INTERMEDIATE MICROECONOMICS

CONSUMER CHOICE

SPRING 2019, PROFESSOR ANH NGUYEN

Consumer Choice

- We will study individual consumer choices where
  1. There are two goods
  2. The consumer chooses how much of each good to consume
  3. The price of each good is fixed
  4. The consumer faces a budget constraint

1. Budget Constraint

Budget Constraint

- Objectives:
  1. Graphically represent a budget constraint that a consumer faces
  2. Understand what the slope of a budget constraint means
- Reading: pp. 25-41

Budget Constraint

People make the best choices they can *given their circumstances.*

"Constraints"

- Current Income
- Assets (from savings)
- Skills
- Time
- Future Income (for borrowing)

Budget Constraint

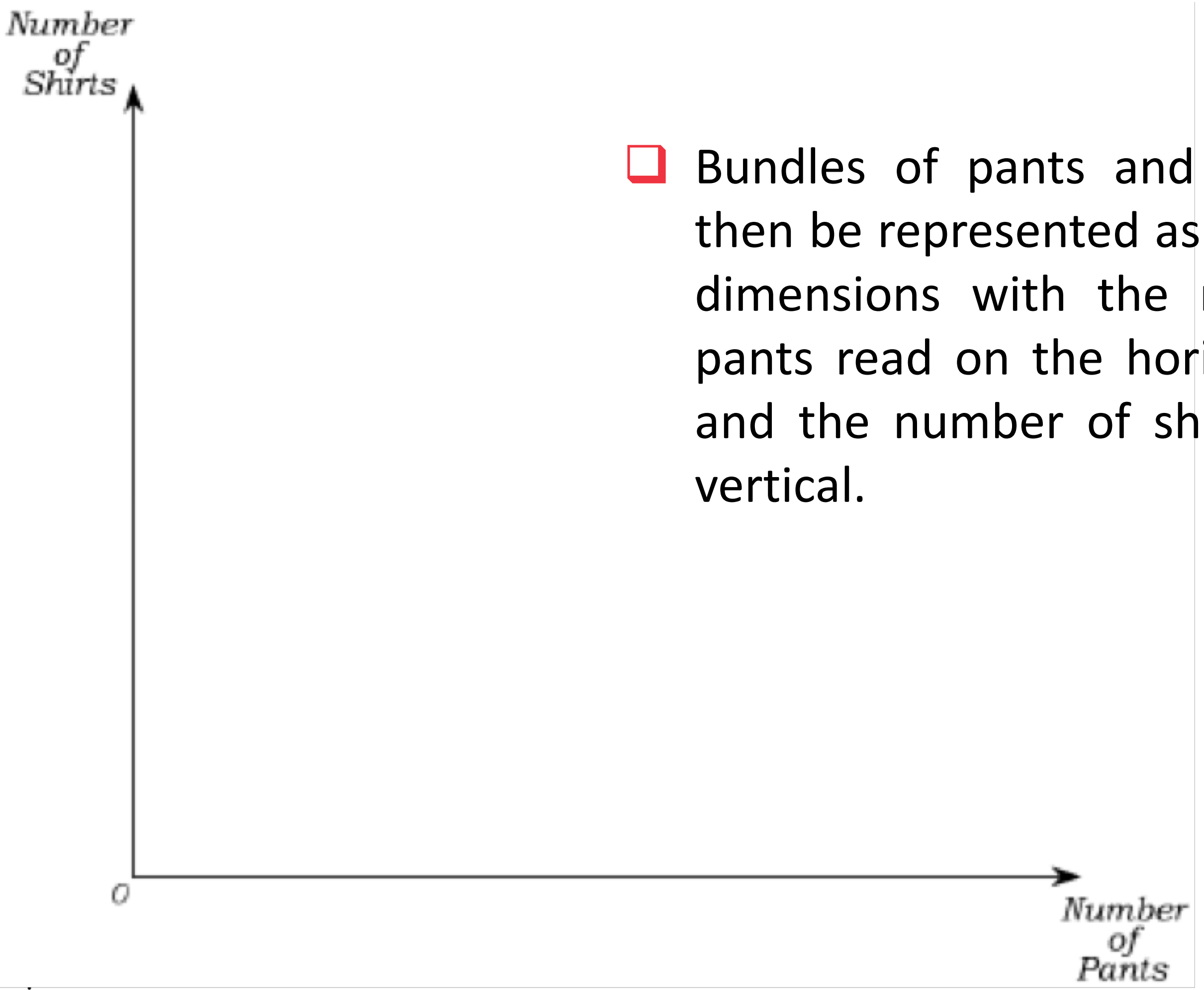
- Consumers are "small", have no power over price.
  - In other words, consumers are **price-takers**.

- We begin by assuming that decisions leading to currently available income are in the past.
  - The choice that remains is how to spend this available income.
  - Such income that the consumer takes as given is called **exogenous**.

Budget Constraint

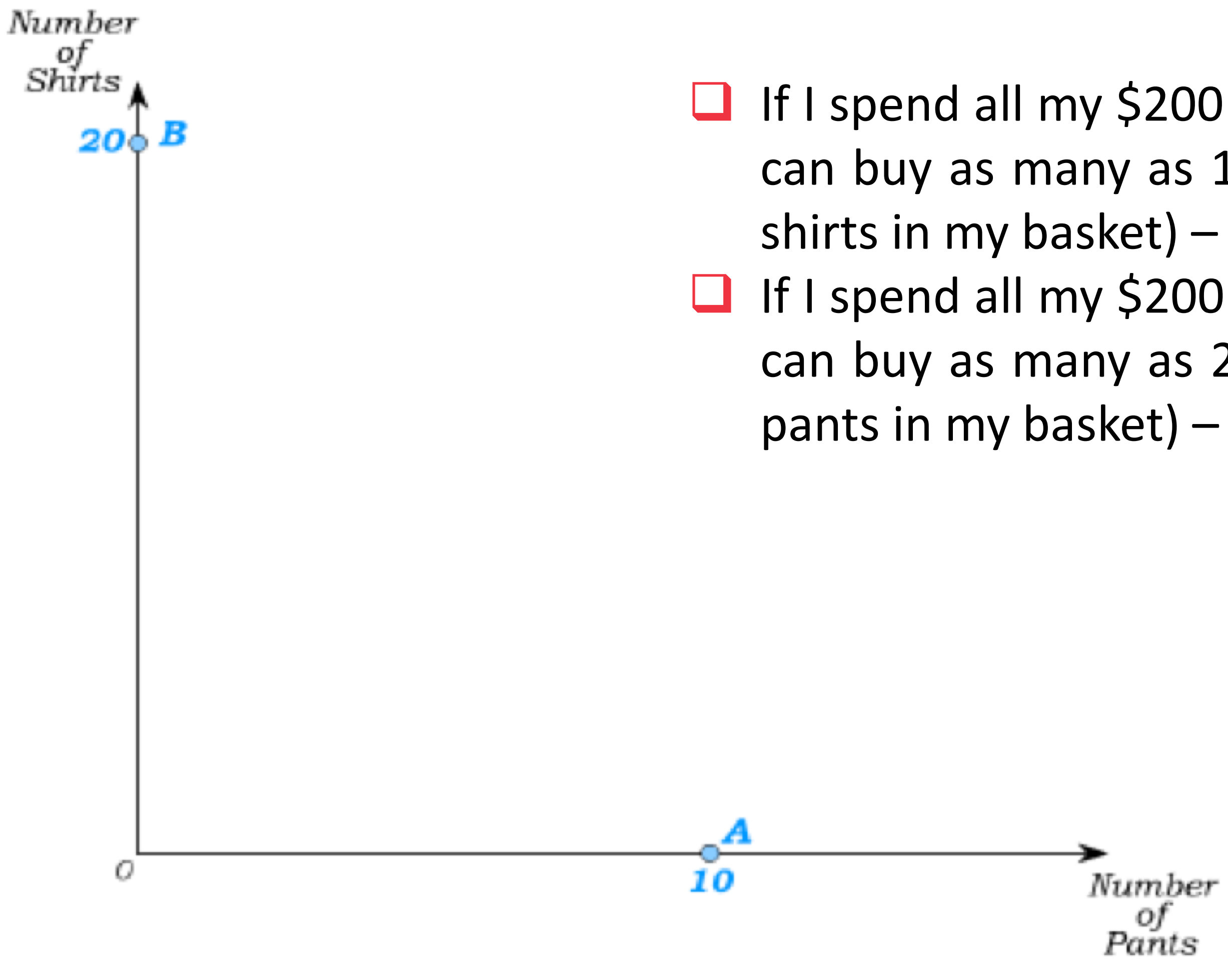
- Example:
  - Price of a pair of pants = $20
  - Price of a shirt = $10
  - Suppose I have $200 to spend on pants and shirts.
- Given those restrictions, I can buy any **bundle** of pants and shirts that I want.
- The set of all “bundles” of pants and shirts that I could buy is then called my **choice set**.

Budget Constraint



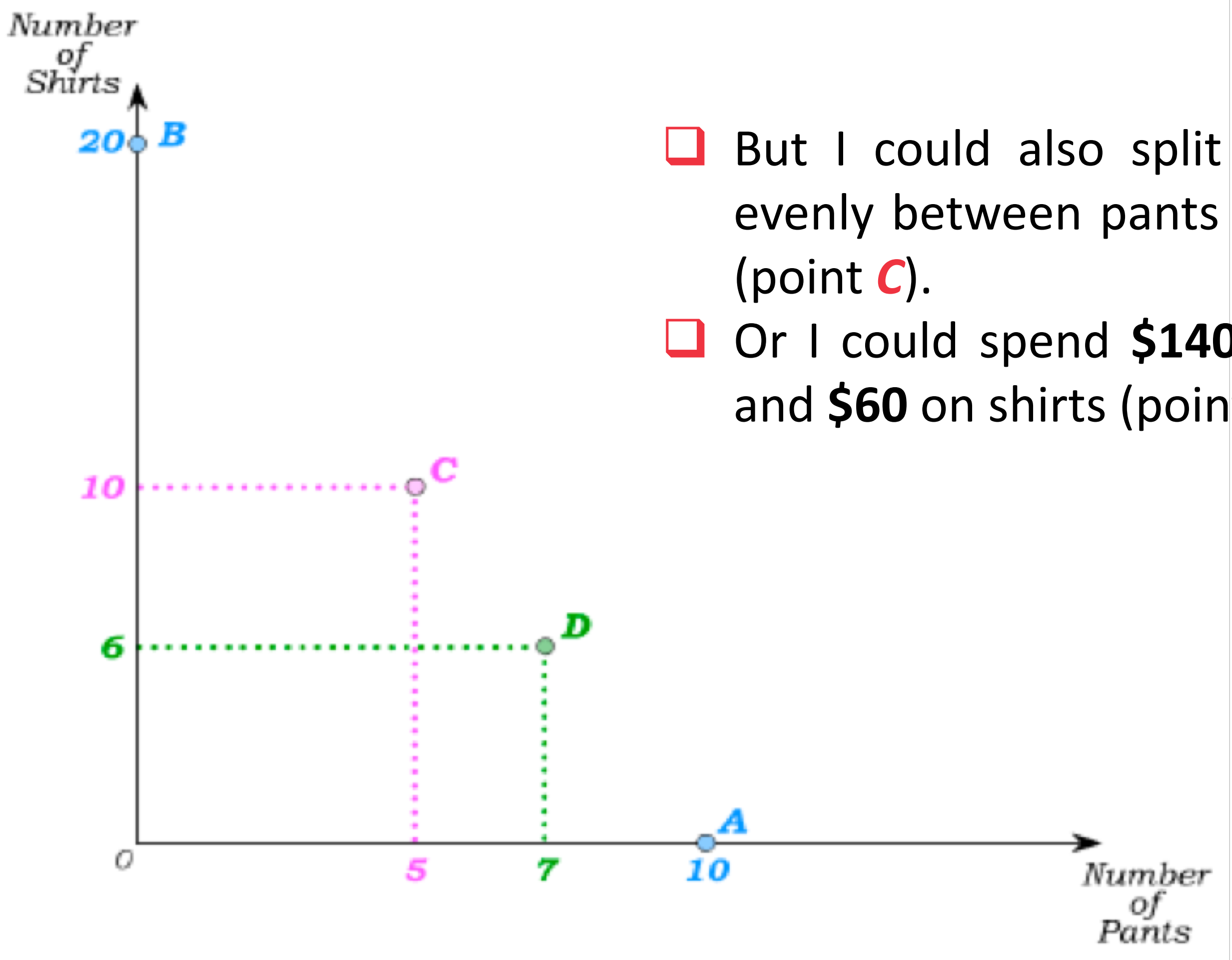
- Bundles of pants and shirts can then be represented as points in 2 dimensions with the number of pants read on the horizontal axis and the number of shirts on the vertical.

Budget Constraint



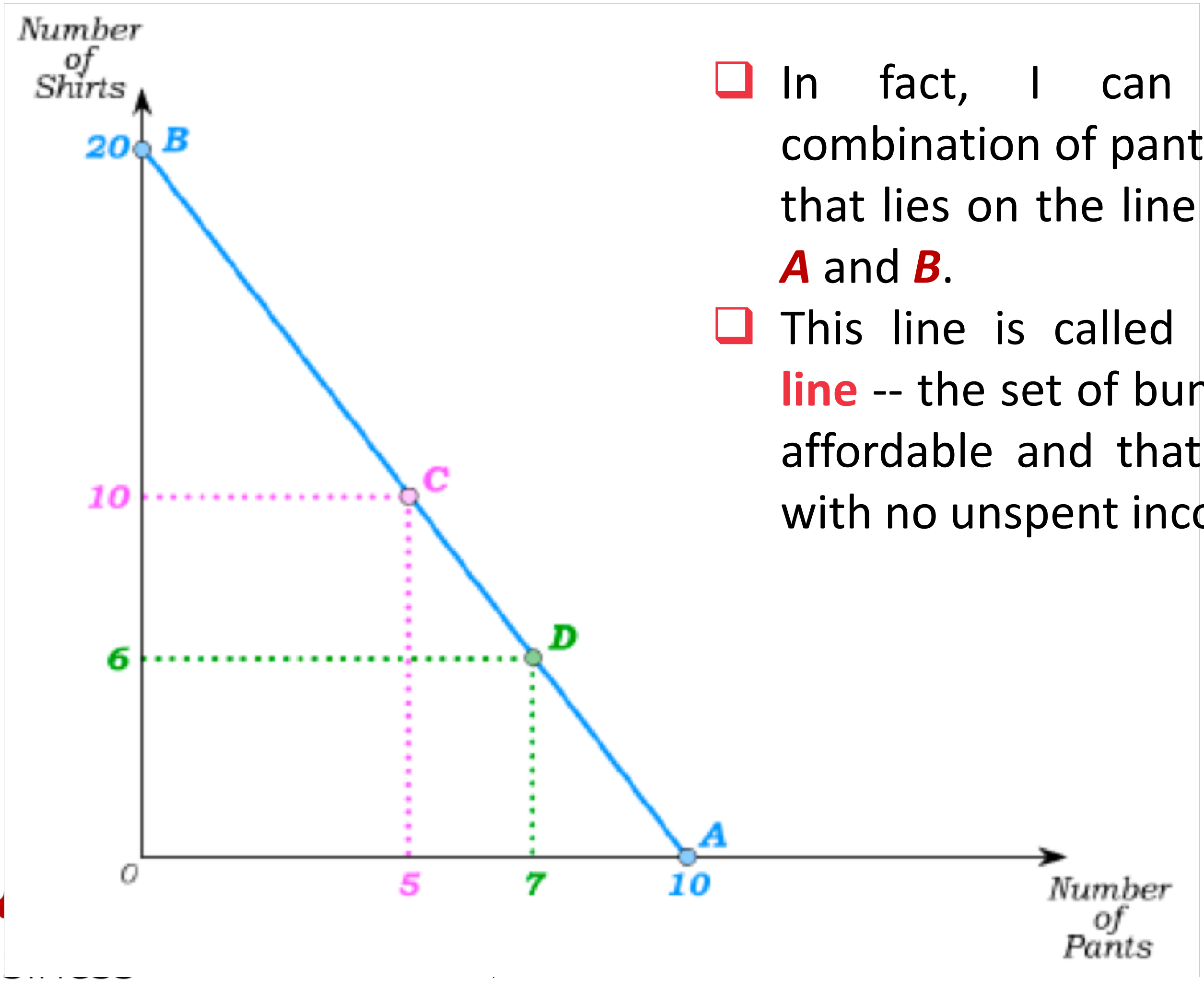
- If I spend all my $200 on pants, I can buy as many as 10 (with no shirts in my basket) – point ***A***.
- If I spend all my $200 on shirts, I can buy as many as 20 (with no pants in my basket) – point ***B***.

Budget Constraint



- But I could also split the **$200** evenly between pants and shirts (point ***C***).
- Or I could spend **$140** on pants and **$60** on shirts (point ***D***).

Budget Constraint



- In fact, I can buy any combination of pants and shirts that lies on the line connecting ***A*** and ***B***.
- This line is called my **budget line** -- the set of bundles that is affordable and that leaves me with no unspent income.

Budget Constraint

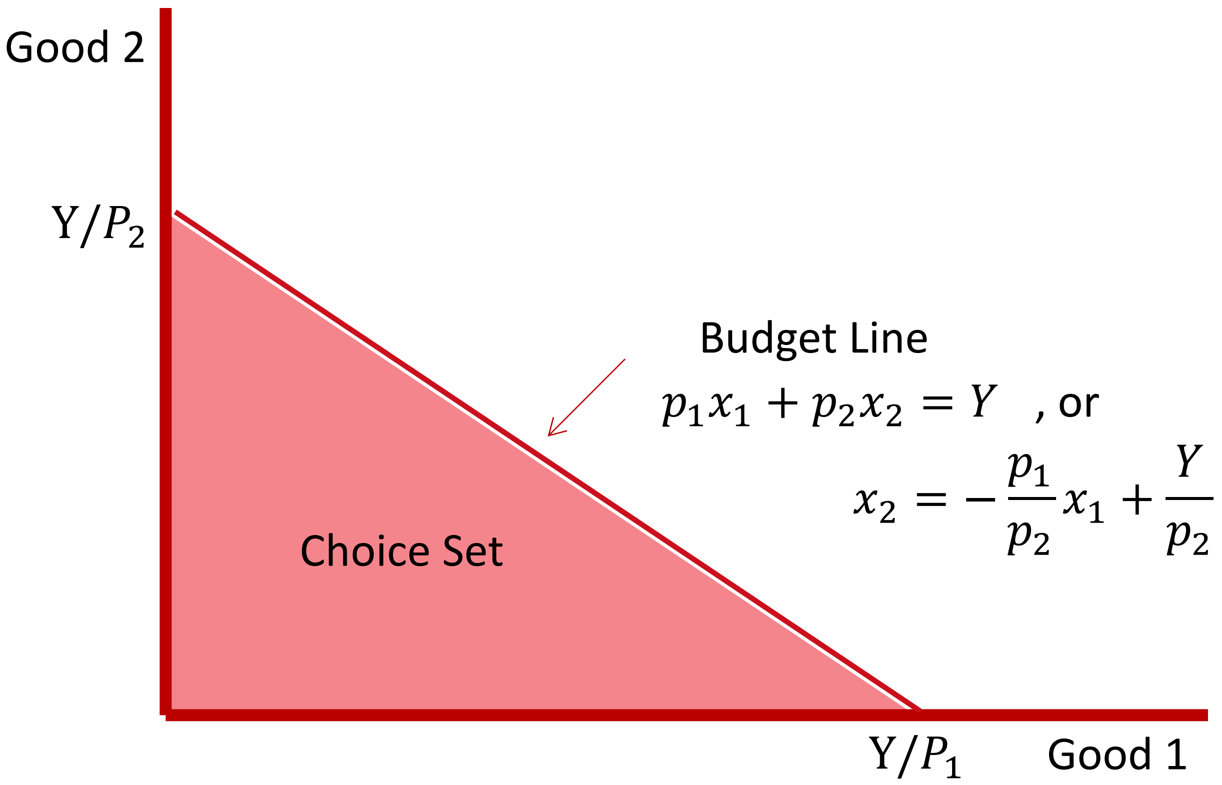
- Example:
  - Price of a pair of pants (p_1) = \$20
  - Price of a shirt (p_2) = \$10
  - Income (Y) = \$200
- The **choice set** is the set of bundles that satisfy the **budget constraint** that the total spending on goods cannot exceed the income

$$p_1 x_1 + p_2 x_2 \leq Y$$

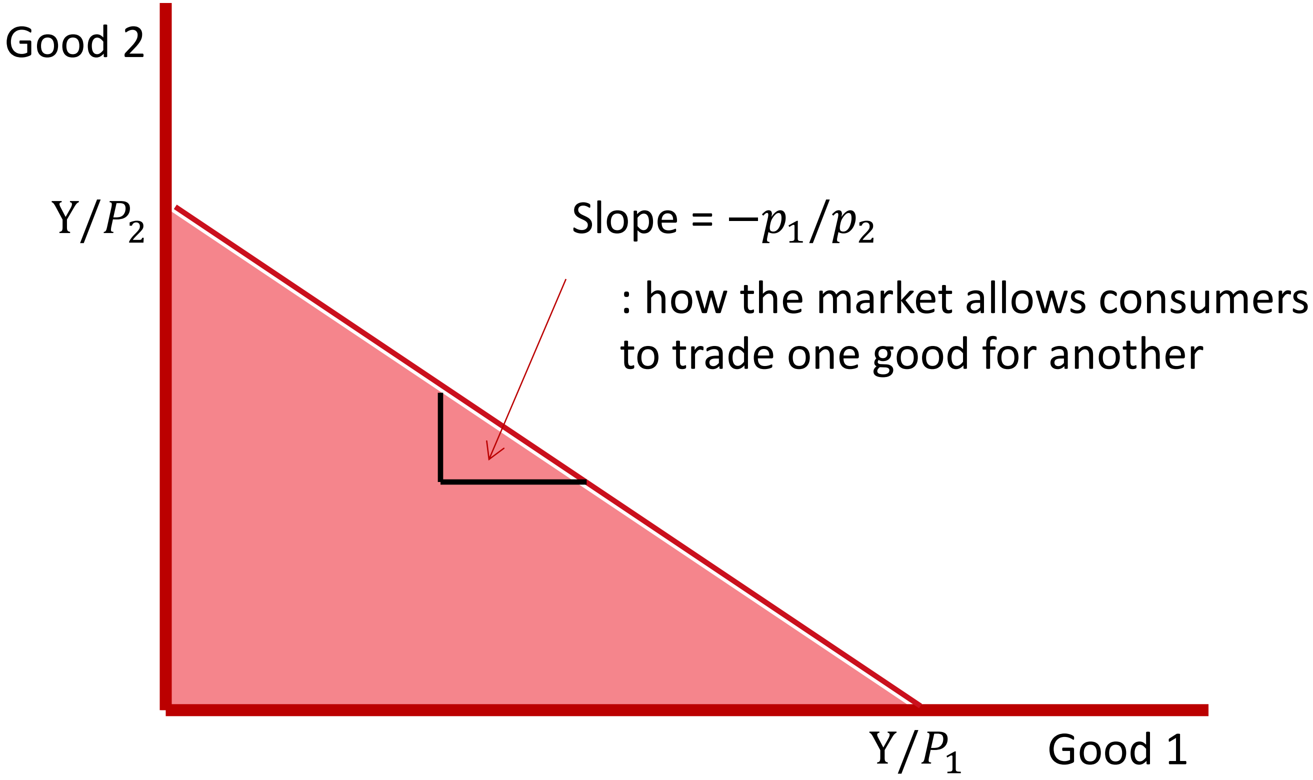
- The **budget line** is the set of bundles that *exactly* satisfy the budget constraint

$$p_1 x_1 + p_2 x_2 = Y$$

Budget Constraint

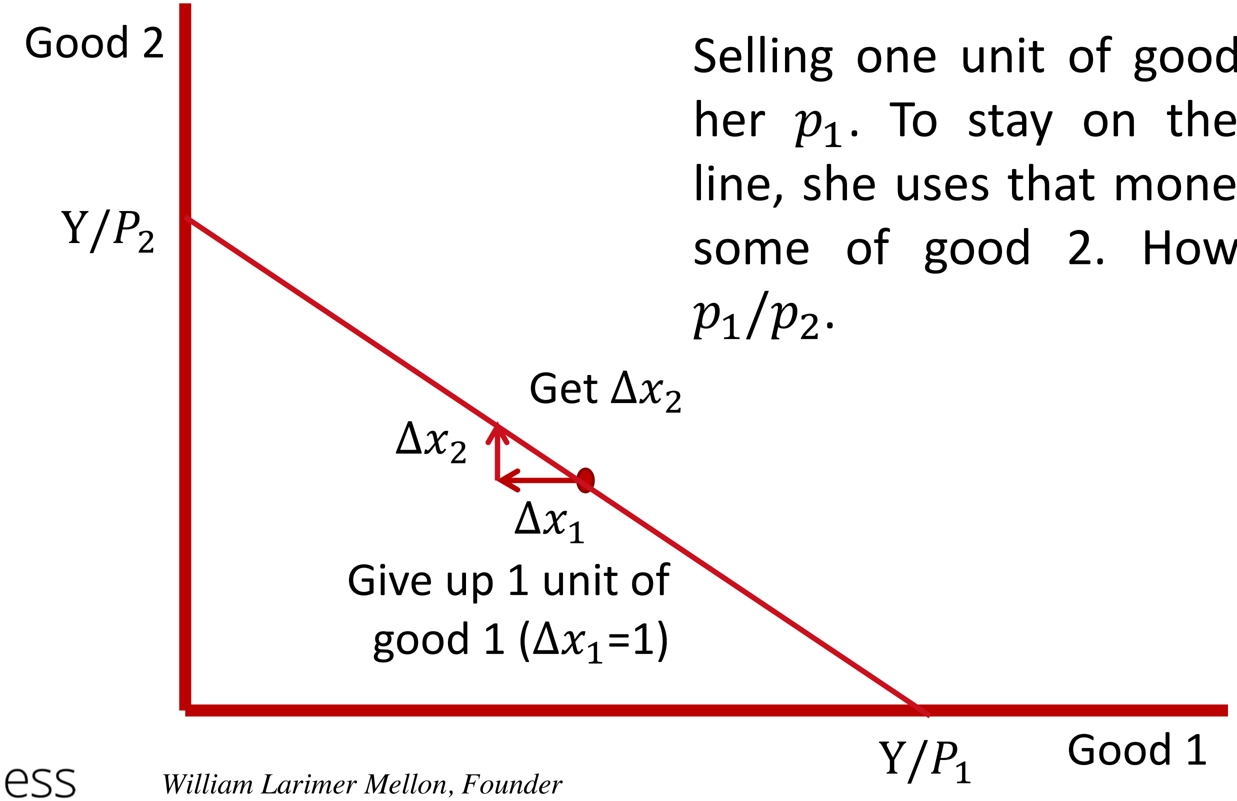


Slope of Budget Line



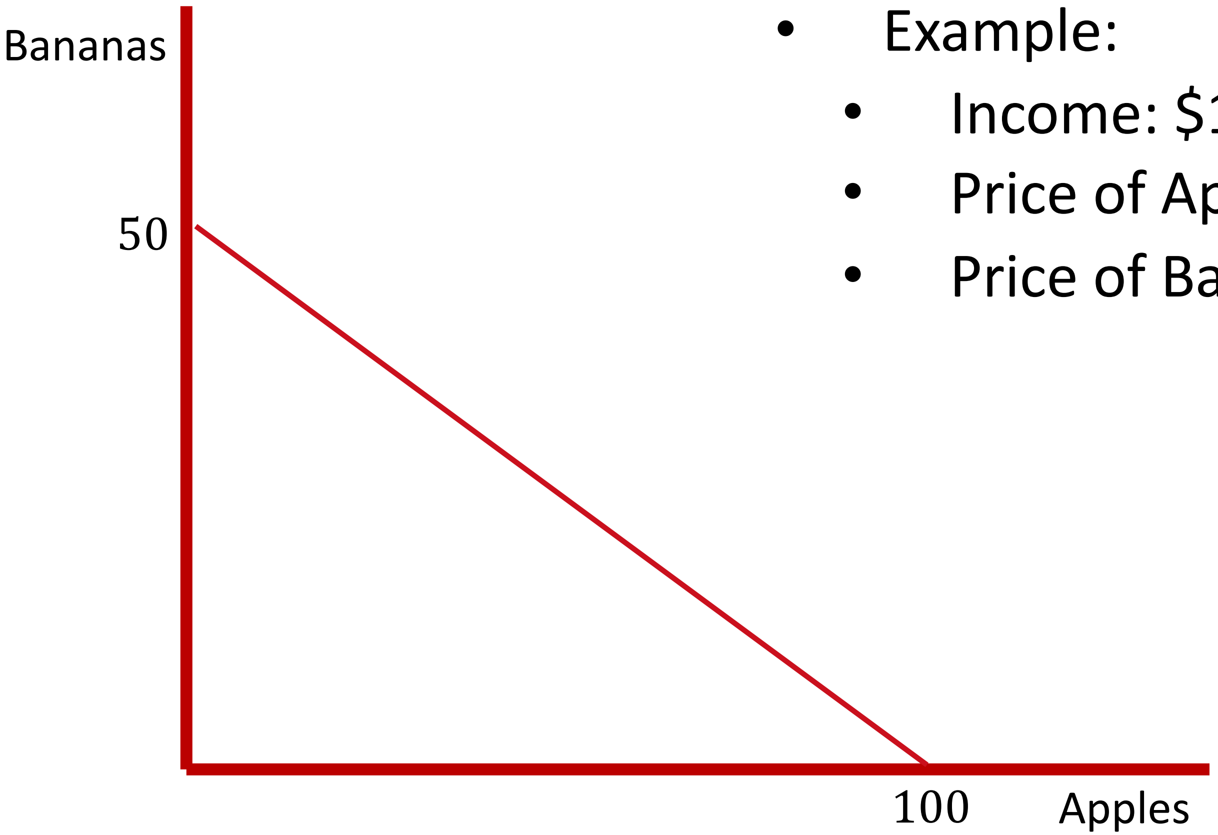
Slope of Budget Line

- How much of good 2 does a consumer need to *get* in exchange for one unit of good 1 *in order to stay on the budget line*?



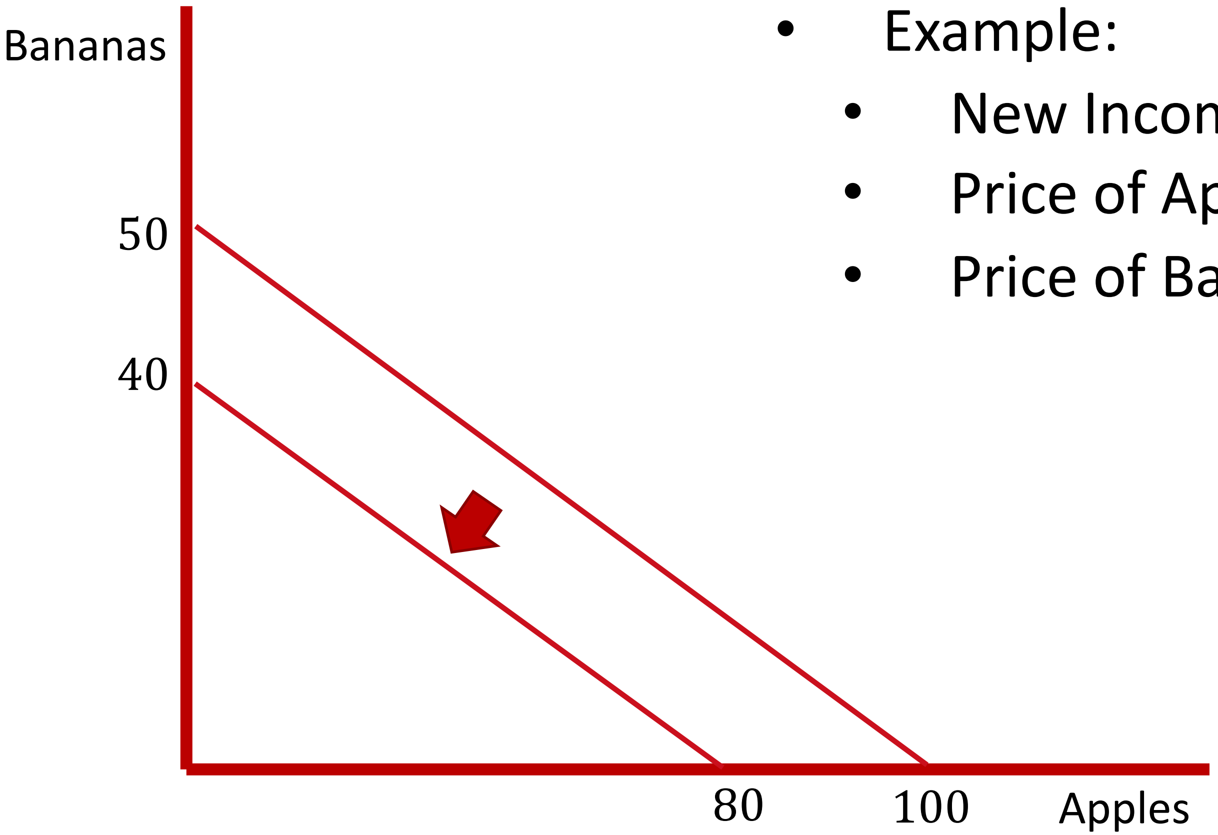
Selling one unit of good 1 gives her p_1. To stay on the budget line, she uses that money to buy some of good 2. How much? p_1/p_2.

Budget Line: Example



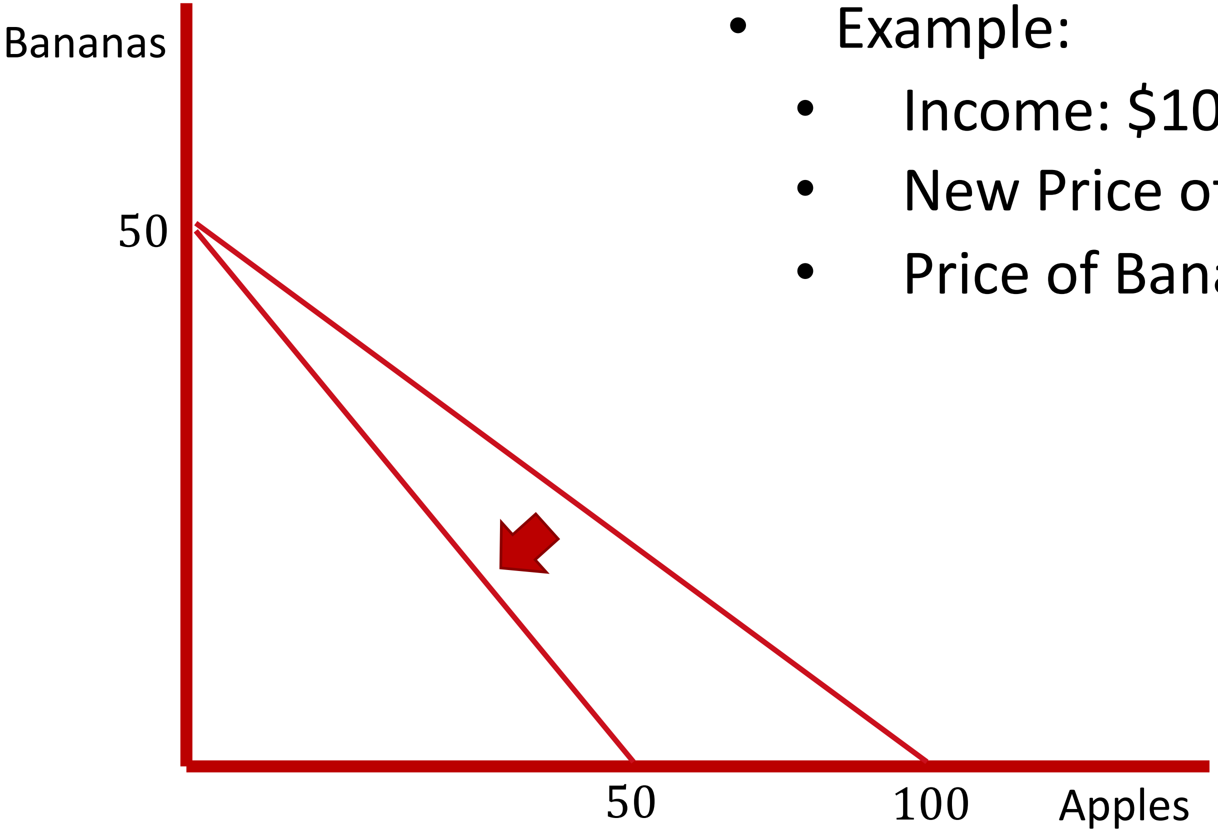
- Example:
  - Income: $100
  - Price of Apples: $1
  - Price of Banana: $2

Budget Line: Example



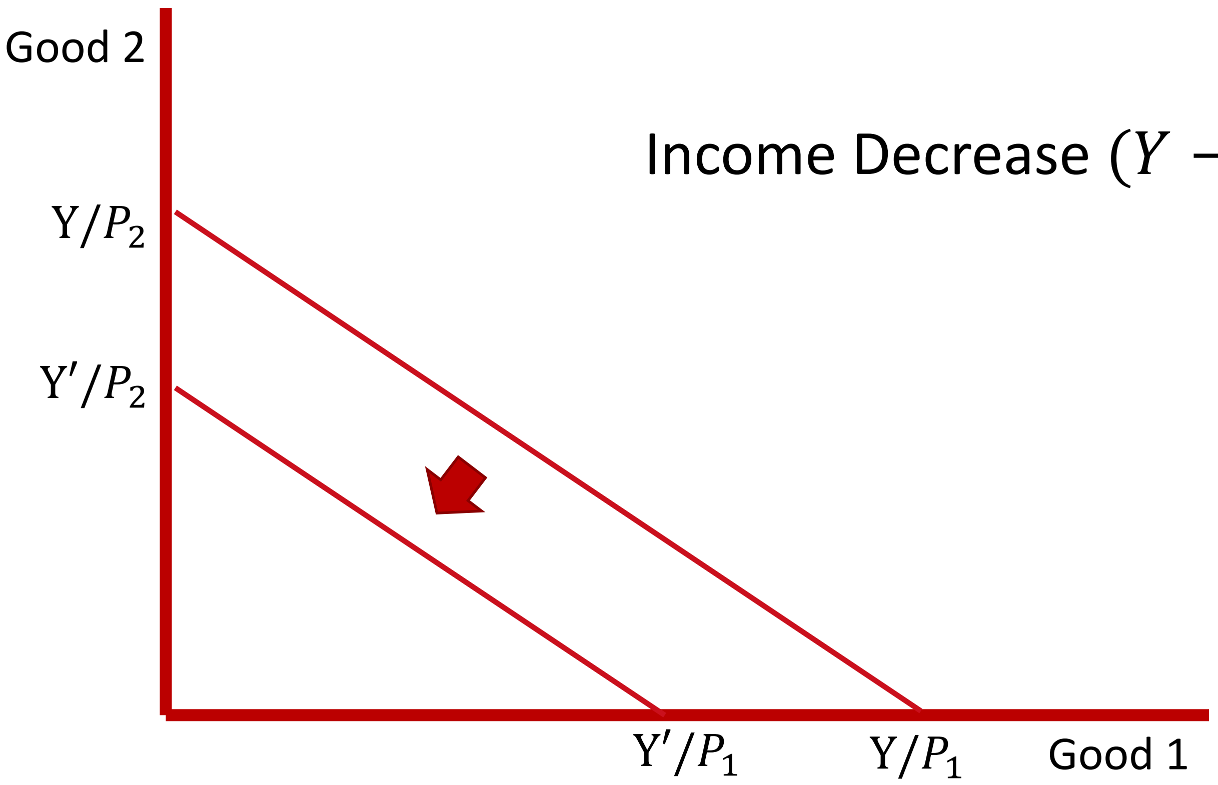
- Example:
  - New Income: $80
  - Price of Apples: $1
  - Price of Banana: $2

Budget Line: Example

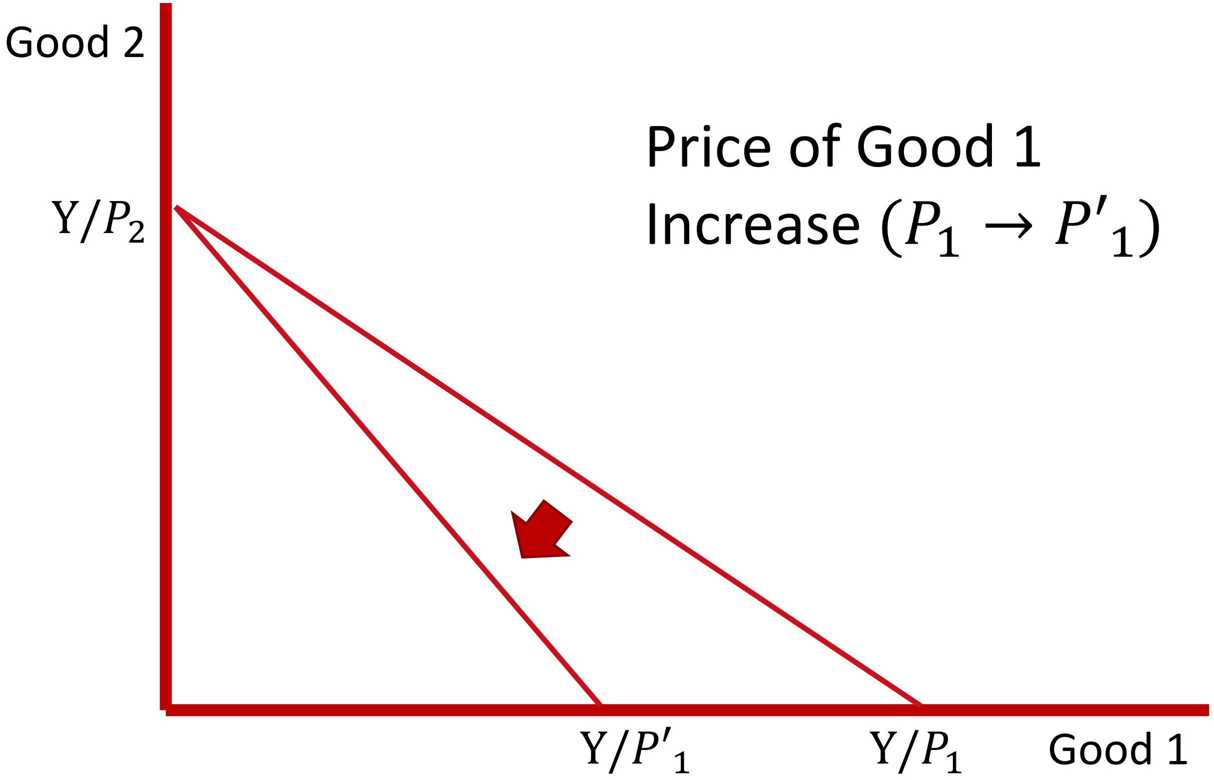


- Example:
  - Income: $100
  - New Price of Apples: $2
  - Price of Banana: $2

Shifts of Budget Line: Income Change



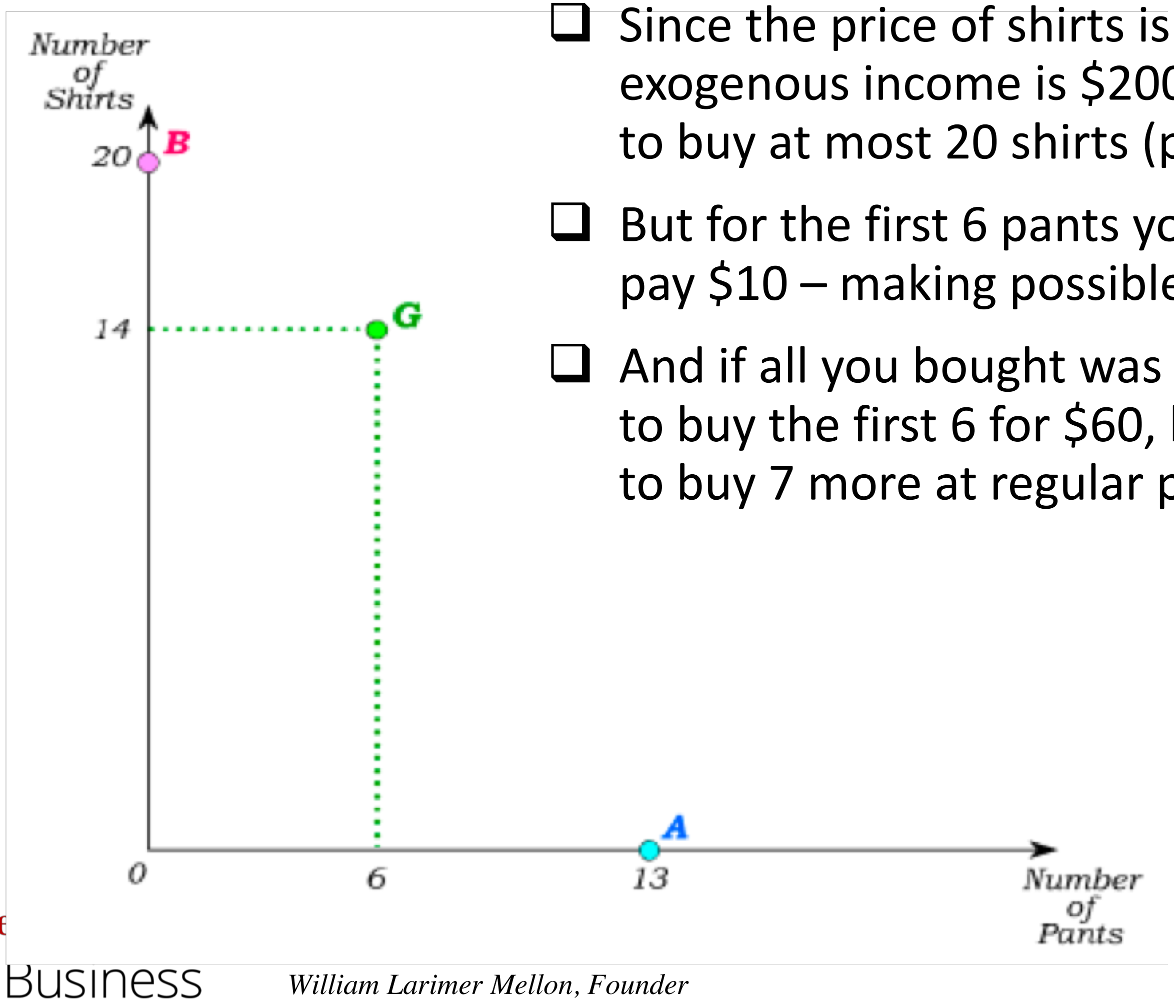
Shifts of Budget Line: Price Change



Kinky Budgets

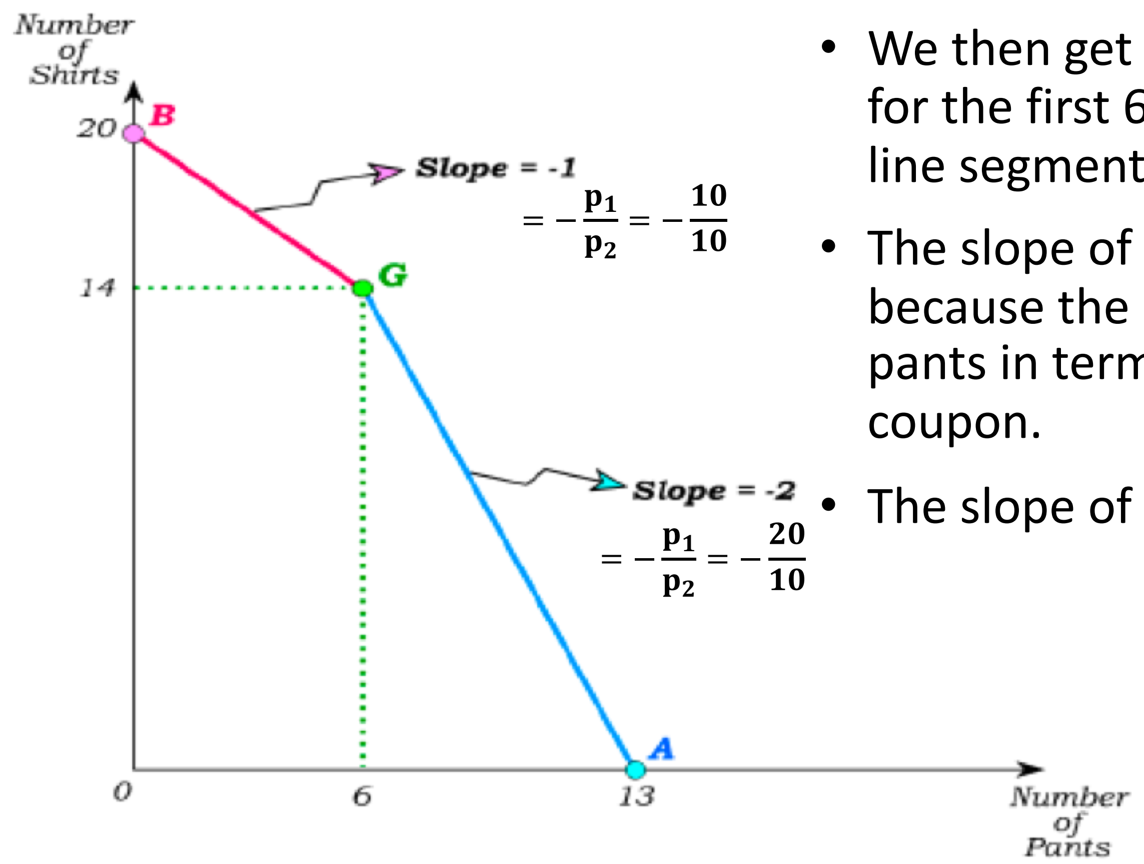
- Budget lines are not always straight lines.
- Suppose, for instance, you held a coupon that gives you 50% off the first 6 pair of pants (when the regular price of pants is $20.)

Kinky Budgets



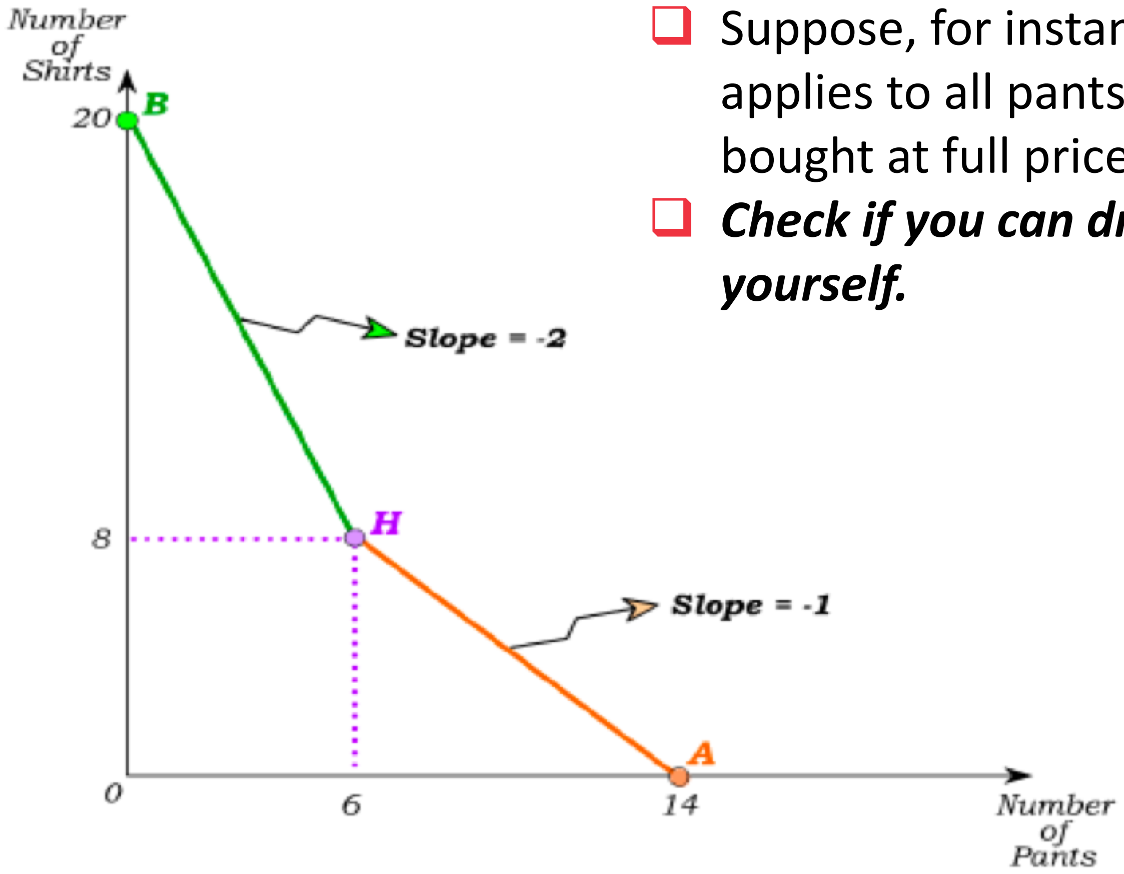
- Since the price of shirts is \$10 and your exogenous income is \$200, you'd still be able to buy at most 20 shirts (point ***B***).
- But for the first 6 pants you buy, you now only pay \$10 – making possible the bundle ***G***.
- And if all you bought was pants, you'd be able to buy the first 6 for \$60, leaving you with \$140 to buy 7 more at regular price (point ***A***).

Kinky Budgets



- We then get the red line segment for the first 6 pants and the blue line segment for the remainder.
- The slope of the red portion is –1, because the opportunity cost of pants in terms of shirts is 1 with the coupon.
- The slope of the blue portion is –2.

Kinky Budgets



- Suppose, for instance, that a 50% discount applies to all pants after the first 6 are bought at full price.
- ***Check if you can draw this budget set by yourself.***

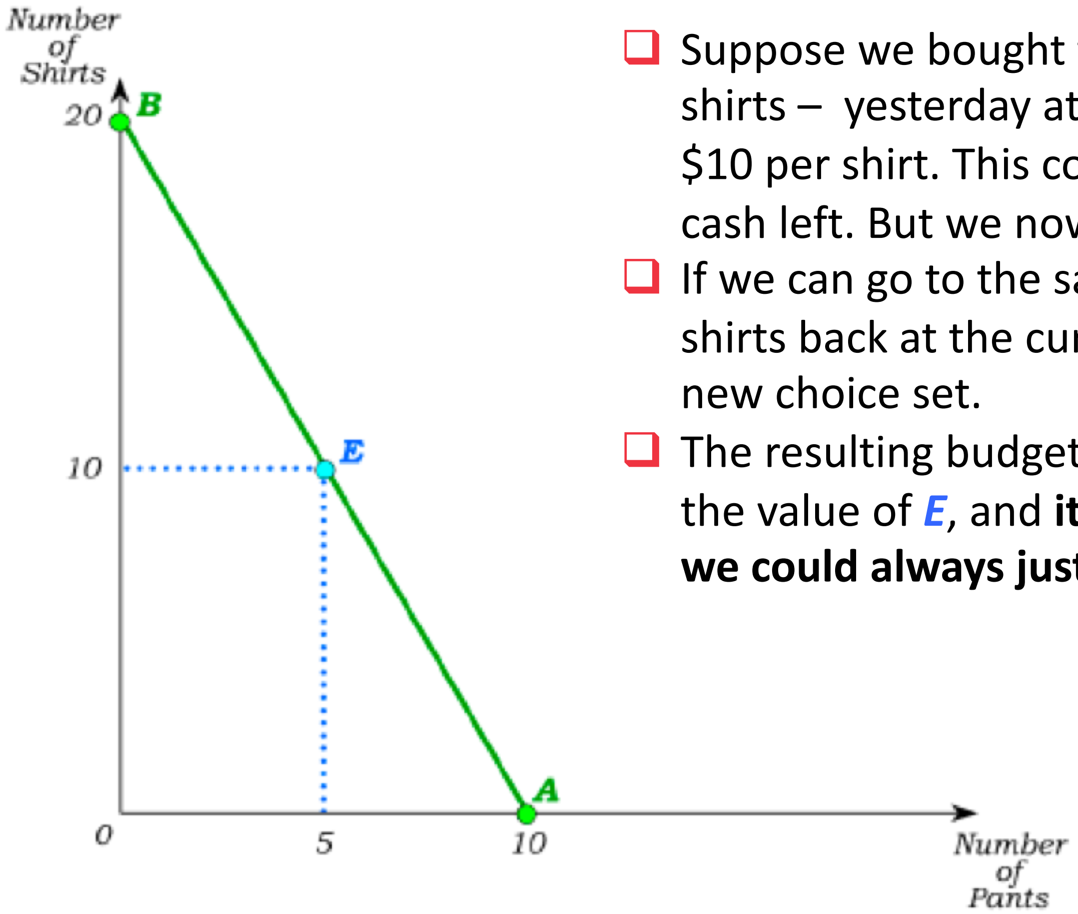
Composite Goods

- There are more than 2 goods to consider.
- We thus often use the trick of aggregating all goods except for one into a **composite good**.
- A composite good is an index of “dollars worth of all other goods”, with **the price of a composite good therefore equal to 1**.
- When a composite good is put on the vertical axis, the slope of the budget is then simply (minus) the price of the good on the horizontal.

Budgets with Endowments

- So far, we have dealt with choice sets that arise from ***exogenous*** income that is simply a dollar amount the consumer takes as given.
- Choice sets may also arise from a consumer's ability to sell something she owns. That "something" is called an **endowment** – and its value depends on the prices at which the "something" can be sold.
- We call the income that arises from a consumer's decision to sell some or all of her endowment ***endogenous***.

Budgets with Endowments



- Suppose we bought the bundle ***E*** – 5 pants an 10 shirts – yesterday at \$20 per pair of pants and \$10 per shirt. This cost us \$200 and we have no cash left. But we now ***own*** the bundle ***E***.
- If we can go to the same store and sell pants and shirts back at the current prices, we can create a new choice set.
- The resulting budget line is as if we had \$200 – the value of ***E***, and **it must pass through *E* since we could always just consume *E*.**

Budgets with Endowments

- But now suppose that the price of pants has dropped from the initial \$20 to \$10 when we return to the store.
- If we now sell our 5 pants in our bundle ***E***, we only collect \$50 in store credit and can therefore only buy 5 additional shirts – which gets us bundle ***B'***. Similarly, if we sell our 10 shirts from ***E*** instead, we get bundle ***A'***.
- **The new choice set must pass through our endowment bundle *E* because it is always possible to simply consume that bundle.**

2. Preferences and Utility

Preferences and Utility

- Objectives:
  1. Understand assumptions about consumer preferences
  2. *Graphically* represent preferences over two goods using ***indifference curves***
  3. Numerically represent preferences using ***utility functions*** and ***marginal rate of substitution***

- Reading: pp. 75-100 and pp. 113-134.

Preference Relations

- Consider any two bundles, X and Y
  - $X = (x_1, x_2)$ and $Y = (y_1, y_2)$.
  - x_1 is the quantity of good 1 of bundle X
  - x_2 is the quantity of good 2 of bundle X

- Preference relations:
  - $X \succ Y$: Consumer strictly prefers X over Y
  - $X \sim Y$: Consumer is indifferent between X and Y
  - $X \succsim Y$: Consumer weakly prefers X over Y

Assumptions on Preferences (1)

- ***Completeness***: For all consumption bundles, a consumer can rank them so that any of the followings is true: $X \succ Y$, $X \sim Y$, or $X \succsim Y$
  - When given a choice between two alternative bundles, the consumer can always make a comparison

Assumptions on Preferences (2)

- ***Transitivity***: If
  1. If $X \succ Y$ and $\mathrm{Y} \succ Z$, then $\mathrm{X} \succ Z$
  2. If $X \sim Y$ and $\mathrm{Y} \sim Z$, then $\mathrm{X} \sim Z$
  3. If $X \succ Y$ and $\mathrm{Y} \sim Z$, then $\mathrm{X} \succ Z$
  4. If $X \sim Y$ and $\mathrm{Y} \succ Z$, then $\mathrm{X} \succ Z$

- This assumption rules out irrational preference cycles

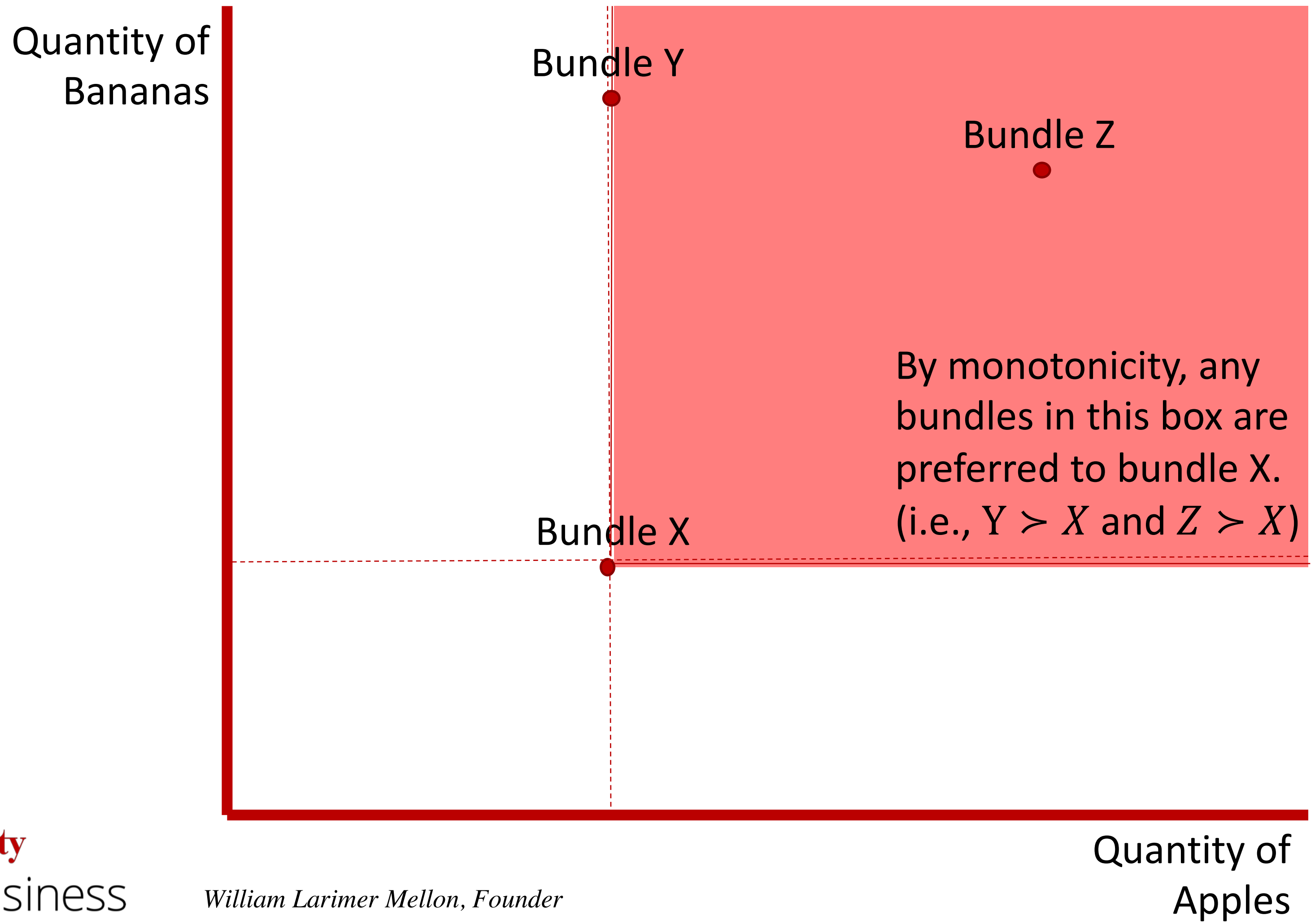
Assumptions on Preferences (3)

- ***Monotonicity***: If
  1. X has more of one good (or both) than Y does, and
  2. X has at least as much of both goods as Y does,

  then $X \succ Y$

- Consumers prefer more of each good to less

Assumptions on Preferences (3)



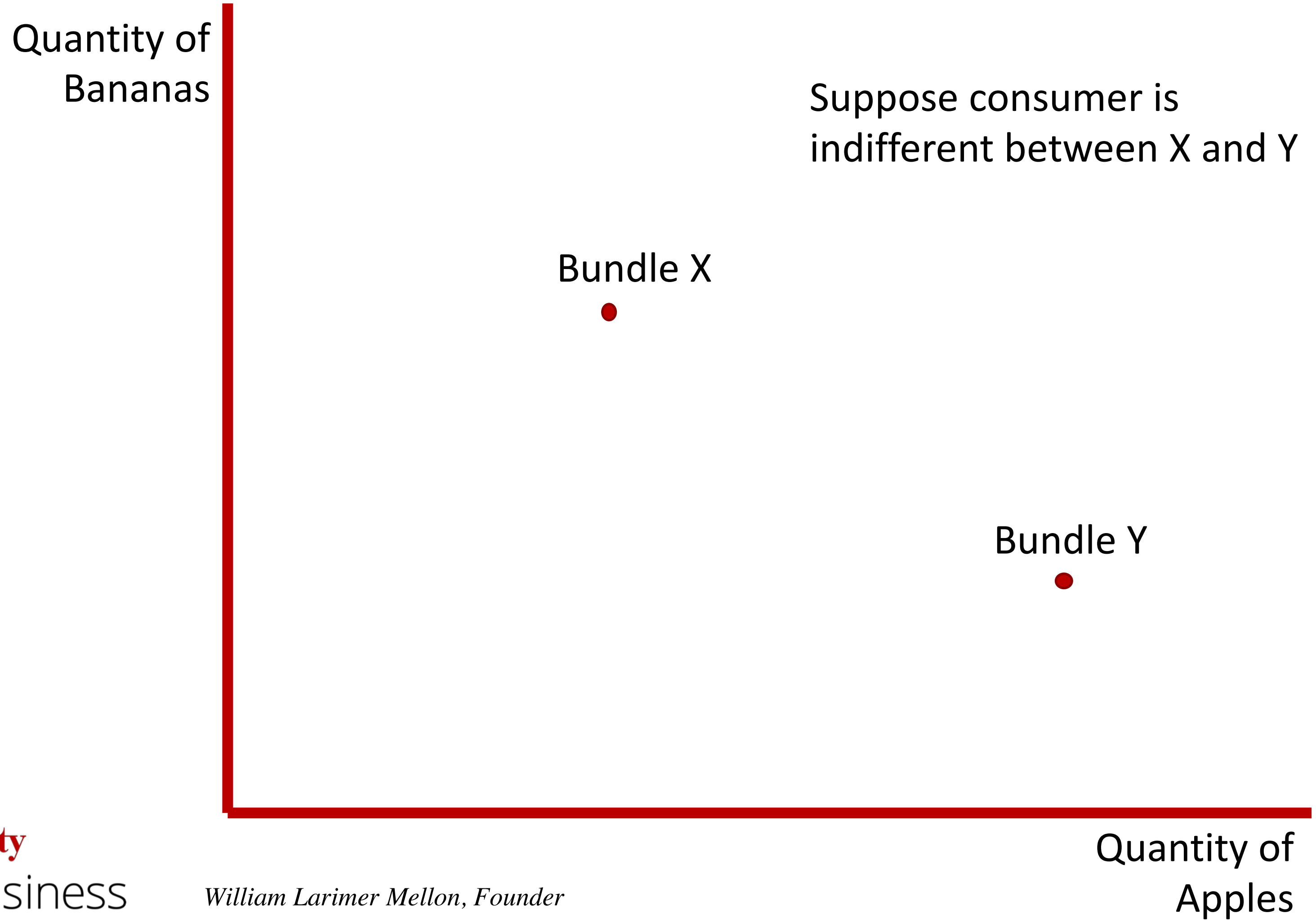
Assumptions on Preferences (4)

- ***Convexity***: If $X \sim Y$, then consumers strictly prefer any weighted average of X and Y to X.
  - If $X \sim Y$, then $aX + (1-a)Y \succ X$ for any $0 < a < 1$.

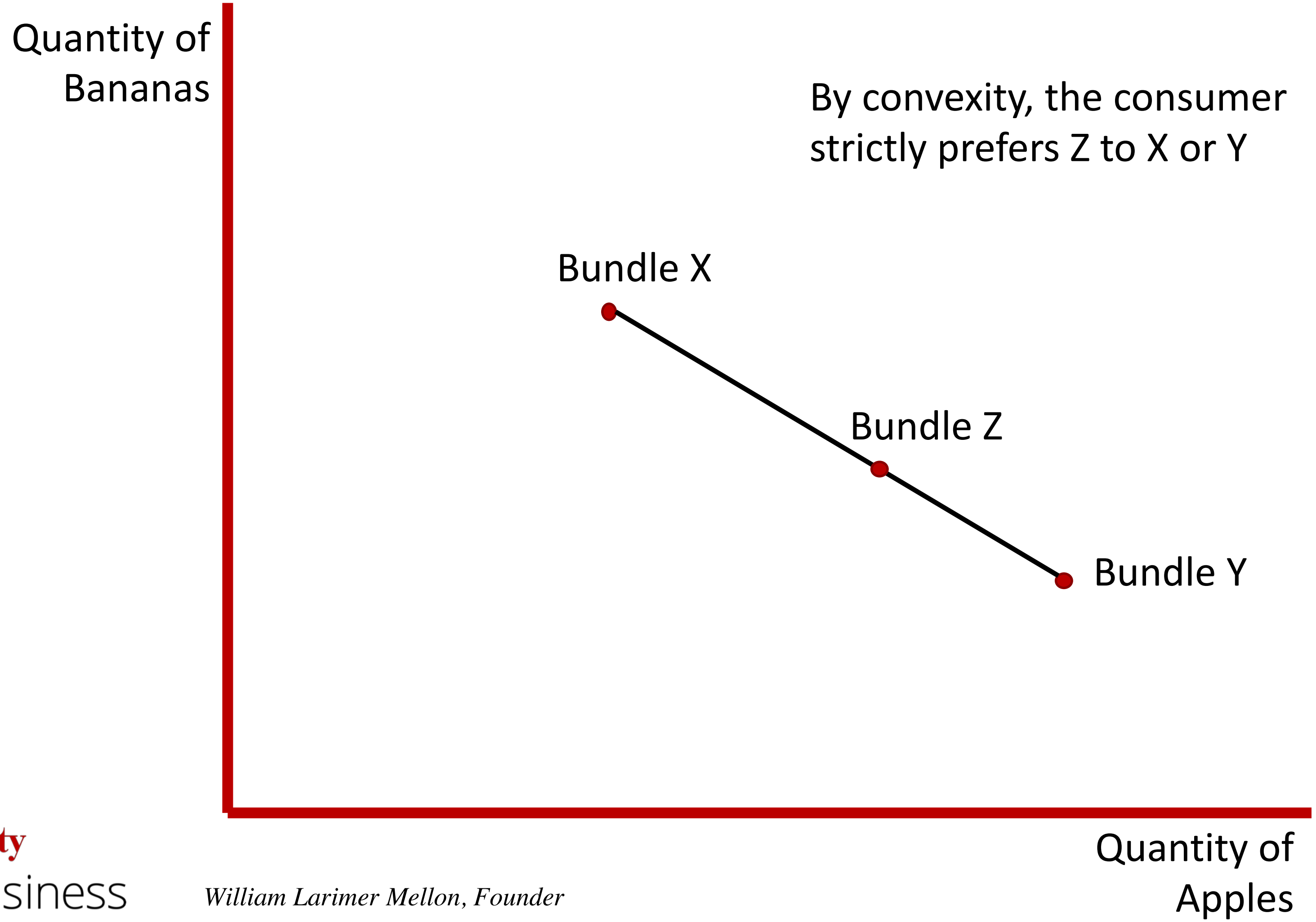
- Consumers prefer averages to extremes

Assumptions on Preferences (4)

Suppose consumer is indifferent between X and Y



Assumptions on Preferences (4)

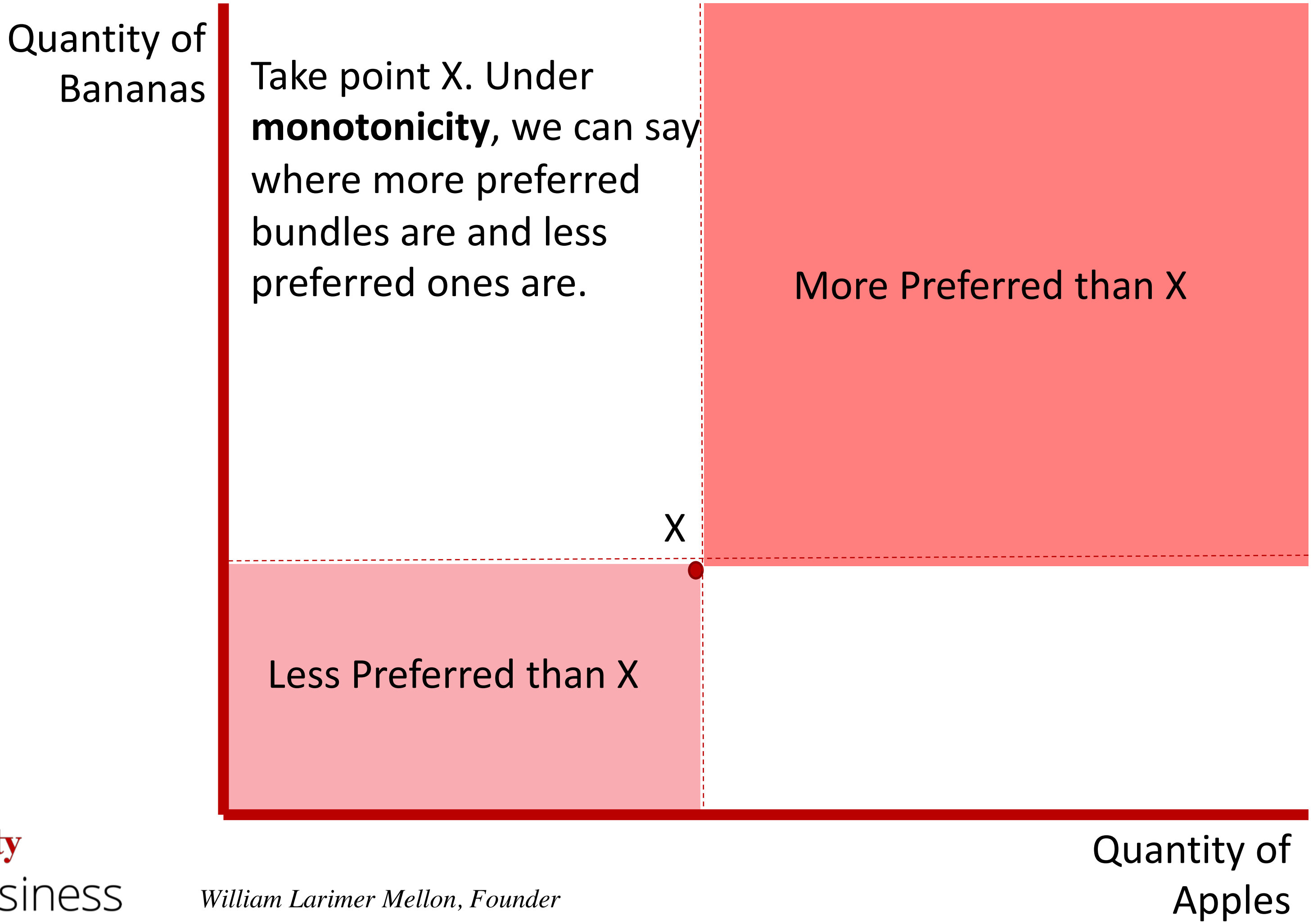


Indifference Curve (IC)

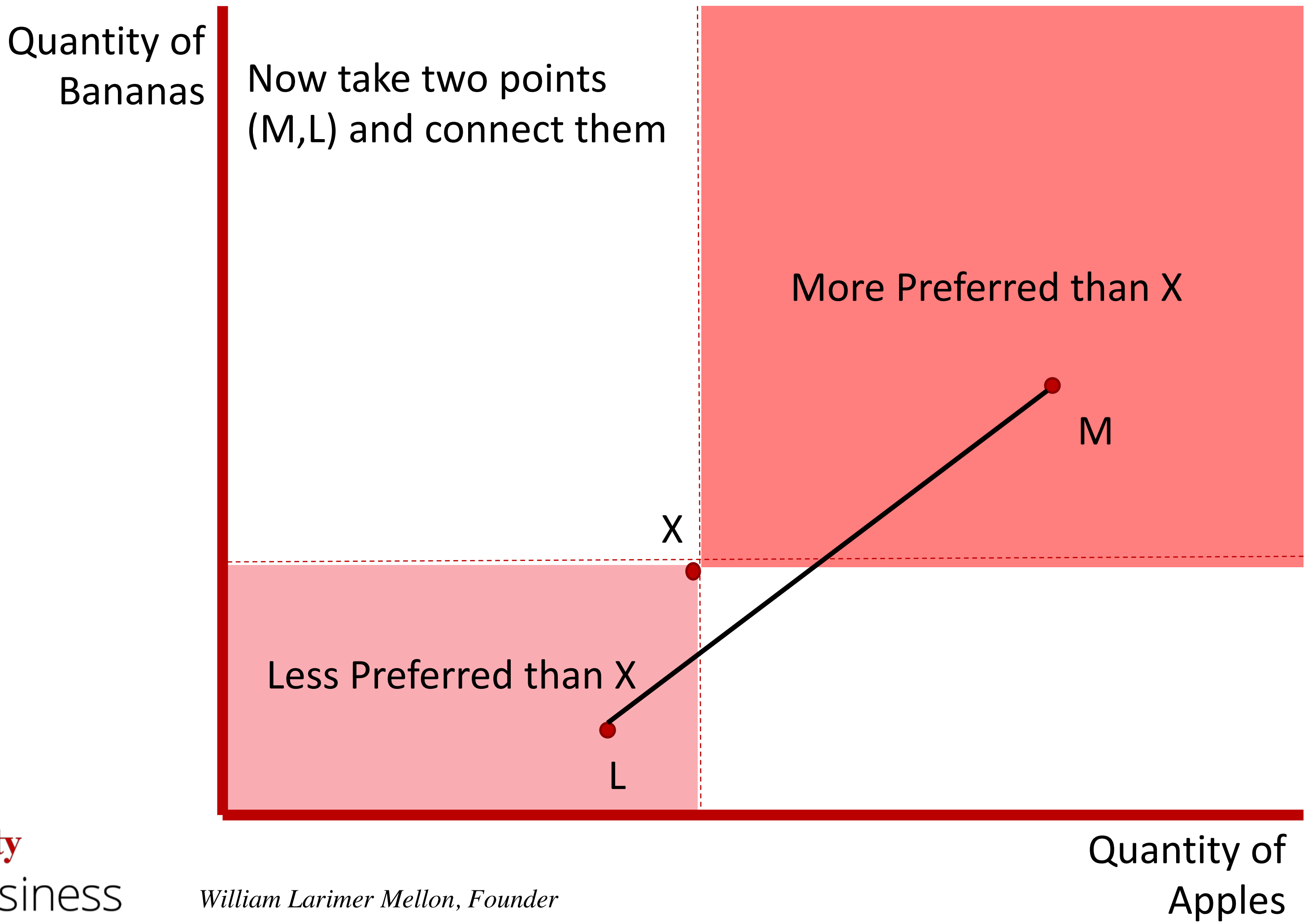
- ***Indifference Curve*: A set of equally desirable consumption bundles**

- If X and Y are on the same indifference curve, then $X \sim Y$

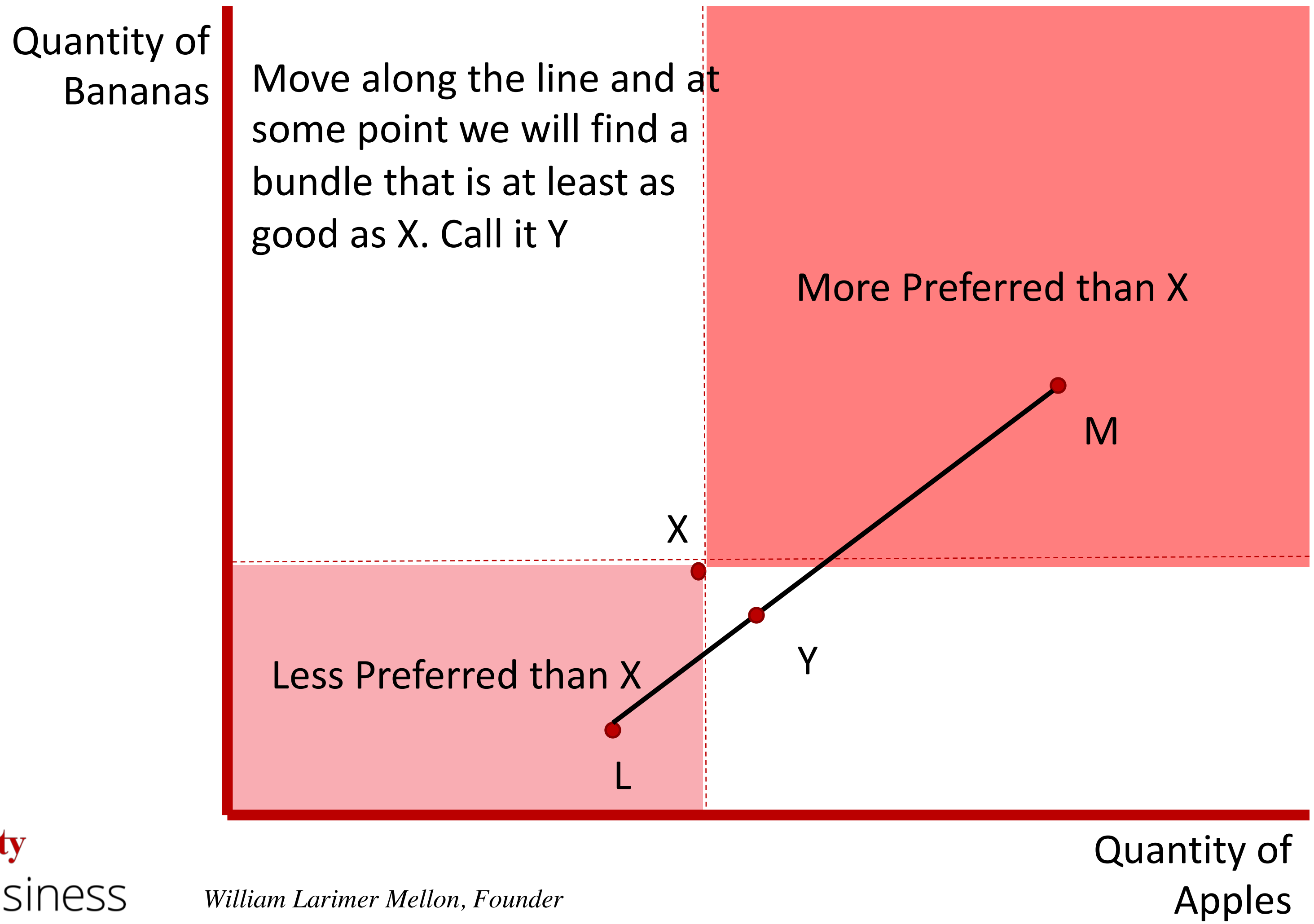
Indifference Curve



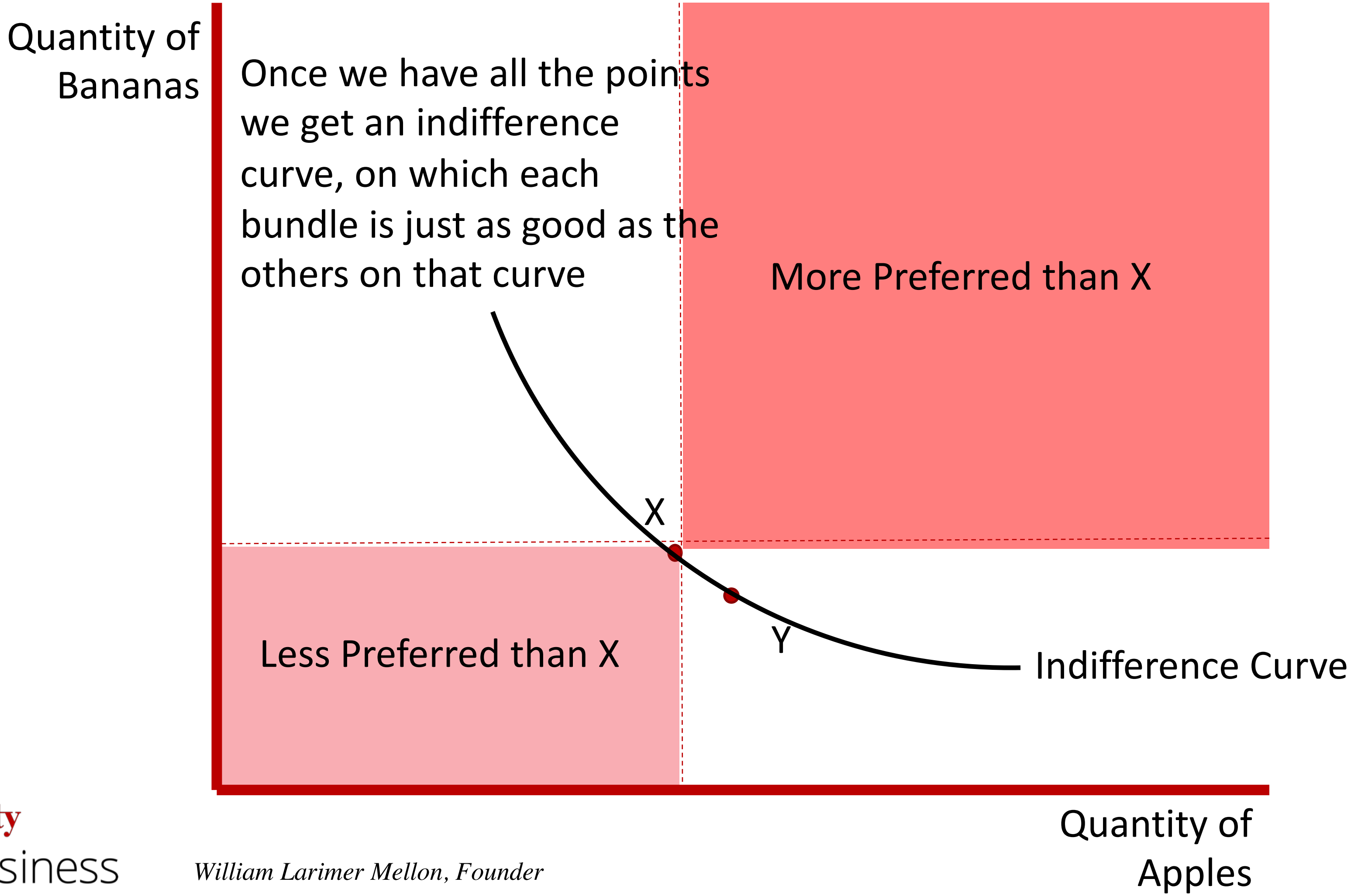
Indifference Curve



Indifference Curve

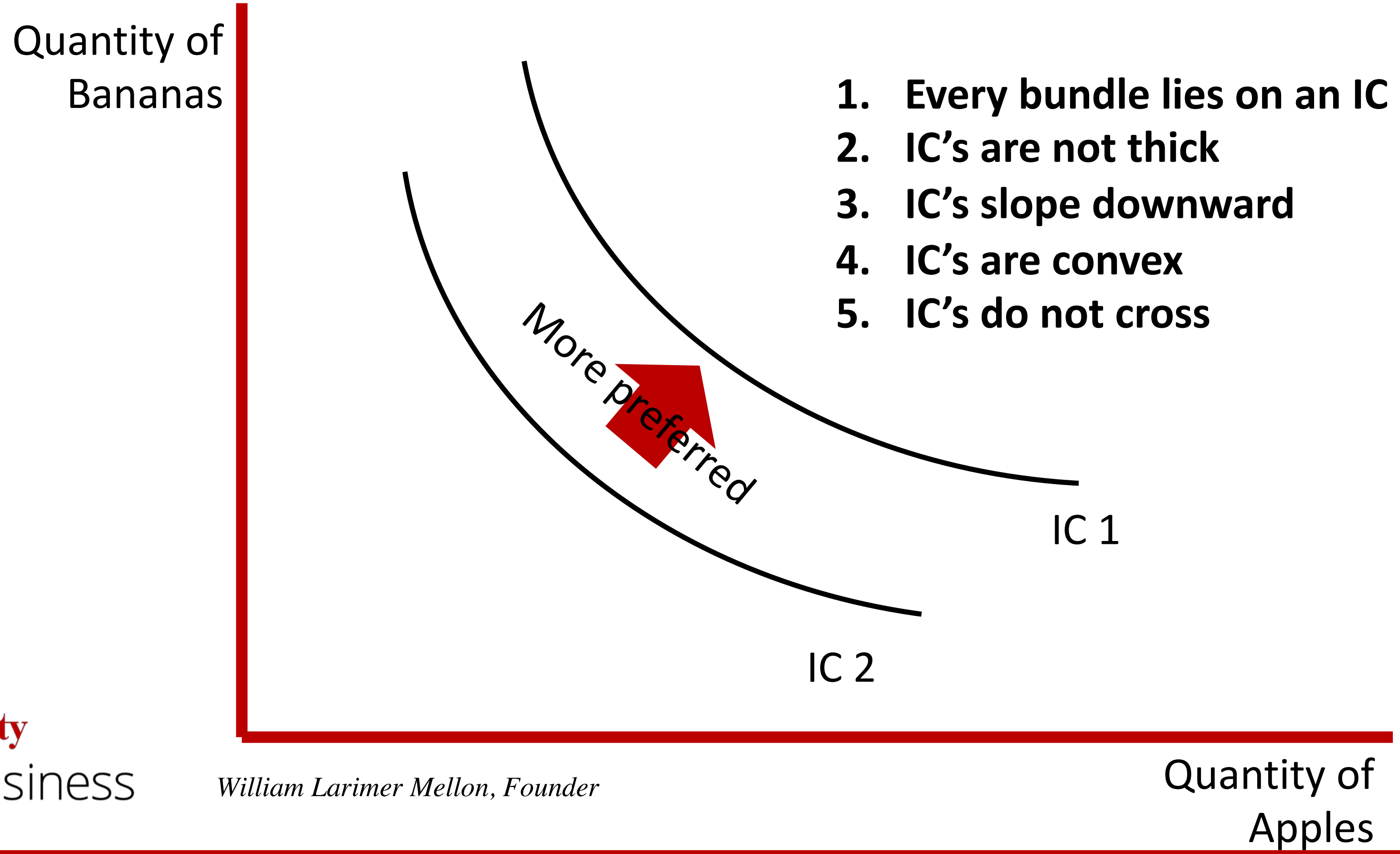


Indifference Curve



Preference Assumptions and IC

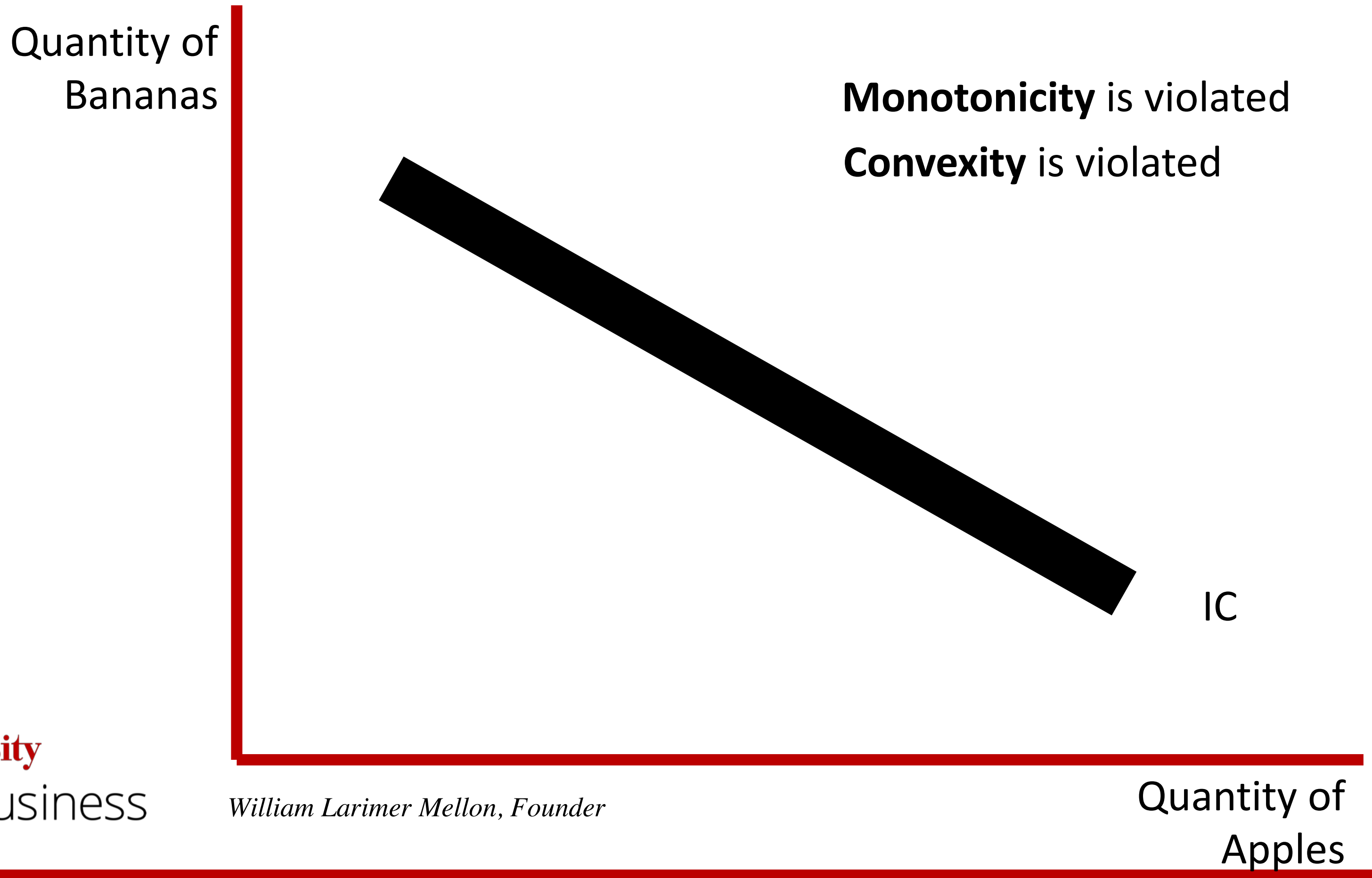
- Under completeness, transitivity, monotonicity, and convexity:



1. **Every bundle lies on an IC**
2. **IC's are not thick**
3. **IC's slope downward**
4. **IC's are convex**
5. **IC's do not cross**

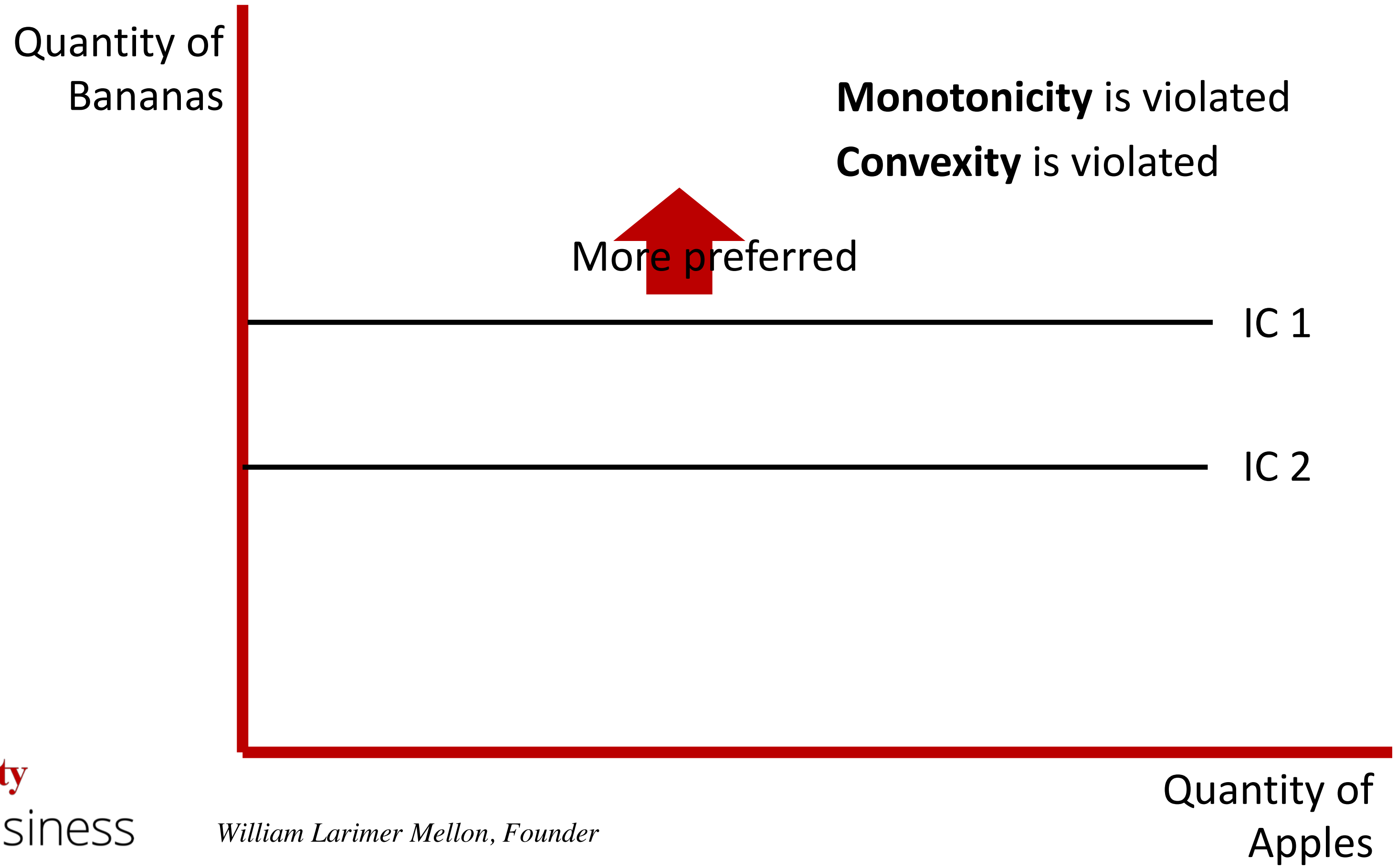
Preference Assumptions and IC (1)

- Under the assumptions, IC's are not thick



Preference Assumptions and IC (2)

- Under the assumptions, IC's slope downward



Preference Assumptions and IC (2)

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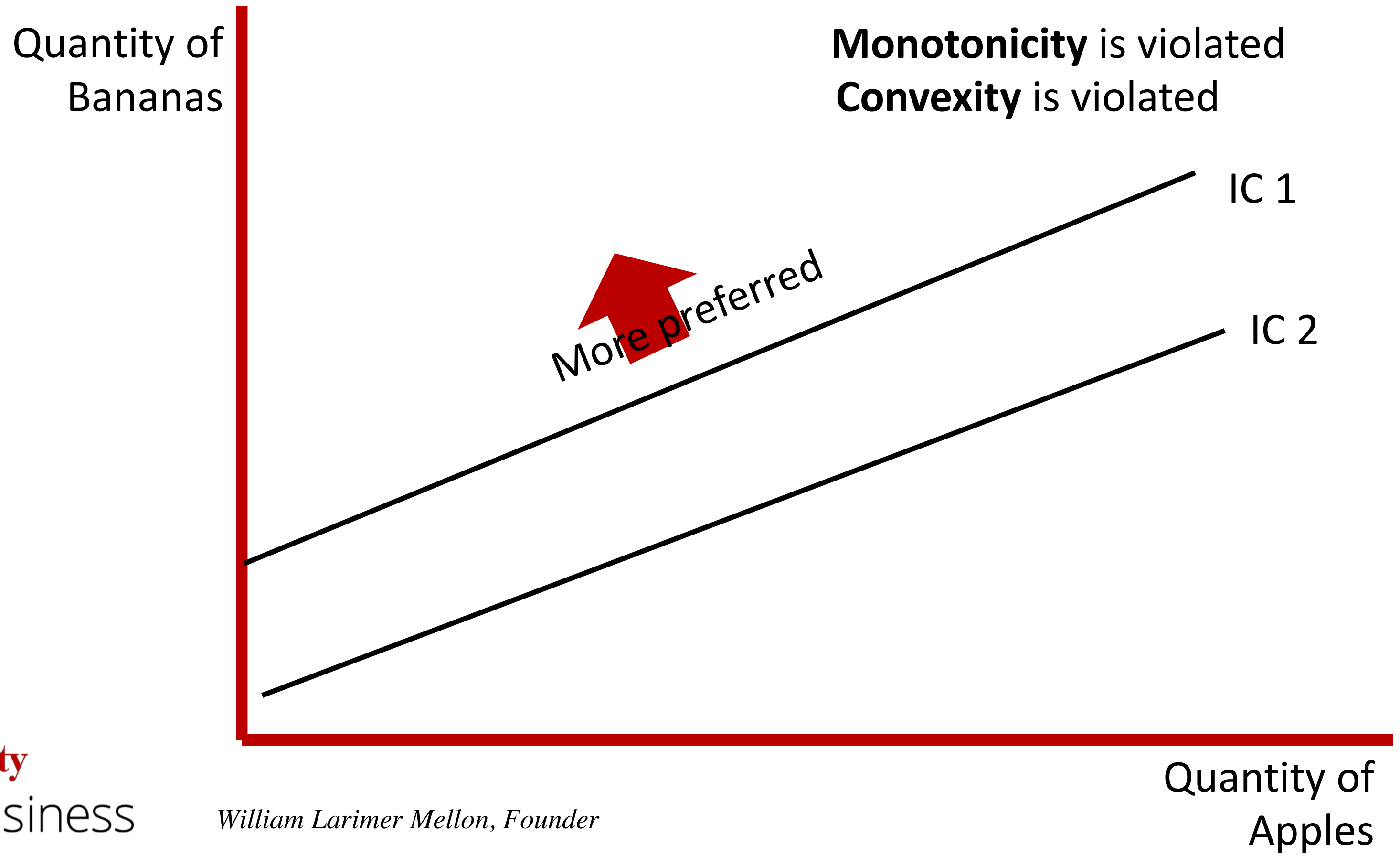
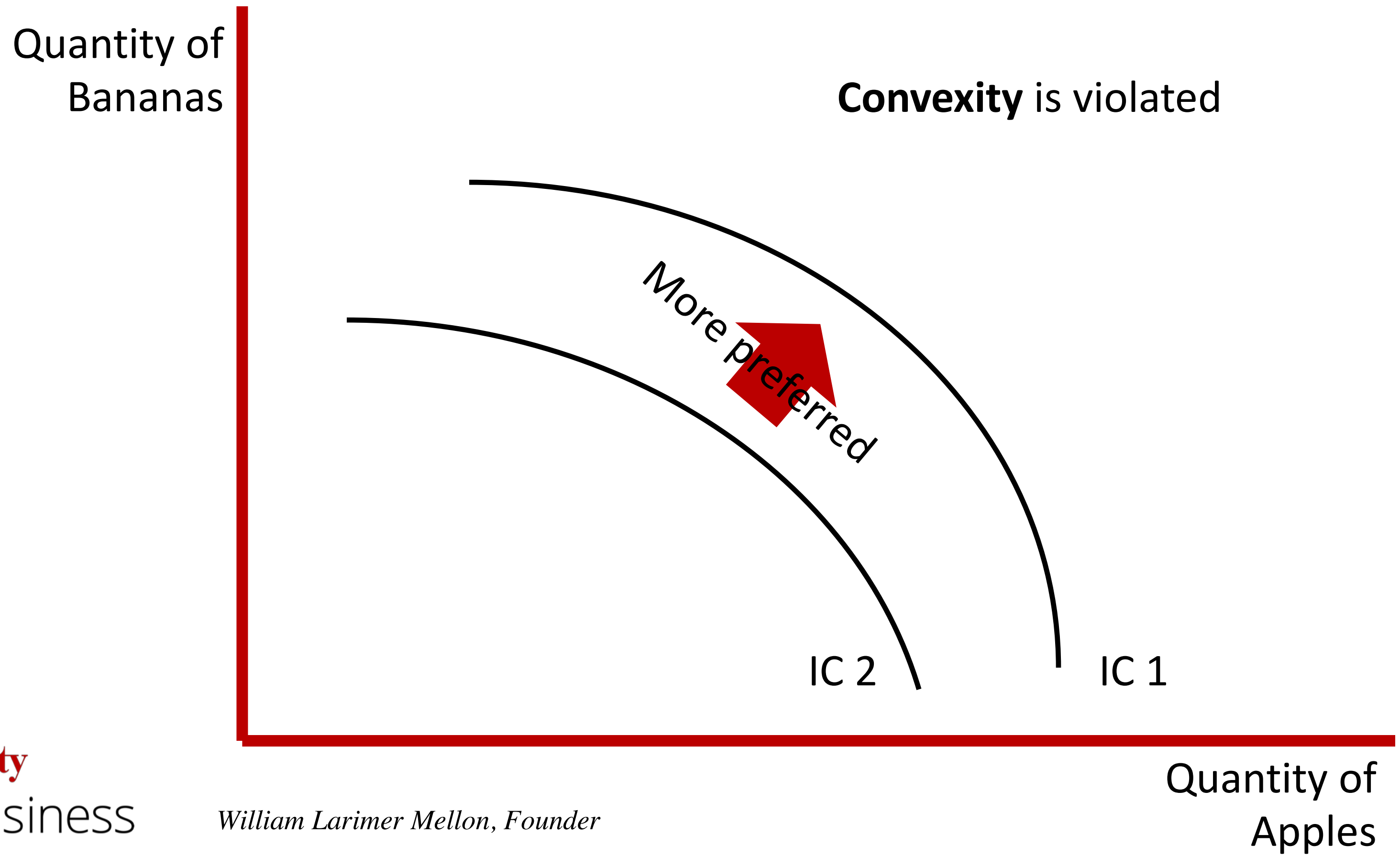


Preference Assumptions and IC (3)

- Under the assumptions, IC's are convex



Preference Assumptions and IC (3)

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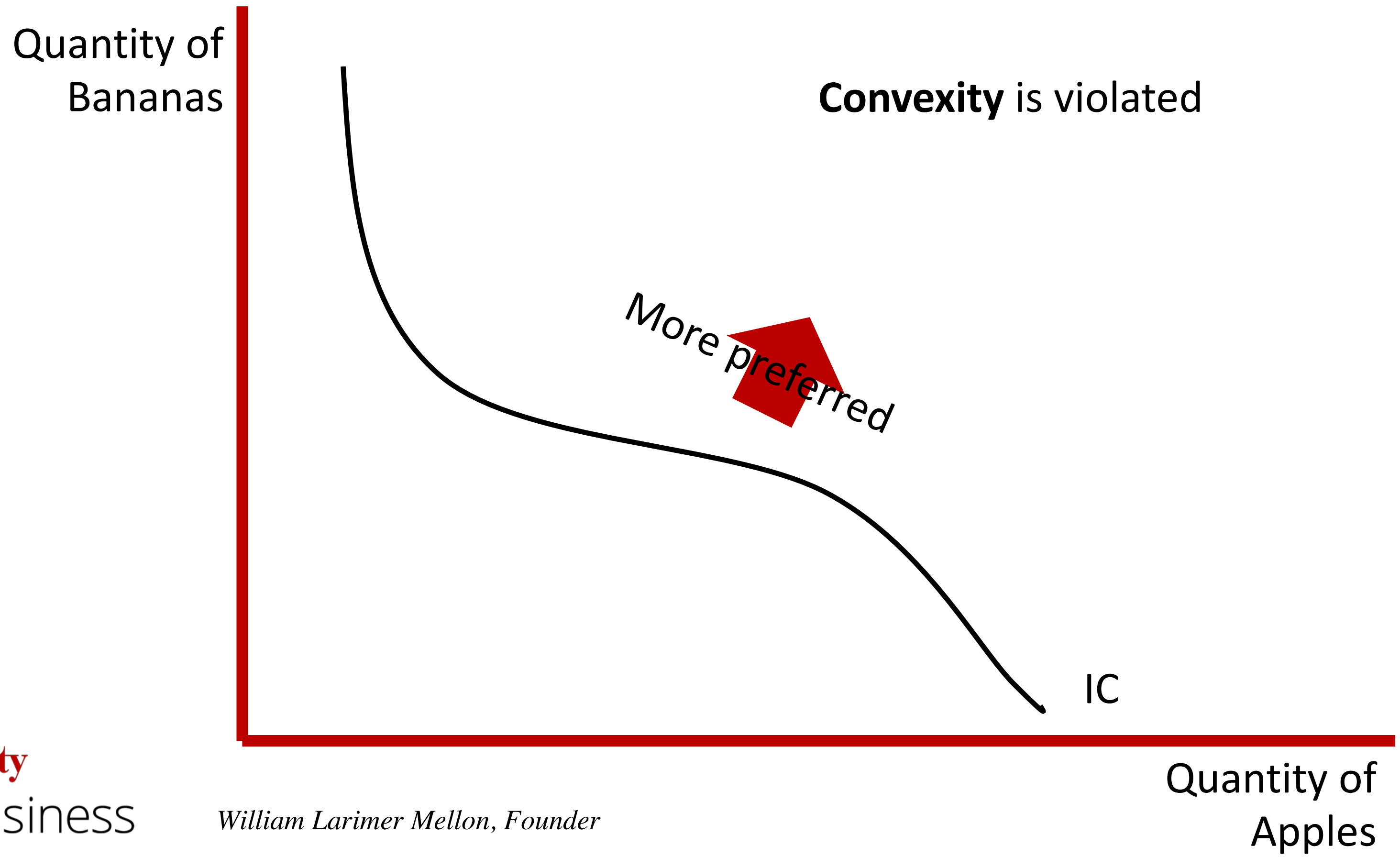
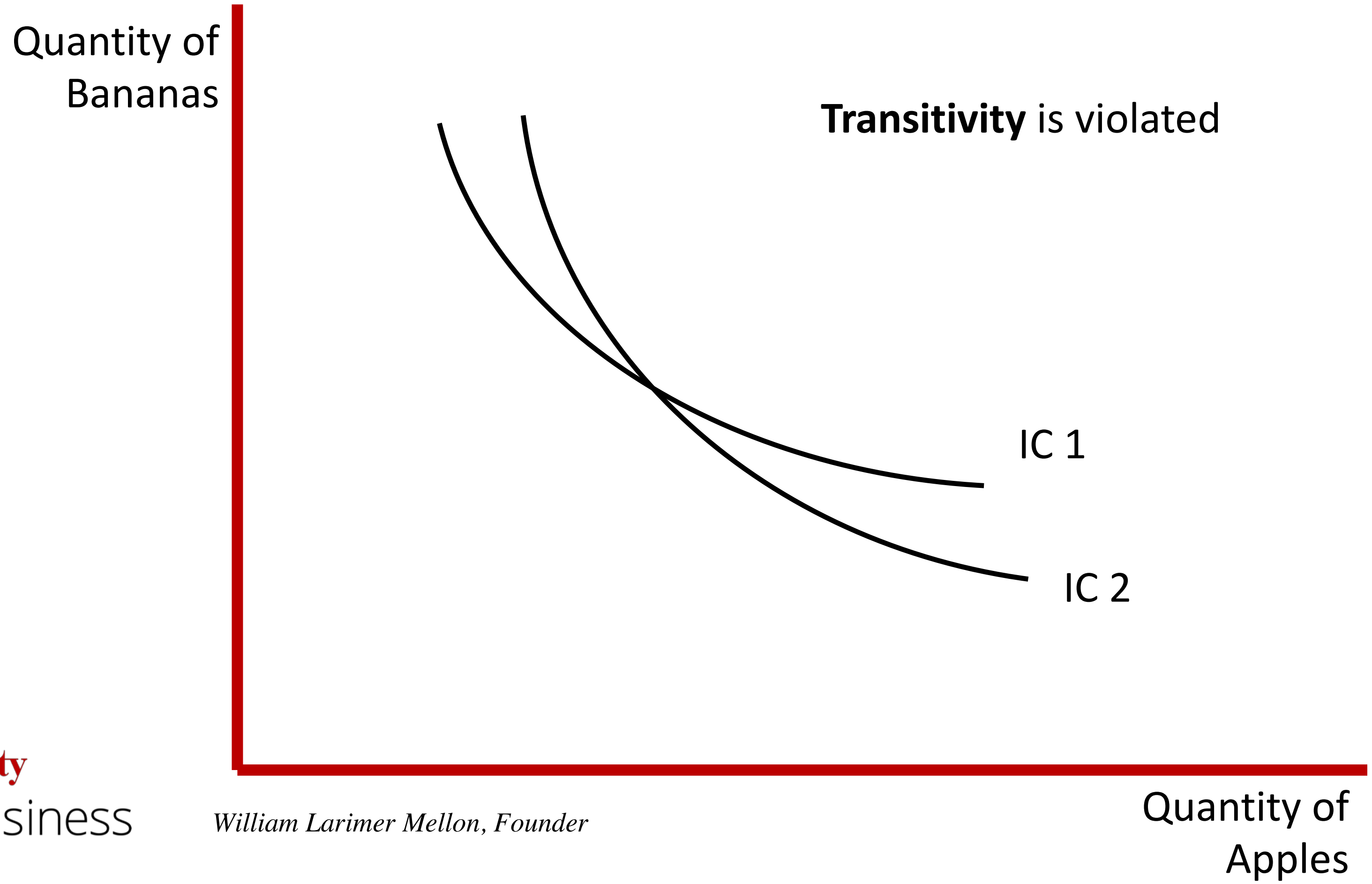


Preference Assumptions and IC (4)

- Under the assumptions, IC's do not cross



Utility Function

- Utility: A set of numerical values that reflect the relative rankings of various bundles of goods

- Utility function:
  - A mapping of a bundle of goods to a real number
    $$u: (x_1, x_2) \to u \in \boldsymbol{R}$$
  - Represents the ranking of bundles
    1. $(x_1, x_2) \succ (y_1, y_2)$ iff $u(x_1, x_2) > u(y_1, y_2)$
    2. $(x_1, x_2) \sim (y_1, y_2)$ iff $u(x_1, x_2) = u(y_1, y_2)$

Utility Function

- Example:
  - x_1 = apples, x_2 = bananas

$$u(x_1, x_2) = \sqrt{x_1 x_2}$$

  - Bundle X contains 16 apples and 9 bananas: *u*(*X*) = *12*
  - Bundle *Y* contains 13 apples and 13 bananas: *u*(*Y*) = *13*
  - Therefore, $\mathrm{Y} \succ X$

Utility Functions and IC's

- A graphical representation of utility functions is three-dimensional.
  - One axis for x_1, another for x_2 and the last for u
  - What if we want to represent utility functions in a two-dimensional graph? *Use a level curve!*

- We use indifference curves to represent constant levels of utility.
  - If X and Y are on the same IC, then $u(X) = u(Y)$
  - IC's are the level curves for a utility function.

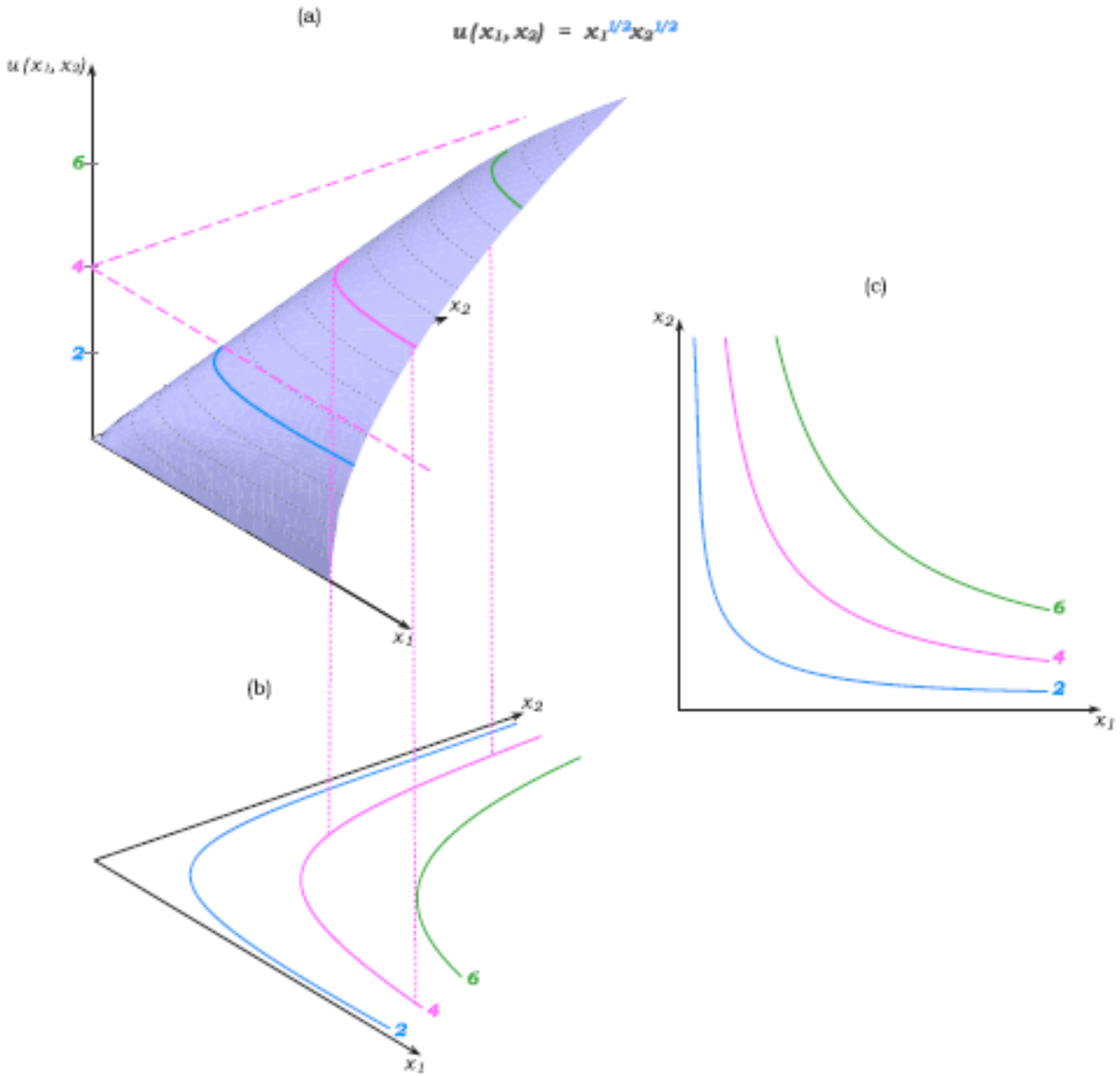
Utility Functions and IC's

- Utility functions:

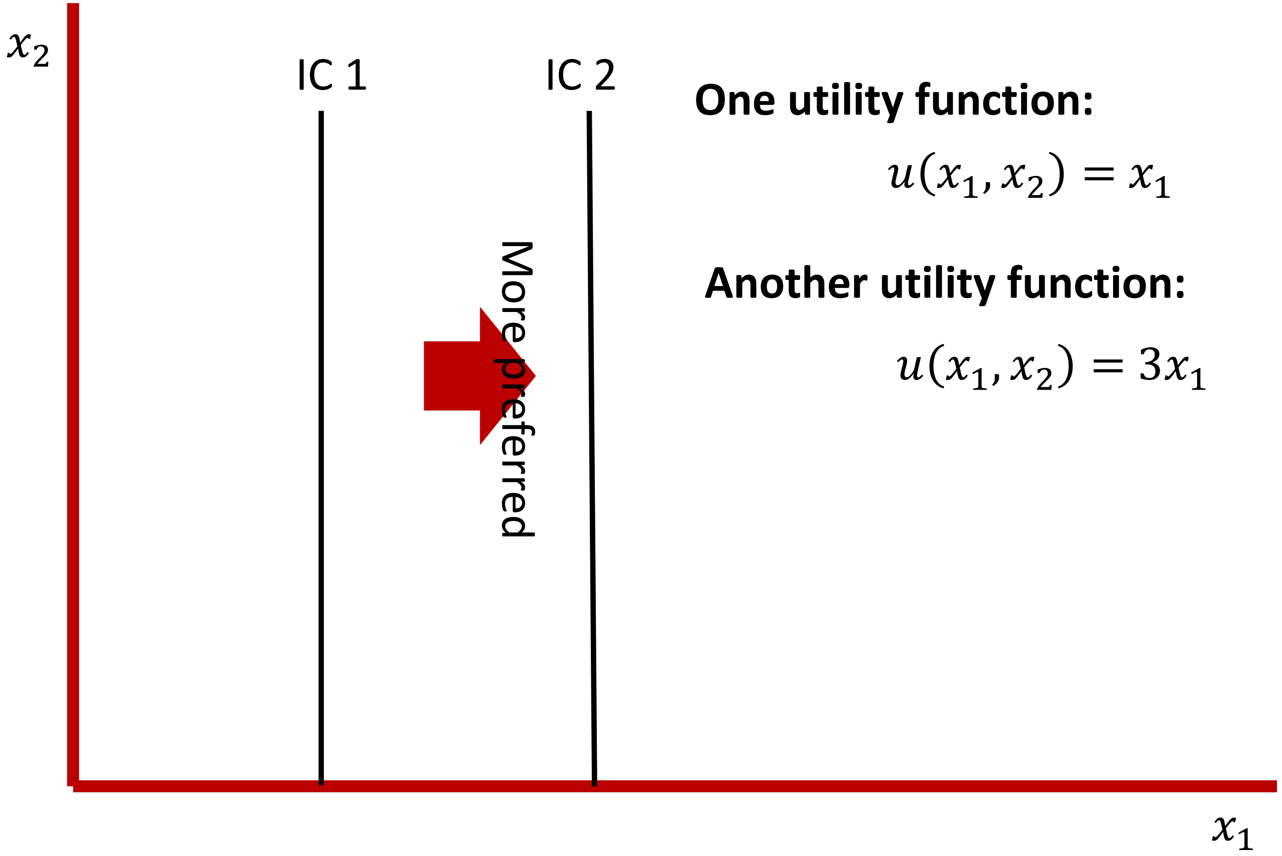
$$u:(x_1, x_2) \rightarrow u \in \boldsymbol{R}$$

  - Dimension 1: Quantity of good 1
  - Dimension 2: Quantity of good 2
  - Dimension 3: Utility
- Indifference curves:
  - Given any value of utility,
  - Dimension 1: Quantity of good 1
  - Dimension 2: Quantity of good 2

Utility Functions and IC's



Utility Functions and IC's



One utility function:

$$u(x_1, x_2) = x_1$$

Another utility function:

$$u(x_1, x_2) = 3x_1$$

Utility Functions and IC's

$$u(x_1, x_2) = \sqrt{x_1 x_2}$$

- Pick a value of utility: $u = 4$

$$\sqrt{x_1 x_2} = 4$$

- Express x_2 as a function of x_1:

$$x_1 x_2 = 16 \rightarrow x_2 = 16/x_1$$

Utility Functions and IC's

$$u(x_1, x_2) = \sqrt{x_1 x_2}$$

- Pick a value of utility: u

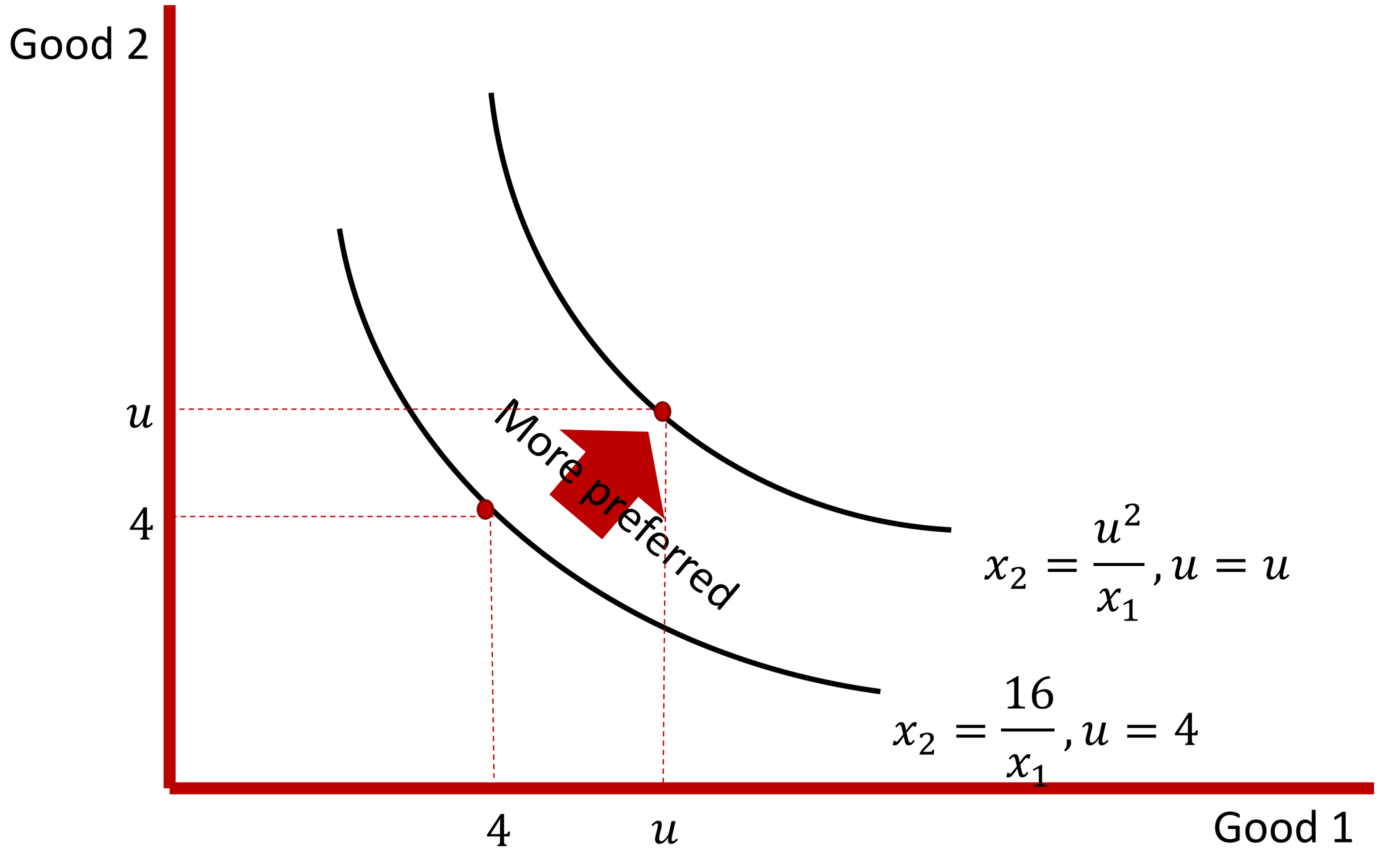
$$\sqrt{x_1 x_2} = u$$

- Express x_2 as a function of x_1:

$$x_1 x_2 = u^2 \rightarrow x_2 = u^2 / x_1$$

Utility Functions and IC's

$$u(x_1, x_2) = \sqrt{x_1 x_2}$$



Utility Functions and IC's

$$v(x_1, x_2) = x_1 x_2$$

- Pick a value of utility: $u = 16$

$$x_1 x_2 = 16$$

- Express x_2 as a function of x_1:

$$x_1 x_2 = 16 \rightarrow x_2 = 16/x_1$$

Utility Functions and IC's

$$v(x_1, x_2) = x_1 x_2$$

- Pick a value of utility: v

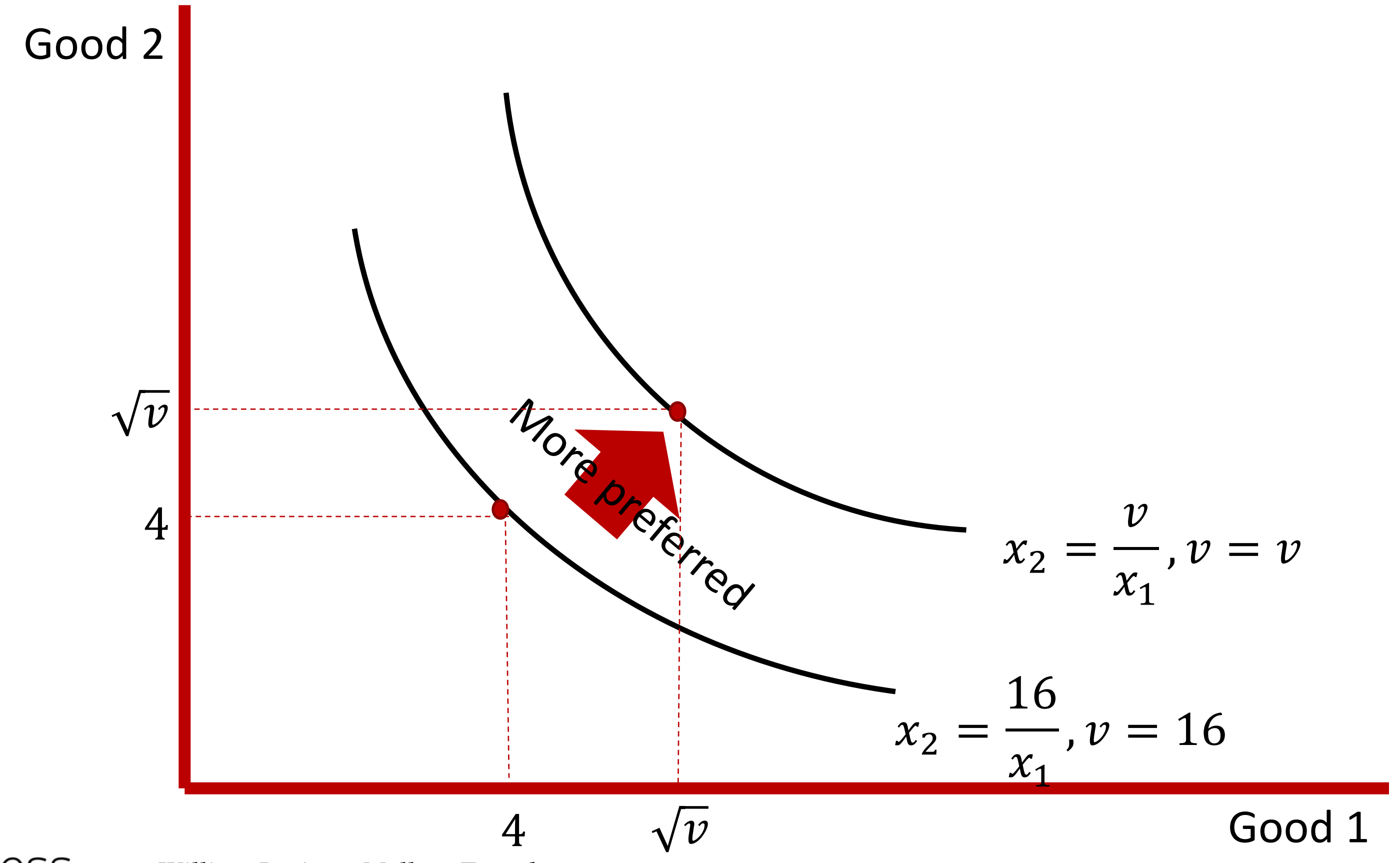
$$x_1 x_2 = v$$

- Express x_2 as a function of x_1:

$$x_1 x_2 = v \rightarrow x_2 = v/x_1$$

Utility Functions and IC's

$$v(x_1, x_2) = x_1 x_2$$



Utility is Ordinal

- Utility is an *ordinal* measure (not *cardinal*)
  - Utility tells us the relative ranking of two things but not how much more one rank is valued than another.
  - We don't really care that *U(X)* =12 and *U(Y)* = 13 in the previous example; we care that $\mathrm{Y} \succ X$
  - Any utility function that generated $\mathrm{Y} \succ X$ would be consistent with these preferences

Many Utility Functions Represent Same Preferences

- A utility function can be transformed into another utility function in such a way that preferences are maintained (*Positive monotonic transformation*)

- Example:
  - $u(x_1, x_2) = \sqrt{x_1 x_2}$
  - $u(x_1, x_2) = x_1 x_2$
  - $u(x_1, x_2) = x_1 x_2 + 10$

Many Utility Functions Represent Same Preferences

$$u(x_1, x_2) = \sqrt{x_1 x_2}$$
$$v(x_1, x_2) = x_1 x_2$$

- The bundles on the IC of $u(x_1, x_2) = 4$ are the same bundles on the IC of $v(x_1, x_2) = 16$.
- The bundles on the IC of $u(x_1, x_2) = u$ are the same bundles on the IC of $v(x_1, x_2) = u^2$.
- Therefore, these two utility functions represent the same preferences.

Many Utility Functions Represent Same Preferences

$$v(x_1, x_2) = x_1 x_2$$
$$w(x_1, x_2) = x_1 + x_2$$

- These two utility functions do NOT represent the same preferences.
- Why?
  - No positive monotone transformation
  - Different IC’s
  - Different marginal rate of substitutions

Marginal Rate of Substitution (MRS)

- Marginal Rate of Substitution (MRS) is the maximum amount of one good that a consumer will sacrifice (trade) to obtain one more unit of another good

Marginal Rate of Substitution (MRS)

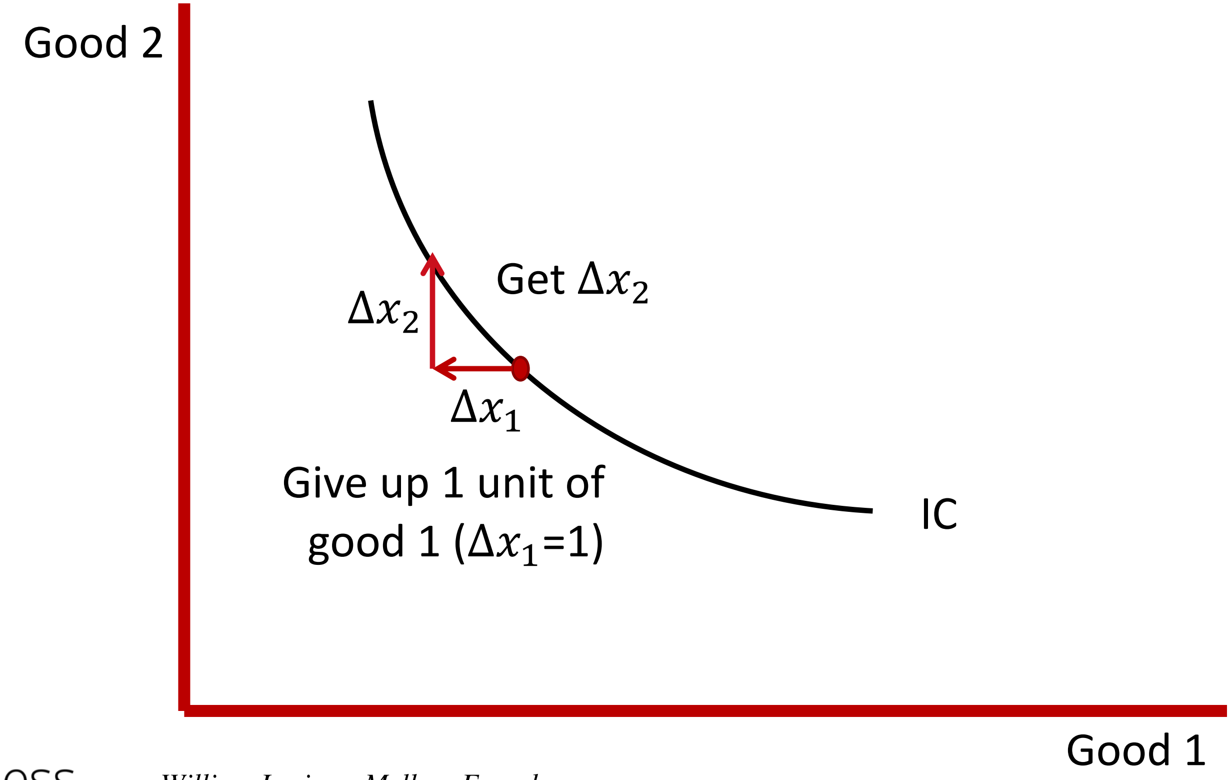
- Suppose a consumer *gives up* one unit of good 1 in exchange for getting some amount of good 2. How much of good 2 does she need to get *in order to end up on the same indifference curve*?

- Consider that she *gets* a unit of good 1 in exchange for giving up some amount of good 2. How much of good 2 can she give up and end up on the same indifference curve?

- The answer to either of these questions is ***a measure of the consumer's valuation of a unit of good 1, in terms of good 2***

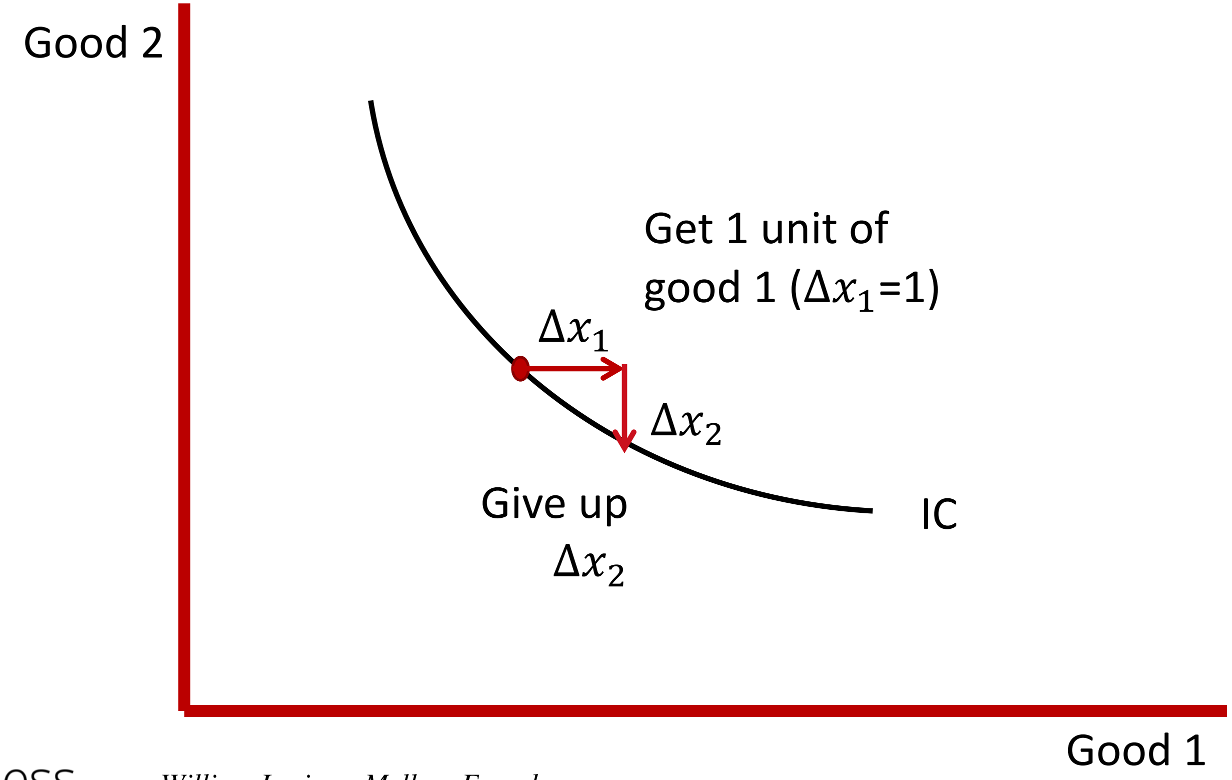
Marginal Rate of Substitution (MRS)

- How much of good 2 does a consumer need to *get* in exchange for one unit of good 1 *in order to end up on the same indifference curve*?



Marginal Rate of Substitution (MRS)

- How much of good 2 does a consumer need to *give up* in exchange for one unit of good 1 *in order to end up on the same indifference curve*?



Indifference Curve and MRS

- Let Δx_1 (Δx_2) represent a change in the consumption of good 1 (2)

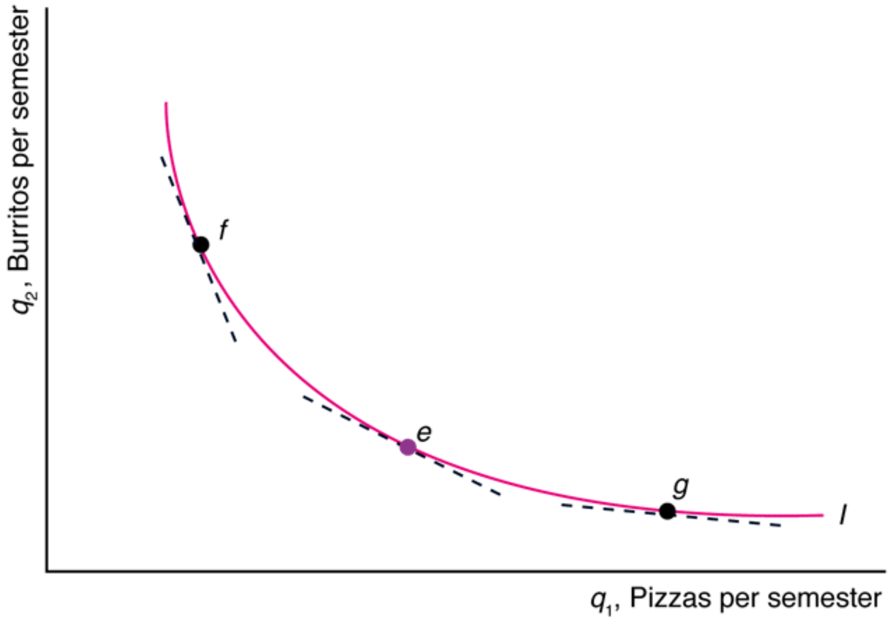
- Consider Δx_2 / Δx_1 :
  - The ratio at which she has to get (give up) good 2 in exchange for giving up (getting) good 1
  - Now, we take the limit as $\Delta x_1 \to 0$

- MRS of good 2 for good 1:

$$MRS_{1,2} = \lim_{\Delta x_1 \to 0} -\frac{\Delta x_2}{\Delta x_1} = -\frac{dx_2}{dx_1} = -IC\ slope$$

Indifference Curve and MRS

- MRS is the slope at a particular point on the IC



□ MRS decreases as good 1 increases (Why: Convexity)

Marginal Utility and MRS

- The MRS depends on how much extra utility a consumer gets from a little more of each good

- **Marginal utility** is the extra utility that a consumer gets from consuming the last unit of a good, holding the consumption of other goods constant

$$MU_1 = \frac{\partial u}{\partial x_1}$$

- Using calculus:

$$MRS_{1,2} = -\frac{dx_2}{dx_1} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{MU_1}{MU_2}$$

Marginal Utility and MRS

- How are they different?
  - They are similar in the sense that they measure the **value** of a good
  - Marginal utility is measured in **utilities**
  - MRS is measured in the **other good**
- Graphical representation for MU and MRS
- MRS
  - MRS is measured for **good 1** in terms of **good 2**
  - MRS is measured for the **good in the X-axis** in terms of the **good in the Y-axis**

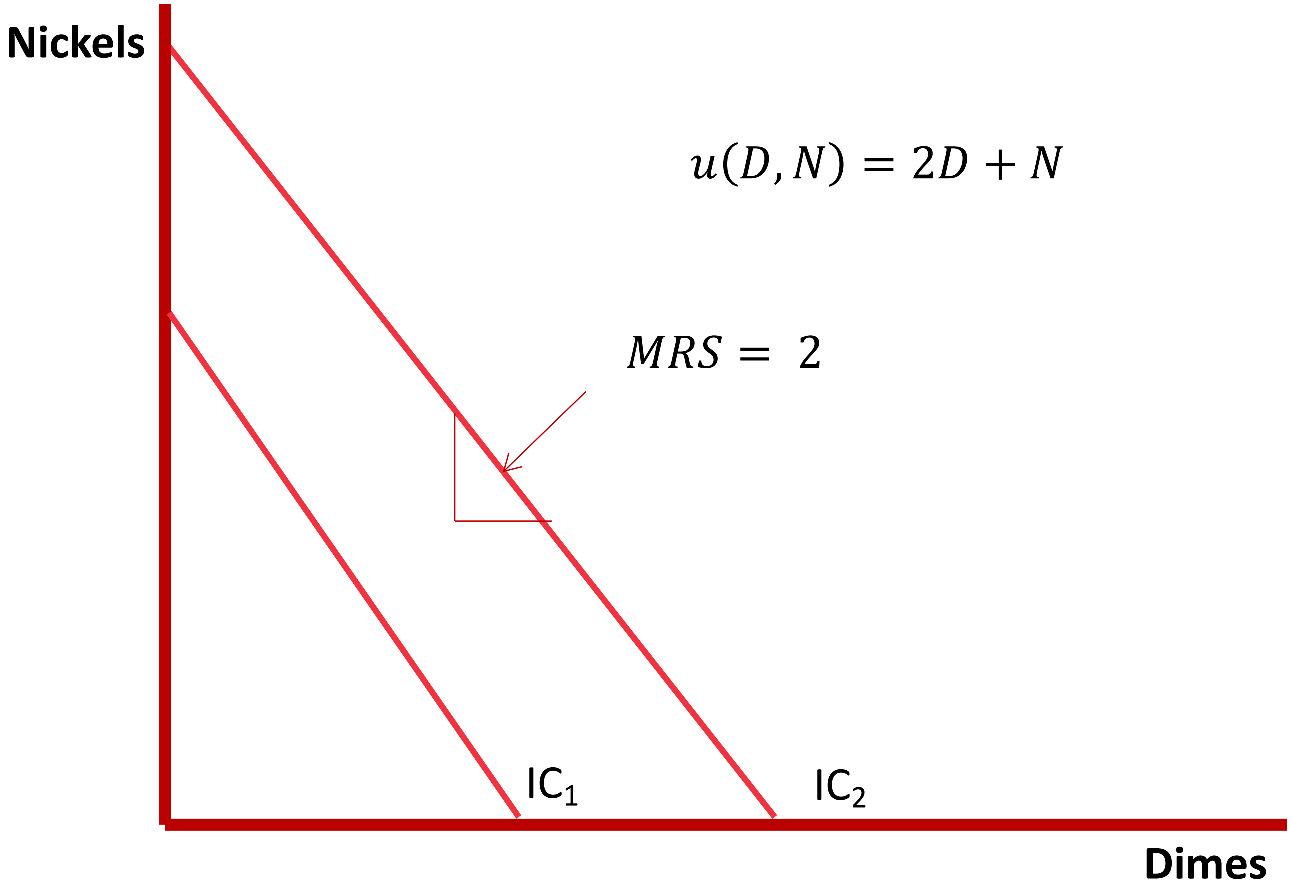
Perfect Substitutes

- ***Perfect Substitutes***: Goods that a consumer is indifferent as to the **proportions** in which she consumes two goods (*constant MRS*)
- Example: Dimes (D) and Nickel (N)

$$u(D, N) = 2D + N$$

  - MRS for Dimes in terms of Nickels = 2 (*constant*)
  - MRS for Nickels in terms of Dimes = ½ (constant)

Perfect Substitutes



Perfect Substitutes

- $U(G,R) = aG + bR$
- $MRS = \frac{MU_1(G,R)}{MU_2(G,R)} = \frac{a}{b}$
  - If the consumer is indifferent between green and red apples, then MRS is 1:
  $$U(G,R) = G + R$$
  - If the consumer is willing to exchange one green apple for three red apples, then MRS is 3:
  $$U(G,R) = 3G + R$$

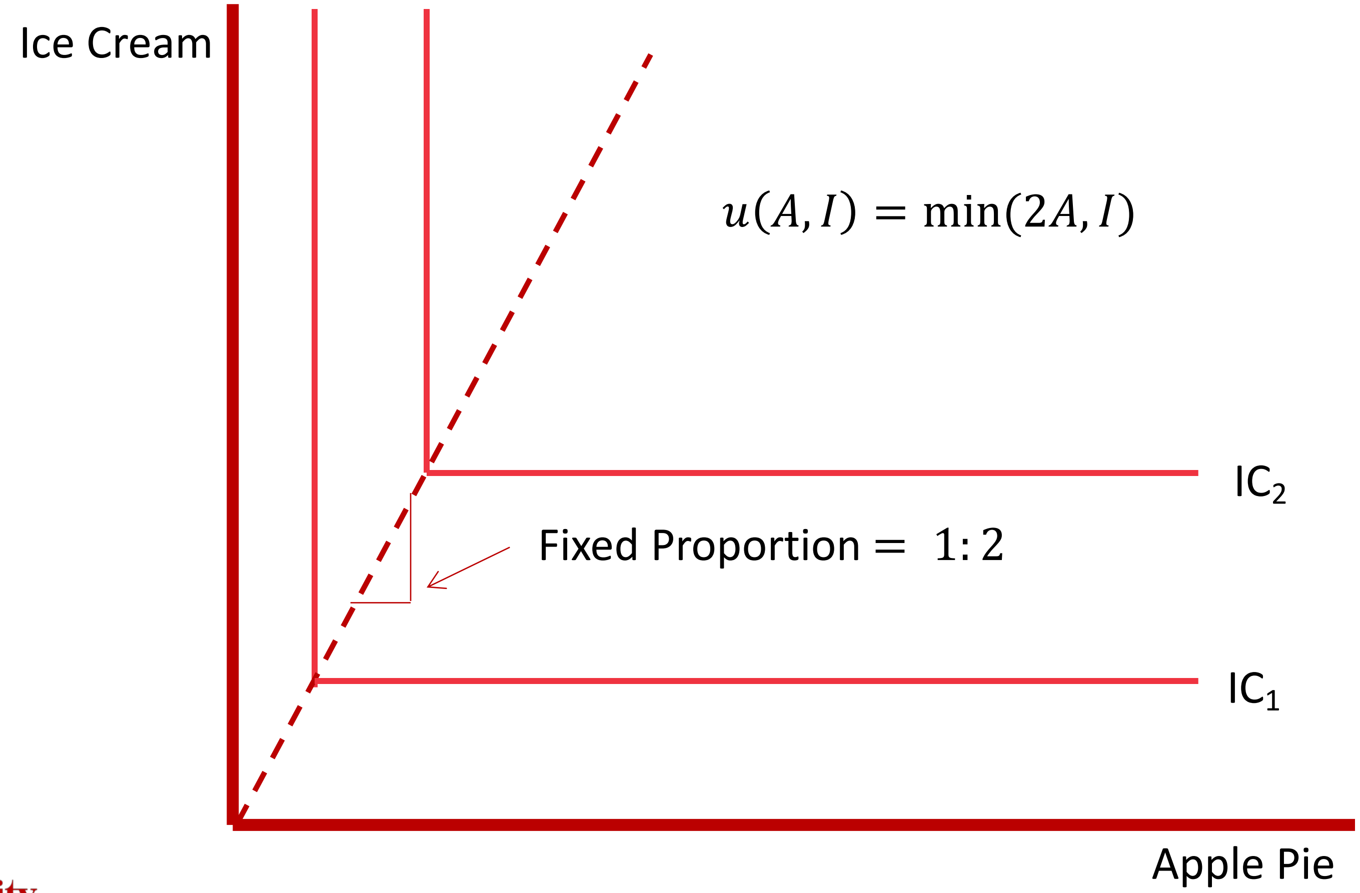
Perfect Complements

- ***Perfect Complements***: Goods that are consumed in fixed proportions
- Example: Apple pie (A) and Ice cream (I)

$$u(A, I) = \min(2A, I)$$

  - One piece of apple pie per 2 spoons of ice cream
  - MRS is *not always* defined

Perfect Complements



Imperfect Substitutes

- Between extreme examples of perfect substitutes and perfect complements are standard-shaped, convex indifference curves
- Examples:
  - Cobb-Douglas utility function

$$u(A, B\ ) = A^{a}\ B^{1-a}$$

  for $0 < a < 1.$

  - Quasi-linear utility function

$$u(A, B\ ) = u(A) + B$$

Examples of Marginal Utilities and Marginal Rate of Substitution

Utility Function	$U(q_1, q_2)$	$U_1 = \frac{\partial U(q_1, q_2)}{\partial q_1}$	$U_2 = \frac{\partial U(q_1, q_2)}{\partial q_2}$	$MRS = -\frac{U_1}{U_2}$
Perfect substitutes	$iq_1 + jq_2$	i	j	$-\frac{i}{j}$
Perfect complements	$\min(iq_1, jq_2)$	0	0	0
Cobb-Douglas	$q_1^a q_2^{1-a}$	$a\frac{U(q_1, q_2)}{q_1}$	$(1-a)\frac{U(q_1, q_2)}{q_2}$	$-\frac{a}{1-a}\frac{q_2}{q_1}$
Constant Elasticity of Substitution (CES)	$(q_1^\rho + q_2^\rho)^{1/\rho}$	$(q_1^\rho + q_2^\rho)^{(1-\rho)/\rho} q_1^{\rho-1}$	$(q_1^\rho + q_2^\rho)^{(1-\rho)/\rho} q_2^{\rho-1}$	$-\left(\frac{q_1}{q_2}\right)^{\rho-1}$
Quasilinear	$u(q_1) + q_2$	$\frac{du(q_1)}{dq_1}$	1	$-\frac{du(q_1)}{dq_1}$

Notes: $i > 0$, $j > 0$, $0 < a < 1$, $\rho \neq 0$, and $\rho < 1$. We are evaluating the prefect complements' indifference curve at its right-angle corner, where it is not differentiable, hence the formula $MRS = -U_1/U_2$ is not well-defined. We arbitrarily say that the $MRS = 0$ because no substitution is possible.

3. Consumer Optimization

Consumer Optimization

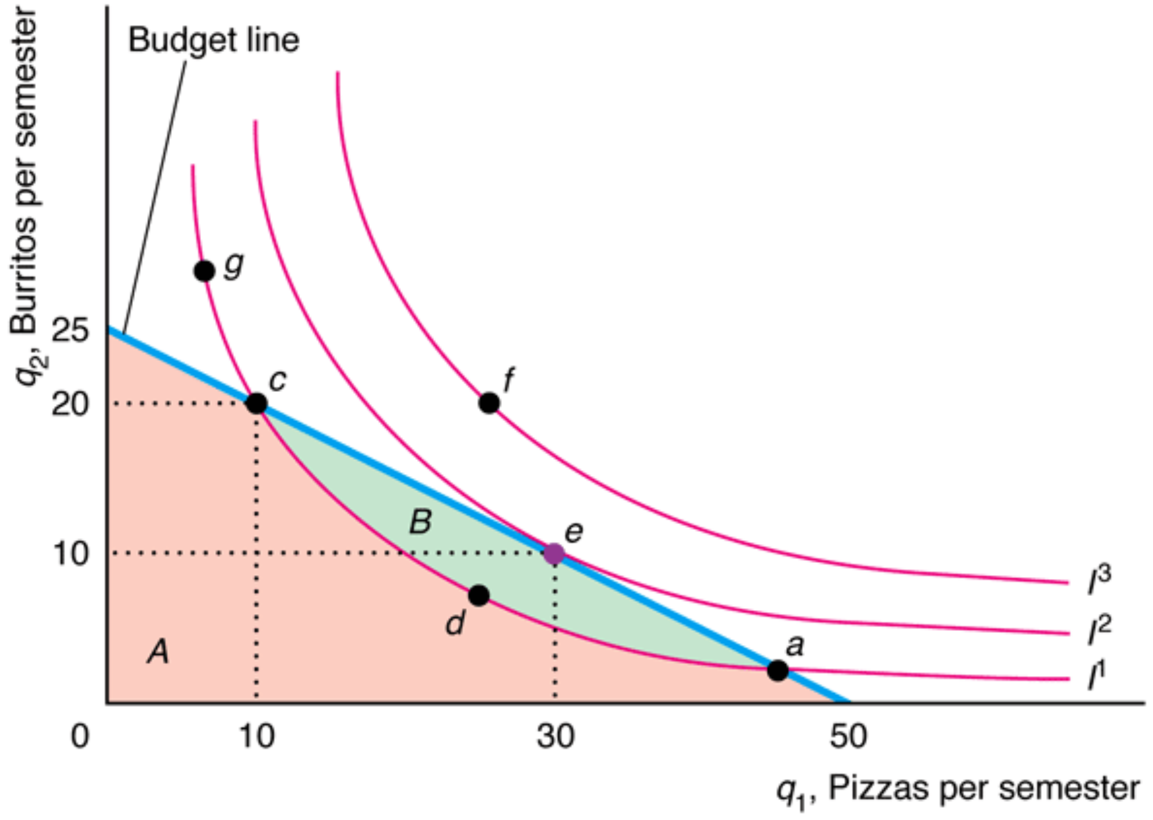
- Objectives:
  1. Graphically represent and solve consumer optimization problem
  2. Mathematically represent and solve consumer optimization problem: ***maximization of utility subject to a budget constraint***
- At an interior solution, $MRS = \frac{MU_1}{MU_2} = \frac{p_1}{p_2}$
- Reading: pp.143-166

Graphical Constrained Optimization

- The consumer will choose the bundle that she most prefers among those that she can afford

- She will find the highest indifference curve that is consistent with his budget

Graphical Constrained Optimization



Graphical Constrained Optimization

- Is point f an optimum? Why not?
- Is point c an optimum? Why not?
- What is the MRS at point c?

$$MRS > \frac{p_1}{p_2} = \frac{1}{2}$$

  - $\frac{p_1}{p_2}$ is the absolute value of the slope of the budget line, $\frac{25}{50} = \frac{1}{2}$.

Graphical Constrained Optimization

- Why isn't bundle c an optimal choice? Suppose the MRS at c is $2 > \frac{1}{2}$
  - $MRS = 2$: Consumer is willing to substitute 2 units of good 2 for 1 unit of good 1
  - $\frac{p_1}{p_2} = \frac{1}{2}$: Market is willing to trade 1 unit of good 2 for 2 units of good 1

Graphical Constrained Optimization

- Why isn't bundle c an optimal choice? Suppose the MRS at c is $2 > \frac{1}{2}$
  - If the consumer offers 2 units of good 2, she can get 4 units of good 1 instead of 1 unit of good 1. Therefore, she will trade, which implies that bundle c cannot be an optimal choice.

Graphical Constrained Optimization

- At e, we have

$$MRS = \frac{p_1}{p_2}$$

  - Are there any profitable trades to be made?

Graphical Constrained Optimization

- When the optimal bundle occurs at a point of ***tangency*** between the indifference curve and budget line, this is called an ***interior solution***.
  - Mathematically,

$$MRS = \frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

  - Rearranging, we can see that the marginal utility per dollar is equated across goods at the optimum:

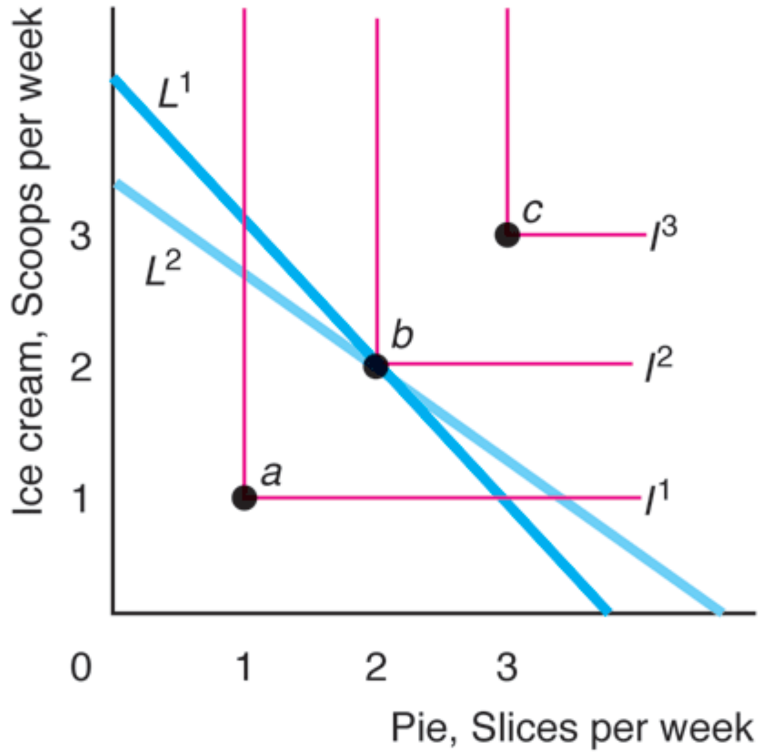
$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

Interior vs. Corner Solutions

- Interior Solution:
  - **Both** goods are consumed with **positive amounts**.
  - Except perfect complements, $MRS = p_1/p_2$.
    - Cobb-Douglas ($x^a y^{1-a}$) and CES ($x^a + y^a$)
    - Perfect substitutes
    - Quasi-linear utility functions ($u(x) + y$)
- Corner Solution:
  - **Only one** good is consumed with a **positive amount**.
  - At a corner solution, $MRS \neq p_1/p_2$
    - Perfect substitutes
    - Quasi-linear utility functions

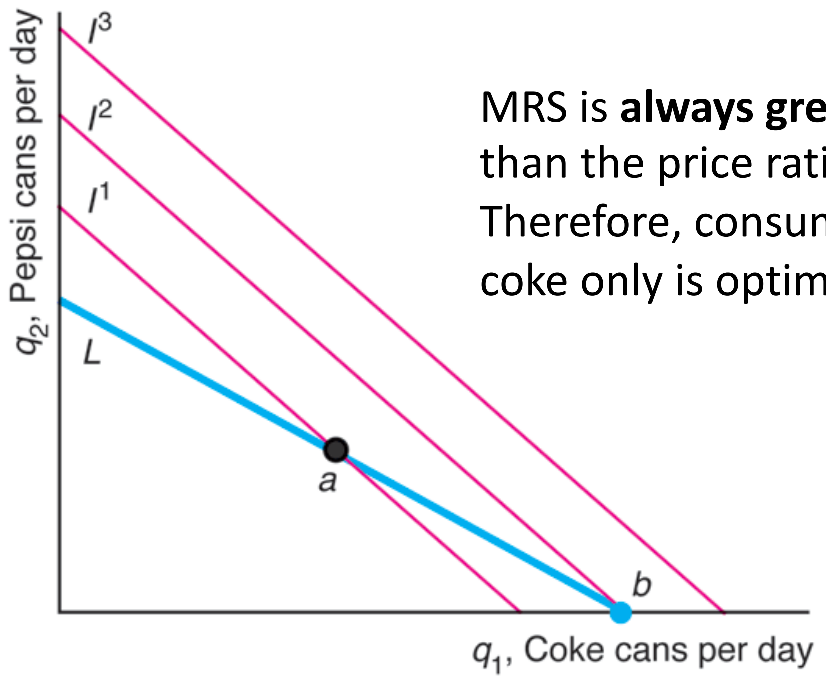
Graphical Constrained Optimization with Perfect Complements

- The optimal bundle is on the budget line and at the right angle (i.e. vertex) of an indifference curve.



Graphical Constrained Optimization with Perfect Substitutes

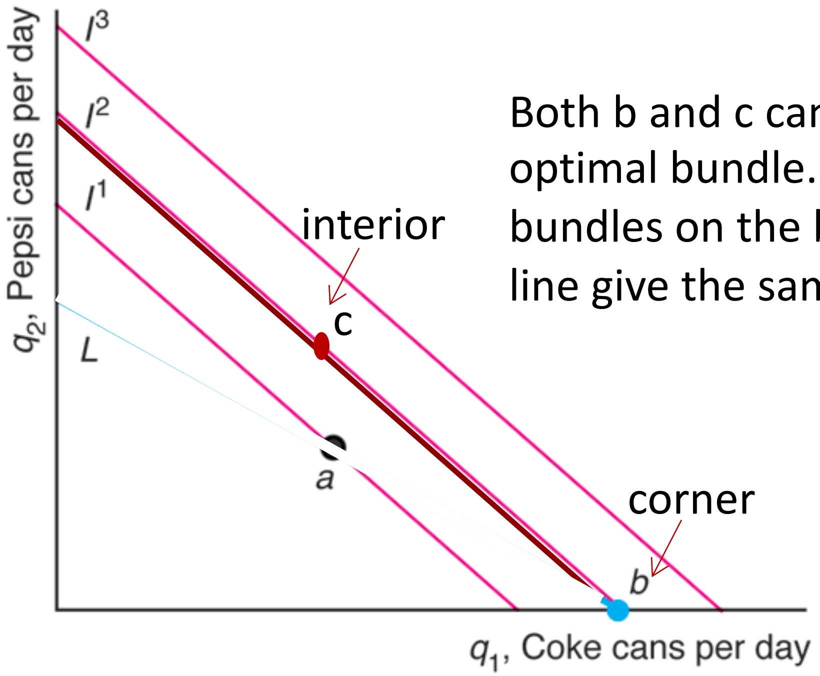
- With perfect substitutes, if MRS does not equal the price ratio (p_1/p_2), then the optimal bundle occurs at a ***corner solution***, bundle b.



MRS is **always greater** than the price ratio. Therefore, consuming coke only is optimal.

Graphical Constrained Optimization with Perfect Substitutes

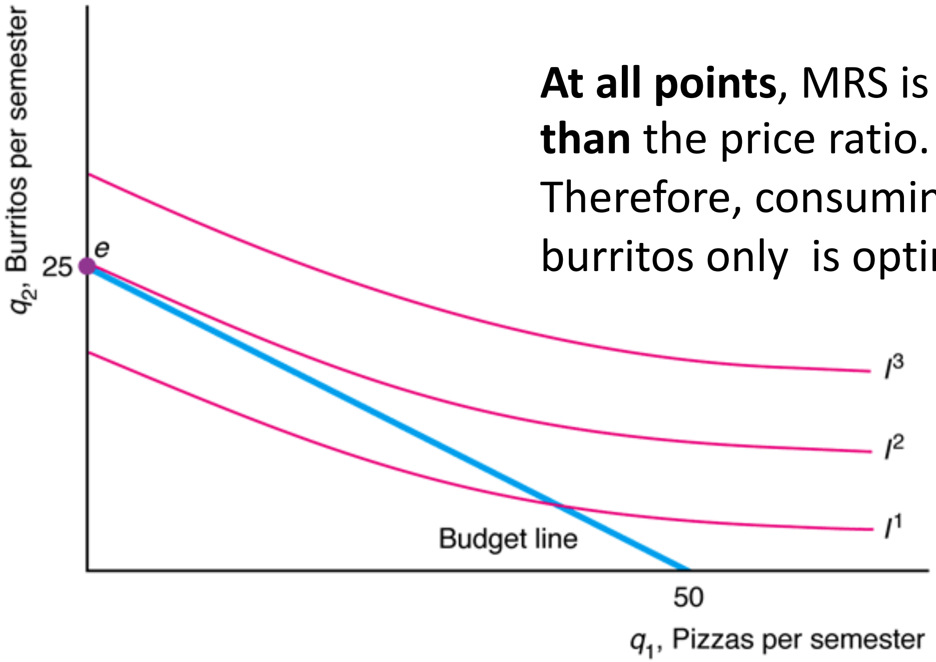
- If MRS is equal the price ratio (p_1/p_2), then the optimal bundle occurs at any point on the budget line.



Both b and c can be an optimal bundle. All bundles on the budget line give the same utility.

Graphical Constrained Optimization with Quasilinear Preferences

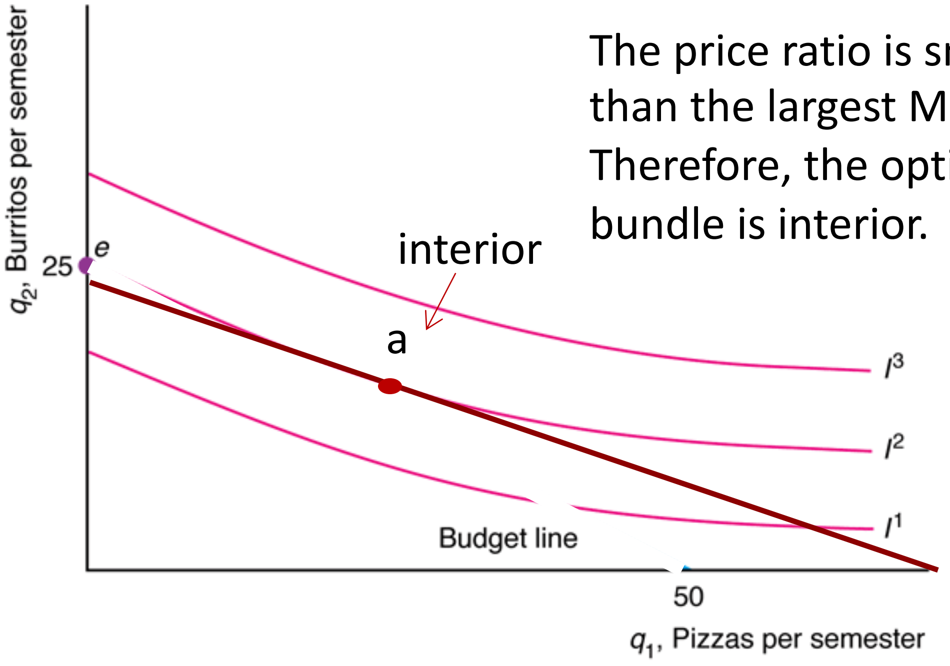
- If the relative price of one good is too high and the preferences are quasilinear, the optimal bundle occurs at a ***corner solution***.



At all points, MRS is **less than** the price ratio. Therefore, consuming burritos only is optimal.

Graphical Constrained Optimization with Quasilinear Preferences

- If the relative price of one good is not too high, the optimal bundle can be an interior solution.



The price ratio is smaller than the largest MRS. Therefore, the optimal bundle is interior.

Consumer Choice with Calculus

- Our graphical analysis of consumers' constrained choices can be stated mathematically:

$$\max_{x_1, x_2} \quad u(x_1, x_2)$$

$$s.t.\, p_1 x_1 + p_2 x_2 = Y$$

- Solution reveals utility-maximizing values of x_1 and x_2 as functions of prices, p_1 and p_2, and income, Y.

Consumer Choice with Calculus

- How to solve this constrained optimization problem (*Brute force method*):
  - **Step 1**: Substitute $x_2 = (Y - p_1 x_1)/p_2$.
  - **Step 2**: Solve for x_1 (*Unconstrained optimization*)

$$\max_{x_1} \quad u(x_1, (Y - p_1 x_1)/p_2)$$

  - **Step 3**: Plug in the optimal x_1 to solve for x_2

$$x_2 = (Y - p_1 x_1)/p_2$$

Consumer Choice with Calculus

- Alternatively, we use the following fact:
  - An interior solution satisfies the two conditions as in the graphical analysis:

$$MRS = \frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$
$$p_1 x_1 + p_2 x_2 = Y$$

Consumer Choice with Calculus

- Alternative way to solve the consumer's problem: (*Use-the-graphs method*)
  - **Step 1**: Using the given utility function, solve for MRS and obtain:
    $$\frac{MU_1(x_1,x_2)}{MU_2(x_1,x_2)} = \frac{p_1}{p_2}$$
  - **Step 2**: Write down the budget constraint:
    $$p_1 x_1 + p_2 x_2 = Y$$
  - **Step 3**: Solve for the two unknowns (x_1 , x_2)

Consumer Choice with Calculus

- Example: Apple (a) and Bananas (b)
  - Utility function: $u(a, b) = a^{0.5}b^{0.5}$
  - Price of an apple is \$2; that of a banana is \$1
  - Weekly budget for apples and bananas: \$10
  - Optimal bundle?

$$\max_{a,b} \quad a^{0.5}b^{0.5}$$
$$s.t.\, 2a + b = 10$$

Consumer Choice with Calculus

- *Brute force method*:
  - **Step 1**: Substitute $x_2 = (Y - p_1 x_1)/p_2$.

$$b = 10 - 2a$$

Consumer Choice with Calculus

- *Brute force method*:
  - **Step 1**: $b = 10 - 2a$
  - **Step 2**: Solve for x_1 (*Unconstrained optimization*)

$$\max_{x_1} \quad u(x_1, (Y - p_1 x_1)/p_2)$$

$$\max_{a} \quad a^{0.5}(10 - 2a)^{0.5}$$

- To solve for a, we take the ***first order condition***:

$$0.5a^{-0.5}(10 - 2a)^{0.5} - a^{0.5}(10 - 2a)^{-0.5} = 0$$

- Rearranging the equation:

$$0.5(10 - 2a) = a \rightarrow a = 5/2$$

Consumer Choice with Calculus

- *Brute force method*:
  - **Step 1**: $b = 10 - 2a$
  - **Step 2**: $a = 5/2$
  - **Step 3:** Plug in the optimal x_1 to solve for x_2

$$x_2 = (Y - p_1 x_1)/p_2$$

    - $b = 10 - 2a = 5.$
  - Final solution: 5/2 apples and 5 bananas

Consumer Choice with Calculus

- *Use-the-graphs method:*
  - **Step 1**: Using the given utility function, solve for MRS and obtain:

$$\frac{MU_1(x_1,x_2)}{MU_2(x_1,x_2)} = \frac{p_1}{p_2}$$

- $\frac{MU_1(x_1,x_2)}{MU_2(x_1,x_2)} = \frac{0.5a^{-0.5}b^{0.5}}{0.5a^{0.5}b^{-0.5}} = \frac{b}{a}$
- $\frac{p_1}{p_2} = \frac{2}{1} = 2$
- Therefore, $b = 2a$

Consumer Choice with Calculus

- *Use-the-graphs method*
  - **Step 1**: $b = 2a$
  - **Step 2**: Write down the budget constraint:

$$p_1 x_1 + p_2 x_2 = Y$$

$$2a + b = 10$$

Consumer Choice with Calculus

- *Use-the-graphs method*
  - **Step 1**: $b = 2a$
  - **Step 2**: $2a + b = 10$
  - **Step 3**: Solve for the two unknowns (x_1, x_2)
    - $a = \frac{5}{2}, b = 5$
  - Final solution: 5/2 apples and 5 bananas

Type of Solution for Five Utility Functions

Utility Function	$U(q_1, q_2)$	Type of Solution
Perfect complements	$\min(iq_1, jq_2)$	interior
Cobb-Douglas	$q_1^a q_2^{1-a}$	interior
Constant Elasticity of Substitution	$(q_1^\rho + q_2^\rho)^{1/\rho}$	interior
Perfect substitutes	$iq_1 + jq_2$	interior or corner
Quasilinear	$u(q_1) + q_2$	interior or corner

Notes: $i > 0$, $j > 0$, $0 < a < 1$, $\rho \neq 0$, and $\rho < 1$.