Ch2. Simple Linear Regression

- Relationship between 2 variables
- The regression model
- Assumptions
- Estimation and method of least squares
- Inferences concerning β_1 and β_0
- Estimation of the mean of the response variable for a given level of x
- Prediction of new observation
- Analysis of variance approach to regression analysis
- Measures of linear association between x and y

Simple Linear Regression Model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, 2, \ldots, n$$

- Assumptions:
  - $\mathrm{E}(\epsilon_i) = 0$,
  - $\mathrm{Var}(\epsilon_i) = \sigma^2$
  - $\mathrm{Cov}(\epsilon_i, \epsilon_j) = 0$
- In matrix notation.

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

Simple Linear Regression Equation

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

- The simple linear regression equation provides an estimate of the population regression line
- $\hat{\beta}_0$ is the estimated average value of y when the value of x is zero
- $\hat{\beta}_1$ is the estimated change in the average values of y as a result of a one-unit change in x

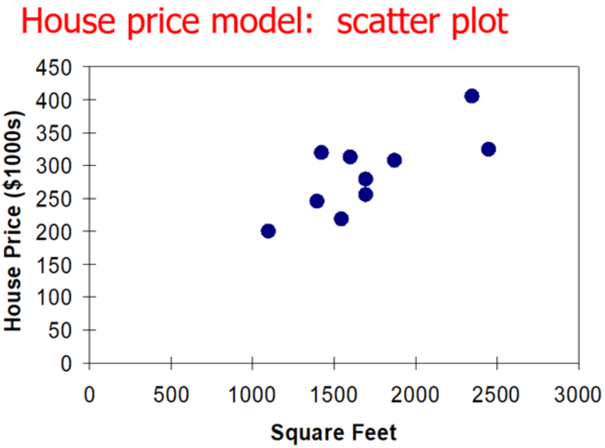
Simple Linear Regression: an example

A real estate agent wishes to examine the relationship between the selling price of a home and its size (measured in square feet)

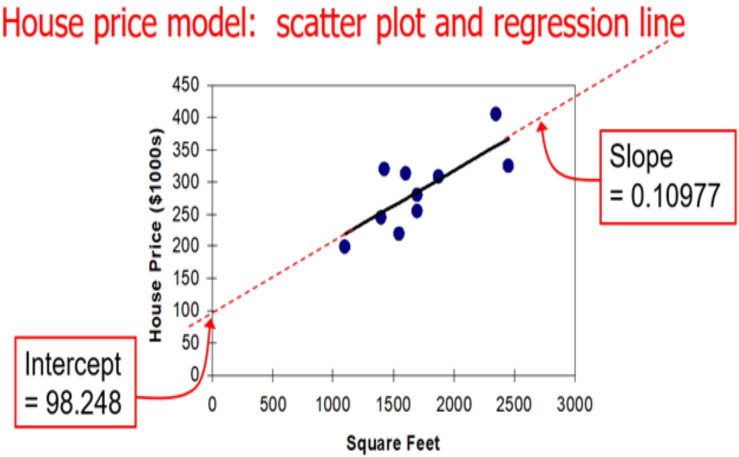
- A random sample of 10 houses is selected
- y= house price in \$1000s, x= square feet

y	x
245	1400
312	1600
279	1700
308	1875
199	1100
219	1550
405	2350
324	2450
319	1425
255	1700

An example: Graphical Presentation



An example: Graphical Presentation



$$\hat{y} = 98.248 + 0.10977x$$

An example: Interpretation of the intercept, $\hat{\beta}_0$

$$\hat{y} = 98.248 + 0.10977x$$

- $\hat{\beta}_0$ is the estimated average value of y when the value of x is zero (if $x = 0$ is in the range of observed x values)
- Here, no houses had 0 square feet, so $\hat{\beta}_0 = 98.248$ just indicates that, for houses within the range of sizes observed, \$98,248 is the portion of the house price not explained by square feet.

An example: Interpretation of the Slope Coefficient, $\hat{\beta}_1$

$$\hat{y} = 98.248 + 0.10977x$$

- $\hat{\beta}_1$ measures the estimated change in the average value of y as a result of a one-unit change in x
  - Here, $\hat{\beta}_1 = .10977$ tells us that the average value of a house increases by .10977(k)=$109.77, on average, for each additional one square foot of size.

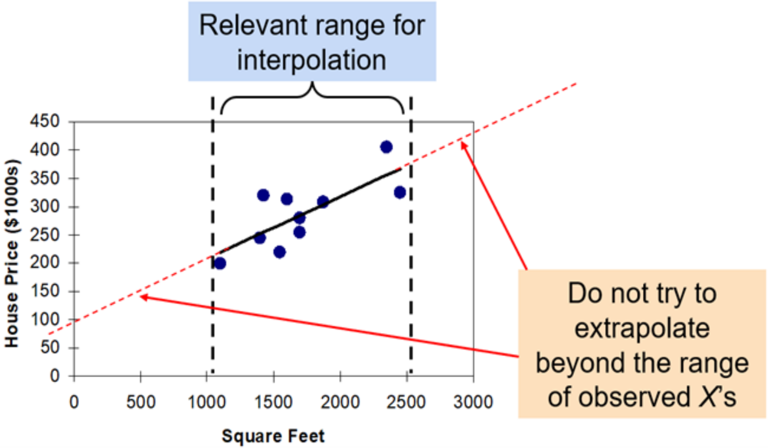
An example: Predictions using Regression Analysis

- Predict the price for a house with 2000 square feet:

$$\hat{y} = 98.25 + 0.10977 \times 2000 = 317.85$$

- The predicted price for a house with 2000 square feet is 317.85($1,000s) = $317,850

An example: Interpolation vs. Extrapolation



When using a regression model for prediction, only predict within the relevant range of data unless you have further information.

Estimation: Method of Least Squares

- $\hat{\beta}_0$ and $\hat{\beta}_1$ are obtained by finding the values of β_0 and β_1 that minimize the sum of the squared differences between y and $\hat{y}$:

$$\min \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

- Solutions:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{S_{xy}}{S_{xx}}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

- Comparing $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$ with $r = \frac{S_{xy}}{\sqrt{S_{xx}}\sqrt{S_{yy}}}$

Estimation of error terms variance σ^2

- The estimator of σ^2 is

$$S^2 = MSE = \frac{SSE}{n-2} = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-2}$$

- S^2 is an unbiased estimator of σ^2

Estimation: Method of Maximum Likelihood

- The simple linear regression model with normal error

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i,\ \ \epsilon_i \sim N(0, \sigma^2)\ \ i = 1, 2, \ldots, n,$$

- The likelihood of the above model
- $\hat{\beta}_0$ and $\hat{\beta}_1$ are obtained by maximising the above likelihood
- MLEs:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

- The estimator of σ^2 is $\frac{SSE}{n} = \frac{n-2}{n} S^2$.

Estimation: Method of Maximum Likelihood

- MLE of β_0 = LSE of β_0 and is unbiased
- MLE of β_1 = LSE of β_1 and is unbiased
- MLE of σ^2 is less than the unbiased estimator of σ^2, but is asymptotically unbiased

Distribution of $\hat{\beta}_1$

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, 2, \ldots, n,$$

- Assumptions
  - x_i's are known constants,
  - $\epsilon_i \sim N(0, \sigma^2)$ independently for $i = 1, 2, \ldots, n$
- Therefore, $y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$
- $$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \sum_{i=1}^n c_i y_i$$

  where $c_i = \frac{(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$, and then $\hat{\beta}_1$ follows a normal distribution.
- $\hat{\beta}_1 \sim N(\beta_1, \sigma^2 / S_{xx})$.

Testing (Two-sided test of β_1)

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, 2, \ldots, n,$$

- $$H_0 : \beta_1 = 0 \text{ (no linear relationship) } v.s.$$
  $$H_1 : \beta_1 \neq 0 \text{ (linear relationship does exist between } x \text{ and } y\text{)}$$
- Test statistic:
  $$t = \frac{\hat{\beta}_1 - \beta_1}{S/S_{xx}^{1/2}} \sim t_{n-2} \text{ if } H_0 \text{ is true}$$
- Decision rule: reject H_0 if $|t| > t_{\alpha/2, n-2}$.

Two-sided test and confidence interval of β_1

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, 2, \ldots, n,$$

- $$H_0 : \beta_1 = k \quad v.s. H_1 : \beta_1 \neq k \text{ (k is a constant)}$$
    - What are the test statistic and decision rule?
- What are the confidence interval of β_1?

Distribution of $\hat{\beta}_0$

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \;\; \epsilon_i \sim N(0, \sigma^2) \;\; i = 1, 2, \ldots, n,$$

- $\hat{\beta}_1 \sim N(\beta_1, \sigma^2 / S_{xx})$.
- $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ also follows a normal distribution

$$\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_i (x_i - \bar{x})^2}\right]\right)$$

Estimation of the mean of the response variable for a given level of x

- Example
  - y (in \$000) – house price, x (square feet) – house size
  - Estimate the average house price for houses with 2000 square feet.
- Let x_h be the level of x for which we wish to estimate the mean response, then

$$y_h = \beta_0 + \beta_1 x_h + \epsilon_h,$$

  the mean response is $\mathrm{E}(y_h) = \beta_0 + \beta_1 x_h$.
- The estimation of $\mathrm{E}(y_h)$ is $\hat{y}_h = \hat{\beta}_0 + \hat{\beta}_1 x_h$, with distribution

$$\hat{y}_h \sim N\left(\beta_0 + \beta_1 x_h, \sigma^2 \left[\frac{1}{n} + \frac{(x_h - \bar{x})^2}{\sum_i (x_i - \bar{x})^2}\right]\right)$$

Confidence interval for $E(y_h)$

$$E(y_h) - \hat{y}_h \sim N\left(0, \sigma^2\left[\frac{1}{n} + \frac{(x_h - \bar{x})^2}{\sum_i (x_i - \bar{x})^2}\right]\right)$$

Two-sided $100(1-\alpha)\%$ C.I. for $E(y_h)$ is

$$\left(\hat{y}_h - t_{\alpha/2,n-2} S\sqrt{\frac{1}{n} + \frac{(x_h - \bar{x})^2}{\sum_i (x_i - \bar{x})^2}}, \hat{y}_h + t_{\alpha/2,n-2} S\sqrt{\frac{1}{n} + \frac{(x_h - \bar{x})^2}{\sum_i (x_i - \bar{x})^2}}\right)$$

Prediction of a new observation y_h

- Example
  - y (in \$000) – house price, x (square feet) – house size
  - Estimate the house price for **an individual** house with 2000 square feet.
- It means we wish to estimate the response y_h given x_h

$$y_h = \beta_0 + \beta_1 x_h + \epsilon_h,$$

- The estimation of y_h is still $\hat{y}_h = \hat{\beta}_0 + \hat{\beta}_1 x_h$, but

$$y_h - \hat{y}_h \sim N\left(0, \sigma^2 \left[1 + \frac{1}{n} + \frac{(x_h - \bar{x})^2}{\sum_i (x_i - \bar{x})^2}\right]\right)$$

Confidence interval for a new observation y_h

Two-sided $100(1-\alpha)\%$ C.I. for y_h is

$$\left(\hat{y}_h - t_{\alpha/2,n-2} S\sqrt{1+\frac{1}{n}+\frac{(x_h-\bar{x})^2}{\sum_i (x_i-\bar{x})^2}},\right.$$

$$\left.\hat{y}_h + t_{\alpha/2,n-2} S\sqrt{1+\frac{1}{n}+\frac{(x_h-\bar{x})^2}{\sum_i (x_i-\bar{x})^2}}\right)$$

We also call it as a **predictive interval**.

Analysis of variance approach to regression analysis

- Partitioning of Total Sum of Squares (SST)

$$\begin{aligned} SST &= \sum_{i=1}^{n}(y_i - \bar{y})^2 \\ &= \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 \\ &= SSE + SSR \end{aligned}$$

  where SSE=sum of squares of residual, SSR=sum of squares due to regression.

- OR

  Total Variation = Unexplained Variation + Explained Variation

Analysis of variance (ANOVA) table

	Sum of Squares (SS)	Degrees of freedom (df)	Mean squares (MS)	F
Regression	SSR	1	$MSR = \frac{SSR}{1}$	$\frac{MSR}{MSE}$
Error	SSE	n-2	$MSE = \frac{SSE}{(n-2)}$	
Total	SST	n-1		

- Test $H_0 : \beta_1 = 0$ (**no linear relationship**) v.s. $H_1 : \beta_1 \neq 0$ (**linear relationship does exist** between y and x)
- Test statistics $F = \frac{MSR}{MSE} \sim_{H_0} F_{1,n-2}$
- Reject H_0 if $F > F_{\alpha,1,n-2}$.

The coefficient of determination

- The coefficient of determination OR R-squared is defined

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

- The proportion of the variation can be explained by the model: $0 \leq R^2 \leq 1$.
- Coefficient of correlation (true for simple linear regression only)

$$r = \pm\sqrt{R^2}$$