MA329 Statistical linear models 22-23

Assignment 3 (Due date: Nov 2, 11pm. For late submission, each day costs 10 percent. The solution will be released at 6pm Nov 4 since midterm test is on Nov 7. This assignment will not be accepted once the solution is released.)

1. (10 marks) Consider a nonsingular $n \times n$ matrix $\boldsymbol{A}$ whose elements are functions of the scalar x. Also consider the full-rank $p \times n$ matrix $\boldsymbol{B}$. Let $\boldsymbol{H} = \boldsymbol{B}'(\boldsymbol{BAB}')^{-1}\boldsymbol{B}$. Show that
$$\frac{\partial \boldsymbol{H}}{\partial x} = -\boldsymbol{H}\frac{\partial \boldsymbol{A}}{\partial x}\boldsymbol{H}$$

2. (20 marks) Let $\boldsymbol{y} = (y_1, y_2, y_3)'$ be distributed as $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where
$$\boldsymbol{\mu} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$
   (a) Find the distribution of $\begin{pmatrix} y_1 - y_2 + y_3 \\ 2y_1 + y_2 - y_3 \end{pmatrix}$;

   (b) The conditional distribution of (y_1, y_2) given y_3;

   (c) The partial correlation between y_1 and y_2 given y_3.

3. (10 marks) Let $\mathbf{Y} = (Y_1, \ Y_2, \ Y_3)'$. $E(\mathbf{Y}) = (2, \ 3, \ 4)'$ and the covariance matrix of $\mathbf{Y}$ is
$$\boldsymbol{\Sigma} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$
   Let $U = \sum_{i=1}^{3}(Y_i - \overline{Y})^2$. Find the expected value of U.

4. (10 marks) Let $\mathbf{Y} = (Y_1, ..., Y_n)'$. $E(\mathbf{Y}) = \mu\mathbf{1}$ and the covariance matrix of $\mathbf{Y}$ is $\sigma^2\mathbf{I}$. Let
$$U = \sum_{i<j}(Y_i - Y_j)^2.$$
   Find the expected value of U, and find a constant k such that kU is an unbiased estimator of σ^2.

Assignment 3 — Solutions.

Q1. Let $\underset{\sim}{V} = \underset{\sim}{B}\underset{\sim}{A}\underset{\sim}{B}'$ then $\dfrac{\partial \underset{\sim}{V}}{\partial x} = \underset{\sim}{B}\dfrac{\partial \underset{\sim}{A}}{\partial x}\underset{\sim}{B}'$ $\quad (*)$

$$\underset{\sim}{H} = \underset{\sim}{B}'(\underset{\sim}{B}\underset{\sim}{A}\underset{\sim}{B}')^{-1}\underset{\sim}{B} = \underset{\sim}{B}'\underset{\sim}{V}^{-1}\underset{\sim}{B}$$

$$\frac{\partial \underset{\sim}{H}}{\partial x} = \underset{\sim}{B}'\frac{\partial \underset{\sim}{V}^{-1}}{\partial x}\underset{\sim}{B}$$

$$\boxed{\frac{\partial \underset{\sim}{V}^{-1}}{\partial x} = -\underset{\sim}{V}^{-1}\frac{\partial \underset{\sim}{V}}{\partial x}\underset{\sim}{V}^{-1}}$$

$$= -\underset{\sim}{B}'\underset{\sim}{V}^{-1}\frac{\partial \underset{\sim}{V}}{\partial x}\underset{\sim}{V}^{-1}\underset{\sim}{B}$$

$$= -\underset{\sim}{B}'\underset{\sim}{V}^{-1}\underset{\sim}{B}\frac{\partial \underset{\sim}{A}}{\partial x}\underset{\sim}{B}'\underset{\sim}{V}^{-1}\underset{\sim}{B} \qquad \boxed{\text{from } (*)}$$

$$= -\underset{\sim}{H}\frac{\partial \underset{\sim}{A}}{\partial x}\underset{\sim}{H}$$

Q2

(a)

$$\begin{pmatrix} y_1 - y_2 + y_3 \\ 2y_1 + y_2 - y_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \sim N(\underset{\sim}{\mu}_1, \underset{\sim}{\Sigma}_1).$$

$$\underset{\sim}{\mu}_1 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{\Sigma}_1 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 2 \\ 9 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 \\ 4 & 23 \end{pmatrix}.$$

(b) $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Big| \, y_3 \sim N(\underset{\sim}{\mu}_2, \underset{\sim}{\Sigma}_2)$.

$$\begin{pmatrix} y_1 \\ y_2 \\ \hline y_3 \end{pmatrix} \begin{matrix} \underset{\sim}{y} \\ \hline x \end{matrix}$$

$$\underset{\sim}{\Sigma} = \left(\begin{array}{cc|c} 4 & 1 & 0 \\ 1 & 2 & 1 \\ \hline 0 & 1 & 3 \end{array}\right) = \begin{pmatrix} \underset{\sim}{\Sigma}_{yy} & \underset{\sim}{\Sigma}_{yx} \\ \underset{\sim}{\Sigma}_{xy} & \underset{\sim}{\Sigma}_{xx} \end{pmatrix}$$

$$\underset{\sim}{\mu}_2 = \mu_y + \underset{\sim}{\Sigma}_{yx} \underset{\sim}{\Sigma}_{xx}^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}_x)$$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} 3^{-1} (y_3 - 3)$$

$$= \begin{pmatrix} 2 \\ \frac{y_3}{3} - 2 \end{pmatrix}$$

$$\underset{\sim}{\Sigma}_2 = \underset{\sim}{\Sigma}_{yy} - \underset{\sim}{\Sigma}_{yx} \underset{\sim}{\Sigma}_{xx}^{-1} \underset{\sim}{\Sigma}_{xy}$$

$$= \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} 3^{-1} (0 \;\; 1) = \begin{pmatrix} 4 & 1 \\ 1 & \frac{5}{3} \end{pmatrix}$$

(c) The partial correlation between y_1 and y_2 given y_3 is:

$$\frac{1}{\sqrt{4 * 5/3}} = 0.3873$$

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Q3. $U = \sum_{i=1}^{3} (y_i - \bar{y})$ $\bar{y}$

$$= \underset{\sim}{y}' \underset{\sim}{A} \underset{\sim}{y}.$$

where

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad \underset{\sim}{A} = (I_3 - \tfrac{1}{n}\underset{\sim}{J}), \quad n = 3, \quad \underset{\sim}{J} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Thus.

$$E(U) = tr(\underset{\sim}{A}\,\underset{\sim}{\Sigma}) + \underset{\sim}{\mu}\,\underset{\sim}{A}\,\underset{\sim}{\mu} \qquad \underset{\sim}{\mu} = E(\underset{\sim}{y})$$

$$= 4 + 2 = 6$$

Q4. $u = \sum_{i<j} (y_i - y_j)^2 = \underset{\sim}{y}' \underset{\sim}{A} \underset{\sim}{y}$

where $\underset{\sim}{A} = n I_n - J_n$, $J_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & & 1 \end{pmatrix}_{n\times n}$ $\underset{\sim}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

Thus.

$$E(u) = tr(\underset{\sim}{A}\underset{\sim}{\Sigma}) + \underset{\sim}{\mu}' \underset{\sim}{A} \underset{\sim}{\mu}$$

$$\underset{\sim}{\mu} = E(\underset{\sim}{y}) = \mu \underset{\sim}{1}_n. \qquad \underset{\sim}{1}_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}$$

$$J_n = \underset{\sim}{1}_n \cdot \underset{\sim}{1}_n'$$

$$\underset{\sim}{\mu}' \underset{\sim}{A} \underset{\sim}{\mu} = \mu^2 \underset{\sim}{1}_n' (n I_n - \underset{\sim}{1}_n \underset{\sim}{1}_n') \underset{\sim}{1}_n \qquad \underset{\sim}{1}_n' \underset{\sim}{1}_n = n$$

$$= \mu^2 (n \underset{\sim}{1}_n' \underset{\sim}{1}_n - \underset{\sim}{1}_n' \underset{\sim}{1}_n \cdot \underset{\sim}{1}_n' \underset{\sim}{1}_n)$$

$$= 0$$

$$tr(\underset{\sim}{A}\underset{\sim}{\Sigma}) = \sigma^2 tr(n I_n - J_n)$$

$$= \sigma^2 (n^2 - n)$$

Then, $E(u) = n(n-1)\sigma^2$.

and ku is an unbiased estimator of σ^2 when $k = \frac{1}{n(n-1)}$.