# MACROECONOMICS
# 73-240

## Lecture 14

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# Plan for This Lecture

1. Basics of Solow Growth Model

# What's ahead

1. We want to look at modern economic growth.
2. But this period was also a period characterized as the Great Divergence
3. So we want a *descriptive* model that can help us identify which factors/traits lead to disparate outcomes across countries.

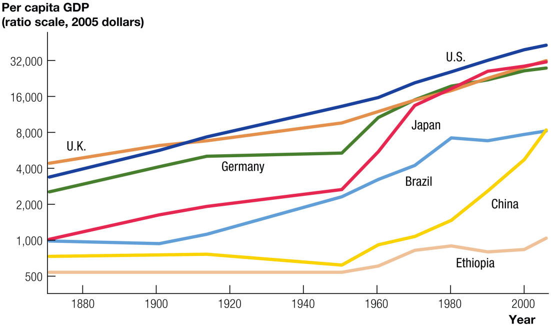
# What's ahead

1. We will take a baby step: we will NOT ask why countries chose those factors/traits
2. We will instead ask if a country is characterized with a particular trait, can we predict what its GDP per capita is
3. Hence the model is descriptive and is a step towards thinking about what factors or traits we perhaps want to encourage to promote growth and GDP per capita.

# Questions we would like the model to address

# Growth rates across countries differ

- Why are some countries richer than others (addressing level of GDP per capita)
- Why some countries grow faster than others ( addressing growth rate of GDP per capita)

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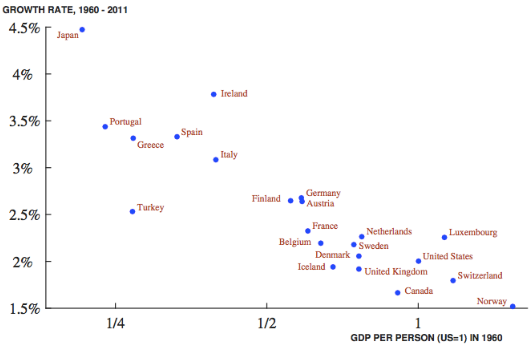
# Growth rates across countries differ

Why does it seem conditional on certain characteristics, countries catch up? (address conditional convergence)

Figure 25: Convergence in the OECD




Can our model also predict the absence of unconditional convergence

# Solow Growth Model

# The Solow model

- Due to Robert Solow, who won Nobel Prize for contributions to the study of economic growth
- A major paradigm:
  - widely used in policy making
  - used by investment banks who want to look at how emerging economies grow
  - benchmark against which most recent growth theories are compared
- Looks at the determinants of economic growth and the standard of living in the long run

# What is different from what we did so far?

- $K$ is no longer fixed:
  - investment causes it to grow, depreciation causes it to shrink
- $N$ grows, due to *exogenous* population growth
- Will find:
  - Continuing increases in productivity will lead to ever-increasing standards of living in per-capita terms
  - Absent productivity growth, no long-run increases in standards of living

# Simplifying Assumptions

We will make several simplifying assumptions

- Population grows at a constant rate
- Households save a constant fraction of disposable income
- Capital depreciates at a constant rate

We will ask with data whether these are good or bad assumptions

# Here we go!

# Key Ingredients

- Dynamic model, time goes on forever. Agents care about today *and* the future.

- Population grows at constant population growth rate $(1 + n)$

$$N' = (1 + n)N$$

  - Note that $n$ is an *exogenous* net population growth rate.
    - Unlike Malthus, $n$ is independent of $C/N$
    - Why? We want to ask if a country has a particular population growth rate (we don't care how it got that population growth), how would its outcome be different

# Key Ingredients

- Households don't care about leisure
  - Have one unit of time and no disutility from labor. Supply all labor $\Longrightarrow N^s = N$=population.
  - Notice that this together with exogenous net population growth implies that the size of the labor force is always changing.

# Key Ingredients

- Households care about consumption today and tomorrow, AND can save
  - Simplification: Households save a constant fraction of income

$$C + S = Y, \quad \text{and } S = sY$$

  - Households save by investing in capital goods $\Longrightarrow$ $S = I$.
  - Why this simplification? Constant savings rate $s$ is a stand-in for an economy's average saving rate
  - In part, we want to ask if countries are characterized by different savings rate, will their outcomes differ?

# Production Function and Capital Goods

- Households save by investing in capital goods
- New Ingredient(!): Capital can be accumulated
  - Capital evolves according to:

    $$K' = (1-d)K + I$$

    where $d$ is the depreciation rate. $(1-d)K$ is then how much capital remains after depreciation.
  - Households save in $K$ because they earn income from renting out $K$ to firms (you own office space, you rent it out to a firm).

We will flip this assumption after mid-term 2 and ask what happens if firms instead made investment decisions.

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# The Representative Firm

- Firms care about profits and makes a decision every period on how much capital and labor to rent/hire each period.

$$\max_{N^d, K} \pi = Y - wN^d - rK$$

where

$$Y = zF(K, N)$$

- Usual assumptions about production function apply:
  - Constant returns to scale
  - More inputs leads to more output
  - Diminishing marginal product

# Variables

- Exogenous Variables:

$$\{d, s, n, z\}$$

  - depreciation rate, savings rate, population growth rate, technology $z$

- Endogenous Variables of interest:

$$\{C, S, Y, I, K, K'\}$$

  - Ultimately, can focus on only $K/N = k$.

# Equilibrium and Steady State

1) Market clearing for goods: $Y = C + I$

2) Market clearing for labor: $N^d = N$

3) Market clearing for assets (NEW!): $S = I$

4) Steady State (NEW!): Capital per Worker is Constant

$$\frac{K'}{N'} = \frac{K}{N}$$

# Steady State

- Unlike Malthus, we may never have a steady state in population size for Solow model
- Why? because in Solow, $n$ is assumed to be exogenous and for any $n > 0$, population grows!
- So instead, use notion of per capita steady state (dividing by $N$ ensures that objects aren't growing just due to $N$ constantly growing)

# Working in per-capita terms

- We will define objects in per-capita terms in the Solow Model.
- Notation: capital per person, output per person, consumption per person given (respectively) by:

$$k = \frac{K}{N}, \quad y = \frac{Y}{N} \quad c = \frac{C}{N}$$

- Under constant returns to scale, we can also write output per person as:

$$y = \frac{Y}{N} = \frac{zF(K,N)}{N} = zF(\frac{K}{N}, 1) = zf(k)$$

- From now on lower case letters denote per-capita quantities.

# Working in per-capita terms

- Observe that the growth rate of $k$ can be written as:

$$g_k = \frac{k' - k}{k} \approx \ln(k') - \ln(k)$$

where the last equality is what we showed in HW 1!

$$g_k = \ln(k') - \ln(k) = \ln(K'/N') - \ln(K/N)$$

Using properties of logarithms:

$$g_k = \ln\frac{K'}{N'} - \ln\frac{K}{N} = \underbrace{\left(\ln K' - \ln K\right)}_{g_K} - \underbrace{\left(\ln N' - \ln N\right)}_{g_N}$$

and so, we have:

$$g_k = g_K - g_N$$

where $g_N = \frac{N'}{N} - 1 = n$

# Steady State in per-capita terms

- Our goal is ask what is the steady state in per-capita terms (fixed point) that the economy will converge to in the long-run absent shocks
- Note the above question is akin to asking if there's no further shocks, can we predict what kind of GDP per capita a country will have
- Essentially, we want to ask absent shocks, are there factors we can identify that will naturally lead one country to be richer than another country

# Solving For the Steady State

- Our goal is now to derive a condition for the steady state value of $k$.

  This is important to understand questions like:

  1. How to foster growth?
  2. Why is the saving rate important?
  3. Is a capital tax good or bad?

# Solving For the Steady State

Our goal: find steady state $k$
Start with how capital involves!

$$K' = (1-d)K + I$$

Subtract $K$ from both sides of equation:

$$\underbrace{K' - K}_{\text{change in } K} = I - dK$$

The right-hand side has a simple intuition: if you invest more than the amount that has depreciated, capital stock grows!

# Solving For the Steady State

From last slide:

$$K' - K = I - dK$$

Substitute $I = S$ from asset market clearing and use fact that $S = sY = szF(K, N)$

$$K' - K = szF(K, N) - dK$$

Growth rate of capital:

$$g_K = \frac{K' - K}{K} = \frac{1}{K} szF(K, N) - d$$

But goal is to find $k^{ss}$. Let's note that $g_k = g_K - g_N$ and $kg_k = k' - k$

$$k' - k = kg_k = k(g_K - g_N)$$

# Solving For the Steady State

From last slide:

$$k' - k = k(g_K - g_N)$$

and we also know $g_N = n$ and the form of $g_K$ (see last slide), so substituting in for $g_K$ and $g_N$

$$k' - k = k\left(\frac{1}{K}szF(K,N) - d - n\right)$$

Finally, since $k = \frac{K}{N} \implies k\frac{1}{K} = \frac{1}{N}$

$$k' - k = \frac{1}{N}szF(K,N) - (d+n)k$$

From CRS of production function:

$$k' - k = szf(k) - (d+n)k$$

# The steady state

- In steady state, $k' = k = k^{ss}$ which implies

$$\underbrace{szf(k^{ss})}_{\text{investment per worker}} = \underbrace{(n+d)k^{ss}}_{\text{break-even investment}}$$

- What adds to the stock of capital per worker:
  - Investment per-worker
- What causes the stock of capital per worker to shrink
  - Depreciation
  - Having to share stock of capital amongst larger work force

- In steady state, $k$ is not changing if invest just enough to replace $k$ that is shrinking (i.e. break-even)

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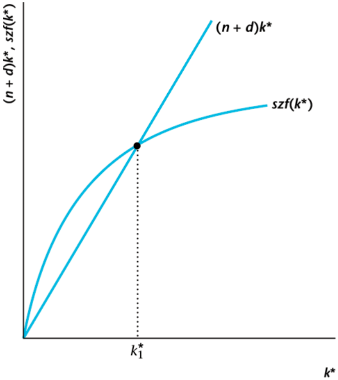
# Analysis of the Steady State

Key equilibrium condition (note $k^{ss} = k^*$):

$$szf(k^*) = (n+d)k^*$$

To understand it, graph both sides of the equation:



# Convergence to Steady State

- Claim: Solow Model economy has tendency to converge to steady state capital per worker, $k^*$

- Proof:
  - Suppose $k_0 < k^*$
    - Then $szf(k_0) > (d+n)k_0$. So $k_1 > k_0$
    - Capital per worker increases towards $k^*$

  - Suppose $k_0 > k^*$
    - Then $szf(k_0) < (d+n)k_0$. So $k_1 < k_0$
    - Capital per worker decreases towards $k^*$

  - Remember what the graph of this looks like!

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# Convergence to Steady State

- Knowing $k^*$ informs us of GDP per capita in the long run, $y^* = zf(k^*)$
- Suppose the following set of countries have the same $s, z, n, d$ but have not yet converged to their steady state. Use the graph to explain what will happen to each country's GDP per capita over time, and what their GDP per capita growth rates are like

 1. Country A currently has $k_A < k^*$, country B currently has $k_B > k^*$.try over time?
 2. Country D and E currently have $k_E < k_D < k^*$ . $k_E$ is far from $k^*$, $k_D$ is close to $k^*$. Which country grows at a faster rate?

# Growth Rates

- Question: what is the growth rate of GDP per capita once we arrive at the steady state?

# Long-Run Growth Properties of the Solow Model

Long-Run Growth in the Solow Model

- In Steady State what is the growth rate of capital per worker?
  Answer: 0!
  $(\Delta(k) = (k' - k)/k = 0$ since $k' = k$ in steady state)

- In Steady State what is the growth rate of output per worker?
  Answer: 0!
  $(\Delta(y) = \Delta(zf(k)) = \Delta(z) + \Delta(f(k)) = 0$ when $z$ and $k$ are constant)

# Growth Rates

- Notice that so far we conditioned on countries who have the same $s, n, z, d$
- We observed that countries who share the same $s, n, z, d$ converge to the same GDP per capita in steady state (conditional convergence)
- Question: Can countries have different outcomes in the long run?

# Growth Rates

- Question: Can countries have different outcomes in the long run?
- Suppose $Y = zK^{\alpha}N^{1-\alpha}$, this implies $y = \frac{Y}{N} = z\left(\frac{K}{N}\right)^{\alpha} = zk^{\alpha}$.
- Capital per person evolves according to:

$$k' - k = szk^{\alpha} - (n+d)k$$

- In steady state, $k' = k \implies k' - k = 0$

$$szk^{ss,\alpha} = (n+d)k^{ss}$$

- which implies:

$$k^{ss} = \left(\frac{sz}{n+d}\right)^{1/(1-\alpha)}$$

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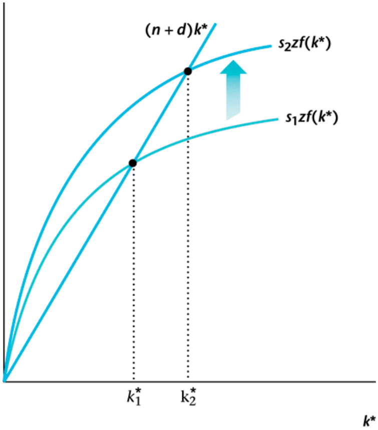
# Other Long-Run Properties of the Solow Model

Other Long-Run Properties:

- Level of capital (and output!) per worker depends on savings rate, population growth rate, and depreciation rate:
  - Higher savings rate implies *higher* $k^*, y^*$
  - Higher population growth rate implies *lower* $k^*, y^*$
  - Higher capital depreciation rate implies *lower* $k^*, y^*$
  - Higher technology implies *higher* $k^*$, $y^*$

# Changing the Saving Rate

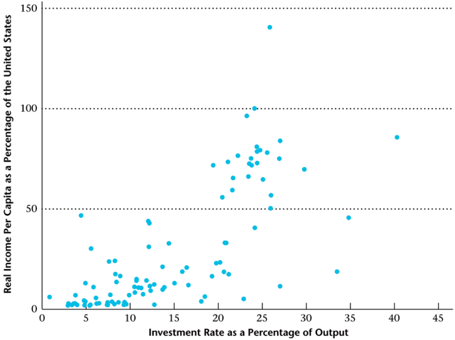
- Suppose $s$ changes: $s_1$ to $s_2$ $(s_2 > s_1)$




- $k_1$ increases to $k_2$
- Question: what happens to $y_1$?

# From Data Lecture

Recall that in equilibrium saving = investment.

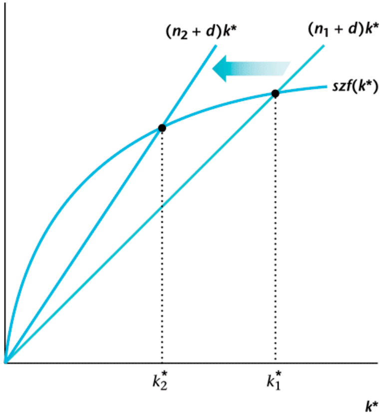



*Source:* A. Heston, R. Summers, and B. Aten, *Penn World Table Version 6.1*, Center for International Comparisons at the University of Pennsylvania (CICUP), October 18, 2002, available at pwt.econ.upenn.edu.

Investment is positively correlated with GDP levels.

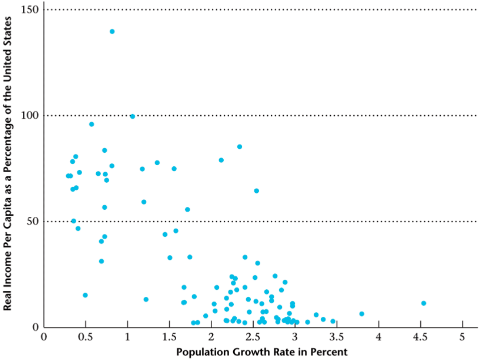
# Changing the Population Growth Rate

- Suppose $n$ changes: $n_1$ to $n_2$ $(n_2 > n_1)$




- $k_1$ decreases to $k_2$

# From Data Lecture




*Source:* A. Heston, R. Summers, and B. Aten, *Penn World Table Version 6.1*, Center for International Comparisons at the University of Pennsylvania (CICUP), October 18, 2002, available at pwt.econ.upenn.edu.

Population growth is negatively correlated with GDP levels.

# Short-Run Growth Properties of the Solow Model

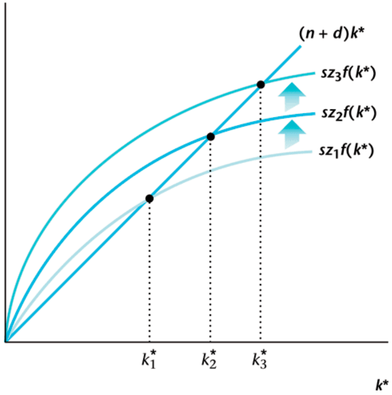
Short-Run Growth in the Solow Model

- If a country has $k_0 < k^*$, then capital per worker increases to the steady state, or $\Delta(k) > 0$
- Suppose two countries are identical except one country has less initial capital per worker. Then country with lower capital per worker will grow faster in the short run.
- Suppose two countries are identical but one country has higher savings rate. If both countries have same initial capital per worker (neither are at steady state), country with higher savings rate grows faster.

# Solow vs. Malthus

# Changing $z$

- Suppose $z$ changes: $z_1$ to $z_2$ to $z_3$ $(z_3 > z_2 > z_1)$




- We observe that increases in $z$ lead to $\uparrow k^{ss}, \uparrow y^{ss} = zf(k^{ss}), \uparrow c^{ss} = (1-s)y^{ss}$.

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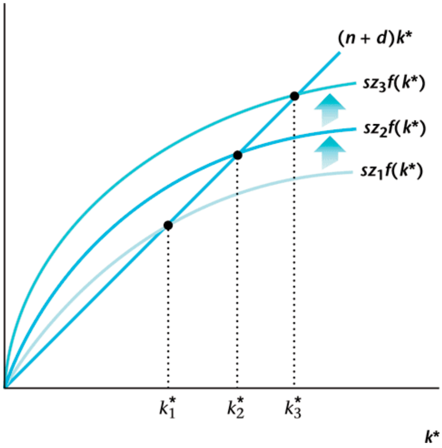
# Sustaining Growth

Claim: To sustain growth, we need increases in $z$!




Increasing $z_1$ to $z_2$ to $z_3$ generates long term growth!

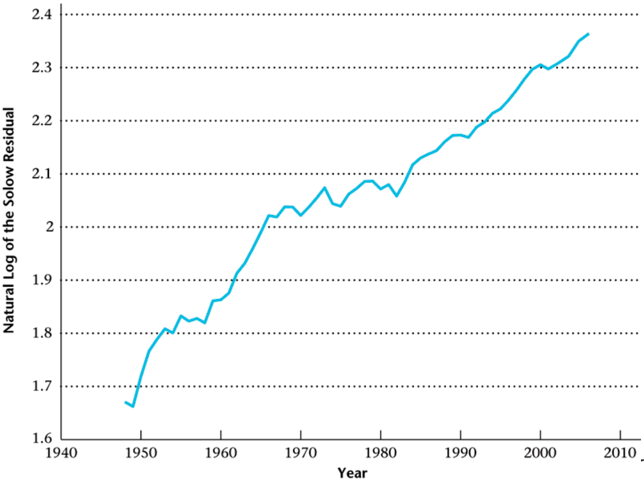
# Looking at data

- We have looked at the Solow Residual before
- Recall, using observed data in $Y$, $K$ and $N$ we can back out what $z$ is

$$z = \frac{Y}{K^{0.36} N^{0.64}}$$

# The Solow Residual

Plotting $z$ (the Solow residual) for the US




*Source:* Bureau of Economic Analysis, Department of Commerce, and Bureau of Labor Statistics.

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# Changing $z$

- Solow Model: Improvements in technology $z$ do lead to sustained improvements in GDP per capita and individual living standards
- Malthus Model: In contrast, improvements in technology $z$ only led to increased population
- If production was according to Malthus prior to 1820s, and production was according to Solow post 1820s, we now can explain why we had stagnant growth and then sudden take-off.

# Review

# Details

- Nov 6th in class (Attend your respective section!)
- 10 short answer questions, 3 long problems
- Coverage: Lecture 10-14.

# What do you need to know?

Unemployment

- Definitions

- How unemployment rate evolves, how to calculate steady state unemployment rate

- Search model of unemployment (Lecture 11).

What you don't need to know: quits rate

# What do you need to know?

Growth

- You should know how to calculate growth rates, how long it takes for a country to catch up

- Malthus (All of Lecture 13)

- Solow (Lecture 14)

What you don't need to know: how to analytically derive from $K' = (1-d)K + I$ the equation defining how capital per worker evolves over time

$$k' - k = szf(k) - (n+d)k$$