Chapter 7
Convexity, Jensen's Inequality

The main purpose of this section is to acquaint the reader with one of the most important theorems, that is widely used in proving inequalities, *Jensen's inequality*. This is an inequality regarding so-called convex functions, so firstly we will give some definitions and theorems whose proofs are subject to mathematical analysis, and therefore we'll present them here without proof.

Also we will consider that the reader has an elementary knowledge of differential calculus.

Definition 7.1 For the function $f : [a, b] \to \mathbb{R}$ we'll say that it is convex on the interval $[a, b]$ if for any $x, y \in [a, b]$ and any $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y). \tag{7.1}$$

If in (7.1) we have strict inequality then we'll say that f is strictly convex.

For the function f we'll say that it is concave if $-f$ is a convex function.

If the function f is defined on $\mathbb{R}$, it can happen that on some interval this function is a convex function, but on another interval it is a concave function. For this reason, we will consider functions defined on intervals.

Example 7.1 The function $f(x) = x^2$ is convex on $\mathbb{R}$, moreover $f(x) = x^n$ is convex on $\mathbb{R}$ for even n. Also $f(x) = x^n$ is convex on $\mathbb{R}^+$ for n odd, and it is concave on $\mathbb{R}^-$.

The function $f(x) = \sin x$ on $(\pi, 2\pi)$ is convex, but on $(0, \pi)$ it is concave.

Now we will state a theorem that will give a criterion for determining whether and when a function is convex, respectively concave.

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Theorem 7.1 *Let $f : (a, b) \to \mathbb{R}$ and for any $x \in (a, b)$ suppose there exists a second derivative $f''(x)$. The function $f(x)$ is convex on (a, b) if and only if for each $x \in (a, b)$ we have $f''(x) \geq 0$. If $f''(x) > 0$ for each $x \in (a, b)$, then f is strictly convex on (a, b).*

Clearly, according to Definition 7.1 and Theorem 7.1 we have that the function $f(x)$ is concave on (a, b) if and only if $f''(x) \leq 0$, for all $x \in (a, b)$.

Example 7.2 Consider the power function $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined as $f(x) = x^\alpha$. For the second derivative we have $f''(x) = \alpha(\alpha - 1)x^{\alpha-2}$, and clearly $f''(x) > 0$ for $\alpha > 1$ or $\alpha < 0$ and $f''(x) < 0$ for $0 < \alpha < 1$. So f is (strictly) convex for $\alpha > 1$ or $\alpha < 0$ and f is (strictly) concave for $0 < \alpha < 1$.

Example 7.3 For the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \ln(1+e^x)$ we have $f'(x) = \frac{e^x}{1+e^x}$, and $f''(x) = \frac{e^x}{(1+e^x)^2} > 0$ for $x \in \mathbb{R}$, and therefore f is convex on $\mathbb{R}$.

Example 7.4 For the function $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = (1 + x^\alpha)^{\frac{1}{\alpha}}$ for $\alpha \neq 0$ we have $f''(x) = (\alpha - 1)x^{\alpha-2}(1 + x^\alpha)^{\frac{1}{\alpha}}$, from where it follows that for $\alpha < 1$ the function f is strictly concave and for $\alpha > 1$ the function f is strictly convex.

Theorem 7.2 *Let $f_1, f_2, \ldots, f_n$ be convex functions on (a, b). Then the function $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ is also convex on (a, b), for any $c_1, c_2, \ldots, c_n \in (0, \infty)$.*

Theorem 7.3 (Jensen's inequality) *Let $f : (a, b) \to \mathbb{R}$ be a convex function on the interval (a, b). Let $n \in \mathbb{N}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in (0, 1)$ be real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Then for any $x_1, x_2, \ldots, x_n \in (a, b)$ we have*

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i),$$

i.e.

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n). \quad (7.2)$$

Proof We'll prove inequality (7.2) by mathematical induction.

For $n = 1$ we have $\alpha_1 = 1$ and since $f(x_1) = f(x_1)$ we get $f(\alpha_1 x_1) = \alpha_1 f(x_1)$, so (7.2) is true.

Let $n = 2$. Then (7.2) holds due to Definition 7.1.

Suppose that for $n = k$, and any real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k \in [0, 1]$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$ and any $x_1, x_2, \ldots, x_k \in (a, b)$, we have

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) \le \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k). \tag{7.3}$$

Let $n = k + 1$, and let $\alpha_1, \alpha_2, \ldots, \alpha_{k+1} \in [0, 1]$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_{k+1} = 1$.

Let $x_1, x_2, \ldots, x_{k+1} \in (a, b)$.

Then we have

$$\begin{aligned}
&\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{k+1} x_{k+1} \\
&\quad = (\alpha_1 x_1 + \cdots + \alpha_k x_k) + \alpha_{k+1} x_{k+1} \\
&\quad = (1 - \alpha_{k+1}) \left(\frac{\alpha_1}{1 - \alpha_{k+1}} x_1 + \frac{\alpha_2}{1 - \alpha_{k+1}} x_2 + \cdots + \frac{\alpha_k}{1 - \alpha_{k+1}} x_k \right) + \alpha_{k+1} x_{k+1}.
\end{aligned} \tag{7.4}$$

Let

$$\frac{\alpha_1}{1 - \alpha_{k+1}} x_1 + \frac{\alpha_2}{1 - \alpha_{k+1}} x_2 + \cdots + \frac{\alpha_k}{1 - \alpha_{k+1}} x_k = y_{k+1}.$$

Then since $x_1, x_2, \ldots, x_k \in (a, b)$ we deduce

$$\begin{aligned}
y_{k+1} &= \frac{\alpha_1}{1 - \alpha_{k+1}} x_1 + \frac{\alpha_2}{1 - \alpha_{k+1}} x_2 + \cdots + \frac{\alpha_k}{1 - \alpha_{k+1}} x_k \\
&< \frac{\alpha_1}{1 - \alpha_{k+1}} b + \frac{\alpha_2}{1 - \alpha_{k+1}} b + \cdots + \frac{\alpha_k}{1 - \alpha_{k+1}} b \\
&< \frac{b}{1 - \alpha_{k+1}} (\alpha_1 + \alpha_2 + \cdots + \alpha_k) = \frac{b}{1 - \alpha_{k+1}} (1 - \alpha_{k+1}) = b.
\end{aligned}$$

Similarly we deduce that $y_{k+1} > a$.

Thus $y_{k+1} \in (a, b)$.

According to Definition 7.1 and by (7.4) we obtain

$$\begin{aligned}
f(\alpha_1 x_1 + \cdots + \alpha_k x_k + \alpha_{k+1} x_{k+1}) &= f((1 - \alpha_{k+1}) y_{k+1} + \alpha_{k+1} x_{k+1}) \\
&\le (1 - \alpha_{k+1}) f(y_{k+1}) + \alpha_{k+1} f(x_{k+1}).
\end{aligned} \tag{7.5}$$

By inequality (7.3) and since

$$\frac{\alpha_1}{1 - \alpha_{k+1}} + \frac{\alpha_2}{1 - \alpha_{k+1}} + \cdots + \frac{\alpha_k}{1 - \alpha_{k+1}} = 1$$

we obtain

$$\begin{aligned}
f(y_{k+1}) &= f\left(\frac{\alpha_1}{1 - \alpha_{k+1}} x_1 + \frac{\alpha_2}{1 - \alpha_{k+1}} x_2 + \cdots + \frac{\alpha_k}{1 - \alpha_{k+1}} x_k \right) \\
&\le \frac{\alpha_1}{1 - \alpha_{k+1}} f(x_1) + \frac{\alpha_2}{1 - \alpha_{k+1}} f(x_2) + \cdots + \frac{\alpha_k}{1 - \alpha_{k+1}} f(x_k).
\end{aligned} \tag{7.6}$$

Finally according to (7.5) and (7.6) we deduce

$$f(\alpha_1 x_1 + \cdots + \alpha_{k+1} x_{k+1}) \leq \alpha_1 f(x_1) + \cdots + \alpha_{k+1} f(x_{k+1}).$$

So by the principle of mathematical induction inequality, (7.2) holds for any positive integer n, any $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1]$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, and arbitrary $x_1, x_2, \ldots, x_n \in (a, b)$. □

Remark If f is strictly convex then equality in *Jensen's inequality* occurs only for $x_1 = x_2 = \cdots = x_n$.

If the function $f(x)$ is concave then in *Jensen's inequality* we have the reverse inequality, i.e.

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \geq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n).$$

It is important to note that *Jensen's inequality* can also be written in the equivalent form:

If $f : I \to \mathbb{R}$ is convex on $I, x_1, x_2, \ldots, x_n \in I$ and $m_1, m_2, \ldots, m_n \geq 0$ are real numbers such that $m_1 + m_2 + \cdots + m_n > 0$. Then

$$f\left(\frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}\right) \leq \frac{m_1 f(x_1) + m_2 f(x_2) + \cdots + m_n f(x_n)}{m_1 + m_2 + \cdots + m_n}.$$

Example 7.5 Consider the function $f(x) = -\ln x$, on the interval $(0, +\infty)$. For the second derivative we have $f''(x) = \frac{1}{x^2} > 0$, which means that $f(x)$ is a strictly convex on $x \in (0, +\infty)$.

By *Jensen's inequality* for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$, and $x_i \in (0, +\infty), i = 1, 2, \ldots, n$, we obtain

$$\begin{aligned}
&-\ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leq -\left(\frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n}\right) \\
\Leftrightarrow \quad &\frac{\ln x_1 + \ln x_2 + \cdots + \ln x_n}{n} \leq \ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \\
\Leftrightarrow \quad &\ln(x_1 x_2 \cdots x_n)^{1/n} \leq \ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right),
\end{aligned}$$

i.e.

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n},$$

which is the well-known inequality $AM \geq GM$.

Example 7.6 Let us consider the function $f(x) = x^2$. Since $f''(x) = 2 > 0$ it follows that f is convex on $\mathbb{R}$. Then by *Jensen's inequality*

$$f\left(\frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}\right) \leq \frac{m_1 f(x_1) + m_2 f(x_2) + \cdots + m_n f(x_n)}{m_1 + m_2 + \cdots + m_n},$$

we obtain

$$\left(\frac{m_1x_1+m_2x_2+\cdots+m_nx_n}{m_1+m_2+\cdots+m_n}\right)^2 \le \frac{m_1x_1^2+m_2x_2^2+\cdots+m_nx_n^2}{m_1+m_2+\cdots+m_n},$$

i.e.

$$(m_1x_1+m_2x_2+\cdots+m_nx_n)^2 \le (m_1x_1^2+m_2x_2^2+\cdots+m_nx_n^2)(m_1+m_2+\cdots+m_n).$$

By taking $m_i = b_i^2, x_i = \frac{a_i}{b_i}$ for $i = 1, 2, \ldots, n$ in the last inequality, we obtain

$$(a_1b_1+a_2b_2+\cdots+a_nb_n)^2 \le (a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2),$$

which is the well-known *Cauchy–Schwarz inequality*.

On this occasion we will present *Popoviciu's inequality*, which will be used in the same manner as *Jensen's inequality*. But we must note that this inequality is stronger then *Jensen's inequality*, i.e. in some cases this inequality can be a powerful tool for proving other inequalities, where *Jensen's inequality* does not work.

Theorem 7.4 (Popoviciu's inequality) *Let* $f : [a, b] \to \mathbb{R}$ *be a convex function on the interval* $[a, b]$. *Then for any* $x, y, z \in [a, b]$ *we have*

$$f\left(\frac{x+y+z}{3}\right) + \frac{f(x)+f(y)+f(z)}{3} \ge \frac{2}{3}\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right).$$

Proof Without loss of generality we assume that $x \le y \le z$.

If $y \le \frac{x+y+z}{3}$ then $\frac{x+y+z}{3} \le \frac{x+z}{2} \le z$ and $\frac{x+y+z}{3} \le \frac{y+z}{2} \le z$.

Therefore there exist $s, t \in [0, 1]$ such that

$$\frac{x+z}{2} = \left(\frac{x+y+z}{3}\right)s + z(1-s) \quad \text{and} \tag{7.7}$$

$$\frac{y+z}{2} = \left(\frac{x+y+z}{3}\right)t + z(1-t). \tag{7.8}$$

Summing (7.7) and (7.8) gives

$$\frac{x+y-2z}{2} = \frac{x+y-2z}{3}(s+t),$$

from which we obtain $s + t = \frac{3}{2}$.

Because the function f is convex, we have

$$f\left(\frac{x+z}{2}\right) \le s \cdot f\left(\frac{x+y+z}{3}\right) + (1-s)f(z),$$
$$f\left(\frac{y+z}{2}\right) \le t \cdot f\left(\frac{x+y+z}{3}\right) + (1-t)f(z)$$

and

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

After adding together the last three inequalities we obtain the required inequality.

The case when $\frac{x+y+z}{3} < y$ is considered similarly, bearing in mind that $x \le \frac{x+z}{2} \le \frac{x+y+z}{3}$ and $x \le \frac{y+z}{2} \le \frac{x+y+z}{3}$. □

Note If f is a concave function on $[a,b]$ then in *Popoviciu's inequality* for all $x, y, z \in [a,b]$ we have the reverse inequality, i.e. we have

$$f\left(\frac{x+y+z}{3}\right) + \frac{f(x)+f(y)+f(z)}{3}$$
$$\le \frac{2}{3}\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right).$$

Theorem 7.5 (Generalized Popoviciu's inequality) *Let $f : [a,b] \to \mathbb{R}$ be a convex function on the interval $[a,b]$ and $a_1, a_2, \ldots, a_n \in [a,b]$. Then*

$$f(a_1) + f(a_2) + \cdots + f(a_n) + n(n-2)f(a)$$
$$\ge (n-1)(f(b_1) + f(b_2) + \cdots + f(b_n)),$$

where $a = \frac{a_1+a_2+\cdots+a_n}{n}$, and $b_i = \frac{1}{n-1}\sum_{i\ne j} a_j$ for all i.

Theorem 7.6 (Weighted *AM–GM* inequality) *Let $a_i \in (0,\infty)$, $i = 1, 2, \ldots, n$, and $\alpha_i \in [0,1]$, $i = 1, 2, \ldots, n$, be such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Then*

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \le a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n. \tag{7.9}$$

Proof For the function $f(x) = -\ln x$ we have $f'(x) = -\frac{1}{x}$ and $f''(x) = \frac{1}{x^2}$, i.e. $f''(x) > 0$, for $x \in (0,\infty)$.

So due to Theorem 7.1 we conclude that the function f is convex on $(0,\infty)$.

Let $a_i \in (0,\infty)$, $i = 1,2,\dots,n$, and $\alpha_i \in [0,1]$, $i = 1,2,\dots,n$, be arbitrary real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$.

By *Jensen's inequality* we deduce

$$
\begin{aligned}
&-\ln\left(\sum_{i=1}^n a_i\alpha_i\right) = f\left(\sum_{i=1}^n a_i\alpha_i\right) \le \sum_{i=1}^n \alpha_i f(a_i) = -\sum_{i=1}^n \alpha_i \ln a_i \\
\Leftrightarrow\quad & -\ln(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) \le -\alpha_1 \ln a_1 - \alpha_2 \ln a_2 - \cdots - \alpha_n \ln a_n \\
\Leftrightarrow\quad & \alpha_1 \ln a_1 + \alpha_2 \ln a_2 + \cdots + \alpha_n \ln a_n \le \ln(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) \\
\Leftrightarrow\quad & \ln a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \le \ln(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) \\
\Leftrightarrow\quad & a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \le a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n,
\end{aligned}
$$

as required. □

Note By inequality (7.9) for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$, we obtain the inequality $AM \ge GM$.

Exercise 7.1 Let α, β, γ be the angles of a triangle. Prove the inequality

$$\sin\alpha \sin\beta \sin\gamma \le \frac{3\sqrt{3}}{8}.$$

Solution Since $\alpha, \beta, \gamma \in (0,\pi)$ it follows that $\sin\alpha, \sin\beta, \sin\gamma > 0$.

Therefore since $AM \ge GM$ we obtain

$$\sqrt[3]{\sin\alpha \sin\beta \sin\gamma} \le \frac{\sin\alpha + \sin\beta + \sin\gamma}{3}. \tag{7.10}$$

Since $f(x) = \sin x$ is concave on $(0,\pi)$, by *Jensen's inequality* we deduce

$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{3} \le \sin\frac{\alpha+\beta+\gamma}{3} = \frac{\sqrt{3}}{2}. \tag{7.11}$$

Due to (7.10) and (7.11) we get

$$\sqrt[3]{\sin\alpha \sin\beta \sin\gamma} \le \frac{\sqrt{3}}{2} \quad\Leftrightarrow\quad \sin\alpha \sin\beta \sin\gamma \le \frac{3\sqrt{3}}{8}.$$

Equality occurs iff $\alpha = \beta = \gamma$, i.e. the triangle is equilateral.

Exercise 7.2 Let $a, b, c \in \mathbb{R}^+$. Prove the inequalities:

(1) $4(a^3 + b^3) \ge (a+b)^3$;
(2) $9(a^3 + b^3 + c^3) \ge (a+b+c)^3$.

Solution (1) The function $f(x) = x^3$ is convex on $(0, +\infty)$, thus from *Jensen's inequality* it follows that

$$\left(\frac{a+b}{2}\right)^3 \le \frac{a^3+b^3}{2} \quad \Leftrightarrow \quad 4(a^3+b^3) \ge (a+b)^3.$$

(2) Similarly as in (1) we deduce that

$$\left(\frac{a+b+c}{3}\right)^3 \le \frac{a^3+b^3+c^3}{3} \quad \Leftrightarrow \quad 9(a^3+b^3+c^3) \ge (a+b+c)^3.$$

Exercise 7.3 Let $\alpha_i > 0, i = 1, 2, \ldots, n$, be real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Prove the inequality

$$\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \cdots \alpha_n^{\alpha_n} \ge \frac{1}{n}.$$

Solution If we take $a_i = \frac{1}{\alpha_i}, i = 1, 2, \ldots, n$, by the *Weighted AM–GM inequality* we get

$$\frac{1}{\alpha_1^{\alpha_1}} \frac{1}{\alpha_2^{\alpha_2}} \cdots \frac{1}{\alpha_n^{\alpha_n}} \le \frac{1}{\alpha_1}\alpha_1 + \frac{1}{\alpha_2}\alpha_2 + \cdots + \frac{1}{\alpha_n}\alpha_n = n,$$

i.e.

$$\frac{1}{n} \le \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \cdots \alpha_n^{\alpha_n}.$$

Exercise 7.4 Find the minimum value of k such that for arbitrary $a, b > 0$ we have

$$\sqrt[3]{a} + \sqrt[3]{b} \le k\sqrt[3]{a+b}.$$

Solution Consider the function $f(x) = \sqrt[3]{x}$.

We have $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ and $f''(x) = -\frac{2}{9}x^{-\frac{5}{3}} < 0$, for any $x \in (0, \infty)$. Thus $f(x)$ is concave on the interval $(0, \infty)$.

By *Jensen's inequality* we deduce

$$\begin{aligned}
&\frac{1}{2}f(a) + \frac{1}{2}f(b) \le f\left(\frac{a+b}{2}\right) \\
\Leftrightarrow \quad &\frac{\sqrt[3]{a} + \sqrt[3]{b}}{2} \le \sqrt[3]{\frac{a+b}{2}} \\
\Leftrightarrow \quad &\sqrt[3]{a} + \sqrt[3]{b} \le \frac{2}{\sqrt[3]{2}}\sqrt[3]{a+b} = \sqrt[3]{4} \cdot \sqrt[3]{a+b}.
\end{aligned}$$

Therefore $k_{\min} = \sqrt[3]{4}$, and for instance we reach this value for $a = b$.

Exercise 7.5 Let $x, y, z \geq 0$ be real numbers. Prove the inequality

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1} \geq \sqrt{6(x+y+z)}.$$

Solution Consider the function $f(t)=\sqrt{t^2+1}, t \geq 0$.

Since $f''(t)=\frac{1}{(\sqrt{t^2+1})^3}>0$, f is convex on $[0,\infty)$.

Therefore by *Jensen's inequality* we have

$$\frac{\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}}{3} \geq \sqrt{\left(\frac{x+y+z}{3}\right)^2+1},$$

i.e.

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1} \geq \sqrt{(x+y+z)^2+9}. \tag{7.12}$$

From the obvious inequality $((x+y+z)-3)^2 \geq 0$ it follows that

$$(x+y+z)^2+9 \geq 6(x+y+z). \tag{7.13}$$

By (7.12) and (7.13) we obtain

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1} \geq \sqrt{(x+y+z)^2+9} \geq \sqrt{6(x+y+z)}.$$

Equality occurs if and only if $x=y=z=1$.

Exercise 7.6 Let x, y, z be positive real numbers. Prove the inequality

$$\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y} \geq 4\left(\frac{z}{x+y}+\frac{x}{y+z}+\frac{y}{z+x}\right).$$

Solution Consider the function $f(x)=x+\frac{1}{x}$.

Since $f'(x)=1-\frac{1}{x^2}$ and $f''(x)=\frac{2}{x^3}>0$ for any $x>0$ it follows that f is convex on $\mathbb{R}^+$.

Now by *Popoviciu's inequality* we can easily obtain the required inequality.