73-240 Handout: Solving a Competitive Equilibrium. An Example

1 Notation

- From Household
  - c = consumption, $C = Nc$ aggregate consumption
  - l = leisure
  - h = total hours
  - n^s = labor supplied, N^s aggregate labor supplied
  - σ risk aversion parameter, ψ parameter governing relative preference for leisure, ν parameter governing concavity of utility from leisure
  - w = wage
  - π = dividend income, and also equal to profits! [Because we have assumed that all households are representative and aggregate to one large household, this implies that all profits go back to the large household. We don't have any Warren Buffetts in this economy, all households are alike which means the household's share of the firm's profits across all households must add to 1]
  - T = lump-sum taxes
- From Firm
  - π = profits
  - z = productivity
  - k = capital of a firm, K = aggregate capital
  - n^d = labor demand of a firm, N^d = aggregate labor demand
  - w = wage!
- From Government
  - G = government spending
  - T = government tax revenue from lump-sum tax

2 Setting up the Household's problem: an example

Household's goal is to maximize utility subject to constraints. Note we can re-write leisure as $l = h - N^s$. Because we are going to solve for a competitive equilibrium in the end, its easier to write leisure in terms of total hours and labor supply.

$$\max_{C,l} U(C,l) = \frac{C^{1-\sigma} - 1}{1-\sigma} + \psi \frac{l^{1-\nu}}{1-\nu}$$

s.t.

$$C = wN^s + \pi - T$$

and we know $h = N^s + l$

Set up the Lagrangian:

$$\mathcal{L} = \max_{C,l,\lambda} \frac{C^{1-\sigma} - 1}{1-\sigma} + \psi \frac{l^{1-\nu}}{1-\nu} + \lambda \left[w(h-l) + \pi - T - C \right]$$

Take first order conditions:

- wrt to consumption: $\frac{\partial \mathcal{L}}{\partial C} = 0$ implies
$$MU_c = C^{-\sigma} = \lambda$$
- wrt to leisure: $\frac{\partial \mathcal{L}}{\partial l} = 0$ implies
$$MU_l = \psi l^{-\nu} = \lambda w$$
- wrt to λ, we get back the budget constraint:
$$C = wN^s + \pi - T \tag{1}$$

Let's write down what $MRS_{l,c}$ is equal to. Note that to find $MRS_{l,c}$, we take MU_l/MU_c. So we get:

$$\underbrace{\frac{MU_l}{MU_c}}_{MRS_{l,c}} = \frac{\psi C^\sigma}{l^\nu} = \frac{\psi C^\sigma}{(h - N^s)^\nu} = w \tag{2}$$

where we have also used the fact that $l = h - N^s$ in the above. Observe that equation 2 is one of our optimality conditions. It tells us that the rate at which the Household desires to substitute between leisure and consumption must be equal to the opportunity cost of leisure which is given by the wage rate. Also observe that equation 1 is our other optimality condition from the household. The household always chooses what is affordable.

These are the two household optimality conditions we need. Note that the household's problem traces out how much consumption and leisure they're willing to trade off given a wage rate, and also tells you how much consumption and leisure they would like to optimally consume given w, π, T and parameters of the model.

3 Setting up the Firm's problem: an example

The Firm's goal is to maximize profits. In the examples we talk about in class, we have that the firm is born with capital K and takes exogenous productivity z as given. Hence, K and z are exogenous variables and cannot be controlled by the firm. What the firm can instead choose is labor demand.

$$\max_{N^d} \pi = zK^{\alpha}N^{d,1-\alpha} - wN^d$$

Take first order conditions:

$$\underbrace{(1-\alpha)zK^{\alpha}N^{d,-\alpha}}_{MPN} = w$$

This gives us the firm's optimality condition. For any wage w, the firm is telling you how much labor he is willing to demand.

4 Government: an example

We'll consider a government that only issues lump-sum tax (this assumption will be relaxed in the future). The government has to spend G, i.e. G is exogenous. In a one period model, to balance the budget when the only tax instrument is lump-sum taxes, the government budget constraint becomes:

$$G = T$$

5 Putting it altogether: See lecture notes for definition of competitive equilibrium

Equilibrium in this economy must be characterized by what agents in the economy find optimal (desirable for each agent given what they take as given) and feasible (what is consumed must be produced, expenditure is equal to output)

We can ask one question, how much labor supply is the household willing to provide and how much labor does the firm want to use? From the household's and firm's problem, we can derive labor supply and demand curves that tell us how much the household (firm) is willing to supply (demand) given a wage rate. We know labor services get traded when firms and workers agree on a price for labor, i.e. the labor market clears at equilibrium wage w^* and at this wage w^*, $N^s = N^d$. We can now combine the household and firm's optimality condition

$$\underbrace{\frac{\psi C^{\sigma}}{(h-N)^{\nu}}}_{MRS_{l,c}} = w^* = \underbrace{(1-\alpha)zK^{\alpha}N^{-\alpha}}_{MPN} = -MRT_{l,c}$$

This is an equation that relates how much the household was willing to trade consumption and labor supplied with how much labor the firm demands. Recall that $MPN = -MRT_{l,c}$. This means that how many consumption goods the economy can transform into leisure is given by MPN. Optimality tells us that the rate at which the household is willing to substitute consumption for leisure must be equal to the rate at which the economy can transform consumption into leisure goods.

The above equation with $MRS = -MRT$ has C and N in terms of exogenous variables (z, K) and parameters of the model. We still only have one equation for C and N so we need another equation to solve. Let's recall that the above equation only talks about desirability and not affordability. To get back to affordability, we need to go back to the household's budget constraint:

$$C = w^*N + \pi - T$$

Let's substitute in the fact that the government budget constraint must balance in equilibrium, i.e. $G = T$

$$C = w^*N + \pi - G$$

Now we still have endogenous objects w and π showing up, so let's use what we know about profits from the firm's problem:

$$C = w^*N + zK^{\alpha}N^{1-\alpha} - w^*N - G$$

In equilibrium, labor income must equal to the firm's wage bill. So our equation simplifies to:

$$C = \underbrace{zK^{\alpha}N^{1-\alpha}}_{Y} - G$$

But this just gives back the goods market clearing condition, which says that total spending, $C + G$, must equal total output Y. Hence the two equations we need to characterize and solve for equilibrium are:

$$\frac{\psi C^{\sigma}}{(h-N)^{\nu}} = (1-\alpha)zK^{\alpha}N^{-\alpha}$$

$$C = zK^{\alpha}N^{1-\alpha} - G$$

Now we have two equations and two unknowns, we can solve for optimal C^* and N^*. Once we have C^* and N^*, and we know $T = G$ in equilibrium, we can work backwards and solve for equilibrium w^*, Y^*, π^*.