Chapter 16

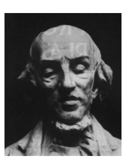
A Selection of Minimal Surfaces

The origins of minimal surface theory can be traced back to 1744 with Euler's paper [Euler2], and to 1760 with Lagrange's paper [Lag]. Euler showed that the catenoid is a minimal surface, and Lagrange wrote down the partial differential equation that must be satisfied for a surface of the form $z = f(x, y)$ to be minimal. In 1776, Meusnier[1] rediscovered the catenoid and also showed that the helicoid is a minimal surface [Meu]. The mathematical world had to wait over 50 years until other examples were found by Scherk. These include the surfaces now called 'Scherk's minimal surface', 'Scherk's fifth minimal surface', and a family of surfaces that includes both the catenoid and the helicoid.

The importance of minimal surfaces as those of least potential surface energy was illustrated by the experiments of Plateau[2], who dipped wires in the form of space curves into a solution of soapy water and glycerin, thus realizing minimal surfaces experimentally. Plateau's problem is that of determining the minimal surfaces through a given curve. It has been formulated and studied in great generality by many mathematicians.

In this chapter, we begin a study of minimal surfaces by showing that a minimal surface is a critical point of the area function in an appropriate sense (Theorem 16.4). In Section 16.2, we provide the first of a series of new examples

[1] Jean Babtiste Meusnier de la Place (1754–1793). French mathematician, a student of Monge. He was a general in the revolutionary army and died of battle wounds.

[2]  Joseph Antoine Ferdinand Plateau (1801–1883). Belgian physicist. His thesis concerned the impressions that light can have on the eye. Unfortunately, this led him to stare into the bright sun, which had an adverse effect on his sight. In 1840, he began a series of experiments with surfaces for which he became famous.

of minimal surfaces by investigating the celebrated deformation from the helicoid to the catenoid. This construction also provides examples of local isometries of surfaces, a concept defined in Chapter 12. In Section 16.3, we prove that any surface of revolution which is also a minimal surface is contained in a catenoid or a plane.

The minimal surfaces of Enneper, Catalan and Henneberg are described in Section 16.4. The second is intimately related to a cycloid curve that it contains, and this example anticipates the definition of geodesic torsion in Section 17.4 and the theory of geodesics in Chapter 18. A discussion of Scherk's surface in Section 16.5 begins with the minimal surface equation for a Monge patch.

In Section 16.6 we show that the Gauss map of a minimal surface is conformal, an important result that lies at the heart of much modern research into minimal surfaces [Os1]. *Isothermal coordinates* for surfaces are introduced in Section 16.7, and provide the foundation for our further study of minimal surfaces using a complex variable in Chapter 22.

16.1 Normal Variation

On page 398, we defined a minimal surface in $\mathbb{R}^3$ as a surface whose mean curvature vanishes. This is Lagrange's 1760 definition. A more intuitive meaning of minimal surface is the surface of least area among a family of surfaces having the same boundary. In order to show how these two definitions coincide, we define the normal variation of a surface $\mathcal{M}$ in $\mathbb{R}^3$ to be a family of surfaces $t \mapsto \mathcal{M}(t)$ representing how $\mathcal{M}$ changes when pulled in a normal direction. Let $A(t)$ denote the area of $\mathcal{M}(t)$. We show that the mean curvature of $\mathcal{M}$ vanishes if and only if the first derivative of $t \mapsto A(t)$ vanishes at $\mathcal{M}$.

First, we make precise the notion of normal variation. The notion of a bounded subset was discussed on page 373, and the general formula for the area of a bounded region of a surface in $\mathbb{R}^n$ was given by (12.24). In the case $n = 3$, (12.22) on the foot of page 372 yields

Lemma 16.1. *Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^3$, and $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ a regular patch. Given a bounded region $\mathcal{Q}$ in $\mathcal{U}$, the area of $\mathbf{x}(\mathcal{Q})$ is given by*

$$\mathsf{area}(\mathbf{x}(\mathcal{Q})) = \iint_{\mathcal{Q}} \|\mathbf{x}_u \times \mathbf{x}_v\| \, du dv.$$

We wish to study how area changes with a small perturbation of a surface. The simplest perturbation is one that is normal to the surface.

Definition 16.2. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ be a regular patch, and choose a bounded region $\mathcal{Q} \subset \mathcal{U}$. Suppose that $h\colon \mathcal{Q} \to \mathbb{R}$ is differentiable and $\varepsilon > 0$. Let $\mathbf{U}$ denote a* *unit vector field such that* $\mathbf{U}(u,v)$ *is perpendicular to* $\mathbf{x}(u,v)$ *for all* $(u,v)\in\mathcal{U}$. *Then the* ***normal variation*** *of* $\mathbf{x}$ *and* $\mathcal{Q}$, *determined by* h, *is the map*

$$\mathbf{X}\colon (-\varepsilon,\varepsilon)\times\mathcal{Q}\longrightarrow\mathbb{R}^3$$

given by

$$\mathbf{X}_t(u,v)=\mathbf{x}(u,v)+t\,h(u,v)\mathbf{U}(u,v) \tag{16.1}$$

for $(u,v)\in\mathcal{Q}$ *and* $-\varepsilon<t<\varepsilon$.

It follows from this definition that $\mathbf{X}_t$ is a patch for each t with $-\varepsilon<t<\varepsilon$ for sufficiently small ε. Let

$$\begin{cases} E(t) &= (\mathbf{X}_t)_u\cdot(\mathbf{X}_t)_u,\\ F(t) &= (\mathbf{X}_t)_u\cdot(\mathbf{X}_t)_v,\\ G(t) &= (\mathbf{X}_t)_v\cdot(\mathbf{X}_t)_v. \end{cases} \tag{16.2}$$

Then $E=E(0)$, $F=F(0)$, $G=G(0)$, and by Lemma 16.1 the area of $\mathbf{X}_t(\mathcal{Q})$ is given by

$$A(t)=\iint_{\mathcal{Q}}\sqrt{E(t)G(t)-F(t)^2}\,du\,dv. \tag{16.3}$$

We compute the derivative at zero of the function $\mathcal{A}(t)$.

Lemma 16.3. *We have*

$$A'(0)=-2\iint_{\mathcal{Q}}h\,H\sqrt{EG-F^2}\,du\,dv, \tag{16.4}$$

where H *denotes the mean curvature of* $\mathcal{M}$.

Proof. Differentiating (16.1) with respect to u and v gives

$$\begin{cases} (\mathbf{X}_t)_u=\mathbf{x}_u+t\,h_u\mathbf{U}+t\,h\,\mathbf{U}_u,\\ (\mathbf{X}_t)_v=\mathbf{x}_v+t\,h_v\mathbf{U}+t\,h\,\mathbf{U}_v. \end{cases} \tag{16.5}$$

From (13.9), page 394, (16.5) and the definitions (16.2), it follows that

$$\begin{aligned} E(t) &= (\mathbf{X}_t)_u\cdot(\mathbf{X}_t)_u\\ &= (\mathbf{x}_u+t\,h_u\mathbf{U}+t\,h\,\mathbf{U}_u)\cdot(\mathbf{x}_u+t\,h_u\mathbf{U}+t\,h\,\mathbf{U}_u)\\ &= E+2t\,h\,\mathbf{x}_u\cdot\mathbf{U}_u+O\big(t^2\big)\\ &= E-2t\,h\,e+O\big(t^2\big). \end{aligned} \tag{16.6}$$

Similarly,

$$\begin{aligned} F(t) &= F-2t\,h\,f+O\big(t^2\big),\\ G(t) &= G-2t\,h\,g+O\big(t^2\big). \end{aligned} \tag{16.7}$$

From (16.6),(16.7) and Theorem 13.25, page 400, we get

$$\begin{aligned} E(t)G(t) - F(t)^2 &= \bigl(E - 2the + O(t^2)\bigr)\bigl(G - 2thg + O(t^2)\bigr) \\ &\qquad\qquad - \bigl(F - 2thf + O(t^2)\bigr)^2 \\ &= EG - F^2 - 2th(Eg - 2Ff + Ge) + O(t^2) \\ &= \bigl(EG - F^2\bigr)(1 - 4thH) + O(t^2)\,, \end{aligned}$$

and hence

$$\begin{aligned} \sqrt{E(t)G(t) - F(t)^2} &= \sqrt{(EG - F^2)(1 - 4thH) + O(t^2)} \\ &= \sqrt{EG - F^2}\sqrt{1 - 4thH + O(t^2)} \\ &= \sqrt{EG - F^2}\,(1 - 2thH) + O(t^2)\,. \end{aligned}$$

Combined with (16.3), we obtain

$$\begin{aligned} A(t) &= \iint_{\mathcal{Q}} j\left(\sqrt{EG - F^2}\,(1 - 2thH) + O(t^2)\right) du dv \\ &= \iint_{\mathcal{Q}} \sqrt{EG - F^2}\, du dv - 2t \iint_{\mathcal{Q}} hH\sqrt{EG - F^2}\, du dv + O(t^2)\,. \end{aligned}$$

When we differentiate with respect to t and evaluate the resulting expression at $t = 0$, we obtain (16.4). ▮

Theorem 16.4. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ *be a regular patch, and choose a bounded region* $\mathcal{Q} \subset \mathcal{U}$. *Then* $\mathbf{x}$ *is minimal on* $\mathcal{Q}$ *if and only if* $A'(0) = 0$ *for a normal variation of* $\mathbf{x}$ *and* $\mathcal{Q}$ *with respect to any* $h\colon \mathcal{Q} \to \mathbb{R}$.

Proof. If H is identically zero for $\mathbf{x}$, then (16.4) implies that $A'(0) = 0$ for any h. Conversely, suppose that $A'(0) = 0$ for any differentiable function $h\colon \mathcal{Q} \to \mathbb{R}$ but that there is $\mathbf{q} \in \mathcal{Q}$ for which $H(\mathbf{q}) \neq 0$. Choose h such that $h(\mathbf{q}) = H(\mathbf{q})$, with h identically zero outside of a small neighborhood of $\mathbf{q}$ on which $hH \geqslant 0$. But now (16.4) implies that $A'(0) < 0$. This contradiction shows that $H(\mathbf{q}) = 0$. Since $\mathbf{q}$ is arbitrary, $\mathbf{x}$ is minimal. ▮

We have said nothing about the second derivative of A at 0, so that a minimal surface, although a critical point of A, may not actually be a *minimum*. More details of this approach, as well as extensions of the results of this section, can be found in [Nits, pages 90–116].

16.2 Deformation from the Helicoid to the Catenoid

We have already considered the helicoid on page 376 and the catenoid on page 464. In fact, these two surfaces constitute the initial and final points of a deformation through isometric minimal surfaces. For each t with $0 \leqslant t \leqslant \pi/2$, define

$$\begin{aligned} \mathbf{z}[t](u,v) = {} & \cos t\,\big(\sinh v \sin u,\ -\sinh v \cos u,\ u\big) \\ & + \sin t\,\big(\cosh v \cos u,\ \cosh v \sin u,\ v\big). \end{aligned} \tag{16.8}$$

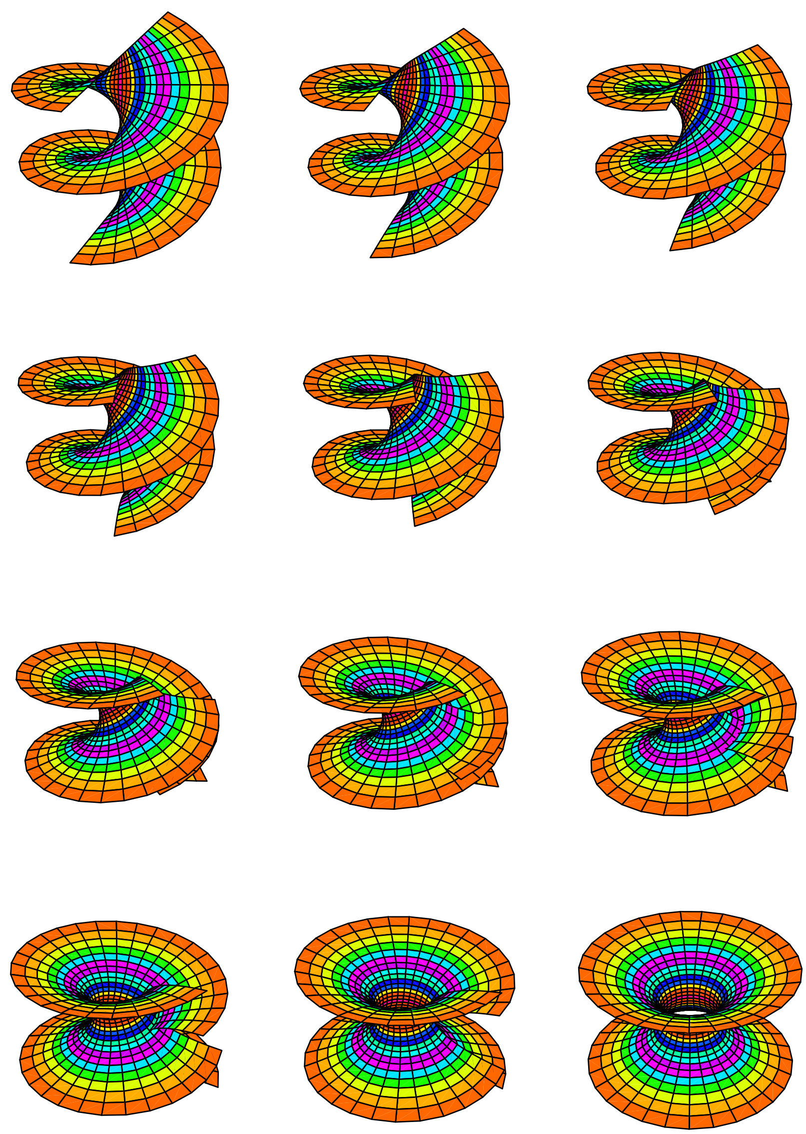


Figure 16.1: $\mathbf{z}\left[\frac{n\pi}{16}\right]$ with $0 \leqslant n \leqslant 8$

The exact relation between this family of surfaces and the helicoid and catenoid is provided by the equations

$$\begin{aligned}\mathbf{z}[0](u,v) &= \mathsf{helicoid}[1,1]\Big(u-\frac{\pi}{2},\ \sinh v\Big)+\big(0,0,\frac{\pi}{2}\big),\\ \mathbf{z}[\frac{\pi}{2}](u,v) &= \mathsf{catenoid}[1](u,v),\end{aligned}$$

verified in Notebook 16.

Theorem 16.5. *The* 1*-parameter family of surfaces* (16.8) *is a deformation from the helicoid to the catenoid such that* $\mathbf{z}[0]$ *is* (*a reparametrization of*) *a helicoid and* $\mathbf{z}[\pi/2]$ *is a catenoid. Furthermore, each* $\mathbf{z}[t]$ *is a minimal surface which is locally isometric to* $\mathbf{z}[0]$. *In particular, the helicoid is locally isometric to the catenoid.*

Proof. Let $E(t),F(t),G(t)$ denote the coefficients of the first fundamental form of $\mathbf{z}[t]$. An easy calculation (by hand or with the help of Notebook 16) shows that

$$E(t)=\cosh^2 v=G(t), \qquad F(t)=0.$$

In particular, $E(t),F(t),G(t)$ are constant functions of t. The result follows from Lemma 12.7 on page 366. ▮

It is now an easy matter to plot accurate pictures of the deformation $t\mapsto\mathbf{z}[t]$ like those of Figure 16.1, that are colored by a function of Gaussian curvature. The immersion $\mathbf{z}[t]$ obviously does depend on t; mathematically, this is actually a special case of a subsequent result on page 724. Neither the catenoid nor the helicoid intersects itself; however, every intermediate surface $\mathbf{z}[t]$ has self-intersections. It is also true that asymptotic curves on the helicoid are gradually transformed into principal curves on the catenoid (see page 390 for the definitions, and page 725 for the result).

Figure 16.2 illustrates two particular regions of the helicoid and catenoid that correspond under the deformation. The boundaries of these regions include a helix and a circle, with respective parametrizations

$$\begin{aligned}\boldsymbol{\alpha}(u) &= \big(\sinh 1\sin u,\ -\sinh 1\cos u,\ u\big), && 0\leqslant u<2\pi,\\ \boldsymbol{\gamma}(u) &= \big(\cosh 1\cos u,\ \cosh 1\sin u,\ 1\big), && 0\leqslant u<2\pi.\end{aligned}$$

The length of the helix is

$$\int_0^{2\pi}|\boldsymbol{\alpha}'(u)|du=2\pi\sqrt{\sinh^2 1+1},$$

and that of the circle is

$$\int_0^{2\pi}|\boldsymbol{\gamma}'(u)|\,du=2\pi\cosh 1.$$

The fact that these two lengths are equal is consistent with the statement that the helix maps isometrically onto the circle.

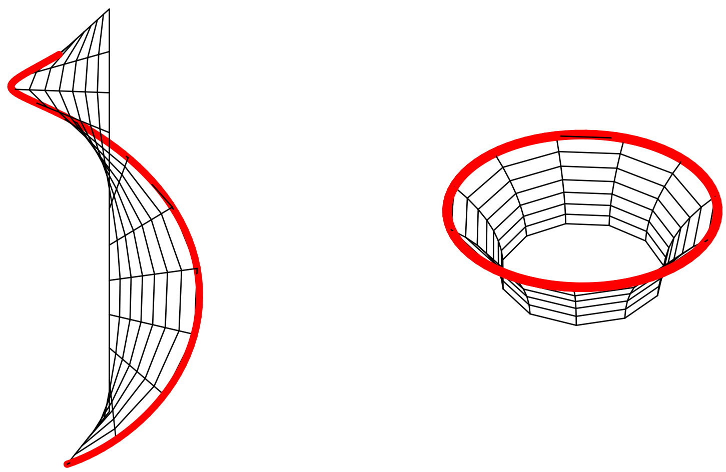


Figure 16.2: Isometric regions of the helicoid and catenoid

16.3 Minimal Surfaces of Revolution

The catenoid is the only member of the family $\mathbf{z}[t]$ that is a surface of revolution. We next prove that the catenoid and the plane are in fact essentially the only surfaces of revolution that are simultaneously minimal surfaces. The following result is therefore an analogue of Exercise 7 of Chapter 15, that dealt with the case of zero *Gaussian* curvature.

Theorem 16.6. *A surface of revolution $\mathcal{M}$ which is a minimal surface is contained in either a plane or a catenoid.*

Proof. Let $\mathbf{x}$ be a patch whose trace is contained in $\mathcal{M}$, and let $\boldsymbol{\alpha} = (\varphi, \psi)$ be the profile curve. Then $\mathbf{x}$ is given by (15.1). There are three cases:

Case 1. ψ' is identically 0. Then ψ is constant, so that $\boldsymbol{\alpha}$ is a horizontal line and $\mathcal{M}$ is part of a plane perpendicular to the axis of revolution.

Case 2. ψ' is never 0. Then by the inverse function theorem, ψ has an inverse ψ^{-1}. Define

$$\widetilde{\boldsymbol{\alpha}}(t) = \boldsymbol{\alpha}\big(\psi^{-1}(t)\big) = \big(h(t),\, t\big),$$

where $h = \varphi \circ \psi^{-1}$, and a new patch $\mathbf{y}$ by

$$\mathbf{y}(u, v) = \big(h(v)\cos u,\ h(v)\sin u,\ v\big).$$

Since $\widetilde{\boldsymbol{\alpha}}$ is a reparametrization of $\boldsymbol{\alpha}$, it follows that $\mathbf{x}$ and $\mathbf{y}$ have the same trace. Thus it suffices to show that the surface of revolution $\mathbf{y}$ is part of a catenoid.

In the notation of page 469, equations (15.11) reduce to

$$\begin{cases} k_m = \dfrac{g}{G} = \dfrac{h''}{(h'^2+1)^{3/2}}, \\ k_p = \dfrac{e}{E} = -\dfrac{1}{h\sqrt{h'^2+1}}. \end{cases} \tag{16.9}$$

It follows from the assumption that $H = 0$ and (16.9) that h must satisfy the differential equation

$$h''h = 1 + h'^2. \tag{16.10}$$

To solve (16.10), we first rewrite it as

$$\frac{2h'h''}{1+h'^2} = \frac{2h'}{h}.$$

Integrating both sides yields

$$\log(1+h'^2) = \log(h^2) - \log(c^2)$$

for some constant $c \neq 0$, and exponentiating we obtain

$$1 + h'^2 = \left(\frac{h}{c}\right)^2. \tag{16.11}$$

The first-order differential equation (16.11) can be written as

$$\frac{h'/c}{\sqrt{(h/c)^2 - 1}} = \frac{1}{c}. \tag{16.12}$$

Both sides of (16.12) can be integrated to yield

$$\operatorname{arccosh} \frac{h}{c} = \frac{v}{c} + b.$$

Thus the solution of (16.10) is

$$h(v) = c\cosh\left(\frac{v}{c} + b\right),$$

and so $\mathcal{M}$ is part of a catenoid.

Case 3. ψ' is zero at some points, but nonzero at others. In fact, this case cannot occur. Suppose, for example, that $\psi'(v_0) = 0$, but $\psi'(v) > 0$ for $v < v_0$. By Case 2, the profile curve is a catenary for $v < v_0$, whose slope is given by φ'/ψ'. Then $\psi'(v_0) = 0$ implies that the slope becomes infinite at v_0. But this is impossible, since the profile curve is the graph of the function cosh. ▮

Nicely complementing Theorem 16.6 is a theorem due to Catalan which assets that the *helicoid* is the only minimal surface other than the plane which is also ruled (see [dC1]).

16.4 More Examples of Minimal Surfaces

Enneper's Minimal Surface

One of the simplest minimal surfaces is the one found by Enneper in 1864 (see [Enn1]). It is defined by

$$\text{enneper}(u,v) = \left(u - \frac{u^3}{3} + u\,v^2,\ -v + \frac{v^3}{3} - v\,u^2,\ u^2 - v^2\right).$$

Determining its first and second fundamental forms was the object of Exercise 6 of Chapter 12 (see Figure 12.13 on page 378) and Exercise 4 of Chapter 13.

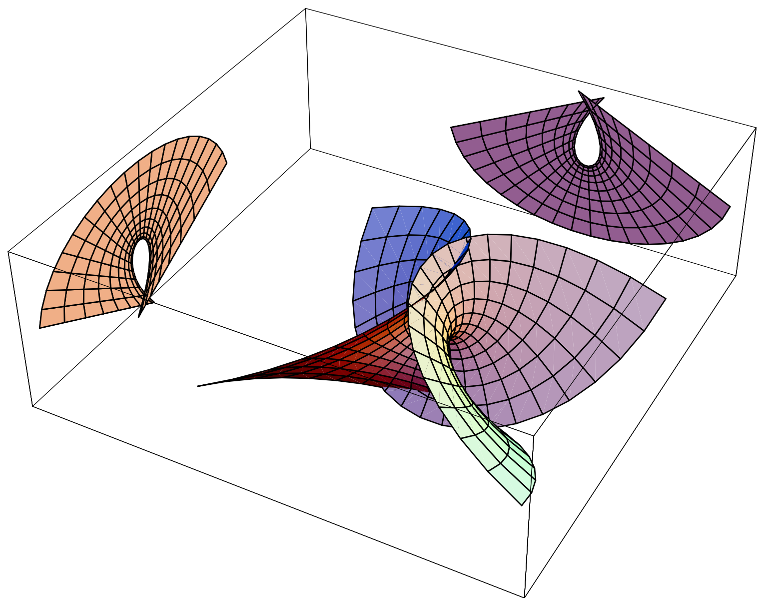


Figure 16.3: Enneper's surface with shadows

It is easy to verify directly that H is identically zero. Firstly,

$$\begin{aligned}\mathbf{x}_u &= (1 - u^2 + v^2,\ -2uv,\ 2u),\\ \mathbf{x}_v &= (2uv,\ -1 - u^2 + v^2,\ -2v),\end{aligned}$$

whence

$$E = \left(1 + u^2 + v^2\right)^2 = G, \qquad F = 0. \tag{16.13}$$

Setting $\rho = 1 + u^2 + v^2$ helps further calculations, such as

$$\mathbf{x}_u \times \mathbf{x}_v = \left(2u\rho,\ 2v\rho,\ \rho^2 - 2\rho\right).$$

However, we do not need this normal vector, since the vanishing of the mean curvature already follows from (16.13) and the equations

$$\mathbf{x}_{uu} = -2(u,\ v,\ -1) = -\mathbf{x}_{vv}.$$

In spite of the simplicity of its definition, Enneper's surface is complicated because of self-intersections. These are best visualized by projecting the surface to the coordinate planes as shown in Figure 16.3.

Catalan's Minimal Surface

The minimal surface of Catalan[3] is parametrized by

$$\mathbf{x}(u,v) = \left(u - \sin u \cosh v,\ 1 - \cos u \cosh v,\ -4 \sin\frac{u}{2} \sinh\frac{v}{2}\right). \tag{16.14}$$

We defer to Notebook 16 for the verification that the mean curvature H is identically zero.

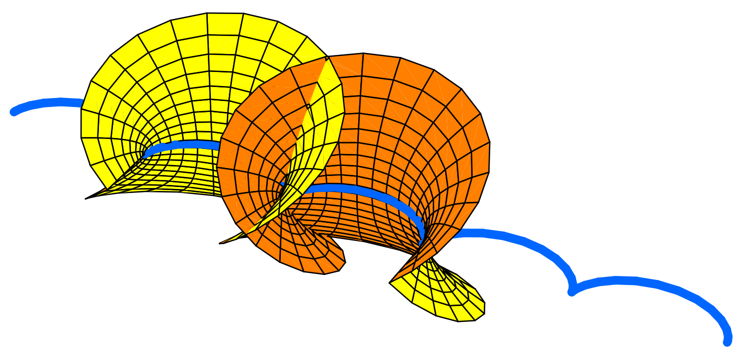


Figure 16.4: Catalan's minimal surface

The intersection of the plane $z = 0$ with the trace of (16.14) contains the coordinate curve

$$\boldsymbol{\alpha}(u) = \mathbf{x}(u,0) = (u - \sin u,\ 1 - \cos u,\ 0), \tag{16.15}$$

that defines a cycloid in the xy-plane. Much of the interest in Catalan's surface derives from its properties in relation to this curve, that we now investigate. We denote by s an arc length function along the curve, whose speed is therefore

$$\frac{ds}{du} = |\boldsymbol{\alpha}'(u)| = \sqrt{2 - 2\cos u} = 2\left|\sin\frac{u}{2}\right|. \tag{16.16}$$

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3  Eugène Charles Catalan (1814–1894). Belgian mathematician, who had difficulty obtaining a position in France because of his left-wing views. Catalan's constant is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.915966.$$

One can use this to write down s as a function of u, though we shall not need an explicit formula.

Consider the vectors $\mathbf{x}_u$, $\mathbf{x}_v$ tangent to the surface at points of the curve. The first is the tangent vector to the curve itself, since

$$\mathbf{x}_u(u,0) = \boldsymbol{\alpha}'(u) = (1-\cos u,\ \sin u,\ 0).$$

By contrast,

$$\mathbf{x}_v(u,0) = \Big(0,\ 0,\ -2\sin\frac{u}{2}\Big),$$

and so this second vector is orthogonal to the plane containing the trace of $\boldsymbol{\alpha}$. The acceleration vector $\boldsymbol{\alpha}''(s)$ is perpendicular to

$$\boldsymbol{\alpha}'(s) = \boldsymbol{\alpha}'(u)\frac{du}{ds},$$

since the latter has constant norm. But $\boldsymbol{\alpha}''(s)$ must lie in the xy-plane, and is thus orthogonal to $\mathbf{x}_v$. We have therefore established the next result without any real calculation.

Lemma 16.7. The acceleration $\boldsymbol{\alpha}''(s)$ of the curve (16.15) is everywhere parallel to the normal vector $\mathbf{x}_u \times \mathbf{x}_v$ of the surface (16.14).

In general, given a curve $\boldsymbol{\alpha}$ whose trace lies on a surface, the tangential component of $\boldsymbol{\alpha}''(s)$ defines the so-called ***geodesic curvature***, which in the next chapter we prove depends only on the first fundamental form. The curve $\boldsymbol{\alpha}$ is called ***pregeodesic*** if its acceleration vector $\boldsymbol{\alpha}''(s)$ satisfies the conclusion of Lemma 16.7. The term ***geodesic*** is reserved for the case in which the curve's parameter is a constant times arc length; for then the mapping $s \mapsto \boldsymbol{\alpha}(s)$ satisfies the so-called geodesic equations, that are further explained in Chapter 18.

An interesting problem in minimal surface theory is the determination of a minimal surface that contains a given curve as a geodesic (see Section 22.6), or a curve tangent to asymptotic or principal directions. See [Nits, page 140].

Henneberg's Minimal Surface

The following elementary parametrization of the minimal surface of Henneberg[4] is given on page 144 of [Nits]:

$$\mathsf{henneberg}(u,v) = \Big(2\sinh u\cos v - \tfrac{2}{3}\sinh 3u\cos 3v,\ 2\sinh u\sin v + \tfrac{2}{3}\sinh 3u\sin 3v,\ 2\cosh 2u\cos 2v\Big).$$

4  Ernst Lebrecht Henneberg (1850–1922). German mathematician. Professor at the University of Darmstadt.

This patch fails to be regular precisely at the points $(0, n\pi/2)$, where n is an integer. Note that (16.4) is periodic in v with period 2π, and that

$$\mathsf{henneberg}(u,v) = \mathsf{henneberg}(-u, v+\pi). \tag{16.17}$$

Hence for any region $\mathcal{U}$ in the right half-plane $\{\,(p,q) \mid p > 0\,\}$ there will be a region $\widetilde{\mathcal{U}}$ in the left half-plane $\{\,(p,q) \mid p < 0\,\}$ whose image under henneberg is the same subset of $\mathbb{R}^3$. But (16.17) implies that the unit normal $\mathbf{U}$ of henneberg satisfies

$$\mathbf{U}(u,v) = -\mathbf{U}(-u, v+\pi).$$

Hence $\mathsf{henneberg}(\mathcal{U})$ and $\mathsf{henneberg}(\widetilde{\mathcal{U}})$ will have opposite orientations. An instance of this is illustrated by the different shadings on the left and right of Figure 16.5.

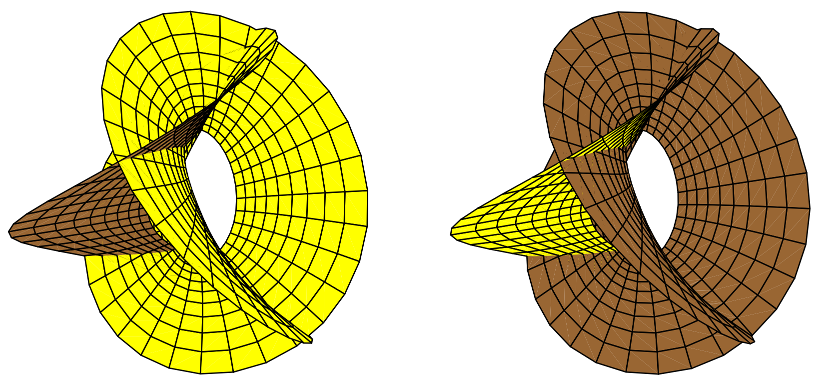

Figure 16.5: The henneberg images of $\{(p,q) \mid \frac{1}{3} \leqslant |p| \leqslant \frac{5}{6},\ -\pi \leqslant q < \pi\,\}$

In Chapter 22, we shall show how to associate to a minimal surface $\mathcal{M}$ a ***conjugate minimal surface*** $\widetilde{\mathcal{M}}$, and an isometric deformation $\mathbf{z}[t]$ from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$ through minimal surfaces, that generalizes equation (16.8). It turns out that the conjugate of Henneberg's surface is given by

$$\begin{aligned}\mathsf{hennebergconj}(u,v) = \Big(&2\cosh u \sin v - \tfrac{2}{3}\cosh 3u \sin 3v,\\ &2\cosh u \cos v + \tfrac{2}{3}\cosh 3u \cos 3v,\ 2\sinh 2u \sin 2v\Big),\end{aligned}$$

with a parametrization that is obtained from henneberg by merely interchanging

$$\sinh u \leftrightarrow \cosh u \qquad \text{and} \qquad \sin v \leftrightarrow \cos v$$

(see Exercise 12 on page 754). The resulting surface is apparently more complicated to visualize (Figure 16.6). Notebook 16 contains a verification that its mean curvature vanishes.

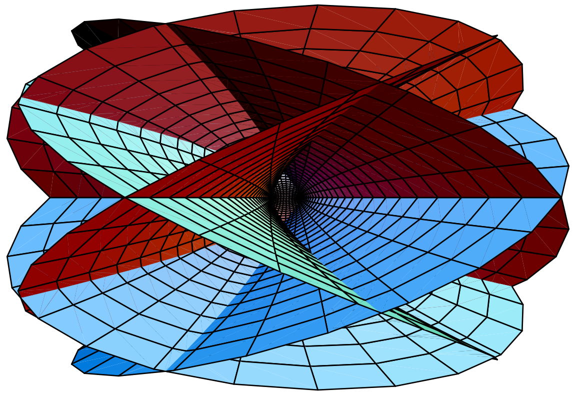

Figure 16.6: A realization of the conjugate Henneberg minimal surface

16.5 Monge Patches and Scherk's Minimal Surface

We have the following immediate consequence of Lemma 13.34, page 409:

Lemma 16.8. *A Monge patch* $(u,v) \mapsto \big(u,v,h(u,v)\big)$ *is a minimal surface if and only if*

$$(1+h_v^2)h_{uu} - 2h_u h_v h_{uv} + (1+h_u^2)h_{vv} = 0. \tag{16.18}$$

One can hope to find interesting examples of minimal surfaces by assuming that h has a special form. In 1835 Scherk[5] determined the minimal surfaces of the form

$$(u,v) \mapsto \big(u,v,f(u)+g(v)\big)$$

(see [Scherk], [BaCo]).

Theorem 16.9. *If a Monge patch* $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ *with* $h(u,v) = f(u)+g(v)$ *is a minimal surface, then either* $\mathcal{M}$ *is part of a plane or there are constants* a, c_1, c_2, c_3, c_4*, with* $a \neq 0$*, such that*

$$\begin{aligned} f(u) &= -\frac{1}{a}\log\big(\cos(au+c_1)\big)+c_2, \\ g(v) &= \frac{1}{a}\log\big(\cos(av+c_3)\big)+c_4. \end{aligned} \tag{16.19}$$

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[5]Heinrich Ferdinand Scherk (1798–1885). German mathematician, who also worked in number theory. He studied with Bessel in Königsberg, and was elected rector of the University of Kiel three times, but was forced to leave in 1848 by the Danes.

Proof. When $h(u,v) = f(u) + g(v)$, we have

$$h_{uu} = f''(u), \qquad h_{uv} = 0, \qquad h_{vv} = g''(v).$$

Hence equation (16.18) reduces to

$$\frac{f''(u)}{1 + f'(u)^2} = \frac{-g''(v)}{1 + g'(v)^2}. \tag{16.20}$$

Here u and v are independent variables, so each side of (16.20) must equal a constant, call it a. If $a = 0$, then both f and g are linear, so that $\mathcal{M}$ is part of a plane. Otherwise, the two equations

$$\frac{f''(u)}{1 + f'(u)^2} = a = \frac{-g''(v)}{1 + g'(v)^2} \tag{16.21}$$

are easily solved by integrating twice. The result is (16.19). ▮

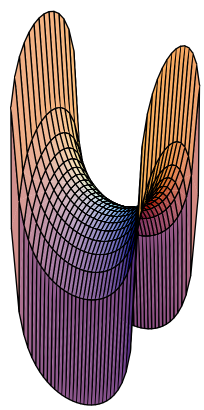

Figure 16.7: A piece of Scherk's minimal surface

As suggested by Theorem 16.9, we define ***Scherk's minimal surface*** by

$$\mathsf{scherk}[a](u,v) = \left(u,\ v,\ \frac{1}{a} \log \left(\frac{\cos a v}{\cos a u} \right) \right).$$

For simplicity, we also take $a = 1$. Figure 16.7 shows the image by this patch of (almost) the open rectangle $-\pi/2 < u, v < \pi/2$. In fact, $\mathsf{scherk}[1][u,v]$ is well-defined on the set

$$\mathcal{R} = \left\{ (u,v) \mid \cos u \cos v > 0 \right\}$$

that can be imagined as the union of the black squares on an infinite chess board. To see this, consider a single square

$$\mathcal{Q}(m,n) = \left\{ (x,y) \,\middle|\, m\pi - \frac{\pi}{2} < x < m\pi + \frac{\pi}{2}, \quad n\pi - \frac{\pi}{2} < y < n\pi + \frac{\pi}{2} \right\}$$

that we color black if $m+n$ is even and white if $m+n$ is odd. Then

$$\mathcal{R} = \bigcup \{\, \mathcal{Q}(m,n) \mid m \text{ and } n \text{ are integers with } m+n \text{ even} \,\}.$$

It is easy to see that

$$\mathsf{scherk}[1](u+2m\pi, v+2n\pi) = \mathsf{scherk}[1](u,v)$$

for all real u and v and all integers m and n. Hence the piece of Scherk's surface over a black square $\mathcal{Q}(m,n)$, $m+n$ even, is a translate of the portion over $\mathcal{Q}(0,0)$. We can now plot identical pieces of the surface over the black squares, and these fit together to produce the entire surface.

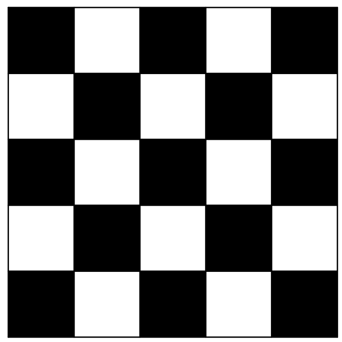

Figure 16.8: Chequered board

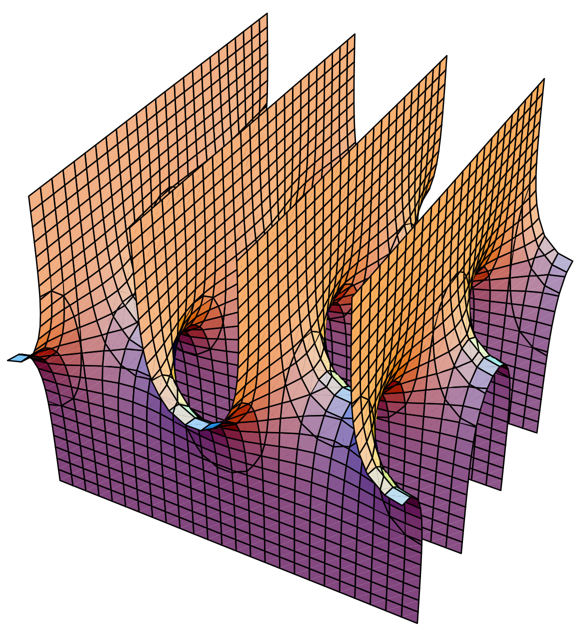

Figure 16.9: Scherk's minimal surface

Actually, Figure 16.9 was obtained in Notebook 16 from an implicit equation of Scherk's surface.

16.6 The Gauss Map of a Minimal Surface

Orientability was defined in Section 11.1 using the notion of a *complex structure* J, and this led to Definition 11.3 of the Gauss map on page 333. We next show that the Gauss map of a *minimal* surface has special properties that leads one to use complex analysis to advantage in the study of minimal surfaces. First, we need the following notion:

Definition 16.10. *Let $\mathcal{M}_1, \mathcal{M}_2$ be oriented regular surfaces in $\mathbb{R}^n$, and let J_1, J_2 be the corresponding complex structures. A map $\Phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is called*

(i) *a* ***complex map*** *if*

$$\Phi_* \circ J_1 = J_2 \circ \Phi_*. \tag{16.22}$$

(ii) *an* ***anticomplex map*** *if*

$$\Phi_* \circ J_1 = -J_2 \circ \Phi_*. \tag{16.23}$$

An elementary but important property of complex and anticomplex maps (also called ***holomorphic*** and ***antiholomorphic maps***) is

Lemma 16.11. *Let $\Phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a complex or anticomplex map for which Φ_* is nowhere zero. Then Φ is a conformal map.*

Proof. Let $\mathbf{p} \in \mathcal{M}_1$. Choose a nonzero tangent vector $\mathbf{v_p}$ to $\mathcal{M}_1$ at $\mathbf{p}$. We can assume that $\|\Phi_*(\mathbf{v_p})\| = \lambda\|\mathbf{v_p}\|$ for some $\lambda > 0$. Since Φ is complex or anticomplex, abbreviating $(J_1)_\mathbf{p}$ to $J_\mathbf{p}$, we have

$$\|\Phi_*(J_\mathbf{p}\mathbf{v_p})\| = \lambda\|\mathbf{v_p}\| \qquad \text{and} \quad \Phi_*(\mathbf{v_p}) \cdot \Phi_*(J_\mathbf{p}\mathbf{v_p}) = 0.$$

It now follows that

$$\|\Phi_*(a\mathbf{v_p} + bJ_\mathbf{p}\mathbf{v_p})\|^2 = \lambda^2\|a\mathbf{v_p} + bJ_\mathbf{p}\mathbf{v_p}\|^2$$

for all $a, b \in \mathbb{R}$. Since $\{\mathbf{v_p}, J_\mathbf{p}\mathbf{v_p}\}$ is a basis of $\mathcal{M}_\mathbf{p}$, we can set $\lambda = \lambda(\mathbf{p})$ in Definition 12.11 on page 370 to deduce that Φ is conformal. ▮

We have a ready-made example of an anticomplex map.

Theorem 16.12. *The Gauss map of an oriented minimal surface $\mathcal{M} \subset \mathbb{R}^3$ is anticomplex, and*

$$J_\mathbf{p}S_\mathbf{p} = -S_\mathbf{p}J_\mathbf{p} \tag{16.24}$$

for all $\mathbf{p} \in \mathcal{M}$, where $S_\mathbf{p}$ is the shape operator of $\mathcal{M}$ at $\mathbf{p}$.

Proof. Let $\mathbf{p} \mapsto \mathbf{U}(\mathbf{p})$ be the Gauss map of $\mathcal{M}$. Let $\mathbf{p} \in \mathcal{M}$, and let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be an orthonormal basis of $\mathcal{M}_{\mathbf{p}}$ which diagonalizes $S_{\mathbf{p}}$, and let k_1, k_2 be the corresponding principal curvatures. The unit vector $\mathbf{U}_{\mathbf{p}}$ determines a complex structure $J_{\mathbf{p}}$ on $\mathcal{M}_{\mathbf{p}}$, via (11.1) on page 332, that satisfies $J_{\mathbf{p}}\mathbf{e}_1 = \pm\mathbf{e}_2$ and $J_{\mathbf{p}}\mathbf{e}_2 = \mp\mathbf{e}_1$. We use the fact that $\mathcal{M}$ is a minimal surface to compute

$$J_{\mathbf{p}}S_{\mathbf{p}}\mathbf{e}_1 = J_{\mathbf{p}}k_1\mathbf{e}_1 = \pm k_1\mathbf{e}_2 = \mp k_2\mathbf{e}_2 = \mp S_{\mathbf{p}}\mathbf{e}_2 = -S_{\mathbf{p}}J_{\mathbf{p}}\mathbf{e}_1.$$

Similarly, $J_{\mathbf{p}}S_{\mathbf{p}}\mathbf{e}_2 = -S_{\mathbf{p}}J_{\mathbf{p}}\mathbf{e}_2$. In this way, we have established (16.24). But Lemma 13.5, on page 388, tells us that $S_{\mathbf{p}}$ is the negative of the tangent map of $\mathbf{U}$ at $\mathbf{p}$. Hence (16.24) implies that the Gauss map is anticomplex. ▌

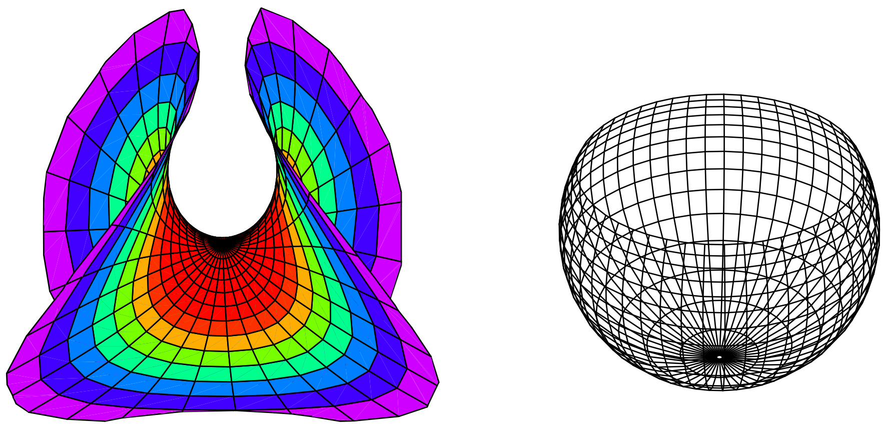

Figure 16.10: Enneper's surface and its Gauss image

The shape operator $S_{\mathbf{p}}$ at a point of $\mathcal{M}$ vanishes if and only if $k_1 = 0 = k_2$, so that (by definition) $\mathbf{p}$ is a planar point. On a minimal surface this is equivalent to asserting that the Gaussian curvature $K(\mathbf{p})$ is zero. Strictly speaking, we need to exclude such points when applying Lemma 16.11 to the conclusion of Theorem 16.12. We then obtain

Corollary 16.13. *The Gauss map of a minimal surface without planar points is conformal.*

Conformality means that the Gauss map preserves the proportions of infinitesimally small rectangles. When we plot a surface in the usual way with a sufficiently fine mesh, it is divided into lots of small, approximately rectangular shapes. The ratios of the lengths of the sides of these rectangles, and also the angles between the sides must be approximately preserved by the Gauss map. This phenomenon can be examined in Figure 16.10 which plots Enneper's surface along with its image under the Gauss map.

16.7 Isothermal Coordinates

In this final section, we shall define an important type of patch that can be found on an arbitrary surface. It has a special significance for minimal surfaces, and will play an important role in the subsequent study in Chapter 22 in which we shall make more explicit the role of complex structures.

Definition 16.14. *Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an open subset. A patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ is called* ***isothermal*** *if there exists a differentiable function $\lambda\colon \mathcal{U} \to \mathbb{R}$ such that*

$$\mathbf{x}_u \cdot \mathbf{x}_u = \mathbf{x}_v \cdot \mathbf{x}_v = \lambda^2 \qquad \textit{and} \qquad \mathbf{x}_u \cdot \mathbf{x}_v = 0. \tag{16.25}$$

We call λ the ***scaling function*** *of the isothermal patch.*

Note that such a patch is regular at all points where λ is nonzero.

The intuitive meaning of an isothermal patch $\mathbf{x}$ can be described as follows. Since $\mathbf{x}_u$ and $\mathbf{x}_v$ have the same length and are orthogonal, an isothermal patch maps an infinitesimal square in $\mathcal{U}$ into an infinitesimal square on its image. A more general patch would transform an infinitesimal square into an infinitesimal quadrilateral. The name 'isothermal' is due to Lamé[6], and dates from 1833.

The following lemma is obvious from the definition of conformal map on page 370.

Lemma 16.15. *A patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ is isothermal if and only if it is conformal, considered as a mapping $\mathbf{x}\colon \mathcal{U} \to \mathbf{x}(\mathcal{U})$.*

Let us first deal with the case of the sphere. Its standard parametrization, given on page 288, is

$$\mathbf{x}(u,v) = a\big(\cos u \cos v,\ \sin u \cos v,\ \sin v\big). \tag{16.26}$$

The metric of this patch was computed to be

$$ds^2 = a^2\big(\cos^2 v\, du^2 + dv^2\big) \tag{16.27}$$

(this is equation (12.26) on page 374), so the patch is certainly *not* isothermal. As a hint of what is in store in Chapter 22, we use complex numbers to factor the right-hand side of (16.27) so as to obtain

$$ds^2 = a^2 \cos^2 v \left(du + \frac{i\,dv}{\cos v}\right)\left(du - \frac{i\,dv}{\cos v}\right). \tag{16.28}$$

6 
Gabriel Lamé (1795–1870). French engineer, mathematician and physicist. He worked on a wide variety of topics, such as number theory, differential geometry (we shall mention his work on triply orthogonal systems in Chapter 19), elasticity (where two elastic constants are named after him) and difusion in crystalline material.

This has the effect of partially separating the variables, and if we set $w = \log(\tan v + \sec v)$, then

$$dw = \frac{dv}{\cos v}.$$

Moreover,

$$e^w + e^{-w} = \frac{\sin v + 1}{\cos v} + \frac{\cos v}{\sin v + 1} = \frac{2}{\cos v},$$

so that $\cos v = \operatorname{sech} w$. Thus, (16.28) becomes

$$ds^2 = \lambda^2(du^2 + dw^2)$$

with $\lambda = a \operatorname{sech} w$. In summary,

Lemma 16.16. *The patch*

$$\mathbf{y}(u, w) = a\big(\cos u \operatorname{sech} w,\ \sin u \operatorname{sech} w,\ \tanh w\big) \tag{16.29}$$

is an isothermal patch on a sphere $S^2(a)$ of radius a.

This patch is a special case of Mercator's parametrization of the ellipsoid given in Exercise 6 of Chapter 10. But a computation in Notebook 16 confirms that it is only isothermal in the spherical case.

It follows from Corollary 16.13 and Lemma 16.16 that the composition

$$\mathbf{p} \mapsto \mathbf{U}(\mathbf{p}) \mapsto \mathbf{y}^{-1}(\mathbf{p}),$$

defined on a suitable open subset of $\mathcal{M}$, is conformal. Its inverse will define an isothermal patch on the minimal surface $\mathcal{M}$. The following well-known result actually guarantees the existence of isothermal patches on an arbitrary surface.

Theorem 16.17. *Let $\mathcal{M}$ be a surface, and suppose ds^2 is a metric on $\mathcal{M}$. Let $\mathbf{p} \in \mathcal{M}$. Then there exists an open set $\mathcal{U} \subset \mathbb{R}^2$ and an isothermal patch $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ such that $\mathbf{p} \in \mathbf{x}(\mathcal{U})$ and*

$$ds^2 = \lambda^2(du^2 + dv^2).$$

For a proof, see [Bers, page 15], [Os1, page 31] or [FoTu, page 54]. It is also possible to argue in the way that led to Lemma 16.16. For we can factor an arbitrary metric $ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$ as

$$ds^2 = \left(\sqrt{E}\,dp + \frac{F + i\sqrt{EG - F^2}}{\sqrt{E}}dq\right)\left(\sqrt{E}\,dp + \frac{F - i\sqrt{EG - F^2}}{\sqrt{E}}dq\right).$$

One can then establish the existence of a complex-valued integrating factor μ for which the differential system

$$\mu(du + i\,dv) = \sqrt{E}\,dp + \frac{F + i\sqrt{EG - F^2}}{\sqrt{E}}dq$$

can be resolved for real-valued functions u, v; see [AhSa, pages 125–126] for details. Setting $\lambda = |\mu|$ gives

$$ds^2 = |\mu|^2(du + i\,dv)(du - i\,dv) = \lambda^2(du^2 + dv^2),$$

and the coordinates u, v provide an isothermal patch.

We finish with a result of special significance for minimal surfaces.

Lemma 16.18. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ be a regular isothermal patch with scaling function λ and mean curvature H. Then we have*

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 2\lambda^2 H\,\mathbf{U}, \tag{16.30}$$

where $\mathbf{U} = (\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|$ is the unit normal.

Proof. Since $\mathbf{x}$ is isothermal, we can differentiate equations (16.25), obtaining

$$\mathbf{x}_{uu} \cdot \mathbf{x}_u = \mathbf{x}_{uv} \cdot \mathbf{x}_v \qquad \text{and} \qquad \mathbf{x}_{vv} \cdot \mathbf{x}_u = -\mathbf{x}_{vu} \cdot \mathbf{x}_v.$$

Therefore,

$$(\mathbf{x}_{uu} + \mathbf{x}_{vv}) \cdot \mathbf{x}_u = \mathbf{x}_{uv} \cdot \mathbf{x}_v - \mathbf{x}_{vu} \cdot \mathbf{x}_v = 0.$$

Similarly, $(\mathbf{x}_{uu} + \mathbf{x}_{vv}) \cdot \mathbf{x}_v = 0$. It follows that $\mathbf{x}_{uu} + \mathbf{x}_{vv}$ is normal to $\mathcal{M}$, and so $\mathbf{x}_{uu} + \mathbf{x}_{vv}$ is a multiple of $\mathbf{U}$. To find out which multiple, we use Theorem 13.25, page 400, and the assumption that $\mathbf{x}$ is isothermal to compute

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{e + g}{2\lambda^2} = \frac{(\mathbf{x}_{uu} + \mathbf{x}_{vv}) \cdot \mathbf{U}}{2\lambda^2},$$

so that we get (16.30). ∎

We denote by

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

the ***Laplacian*** of $\mathbb{R}^2$, as defined in Section 9.5. Extending this to vector-valued functions, we may write

$$\Delta \mathbf{x} = \mathbf{x}_{uu} + \mathbf{x}_{vv},$$

and conclude the chapter with

Corollary 16.19. *A* minimal *isothermal patch satisfies the Laplace equation*

$$\Delta \mathbf{x} = 0.$$

.

16.8 Exercises

1. Prove that the unit normal of the patch $\mathbf{z}[t]$ defined at the top of page 505 does not depend upon the variable t. Show that the coefficients $e(t)$, $f(t)$ and $g(t)$ of the second fundamental form of $\mathbf{z}[t]$ satisfy

$$e(t)(u,v) = -g(t)(u,v) = -\sin t, \qquad f(t)(u,v) = \cos t.$$

More general results can be found in Section 22.1.

M **2.** Check that the surfaces of Henneberg and Catalan are indeed minimal surfaces and compute the Gaussian curvature of each.

3. Prove that the acceleration $\boldsymbol{\alpha}''(s)$ of the curve (16.15) is parallel to the vector $(\cos\frac{u}{2}, \sin\frac{u}{2})$, where the parameters u, s are related by (16.16).

4. Verify that the image of the patch (16.29) is contained in the sphere $S^2(a)$. Compute $\mathbf{x}_u$ and $\mathbf{x}_v$ and prove directly that the patch is isothermal.

M **5.** ***Scherk's fifth minimal surface*** is defined implicitly by

$$\sin z = \sinh x \sinh y. \tag{16.31}$$

Show that it indeed is a minimal surface.

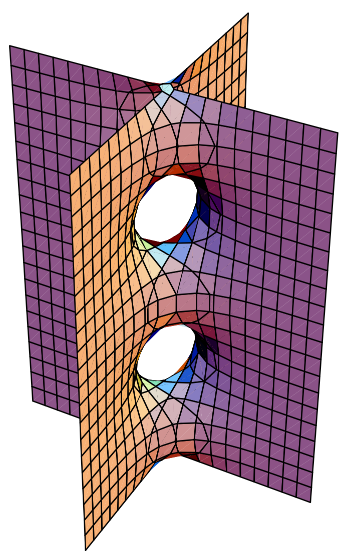

Figure 16.11: Scherk's fifth minimal surface plotted implicitly

6. Show that a surface of revolution is isothermal if and only if $\psi'^2 = \varphi^2 - \varphi'^2$, with notation from page 462. Hence the surface catenoid$[a]$ is isothermal if and only if $a = \pm 1$, even though it is a minimal surface for all a.

M **7.** Portions of Scherk's fifth minimal surface are parametrized by

$$\text{scherk5}[a,b,c](u,v) = \Big(a \operatorname{arcsinh} u,\ b \operatorname{arcsinh} v,\ c \arcsin(uv)\Big),$$

where a, b, c are each equal to ± 1. Show that scherk5$[a,b,c]$ has zero mean curvature H, and compute its Gaussian curvature K.

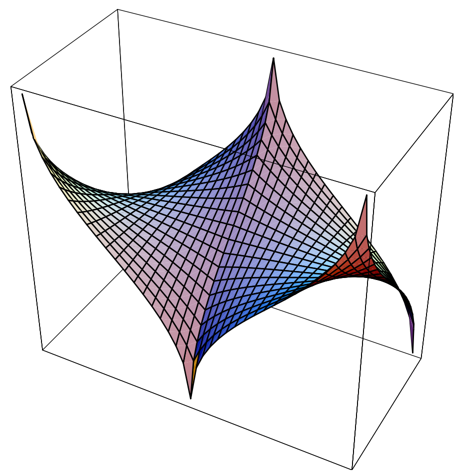

Figure 16.12: Part of Scherk's fifth minimal surface

8. Let $\mathbf{x}$ be an isothermal patch. Show that the Weingarten equations (13.10), page 394, simplify to

$$(16.32) \qquad \begin{cases} S(\mathbf{x}_u) &= \dfrac{e}{\lambda^2}\mathbf{x}_u + \dfrac{f}{\lambda^2}\mathbf{x}_v, \\[2ex] S(\mathbf{x}_v) &= \dfrac{f}{\lambda^2}\mathbf{x}_u + \dfrac{g}{\lambda^2}\mathbf{x}_v. \end{cases}$$

M **9.** A twisted generalization of Scherk's minimal surface is given by

$$\begin{aligned} \text{scherk}[a,\theta](u,v) \;=\; \Bigg(&\sec 2\theta(u\cos\theta + v\sin\theta),\ \sec 2\theta(u\sin\theta + v\cos\theta), \\ &\frac{1}{a}\log\left(\frac{\cos\big(\sec 2\theta(u\sin\theta + v\cos\theta)\big)}{\cos\big(\sec 2\theta(u\cos\theta + v\sin\theta)\big)}\right)\Bigg). \end{aligned}$$

Show that scherk$[a,\theta]$ *is* minimal, and compute its Gaussian curvature.