Ch3. Matrix algebra

3.1 Vector and matrix

- A brief review of vector and matrix

Remark 3.1. $\underset{\sim}{x}_{n\times 1} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\underset{\sim}{A}_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$

$rank(\underset{\sim}{A})$

* square matrix $\underset{\sim}{A}_{n\times n}$, symmetric $\underset{\sim}{A} = \underset{\sim}{A}'$

Ex. covariance matrix : $\underset{\sim}{A} = cov(\underset{\sim}{x}) = \begin{pmatrix} cov(x_1,x_1) & cov(x_1,x_2) & \cdots & cov(x_1,x_n) \\ \cdots & & & \\ cov(x_n,x_1) & \cdots & & cov(x_n,x_n) \end{pmatrix}$

$cov(x_1, x_1) = Var(x_1)$, $\underset{\sim}{A} \geq 0$, $\underset{\sim}{A}$ ← symmetric

* If $Rank(\underset{\sim}{A}) = n$ (full-rank)

$\Rightarrow$ unique $\underset{\sim}{A}^{-1}$: $\underset{\sim}{A}\underset{\sim}{A}^{-1} = \underset{\sim}{A}^{-1}\underset{\sim}{A} = \underset{\sim}{I}_n$

- $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{c} \Rightarrow \underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{c}$

- $(\underset{\sim}{A}\underset{\sim}{B})^{-1} = \underset{\sim}{B}^{-1}\underset{\sim}{A}^{-1}$

* Partion of a matrix $\underset{\sim}{A} = \begin{pmatrix} \underset{\sim}{A}_{11} & \underset{\sim}{A}_{12} \\ \underset{\sim}{A}_{21} & \underset{\sim}{A}_{22} \end{pmatrix}$

$$\underset{\sim}{A}^{-1} = \begin{pmatrix} \underset{\sim}{A}_{11}^{-1} + \underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}\underset{\sim}{B}^{-1}\underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1} & -\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}\underset{\sim}{B}^{-1} \\ -\underset{\sim}{B}^{-1}\underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1} & \underset{\sim}{B}^{-1} \end{pmatrix},$$

$\underset{\sim}{B} = \underset{\sim}{A}_{22} - \underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}$

$$= \begin{pmatrix} \underset{\sim}{D}^{-1} & -\underset{\sim}{D}^{-1}\underset{\sim}{A}_{12}\underset{\sim}{A}_{22}^{-1} \\ -\underset{\sim}{A}_{22}^{-1}\underset{\sim}{A}_{21}\underset{\sim}{D}^{-1} & \underset{\sim}{A}_{22}^{-1} + \underset{\sim}{A}_{22}^{-1}\underset{\sim}{A}_{21}\underset{\sim}{D}^{-1}\underset{\sim}{A}_{12}\underset{\sim}{A}_{22}^{-1} \end{pmatrix}$$

$$\underset{\sim}{D} = \underset{\sim}{A}_{11} - \underset{\sim}{A}_{12}\underset{\sim}{A}_{22}^{-1}\underset{\sim}{A}_{21}$$

special cases

$$- \quad \underset{\sim}{A} = \begin{pmatrix} \underset{\sim}{A}_{11} & \underset{\sim}{a}_{12} \\ \underset{\sim}{a}_{12}' & a_{22} \end{pmatrix}$$

a_{22}: 1×1

$\underset{\sim}{A}_{11}$ – symmetric

$$\underset{\sim}{A}^{-1} = \frac{1}{b}\begin{pmatrix} b\underset{\sim}{A}_{11}^{-1} + \underset{\sim}{A}_{11}^{-1}\underset{\sim}{a}_{12}\underset{\sim}{a}_{12}'\underset{\sim}{A}_{11}^{-1} & -\underset{\sim}{A}_{11}^{-1}\underset{\sim}{a}_{12} \\ -\underset{\sim}{a}_{12}'\underset{\sim}{A}_{11}^{-1} & 1 \end{pmatrix}$$

$$b = a_{22} - \underset{\sim}{a}_{12}'\underset{\sim}{A}_{11}^{-1}\underset{\sim}{a}_{12}$$

$$- \quad B^{-1} = \left(\underset{\sim}{A}_{22} - \underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12}\right)^{-1}$$

$$= \underset{\sim}{A}_{22}^{-1} + \underset{\sim}{A}_{22}^{-1}\underset{\sim}{A}_{21}\left(\underline{\underset{\sim}{A}_{11} - \underset{\sim}{A}_{12}\underset{\sim}{A}_{22}^{-1}\underset{\sim}{A}_{21}}\right)^{-1}\underset{\sim}{A}_{12}\underset{\sim}{A}_{22}^{-1}$$

Ex. $\left(\underset{n\times n}{\underset{\sim}{A}} + \underset{n\times 1}{\underset{\sim}{c}}\,\underset{1\times n}{\underset{\sim}{c}'}\right)^{-1} = \underset{\sim}{A}^{-1} - \dfrac{\underset{\sim}{A}^{-1}\underset{\sim}{c}\,\underset{\sim}{c}'\,\underset{\sim}{A}^{-1}}{\left(1 + \underset{\sim}{c}'\underset{\sim}{A}^{-1}\underset{\sim}{c}\right)}$

$$\left(\begin{array}{c|c} -1 & \underset{\sim}{c}' \\ \hline \underset{\sim}{c} & \underset{\sim}{A}_{n\times n} \end{array}\right)$$

$\underset{\sim}{A}_{22} = \underset{\sim}{A}$, $\underset{\sim}{A}_{21} = \underset{\sim}{c}$

$\underset{\sim}{A}_{12} = \underset{\sim}{c}'$, $\underset{\sim}{A}_{11} = -1$

3.1 Vector and matrix – Full Rank Factorization

Theorem : $A_{p\times q}$ of rank r can always be factorized as

$$A = K_{p\times r} L_{r\times q}$$

$p\times q$

where K and L have full column and full row rank respectively.

Proof: There exist nonsingular matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad A = P^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}.$$

Partition P^{-1} and Q^{-1} as

$$P^{-1} = [K_{p\times r} \quad W_{p\times(p-r)}], \quad Q^{-1} = \begin{bmatrix} L_{r\times q} \\ Z_{(q-r)\times q} \end{bmatrix}.$$

3.1 Vector and matrix – Full Rank Factorization

Then,

$$\begin{aligned} \boldsymbol{A} &= [\boldsymbol{K} \quad \boldsymbol{W}] \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{L} \\ \boldsymbol{Z} \end{bmatrix} \\ &= [\boldsymbol{K} \quad \boldsymbol{0}] \begin{bmatrix} \boldsymbol{L} \\ \boldsymbol{Z} \end{bmatrix} \\ &= \boldsymbol{K}\boldsymbol{L}. \end{aligned}$$

Note: the concept of left inverse and the right inverse .

$\underset{\sim p\times p}{P^{-1}} \triangleq [\underset{p\times r}{\underset{\sim}{K}} \quad \underset{p\times(p-r)}{\underset{\sim}{W}}]$ $\quad$ $\mathrm{rank}(\underset{\sim}{K}) = r$

$$\underset{\sim}{P}_{p\times p} \triangleq \begin{pmatrix} U_{r\times p} \\ V_{(p-r)*p} \end{pmatrix} \Rightarrow \underset{\sim}{P}\,\underset{\sim}{P}^{-1} = \begin{pmatrix} \underset{\sim}{U}\underset{\sim}{K} & \underset{\sim}{U}\underset{\sim}{W} \\ \underset{\sim}{V}\underset{\sim}{K} & \underset{\sim}{V}\underset{\sim}{W} \end{pmatrix} = I_{p\times p}$$

$$\Rightarrow \underset{r*p}{U}\ \underset{p\times r}{\underset{\sim}{K}} = \underset{\sim}{I}_r \qquad (p \geq r)$$

we define U_{r*p} is a left Inverse of $\underset{\sim}{K}_{p\times r}$

similarly, $\underset{\sim}{R}$ is a right Inverse of L if

$$\underset{r*q}{L}\ \underset{q*r}{R} = \underset{\sim}{I}_r.$$

$$(q \geq r)$$

3.1 Vector and matrix – Idempotent Matrices

$$A^2 = A$$

- All idempotent matrices (except I) are singular.
  Proof: Since $A^2 = A$ and if A is nonsingular,
  $$A = A^{-1}AA = A^{-1}A = I.$$
- $r(A) = \mathrm{tr}(A)$.
  Proof: Consider the full rank factorization, let
  $$A = BC \quad \text{and} \quad A^2 = BCBC = BC.$$
  But B has a left inverse U and C has a right inverse R, then
  $$\begin{aligned} UBCBCR &= UBCR \\ \Rightarrow \qquad CB &= I_{r\times r} \end{aligned}$$
  So, $\mathrm{tr}(A) = \mathrm{tr}(BC) = \mathrm{tr}(CB) = \mathrm{tr}(I_{r\times r}) = r = r(A)$.

3.1 Vector and matrix – Idempotent Matrices

- Eigenvalues of idempotent matrices are either 0 or 1.

Proof: Let $\lambda, \boldsymbol{x}$ be a pair of eigenvalue and eigenvector.

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$$

$$\Rightarrow \boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}(\lambda\boldsymbol{x}) = \lambda\boldsymbol{A}\boldsymbol{x} = \lambda^2\boldsymbol{x}.$$

But also

$$\boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x},$$

thus,

$$\lambda^2\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow \lambda(\lambda - 1)\boldsymbol{x} = 0.$$

Since $\boldsymbol{x} \neq 0 \;\Rightarrow\; \lambda = 0 \;\; or \;\; 1.$

3.1 Vector and matrix – Idempotent Matrices

- For symmetric matrix $\boldsymbol{A}$, if all eigenvalues are 1 or 0, $\boldsymbol{A}$ is idempotent

Proof: For $\boldsymbol{A}$ symmetric, there exists an orthogonal matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}'\boldsymbol{A}\boldsymbol{P} \;=\; \boldsymbol{D},$$

where $\boldsymbol{D}$ is a diagonal matrix with eigenvalues of $\boldsymbol{A}$ on the diagonal. So, $\boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}\boldsymbol{P}'\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}^2$, But, $\boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}\boldsymbol{P}'\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}'\boldsymbol{A}\boldsymbol{A}\boldsymbol{P}$. However if all eigenvalues are 1 or 0,

$$\begin{aligned}
\Rightarrow \qquad & \boldsymbol{D} = \boldsymbol{D}^2 \\
\Rightarrow \qquad & \boldsymbol{P}'\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}'\boldsymbol{A}\boldsymbol{A}\boldsymbol{P} \\
\Rightarrow \qquad & \boldsymbol{A} = \boldsymbol{A}\boldsymbol{A} \\
\Rightarrow \qquad & \boldsymbol{A} \text{ is idempotent}
\end{aligned}$$

3.2 Generalized Inverse

$n \times m$

Definition: Let $\boldsymbol{A}$ be $m \times n$ and the generalized inverse $\boldsymbol{A}^-$ satisfies

$$\underset{m\times n}{\boldsymbol{A}}\boldsymbol{A}^-\underset{m\times n}{\boldsymbol{A}} = \underset{m\times n}{\boldsymbol{A}}$$

- g-inverse may not be unique.

Example 3.1

$$\boldsymbol{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}_{4\times 1}$$

$A_1^- = (1, 0, 0, 0)$ is $\underset{\sim}{A}$'s g-inverse

$$\underset{\sim}{A}\underset{\sim}{A_1^-}\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \underset{\sim}{A}$$

$$\left.\begin{aligned} A_2^- &= \left(0,\ \tfrac{1}{2},\ 0,\ 0\right) \\ A_3^- &= \left(0,\ 0,\ \tfrac{1}{3},\ 0\right) \\ A_4^- &= \left(0,\ 0,\ 0,\ \tfrac{1}{4}\right) \end{aligned}\right\} \text{are also g-inverse of } \underset{\sim}{A}$$

3.2 Generalized Inverse

Theorem. Suppose $\boldsymbol{A}$ is $n \times p$ of rank r and $\boldsymbol{A}$ is partitioned by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix}$$

where $\boldsymbol{A}_{11}$ is $r \times r$ of rank r (full rank). Then a generalized inverse of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{-} = \begin{pmatrix} \boldsymbol{A}_{11}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}.$$

Proof.

Remark 3.2

Corollary. Suppose $\boldsymbol{A}$ is $n \times p$ of rank r, and $\boldsymbol{A}_{22}$ is $r \times r$ of rank r (full rank). Then a generalized inverse of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{-} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_{22}^{-1} \end{pmatrix}.$$

Remark 3.2.

$$\underset{\sim}{A}\,\underset{\sim}{A}^{-}\,\underset{\sim}{A} = \begin{pmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1} & \underset{\sim}{0} \end{pmatrix} \begin{pmatrix} \underset{\sim}{A}_{11} & \underset{\sim}{A}_{12} \\ \underset{\sim}{A}_{21} & \underset{\sim}{A}_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \underset{\sim}{A}_{11} & \underset{\sim}{A}_{12} \\ \underset{\sim}{A}_{21} & \underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12} \end{pmatrix}$$

Need to prove $\underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12} = \underset{\sim}{A}_{22}$

Let $\underset{\sim}{B} = \begin{pmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ -\underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1} & \underset{\sim}{I} \end{pmatrix}$ — non-singular

$$\underset{\sim}{B}\underset{\sim}{A} = \begin{pmatrix} \underset{r\times r}{\underset{\sim}{A}_{11}} & \underset{\sim}{A}_{12} \\ 0 & \underset{\sim}{A}_{22} - \underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12} \end{pmatrix}$$

rank $(\underset{\sim}{B}\underset{\sim}{A})$

$=$ rank(A)

$= r$

it is a linear combination of the first r columns.

$$\Rightarrow \begin{pmatrix} \underset{\sim}{A}_{12} \\ \underset{\sim}{A}_{22} - \underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{A}_{11} \\ 0 \end{pmatrix} \cdot \underset{\sim}{Q} = \begin{pmatrix} \underset{\sim}{A}_{11}Q \\ 0 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{A}_{22} - \underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12} = \underset{\sim}{0}$$

$$\Rightarrow \underset{\sim}{A}_{21}\underset{\sim}{A}_{11}^{-1}\underset{\sim}{A}_{12} = \underset{\sim}{A}_{22}$$

$$\Rightarrow \underset{\sim}{A}\underset{\sim}{A}^{-}\underset{\sim}{A} = \underset{\sim}{A}$$ thus $\underset{\sim}{A}^{-} = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$

is a g-inverse of $\underset{\sim}{A}$. !

3.2 Generalized Inverse

Example 3.2

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}$$

$\text{rank}(\underset{\sim}{A}) = 2$

$$\underset{\sim}{A} = \left(\begin{array}{cc|c} 2 & 2 & 3 \\ 1 & 0 & 1 \\ \hline 3 & 2 & 4 \end{array}\right)$$

$$A_{11} = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} \qquad A_{11}^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -1 \end{pmatrix} \qquad A_1^{-} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ \frac{1}{2} & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{A} = \left(\begin{array}{c|cc} 2 & 2 & 3 \\ \hline 1 & 0 & 1 \\ 3 & 2 & 4 \end{array}\right)$$

$$A_{22} = \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix}$$

$$A_{22}^{-1} = \begin{pmatrix} -2 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$A_2^{-} = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & -2 & \frac{1}{2} \\ 0 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{A} = \left(\begin{array}{cc|c} 2 & 2 & 3 \\ \hline 1 & 0 & 1 \\ 3 & 2 & 4 \end{array}\right)$$

$$\underset{\sim}{A}_{21} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$$

$$\underset{\sim}{A}_{21}^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

$$A_3^{-} = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} \\ \hline 0 & 0 & 0 \end{array}\right)$$

3.2 Generalized Inverse – properties

- Let $\boldsymbol{X}$ be $m \times n$, $r(\boldsymbol{X}) = k > 0$,
  - (i) $r(\boldsymbol{X}^-) \geq k$;
  - (ii) $\boldsymbol{X}^-\boldsymbol{X}$ and $\boldsymbol{X}\boldsymbol{X}^-$ are idempotent;
  - (iii) $r(\boldsymbol{X}^-\boldsymbol{X}) = r(\boldsymbol{X}\boldsymbol{X}^-) = k$;
  - (iv) $\boldsymbol{X}^-\boldsymbol{X} = \mathbf{I}$ if and only if $r(\boldsymbol{X}) = n$;
  - (v) $\boldsymbol{X}\boldsymbol{X}^- = \mathbf{I}$ if and only if $r(\boldsymbol{X}) = m$;
  - (vi) $\text{tr}(\boldsymbol{X}^-\boldsymbol{X}) = tr(\boldsymbol{X}\boldsymbol{X}^-) = k = r(\boldsymbol{X})$;
  - (vii) If $\boldsymbol{X}^-$ is any g-inverse of $\boldsymbol{X}$,then $(\boldsymbol{X}^-)'$ is a g-inverse of $\boldsymbol{X}'$.

3.2 Generalized Inverse – properties

Remark 3.3

- Let $\boldsymbol{K} = \underline{\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'}$, $\boldsymbol{K}$ is invariant for any g-inverse of $\boldsymbol{X}'\boldsymbol{X}$. ✱
- For $\boldsymbol{K} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$,
  - (i) $\boldsymbol{K} = \boldsymbol{K}'$, $\boldsymbol{K} = \boldsymbol{K}^2$ (So, Symmetric Idempotent);
  - (ii) rank$(\boldsymbol{K})$ =rank$(\boldsymbol{X}) = r$ (rank$(\boldsymbol{K})$ = tr$(\boldsymbol{K})$ = rank$(\boldsymbol{X})$);
  - (iii) $\boldsymbol{K}\boldsymbol{X} = \boldsymbol{X}$, $\boldsymbol{X}'\boldsymbol{K} = \boldsymbol{X}'$; homework 2
  - (iv) $(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$ is a g-inverse of $\boldsymbol{X}$ for any g-inverse of $\boldsymbol{X}'\boldsymbol{X}$;
  - (v) $\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}$ is a g-inverse of $\boldsymbol{X}'$ for any g-inverse of $\boldsymbol{X}'\boldsymbol{X}$.

Remark 3.3 To prove $\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'$ is invariant

Proof. From the definition of g-inverse

$$\underset{\sim}{X}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}(\underset{\sim}{X}'\underset{\sim}{X}) = \underset{\sim}{X}'\underset{\sim}{X}$$

$$\underset{\sim}{L}'\underset{\sim}{J}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}' \qquad \underset{\sim}{J}\underset{\sim}{L} = \underset{\sim}{L}'\underset{\sim}{J}'\underset{\sim}{J}\underset{\sim}{L}$$

R' $\quad R \quad R' \quad R$

$\Rightarrow$

$$\underset{\sim}{J}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{J} = \underset{\sim}{J}'\underset{\sim}{J}$$

$$\Rightarrow \underset{\sim}{J}'\underset{\sim}{J}\underset{\sim}{L}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{L}'\underset{\sim}{J}'\underset{\sim}{J} = \underset{\sim}{J}'\underset{\sim}{J}$$

$$\Rightarrow \underset{\sim}{L}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{L}' = (\underset{\sim}{J}'\underset{\sim}{J})^{-1}$$

$$\Rightarrow \underset{\sim}{J}\underset{\sim}{L}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{L}'\underset{\sim}{J}' = \underset{\sim}{J}(\underset{\sim}{J}'\underset{\sim}{J})^{-1}\underset{\sim}{J}'$$

$$\Rightarrow \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}' = \underset{\sim}{J}(\underset{\sim}{J}'\underset{\sim}{J})^{-1}\underset{\sim}{J}'$$

$\underset{\sim}{X}_{m\times n} = \underset{\sim}{J}_{m\times k}\,\underset{\sim}{L}_{k\times n}$

$\text{rank}(\underset{\sim}{X}) = k$

$= \underset{\sim}{J}^{*}_{m\times k}\,\underset{\sim}{L}^{*}_{k\times n}$

$J_{m\times k} = J^{*}_{m\times k}\,\underset{\sim}{S}_{k\times k}$

$\underset{\sim}{S}$ – nonsingular

$\underset{\sim}{L}\underset{\sim}{R} = \underset{\sim}{I}_k$

$\underset{\sim}{R}$ is a right inverse of $\underset{\sim}{L}$

invariant!

Because

$$\underset{\sim}{J}(\underset{\sim}{J}'\underset{\sim}{J})^{-1}\underset{\sim}{J}' = \underset{\sim}{J}^*\underset{\sim}{S}(\underset{\sim}{S}'\underset{\sim}{J}^{*\prime}\underset{\sim}{J}^*\underset{\sim}{S})^{-1}\underset{\sim}{S}'\underset{\sim}{J}^{*\prime}$$

$$= \underset{\sim}{J}^*(\underset{\sim}{J}^{*\prime}\underset{\sim}{J}^*)^{-1}\underset{\sim}{J}^{*\prime}$$

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

3.3 Moore–Penrose Inverse

Definition: Let $\boldsymbol{A}$ be an $m \times n$ matrix. If a matrix $\boldsymbol{A}^+$ exists and satisfies

$$\left.\begin{array}{ll} (1) & \boldsymbol{A}\boldsymbol{A}^+ \textit{ is symmetric} \\ (2) & \boldsymbol{A}^+\boldsymbol{A} \textit{ is symmetric} \\ (3) & \boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A} \\ (4) & \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+ \end{array}\right\}$$

$\boldsymbol{A}^+$ is defined as a Moore–Penrose inverse of $\boldsymbol{A}$.

Theorem Each matrix $\boldsymbol{A}$ has an $\boldsymbol{A}^+$.

Proof:. If $\boldsymbol{A} = 0$ then $\boldsymbol{A}^+ = 0$.

If $\boldsymbol{A} \neq 0$, $\boldsymbol{A}$ can be factored by full rank factorization $\boldsymbol{A} = \boldsymbol{A_L}\boldsymbol{A_R} = \boldsymbol{B}\boldsymbol{C}$, where $\boldsymbol{B}$ is $m \times r$ of rank r and $\boldsymbol{C}$ is $r \times n$ of rank r. Hence, $\boldsymbol{B}'\boldsymbol{B}$ and $\boldsymbol{C}\boldsymbol{C}'$ are both nonsingular.

Define

$$\boldsymbol{A}^+ = \mathbf{C}'(\mathbf{C}\mathbf{C}')^{-1}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'$$

and it can be shown that $\boldsymbol{A}^+$ satisfies the 4 conditions above.

3.3 Moore–Penrose Inverse: properties

- The Moore–Penrose inverse is unique;
- $(\boldsymbol{A}')^+ = (\boldsymbol{A}^+)'$;
- $(\boldsymbol{A}^+)^+ = \boldsymbol{A}$;
- $r(\boldsymbol{A}^+) = r(\boldsymbol{A})$;
- If $\boldsymbol{A} = \boldsymbol{A}'$, then $\boldsymbol{A}^+ = (\boldsymbol{A}^+)'$;
- If $\boldsymbol{A}$ is nonsingular, $\boldsymbol{A}^{-1} = \boldsymbol{A}^+$;
- If $\boldsymbol{A}$ is symmetric idempotent, $\boldsymbol{A}^+ = \boldsymbol{A}$;
- If $r(\boldsymbol{A}_{m\times n}) = m$,then $\boldsymbol{A}^+ = \boldsymbol{A}'(\boldsymbol{A}\boldsymbol{A}')^{-1}$, $\boldsymbol{A}\boldsymbol{A}^+ = \mathbf{I}$,
  If $r(\boldsymbol{A}_{m\times n}) = n$,then $\boldsymbol{A}^+ = (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'$, $\boldsymbol{A}^+\boldsymbol{A} = \mathbf{I}$;
- The matrices $\boldsymbol{A}\boldsymbol{A}^+$, $\boldsymbol{A}^+\boldsymbol{A}$, $\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^+$ and $\boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A}$ are all symmetric idempotent.

3.4 Vector and matrix calculus

3.3.1 $u = f(\underset{\sim}{x})$, $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$

$$\frac{\partial u}{\partial \underset{\sim}{x}} = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_p} \end{pmatrix}$$

– Let $u = \underset{\sim}{a}'\underset{\sim}{x} = \underset{\sim}{x}'\underset{\sim}{a}$ $\quad$ $\underset{\sim}{a}$ – Vector of constants

$$\frac{\partial u}{\partial \underset{\sim}{x}} = \frac{\partial(\underset{\sim}{a}'\underset{\sim}{x})}{\partial \underset{\sim}{x}} = \underset{\sim}{a}$$

– Let $u = \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ $\quad$ $\underset{\sim}{A}$ – symmetric

$$\frac{\partial u}{\partial \underset{\sim}{x}} = \frac{\partial(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x})}{\partial \underset{\sim}{x}} = 2\underset{\sim}{A}\underset{\sim}{x}$$

3.3.2 $u = f(\underset{\sim}{x})$, $\underset{\sim}{x} = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & & \\ x_{p1} & \cdots & x_{pp} \end{pmatrix}$

$$\frac{\partial u}{\partial \underset{\sim}{x}} = \begin{pmatrix} \frac{\partial u}{\partial x_{11}} & \cdots & \frac{\partial u}{\partial x_{1p}} \\ \vdots & & \\ \frac{\partial u}{\partial x_{p1}} & \cdots & \frac{\partial u}{\partial x_{pp}} \end{pmatrix}$$

— Let $u = tr(\underset{\sim}{X}\underset{\sim}{A})$

$\underset{\sim}{X}_{p\times p}$ — symmetric positive definite

$\underset{\sim}{A}_{p\times p}$ — matrix of constants

$$\frac{\partial u}{\partial \underset{\sim}{X}} = \frac{\partial\, tr(\underset{\sim}{X}\underset{\sim}{A})}{\partial \underset{\sim}{X}}$$

$$= \underset{\sim}{A} + \underset{\sim}{A}' - diag(\underset{\sim}{A})$$

- $\dfrac{\partial \log |\underset{\sim}{X}|}{\partial \underset{\sim}{X}} = 2\underset{\sim}{X}^{-1} - \operatorname{diag}(\underset{\sim}{X}^{-1})$

3.3.3 $\underset{\sim}{A}_{n\times n} = (a_{ij})$ a_{ij} - function of $x_{1\times 1}$

- $\dfrac{\partial \underset{\sim}{A}^{-1}}{\partial x} = -\underset{\sim}{A}^{-1} \dfrac{\partial \underset{\sim}{A}}{\partial x} \underset{\sim}{A}^{-1}$

$$\boxed{\begin{aligned} &\underset{\sim}{A}^{-1}\underset{\sim}{A} = \underset{\sim}{I} \\ &\frac{\partial \underset{\sim}{A}^{-1}}{\partial x}\underset{\sim}{A} + \underset{\sim}{A}^{-1}\frac{\partial \underset{\sim}{A}}{\partial x} = 0 \end{aligned}}$$

- $\dfrac{\partial \log |\underset{\sim}{A}|}{\partial x} = \operatorname{tr}\left(\underset{\sim}{A}^{-1} \dfrac{\partial \underset{\sim}{A}}{\partial x}\right)$