Maximum Likelihood Estimator

Exercise 1 (Laplace Model)

Let $X_1,\dots,X_n$ be a n-sample. We model our date using Laplace distribution with the unknown parameter $\theta \in \Theta = \mathbb{R}$. The density of Laplace distribution is :

$$f(\theta,x) = \frac{1}{2\sigma}\exp\big(-\frac{|x-\theta|}{\sigma}\big),$$

where $\sigma > 0$ is **known** (for example, you can take $\sigma = 1$).

1) Write the likelihood function of the sample.

2) Write the associated log-likelihood.

3) Calculate the maximum likelihood estimator.

Exercise 2 (Cauchy Model)

Let $X_1,\dots,X_n$ be a n-sample. We model our date using Cauchy distribution of unknown parameter $\theta \in \Theta = \mathbb{R}$. The density of Cauchy distribution is :

$$f(\theta,x) = \frac{1}{\pi(1+(x-\theta)^2)}.$$

1) Write the likelihood function of the sample.

2) Write the likelihood equation.

3) Is there a single maximum likelihood estimator ?

Exercise 3 (Discrete distribution)

Let $\theta \in]0,1[$ be an unknown parameter. We denote by X a random variable with the following distribution

$$\mathbb{P}_\theta(X=k) = (k+1)(1-\theta)^2\theta^k, \text{ for all } k \in \mathbb{N}.$$

We have that

$$\mathbb{E}_\theta[X] = \frac{2\theta}{1-\theta} \text{ and } \mathrm{Var}_\theta[X] = \frac{2\theta}{(1-\theta)^2}.$$

We want to estimate θ from a sample $X_1, \ldots, X_n$ with the same distribution as X.

1) Provide an estimator of θ using the method of moments.

2) Is the maximum likelihood estimator well defined ?

Exercise 4 (Step-function density)
For any real θ, we define the function

$$f_\theta : x \in \mathbb{R} \mapsto \begin{cases} 1-\theta & \text{if } -1/2 < x \leq 0, \\ 1+\theta & \text{if } 0 < x \leq 1/2 \\ 0 & \text{if not.} \end{cases}$$

1) What conditions must θ satisfy for f_θ to be a probability density ?

2) Let $X_1, \ldots, X_n$ be a sample of density f_θ. Is the maximum likelihood estimator θ_n of θ well defined ?

3) Under the conditions of question 1, is $\hat{\theta}_n$ unbiased ? consistent ? asymptotically normal ?