Lab 3

Kater's Pendulum and a Precise Value of g

Introduction

The measurement of a quantity with an accuracy of better than 1% is a challenge. The precise measurement of g will give you some insight into the difficulties associated with precision measurements. With the equipment available you should be able to find the value of g to better than 1% during the lab period.

This experiment illustrates two very important techniques:

1. The measurement of the period of a high quality oscillator is one of the most precise measuring techniques in physics. The accuracy obtainable is often limited by the length of time you have to observe the oscillation. In this experiment the error in measuring the period is reduced to a very low level.

2. By combining two such precise measurements we can greatly reduce the effect of systematic errors that are intractable or difficult to measure.

Difficulties with a Simple Pendulum

One method of measuring the value of g is by using a simple pendulum. For small oscillations, the period T could be calculated according to

$$T^2 = \frac{4\pi^2 L}{g} \tag{1}$$

where L is the length of the pendulum. However, this method suffers from several complications if high precision is desired. Among them are the following:

1. the centre of gravity of the bob is difficult to determine in the inevitably inhomogeneous material of the bob.

2. the effect of the buoyancy of the air on the bob.

3. the stretching of the string, so that L does not remain constant.

4. the support will either be simple, leading to excessive damping that limits the accuracy of the period measurement, or complicated, such as a knife edge, in which case a correction must be made for its moment of inertia.

The Compound Pendulum

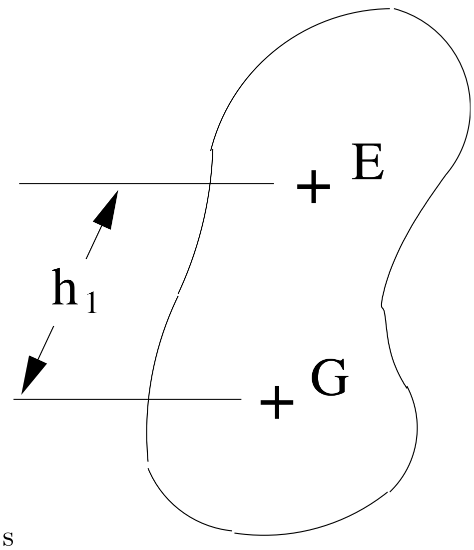

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Figure 1: Compound Pendulum

As soon as you try to modify a simple pendulum to improve its performance it becomes a compound pendulum which we discuss here.

Suppose E is the pivot axis of the compound pendulum and G is its center of gravity. Its moment of inertia I about an axis through G parallel to E is $I = Mk^2$ where M is the pendulum's mass and k is its radus of gyration. By the parallel axis theorem the moment of inertia about E is $I + Mh_1^2$.

The period of the oscillation about E is obtained from

$$T^2 = \frac{4\pi^2 I_E}{gh_1M} = \frac{4\pi^2(I + Mh_1^2)}{h_1 \cdot g \cdot M} \tag{2}$$

or

$$T^2 = \frac{4\pi^2(k^2 + h_1^2)}{h_1 \cdot g} \tag{3}$$

Kater's Pendulum

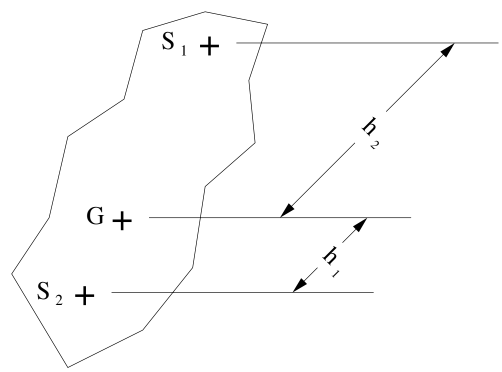

Figure 2: Kater's Pendulum

As described in the background document, Kater's pendulum is a compound pendulum in which we have two knife edges S_1 and S_2, located on opposite sides of the centre of gravity G. As seen in Fig. **??**, G is located at distances h_1 and h_2 from each knife edge. The separation between the two edges is $H = h_1 + h_2$. Defining T_1 and T_2 as the periods of oscillation about S_1 and S_2, we have

$$T_1^2 = \frac{4\pi^2(k^2 + h_1^2)}{g \cdot h_1} \tag{4}$$

$$T_2^2 = \frac{4\pi^2(k^2 + h_2^2)}{g \cdot h_2} \tag{5}$$

and

$$\frac{4\pi^2}{g} = A + B \tag{6}$$

where

$$A = \frac{T_1^2 + T_2^2}{2(H)} \tag{7}$$

$$B = \frac{T_1^2 - T_2^2}{2(h_1 - h_2)} \tag{8}$$

If we could make $T_1 = T_2 = T$ without making $h_1 = h_2$ Then we could have

$$\frac{4\pi^2}{g} = \frac{T^2}{H} \tag{9}$$

and g could be obtained from a measurement of the period T and the distance $H = h_1 + h_2$. The advantage of this method is that you don't have to find the location of the centre of gravity, G, which is difficult to do precisely. Once the periods T_1 and T_2 are equal, the pendulum can be inverted so that the second pivot point becomes the centre of oscillation.

This way it is only necessary to measure the distance between both pivot points instead of between the pivot point and G.

In practice you can't make $T_1 = T_2$ exactly equal, but by varying the position of the centre of gravity G, you can experimentally find the variation of T_1 and T_2 as a function of h_1 and h_2. With this information you can eliminate the effect of the B term (Eqs 6 and 8) to a high degree of accuracy. A simple version of Kater's pendulum is shown in figure 3. S_1 and S_2 are knife edges on which you can swing the pendulum. M_1 and M_2 are large and small masses, which are adjustable in position. They are moved to make T_1 and T_2 as close as possible. The distance $S_1S_2 = H$ is measured precisely and the distances $S_1G = h_1$ and $S_2G = h_2$ are measured as accurately as possible.

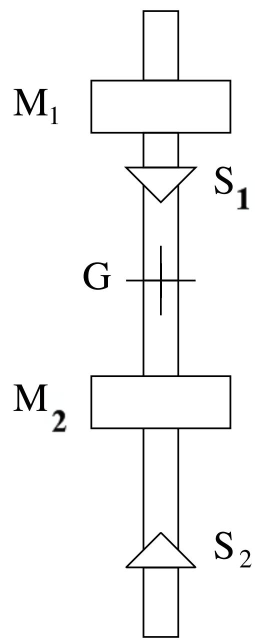

Figure 3: Simple Kater's Pendulum

This form of Kater's pendulum is inherently inaccurate because the masses M_1 and M_2 displace a considerable amount of air as they swing. This air moves with the pendulum and contributes to its moment of inertia in unknown (and essentially impossible to determine) ways. Since the amount of air displaced depends on the velocity of the masses, the radius of gyration is different when the pendulum is suspended on S_1 than it is when the pendulum is suspended on S_2. You might do the experiment in a vacuum to avoid this problem; however, a clever modification of the apparatus yields the same result. The trick is to use two large masses and two small adjusting masses having identical physical dimensions but differing densities. These are arranged symmetrically about the knife edges as shown in figure 4. Q and P are the large masses, while S and R are the corresponding small masses. Provided the masses are symmetrically arranged about the knife edges the effect of the air will be the same when pivoting about S_1 and S_2. The following graph represents the pendulum that we will be using in the lab.

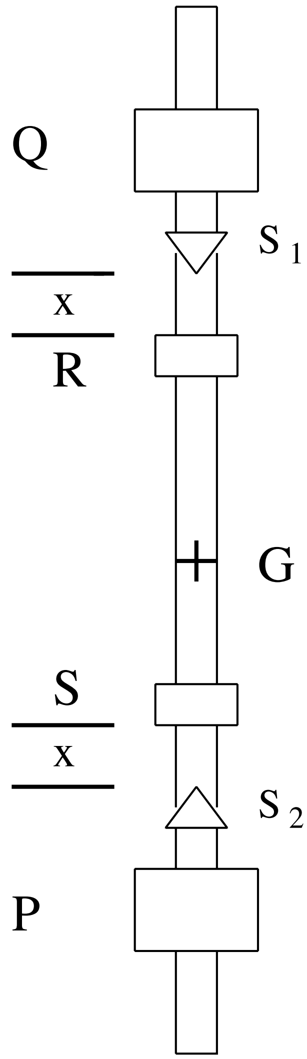

Figure 4: Modified Kater's Pendulum

Overview of the Experiment

The easiest way to adjust the pendulum used in this experiment is to leave Q, P, S_1 and S_2 fixed and to move R and S (the small masses) until the periods T_1 and T_2 are nearly equal to say within 0.01%. Everything must be kept symmetrical to eliminate the effect of uneven air buoyancy.

To get the most accurate position for R and S, data must be taken for values of x that make $T_1 > T_2$ and for values of x that make $T_2 > T_1$. Note that x is the distance between the knife blades and the small masses, and must be kept symmetric on both ends of the pendulum. Plot a graph of T_1 and T_2 against x. It should look something like the graph plotted in figure 5. The value of x that makes T_1 most nearly equal to T_2 is obtained where the two lines intersect. Set R and S to this position and measure the period.

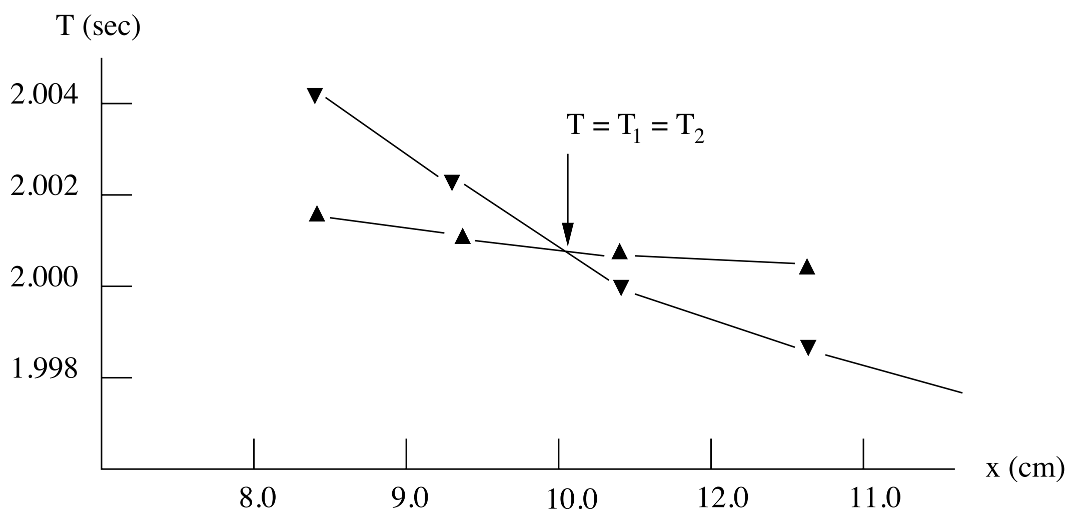

Figure 5: Graph of periods of oscillation for the two ends of the pendulum

The distance between the knife edges should be measured as precisely as possible; this is H. G is located by balancing the pendulum on the knife edge provided in the lab. Measure h_1 and h_2 on either side of G. Compare the sum of h_1 and h_2 with H (in your Discussion, you will want to comment on the discrepancy between and relative accuracy of these two measurements).

Once the x that equalizes T_1 and T_2 is determined you will set about making accurate, repeated measurements of T_1 and T_2 in order to determine g. Details of the procedure are described below.

Detailed Procedure

Set up

1. Set up the circuit for the experiment as in the diagram.

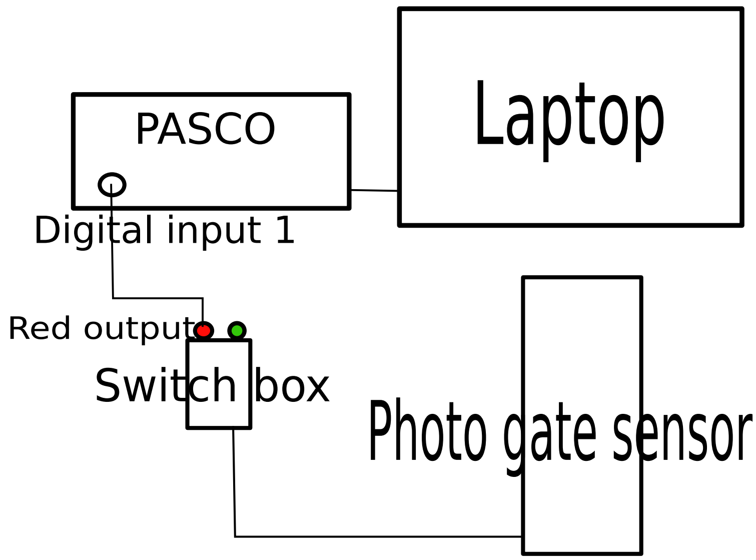

Figure 6: Schematic diagram of the set up.

2. Open the Capstone file for this experiment, the path to the file directory will be given to you.
3. Place the two small masses at equal distance from their respective blades by using the measurement tool "vernier caliper". Take note of this distance in the Measuring Period tab (first tab) of the capstone file. **Note** that the vernier caliper measures in *mm*.
4. Hang the pendulum from the support. Make sure that the knife blades of the pendulum are properly put on the black rock and ensure the safety screw is secured.
5. Now you are ready to take data!

Taking Data

Before you start, make sure you have recorded the distances between the masses and their corresponding blades. Also, ensure that you know which big mass is metal and which one is plastic.

1. Make sure the photo gate sensor is aligned perpendicularly to the plane of oscillation of the pendulum.
2. Make sure the shadow casted on the photo detector by the light bulb is roughly centered when the pendulum is at rest. Also, the light should shine on the rod and not the movable mass.

3. Given the pendulum a gentle swing (small angle) and make sure the top is not wobbling or hitting any of the support structure. Also, to ensure the consistency of your measurements,make sure you the angle of your swing is roughly the same for each trial.

4. Press the monitor button located at the bottom left of the screen and count 11 data points on the graph to the right. Then press stop. Note that the 11 data points represent exactly 5 periods of oscillation.

5. Use the delta tool and move the cursor on the first and the 11^{th} data points to know the time at each point.

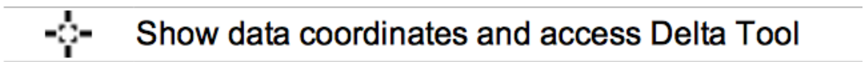

Figure 7: Symbol of the delta tool found on top of the graph.

6. Compute Δt (difference of time) manually and record the value in the table to the left in the corresponding column.

7. Flip over Kater's pendulum and repeat steps 1 - 6.

8. Repeat all previous steps for about 6 different distances while constantly check the Crossover point tab (second tab). (We want to find the crossover point to determine the equilibrium distance where T1 = T2, regardless the orientation of the pendulum).

9. Find the crossover point of the graph in the second tab.

10. Use the delta tool to find the distance at which the point occurred by zooming in on this point and moving the cursor over it (zoom and center the cursor until the last digit of the cursors x-value no longer changes.) Record this value.

11. Set the two masses at the crossover distance from the blades.

12. Measure the period of 5 oscillations (11 data points) of the pendulum in both orientations.

13. Take an average of the two measurements obtained.

14. Find the value of a single period which corresponds to the value of T.

15. Measure the distance between the two knife blades using the custom rail for the vernier caliper(lab technician will show you how to use it). This is your value of H.

Analysis

1. Calculate g as $g = \frac{4\pi^2 H}{T^2}$ with uncertainty values and show your full error analysis.

Sources of Error

The precision of this experiment depends on the accurate measurement of the periods T_1 and T_2 as well as the distance, x, between the knife edges and the small masses.

The period of the pendulum depends on the amplitude of the swing. Simple harmonic motion and the above theory apply for only very small amplitudes of oscillation. Call this period $T_\circ$.

The variation of the period with amplitude θ is shown in table 1.

In this experiment a swing of 5 or 6 degrees will result in negligible error.

θ (degrees)	$T/T_\circ$
0	1.00000
5	1.00048
10	1.00191
15	1.00430

Table 1: Change of period with respect to amplitude of oscillation

Possible Project Experiment

This experiment is capable of giving an accuracy of $\sim 0.1\%$ but would then take longer than one afternoon. It could be considered as a possible project in the second term.

References

[1] Searle, Experimental Physics. Cambridge University Press, pages 7 to 15.

[2] A description of a very precise measurement of g can be found in a paper by J.S.Clark in The Transactions of the Royal Society of London, A, vol.238, pg. 265 (1939).

[3] Henry Kater's original paper can be found in the Philosophical Transactions of the Royal Society, pg. 337(1819).

Hand-in instructions

Data and data analysis:

1. Data table: T1, T2 and x (with uncertainties).

2. Figure of T1 VS. x and T2 VS. x used to determine cross over point.

   (a) Plot the figures using Python.

   (b) *Optional: for 5 bonus points.*

   - Use the least-squares method (see Text sections 5.2 5.6 and Thursdays lecture) to determine the best-fit lines to your data (T1 and T2).
   - Include these lines on your figure along with your data (with error bars).
   - Include the max/min lines in this figure to demonstrate your uncertainty in the parameters you determine (i.e. slope and intercept) using the least-squares method.
   - Determine the cross over point from your best fit lines. Does this determination agree (within error) to the determination you made in lab? Explain any discrepancy.

   ***Note**: You must include your least-squares calculations for the optional part. These can be done by hand or using Python/Jupyter. Excel spreadsheets will not be accepted.

3. Data table that contains the values of H ,T1 (at crossover x), T2 (at crossover x), T, g.

4. Discussion

   (a) Does your result agree with the accepted value for "g" in the lab? Explain. (*We use Helmerts equation for the variation of "g" with latitude together with an additional small correction for the altitude above sea level at which your measurements were made as the accepted value.*) **See supplementary material at the end of this manual**

   (b) Are we justified in our approximation to ignore the "B" term in Equation 6 (see lab manual)? How small would h1 h2 have to be for the error associated with this approximation to be equal to your determined uncertainty in "g"? Explain.

Report writing:

We are going to start working towards the writing of a complete paper based on your experiments and results. The final two experiments of the term will yield complete journal article type reports, rather than the problem-set style reports you have been handing in to date.

As a stepping stone to that, you are to include the following as part of your hand-in for the Katers pendulum experiment:

- Write an "Abstract" for this experiment based on your results .
  1. What was done.
  2. How it was done (experimental technique used with the relevant experimental conditions).
  3. What were the final results (including the error estimate).
  4. What are the main conclusions (e.g., if the results are consistent with a particular model, literature data, etc.)
- Write an "Experimental methods" section for this experiment. If you make a schematic, make sure to put appropriate labels and a descriptive figure caption.

Note:

- You can use any word-processor that you are comfortable with to write the report. If you are uncertain how to write a certain section of lab report, refer to the handout for description and examples of report writing.
- You must include sample calculations for EVERY computed quantity that appears in a data table or figure; i.e. show sample calculations for each calculated quantity AND its associated uncertainty. It is really impossible to mark your work without these sample calculations.
- The relevant sample calculations for the data tables and figures should follow the data table or figure in your report in a sensible and well-organized way.
- Tables and Figures must be properly formatted. This is the last time I will mention this. It will be assumed from now on.

Kater's Pendulum

SUPPLEMENTARY INFORMATION

Figure 1: *Henry Kater (1777 - 1835)*

Invented in 1817 by British physicist Henry Kater, Kater's Pendulum was created as a method of accurately measuring the local gravitational acceleration, g. Kater's original pendulum consisted of a metal bar with two knife edges acting as pivot points (one at each end), that can be reversed. The knife edges were adjustable and masses were attached to each end. (See Figure 2). The revised version of this pendulum is discussed below.

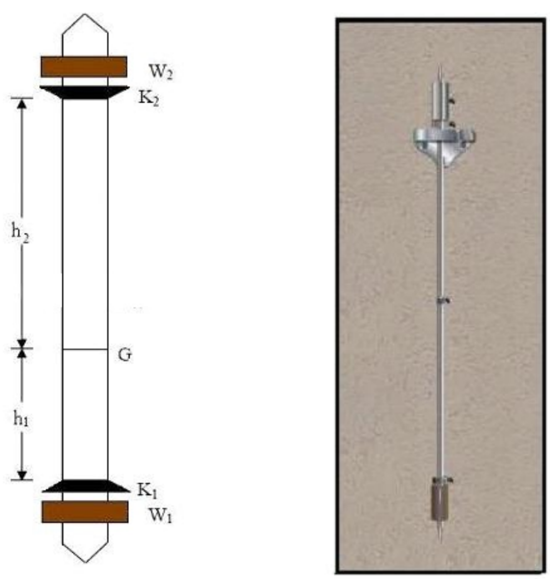

Figure 2: *Kater's pendulum. K_1 and K_2 are adjustable knife edges, W_1 and W_2 are masses and G is the center of gravity.*

Figure 3: *Christiaan Huygens (1629 - 1695)*

Christiaan Huygens proved that the pivot point of a compound pendulum - such as Kater's pendulum - and its center of oscillation are interchangeable (the center of oscillation is defined as the mass position of a simple pendulum that has the same period as the physical/compound pendulum). This means that if a compound pendulum was hung upside down from its center of oscillation, it would have the same period as it did from the original pivot point! Furthermore, the first pivot point would become the new center of oscillation. Kater realized that this principle could be used to create a pendulum with two adjustable pivot points at opposite ends so that each pivot would be located at the other's center of oscillation. This is true only if the periods of oscillation T_1 and T_2 are equal when the pendulum is swung from either end.

In this way, he created a pendulum that would have the same period as a simple pendulum, where the distance between the two pivot points was equivalent to the length of a simple pendulum. Previously, it was necessary to locate the position of the center of gravity in order to achieve accurate results, which was often very difficult to do. Now, however, the distance between each pivot point and the center of oscillation was known: the distance between the knife edges. The period of a simple pendulum is given by the equation $T = 2\pi\sqrt{\frac{L}{g}}$ and so the value of g could then be calculated:

$$g = \frac{4\pi^2 L}{T^2} \tag{1}$$

where L is the length of the pendulum (distance between knife egdes) and $T_1 = T_2 = T$.

Although the original experiment had two adjustable pivot points, Kater realized it was advantageous to keep the knife edges stationary and to introduce small masses with differing densities that would be adjustable in position. (See Figure 4). It is possible to adjust these weights so that periods of oscillation remain equal to one another. This revised pendulum reduced the effects of air resistance on the masses, which would increase the pendulum's moment of inertia and decreased the accuracy of the equipment.

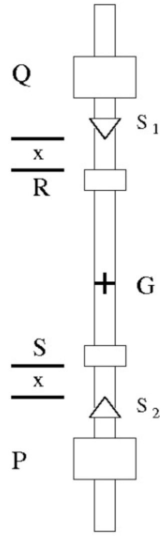

Figure 4: *Modified Kater's Pendulum*

A further advantage of Kater's pendulum was that it would swing as a rigid body, whereas the simple pendulum did not, meaning that the elasticity of the wire caused its length to change slightly as it swung. The result was that Kater's pendulum was much more accurate than previous pendulums and this method remained the standard method of determining g for about a century.

The increased accuracy made possible by Kater's pendulum helped make gravimetry, the measurement of the Earth's gravity field, a standard part of geodesy. Geodesy is the measurement and representation of the Earth, as well as its motion, so that it may be represented on maps. Pendulums were taken on geodetic surveys of the 18th century, particularly the Great

Trigonometric Survey of India, where many important features were measured, such as the height of Everest, as well as measurements of the "geodesic anomaly" caused by the deflection of pendulums due to large masses such as mountains or irregular densities within the earth. Today, Kater's pendulum is mostly used as an instructional tool in laboratories rather than a practical instrument for measuring variations in g, where Kater's pendulum has been replaced by modern gravimeters/gravitometers

Modern gravimeters (or gravitometers) can be configured to measure either the absolute value of g in a specific location of the difference in g between two locations (or the same location as a function of time). Relative gravitometers are capable of measuring changes in g to 1 part in 10^{12}! Sensitive enough to detect snow being removed from the roof of a laboratory in which the instrument is located.

Mathematical models for gravitational acceleration near the surface of the earth

Latitude model

If the terrain is at sea level, we can estimate $g\{\phi\}$, the acceleration at latitude ϕ:

$$\begin{aligned} g\{\phi\} &= 9.780327\ \mathrm{m\cdot s^{-2}}\ \left(1 + 0.0053024\ \sin^2\phi - 0.0000058\ \sin^2 2\phi\right), \\ &= 9.780327\ \mathrm{m\cdot s^{-2}}\ \left(1 + 0.0052792\ \sin^2\phi + 0.0000232\ \sin^4\phi\right), \\ &= 9.780327\ \mathrm{m\cdot s^{-2}}\ \left(1.0053024 - 0.0053256\ \cos^2\phi + 0.0000232\ \cos^4\phi\right), \\ &= 9.780327\ \mathrm{m\cdot s^{-2}}\ \left(1.0026454 - 0.0026512\ \cos 2\phi + 0.0000058\ \cos^2 2\phi\right) \end{aligned} .$$

This is the International Gravity Formula 1967, the 1967 Geodetic Reference System Formula, Helmert's equation or Clairaut's formula.[15]

An alternate formula for *g* as a function of latitude is the WGS (World Geodetic System) 84 Ellipsoidal Gravity Formula:[16]

$$g\{\phi\} = \mathbb{G}_e \left[\frac{1 + k\sin^2\phi}{\sqrt{1 - e^2\sin^2\phi}}\right],$$

where,

- $a,\, b$ are the equatorial and polar semi-axes, respectively;
- $e^2 = 1 - (b/a)^2$ is the spheroid's eccentricity, squared;
- $\mathbb{G}_e,\ \mathbb{G}_p$ is the defined gravity at the equator and poles, respectively;
- $k = \dfrac{b\,\mathbb{G}_p - a\,\mathbb{G}_e}{a\,\mathbb{G}_e}$ (formula constant);

then, where $\mathbb{G}_p = 9.8321849378\ \mathrm{m\cdot s^{-2}}$,[16]

$$g\{\phi\} = 9.7803253359\ \mathrm{m\cdot s^{-2}} \left[\frac{1 + 0.00193185265241\ \sin^2\phi}{\sqrt{1 - 0.00669437999013\ \sin^2\phi}}\right].$$

The difference between the WGS-84 formula and Helmert's equation is less than 0.68 $\mu\mathrm{m\cdot s^{-2}}$.

Free air correction

The first correction to be applied to the model is the free air correction (FAC) that accounts for heights above sea level. Near the surface of the Earth (sea level), gravity decreases with height such that linear extrapolation would give zero gravity at a height of one half of the earth's radius - (9.8 $\mathrm{m \cdot s^{-2}}$ per 3,200 km.)[17]

Using the mass and radius of the Earth:

$$\begin{aligned} r_{\text{Earth}} &= 6.371 \cdot 10^6 \text{ m} \\ m_{\text{Earth}} &= 5.9722 \cdot 10^{24} \text{ kg} \end{aligned}$$

The FAC correction factor (Δg) can be derived from the definition of the acceleration due to gravity in terms of G, the Gravitational Constant (see Estimating g from the law of universal gravitation, below):

$$g_0 = G\, m_{\text{Earth}} / r_{\text{Earth}}^2 = 9.8196\ \frac{\text{m}}{\text{s}^2}$$

where:

$$G = 6.67384 \cdot 10^{-11}\ \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}.$$

At a height h above the nominal surface of the earth g_h is given by:

$$g_h = G\, m_{\text{Earth}} / (r_{\text{Earth}} + h)^2$$

So the FAC for a height h above the nominal earth radius can be expressed:

$$\Delta g_h = \left[G\, m_{\text{Earth}} / (r_{\text{Earth}} + h)^2 \right] - \left[G\, m_{\text{Earth}} / r_{\text{Earth}}^2 \right]$$

This expression can be readily used for programming or inclusion in a spreadsheet. Collecting terms, simplifying and neglecting small terms ($h \ll r_{\text{Earth}}$), however yields the good approximation:

$$\Delta g_h \approx -\frac{G\, m_{\text{Earth}}}{r_{\text{Earth}}^2} \cdot \frac{2\, h}{r_{\text{Earth}}}$$

Using the numerical values above and for a height h in metres:

$$\Delta g_h \approx -3.086 \cdot 10^{-6}\, h$$

Grouping the latitude and FAC altitude factors the expression most commonly found in the literature is:

$$g\{\phi, h\} = g\{\phi\} - 3.086 \cdot 10^{-6} h$$

where $g\{\phi, h\}$ = acceleration in $\mathrm{m \cdot s^{-2}}$ at latitude ϕ and altitude h in metres. Alternatively (with the same units for h) the expression can be grouped as follows:

$$g\{\phi, h\} = g\{\phi\} - 3.155 \cdot 10^{-7} h \, \frac{\mathrm{m}}{\mathrm{s}^2}$$