Optimization

Tutorial 1

Problem 1.

Consider the following objective function to be minimized:

$$f(x) = x^2 + \exp(x)$$

The optimization variable is a real: $x \in \mathbb{R}$.

1. Calculate the gradient and the Hessian of the objective function.
2. Is the objective function convex ? strictly convex ?
3. Determine the critical point(s) of this objective function.
4. Does the problem admits a solution ?

We want to solve this problem using Nelder and Mead method. The folder **Probem1** provides the associated Matlab files. To simplify the study, we will assume that $-10 \le x \le 5$.

5. Plot the evolution of the function with x. Analyze this plot and comment it (regarding the obtained previous results).
6. Solve the problem using Nelder and Mead simplex and determine the value of optimal cost.

Problem 2.

Consider the following objective function to be minimized:

$$J(x, y) = (x-5)^2 + (y-1)^2 + xy$$

Where $(x, y) \in \mathbb{R}^2$.

1. Calculate the gradient and the Hessian of the objective function.
2. Is the objective function convex ? strictly convex ?
3. Determine the critical point(s) of this objective function.
4. Show that we can rewrite the objective function as:

$$J(x, y) = \begin{pmatrix} x & y \end{pmatrix} \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{c}^T \begin{pmatrix} x \\ y \end{pmatrix} + d$$

   Where $\mathbf{M}$ is a matrix, $\mathbf{c}$ a vector and $d \in \mathbb{R}$ to be determined.

5. Using the last formulation of the objective function, study the convexity and critical point of the function. Use the derivation in case of vectors:

For a vector $z \in \mathbb{R}^n$, we have (with $\mathbf{M}$ is a matrix, $\mathbf{c}$ a vector):

$$\frac{d}{dz}(z^T \mathbf{M} z) = \left(\mathbf{M} + \mathbf{M}^T\right) z$$

$$\frac{d}{dz}(z^T \mathbf{c}) = \frac{d}{dz}(\mathbf{c}^T z) = \mathbf{c}$$

6. Compare the results obtained in questions 1-3 and 5.

We want to solve this problem using Matlab, considering the first formulation of the cost function. The folder **Probem2** provides the associated Matlab files.

7. Solve the optimization problem and give the optimal cost. Test several optimization methods, initialization,…

Problem 3.

We want to minimize the objective function provided in folder **Probem3.**

1. Analyze the cost function (file *Cost_1.m*). Characterize the minimizers.
2. Solve the optimization problem (using the method of your choice), using the file *Problem3.m*.
3. Study the influence of the initialization on the optimal solution (e.g. by a random initialization, using `randn`).
4. Consider the new objective function (file *Cost_2.m*). Analyze this function.
5. Solve the problem and comment the obtained result.
6. In the cost function *Cost_2.m*, test several values of parameter c (e.g. 0.1, 1, 10). Analyze the obtained solution (optimization variable and cost). Comment the result.

Problem 4.

We want to minimize the objective function provided in folder **Probem4.**

1. Analyze the cost function (file *Cost.m*).
2. Solve the optimization problem (using the method of your choice), using the file *Problem4.m*, considering the 3 cases.
3. Comment the obtained results.

Problem 5.

Consider the following objective function to be minimized:

$$f(x) = x^2(x-2)(x+2)$$

The optimization variable is a real: $x \in \mathbb{R}$.
To simplify the study, we will assume that $-5 \le x \le 5$.

1. Calculate the gradient and the Hessian of the objective function.
2. Is the objective function convex? strictly convex ?
3. Determine the critical point(s) of this objective function.
4. Does the problem admit a solution? If so, calculate the optimal solutions.

We want to solve this problem using Nelder and Mead method. Plot the evolution of the function with x.

5. Analyze this plot and comment it (regarding the obtained previous results).
6. Solve the problem using Nelder and Mead simplex and determine the value of optimal cost.

Hint: you can adapt the files used in the Problem 1.

Problem 6.

Consider the objective function to minimize:

$$J(x,y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2$$

1. Plot the function and its contour. How many minimizers does the problem has?
2. From this plot, is the function convex ?
3. Using Matlab, determine the optimal solution(s).
    - Compare performances of several optimization methods (Nelder and Mead, BFGS, DFP, steepest descent).
    - Consider several initializations.

Hint: you can adapt the files used for the minimization of the Rosenbrock function provided in the course.