Principles of Finance
Risk Aversion and Certainty Equivalent

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Utility theory

- In utility theory, when there is uncertainty, people make choices by maximizing the expected utility of future wealth:

$$E\big(u(W)\big) = \sum_s \pi_s u(w_s)$$

- where W is a random variable that represents wealth, w_s is its future value in state s, π_s is the probability of state s, and $u(\cdot)$ is a utility function which is assumed to be monotonically increasing.
- The utility function represents the **preferences** of investors.

Keyword:
Expected utility

Expected utility: an example

Suppose your utility function of wealth is $u(W) = \sqrt{W}$ and you have two investment opportunities:

- A: Your future wealth will be \$100 with certainty (100%);
- B: Your future wealth will be \$50 with 50% possibility and \$150 with 50% possibility.

Questions:

1. What is your expected wealth with each opportunity?
2. What are your expected utilities of future wealth?
3. Which opportunity will you choose?

Expected utility: an example

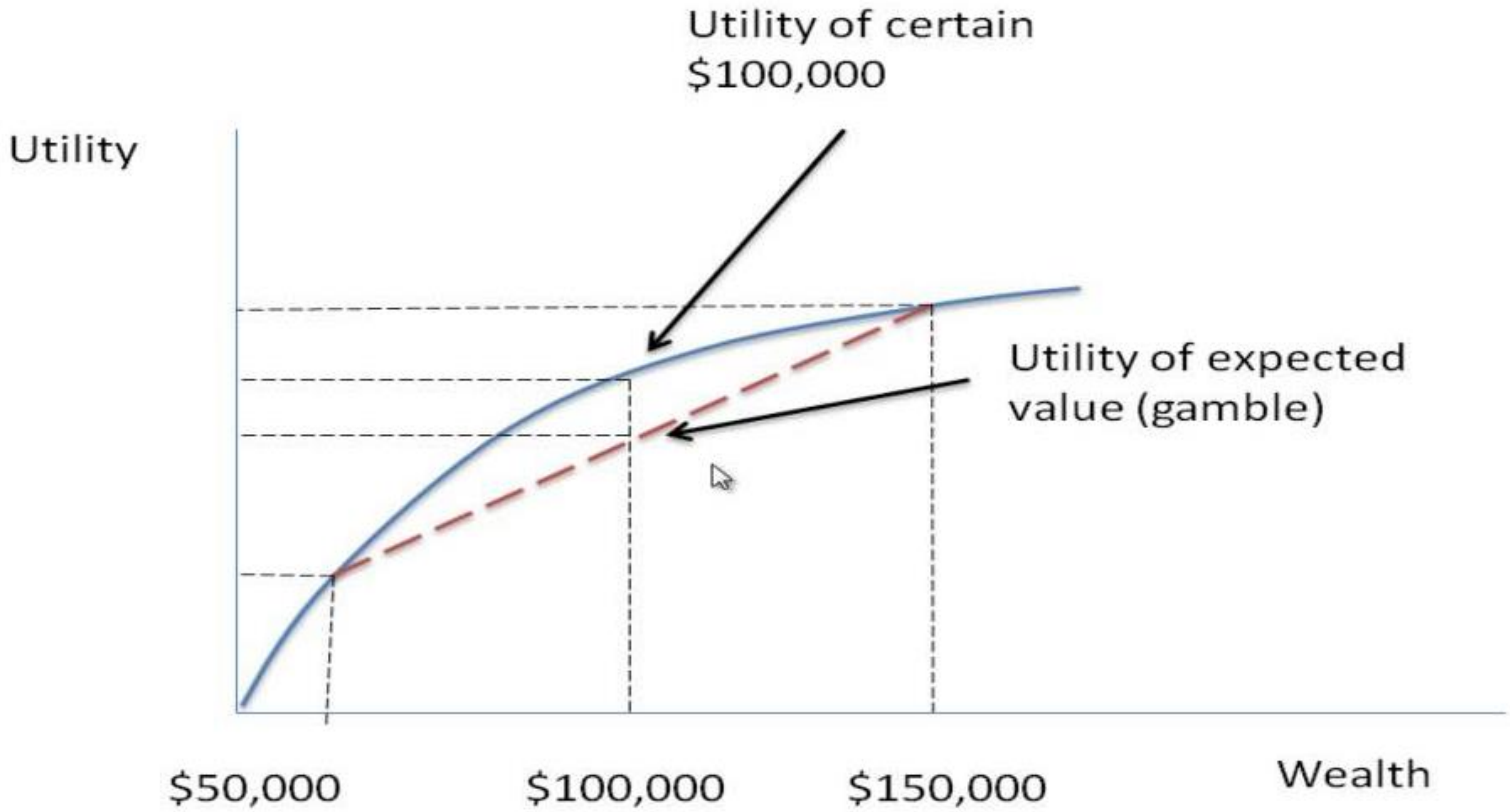
Suppose your utility function of wealth is $u(W) = \sqrt{W}$ and you have two investment opportunities:

- The expected wealth levels are
  - A: $E[W] = 100$
  - B: $E[W] = 0.5 \times 50 + 0.5 \times 150 = 100.$

- The expected utilities of future wealth are:
  - A: $E[u(W)] = \sqrt{100} = 10$
  - B: $E[u(W)] = 0.5 \times \sqrt{50} + 0.5 \times \sqrt{150} \approx 9.66.$

  $\rightarrow$ **Choose opportunity A**.

Expected utility: an example



Risk aversion

- An agent is **risk-averse** if she prefers getting the expected value of a bet to the bet itself, i.e., $u\big(E(W)\big) > E\big(u(W)\big)$. In this case, her utility function is **concave** ($u'' < 0$)

- An agent is **risk-neutral** if she is indifferent between the two, i.e., $u\big(E(W)\big) = E\big(u(W)\big)$. In this case, her utility function is linear ($u'' = 0$)

- An agent is **risk-loving** if she prefers getting the bet itself, i.e., $u\big(E(W)\big) < E\big(u(W)\big)$. In this case, her utility function is convex ($u'' > 0$).

Back to our example

- Calculate $u\big(E(W)\big)$ and $E\big(u(W)\big)$ for the uncertain opportunity:
  - $u\big(E(W)\big) = \sqrt{100} = 10$
  - $E\big(u(W)\big) \approx 0.66 < u\big(E(W)\big)$

- This is an example of risk-averse agent.

- Check the concavity of the utility function:
  - $U(W) = \sqrt{W} \;\rightarrow U'(W) = \frac{1}{2\sqrt{W}} = \frac{1}{2W^{1/2}}$
  - $\rightarrow U''(W) = \frac{1}{2} \times \frac{1}{2} W^{-1/2} \times \left(-\frac{1}{W}\right) = -\frac{1}{4W^{3/2}} < 0$ for $W > 0$.
  - $\rightarrow U$ is concave.

$$\left(\frac{1}{x^k}\right)' = -\frac{k}{x^{k+1}}$$

Risk aversion

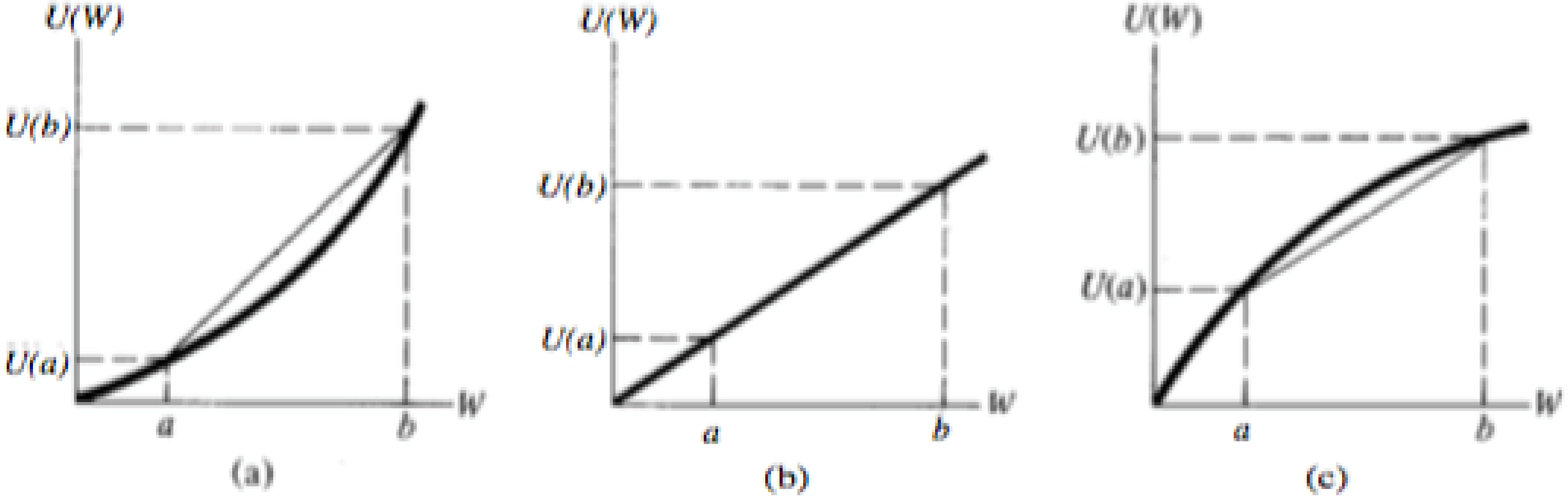


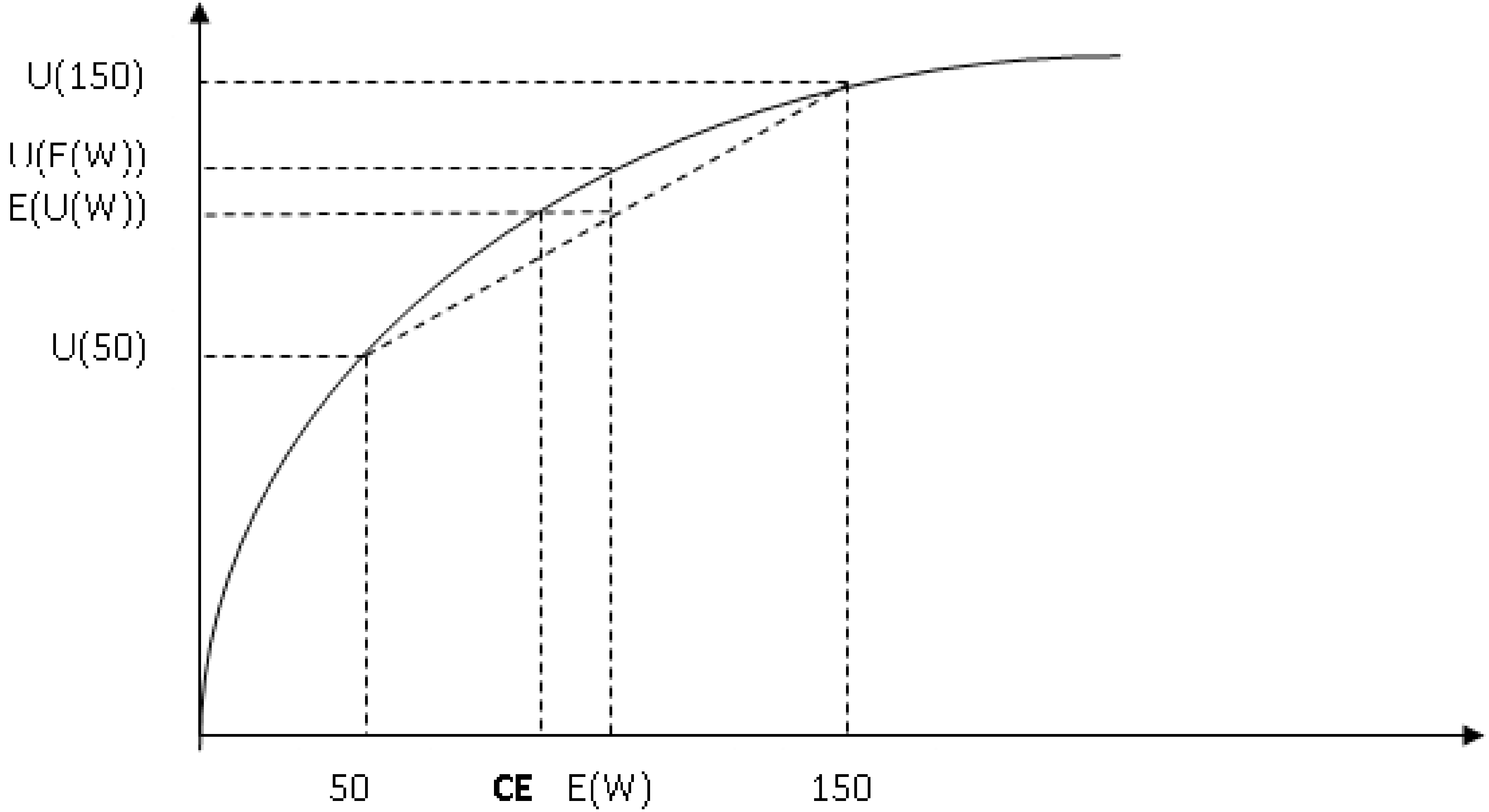
Figure 4.6
Three utility functions with positive marginal utility: (a) risk lover; (b) risk neutral; (c) risk averter.

Certainty equivalent

- The **certainty equivalent** of a bet is the amount of wealth CE that will make the agent indifferent between A: receiving this amount for sure and B: making the bet.
- If the agent chooses A, her expected utility is $u(CE)$
- If she chooses the bet W, her expected utility is $E\big(u(W)\big)$
- Therefore the certainty equivalent is such that

$$u(CE) = E\big(u(W)\big)$$

Certainty equivalent



Risk premium

- Recall that the **risk premium** is given by
$$RP = E[R] - r_f$$
- Here the risky return is
$$E[R] = \frac{E[W] - W_0}{W_o}$$
- Obtaining the certainty equivalent is by definition risk-free, therefore
$$r_f = \frac{CE - W_0}{W_0}$$
- Putting everything together we have
$$RP = \frac{E[W] - CE}{W_0}$$

Risk premium

- Risk aversion implies

$$E[W] - CE > 0$$

and therefore

$$RP = \frac{E[W] - CE}{W_0} > 0$$

- Insurance premium: how much are you ready to pay to be hedged against the uncertainty of the lottery?

$$W_0 + IP = CE$$

Back to our example

If $u(x) = y$, the inverse of u satisfies $x = u^{-1}(y)$. In the case where
$$u(x) = \sqrt{x} = y,$$
the inverse can be found by taking the square:
$$x = y^2.$$
Therefore $u^{-1}(y) = y^2$.

What is the certainty equivalent and the risk premium?

- Remember:
  - A: $E[u(W)] = \sqrt{100} = 10$
  - B: $E[u(W)] = 0.5 \times \sqrt{50} + 0.5 \times \sqrt{150} \approx 9.66$.
- The CE satisfies
  - A: $u(CE) = E[u(W)] = 10$ therefore $CE = u^{-1}(10) = 10^2 = 100$
  - B: $u(CE) = E[u(W)] \approx 9.66$ therefore $CE \approx u^{-1}(9.66) = 9.66^2 \approx 93.32$.
- The risk premium is proportional to $E[W] - CE$
  - A: $RP = E[W] - CE = 0$ (no risk)
  - B: $RP \propto E[W] - CE \approx 100 - 93.32 = 6.68$.

Example

Consider two gambles:

- Gamble X: equal chance of winning 1 or 5 dollars
- Gamble Y: 5/6 chance of winning 2, 1/6 chance of winning 8

Which gamble is preferred if $u(W) = ln(W)$?

What if $u(W) = 9W - W^2$?

Steps?

Example

Calculate the expected utility of final wealth:

- Gamble X: $E[u(W)] = 0.5 \times u(\$1) + 0.5 \times u(\$5)$
  $$= 0.5 \times \ln(1) + 0.5 \times \ln(5) = 0.8047.$$
- Gamble Y: $E[u(W)] = \frac{5}{6} \times ln(2) + \frac{1}{6} ln(8) = 0.9242.$

→ Gamble Y is preferred.

Do the same with $u(W) = 9W - W^2$:

- Gamble X: $E[u(W)] = 0.5 \times u(\$1) + 0.5 \times u(\$5) = 14.$
- Gamble Y: $E[u(W)] = \frac{5}{6} \times ln(2) + \frac{1}{6} ln(8) = 13.$

→ Gamble X is preferred.

Special cases

- A particular case of risk-aversion is when only the mean and the variance of the lottery matters, i.e,

$$E\big(u(W)\big) = f(E(W), Var(W)), \qquad \frac{\partial f}{\partial E(W)} > 0, \frac{\partial f}{\partial Var(W)} < 0$$

- This representation holds if
  - Wealth is normally distributed **or** if
  - The utility function is quadratic: $u(W) = a + bW + cW^2$. Indeed, in this case,
  - $E\big(u(W)\big) = a + bE(W) + cE(W^2) = a + bE(W) + c(Var(W) + E(W)^2)$
    - $\frac{\partial f}{\partial E(W)} = b + 2cE[W] > 0 \ \rightarrow \ \frac{-b}{2c} < E[W]$
    - $\frac{\partial f}{\partial Var(W)} = c < 0$

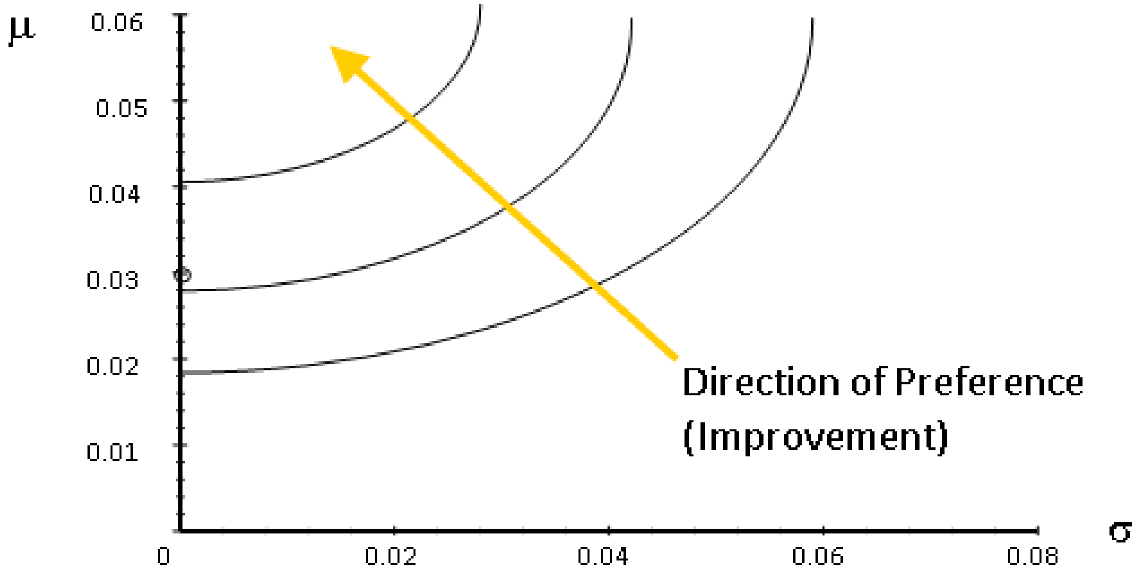
Use here that
$Var(X) = E(X^2) - E(X)^2$
Therefore
$E(X^2) = Var(X) + E(X)^2$

Special case of normally distributed wealth

- Example:

$$E[u(W)] = E[W] - 0.5A \times Var(W)$$

- Investors have a preference for higher expected return and lower standard deviation (variance).



Example

Consider 3 types of portfolios

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

- Consider an expected utility of the form $E[u(W)] = E[W] - 0.5AVar(W)$. Calculate the expected utility of each portfolio for $A = 2, 3.5$ and 5.

Example

Investor Risk Aversion (A)	Utility Score of Portfolio L [$E(r)$ = .07; σ = .05]	Utility Score of Portfolio M [$E(r)$ = .09; σ = .10]	Utility Score of Portfolio H [$E(r)$ = .13; σ = .20]
2.0	$.07 - ½ \times 2 \times .05^2 = .0675$	$.09 - ½ \times 2 \times .1^2 = .0800$	**$.13 - ½ \times 2 \times .2^2 = .09$**
3.5	$.07 - ½ \times 3.5 \times .05^2 = .0656$	**$.09 - ½ \times 3.5 \times .1^2 = .0725$**	$.13 - ½ \times 3.5 \times .2^2 = .06$
5.0	$.07 - ½ \times 5 \times .05^2 = .0638$	**$.09 - ½ \times 5 \times .1^2 = .0650$**	$.13 - ½ \times 5 \times .2^2 = .03$

- A more risk-averse investor will get less utility from a bet than a less risk-averse investor.
- When the risk is low, the difference in utility is small. But when the risk is high, the difference is large.

Challenges

- **Modelling challenge**:
  - In general, investors do not just care about mean and variance of the risk
  - It is tricky to find a utility function that is tractable and realistic

- **Estimation challenge**
  - How does one estimate risk aversion?