Ch5. Quadratic Forms

5.1 Quadratic Form $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$

Let $\boldsymbol{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and assume $\boldsymbol{A}$ symmetric, then m.g.f. of $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ is

$$M_{\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}}(t) = |\boldsymbol{I} - 2t\boldsymbol{A}\boldsymbol{\Sigma}\,|^{-\frac{1}{2}} \cdot e^{\{-\frac{1}{2}\boldsymbol{\mu}'[\boldsymbol{I}-(\boldsymbol{I}-2t\boldsymbol{A}\boldsymbol{\Sigma})^{-1}]\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\}}$$

Ch 5.1 Quadratic Form $x'Ax$

- $\mathrm{E}(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) = \mathrm{tr}(\boldsymbol{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$
- $\mathrm{Var}(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}) = 2\mathrm{tr}[(\boldsymbol{A}\boldsymbol{\Sigma})^2] + 4\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}\boldsymbol{\mu}$

Ch 5.2 Non-Central χ^2, F and t distributions

I. Non-Central χ^2

- Let $\boldsymbol{x} \sim N(\mathbf{0}, \mathbf{I}_n)$, then $\boldsymbol{x}'\boldsymbol{x} \sim \chi^2_{(n)}$;
- Let $\boldsymbol{x} \sim N(\boldsymbol{\mu}, \mathbf{I}_n)$, then

$$u = \boldsymbol{x}'\boldsymbol{x} \sim \chi^2_{(n,\,\lambda)}$$

where $\lambda =$ non-centered parameter $= \frac{1}{2}\boldsymbol{\mu}'\boldsymbol{\mu}$;

- Density is

$$f(u) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{u^{\frac{n}{2}+k-1} e^{-\frac{u}{2}}}{2^{\frac{n}{2}+k}\Gamma(\frac{n}{2}+k)}, \qquad \mu > 0, \lambda \geq 0$$

Note : Define $\lambda^k = 1$ when $\lambda = 0, k = 0$, density function of $u \sim \chi^2_{(n,0)}$ is

$$f(u) = \frac{u^{\frac{n}{2}-1} e^{-\frac{u}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}.$$

Ch 5.2 Non-Central χ^2, F and t distributions

I. Non-Central χ^2

- m.g.f of $u \sim \chi^2_{(n,\lambda)}$ is

$$(1-2t)^{-\frac{n}{2}} e^{-\lambda[1-(1-2t)^{-1}]}.$$

Note : for $\lambda = 0$, $\Rightarrow M_u(t) = (1-2t)^{-\frac{n}{2}}$ which is m.g.f of $\chi^2_{(n)}$

- $\mathrm{E}(u) = n + 2\lambda$ and $\mathrm{Var}(u) = 2n + 8\lambda$;
- If $u_i \sim \chi^2_{(n_i,\lambda_i)}$ independently for $i = 1, \ldots, k$, then

$$\sum_{i=1}^{k} u_i \sim \chi^2_{(\sum_{i=1}^{k} n_i, \sum_{i=1}^{k} \lambda_i)}.$$

Ch 5.2 Non-Central χ^2, F and t distributions

II. Non-Central F

Let $u_1 \sim \chi^2_{(p_1,\lambda)}$, $u_2 \sim \chi^2_{(p_2,0)}$, and Let u_1 be independent of u_2, then

$$w = \frac{u_1/p_1}{u_2/p_2} \sim F_{(p_1,p_2,\lambda)}$$

and

$$\mathrm{E}(w) = \frac{p_2}{p_2 - 2}\left(1 + \frac{2\lambda}{p_1}\right).$$

Ch 5.2 Non-Central χ^2, F and t distributions

III. Non-Central t

Let $z \sim N(\mu, 1), u \sim \chi^2_{(n)}$, z is independent of u, then

$$t = \frac{z}{\sqrt{u/n}} \sim \text{ non-centered } t \text{ distribution.}$$

Ch 5.2 Non-Central χ^2, F and t distributions

Theorem 5.1 Let $\boldsymbol{x}_{p\times 1} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{A}$ be symmetric, then $q = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} \sim \chi^2_{(r,\,\lambda)}$ where r denoting the rank of $\boldsymbol{A}$ and $\lambda = \frac{\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}}{2}$ if and only if $\boldsymbol{A\Sigma}$ is idempotent.

Ch 5.2 Non-Central χ^2, F and t distributions

Corollaries

- If $\boldsymbol{x} \sim N(\mathbf{0}, \mathbf{I})$, then $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ is χ^2_r if and only if $\boldsymbol{A}$ is idempotent of rank r.
- If $\boldsymbol{x} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, then $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ is χ^2_r if and only if $\boldsymbol{A}\boldsymbol{\Sigma}$ is idempotent of rank r.
- If $\boldsymbol{x}$ is $N(\boldsymbol{\mu}, \sigma^2\mathbf{I})$, then $\frac{\boldsymbol{x}'\boldsymbol{x}}{\sigma^2}$ is $\chi^2_{(n, \frac{1}{2}\frac{\boldsymbol{\mu}'\boldsymbol{\mu}}{\sigma^2})}$.
- If $\boldsymbol{x} \sim N(\boldsymbol{\mu}, \mathbf{I})$, then $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ is $\chi^2_{(r, \frac{1}{2}\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu})}$ if and only if $\boldsymbol{A}$ is idempotent of rank r.

Ch 5.3 Independence

Theorem 5.2 When $\boldsymbol{x} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{A}$ is symmetric, then $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{B}\boldsymbol{x}$ are distributed independently if and only if $\boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{A} = 0$.

Ch 5.3 Independence

Theorem 5.3 Let $\boldsymbol{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric, then $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x}$ are distributed independently if and only if $\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{B} = 0$ (or equivalently $\boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{A} = 0$).

Ch 5.4 Additional results and examples

Let the $n \times 1$ vector $\mathbf{x} = (x_1, ..., x_n)' \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $q_1 = \mathbf{x}'\mathbf{A}_1\mathbf{x}$, $q_2 = \mathbf{x}'\mathbf{A}_2\mathbf{x}$ and $\mathbf{T} = \mathbf{B}\mathbf{x}$ where $\mathbf{B}$ is $r \times n$ and $\mathbf{A}_1, \mathbf{A}_2$ are symmetric.

- $\mathrm{E}(q_1) = \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\mathbf{A}_1\boldsymbol{\mu}$.
- $\mathrm{Var}(q_1) = 2\ \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\Sigma}) + 4\ \boldsymbol{\mu}'\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\mu}$.
- $\mathrm{Cov}(q_1, q_2) = 2\ \mathrm{tr}(\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_2\boldsymbol{\Sigma}) + 4\ \boldsymbol{\mu}'\mathbf{A}_1\boldsymbol{\Sigma}\mathbf{A}_2\boldsymbol{\mu}$.
- $\mathrm{Cov}(\mathbf{x}, q_1) = 2\ \boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\mu}$.
- $\mathrm{Cov}(\mathbf{T}, q_1) = 2\ \mathbf{B}\boldsymbol{\Sigma}\mathbf{A}_1\boldsymbol{\mu}$.

Ch 5.4 Additional results and examples

Example 5.1 Let the $n \times 1$ vector $\mathbf{Y} = (Y_1, ..., Y_n)' \sim N(\alpha\mathbf{1}, \sigma^2\mathbf{I})$. Define $U = \sum_{i=1}^{n}(Y_i - \overline{Y})^2/\sigma^2$ and $V = n(\overline{Y} - \alpha)^2/\sigma^2$. Find the distributions of U and V and show that these two random variables are independent.

Ch 5.4 Additional results and examples

Example 5.2 Let the $n \times 1$ vector $\mathbf{Y} = (Y_1, ..., Y_n)' \sim N(\mu\mathbf{1}, \sigma^2\mathbf{I})$. Let

$$\begin{aligned} \overline{Y} &= \frac{\sum_{i=1}^n Y_i}{n} \\ Q_1 &= n\overline{Y}^2 \\ Q_2 &= \sum_{i=1}^n (Y_i - \overline{Y})^2 \end{aligned}$$

1. Prove that $\overline{Y}$ and Q_2 are independent.
2. Prove that Q_1 and Q_2 are independent.
3. Find the distributions of Q_1 and Q_2.

Ch 5.4 Additional results and examples

Example 5.3 Suppose $\boldsymbol{y}$ is $N_3(\boldsymbol{\mu}, \sigma^2\mathbf{I})$ and let $\boldsymbol{\mu}' = (3, -2, 1)$ and

$$\mathbf{A} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\mathbf{B} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

1. Find the distribution of $\boldsymbol{y}'\mathbf{A}\boldsymbol{y}/\sigma^2$.
2. Are $\boldsymbol{y}'\mathbf{A}\boldsymbol{y}$ and $\mathbf{B}\boldsymbol{y}$ independent?
3. Are $\boldsymbol{y}'\mathbf{A}\boldsymbol{y}$ and $y_1 + y_2 + y_3$ independent?

Ch 5.4 Additional results and examples

Example 5.4 Suppose $\mathbf{y}$ is $N_n(\mu\mathbf{1}, \mathbf{\Sigma})$ where

$$\mathbf{\Sigma} = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \rho & \rho & \cdots & 1 \end{pmatrix}.$$

Derive the distribution of

$$\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{\sigma^2(1-\rho)}.$$