PRINCIPLES OF FINANCE

WEEK 4

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Last lecture

Do you remember

- The trade-off between risk and return
- The difference between portfolios and individual stocks
- Which type of risk can be diversified and which one cannot?

Video

- How investors’ preferences can be modelled
- How an investor makes decisions based on her preferences?

What is left to investigate?

- To be able to estimate a precise discount rate for risky projects, we now need to answer these questions:

1. What is the economic rationale for the strength of the relationship between systematic risk and returns?
2. Can we use simple economics to derive a precise relationship between the systematic variance of an investment and the return investors require from it?

- To answer these questions, we need to introduce a model called the Capital Asset Pricing Model (the CAPM)
  - → In this lecture we will aim to understand better what efficient portfolios are
  - → In the next lecture we will get to the CAPM and see how it relates expected returns and risk of an individual stock (or project), using the equivalent efficient portfolio.

Outline of today's lecture

In class

- Portfolio diversification
- Efficient frontier

Video: Optimal portfolio allocation

Real-life example: Warren Buffett

Portfolio diversification

Portfolio allocation and expected return from risky investments

- At the end of the day, return on a risky investment is determined by the demand for this investment
  - Higher demand, higher price, lower expected return
  - Lower demand, lower price, higher expected return
- Last session, we have seen that expected returns are a positive function of systematic risk.
- This must mean that there is higher (lower) demand for investments with lower (higher) systematic risk
- Why do investors behave like this?
  - To answer this question, we need to study how investors should choose their investments.
  - This is called in finance the **optimal portfolio allocation** question

Returns on portfolios with many assets

- Portfolio weights x_i
  - Asset *i*'s fraction of total investment in the portfolio
  - The portfolio weights must add up to 1 (100%).

$$x_{\mathrm{i}} = \frac{\text{Value of investment } i}{\text{Total value of portfolio}}$$

  - For simplicity, let us consider that the only available investments are stocks and the Treasury Bill (T-Bill).
  - The logic of what follows applies if we also include risky bond investments

Returns on portfolios with many assets

- Portfolio Return R_p : weighted average of the returns on the investments in the portfolio

$$R_P = x_1 R_1 + x_2 R_2 + \cdots + x_n R_n = \sum_i x_i R_i$$

- Expected portfolio returns:

$$E(R_P) = \sum_i x_i E(R_i)$$

Returns on portfolios with many assets

Problem

Suppose you invest \$10,000 in Ford stock, and \$30,000 in Tyco International stock. You expect a return of 10% for Ford and 16% for Tyco. What is your portfolio's expected return?

Solution

You invested \$40,000 in total, so your portfolio weights are 10,000/40,000 = 0.25 in Ford and 30,000/40,000 = 0.75 in Tyco. Therefore, your portfolio's expected return is

$$E[R_P] = x_F E[R_F] + x_T E[R_T] = 0.25 \times 10\% + 0.75 \times 16\% = 14.5\%$$

Variance and covariance

- Variance of portfolio returns:

$$Var(R_P) = \sum_i x_i^2 Var(R_i) + \sum_{i,j} 2x_i x_j Cov(R_i, R_j)$$

- Covariance between returns R_i and R_j

$$Cov(R_i, R_j) = E\left[(R_i - E[R_i])(R_j - E[R_j])\right]$$

- Estimate of the covariance from historical data

$$Cov(R_i, R_j) = \frac{1}{T-1}\sum_t (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)$$

Variance, covariance and correlation

- Correlation

$$Corr\left(R_i, R_j\right) = \frac{Cov(R_i, R_j)}{SD(R_i)SD(R_j)}$$

- Case of 2 assets:

$$Var(R_P) = Var(\omega_1 R_1) + Var(\omega_2 R_2) + 2Cov(\omega_1 R_1, \omega_2 R_2)$$

$$= \omega_1^2 Var(R_1) + \omega_2^2 Var(R_2) + 2\omega_1 \omega_2 Cov(R_1, R_2)$$

$$= \boldsymbol{\omega_1^2} Var(R_1) + \boldsymbol{\omega_2^2} Var(R_2) + 2\omega_1 \omega_2 \underbrace{SD(R_1)}_{= \sigma(R_1)} \underbrace{SD(R_1)}_{= \sigma(R_2)} \underbrace{Corr\left(R_i, R_j\right)}_{= \rho}$$

Covariance and correlation

The Covariance and Correlation of a Stock with Itself

Problem

What are the covariance and the correlation of a stock's return with itself?

Solution

Let R_s be the stock's return. From the definition of the covariance,

$$
\begin{aligned}
Cov(R_s, R_s) &= E[(R_s - E[R_s])(R_s - E[R_s])] = E[(R_s - E[R_s])^2] \\
&= Var(R_s)
\end{aligned}
$$

where the last equation follows from the definition of the variance. That is, the covariance of a stock with itself is simply its variance. Then,

$$
Corr(R_s, R_s) = \frac{Cov(R_s, R_s)}{SD(R_s)\ SD(R_s)} = \frac{Var(R_s)}{SD(R_s)^2} = 1
$$

where the last equation follows from the definition of the standard deviation. That is, a stock's return is perfectly positively correlated with itself, as it always moves together with itself in perfect synchrony.

Diversification effect: illustration

Diversification: two-asset example

	Asset 1	Asset 2
Expected return	0.10	0.20
Standard deviation	0.10	0.30
Variance	0.01	0.09

Scenario 1 | **Correlation** **1** | *Covariance* *0.03*

		Portfolio	
		Expected return	Standard deviation
Weight of asset 1	100%	0.100	0.1000
	75%	0.125	0.1500
	50%	0.150	0.2000
	25%	0.175	0.2500
	0%	0.200	0.3000

Scenario 2 | **Correlation** **0.2** | *Covariance* *0.006*

		Portfolio	
		Expected return	Standard deviation
Weight of asset 1	100%	0.100	0.1000
	75%	0.125	0.1162
	50%	0.150	0.1673
	25%	0.175	0.2313
	0%	0.200	0.3000

Same expected returns but lower standard deviation

Diversification effect

- When the weights on individual assets are positive, the standard deviation of a portfolio of two securities is less than the weighted average of the standard deviations of the individual securities, as long as the correlation coefficient is less than 1.

$$\rho < 1 \ \rightarrow \sigma(R_{1\&2}) < \omega_1\sigma(R_1) + \omega_2\sigma(R_2)$$

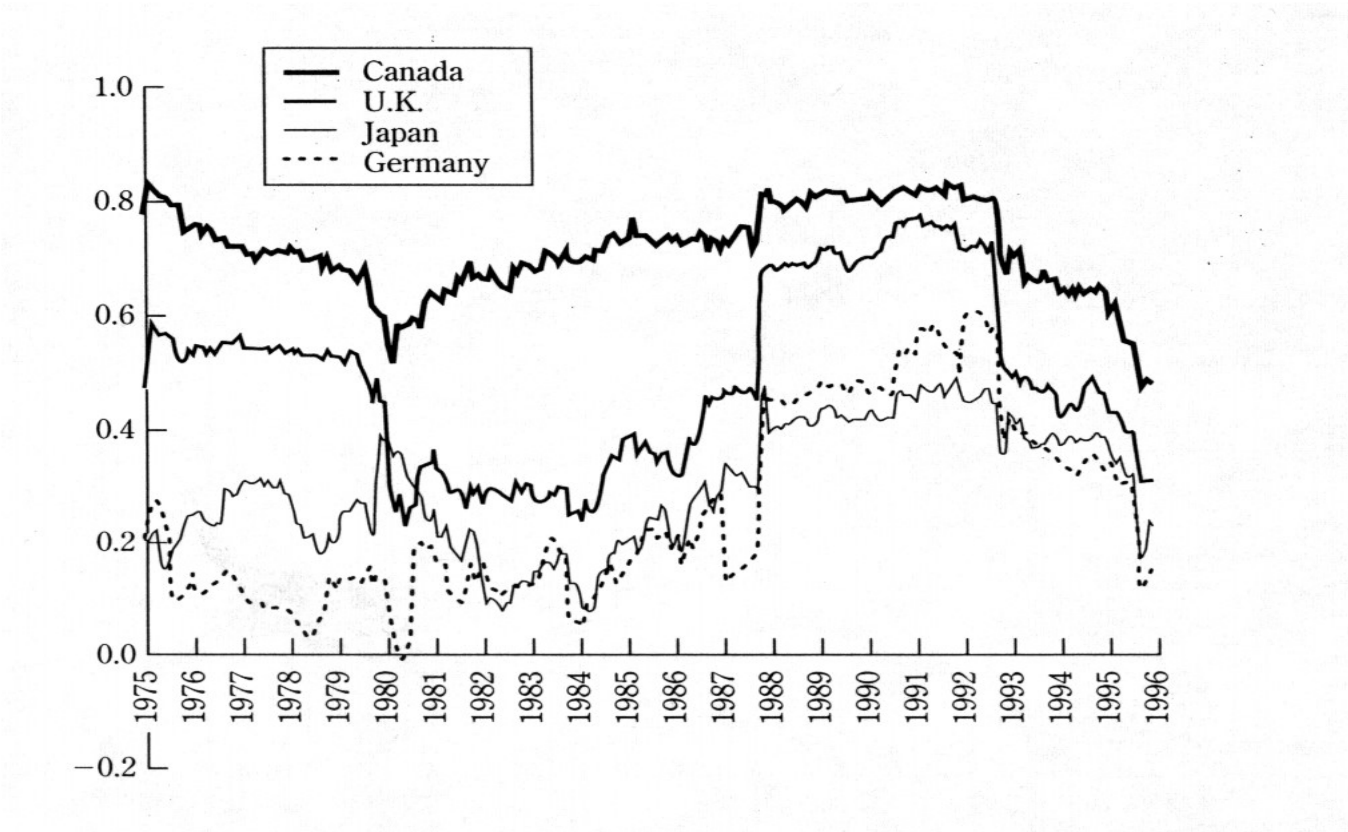
- Investors can obtain the same level of expected return with lower risk. The lower the correlation is, the larger the diversification effect.

Portfolio diversification

- Traditionally, it was thought that a portfolio of about 20 stocks was enough to obtain virtually 90% of all benefits of diversification
- In recent years, this number has increased to about 50 stocks, due to a substantial increase in idiosyncratic volatility.
- Diversification benefits can be achieved by investing internationally if domestic and foreign returns have a low correlation.
- Similarly, diversification can be achieved by investing in different asset classes (e.g., equities, commodities, cryptocurrencies…), or through different channels (e.g., public equity vs. private equity).
- It is typically easier for large institutions to be diversified, but households can do it through funds.

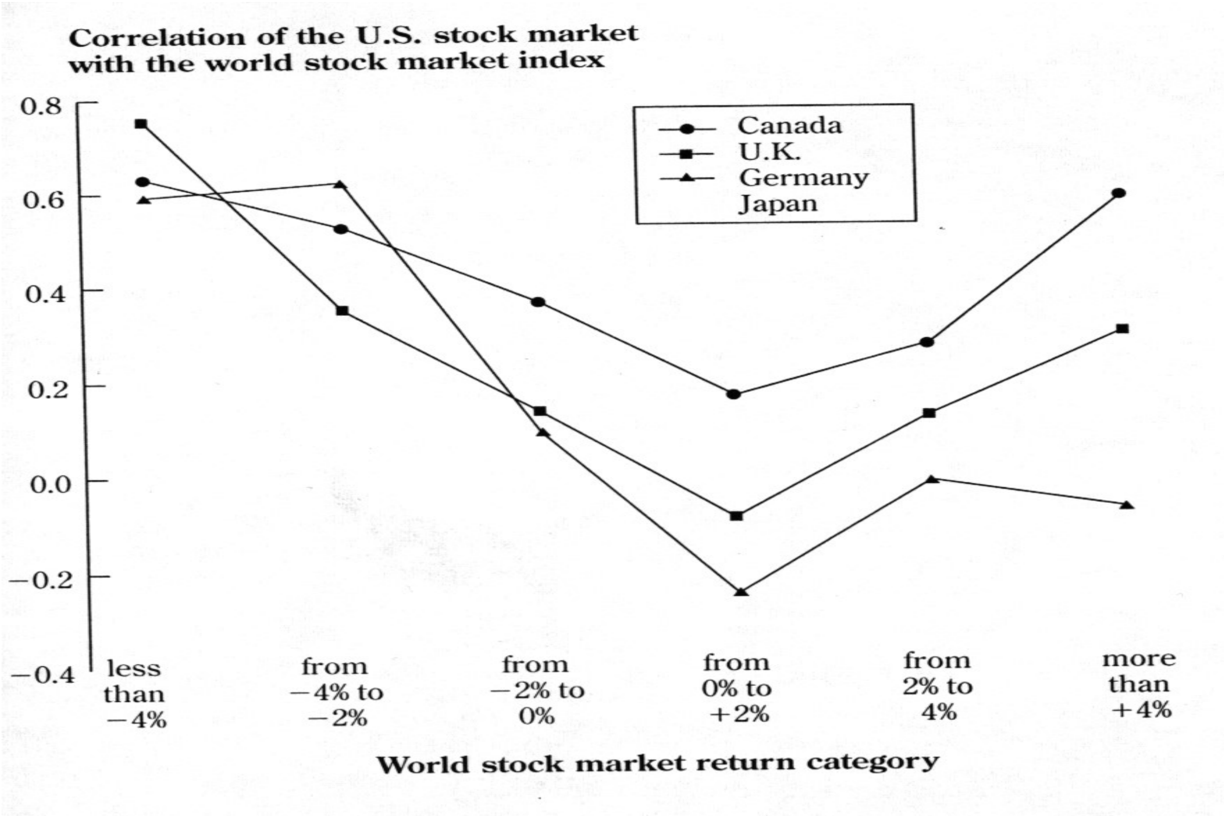
Portfolio diversification

- In practice, correlations change over time



Portfolio diversification

- Unfortunately, correlations are lower in normal times but increase in bad times (when diversification is most needed)



Portfolio diversification

Example: Crash of 87

Pencentage changes in stock price indices in October 1987. The second column shows the return to a US investor in these markets.

Country	Local Currency	U.S. Dollars
Australia	−41.8	−44.9
Austria	−11.4	−5.8
Belgium	−23.2	−18.9
Canada	−22.5	−22.9
Denmark	−12.5	−7.3
France	−22.9	−19.5
Germany	−22.3	−17.1
Hong Kong	−45.8	−45.8
Ireland	−29.1	−25.4
Italy	−16.3	−12.9
Japan	−12.8	−7.7
Malaysia	−39.8	−39.3
Mexico	−35.0	−37.6
Netherlands	−23.3	−18.1
New Zealand	−29.3	−36.0
Norway	−30.5	−28.8
Singapore	−42.2	−41.6
South Africa	−23.9	−29.0
Spain	−27.7	−23.1
Sweden	−21.8	−18.6
Switzerland	−26.1	−20.8
United Kingdom	−26.4	−22.1
United States	−21.6*	−21.6*

*Standard and Poor's 500 index.

Source: R. Roll, "The International Crash of October 1987," in R. Kamphis (ed.), *Black Monday and the Future of Financial Markets*, Richard D. Irwin, Inc., Homewood, Ill., 1989. See table 1, p. 37.

The efficient frontier

The 3 basic questions in optimal portfolio allocation

- Since the weights x can take any value, there is an infinity of portfolios an investor can form and have preferences over
- Most likely however, not all portfolios are equal: some may be better than others
   1. Is there a clear way of assessing what a “better” portfolio means?
   2. Is there just one best stock portfolio?
   3. Can we improve on the best stock portfolios by adding a risk-free asset to the range of options available to an investor?

The 3 basic questions in optimal portfolio allocation

1. Is there a clear way of assessing what a "better" portfolio means?
   - Yes: out of an available set of portfolios, some are *efficient*, others are *inefficient*
2. Is there just one best stock portfolio?
   - No: there are several efficient stock portfolios, which form the *efficient frontier* of stock portfolios
3. Can we improve on the best pure stock portfolios by adding a risk-free asset to the range of options available to an investor?
   - Yes: in fact, investors should always mix the risk-free asset and a basket of stocks called the tangent portfolio. This set of portfolios forms the efficient frontier of all feasible portfolios; it is also called the *Capital Market Line.*

Risk vs. Return: Choosing efficient portfolios

- For any given set of stocks that we can choose from, there are indeed portfolios of these stocks that are more attractive (in a risk-return sense) than others
  - Inefficient Portfolios
    - It is possible to find another portfolio that has higher expected return with the same volatility, or lower volatility for the same expected return
  - Efficient Portfolios
    - There is no way to reduce the volatility of the portfolio without lowering its expected return
- An investor should only choose efficient portfolios. Why?
  - Because investors are risk-averse

Risk vs. Return: Choosing efficient portfolios

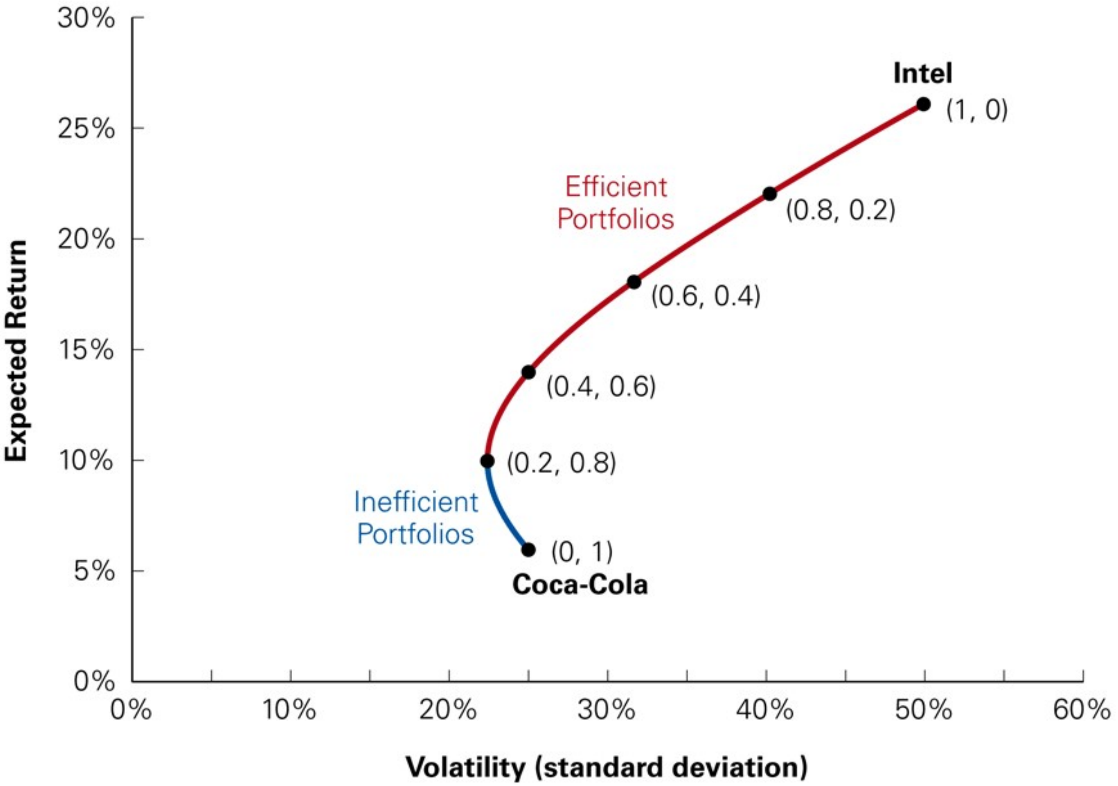
Example

- Suppose we have two stocks (Intel, Coca-Cola)
- Suppose – for now – that the returns of these stocks are uncorrelated
- We can form different portfolios with these stocks (e.g. invest 10% of our money in Intel, rest in Coca-Cola; or invest 70% in Intel, rest in Coca-Cola)
- By varying the portfolio weights, we can identify the set of efficient portfolios of these two stocks

Stock	Expected Return	Volatility
Intel	26%	50%
Coca-Cola	6%	25%

Portfolio Weights		Expected Return (%)	Volatility (%)
x_I	x_C	$E[R_P]$	$SD[R_P]$
1.00	0.00	26.0	50.0
0.80	0.20	22.0	40.3
0.60	0.40	18.0	31.6
0.40	0.60	14.0	25.0
0.20	0.80	10.0	22.4
0.00	1.00	6.0	25.0

Risk vs. Return: Choosing efficient portfolios



The red curve with the set of all efficient portfolios of Intel and Coca-Cola is called the **Efficient Frontier**

Keyword:
Efficient frontier

Risk vs. Return: Choosing efficient portfolios

- For each level of risk (measured by standard deviation), find the set of portfolios with highest expected return.

- If investors only care about risk and return, every rational, risk–averse investor prefers a portfolio on the efficient set to any other portfolio.

The equation of the efficient frontier

- Expected returns of the portfolio

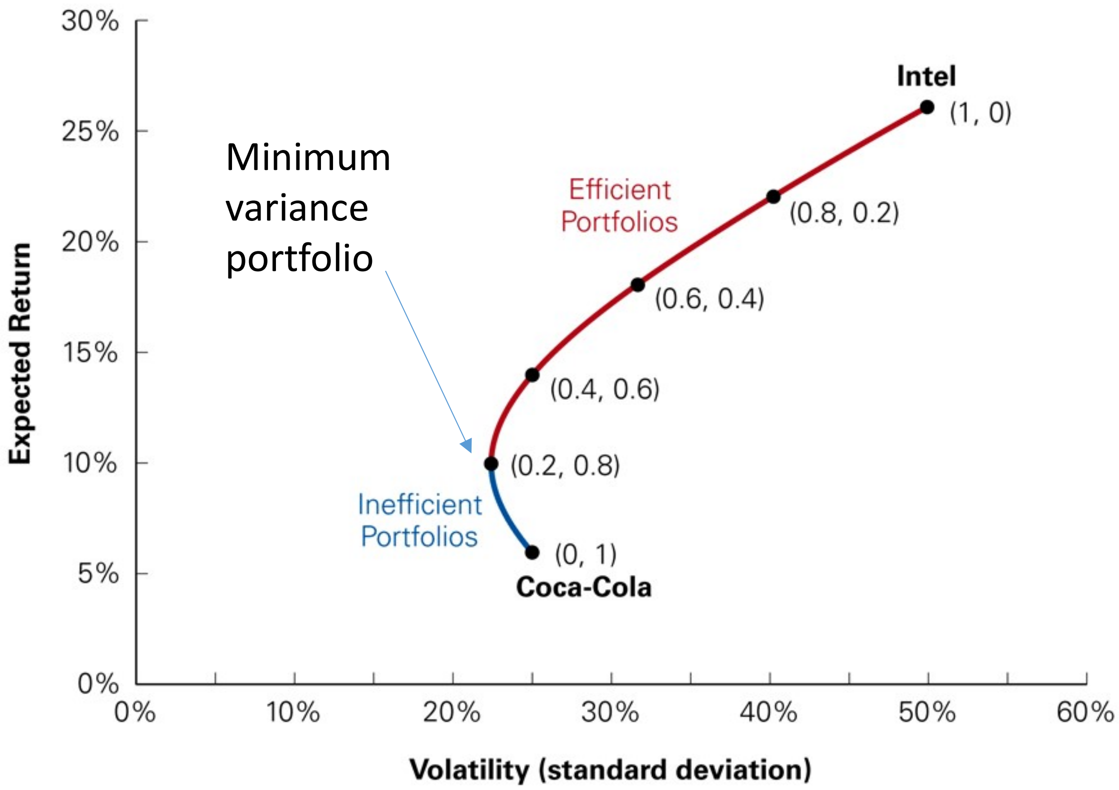
$$\mu_P = \omega_1\mu_1 + \omega_2\mu_2 = \omega_1\mu_1 + (1-\omega_1)\mu_2 \Rightarrow \omega_1 = \frac{\mu_P - \mu_2}{\mu_1 - \mu_2}$$

- Variance of the portfolio

$$\begin{aligned}\sigma_P^2 &= \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_{12} \\ &= \omega_1^2\sigma_1^2 + (1-\omega_1)^2\sigma_2^2 + 2\omega_1(1-\omega_1)\sigma_{12} \\ &= (\sigma_1^2 + \sigma_2^2 - 2\sigma_{12})\omega_1^2 + 2\omega_1(\sigma_{12} - \sigma_2^2) + \sigma_2^2\end{aligned}$$

$$=> \boldsymbol{\sigma_P^2} = (\sigma_1^2 + \sigma_2^2 - 2\sigma_{12})\left(\frac{\boldsymbol{\mu_P} - \mu_2}{\mu_1 - \mu_2}\right)^2 + 2\frac{\boldsymbol{\mu_P} - \mu_2}{\mu_1 - \mu_2}(\sigma_{12} - \sigma_2^2) + \sigma_2^2$$

Minimum variance portfolio



To find the minimum variance portfolio:

$$\min_{\omega_1} \ \omega_1^2\sigma_1^2 + (1-\omega_1)^2\sigma_2^2 + 2\omega_1(1-\omega_1)\sigma_1\sigma_2\rho_{12}$$

Take the first order condition by differentiating w.r.t ω_1:

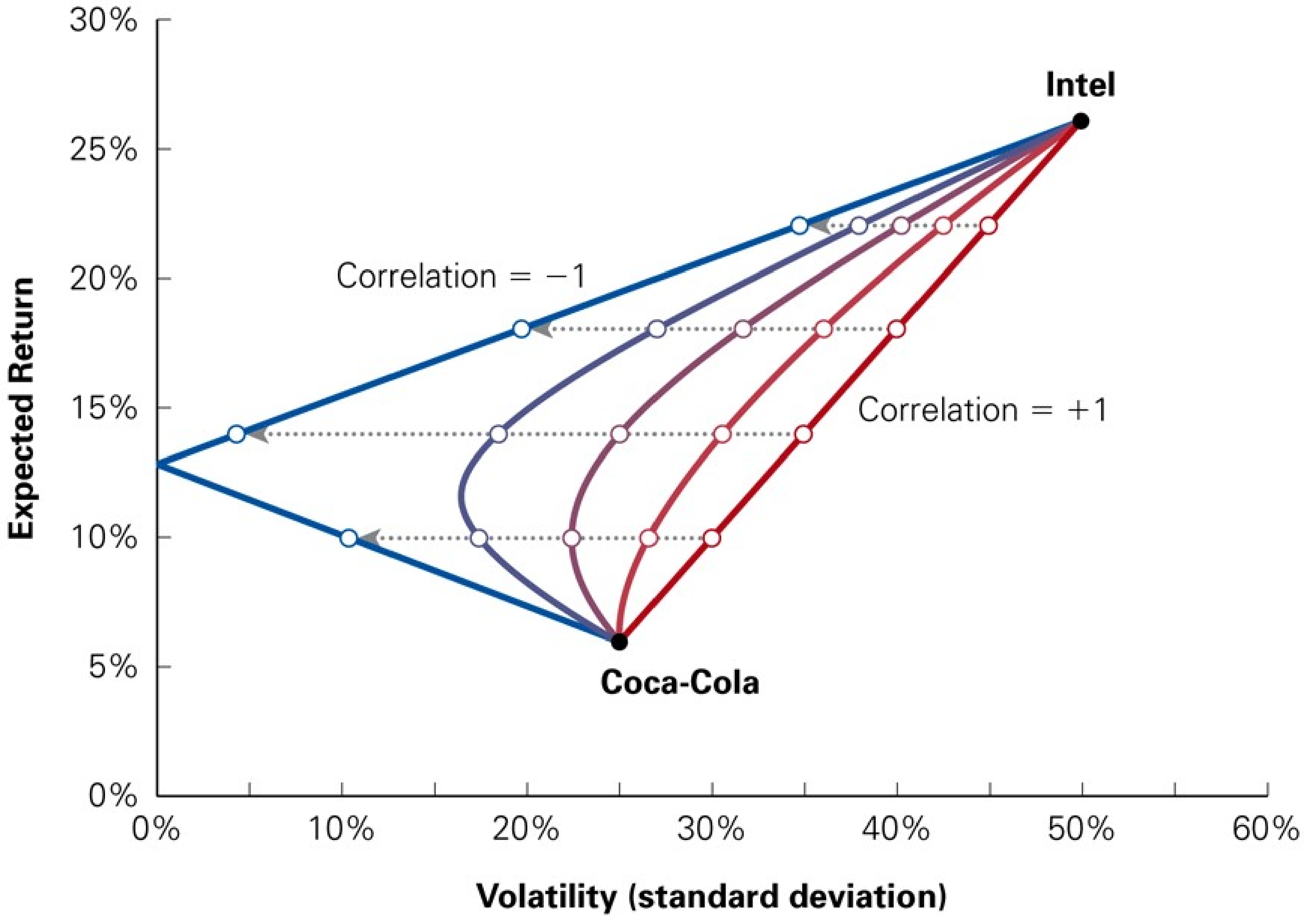
$$2\omega_1\sigma_1^2 - 2(1-\omega_1)\sigma_2^2 + 2\sigma_1\sigma_2\rho_{12} - 4\omega_1\sigma_1\sigma_2\rho_{12} = 0$$

$$\Rightarrow \omega_1^{MV} = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

Effect of correlation between assets

- Correlation has no effect on the expected return of a portfolio
- However, the volatility of the portfolio will differ depending on the correlation between its individual assets
- The lower the correlation between the individual assets, the lower the portfolio volatility
- On the next slide, we plot expected returns and volatility of portfolios of Intel and Coca-Cola
  - We assume different values for the correlation between Intel and Coca-Cola
  - Each curve represents a different correlation

Effect of correlation between assets



Short sales

- Long position
  - A positive investment in a security
  - Positive portfolio weight
- Short position
  - A “negative investment” in a security
  - Negative portfolio weight
  - In a short sale, you sell a stock that you do not own – you borrow it from a bank - and then buy that stock back in the future to return it to the bank
  - Short selling is an advantageous strategy if you expect a stock price to decline in the future
- Allowing short positions in stocks increases our set of feasible portfolios

Short sales - Example

- Suppose we have $20,000 to invest
- We short sell $10,000 worth of Coca-Cola and invest the proceeds from the short-sale plus the $20,000 in Intel
- Calculate the expected return and volatility of the portfolio if Intel and Coca-Cola stocks are uncorrelated

Stock	Expected Return	Volatility
Intel	26%	50%
Coca-Cola	6%	25%

Short sales - Example

- The short sale is like a “negative investment” of -$10,000 in Coca-Cola
  - We “borrow” $10,000 worth of Coca-Cola stocks
  - We will have to return the stocks next year
    - We have to purchase and return the borrowed Coca-Cola stocks next year for an *expected* price of $10,000*1.06 = $10,600 next year
    - *Note that while we can form an expectation of the Coca-Cola price next year, it is not certain!*
- To calculate expected returns and volatility of our portfolio, we need the portfolio weights
- For the portfolio weights we need the total *current* value of the portfolio
- *Currently,* the total value of our portfolio is $20,000
  - That is the $30,000 investment in Intel, plus the - $10,000 that we “borrowed” by short selling Coca-Cola shares

Short sales - Example

- The portfolio weights in Intel and Coca-Cola are thus:

$$x_I = \frac{\text{Value of investment in Intel}}{\text{Total value of portfolio}} = \frac{30{,}000}{20{,}000} = 150\%$$

$$x_C = \frac{\text{Value of investment in Coca-Cola}}{\text{Total value of portfolio}} = \frac{-10{,}000}{20{,}000} = -50\%$$

- Note that the weights still add up to 100% (as they should!)
- Now let us calculate the expected return and volatility of the portfolio:

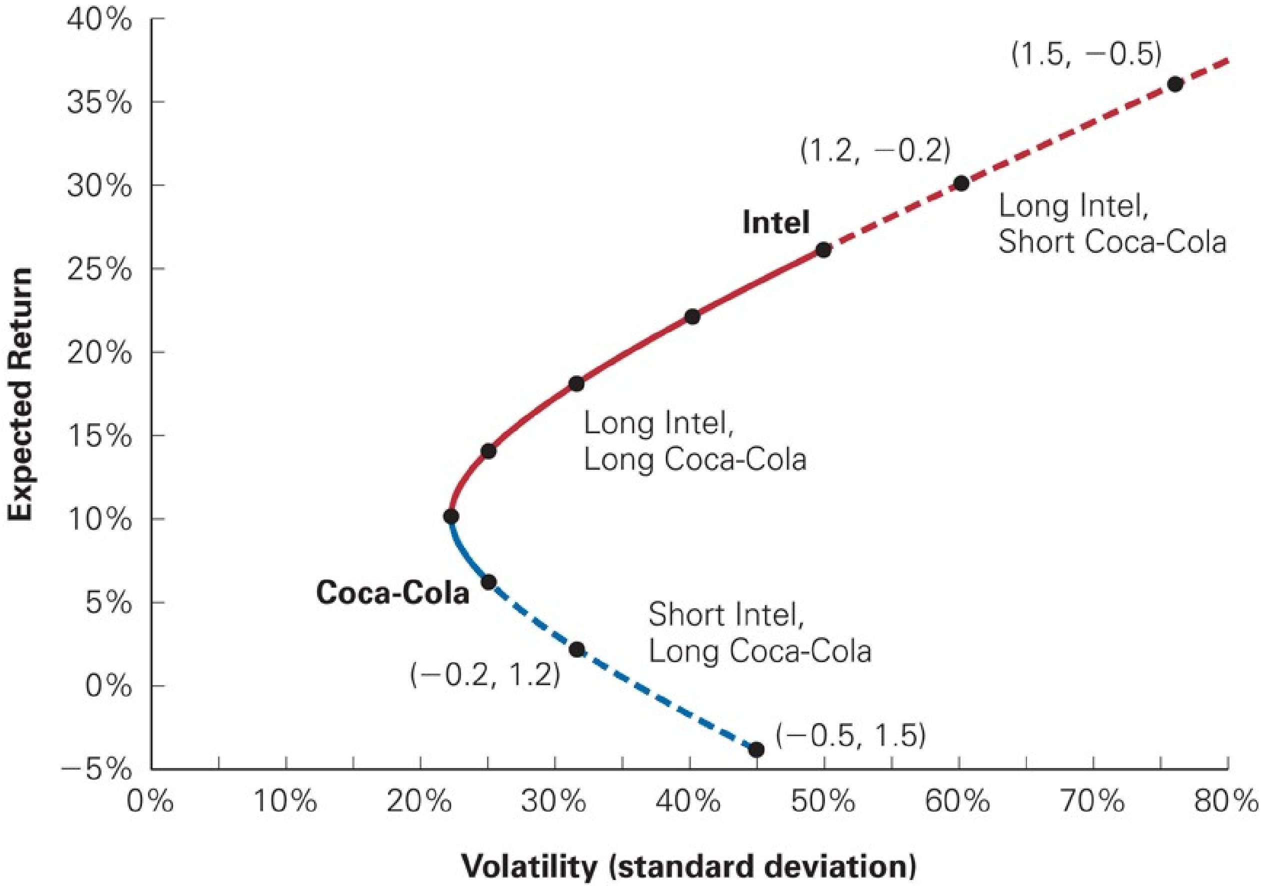
$$E[R_P] = x_I E[R_I] + x_C E[R_C] = 1.50 \times 26\% + (-0.50) \times 6\% = 36\%$$

$$SD(R_P) = \sqrt{Var(R_P)} = \sqrt{x_1^2 Var(R_I) + x_C^2 Var(R_C) + 2 x_I x_C Cov(R_I, R_C)}$$

$$= \sqrt{1.5^2 \times 0.50^2 + (-0.5)^2 \times 0.25^2 + 2(1.5)\,(-0.5)\,(0)} = 76.0\%$$

Increases the expected return, but also the volatility, both above the values for the two individual stocks

Portfolios of Intel and Coca-Cola with short sales



Optimal risky portfolio

- Which risky portfolio gives us the best risk-return combination?
  - Find the portfolio that maximizes the Sharpe ratio (compensation per unit of risk)
  - The **Sharpe ratio** for any portfolio is defined as the excess return per unit of risk

$$Sharpe\ Ratio(SR) = \frac{Portfolio\ Excess\ Return}{Portfolio\ Volatility} = \frac{E(R_P) - r_f}{SD(R_P)}$$

  - The Sharpe ratio is the measure of performance used by most stock mutual funds

Optimal risky portfolio

- For the two-asset case, choose ω_1 and ω_2 to maximize SR.

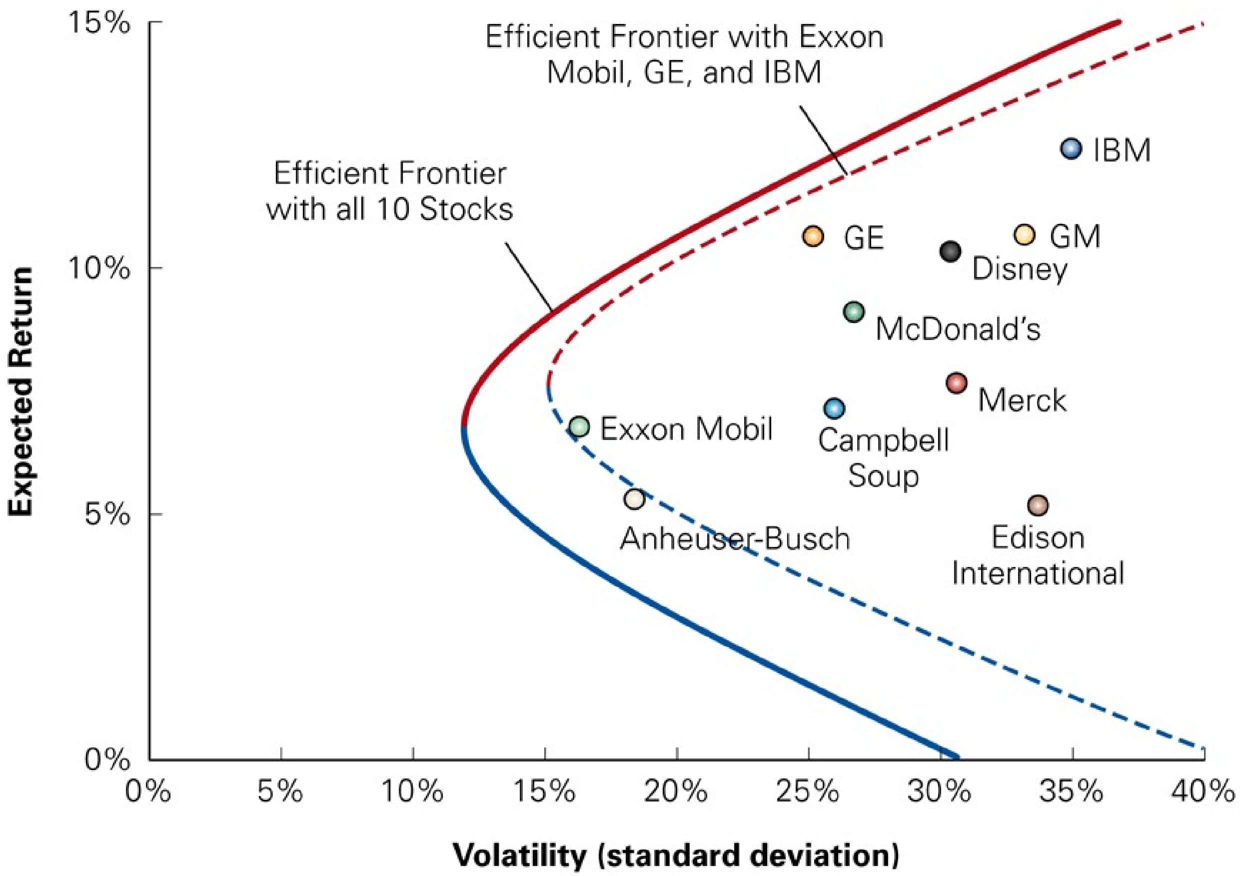
$$Max_{\omega}\ SR = \frac{\mu_P - r_f}{\sigma_P}$$

- The solution to the maximization problem is given by

$$\omega_1 = \frac{(\mu_1 - r_f)\sigma_2^2 - (\mu_2 - r_f)Cov(r_1, r_2)}{(\mu_1 - r_f)\sigma_2^2 + (\mu_2 - r_f)\sigma_1^2 - [(\mu_1 - r_f) + (\mu_2 - r_f)]Cov(r_1, r_2)}$$

$$\omega_2 = 1 - \omega_1$$

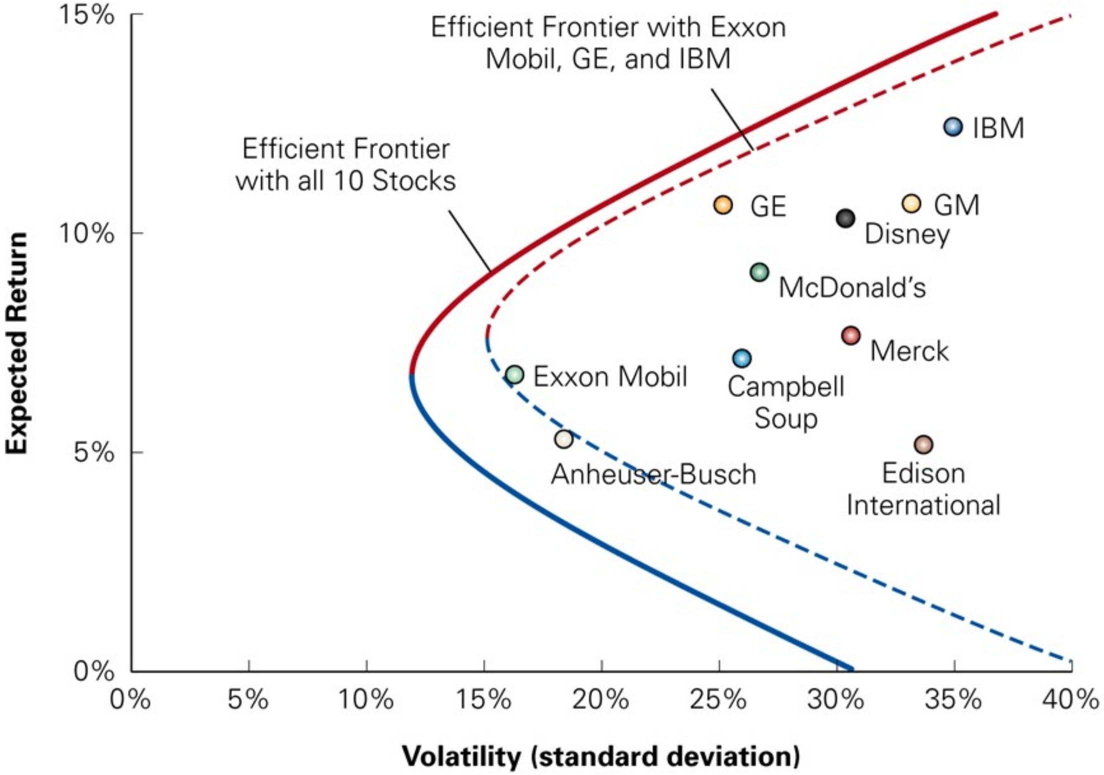
Efficient frontier with 10 stocks vs. 3 stocks



With more assets

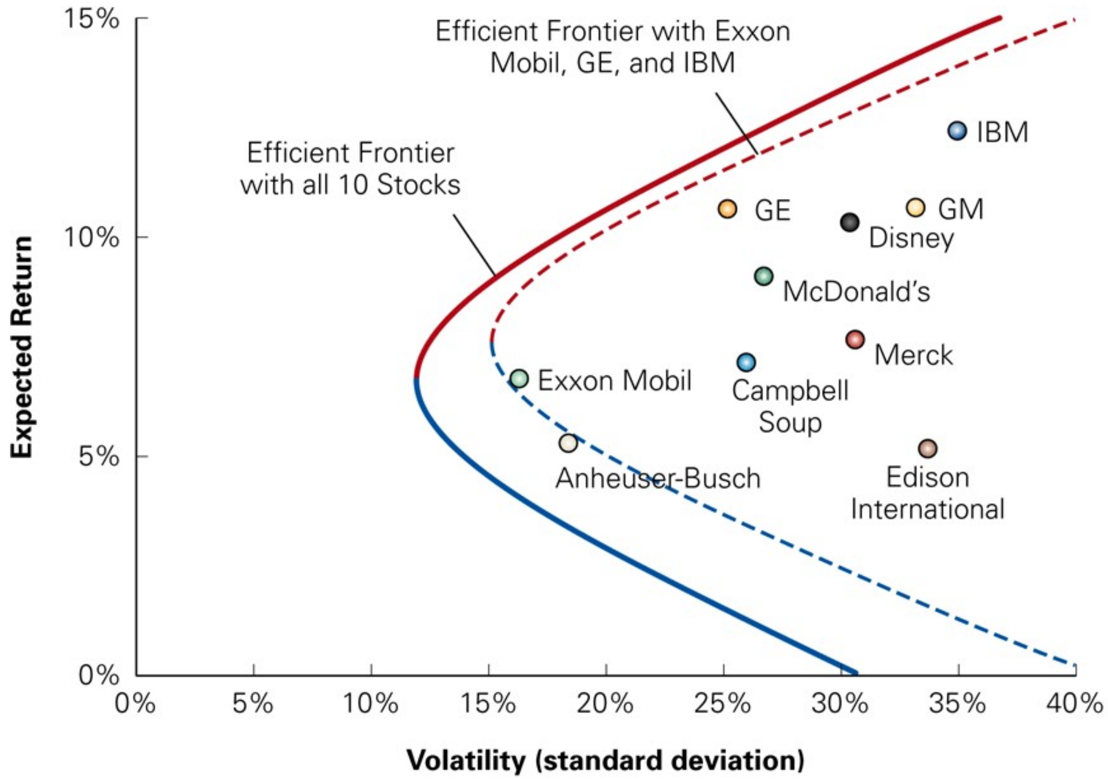
- If investors only care about risk and return, every rational, risk–averse investor prefers a portfolio on the efficient set to any other portfolio.
- With many assets, suppose the weights of all securities are positive. As long as the correlations between pairs of securities are less than 1, the standard deviation of a portfolio of many assets is less than the weighted average of the standard deviations of the individual securities.
- Where does the diversification effect from? Recall that when adding assets, one can decrease exposure to idiosyncratic risk. Only the exposure to systematic risk remains.

Efficient frontier with 10 stocks vs. 3 stocks



- As long as they are not perfectly and positively correlated with any combination of existing securities, adding new stocks will always make formerly efficient stock portfolios inefficient
- **Question:** why is it that holding GE only does not belong to the efficient frontier of stock portfolios?

Efficient frontier with 10 stocks vs. 3 stocks



- **Question:** why is it that holding GE only does not belong to the efficient frontier of stock portfolios?
- **Answer:** Because, as long as GM and GE are not perfectly positively correlated, I can reduce the variance while keeping the expected return constant (since GE and GM have similar expectations) by putting GE and GM together (i.e., by diversifying my portfolio).

Adding a risk-free asset to a risky portfolio

- So far, we have considered only portfolios of risky assets
- Risk can be further reduced by investing a portion of a portfolio in a risk-free investment (e.g., U.S. Treasury-Bills)
  - However, doing so will reduce the expected return of the portfolio (why?)
- On the other hand, an aggressive investor who is seeking high expected returns might decide to borrow money at the risk-free rate (short-sell T-Bills) to invest even more in the stock market.

Adding a risk-free asset to a risky portfolio

- Consider an arbitrary risky portfolio (*P*) and a risk-free asset
- We invest an arbitrary fraction *x* into the risky portfolio and a corresponding fraction (*1-x*) in the risk-free asset
    - The expected return of this portfolio would be:

$$
\begin{aligned}
E[R_{xP}] &= (1 - x)r_f + xE[R_P] \\
&= r_f + x\left(E[R_P] - r_f\right)
\end{aligned}
$$

- $x < 1$ means that you lend at the risk-free rate.
- $x > 1$ means that you borrow at the risk-free rate.

Risk-free rate

Risky share of total portfolio

Expected excess return on risky portfolio P

Adding a risk-free asset to a risky portfolio

- The standard deviation of this portfolio would be:

$$SD[R_{xP}] = \sqrt{(1-x)^2 Var(r_f) + x^2 Var(R_P) + 2(1-x)xCov(r_f, R_P)}$$

$$= \sqrt{x^2 Var(R_P)}$$

$$= xSD(R_P)$$

0

- Note:
  - We know for sure what the risk-free asset is worth next period, so its return is a constant
  - The variance of a constant is zero
  - The correlation / covariance of a constant with anything else is zero
- Therefore, the standard deviation of our portfolio of risky asset (*P*) and risk-free asset is only a fraction of the volatility of the risky portfolio, based on the amount invested in the risky portfolio.

Risky portfolio plus risk-free asset - Example

- Suppose you can invest in a diversified portfolio *P* of risky assets
  - Its expected return is 11%, and its volatility is 8%
- There is also a risk-free asset with a return of 5%
- What is the expected return and volatility of a new portfolio consisting of 50% investment in the risky portfolio *P* and 50% in the risk-free asset?

Risky portfolio plus risk-free asset - Example

- Expected return:

$$E[R_{xP}] = (1-x)r_f + x\,E[R_P]$$

$$= r_f + x\big(E[R_P] - r_f\big) \qquad (*)$$

$$E[R_{xP}] = 5\% + 0.5 \cdot (11\% - 5\%) = 8\%$$

- Volatility:

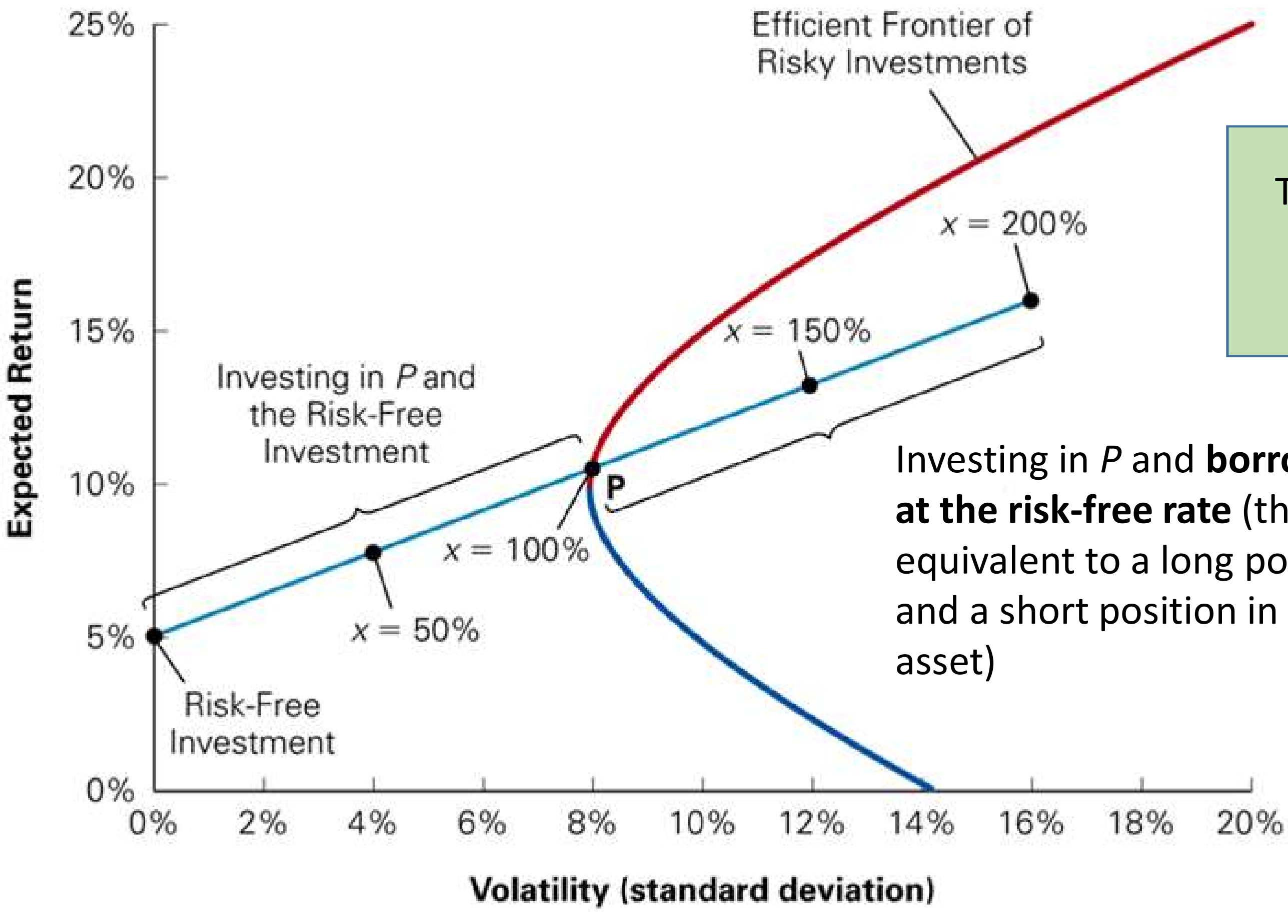
$$SD(R_{xP}) = x\,SD(R_P) \;\; (**)$$

$$SD(R_{xP}) = 0.5 \;\cdot 8\% = 4\%$$

- Insert (**) into (*)

$$E[R_{xP}] = r_f + \frac{SD(R_{xP})}{SD(R_P)}(E[R_P] - r_f)$$

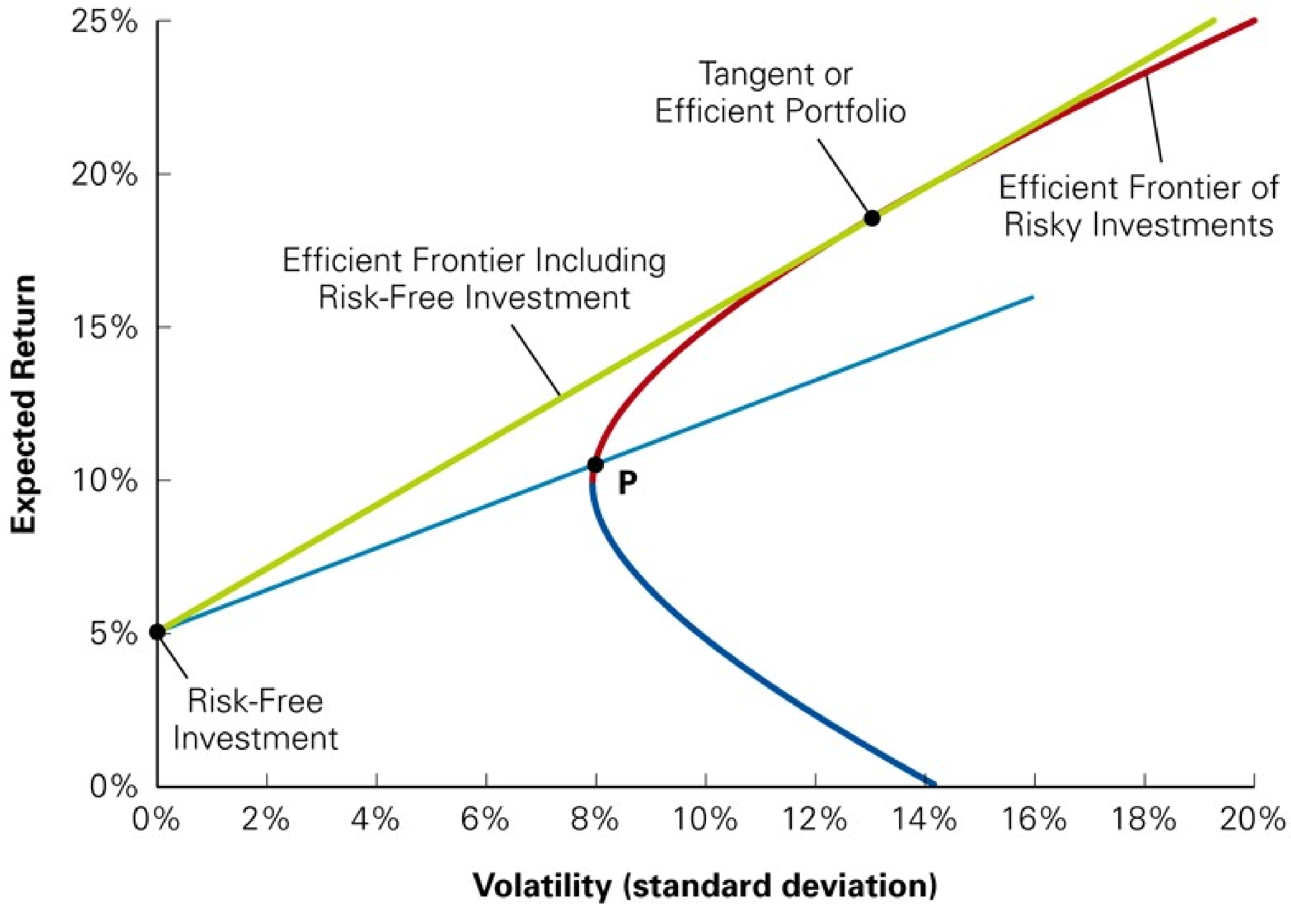
Risky portfolio plus risk-free asset



This line is called the Capital Allocation Line (CAL) of portfolio *P.*

Investing in *P* and **borrowing at the risk-free rate** (this is equivalent to a long position in *P* and a short position in the risk-free asset)

The tangent or efficient portfolio



Note that with the given set of risky assets, instead of having invested in portfolio *P*, we could have also invested in the **tangent portfolio.** It is the **optimal risky portfolio.**

The tangent portfolio has one last interesting property: it has the highest *Sharpe Ratio* of all the portfolios formed of available stocks