Advanced Optimization
Lecture 4: Gradient Descent, Constrained and Linear Optimization

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CentraleSupélec / ESSEC Business School

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Course Overview

		Topic
Wed, 13.10.2021	PM	Introduction, examples of problems, problem types
Wed, 20.10.2021	PM	Continuous (unconstrained) optimization: convexity, gradients, Hessian, … [technical test Evalmee]
Wed, 27.10.2021	PM	Continous optimization II: [1st mini-exam] Constrained optimization: Lagrangian, optimality conditions
Wed, 03.11.2021	PM	gradient descent, Newton direction, quasi-Newton (BFGS) Linear programming: duality, maxflow/mincut, simplex algo
Wed, 10.11.2021	PM	Gradient-based and derivative-free stochastic algorithms: SGD and CMA-ES
Wed, 17.11.2021	PM	Other blackbox optimizers: Nelder-Mead, Bayesian optimization *[2nd mini-exam]*
Wed, 24.11.2021	PM	Benchmarking solvers: runtime distributions, performance profiles
Tue, 30.11.2021	23:59	Deadline open source project (PDF sent by email)
Wed, 01.12.2021	PM	Discrete optimization: branch and bound, branch and cut, k-means clustering
Wed, 15.12.2021	PM	Exam

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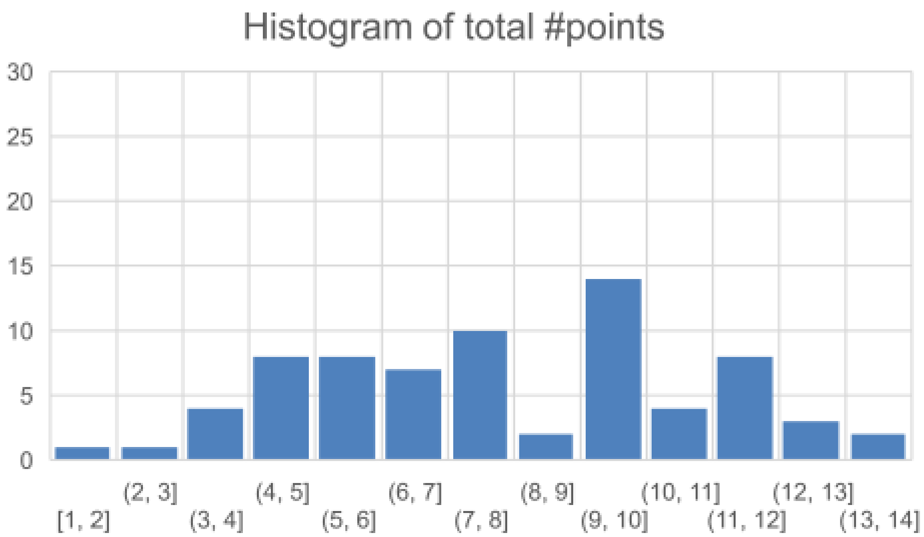
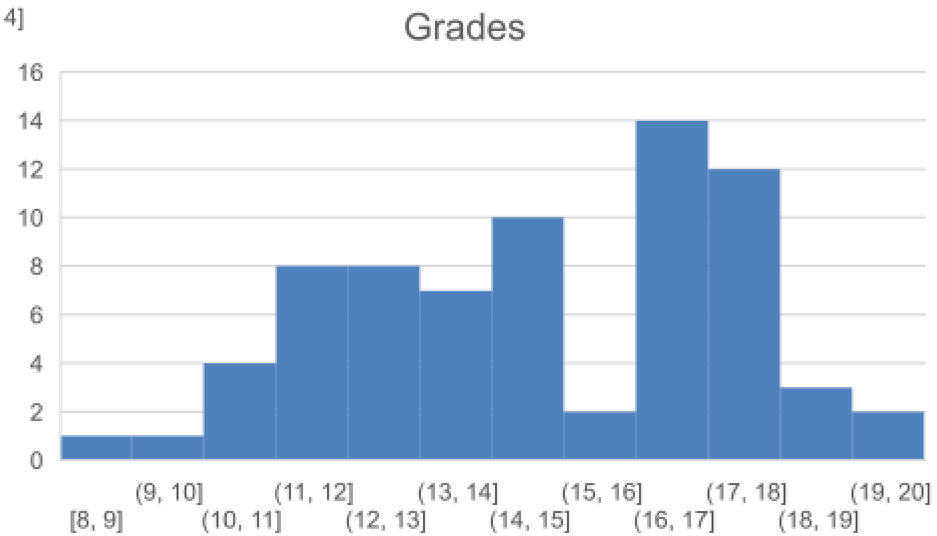
speaking of the mini-exam…

Grading Almost Finalized…





Details on Continuous Optimization Lectures

Introduction to Continuous Optimization

- examples and typical difficulties in optimization

Mathematical Tools to Characterize Optima

- reminders about differentiability, gradient, Hessian matrix
- unconstraint optimization
  - first and second order conditions
  - convexity
- constraint optimization
  - Lagrangian, optimality conditions

Gradient-based Algorithms

- gradient descent
- quasi-Newton method (BFGS) and invariances

Linear programming, duality

Learning in Optimization / Optimization in Machine Learning

- Stochastic gradient descent (SGD) + Adam
- CMA-ES (adaptive algorithms / Information Geometry)
- Other derivative-free algorithms: Nelder-Mead, Bayesian opt.

back to
optimality conditions
in constrained optimization

Reminder: Euler-Lagrange Equation

Theorem:

Be U an open set of $(E, ||\,.\,||)$, and $f: U \to \mathbb{R}$, $g: U \to \mathbb{R}$ in $\mathcal{C}^1$.

Let $a \in E$ satisfy

$$\begin{cases} f(a) = \inf \{f(x) \mid x \in \mathbb{R}^n, g(x) = 0\} \\ g(a) = 0 \end{cases}$$

i.e. a is optimum of the problem

If $\nabla g(a) \neq 0$, then there exists a constant $\lambda \in \mathbb{R}$ called *Lagrange multiplier*, such that

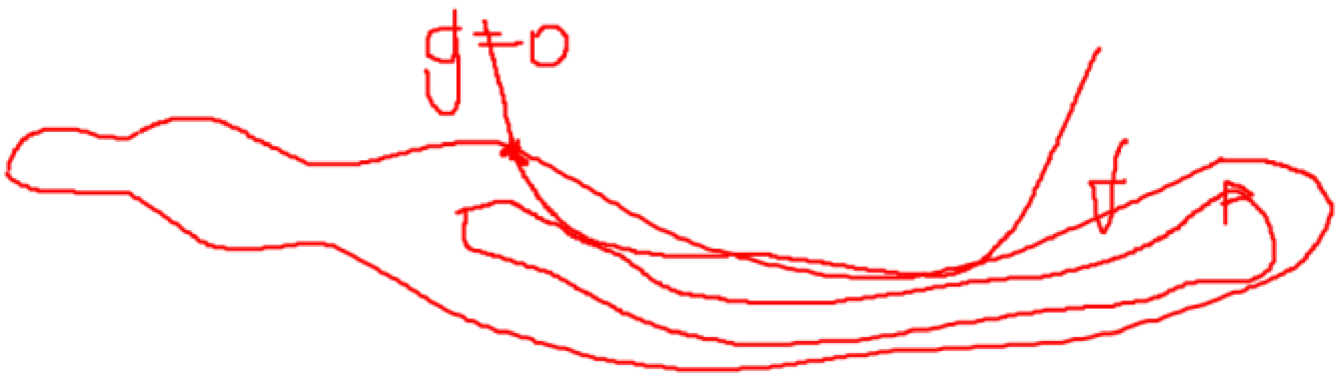
$$\underbrace{\nabla f(a) + \lambda \nabla g(a) = 0}_{} \quad \text{Euler} - \text{Lagrange equation}$$

i.e. gradients of f and g in a are colinear

Interpretation of Euler-Lagrange Equation

Intuitive way to retrieve the Euler-Lagrange equation:

- In a local minimum a of a constrained problem, the hypersurfaces (or level sets) $f = f(a)$ and $g = 0$ are necessarily tangent (otherwise we could decrease f by moving along $g = 0$).



- Since the gradients $\nabla f(a)$ and $\nabla g(a)$ are orthogonal to the level sets $f = f(a)$ and $g = 0$, it follows that $\nabla f(a)$ and $\nabla g(a)$ are colinear.

Generalization to More than One Constraint

Theorem

- Assume $f: U \to \mathbb{R}$ and $g_k: U \to \mathbb{R}$ $(1 \leq k \leq p)$ are $\mathcal{C}^1$.
- Let a be such that

$$\begin{cases} f(a) = \inf \{f(x) \mid x \in \mathbb{R}^n, \quad g_k(x) = 0, \quad 1 \leq k \leq p\} \\ \quad g_k(a) = 0 \ \text{ for all } 1 \leq k \leq p \end{cases}$$

- If $\left(\nabla g_k(a)\right)_{1 \leq k \leq p}$ are linearly independent, then there exist p real constants $(\lambda_k)_{1 \leq k \leq p}$ such that

$$\nabla f(a) + \sum_{k=1}^{p} \lambda_k \nabla g_k(a) = 0$$

Lagrange multiplier

again: a does not need to be global but local minimum

The Lagrangian

- Define the Lagrangian on $\mathbb{R}^n \times \mathbb{R}^p$ as

$$\mathcal{L}(x, \{\lambda_k\}) = f(x) + \sum_{k=1}^{p} \lambda_k g_k(x)$$

- To find optimal solutions, we can solve the optimality system

$$\begin{cases} \text{Find } (x, \{\lambda_k\}) \in \mathbb{R}^n \times \mathbb{R}^p \text{ such that } \nabla f(x) + \sum_{k=1}^{p} \lambda_k \nabla g_k(x) = 0 \\ g_k(x) = 0 \;\text{ for all } 1 \le k \le p \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{Find } (x, \{\lambda_k\}) \in \mathbb{R}^n \times \mathbb{R}^p \text{ such that } \nabla_x \mathcal{L}(x, \{\lambda_k\}) = 0 \\ \nabla_{\lambda_k} \mathcal{L}(x, \{\lambda_k\})(x) = 0 \;\text{ for all } 1 \le k \le p \end{cases}$$

Inequality Constraint: Definitions

Let $\mathcal{U} = \{x \in \mathbb{R}^n \mid g_k(x) = 0 \text{ (for } k \in E),\ g_k(x) \leq 0 \text{ (for } k \in I)\}$.

Definition:

The points in $\mathbb{R}^n$ that satisfy the constraints are also called *feasible* points.

Definition:

Let $a \in \mathcal{U}$, we say that the constraint $g_k(x) \leq 0$ (for $k \in I$) is *active* in a if $g_k(a) = 0$.

The definition of active constraints
was correct in the slides during the
lecture, but the examples were probably not all.
I apologize and will give some more examples next time.

Theorem (Karush-Kuhn-Tucker, KKT):

Let U be an open set of $(\mathbb{R}^n, || \, ||)$ and $f: U \to \mathbb{R}$, $g_k: U \to \mathbb{R}$, all $\mathcal{C}^1$

Furthermore, let $a \in U$ satisfy

$$\begin{cases} f(a) = \inf(f(x) \mid x \in \mathbb{R}^n, g_k(x) = 0 \ (\text{for } k \in E), g_k(x) \leq 0 \ (\text{for } k \in \mathrm{I}) \\ g_k(a) = 0 \ (\text{for } k \in E) \\ g_k(a) \leq 0 \ (\text{for } k \in I) \end{cases}$$

also works again for a being a local minimum

Let I_a^0 be the set of constraints that are active in a. Assume that $\left(\nabla g_k(a)\right)_{k \in E \cup I_a^0}$ are linearly independent.

Then there exist $(\lambda_k)_{1 \leq k \leq p}$ that satisfy

$$\begin{cases} \nabla f(a) + \sum_{k=1}^{p} \lambda_k \nabla g_k(a) = 0 \\ g_k(a) = 0 \ (\text{for } k \in E) \\ g_k(a) \leq 0 \ (\text{for } k \in I) \\ \lambda_k \geq 0 \ (\text{for } k \in I_a^0) \\ \lambda_k g_k(a) = 0 \ (\text{for } k \in E \cup I) \end{cases}$$

Inequality Constraint: Karush-Kuhn-Tucker Theorem

Theorem (Karush-Kuhn-Tucker, KKT):

Let U be an open set of $(\mathbb{R}^n, ||\ ||)$ and $f: U \to \mathbb{R}$, $g_k: U \to \mathbb{R}$, all $\mathcal{C}^1$

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either active constraint or $\lambda_k = 0$

Descent Methods

[methods for unconstrained problems]

Descent Methods

General principle

1. choose an initial point $\boldsymbol{x}_0$, set $t = 0$
2. while not happy
   - choose a descent direction $\boldsymbol{d}_t \neq 0$
   - line search:
     - choose a step size $\sigma_t > 0$
     - set $\boldsymbol{x}_{t+1} = \boldsymbol{x}_t + \sigma_t \boldsymbol{d}_t$
   - set $t = t + 1$

Remaining questions

- how to choose $\boldsymbol{d}_t$?
- how to choose σ_t?

Gradient Descent

Rationale: $\boldsymbol{d}_t = -\nabla f(\boldsymbol{x}_t)$ is a descent direction

indeed for f differentiable

$$f\big(x - \sigma\nabla f(x)\big) = f(x) - \sigma||\nabla f(x)||^2 + o(\sigma||\nabla f(x)||)$$
$$< f(x) \text{ for } \sigma \text{ small enough}$$

Step-size

- optimal step-size: $\sigma_t = \underset{\sigma}{\operatorname{argmin}} f(\boldsymbol{x}_t - \sigma\nabla f(\boldsymbol{x}_t))$
- **Line Search:** total or partial optimization w.r.t. σ
  Total is however often too "expensive" (needs to be performed at each iteration step)
  Partial optimization: execute a limited number of trial steps until a loose approximation of the optimum is found. Typical rule for partial optimization: Armijo rule (see next slides)

Typical stopping criterium:

norm of gradient smaller than ϵ

The Armijo-Goldstein Rule

Choosing the step size:

- Only decreasing f-value is not enough to converge (quickly)
- Want to have a reasonably large decrease in f

Armijo-Goldstein rule:

- also known as backtracking line search
- starts with a (too) large estimate of σ and reduces it until f is reduced enough
- what is enough?
  - assuming a linear f e.g. $m_k(x) = f(x_k) + \nabla f(x_k)^T(x - x_k)$
  - expected decrease if step of σ_k is done in direction $\boldsymbol{d}$: $\sigma_k \nabla f(x_k)^T \boldsymbol{d}$
  - actual decrease: $f(x_k) - f(x_k + \sigma_k \boldsymbol{d})$
  - stop if actual decrease is at least constant times expected decrease (constant typically chosen in [0, 1])

The Armijo-Goldstein Rule

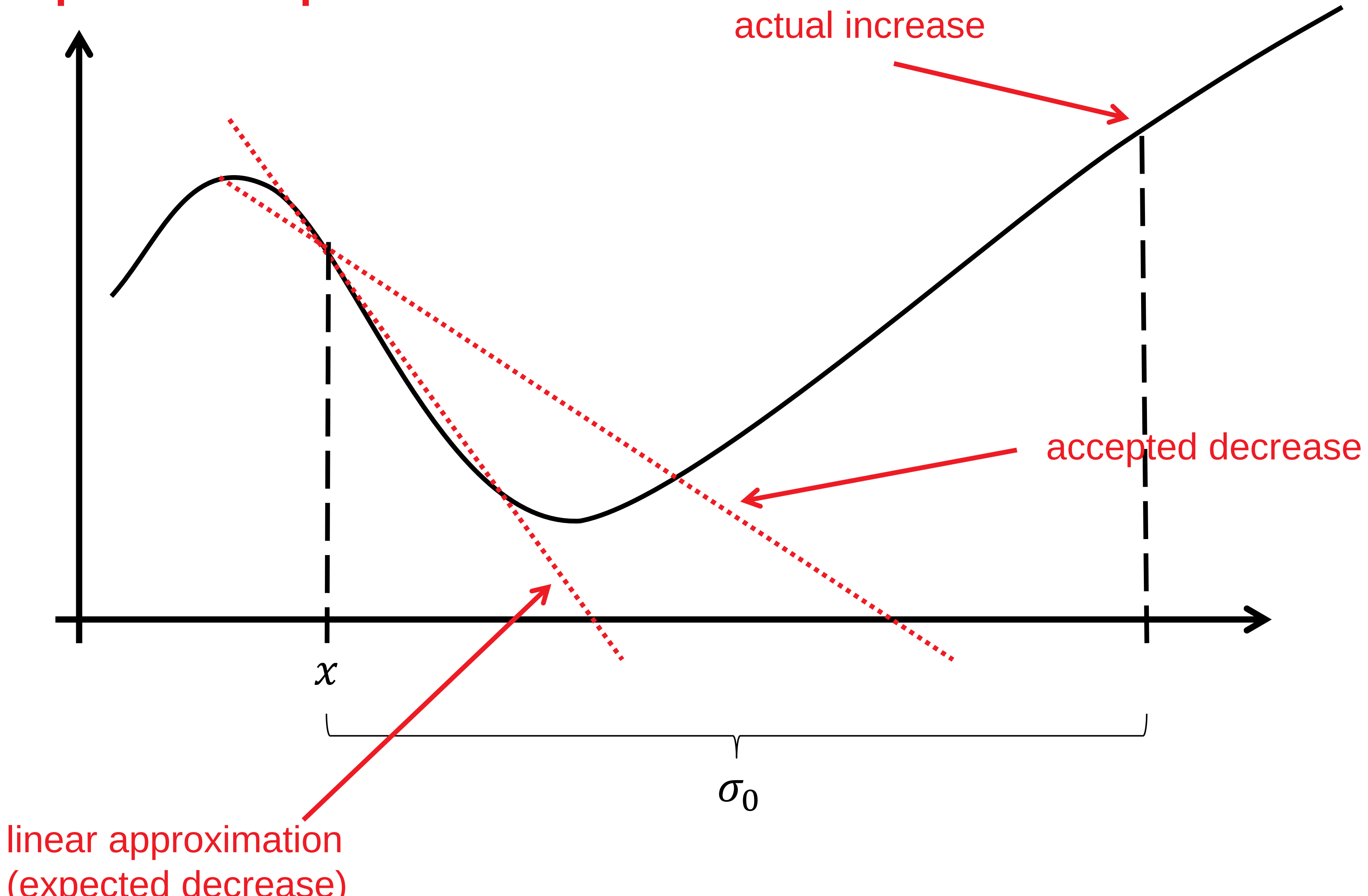
The Actual Algorithm:

Input: descent direction $\mathbf{d}$, point $\mathbf{x}$, objective function $f(\mathbf{x})$ and its gradient $\nabla f(\mathbf{x})$, parameters $\sigma_0 = 10$, $\theta \in [0,1]$ and $\beta \in (0,1)$
Output: step-size σ

Initialize σ: $\sigma \leftarrow \sigma_0$
while $f(\mathbf{x} + \sigma\mathbf{d}) > f(\mathbf{x}) + \theta\sigma\nabla f(\mathbf{x})^T\mathbf{d}$ **do**
    $\sigma \leftarrow \beta\sigma$
end while

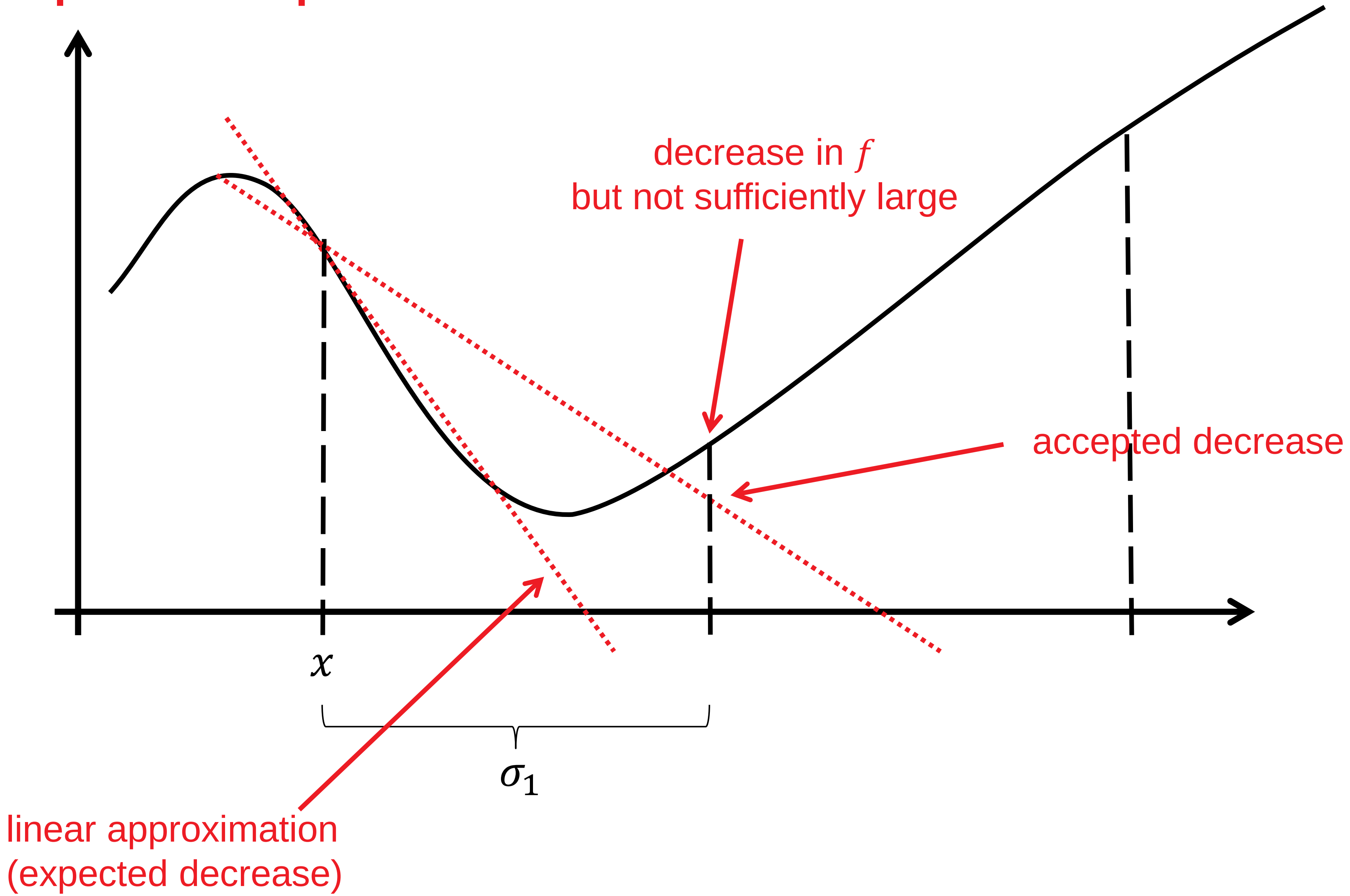
Armijo, in his original publication chose $\beta = \theta = 0.5$.
Choosing $\theta = 0$ means the algorithm accepts any decrease.

Graphical Interpretation



The Armijo-Goldstein Rule

Graphical Interpretation



The Armijo-Goldstein Rule

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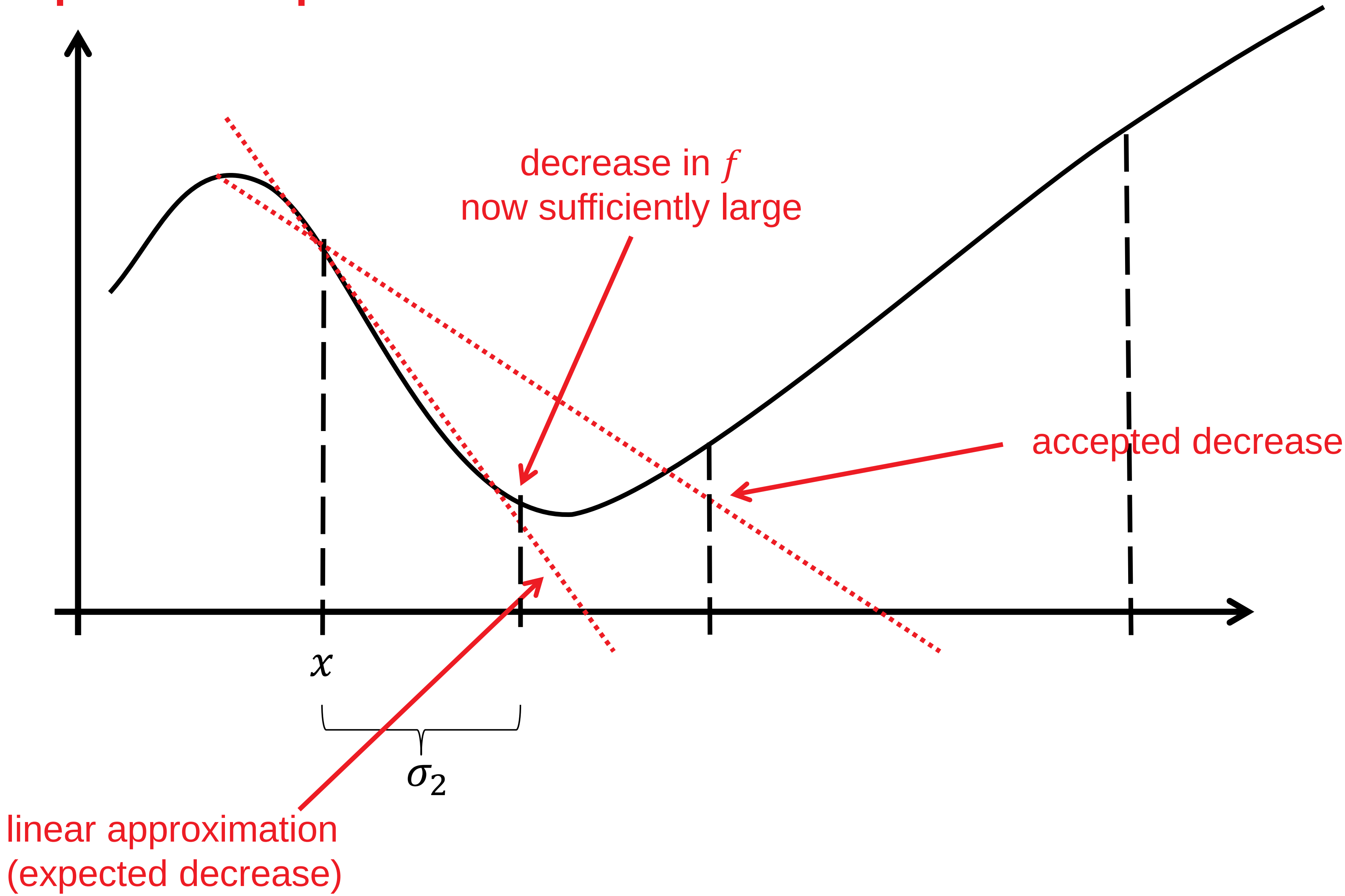


Gradient Descent: Simple Theoretical Analysis

Assume f is twice continuously differentiable, convex and that $\mu I_d \preccurlyeq \nabla^2 f(x) \preccurlyeq L I_d$ with $\mu > 0$ holds, assume a fixed step-size $\sigma_t = \frac{1}{L}$

Note: $A \preccurlyeq B$ means $x^T A x \leq x^T B x$ for all x

$$x_{t+1} - x^* = x_t - x^* - \sigma_t \nabla^2 f(y_t)(x_t - x^*) \text{ for some } y_t \in [x_t, x^*]$$

$$x_{t+1} - x^* = \left(I_d - \frac{1}{L} \nabla^2 f(y_t) \right) (x_t - x^*)$$

$$\text{Hence } ||x_{t+1} - x^*||^2 \leq |||I_d - \tfrac{1}{L} \nabla^2 f(y_t)|||^2 \; ||x_t - x^*||^2$$

$$\leq \left(1 - \frac{\mu}{L}\right)^2 ||x_t - x^*||^2$$

Linear convergence: $||x_{t+1} - x^*|| \leq \left(1 - \frac{\mu}{L}\right) ||x_t - x^*||$

algorithm slower and slower with increasing condition number

Non-convex setting: convergence towards stationary point

Newton Method

- descent direction: $-[\nabla^2 f(x_k)]^{-1}\nabla f(x_k)$ [so-called Newton direction]
- The Newton direction:
  - minimizes the best (locally) quadratic approximation of f: $\tilde{f}(x+\Delta x) = f(x) + \nabla f(x)^T \Delta x + \frac{1}{2}(\Delta x)^T \nabla^2 f(x) \Delta \mathrm{x}$
  - points towards the optimum on $f(x) = (x - x^*)^T A (x - x^*)$
- however, Hessian matrix is expensive to compute in general and its inversion is also not easy

quadratic convergence

$$\left(\text{i.e.} \quad \lim_{k \to \infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^2} = \mu > 0\right)$$

Remark: Affine Invariance

Affine Invariance: same behavior on $f(x)$ and $f(Ax+b)$ for $A \in \mathrm{GLn}(\mathbb{R})$ = set of all invertible $n \times n$ matrices over $\mathbb{R}$

- Newton method is affine invariant

  see **http://users.ece.utexas.edu/~cmcaram/EE381V_2012F/Lecture_6_Scribe_Notes.final.pdf**

- same convergence rate on all convex-quadratic functions
- Gradient method not affine invariant

Quasi-Newton Method: BFGS

$x_{t+1} = x_t - \sigma_t H_t \nabla f(x_t)$ where H_t is an approximation of the inverse Hessian

Key idea of Quasi Newton:

successive iterates x_t, x_{t+1} and gradients $\nabla f(x_t)$, $\nabla f(x_{t+1})$ yield second order information

$$q_t \approx \nabla^2 f(x_{t+1}) p_t$$

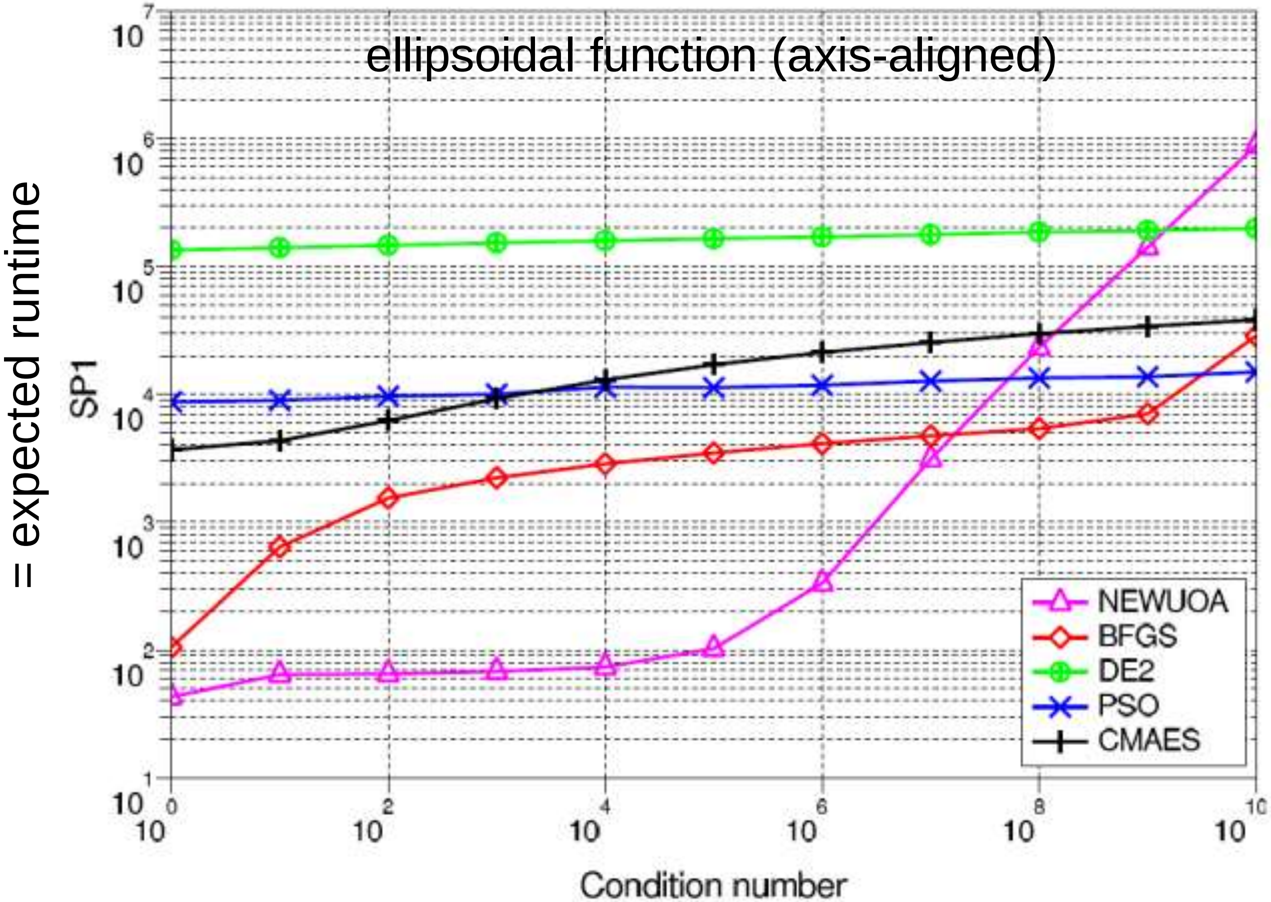
where $p_t = x_{t+1} - x_t$ and $q_t = \nabla f(x_{t+1}) - \nabla f(x_t)$

Most popular implementation of this idea: Broyden-Fletcher-Goldfarb-Shanno (BFGS)

- default in MATLAB's **`fminunc`** and python's **`scipy.optimize.minimize`**

Once Again: Invariances

- Newton as well as BFGS are affine invariant in theory
- However, numerics might play an important role



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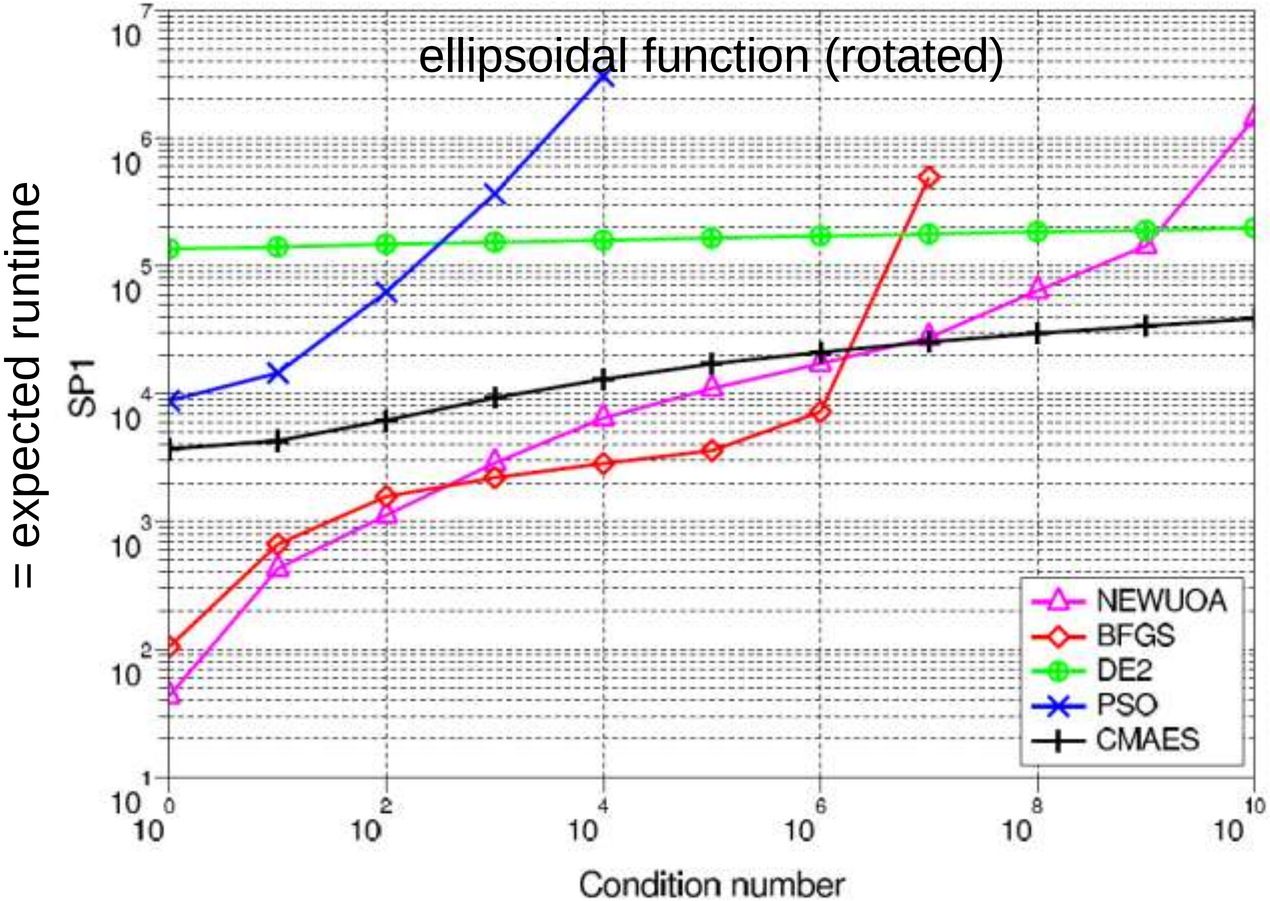

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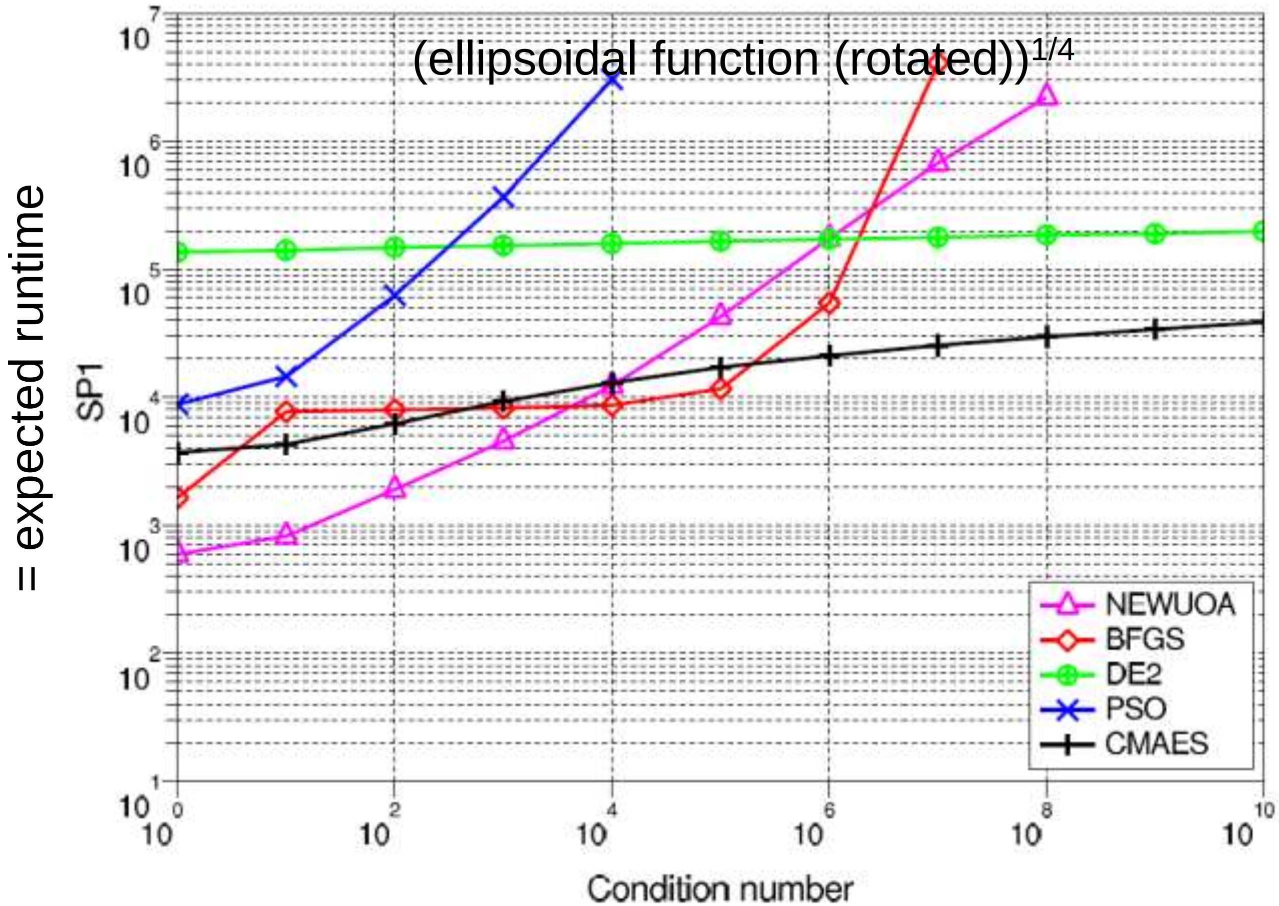
Once Again: Invariances

Other invariances might be nice as well, for example in function-value-free algorithms…



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discussion advanced exercise on invariance

see jupyter notebook in Edunao

Conclusions

I hope it became clear so far...

...what is the difference between gradient and Newton direction and gradient descent and (quasi-)Newton algorithms

... that adapting the step size in descent algorithms is crucial and

… what invariances are and why they are important in optimization (more details on that in the CMA-ES lecture)

Linear Optimization

[optimization with linear objective and linear constraints functions]

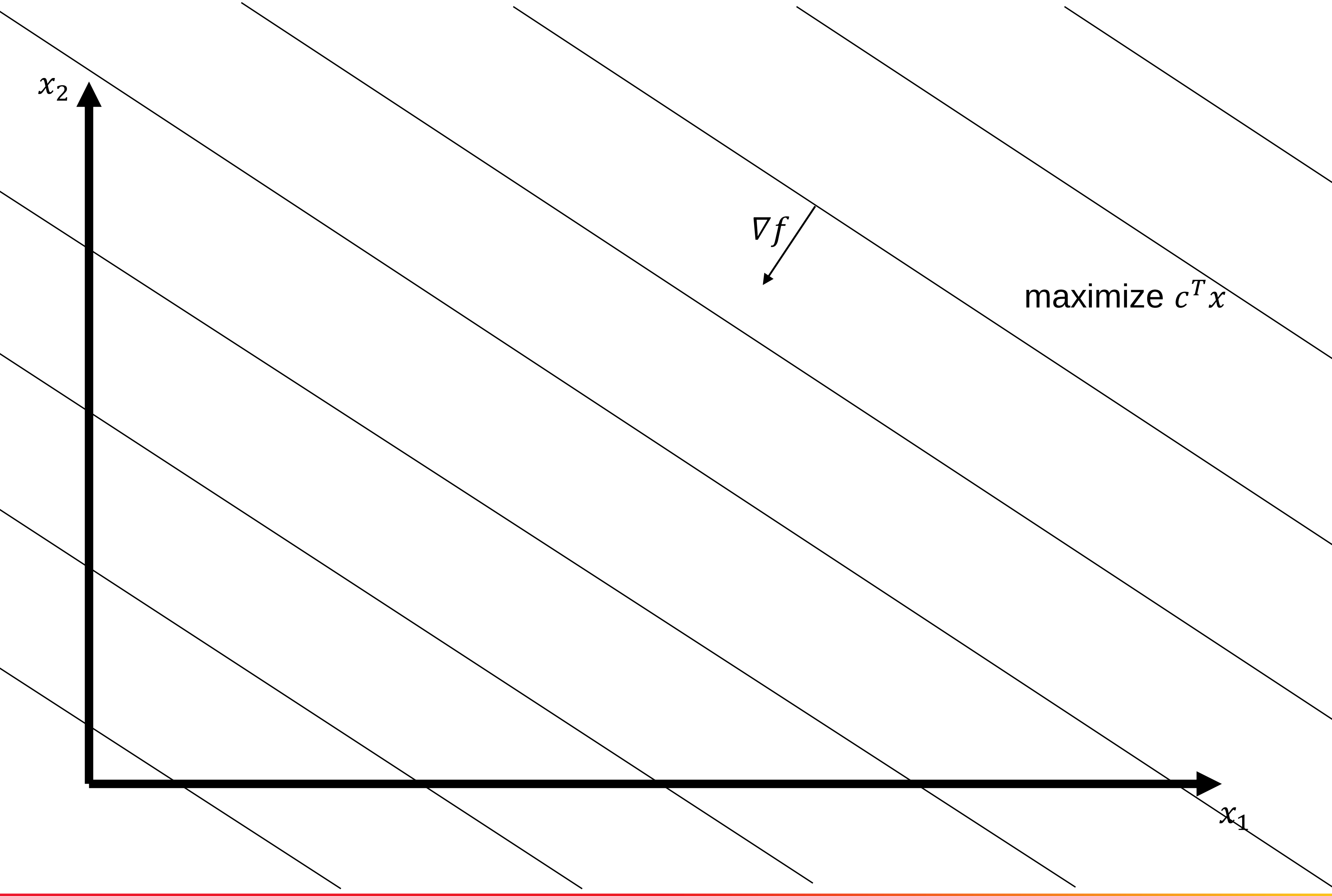
Linear Programming

Linear programming = linear optimization

Find a vector x that

- maximizes $c^T x$
- s.t. $Ax \leq b$
- and $x \geq 0$

Graphical Illustration



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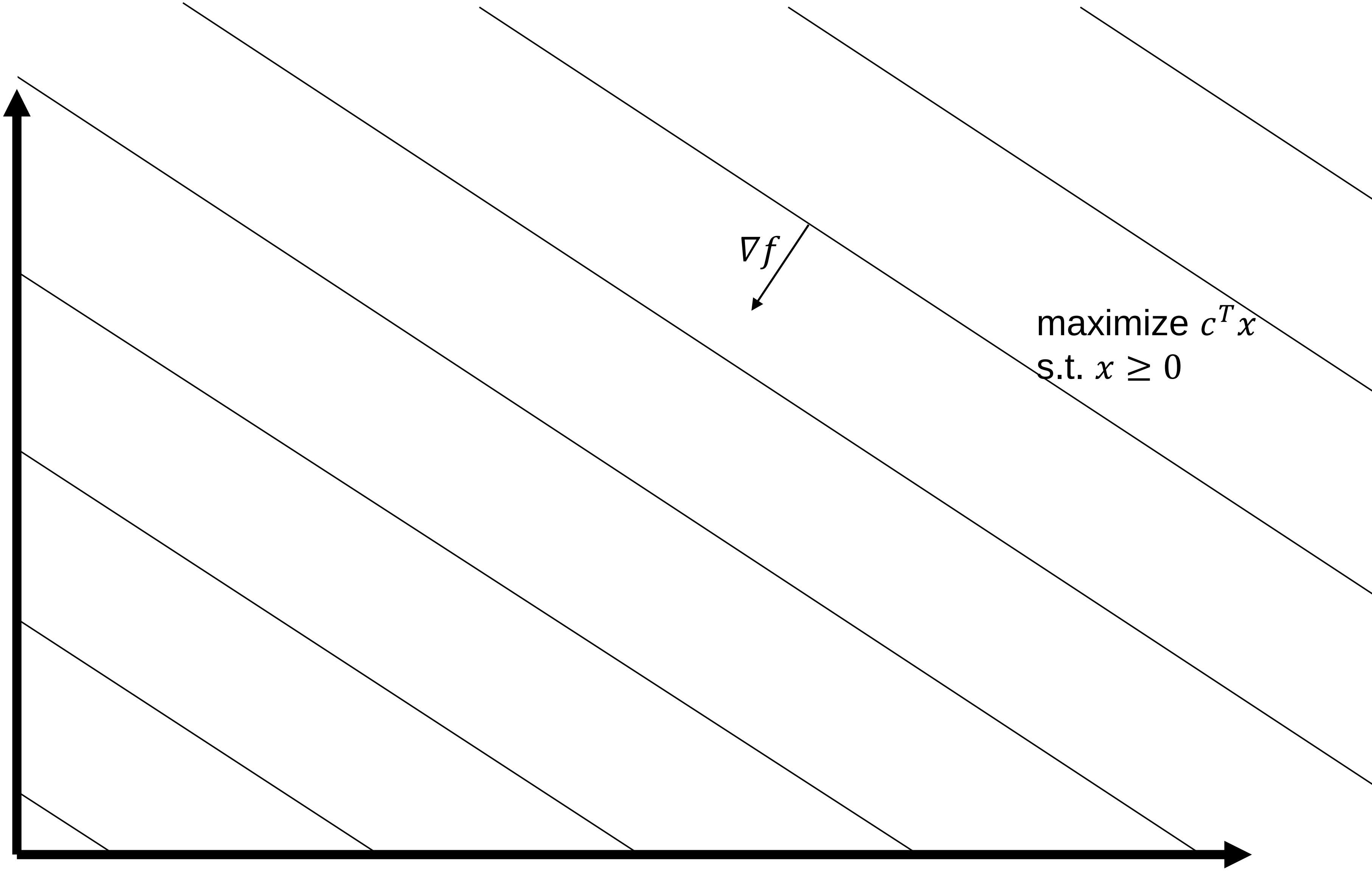
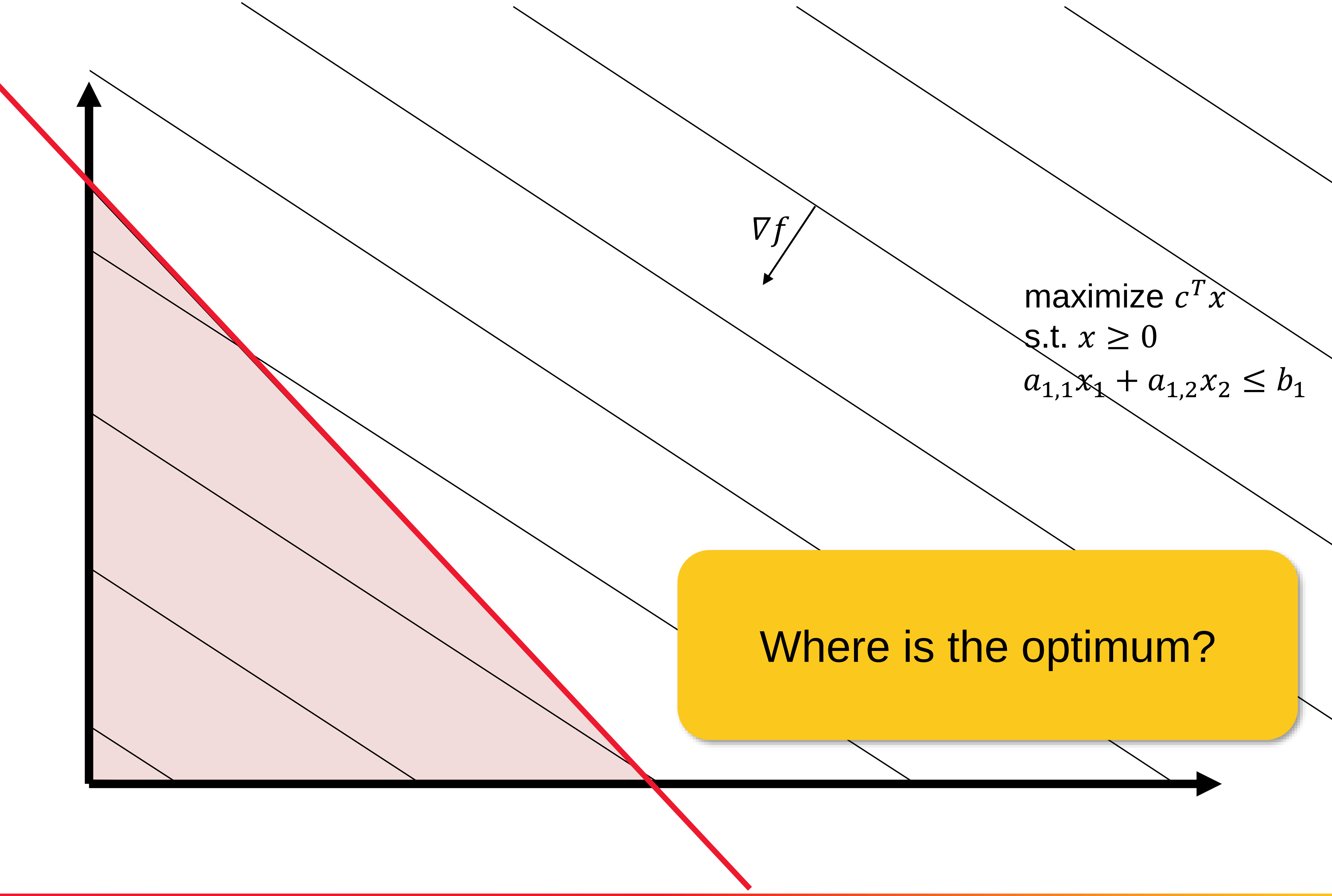
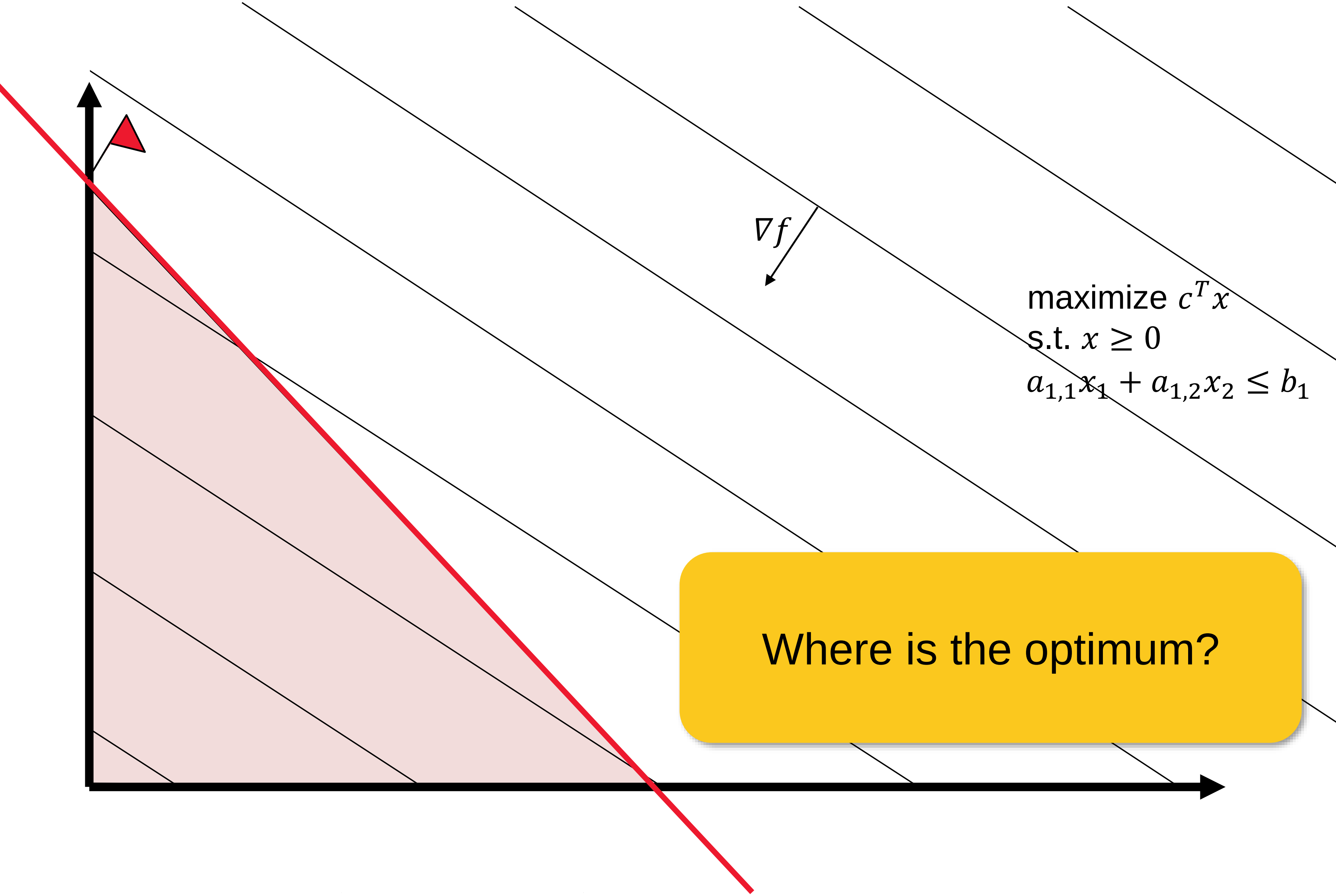




maximize $c^T x$
s.t. $x \geq 0$
$a_{1,1}x_1 + a_{1,2}x_2 \leq b_1$

Where is the optimum?

Graphical Illustration



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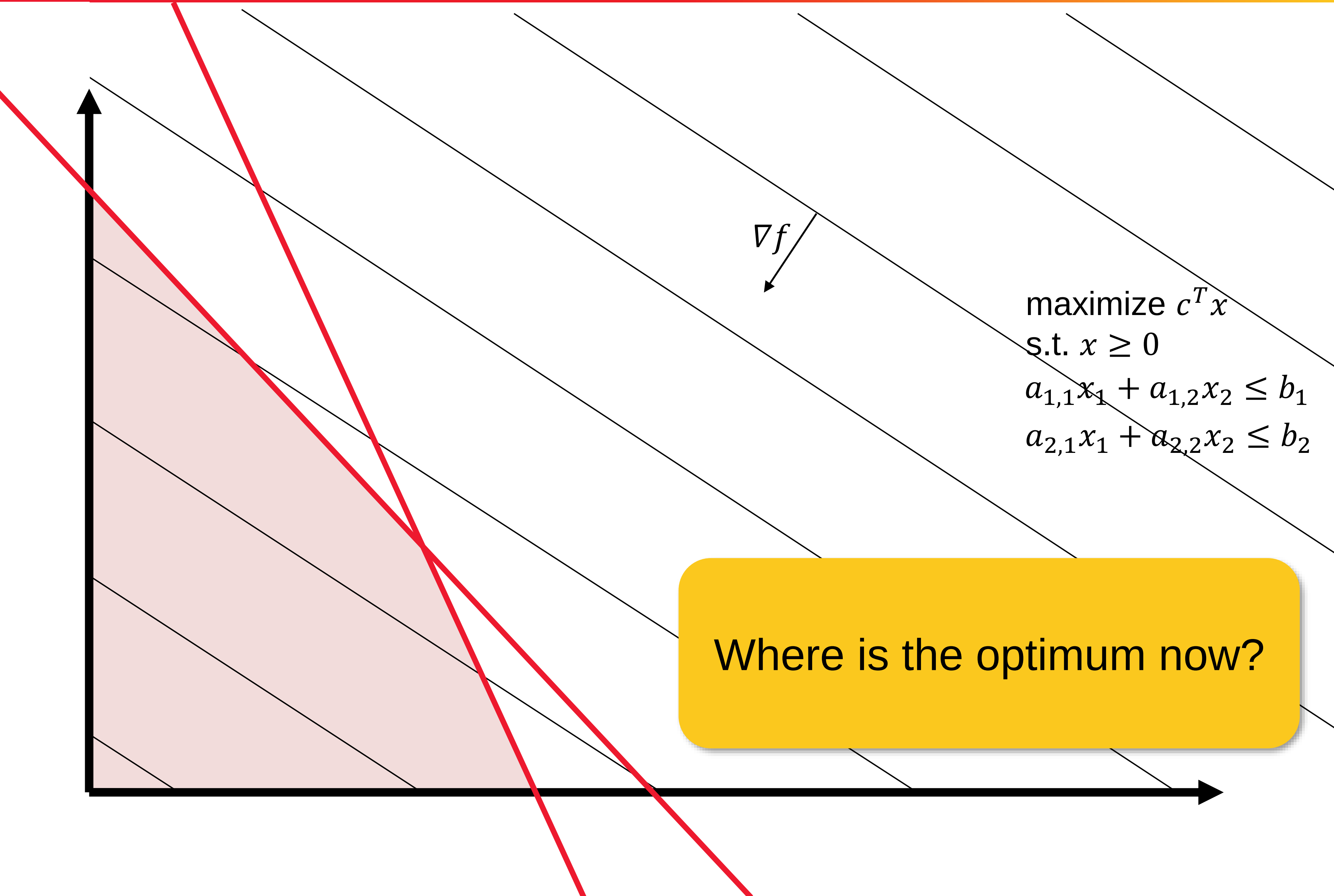

maximize $c^T x$
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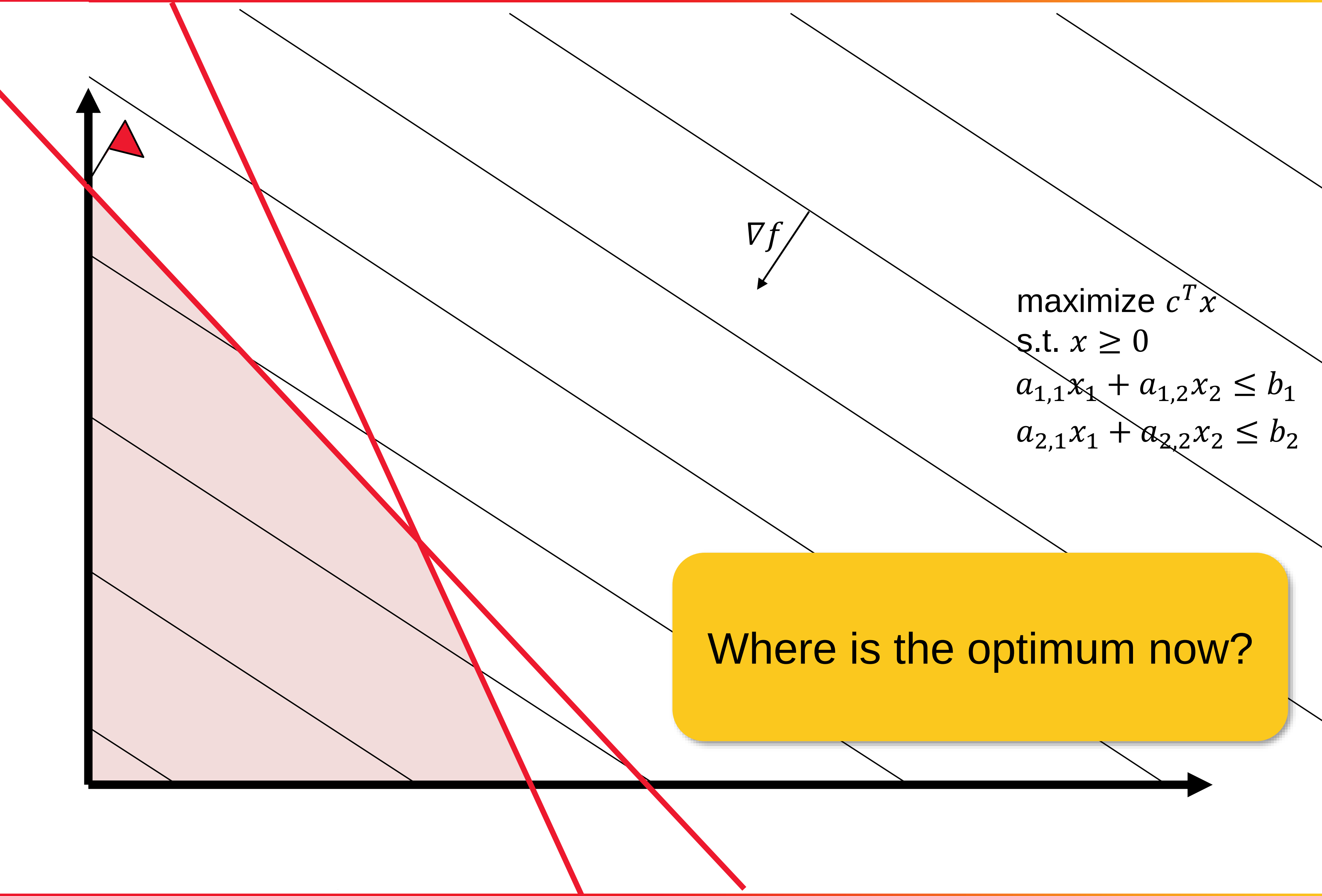
Where is the optimum now?

Graphical Illustration



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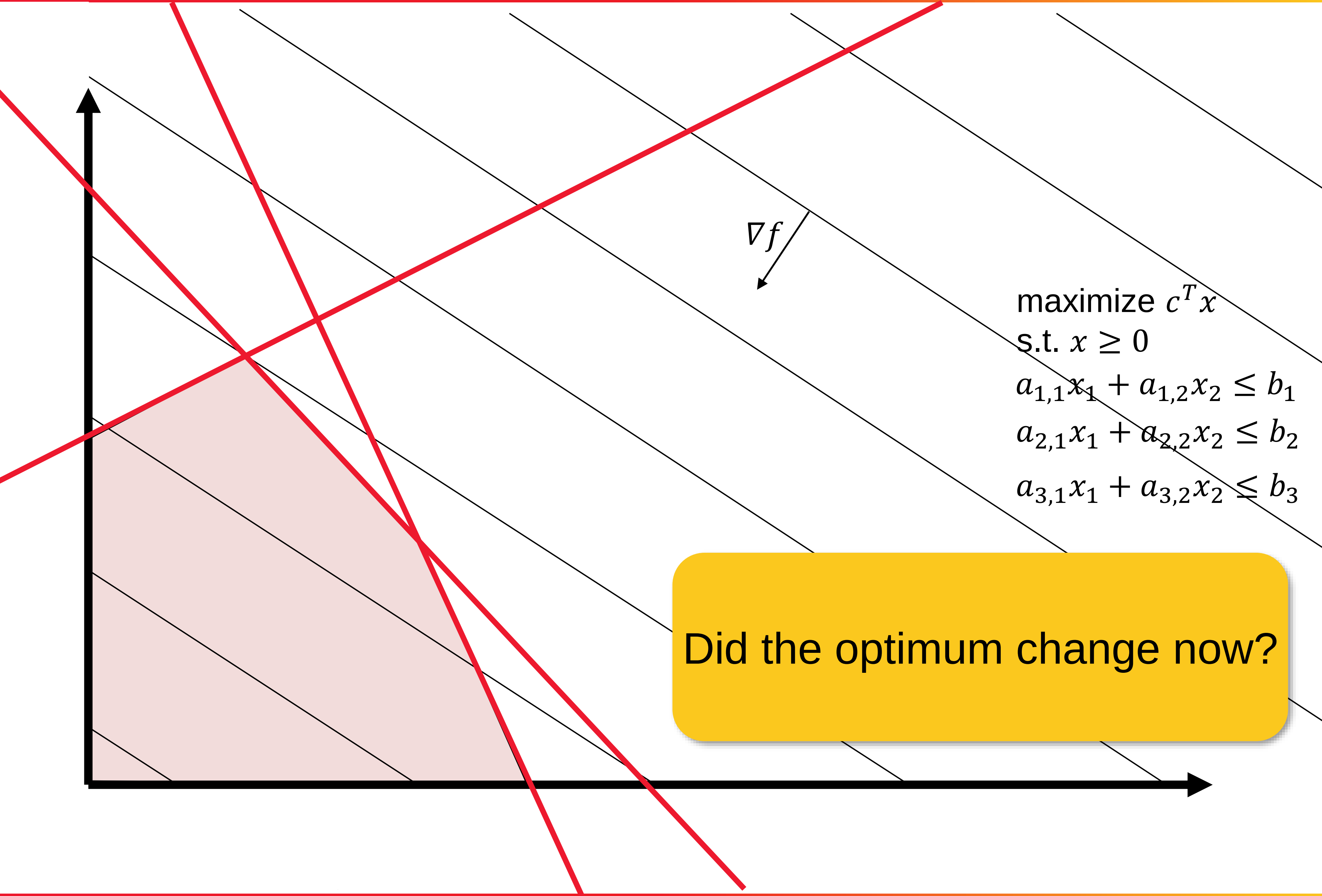
Graphical Illustration



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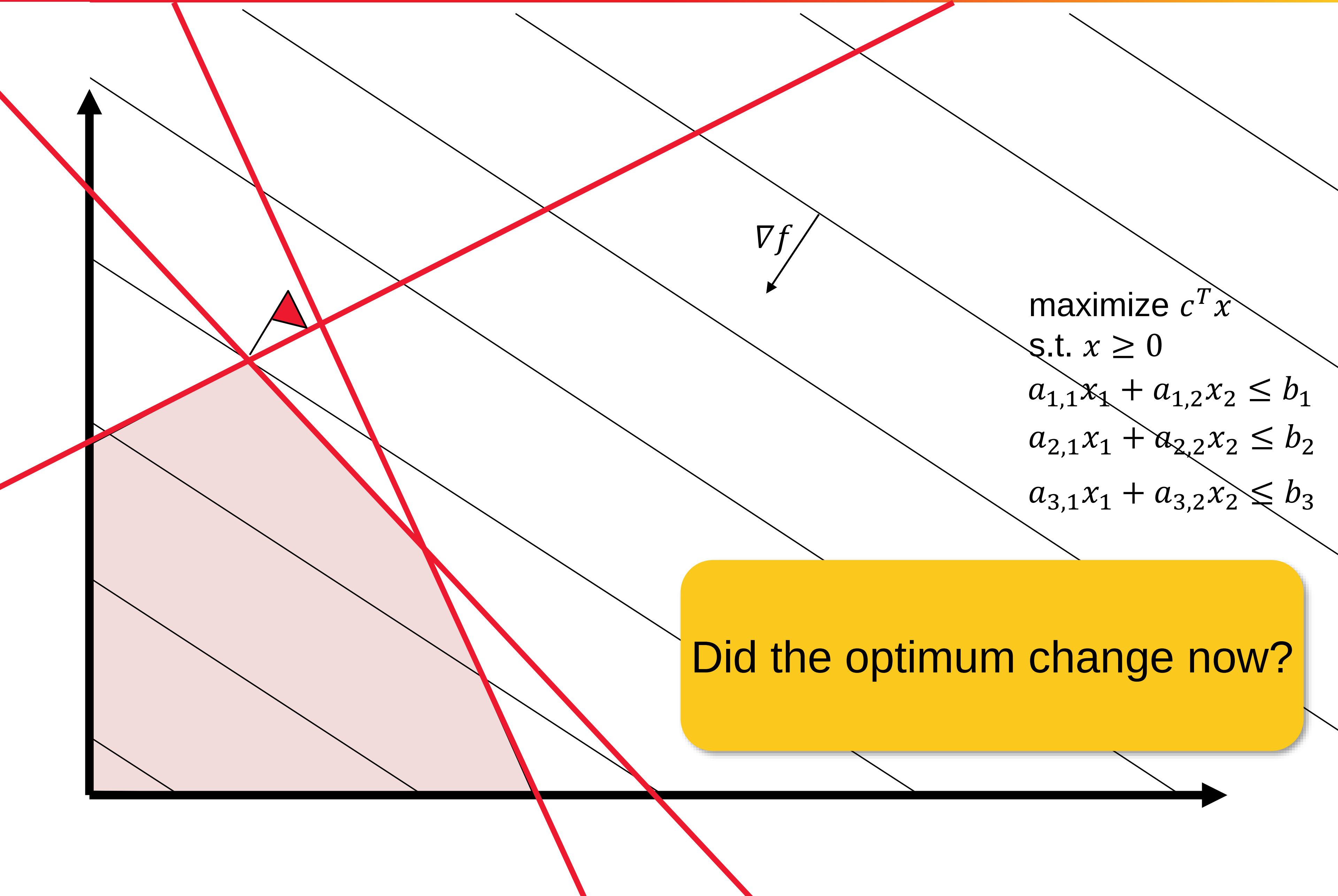
Did the optimum change now?

Graphical Illustration



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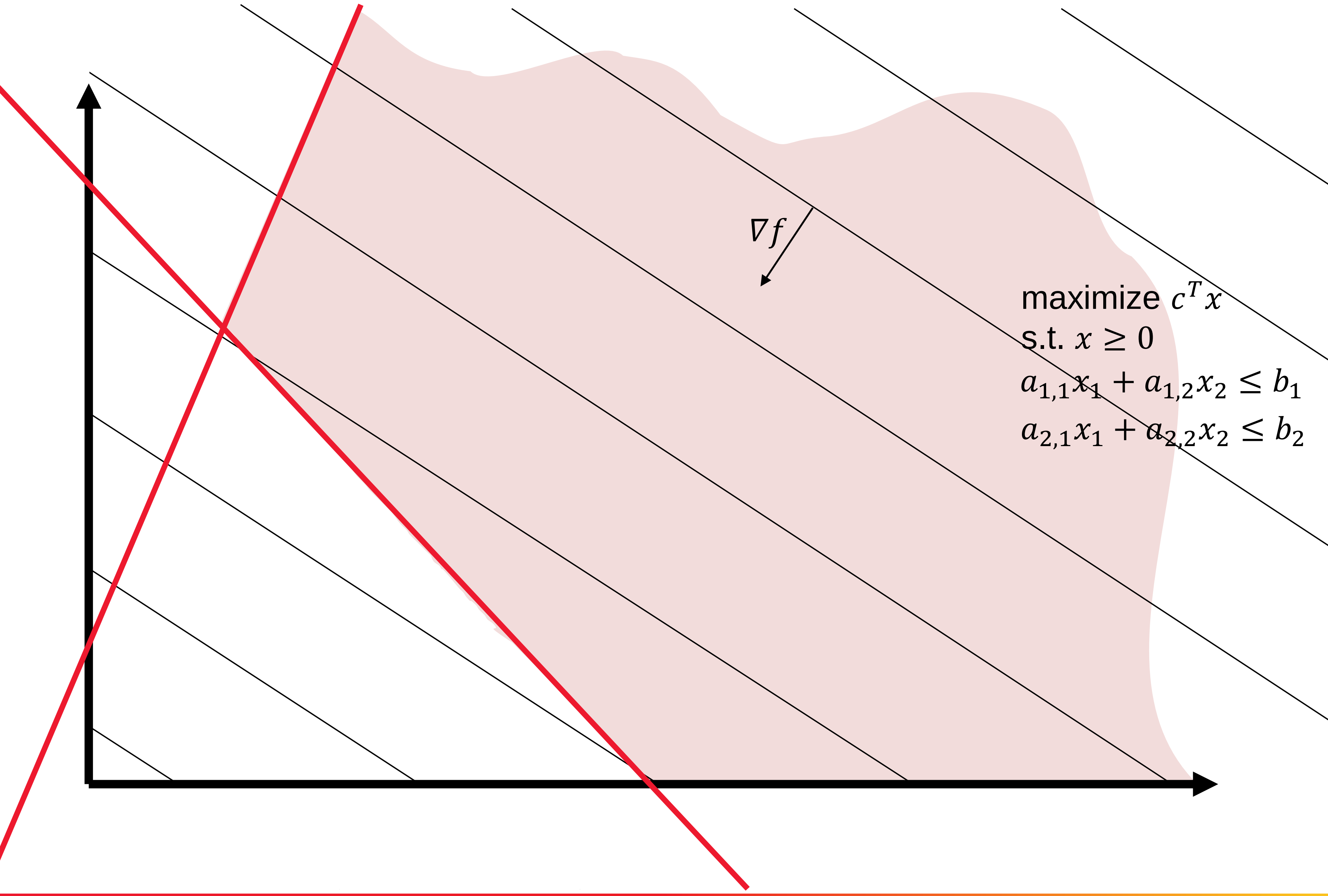
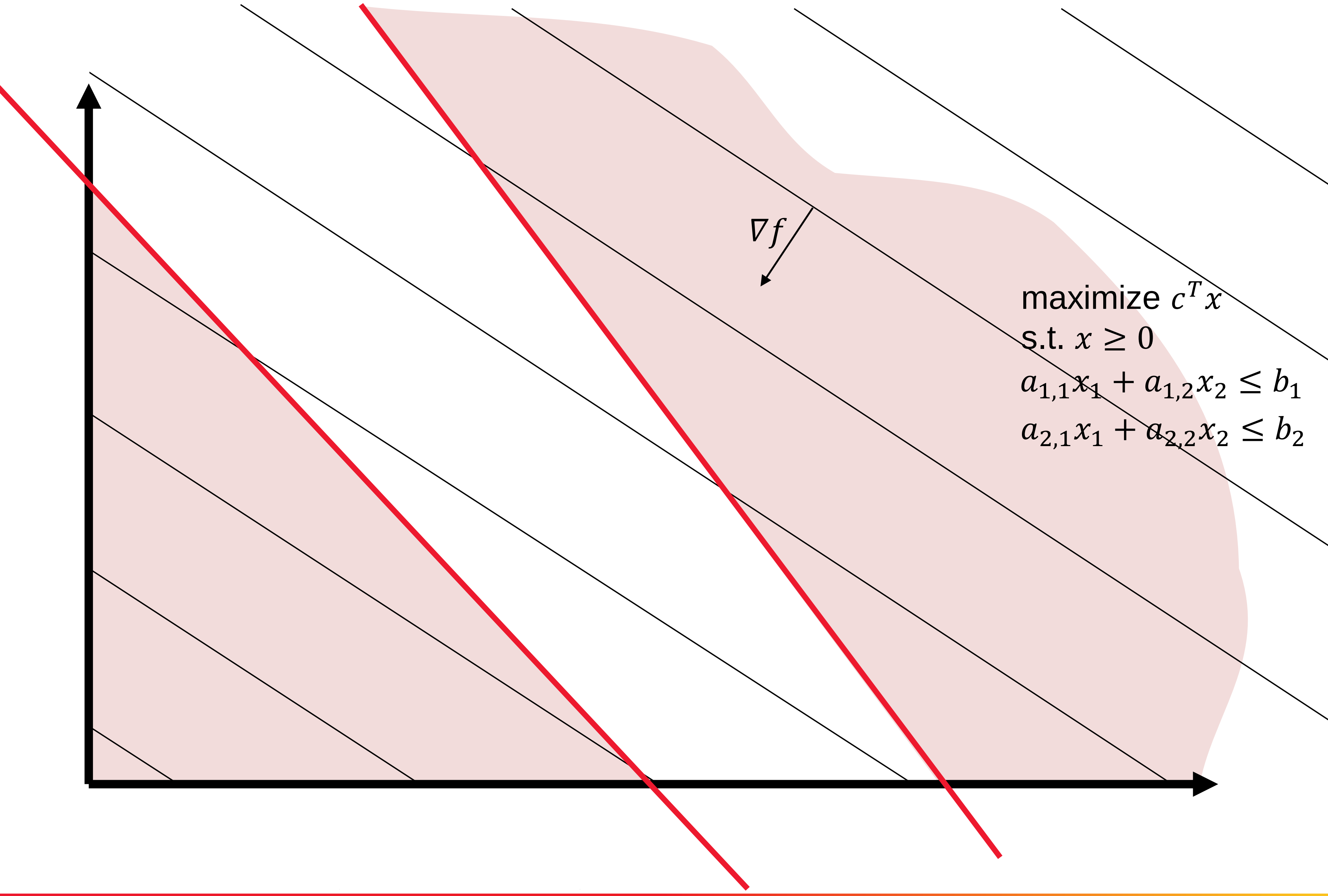
Did the optimum change now?

What Can Happen: Unboundedness



What Can Happen: Infeasibility



How to Solve Linear Programs?

Simplex method (Dantzig, 1940s)

- fast in practice, but exponential in worst case

Interior point methods

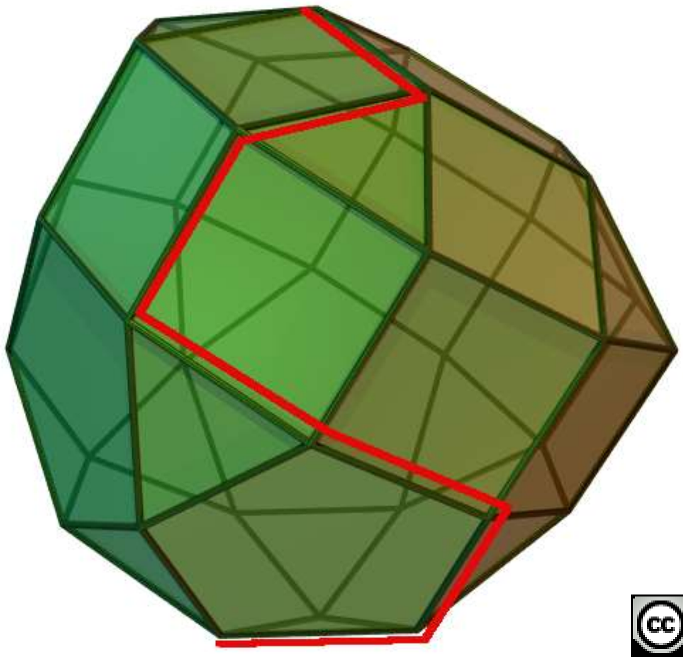
- Khachiyan, 1979: first polynomial algorithm, $O(n^6 L)$

  n: #variables, L: #input bits
- Karmarkar, 1984: $O(n^{3.5} L)$
- Vaidya, 1989: $O(n(n+d)^{1.5} L) = O(n^{2.5})$ for constant d

  d: #constraints

Idea Behind Simplex Algorithm

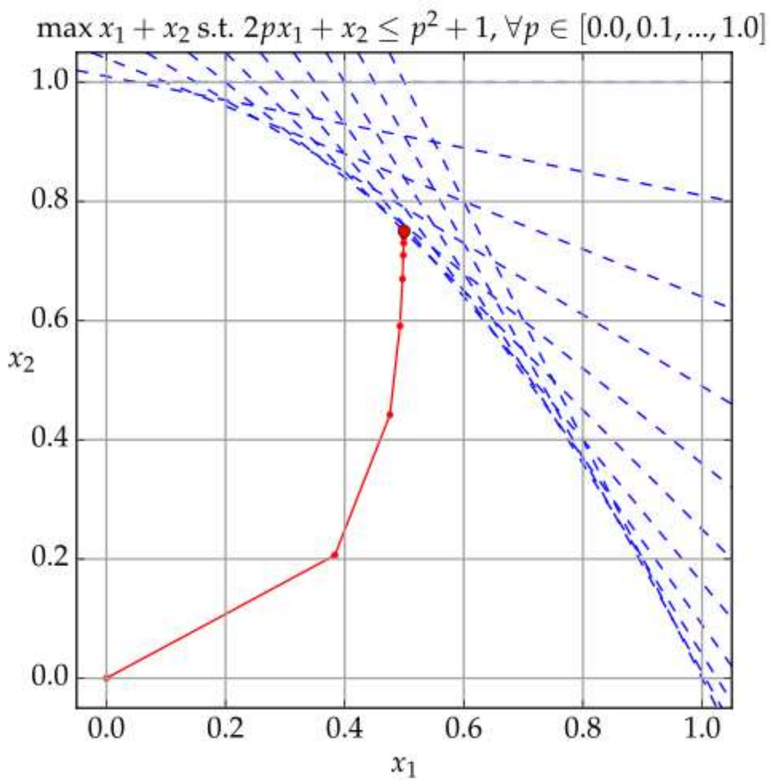
- Move along linear facets from corner to corner
- If corner not optimal, there is always a neighbor which is better
- Corresponds to equality constraints (inequality constraints need to be transformed accordingly via “slack variables”)



Idea Behind Interior Point Methods

- evaluate inside the simplex and move towards the edges
- works with inequality constraints
- solve $f(x) - 1/t \sum_{i=1}^{m} \log(g_i(x))$ iteratively with increasing t

given m inequality constraints $g_i(x) \geq 0$



Conclusions

I hope it became clear...

... what linear programming is and

... what are the ideas behind the simplex algorithm and interior point methods

Next time:

idea of duality

stochastic algorithms: stochastic gradient descent and CMA-ES