73-240 – Problem Set 5

Due Monday Nov. 25th

From the Syllabus:

1. Homework must be turned in on the day it is due (Monday Nov 25th) in class. Late homework will NOT be accepted unless you are sick and have a doctor's note.
2. If you would like to submit your HW early, please make arrangements with your TA to submit your HW during their office hours.
3. Homework regrading: There is a statute of limitations on regrades. If you believe a question has been incorrectly graded, please take your homework to your TA within 2 weeks of it being returned.
4. Working in groups: You may work in groups of up to 4. BUT: You MUST put names of other group members on your homework. You MUST write up your own set of answers. Do NOT simply copy some other person's work.
5. TYPE your work. Long equations may be hand written. Buy a stapler!
6. Write your first and last name on the title of each graph.
7. Carefully explain your work.

Problem 1: The Two Period Model (50 points)

Suppose the both Arthur and his son Adam have preferences over consumption today (c) and consumption tomorrow (c') given by

$$\sqrt{c} + 0.95\sqrt{c'}$$

so that their marginal rate of substitution is

$$MRS = \frac{\sqrt{c'}}{0.95\sqrt{c}}.$$

Arthur's income today (y) and tomorrow (y') are $y = 500, y' = 210$. Adam's income today and tomorrow are $y = 200$ and $y' = 525$. Suppose there are no lump-sum taxes.

A) Suppose that the interest rate is 5% ($r = 0.05$). What are the present values of Arthur and Adam's income?
Answer
For any level of current income, y, future income, y', and interest rate r, the present value of income is given by

$$PV = y + \frac{y'}{1+r}.$$

Therefore, Arthur's present value of income is

$$PV^{\texttt{Arthur}} = 500 + \frac{210}{1+0.05} = 700,$$

and Adam's present value of income is

$$PV^{\texttt{Adam}} = 200 + \frac{525}{1+0.05} = 700.$$

B) How much would each of them optimally consume in each period?
Answer
To find optimal consumption, recall that the optimality conditions from our two-period model are

$$MRS_{c,c'} = 1 + r,$$

which says that at an optimal choice, the slope of the indifference curve ($-MRS_{c,c'}$) should equal the slope of the (lifetime wealth) budget constraint ($-(1+r)$) and

$$c + \frac{c'}{1+r} = y + \frac{y'}{1+r}$$

which says that an optimal choice must lie *on* the (lifetime wealth) budget constraint. Given Arthur and Adam's preferences, we can find the $MRS_{c,c'}$:

$$MRS_{c,c'} = \frac{\frac{1}{2\sqrt{c}}}{0.95\frac{1}{2\sqrt{c'}}} = \frac{\sqrt{c'}}{0.95\sqrt{c}}.$$

The first optimality condition, then, tells us that

$$\frac{\sqrt{c'}}{0.95\sqrt{c}} = 1 + 0.05$$

or, after cross-multiplying and squaring both sides that

$$c' = [(0.95)(1.05)]^2 c.$$

Substituting for c' in the lifetime wealth budget constraint, we can find the optimal choice of c:

$$c + \frac{[(0.95)(1.05)]^2}{1.05} c = 700$$

(where I have substituted for Arthur and Adam's present value income. Solving for c we find $c = 359.4$. Then using the fact that $c' = [(0.95)(1.05)]^2 c$ we find $c' = 357.6$. Of course, since Adam and Arthur have the same present value of income and the same preferences, they choose the same optimal (c, c') bundle.

C) How much would each of them optimally save or borrow in the first period?
Answer Arthur will save $s = y - c = 500 - 359.4 = 140.6$ in the first period; Adam will save $s = y - c = 200 - 359.4 = -159.4$. Since Adam's savings is negative, we could also say Adam will *borrow* 159.4.

D) Suppose now that the interest rate 8%. What are the present values of their income?
Answer
At the new interest rate, Arthur's present value of income is

$$PV^{\texttt{Arthur}} = 500 + \frac{210}{1 + 0.08} = 694.4,$$

and Adam's present value of income is

$$PV^{\texttt{Adam}} = 200 + \frac{525}{1 + 0.05} = 686.1.$$

E) How much would each of them consume in each period in this case? How much do they save or borrow in the first period now? Do they consume more or less in the first period versus when the interest rate was 5%? How is their savings or borrowings affected?

Answer

Following the same steps as in part (B), we find Arthur will choose the consumption bundle $(c, c') = (351.7, 370.2)$, and Adam will choose the consumption bundle $(c, c') = (347.5, 365.8)$. Arthur will now save $500 - 351.7 = 148.3$, and Adam will now save $(200 - 347.5) = -147.5$, in other words Adam borrows 147.5. Compared with the case of risk free rate at 5%, both decrease their consumption in the current period. Arthur continues to save, but saves more. Adam continues to borrow, but borrows less.

F) Who is better off after the interest rate increase? Explain why a change of interest rate affects Arthur and Adam differently using graphs which illustrate the income and substitution effects for both Arthur and Adam.

Answer

We can calculate the utility of the two agents before and after the interest rate change to decide whether they are better off or worse off. At r = 5%, Arthur and Adam share the same utility

$$u(c, c') = \sqrt{c} + 0.95\sqrt{c'} = \sqrt{359.4} + 0.95\sqrt{357.6} = 36.92.$$

At r = 8%, Arthur's utility is

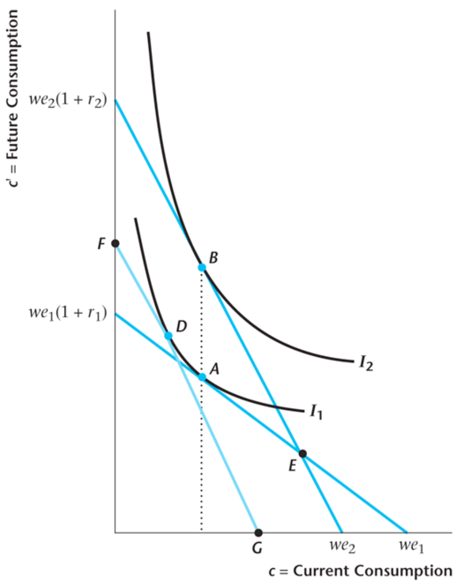
$$u(c, c') = \sqrt{351.7} + 0.95\sqrt{370.2} = 37.03 > 36.92.$$

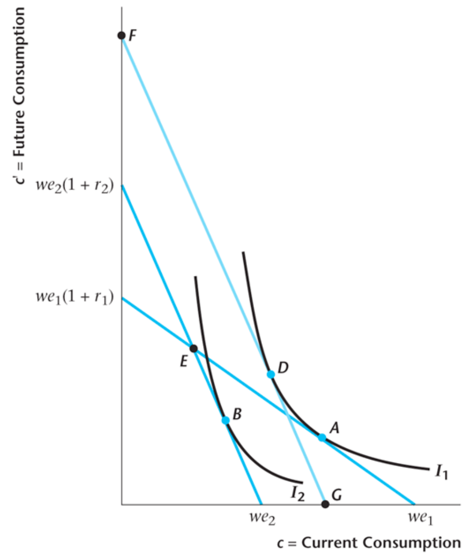
Since Arthur's utility rises, he is better off. Adam's utility is

$$u(c, c') = \sqrt{347.5} + 0.95\sqrt{365.8} = 36.80 < 36.92.$$

Since Adam's utility falls, Adam is worse off. We notice that the rise in interest rate affects the utility of a lender and a borrower differently. Notice that a change in interest rate changes the slope of the budget constraint, which results in both a substitution effect and an income effect. When r rises, the price of future consumption in terms of current consumption falls inducing a substitution effect that causes both a borrower and a lender to reduce current consumption and increase future consumption. When r rises, the present value of income also changes, inducing an income effect. The income effect makes lender richer and causes the lender to increase both current and future consumption. However, the income effect makes borrower poorer and causes the borrower to decrease both current and future consumption.

To see these effects graphically, consider the following figures:





The figure on the left shows the impact of an increase in r on a lender like Arthur. The movement from A to D represents the pure substitution effect while the movement from D to B represents the pure income effect. notice how the income effect shifts out (away from the origin) reflecting the beneficial nature of the increase in r for Arthur.

The figure on the right shows the impact of an increase in r on a borrower like Adam. The movement from A to D represents the pure substitution effect while the movement from D to B represents the pure income effect. notice how the income effect shifts in (towards the origin) reflecting the harmful nature of the increase in r for Adam.

Given the preferences an numbers in the figure, the new optimum for Arthur (the left figure) should lie along the new budget constraint but to left of point B (since his current consumption falls and savings rises).

Problem 2: Consumption-Savings (50 pts)

Consider the following endowment econonmy. Households receive after-tax income $y - t$ and $y' - t'$ in the first and second period respectively. Households can save in a bond that promises to return $(1 + r)$ units in the future. Utility is given by:

$$U(c, c') = \frac{c^{1-\gamma}}{1-\gamma} + \beta \frac{c'^{1-\gamma}}{1-\gamma}$$

A) Write down the household's problem.

Answer

$$\max_{c,c'} U(c, c') = \frac{c^{1-\gamma}}{1-\gamma} + \beta \frac{c'^{1-\gamma}}{1-\gamma}$$

s.t.

$$c + \frac{c'}{1+r} = y - t + \frac{y' - t'}{1+r}$$

B) Characterize the household's optimality conditions

Answer

We can write the Lagrangian as:

$$\max_{c,c'} \mathcal{L}(c, c', \lambda) = \frac{c^{1-\gamma}}{1-\gamma} + \beta \frac{c'^{1-\gamma}}{1-\gamma} + \lambda \left[y - t + \frac{y' - t'}{1+r} - c - \frac{c'}{1+r} \right]$$

Taking first order conditions (FOC) with respect to c we have:

$$c^{-\gamma} = \lambda$$

taking FOC with respect to c':

$$\beta c^{', -\gamma} = \frac{\lambda}{1+r}$$

and with respect to λ, we get the lifetime budget constraint:

$$c + \frac{c'}{1+r} = y - t + \frac{y' - t'}{1+r}$$

The lifetime budget constraint is one of the optimality conditions, the other is the dynamic optimality condition. Using our two first order conditions with respect to c and c', we have:

$$c^{-\gamma} = \beta(1+r)c^{', -\gamma}$$

where the above shows marginal benefit of consumption equal to its marginal cost.

C) Using your dynamic optimality condition from part B), explain what happens to the ratio c'/c if the household becomes more patient, β rises. Explain what happens to the ratio c'/c if the interest rate rises.

Answer

Re-arranging our dynamic optimality condition, we have:

$$\frac{c'}{c} = [\beta(1+r)]^{1/\gamma}$$

For $\gamma \geq 0$, we have that as β rises, c'/c rises, For $\gamma \geq 0$, we have that as r rises, c'/c rises. Intuitively, if the household becomes more patient, i.e. β rises, it wants to allocate more consumption towards tomorrow, so c'/c rises. If interest rate goes up and consumption today becomes more expensive, the substitution effect makes the household want to raise consumption tomorrow relative to today, c'/c rises.

D) The intertemporal elasticity of substitution (IES) is given by:

$$IES = -\frac{d\ln(c'/c)}{d\ln(u'(c')/u'(c))}$$

In words, the intertemporal elasiticity of substitution measures the percentage change in the growth rate of consumption (represented by $\ln(c'/c)$) in response to a 1% change in marginal utilities (represented by $\ln(u'(c')/u'(c))$). The IES measures how willing individuals are to intertemporally substitute consumption today and consumption next period. Prove that the IES in this problem is given by $\frac{1}{\gamma}$

Answer

Observe that $u'(c')/u'(c)$ can be expressed as:

$$\frac{u'(c')}{u'(c)} = \left(\frac{c'}{c}\right)^{-\gamma}$$

which implies:

$$\frac{c'}{c} = \left[\frac{u'(c')}{u'(c)}\right]^{-1/\gamma}$$

Also note that the following is true:

$$-\frac{d\ln(x^{-1/\gamma})}{d\ln x} = \frac{1}{\gamma}\frac{d\ln x}{d\ln x} = \frac{1}{\gamma}$$

So now we can denote $\frac{u'(c')}{u'(c)} = x$ and using the above information, we can show that the IES is given by:

$$IES = -\frac{d\ln(c'/c)}{d\ln(u'(c')/u'(c))} = -\frac{d\ln(x^{-1/\gamma})}{d\ln x} = \frac{1}{\gamma}\frac{d\ln x}{d\ln x} = \frac{1}{\gamma}$$

E) Apply the natural log to your dynamic optimality equation that you derived in part B). State whether the growth rate of consumption is more sensitive to changes in the interest rate as γ falls. Give an explanation as to why this occurs. [Hint: what happens to the IES as γ falls?]

Answer

Applying the natural log to our dynamic optimality condition, we have:

$$\ln\left(\frac{c'}{c}\right) = \frac{1}{\gamma}\ln\beta + \frac{1}{\gamma}\ln(1+r)$$

Since $\ln\left(\frac{c'}{c}\right) = \ln(c') - \ln(c)$ which is equivalent to the growth rate of consumption, we can see that as γ falls, consumption growth rates become more sensitive to changes in r. Intuitively, a fall in γ raises the IES $(\frac{1}{\gamma})$. When households have a higher intertemporal elasticity of substitution, they are more willing to substitute tomorrow's consumption for today. As such, if r rises, they respond more to the interest rate and substitute towards tomorrow's consumption.

F) Now suppose $\gamma = 1$[1]. Starting from the optimality conditions your derived in part B), solve for optimal consumption today c. Solve for optimal savings s.

Answer

In this case, we get from our dynamic optimality condition:

$$c' = \beta(1+r)c$$

plugging into our budget constraint, we get:

$$c + \frac{\beta(1+r)c}{1+r} = y - t + \frac{y'-t'}{1+r}$$

so we arrive at:

$$c^* = \frac{y-t}{1+\beta} + \frac{y'-t'}{(1+r)(1+\beta)}$$

optimal savings is given by $s = y - t - c$, and so we have:

$$s^* = \frac{\beta}{1+\beta}(y-t) - \frac{y'-t'}{(1+r)(1+\beta)}$$

[1]You do not need the following information to solve part F), but for those who are interested: in the limit when $\gamma = 1$, we have:

$$\lim_{\gamma\to 1}\frac{c^{1-\gamma}}{1-\gamma} = \ln c$$