73-240 – Problem set 2

Due Friday Sep. 27th

From the Syllabus:

1. Homework must be turned in on the day it is due (usually on a Friday) in the undergraduate program office in Tepper 2400 by **430pm** sharp. Do NOT leave it under my office door. Late homework will NOT be accepted unless you are sick and have a doctor's note.
2. Homework regrading: There is a statute of limitations on regrades. If you believe a question has been incorrectly graded, please take your homework to your TA within 2 weeks of it being returned.
3. Working in groups: You may work in groups of up to 4. BUT: You MUST put names of other group members on your homework. You MUST write up your own set of answers. Do NOT simply copy some other person's work.
4. TYPE your work. Long equations may be hand written. Buy a stapler!
5. Write your first and last name on the title of each graph. Graphs may be hand drawn.
6. Carefully explain your work.

1 Problem 1: Working with the Utility Function

Consider the household with the following utility function:

$$U(c, \ell) = \sqrt{c\ell}$$

a Derive marginal utility of leisure. Derive marginal utility of consumption.
Answer:

$$\frac{\partial U(c,\ell)}{\partial \ell} = \frac{1}{2}c^{1/2}\ell^{-1/2}$$
$$\frac{\partial U(c,\ell)}{\partial c} = \frac{1}{2}c^{-1/2}\ell^{1/2}$$

b Show whether the utility function exhibits diminishing marginal utility in leisure.
Answer:

$$\frac{\partial^2 U(c,\ell)}{\partial \ell^2} = -\frac{1}{4}c^{1/2}\ell^{-3/2} < 0$$

Yes the utility function exhibits diminishing marginal utility in leisure.

c Derive the marginal rate of substitution between leisure and consumption, $MRS_{\ell,c}$. Show what this implies for the shape of the indifference curve (assuming that indifference curves slope downwards because our household prefers diversity).
Answer:

$$MRS_{\ell,c} = \frac{\partial U(c,\ell)/\partial \ell}{\partial U(c,\ell)/\partial c} = \frac{c}{\ell}$$

Note that this implies the slope of our indifference curve is not constant and that the indifference curve is actually convex (the household prefers diversity). Note that when c is high and ℓ is low, the household's marginal rate of substitution between leisure and consumption is high. When ℓ is low, the household's marginal utility of leisure is very high (a little more leisure makes the household observe a big increase in her utility). As such, the household is willing to substitute a lot of consumption to raise its leisure, as such $MRS_{\ell,c}$ is high when ℓ is low and c is high. In contrast when ℓ is very large and c is low, $MRS_{\ell,c}$ is small. In this case, because of diminishing marginal utility, an additional unit of leisure does not give the household a large increase in utility and instead causes utility to increase by a small amount. As such, the household is only willing to substitute a small amount of consumption for 1 more unit of leisure.

d Show that the utility function is homogeneous of degree 1.
Answer:
If we scale the arguments of the utility function by x factor, the total utility is also scaled by x factor:

$$U(xc, x\ell) = (xc)^{1/2}(x\ell)^{1/2} = x(c\ell)^{1/2} = xU(c,\ell)$$

2 Problem 2: Household maximization

Using the same utility function as in Problem 1, now consider the household that receives zero non-labor income (i.e. dividends less lump-sum taxes is equal to zero) and is paid a wage w for every unit of labor supplied. The household has a total time endowment of $h = 1$.

a Write down the household's problem. In your answer, state what is the household's objective and what constraints it may face.
Answer:

$$\max_{c,\ell} U(c,\ell) = (c\ell)^{1/2}$$

s.t.

$$c = w(1-\ell)$$

b Solve for optimal consumption and leisure of the household.
Answer:

$$\max_{c,\ell,\lambda} \mathcal{L} = (c\ell)^{1/2} + \lambda[w(1-\ell) - c]$$

Take first order conditions wrt c

$$\frac{1}{2}c^{-1/2}\ell^{1/2} = \lambda$$

Take first order conditions wrt ℓ

$$\frac{1}{2}c^{1/2}\ell^{-1/2} = \lambda w$$

Take first order conditions wrt λ

$$c = w(1-\ell)$$

Note that optimality requires what the household chooses is affordable:

$$c = w(1-\ell)$$

and that the household optimally trades off consumption and leisure such that $MRS_{\ell,c} = w$

$$\frac{c}{\ell} = w \implies c = w\ell$$

Plug the above into the budget constraint and re-arrange to solve for ℓ

$$\ell = \frac{1}{2}$$

The household spends half its time in leisure (since $h = 1$, $\ell = 1/2$ implies half the time is spend in leisure). The household optimally consumes $c = 1/2w$.

c Suppose all households are identical, and there are M households in the economy. What is aggregate consumption spending and aggregate labor supply equal to?
Answer:
Since all households are identical, we have: $C = M \times c = 1/2wM$. Similarly, the time spent in labor by a single household is given by $h - \ell = 1 - 1/2 = 1/2$. So the aggregate labor supply of M households is given by $N^s = M \times n^s = 1/2M$

d Suppose the wage rate in this economy rises. What happens to aggregate consumption and aggregate labor supplied? Given your results, suggests what this implies about the relative magnitudes of the income and substitution effects
Answer:

In this case if the wage rate rises, we observe that aggregate consumption C rises with the wage rate since $C = 1/2wM$. Since the aggregate labor supply $N^s = 1/2M$, it does not change with the wage rate. This suggests that the income effect is equal in size to the substitution effect.

3 Problem 3: Firms and Outlook

Recent news suggests that manufacturing firms are pessimistic about their revenue outlooks with the looming trade war. Consider a firm with production function given by

$$zF(k, n^d) = zk^{\alpha}n^{d,1-\alpha}$$

Suppose the firm is born with capital k and pays a real wage w for each unit of labor demanded.

a Write down the firm's problem. State what variables the firm treats as exogenous. State the firm's choice variables.

Answer:
Firm's problem is to maximize profits by choosing labor demand:

$$\max_{n^d} \pi = zk^\alpha n^{d,1-\alpha} - wn^d$$

The exogenous variables are z, k. The firm chooses n^d.

b Solve for optimal labor demanded.
Answer:
Taking first order conditions and setting them to zero, we have:

$$\frac{d\pi}{dn^d} = 0 \implies \underbrace{(1-\alpha)zk^\alpha n^{d,-\alpha}}_{MPN} = w$$

Re-arranging to make labor demand the subject of the equation:

$$n^d = \left(\frac{(1-\alpha)zk^\alpha}{w}\right)^{1/\alpha}$$

c Suppose there are M representative firms in the economy. Write down what is aggregate labor demanded.
Answer:
Because of constant returns to scale and the fact that we have representative firms (identical), we know that the above is equivalent to solving the problem *as if* 1 firm decides aggregate labor demand:

$$N^d = Mn^d = \left(\frac{(1-\alpha)zK^\alpha}{w}\right)^{1/\alpha}$$

where $K = Mk$

d Suppose a decline in revenue outlook can be associated with a decline in z. Explain why manufacturing firms are expected to scale back their hiring in the presence of poor revenue outlooks from a looming trade war.
Answer:
Since z scales our production function, this implies that when firms expect total factor productivity (or a factor that affects profitability separate from inputs) to fall, they also expect marginal product of labor to be lower:

$$MPN = (1-\alpha)zK^\alpha N^{d,-\alpha}$$

Since additional each unit of labor produces a smaller increment in output than before, given no change in w, this implies that the old (original) choice of N^d (prior to the decline in z) now gives us marginal cost (w) greater than marginal benefit (as measured by MPN). Thus since the marginal benefit from using 1 more unit of labor is lower, firms would cut back on labor demand until we again get $MPN = w$.

4 Problem 4: Are the Robots Replacing Labor?

Consider the following firm who uses two types of labor in production, high-skilled labor (N_H) and low-skilled labor (N_L). The firm also uses capital in production. Unlike the standard problem considered in lecture, consider the firm who must rent capital at a real rental rate r, and who pays a real wage rate of w_H and w_L to each unit of high-skilled and low-skilled labor demanded respectively. Let the firm's production function be given by:

$$F(K, N_H, N_L) = (K + N_L)^\alpha N_H^{1-\alpha}$$

a Show which input is capital a complement to, and which input capital acts as a substitute.
[Hint: Remember if two inputs are complements, increasing K would lead to an increase in the other input's marginal product]
Answer:
We take the cross-derivative to examine if two inputs are complements. Observe that the marginal product of low-skilled labor is given by:

$$MPN_L = \frac{dF(K, N_H, N_L)}{dN_L} = \alpha(K + N_L)^{\alpha-1} N_H^{1-\alpha}$$

Differentiating MPN_L with respect to K, we get:

$$\frac{d}{dK} MPN_L = \alpha(\alpha - 1)(K + N_L)^{\alpha-2} N_H^{1-\alpha} < 0$$

Since α is a share bounded between 0 and 1, this implies that the marginal product of low-skilled labor is decreasing in increased usage of capital. So N_L and K are substitutes.

Conversely, observe that the marginal product of high-skilled labor is given by:

$$MPN_H = \frac{dF(K, N_H, N_L)}{dN_H} = (1 - \alpha)(K + N_L)^\alpha N_H^{-\alpha}$$

Differentiating MPN_H with respect to K, we get:

$$\frac{d}{dK}MPN_H = \alpha(1-\alpha)(K+N_L)^{\alpha-1}N_H^{-\alpha} > 0$$

ince α is a share bounded between 0 and 1, this implies that the marginal product of high-skilled labor is increasing in increased usage of capital. So N_H and K are complements.

b Write down the firm's problem.
Answer:
The firm's problem is to maximize profits by choosing K, N_H, N_L:

$$\max_{K,N_H,N_L} \pi = (K+N_L)^{\alpha}N_H^{1-\alpha} - rK - w_L N_L - w_H N_H$$

c State what would happen to the firm's choice of N_L if $w_L < r$. What does the model imply about the use of low skilled labor as robots become relatively cheaper (in other words if r falls relative to w_L.)
[Hint: You don't need math to answer this question, your answer depends on how you answered part (a).]
Answer:
Because K and N_L are perfect substitutes, if $w_L < r$, the firm would only choose N_L in production. If r keeps falling such that at some point this relationship reverses, i.e. $w_L > r$, then the firm would switch and only use capital as opposed to low-skilled labor in production.

d Suppose $w_L = r$, derive the firm's optimality conditions. In your answer, derive an expression for N_H in terms of $(K+N_L), w_H$ and α. Holding all else constant, what would happen to the firm's demand for high skilled labor if $(K+N_L)$ increases because of a fall in w_L and r
Answer:
The firm's optimality conditions can be found by taking first order conditions wrt K, N_L and N_H. Note that taking first order conditions with respect to capital, we get $MPK = r$ or

$$\alpha(K+N_L)^{\alpha-1}N_H^{1-\alpha} = r$$

With respect to low-skilled labor, we get $MPN_L = w_L$

$$\alpha(K+N_L)^{\alpha-1}N_H^{1-\alpha} = w_L$$

Observe that the first two optimality conditions tells us that for the firm to use both N_L and K in production, we require that $w_L = r$, otherwise for any $w_L > r$ or $r > w_L$, the firm has a strict preference for using the cheaper input.

With respect to high-skilled labor, we get $MPN_H = w_H$

$$(1-\alpha)(K+N_L)^{\alpha}N_H^{-\alpha} = w_H$$

Note using the firm's optimality condition:

$$N_H = \left[\frac{(1-\alpha)(K+N_L)^{\alpha}}{w_H}\right]^{1/\alpha}$$

Thus, if $w_L = r$ falls, promoting higher usage of $K + N_L$ because the input is now much cheaper, then N_H will also rise since it is a complement to both K and N_L.

e Given what your previous answers, argue if the following claim is true: "Because robots are cheaper, increased usage of robots makes all labor obsolete."
Answer:
This is not necessarily true. From the above, we see that increased usage of K (where capital can take the form of robots) can encourage higher demand for high-skilled labor since high-skilled labor is a complement to capital. It is however, possible that if increased usage of K stems from a fall in r and if wage rates of low-skilled labor do not fall, then increased usage of K can displace low-skilled labor for which it is a substitute.