Ch4. Random Vector and Multivariate Normal Distribution

4.1 Random vector and matrix

- **Expectation**: Let $\boldsymbol{Y}$ and $\boldsymbol{X}$ be $p \times 1$ random vectors. The expected value of

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{pmatrix} \text{ is given by } \mathrm{E}(\boldsymbol{Y}) = \begin{pmatrix} \mathrm{E}(Y_1) \\ \mathrm{E}(Y_2) \\ \vdots \\ \mathrm{E}(Y_p) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix} = \boldsymbol{\mu}$$

- $\mathrm{E}(a\boldsymbol{X} + b\boldsymbol{Y}) = a\mathrm{E}(\boldsymbol{X}) + b\mathrm{E}(\boldsymbol{Y})$.

4.1 Random vector and matrix

- **Covariance Matrix**: $\boldsymbol{\Sigma} = \mathrm{Cov}(\boldsymbol{Y})$ is defined by

$$\mathrm{E}\{[\boldsymbol{Y} - \mathrm{E}(\boldsymbol{Y})][\boldsymbol{Y} - \mathrm{E}(\boldsymbol{Y})]'\} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{pmatrix}$$

  - Let $\boldsymbol{A}$ be a constant matrix, then

$$\mathrm{Cov}(\boldsymbol{AY}) = \boldsymbol{A}[\mathrm{Cov}\,\boldsymbol{Y}]\boldsymbol{A}'$$

  - Let $\boldsymbol{A}, \boldsymbol{B}$ be constant matrices, then

$$\mathrm{Cov}(\boldsymbol{AX}, \boldsymbol{BY}) = \boldsymbol{A}\mathrm{Cov}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{B}'$$

4.1 Random vector and matrix

- **Generalized variance**: overall measure of variability

$$\text{Generalized variance} = |\boldsymbol{\Sigma}|.$$

4.1 Random vector and matrix

Correlation matrix:

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{pmatrix}$$

where

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}}$$

for $i \neq j$.

4.1 Random vector and matrix

- Partitioned random vectors

$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{pmatrix}$$

$$\boldsymbol{\mu} = E(\boldsymbol{V}) = E\begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} E(\boldsymbol{Y}) \\ E(\boldsymbol{X}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{\boldsymbol{Y}} \\ \boldsymbol{\mu}_{\boldsymbol{X}} \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \mathrm{Cov}(\boldsymbol{V}) = \mathrm{Cov}\begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{Y}\boldsymbol{Y}} & \boldsymbol{\Sigma}_{\boldsymbol{Y}\boldsymbol{X}} \\ \boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{Y}} & \boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{X}} \end{pmatrix}$$

Q: how to find the inverse of $\boldsymbol{\Sigma}$?

4.1 Random vector and matrix

- Let $\boldsymbol{Y}$ be a random vector with mean $\boldsymbol{\mu} = \mathrm{E}(\boldsymbol{Y})$ and $\boldsymbol{\Sigma} = \mathrm{Cov}(\boldsymbol{Y})$, then

$$\mathrm{E}(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}) = \mathrm{tr}(\boldsymbol{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$$

where $\boldsymbol{A}$ is a symmetric matrix.

4.1 Random vector and matrix

- **MGF: The moment generating function** of a random vector $\boldsymbol{Y}$ is given by

$$M_{\boldsymbol{Y}}(\boldsymbol{t}) = E(e^{\boldsymbol{t}'\boldsymbol{Y}})$$

  where $\boldsymbol{t}' = (t_1, t_2, \cdots, t_p)$, if the expectation exists for $-h < t_j < h$ where $h > 0$ and $j = 1, \cdots, p$.
- **Theorem**. Let $g_1(\boldsymbol{Y}_1), \cdots, g_m(\boldsymbol{Y}_m)$ be m functions of the random vectors $\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_m$, respectively. If $\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_m$ are mutually independent, then $g_1(\boldsymbol{Y}_1), \cdots, g_m(\boldsymbol{Y}_m)$ are mutually independent.

4.2. Multivariate Normal Distribution

Let $\boldsymbol{Y}_{p\times 1} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

- Density

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = |\boldsymbol{\Sigma}|^{-\frac{1}{2}}(2\pi)^{-\frac{p}{2}} e^{-\frac{1}{2}\{(\boldsymbol{y}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\}}$$

- MGF: $M_{\boldsymbol{Y}}(\mathbf{t}) = \mathbf{e}^{\boldsymbol{t}'\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}}$

4.2. Multivariate Normal Distribution

- Let $\boldsymbol{B}$ be a constant matrix and $\boldsymbol{C}$ be a constant vector

$$\boldsymbol{B}\boldsymbol{Y} + \boldsymbol{C} \sim N(\boldsymbol{B}\boldsymbol{\mu} + \boldsymbol{C}, \boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{B}')$$

4.2. Multivariate Normal Distribution

- Marginal Distribution, Condition Distribution and independence: Let

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix} \sim N\left[\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}\right]$$

  then

  (i) $\boldsymbol{Y}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$

  (ii) $\boldsymbol{Y}_1 | \boldsymbol{Y}_2 = \boldsymbol{y}_2 \sim N(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{y}_2 - \boldsymbol{\mu}_2),$
  $\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})$

  (iii) $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are independent iff $\boldsymbol{\Sigma}_{12} = \mathbf{0}$

4.2. Multivariate Normal Distribution

- Partial Correlation: Let $\boldsymbol{v} \sim N_q(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{x} \end{pmatrix}; \ \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_y \\ \boldsymbol{\mu}_x \end{pmatrix}; \ \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{yy} & \boldsymbol{\Sigma}_{yx} \\ \boldsymbol{\Sigma}_{xy} & \boldsymbol{\Sigma}_{xx} \end{pmatrix}$$

where $\boldsymbol{y} = (y_1, y_2, ..., y_{r-1})'$ and $\boldsymbol{x} = (x_r, ..., x_q)'$. Let $\rho_{ij.r...q}$ be the partial correlation between y_i and y_j, $1 \leq i < j \leq r-1$, in the conditional distribution of $\boldsymbol{y}$ given $\boldsymbol{x}$. By the definition of correlation, we have

$$\rho_{ij.r...q} = \frac{\sigma_{ij.r...q}}{\sqrt{\sigma_{ii.r...q}\sigma_{jj.r...q}}}.$$

- Matrix of partial correlations

$$\boldsymbol{\Omega}_{y.x} = \boldsymbol{D}_{y.x}^{-1}\boldsymbol{\Sigma}_{y.x}\boldsymbol{D}_{y.x}^{-1}$$

where $\boldsymbol{\Sigma}_{y.x} = \boldsymbol{\Sigma}_{yy} - \boldsymbol{\Sigma}_{yx}\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xy}$ and $\boldsymbol{D}_{y.x} = [diag(\boldsymbol{\Sigma}_{y.x})]^{1/2}$.

4.2. Multivariate Normal Distribution

Example 4.1

$$\boldsymbol{v} = \begin{pmatrix} y \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{pmatrix} \right)$$

To find the conditional distribution of y given $\boldsymbol{x} = (x_1, x_2, x_3)'$.

4.2. Multivariate Normal Distribution

Example 4.2 For $\boldsymbol{v}$ defined in Example 4.1, to find the partial correlation between y and x_1 given (x_2, x_3).