknown magnetic field. As the state evolves, one infers the magnetic field parameter that generates its dynamics. The quantum state ρ contains metrological information called the quantum Fisher information F_Q – and some states can increase the quantity F_Q. This section connects quantum dynamics, information theory, and control.

In metrology, the goal is to maximise the quantum Fisher information F_Q of the state concerning the external field. For multiple parameter estimation $\vec{x}$, I define the quantum Fisher information matrix elements by

$$[F_Q]_{ab} = \frac{1}{2}\mathrm{Tr}\left[\rho\{L_a, L_b\}\right], \tag{1}$$

where L_a is the symmetric derivative satisfying $\partial_a \rho = \frac{1}{2}\{\rho, L_a\}$. For our purposes, the single parameter I want to estimate is x. Note that the symmetric derivative admits an integral representation that can be useful later

$$L_a = \int_0^\infty ds\, e^{-\frac{\rho s}{2}}[\partial_a \rho] e^{-\frac{\rho s}{2}}.$$

By diagonalising the density matrix $\rho = U\Lambda U^\dagger$, I can write the quantum Fisher information F_Q in a useful representation:

$$F_Q = 4\sum_{ij} \frac{\lambda_i}{(\lambda_i + \lambda_j)^2}|\mathcal{D}_{ij}|^2, \tag{2}$$

where $\mathcal{D} = U^\dagger[\partial_x \rho]U$. In this form, the information is calculated with two objects: the density matrix ρ and its derivative for the parameter $\partial_x \rho$.

Let us shift our attention to the system's dynamics. I typically couple the quantum state ρ to the external field using a Hamiltonian for quantum sensing purposes. The quantum state evolves according to the master equation $\partial_t \rho = \mathcal{L}[\rho]$, which accounts for noisy processes. The corresponding master equation reads

$$\partial_t \rho = - i[H, \rho] + \sum_i \left(K_i \rho K_i^\dagger - \frac{1}{2}\{K_i^\dagger K_i, \rho\} \right), \tag{3}$$

where K_i are the channel's Kraus operators. In general, I'll have access to a time-dependent control field, so the total Hamiltonian becomes

$$H = H_0(\vec{x}) + \sum_{i=1}^{p} V_k(t) H_k. \tag{4}$$

The first term encodes the external field x I want to sense, and the last is our control.

In this note, I'll consider evolving a single spin-1/2 state in the presence of an external magnetic field and depolarising noise but with a control field:

$$H = \frac{1}{2}\omega_0 \sigma_3 + \vec{V}(t) \cdot \vec{\sigma} \tag{5}$$

$$K_i = \sqrt{\gamma}\sigma_i. \tag{6}$$

The parameter ω_0 is what I want to estimate, $\gamma > 0$ is the dissipation rate, and $\vec{V}(t)$ is the control.

Performing the time-evolution of the density matrix $\rho(t)$ is non-trivial. One approach is discretising time and stepping in small intervals Δt from the last state to the next using the master equation $\partial_t \rho = \mathcal{L}[\rho]$. Since the Lindblad operator $\mathcal{L}$ generates time translations, exponentiation allows us to perform the time evolutions. Defining the operator $K(t_i, t_j)$ as the propagator from one time (t_j) to a future time (t_i) allows us to write

$$\rho(t_n) = K(t_n, t_0)\rho(t_0) \tag{7}$$

$$= K(t_n, t_{n-1})\rho(t_{n-1}) \tag{8}$$

$$= e^{\Delta t \mathcal{L}} \rho(t_{n-1}). \tag{9}$$

In this form, we are on our way to calculate the derivative $\partial_x \rho$ at each time step, a necessary ingredient for knowing the state's quantum Fisher information. The key idea is to calculate the derivative recursively. Turning the crank, I get

$$\partial_x \rho(t_n) = \left[\partial_x e^{\Delta t \mathcal{L}}\right] \rho(t_{n-1}) + e^{\Delta t \mathcal{L}} \left[\partial_x \rho(t_{n-1})\right]. \tag{10}$$

Terms with $\rho(t_{n-1})$ are known from the last time step, while terms with $\mathcal{L}$ are calculated on the fly.

Let's explicitly calculate the derivative $\partial_a e^{\Delta t \mathcal{L}}$. A key integral representation of the derivative of the exponential map is key for any calculation. Namely, I have

$$\partial_a e^{O(x)} = \int_0^1 ds\, e^{sO(x)} \left[\partial_a O(x)\right] e^{(1-s)O(x)}. \tag{11}$$

Setting $\mathcal{L}\Delta t = O(x)$ and $\mathcal{L}_a \Delta t = \partial_a O(x)$ provides us with

$$\partial_a e^{\Delta t \mathcal{L}} = \Delta t \int_0^1 ds\, e^{s\mathcal{L}\Delta t} \left[\mathcal{L}_a\right] e^{(1-s)\mathcal{L}\Delta t}. \tag{12}$$

The integral is evaluated by diagonalising the Lindbladian $\mathcal{L} = VDV^\dagger$:

$$\partial_a e^{\Delta t \mathcal{L}} = VAV^\dagger \tag{13}$$

$$[A]_{ij} = \frac{e^{\Delta t d_i} - e^{\Delta t d_j}}{d_i - d_j} \left[V^\dagger \mathcal{L}_a V\right]_{ij}. \tag{14}$$

The operator $\mathcal{L}_a$ is typically the dynamics generated by the conjugate operator H_a to the field x_a. That is,

$$\mathcal{L}_a \rho = -i[H_a, \rho]. \tag{15}$$

All these intermediate calculations allow for evaluating the derivative $\partial_a e^{\Delta t \mathcal{L}}$.

In this section, I described the quantum Fisher information matrix F_Q. I also reviewed the dynamics of dissipative systems using the Lindblad master equation. Finally, I showed how to evaluate the quantum Fisher information with dynamics.

1.1 Example (analytical): spin-1/2 particle with Pauli noise

In this example, I analyse the quantum information of a spin-1/2 when it undergoes noisy dynamics. Consider an arbitrary quantum state initialised at ρ with the corresponding Lindblad equation

$$\partial_t \rho = -i[H, \rho] + \sum_{i=1}^{3} \left(K_i \rho K_i^\dagger - \frac{1}{2}\{K_i^\dagger K_i, \rho\} \right), \tag{16}$$

$$H = \frac{1}{2}\omega \sigma_3, \tag{17}$$

$$K_i = \sqrt{\gamma}\sigma_i. \tag{18}$$

The time evolution of the state resembles that of a damped oscillator:

$$\rho(t) = \begin{pmatrix} \frac{1}{2} + c_1 e^{-4\gamma t} & c_2 e^{-(4\gamma + i\omega)t} \\ c_2^* e^{-(4\gamma - i\omega)t} & \frac{1}{2} - c_1 e^{-4\gamma t} \end{pmatrix}, \tag{19}$$

where the initial conditions fix c_1, c_2. A straightforward calculation shows the purity of the state decays exponentially

$$\mathrm{Tr}[\rho^2] = 2\left(c_1{}^2 + |c_2|^2\right) e^{-8\gamma t} + \frac{1}{2}. \tag{20}$$

I calculate the state's quantum Fisher information F_Q. The information with respect to ω is

$$F_Q = 4|c_2|^2 t^2 e^{-8\gamma t}. \tag{21}$$

There are competing effects. In the early times, F_Q grows quadratically but decays exponentially in the late times. The maximum amount of information in the system is $|c_2|^2 e^4/\gamma^2$, which is attained at time $t = 1/(4\gamma)$. Moreover, note the importance of the cross term c_2. If it were not present, the state would contain zero information. The inflection points are located at $t_\pm = (2 \pm \sqrt{2})/(8\gamma)$.

I also have information about the system's dissipation rate, γ. The quantum Fisher information for γ is

$$F_Q = \frac{16\,(c_1^2 + |c_2|^2)\,t^2}{e^{8\gamma t} - 4(c_1^2 + |c_2|^2)} \tag{22}$$

$$= 2t^2 \left(\frac{1}{\mathrm{Tr}[\rho^2]} - 2 \right). \tag{23}$$

and is approximately maximised at $t = (4(c_1^2 + |c_2|^2) - 1)/(12\gamma)$.

1.2 Example (numerics): spin-1/2 particle with Pauli noise

In this example, I examine the same system with numerical techniques. My results serve to validate the matrix formulas I derived and their stability.

The dynamics are represented on the Bloch sphere and coincide with the theory.

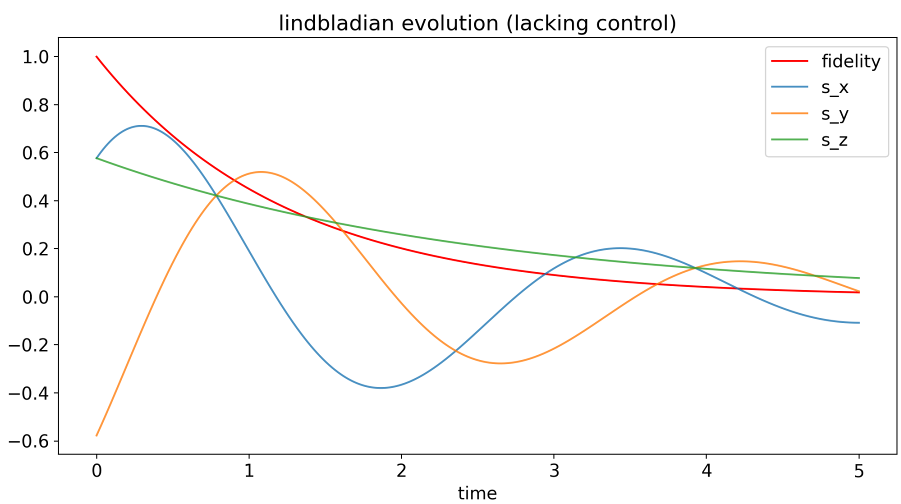

Figure 1: Bloch sphere dynamics and fidelity.

I calculate the state's quantum Fisher information F_Q for ω. My numerical technique coincides with the theory.

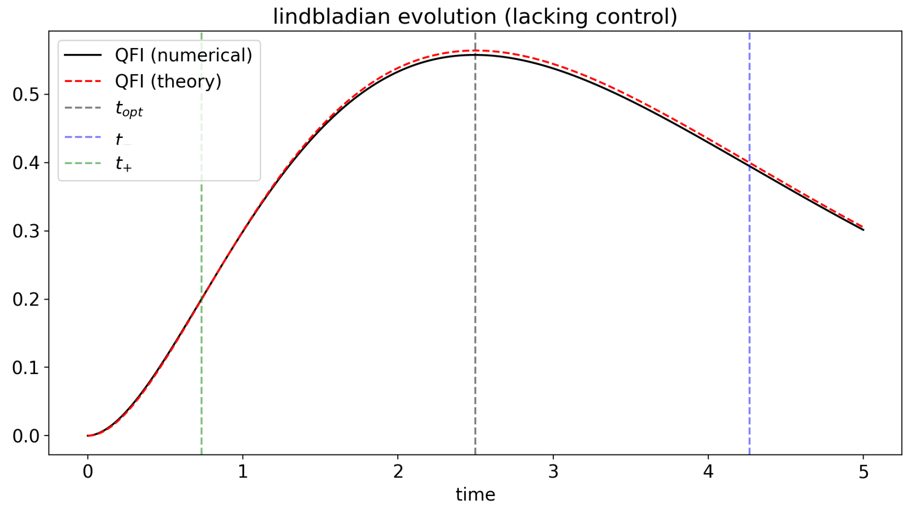

Figure 2: Quantum Fisher information with respect to ω.

1.3 Example (numerics): controlling a spin-1/2 particle with Pauli noise

I examine the same system with a control field $\vec{V}(t)$ in this example. Solving the time-dependent master equation is usually impossible so that I will use my numerical technique. The idea of a control field to improve a reward function is old. It has striking similarities to the concept of dynamical decoupling.

The first control is

$$\vec{V}(t) = 0.5(\cos(5t), \cos(0.1t), 0), \tag{24}$$

which I found via trial and error. Its dynamics are solved and shown in 3. I calculate the state's quantum Fisher information F_Q for ω. The control $\vec{V}(t)$ was chosen to improve the F_Q. (I tried many other controls, but most reduced the quantum Fisher information.) The improvement is shown in 4.

The second control that I found is

$$\vec{V}(t) = 0.5(\cos(t), \cos(0.1t), 1 - e^{-0.1t}). \tag{25}$$

Its dynamics are solved and shown in 5. The quantum Fisher information improvement is shown in 6.

My numerical results show that it is possible to choose controls that improve the quantum Fisher information in two ways: (i) F_Q reaches a higher value, and (ii) F_Q attains its maximum in a shorter time. For quantum sensing purposes, one should judiciously choose a control protocol $\vec{V}(t)$ to improve performance. The caveat is that the space of possible controls is too large! Brute force will fail, so one needs to resort to a more efficient method of determining the optimal control $\vec{V}(t)$.

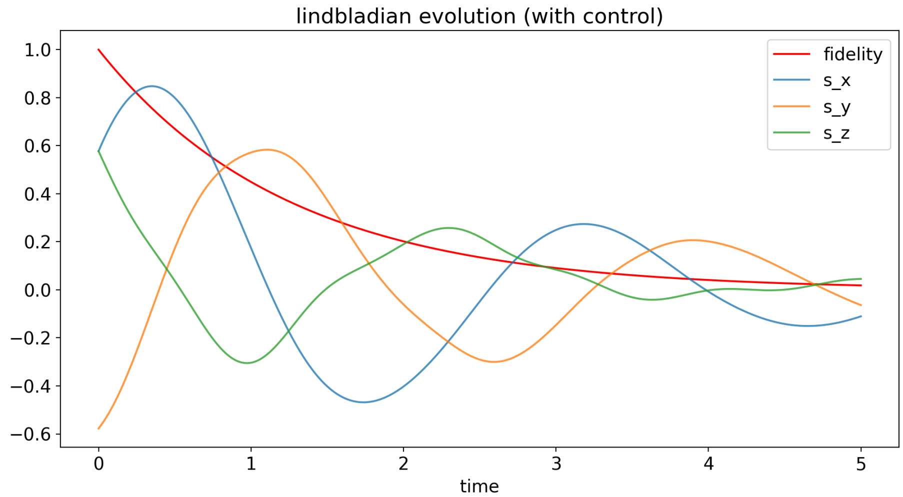

Figure 3: Bloch sphere dynamics and fidelity.

2 Reinforcement learning

Reinforcement learning provides a platform for performing optimal control. This note presents a scheme for sensing an external field $\vec{x}$ via controls $\vec{V}(t)$. In an ideal setting, one tries to maximise the quantum Fisher information over infinite-time dynamics

$$F(\rho_0) = \max_{\vec{V}_t} \left[\int dt\, \gamma^t F_Q(\rho(t), \vec{V}(t)) \right] \tag{26}$$

$$\text{subject to:} \quad \partial_t \rho(t) = \mathcal{L}[\rho(t)] \tag{27}$$

I will make two simplifications for our purposes: (i) discretise the evolution and (ii) set a limit on the evolution. The simplifications render the problem of finding the optimal control $\vec{V}_t$ amenable to a digital computer. The new reward function is

$$F(\rho_0) = \max_{\vec{V}_0, \cdots \vec{V}_{N-1}} \left[\sum_{k=0}^{N-1} \gamma^k F_Q(\rho_k, \vec{V}_k) \right] \tag{28}$$

$$\text{subject to:} \quad \rho_{k+1} = e^{\Delta t \mathcal{L}_k} \rho_k, \tag{29}$$

where $T/N = \Delta t$.

Two broad methods of determining the optimal control are off-policy and on-policy. Q-learning is a famous off-policy method, while policy-gradient is a popular on-policy method. I find the optimal control sequence $\vec{V}_t$ using on-policy-based methods because they are robust to errors and amenable to continuous actions.

I parameterise the policy π in policy gradient using a flexible function approximation, such as a deep neural network. Then, I update the parameters to optimise the

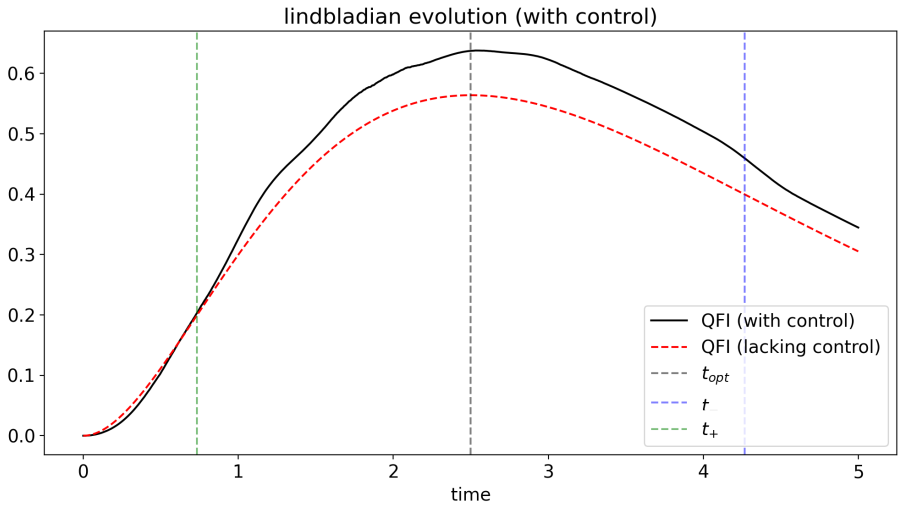

Figure 4: Quantum Fisher information with respect to ω.

reward. For example, I want to maximise the expected reward of the policy

$$\begin{aligned} J(\theta) =& \mathbb{E}_{\pi_\theta}[F_Q] && (30) \\ =& \int_{\mathcal{S}} d\mu(s) \int_{\mathcal{A}} d\nu(a)\, \pi_\theta(a|s) R(s,a) && (31) \\ =& \int d\mu(\rho) \int d^3\vec{V}\, \pi_\theta(\vec{V}|\rho) F_Q(\rho, \vec{V}). && (32) \end{aligned}$$

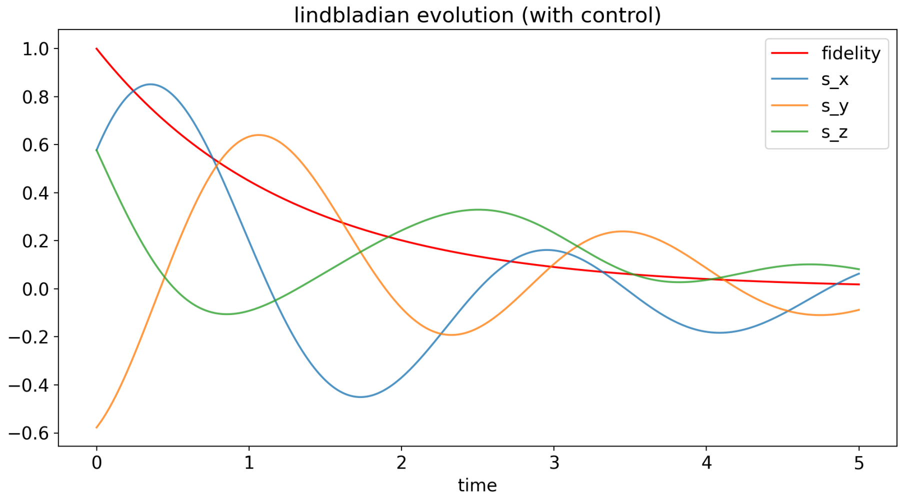

Figure 5: Bloch sphere dynamics and fidelity.

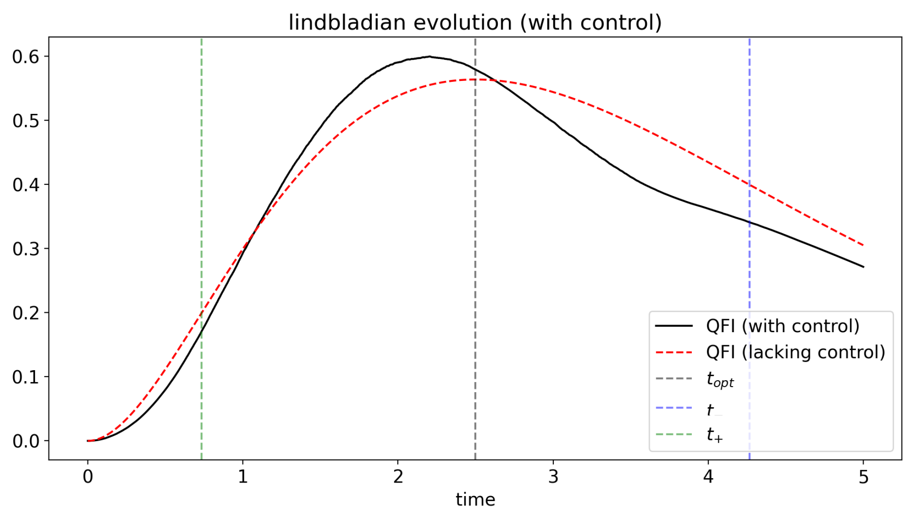

Figure 6: Quantum Fisher information with respect to ω.