Statistical modeling

Exercise 1. Small questions

1. Give a possible sample of size 4 from each of the following populations :
   - — all daily newspapers published in the world,
   - — all distances that may result when you throw a football,
   - — all grades of students at the probability course,
   - — all possible daily numbers of cars going through a crossing.
2. In 1882, Michelson and Newcomb measured the traveling time of light going from and to their lab through a mirror. Their first measurements were : 28, 26, 33, 24, 34, −44, 27, 16, 40, −2, 29, 24, 21, 25 ($*0.001 + 24.8$ in millionths of a second). Why are these measurements not identical ? How do we model this variability in statistics ?
3. Are the following statistical models identifiable
   - — $\big((\mathcal{X} = \mathbb{R}, \mathcal{B}(\mathbb{R})), \{\mathcal{N}(\mu + a, \sigma^2), (\mu, a, \sigma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+\}\big)$,
   - — $\big((\mathcal{X} = \mathbb{R}, \mathcal{B}(\mathbb{R})), \{\mathcal{N}(\mu, \sigma^2), (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+\}\big)$,
   - — $\big((\mathcal{X} = \mathbb{R}, \mathcal{B}(\mathbb{R})), \{\mathcal{N}(\mu, \sigma^2), (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}\}\big)$?

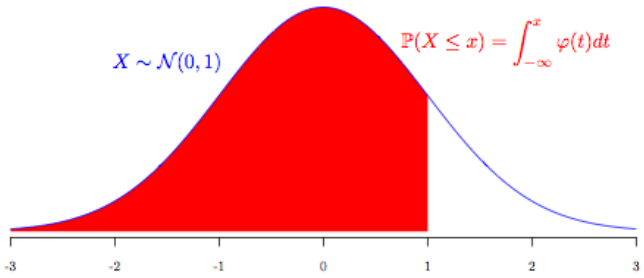
Solution

1. — Le Monde, The New York Times, The Guardian, El País
   — 19, 5, 13, 11
   — 10.45m, 3.23m, 7.53m, 15.74m
   — 102, 84, 26, 73
2. The measurements are not identical because of the uncertainty of the experiment and the measurements, small errors, perturbations happen for each light traveling. We model this variability in statistics thanks to the probability theory (theory of the randomness), saying that these observed values $x_1 = 28$, $x_2 = 26$, $x_3 = 33$, ... are the realization of i.i.d. random variables X_1, X_2, ...
3. — no, $\mathcal{N}(\mu_1, \sigma^2) = \mathcal{N}(\mu_2 + \delta - \delta, \sigma^2)$ for any real δ. So that the same distribution is obtained with $\mu = m + \delta$, $a = -\delta$, for any real δ.
   — yes, since the first moment and the second moment completely identify the two parameters. Assume that $\mathcal{N}(\mu_1, \sigma_1^2) = \mathcal{N}(\mu_2, \sigma_2^2)$ then $\mu_1 = \mathsf{E}_{\mathcal{N}(\mu_1, \sigma_1^2)}(X) = \mathsf{E}_{\mathcal{N}(\mu_2, \sigma_2^2)}(X) = \mu_2$ and $\sigma_1^2 = Var_{\mathcal{N}(\mu_1, \sigma_1^2)}(X) = Var_{\mathcal{N}(\mu_2, \sigma_2^2)}(X) = \sigma_2^2$.
   — No, $\mathcal{N}(\mu, \sigma^2) = \mathcal{N}(\mu, (-\sigma)^2)$

Exercise 2. Gaussian measurements

For this exercise, you can use the table of the cdf of the standard normal distribution in Figure 1. The measurement of atmospheric ozone concentration (in $\mu g/m^3$) is modeled by a random variable X with distribution $\mathcal{N}(m, \sigma^2)$ with $\sigma^2 = 3.1$.

1. Write the statistical model.
2. In many applications, data are often modeled with the Normal distribution, while often the observed values are by definition positive (e.g. weight, size, speed, duration). Can you explain why ?
3. What are the units of m and σ ? What do m and σ represent ?
4. The ozone concentration is considered dangerous for humans when it is greater than $180 \mu g/m^3$.
   (a) Assuming that $m = 178$, what is the probability that the measurement is greater than 180 ? Comment the result.
   (b) Assuming that $m = 183$, what is the probability that the measurement is smaller than 180 ? Comment the result.



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

FIGURE 1 – Table of the cdf of the standard normal distribution

(c) Assuming that $m = 180$, find a real number δ such that the probability $P(180 - \delta \leq X \leq 180 + \delta)$ is bigger than 95%.

5. Are the three last questions statistical or probabilistic questions ? Find a question that a statistician could ask from this experience.

6. Some day, we make some measurements and we assume that this day the ozone concentration is $178\mu g/m^3$ (yet the experimenter doesn't know this concentration otherwise he wouldn't need measurements).
   (a) Compute the probability that a unique measurement is greater than 180 ?
   (b) What is the probability that the mean of three measurements is greater than 180 ?
   (c) How many measurements are necessary for the probability that the mean of these measurements is greater than 180 being less than 1% ?

Solution

1. $\left((\mathbb{R}, \mathcal{B}(\mathbb{R}), \{\mathcal{N}(m, \sigma^2), \theta = (m, \sigma) \in \mathbb{R} \times \mathbb{R}_+\}\right)$ we assume that the measurement of the atmospheric ozone concentration X is distributed from a Gaussian distribution $\mathcal{N}(m, \sigma^2)$ for some unknown $\theta = (m, \sigma) \in \mathbb{R} \times \mathbb{R}_+$. Here the population, all the possible measurements, is supposed to be $\mathbb{R}$, and we have a sample, one measurement x which is a realization of X.

2. In many applications, the data are concentrated around one value and look symmetric. The normal distribution being unimodal and symmetric is then a good candidate.
   In this exercise the population is supposed to be $\mathbb{R}$, the support of the Normal distribution is also $\mathbb{R}$, while the concentration cannot be negative. Worse, for any m, σ, the probability that X is negative is always positive. Yet given that the Gaussian distribution has light tails this probability is usually not high. For instance, when $m = 180$, for all $\sigma \leqslant 22$,

$$\mathbb{P}(X \leqslant 0) = \mathbb{P}\left(\frac{X - m}{\sigma} \leqslant \frac{-m}{\sigma}\right) = F_{\mathcal{N}(0,1)}(-m/\sigma) \leqslant 10^{-15}.$$

   We have to remember that 'all models are wrong but some are useful'. Then a good model is a model which is a good approximation of the reality and is tractable. You have to find a balance between the complex reality and an easy mathematical model which is useful in practice.

3. m and σ are in $\mu g/m^3$. m represents the real atmospheric concentration and σ the imprecision of the measure.

4. (a) This probability is

$$\begin{aligned} P_{\mathcal{N}(178,3.1)}(X \geq 180) &= P_{\mathcal{N}(178,3.1)}\left(\frac{X - 178}{\sqrt{3.1}} \geq \frac{180 - 178}{\sqrt{3.1}}\right) \\ &= P_{U \sim \mathcal{N}(0,1)}\left(U \geq \frac{2}{\sqrt{3.1}}\right) = 1 - F_{\mathcal{N}(0,1)}\left(\frac{2}{\sqrt{3.1}}\right) \sim 12.74\%, \end{aligned}$$

   where $F_{\mathcal{N}(0,1)}$ is the cdf of a standard normal distribution.

   (b)

$$\begin{aligned} P_{\mathcal{N}(183,3.1)}(X \leq 180) &= P_{\mathcal{N}(183,3.1)}\left(\frac{X - 183}{\sqrt{3.1}} \leq \frac{180 - 183}{\sqrt{3.1}}\right) \\ &= P_{U \sim \mathcal{N}(0,1)}\left(U \leq -\frac{3}{\sqrt{3.1}}\right) = P_{U \sim \mathcal{N}(0,1)}\left(U \geq \frac{3}{\sqrt{3.1}}\right) \\ &= 1 - F_{\mathcal{N}(0,1)}\left(\frac{3}{\sqrt{3.1}}\right) \sim 4.46\%. \end{aligned}$$

   (c)

$$\begin{aligned} P_{\mathcal{N}(180,3.1)}(180 - \delta \leq X \leq 180 + \delta) &= P_{\mathcal{N}(180,3.1)}\left(\frac{180 - \delta - 180}{\sqrt{3.1}} \leq \frac{X - 180}{\sqrt{3.1}} \leq \frac{180 + \delta - 180}{\sqrt{3.1}}\right) \\ &= P_{U \sim \mathcal{N}(0,1)}\left(-\frac{\delta}{\sqrt{3.1}} \leq U \leq \frac{\delta}{\sqrt{3.1}}\right) \\ &= 1 - 2P_{U \sim \mathcal{N}(0,1)}\left(U \geq \frac{\delta}{\sqrt{3.1}}\right) \leq 0.95 \\ &\Rightarrow 2F_{\mathcal{N}(0,1)}\left(\frac{3}{\sqrt{3.1}}\right) - 1 \leq 0.95, \end{aligned}$$

   hence, using the table of the cdf of the standard normal distribution, we see that $\delta = 3.45$ is a suitable choice.

5. The three last questions are probability questions, since we assume that we know the population, $m = ...$ and we search for the properties of a sample from this population. In inferential statistics, we do the opposite. We have a sample and we try to infer from this sample some information about the population. For instance, we could try to infer m, that is the average ozone concentration from a sample $x_1, \ldots, x_n$. One could infer m from the empirical mean of the sample, that $1/n \sum_{i=1}^n x_i$ if he has n observed measurements.

6. (a) The probability that a unique measure is greater than 180 is $1-F_{\mathcal{N}(0,1)}\left(\frac{2}{\sqrt{3.1}}\right) \sim 12.74\%$ (question 2.a).

   (b) Let X_1, X_2 and X_3 be the three measurements. The mean of these three measurements is $Z = \bar{X}_3 = 1/3\sum_{i=1}^{3} X_i$, it is distributed according to a normal distribution. Its expectation is $E(Z) = 178$ and its variance $Var(Z) = 3.1/3$. So that Z is distributed from $\mathcal{N}(178, 3.1/3)$. Then the probability that the mean of three measurements is greater than 180 is

$$P_{\mathcal{N}(178,3.1/3)}(Z \geq 180) = P_{\mathcal{N}(178,3.1/3)}\left(\frac{Z-178}{\sqrt{3.1/3}} \geq \frac{2}{\sqrt{3.1/3}}\right) = 1 - F_{\mathcal{N}(0,1)}\left(\frac{2\sqrt{3}}{\sqrt{3.1}}\right) \sim 2.44\%.$$

   (c) We assume that we have n measurements $X_1, \ldots, X_n$. The mean of these measurements is the random variable $\bar{X}_n = 1/n\sum_{i=1}^{n} X_i$. $\bar{X}_n$ is distributed from a normal distribution $\mathcal{N}(178, 3.1/n)$. We search for n such that $P\left(\bar{X}_n \geq 180\right) \leq 0.01$.

$$P_{\mathcal{N}(178,3.1/3)}\left(\bar{X}_n \geq 180\right) = P_{\mathcal{N}(178,3.1/3)}\left(\frac{\bar{X}_n - 178}{\sqrt{3.1/n}} \geq \frac{180-178}{\sqrt{3.1/n}}\right) = 1 - F_{\mathcal{N}(0,1)}\left(\frac{2\sqrt{n}}{\sqrt{3.1}}\right).$$

   From the table, we search n such that $\frac{2\sqrt{n}}{\sqrt{3.1}} \geq 2.33$. So that $n \geq 5$.

Exercise 3. We are in front of a black urn which contains N balls which are numbered from 1 to N. We don't know N the number of balls but we can draw as many balls as we wish if we put it in the urn before drawing another one.

1. Write the statistical model.
2. How can you guess the value of N with the observed numbers $x_1, \ldots, x_n$ during the sampling ?
3. What is the distribution of the greatest number $\hat{N} = \max(X_1, \ldots X_n)$?
4. How many balls do you want to draw ? Help : you can compute the probability that $\hat{N} = N$.

Solution

1. If we draw one ball and see the number X_1, the statistical model associated is

$$(\mathbb{N}, \mathcal{P}(\mathbb{N}), P_N, N \in \mathbb{N})$$

   where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$ and P_N, for $N \in \mathbb{N}$ is the uniform distribution on $\{1, \ldots, N\}$.
   If we draw n balls with numbers $X_1, \ldots, X_n$, the associated statistical model is

$$\left(\mathbb{N}^n, (\mathcal{P}(\mathbb{N}))^n, (P_N)^{\otimes n}, N \in \mathbb{N}\right)$$

   where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$ and P_N, for $N \in \mathbb{N}$ is the uniform distribution on $\{1, \ldots, N\}$. In the following we consider that we have observed n numbers $x_1, \ldots, x_n$ which are realizations of $X_1, \ldots, X_n$.
2. $\hat{N}_{x_1,\ldots,x_n} = \max(x_1, \ldots, x_n)$. Here we want to find the distribution of a maximum, we then use the cdf to characterize the distribution of $\hat{N}$. We assume that the observations come from P_N for some $N \in \mathbb{N}$. Let $t \in \mathbb{R}$ (common error : $t \in \mathbb{N}$ while the cdf is defined on $\mathbb{R}$, make a picture if necessary),

$$\begin{aligned} P_N(\hat{N} \leq t) &= P_N(\forall 1 \leq i \leq n,\ X_i \leq t) \\ &= \left(P_N(X_1 \leq t)\right)^n = \begin{cases} \left(\frac{\lfloor t \rfloor}{N}\right)^n & \text{if } 0 \leq t < N \\ 0 & \text{if } t < 0 \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

3. We want to draw as many balls as we can because for all N,

$$P_N(\hat{N} = N) = P_N(\hat{N} \leq N) - P(\hat{N} \leq N-1) = 1 - \left(\frac{N-1}{N}\right)^n \xrightarrow{n\to\infty} 1.$$