1.1 $\mathcal{P} = \{Bin(10,\theta) : \theta \in [0,1]\}$

$H_0 : \theta = \frac{1}{2} \qquad H_1 : \theta \neq \frac{1}{2}$

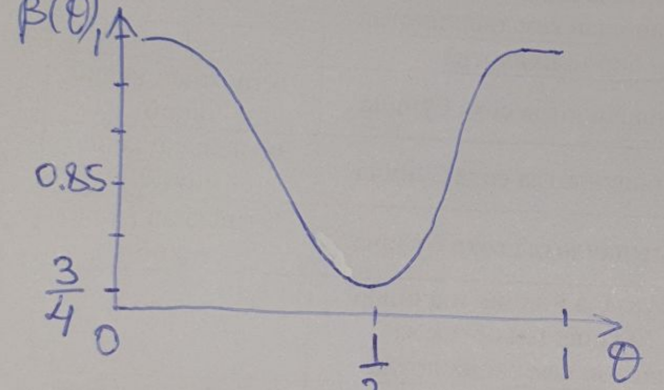
$R = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ # for $\hat{\theta}$, $R = [0,5) \cup (5,10]$ for X

$R^c = \frac{1}{2}$ # for $\hat{\theta}$ or $R^c = 5$ for x.

$\beta(\theta) = P_\theta(X \in R) = 1 - P_\theta(X \notin R) = 1 - P_\theta(X=5) =$

$= 1 - \binom{10}{5} \theta^5 (1-\theta)^5 = 1 - \frac{10!}{5!5!} [\theta(1-\theta)]^5 =$

$= 1 - \frac{\overset{2}{\cancel{6}} \cdot 7 \cdot \overset{2}{\cancel{8}} \cdot 9 \cdot \cancel{10}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5}} \cdot (\theta(1-\theta))^5 = 1 - 252 (\theta(1-\theta))^5$



size of the test

$\sup_{\theta \in \Theta_0} \beta(\theta) =$

$= \sup_{\theta \in \{\frac{1}{2}\}} \beta(\theta) =$

$= \beta(\frac{1}{2}) = \underline{0.7539}$

1.2 $R = \{0, 1, 9, 10\} \qquad R^c = \{2, 3, \dots, 8\}$ # for x

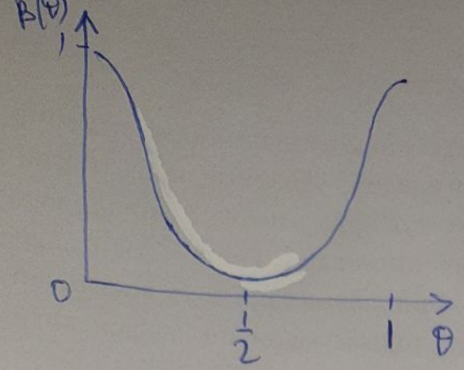
$\sup_{\theta \in \Theta_0} \beta(\theta)$ is the significance level

$\beta(\theta) = P_\theta(X \in R) = \cancel{1 - P_\theta(X \notin R) =} \sum_{x \in R} P(X=x)$

$= \theta^0 (1-\theta)^{10} + \theta^{10} (1-\theta)^0 + 10\theta(1-\theta)^9 + 10\theta^9(1-\theta) =$

$= (1-\theta)^{10} + \theta^{10} + 10(\theta(1-\theta)^9) + 10\theta^9(1-\theta)$

$\sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\frac{1}{2}) = (\frac{1}{2})^{10} + (\frac{1}{2})^{10} + 10\frac{1}{2^{10}} + 10\frac{1}{2^{10}} = \frac{22}{2^{10}} \approx \underline{0.0215}$



$$\beta(\theta=\tfrac{3}{4}) = \beta(\theta=\tfrac{1}{4}) = \frac{1}{4^{10}} + \left(\frac{3}{4}\right)^{10} +$$

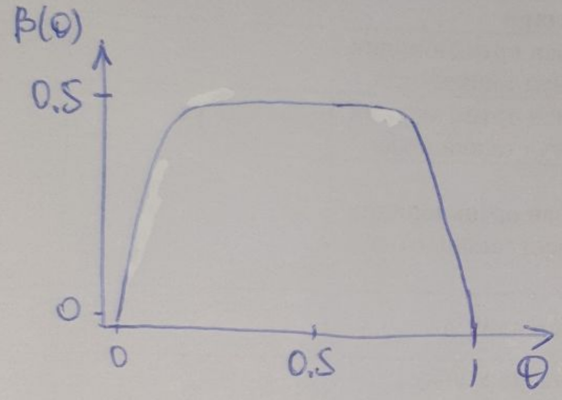
$$+\, 10 \cdot \frac{3^9 \cdot 1 + 3 \cdot 1^9}{4^{9+1}} \approx 0.2440$$

$$P(X \in R^C) = 1 - \beta(\theta) = \underline{0.7560}$$

(1.3) $R = \{x : x \,\%\, 2 = 0\}$ for $X \in [0, 10]$

$$R^C = \{x : x \,\%\, 2 = 1,\ 0 \leq X \leq 10\}$$

$$\beta(\theta) = P_\theta(X \in R) = \sum_{X \in R} \binom{10}{x} \theta^x (1-\theta)^x$$



$$\beta(\tfrac{1}{2}) = \underline{0.5}$$

$$1 - \beta(0.25) = 1 - \beta(0.75) \approx$$

$$\approx 1 - 0.4995 \approx \underline{0.5005}$$

(1.4) $P_{0.5}(|X - 15| > \delta) \leq 0.05 \quad | \; n\theta_0 = 15 \quad \sqrt{n\theta_0(1-\theta_0)} = 2.739$

Since $n\theta_0 = 30 \cdot \frac{1}{2} = 15 \geq 10$ and $n(1-\theta_0) = 15 \geq 10$, we can use normal approximation to a binomial distribution with a continuity correction. Thus,

$$P_{0.5}(|X-15| > \delta) = P(X - 15 > \delta) + P(X - 15 < -\delta) =$$

$$= P(X > 15 + \delta) + P(X < 15 - \delta) = P(X^N \geq 15.5 + \delta) +$$

$$+ P(X^N \leq 14.5 - \delta) = P(X^N \geq 15 + (\delta + 0.5)) + P(X^N \leq 15 - (\delta + 0.5))$$

$$= 2\, P(X^N \leq 15 - (\delta + 0.5)) = 2\left(\frac{X^N - E_{\theta_0} X^N}{Var_{\theta_0} X^N} \leq \frac{15 - (\delta + 0.5) - 15}{2.739} \right) =$$

$$= 2P\left(Z \leq -\frac{\delta + 0.5}{2.739}\right) \leq 0.05$$

$$P\left(Z \le -\frac{\delta+0.5}{2.739}\right) \le 0.025$$

$$\Phi\left(-\frac{\delta+0.5}{2.739}\right) \le 0.025$$

$$-\frac{\delta+0.5}{2.739} \le \Phi^{-1}(0.025) = -1.9599$$

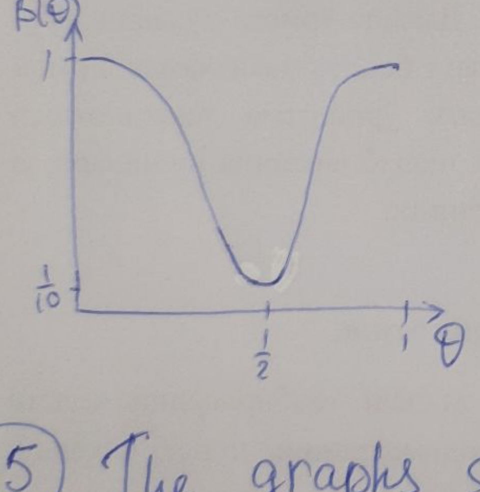
$$\delta+0.5 \ge 5.368$$

$$\delta \ge 4.868$$

We will choose the $\delta = 4.9$

$$R = \{x : x \in [0,10] \cup [20,30]\}$$

$$\beta(\theta) = P_\theta(X \in R) = \Sigma_{X \in R} \binom{30}{x} \theta^x (1-\theta)^x$$



$1-\beta(\frac{1}{4}) = \underline{0.1057}$

$4 \in R$. So, we reject H_0 and conclude that the coin is biased.

(1.5) The graphs show that the power function is very close to 0 ($\beta(0.5)=0.0987$) when $\theta=\theta_0$ and approaches 1 faster than others as θ deviates from θ_0. Therefore, this test offers the best tradeoff, among the given 4 tests, between Type I and Type II errors.

Q2

① $\mathcal{P} = \{ Bin(120, \theta) : \theta \in [0,1] \}$

$H_0: \theta = 0.65 \qquad H_1: \theta < 0.65$

θ - % of male voters

$\hat{\theta} = \frac{X}{120}$

$Var_{\theta_0} \hat{\theta} = 120 \cdot 0.65 \cdot 0.35$

② Usually, standardized test scores are very close in terms of distribution to a normal distribution. So, one might want to use a normal distribution for the underlying distribution. Thus,

$$\mathcal{P} = \{ \mathcal{N}(\mu, \sigma^2) : (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+ \}$$

Under H_0, $\mu = 95$ and $\sigma^2 = 225$

We compute the average score of those 22 students. Using this sample mean, we can conduct the following hypothesis testing.

$H_0: \mu = 95 \qquad H_a: \mu \neq 95$

$H_a: \mu > 95$ If the researcher hopes for a positive effect.

$$Var_{\theta_0} \bar{X} = \frac{\sigma^2}{n} = \frac{225}{22}$$

③ $\mathcal{P} = \{ X \in \{0, 1, \dots 9\} \}$

If $P(X = x) = p_x$ for $x \in \mathcal{P}$, and $\mathcal{A} = \{p_0, p_1, p_2, p_3, p_4\}$ and $\mathcal{B} = \{p_5, p_6, p_7, p_8, p_9\}$

$H_0: p_0 = p_1 = \dots = p_9$

H_1: at least one $p \in \mathcal{A}$ > than at least one $p \in \mathcal{B}$.