Principles of Finance (Fine 31123)

Exercise Set

Solutions

Assistantship I

1 First Principles of Valuation

Question 1

a) The equation for the bond price with annual coupons is given by:

$$\begin{aligned} P &= \frac{C}{r}\left[1-\frac{1}{(1+r)^N}\right]+\frac{NV}{(1+r)^N} \\ &= \frac{100}{0.085}\left[1-\frac{1}{(1.085)^{15}}\right]+\frac{1\ 000}{(1.085)^{15}} \\ &= 1\ 124.56 \end{aligned}$$

b) The equation for the bond price with semi-annual coupons is given by:

$$\begin{aligned} P &= \frac{C_s}{r_s}\left[1-\frac{1}{(1+r_s)^{2N}}\right]+\frac{NV}{(1+r_s)^{2N}} \\ &= \frac{50}{0.0425}\left[1-\frac{1}{(1.0425)^{30}}\right]+\frac{1\ 000}{(1.0425)^{30}} \\ &= 1\ 125.84 \end{aligned}$$

c) If you require a 9% rate of return, the value of the bond should be:

$$\begin{aligned} P &= \frac{100}{0.09}\left[1-\frac{1}{(1.09)^{15}}\right]+\frac{1\ 000}{(1.09)^{15}} \\ &= 1\ 080.61 \end{aligned}$$

Question 2

a) Annual return for each security

The general solution is given by:

$$P=\sum_{t=1}^{N}\frac{CF_t}{(1+r)^t}$$

Security 1:

$$98=\frac{110}{(1+r)} \qquad \Longrightarrow \qquad r=12.24\%$$

Security 2:

$$90 = \frac{110}{(1+r)^2} \qquad \Longrightarrow \qquad r = 10.55\%$$

Security 3:

$$85 = \frac{110}{(1+r)^3} \qquad \Longrightarrow \qquad r = 8.97\%$$

b) A security that offers 110 in year 1 and 110 in year 2 is simply a combination of securities 1 and 2. Therefore, the no-arbitrage price of that security is equal to the sum of the market price for the two securities: P = 98 + 90 = 188.

c) A security that offers 110 in year 1, 110 in year 2 and 110 in year 3 is simply a combination of securities 1, 2 and 3. Therefore, the no-arbitrage price of that security is equal to the sum of the market price for the three securities: P = 98 + 90 + 85 = 273.

d) A security that offers 300 in three years is similar to buying three S3. Therefore, the no-arbitrage price of that security is equal to three times the price of S3: $P = 3 \times 85 = 255$.

e) A security that offers 110 in year 1, 220 in year 2 and 330 in year 3 is simply a combination of security 1, two times security 2 and three times security 3. Therefore, the no-arbitrage price of that security is equal to the sum of the market price for the three securities: $P = 98 + (2 \times 90) + (3 \times 85) = 533$.

f) A security that offers 55 in year 1 should be worth half the value of security 1. Combining this to security 2, the no-arbitrage price should be: P = 49 + 90 = 139. If that security is trading at 135, there exist an arbitrage opportunity because its price is under the no-arbitrage price.

- short sell one S1 with CF = 110 at t=1
- short sell two S2 with CF = 110 at t=2
- buy two securities S4 with CF = 55 at t=1 and 110 at t=2

Strategy	t=0	t=1	t=2
Short sell one S1	98	-110	0
Short sell two S2	180		-220
Buy two S4	-270	110	220
Net gain	8	0	0

Question 3

a) According to the Dividend Discount Model of Gordon-Shapiro, we can write:

$$P = \frac{D_0(1+E(g))}{E(r)-E(g)} \quad \implies \quad E(r) = \frac{D_0(1+E(g))}{P} + E(g)$$

Based on the information, we get

$$E(r) = \frac{D_0(1+E(g))}{P} + E(g) = \frac{5(1.043)}{95} + 0.043 = 0,0979$$

b) The equation for the bond price with semi-annual coupons is given by:

$$\begin{aligned} P &= \frac{C_s}{r_s}\left[1 - \frac{1}{(1+r_s)^{2N}}\right] + \frac{NV}{(1+r_s)^{2N}} \\ &= \frac{80}{0.05}\left[1 - \frac{1}{(1.05)^{30}}\right] + \frac{2\ 000}{(1.05)^{30}} \\ &= 1\ 692.55 \end{aligned}$$

2 Capital Budgeting Decisions

Question 1

Production cash flows:

	Existing Oven	New Oven
Sales	20 000 000	32 000 000
Raw material	-10 000 000	-16 000 000
Labour	-4 000 000	-8 000 000
Fixed costs	-1 000 000	-1 000 000
EBITDA	5 000 000	7 000 000
Income tax (30%)	-1 500 000	-2 100 000
Net cash flows	3 500 000	4 900 000

Option 1: Existing Oven (EO)

- Net cash flows from production: 3 500 000 / year over 10 years
- Resale of the oven after 10 years: 5 000 000 (t=10)
- Tax on capital gain (CG): - 0,3 × 5 000 000 = - 1 500 000 (t=10)

- Tax savings on depreciation: 0,3 × 2 500 000 = 750 000 / year over 10 years

$$\begin{aligned} \text{NPV(EO)} &= \frac{3\ 500\ 000}{0,12}\left[1-\frac{1}{(1,12)^{10}}\right]+\frac{750\ 000}{0,12}\left[1-\frac{1}{(1,12)^{10}}\right]+\frac{(5\ 000\ 000-1\ 500\ 000)}{(1,12)^{10}} \\ &= 25\ 140\ 354 \end{aligned}$$

Option 2: New Oven (NO)

- Net cash flows from production: 4 900 000 / year over 10 years
- Cost of the new oven: - 45 000 000 (t=0)
- Tax savings on depreciation for the new oven: 0,3 × 4 500 000 = 1 350 000 / year over 10 years
- Resale of the new oven in 10 years 25 000 000 (t=10)
- Tax on capital gain (CG): - 0,3 × 25 000 000 = - 7 500 000 (t=10)
- Cost of patent: - 5 000 000 (t=0)
- Tax savings on the patent: 0,3 × 500 000 = 150 000 / year over 10 years
- Resale of the existing oven: 30 000 000 ($t = 0$)
- Tax on capital gain (CG) on the existing oven: - 0,3 × (30 000 000 - 25 000 000) = - 1 500 000 (t=1)

$$\begin{aligned} \text{NPV(NO)} &= -45\ 000\ 000+30\ 000\ 000-\frac{1\ 500\ 000}{1,12}+\frac{4\ 900\ 000}{0,12}\left[1-\frac{1}{(1,12)^{10}}\right] \\ &= +\frac{1\ 350\ 000}{0,12}\left[1-\frac{1}{(1,12)^{10}}\right]-5\ 000\ 000+\frac{150\ 000}{0,12}\left[1-\frac{1}{(1,12)^{10}}\right] \\ &= +\frac{(25\ 000\ 000-7\ 500\ 000)}{(1,12)^{10}} \\ &= 20\ 456\ 673 \end{aligned}$$

Conclusion: NPV(EO) > NPV(NO) $\Rightarrow$ Keep the existing oven.

Question 2

a)

$$\text{NPV(X)} = -3\ 000+\frac{1\ 100}{0,09}\left(1-\frac{1}{(1,09)^{5}}\right) = 1\ 278,6$$

$$\text{NPV(Y)} = -3\ 000 + \frac{300}{(1,09)} + \frac{500}{(1,09)^2} + \frac{800}{(1,09)^3} + \frac{2\ 200}{(1,09)^4} + \frac{2\ 800}{(1,09)^5} = 1\ 692,1$$

$\implies \quad Y \succ X$

b) The IRR is the interest rate which sets the NPV to zero.

$$\text{IRR(X)}: \quad -3\ 000 + \frac{1\ 100}{r}\left(1 - \frac{1}{(1+r)^5}\right) = 0$$

If $r_1 = 0.20 \implies NPV_1 = 289.67$
If $r_2 = 0.25 \implies NPV_2 = -41.8$

We can then write:

$$\frac{289.67}{289.67 + 41.8} = \frac{IRR - 0.20}{0.25 - 0.20} \qquad \implies \qquad IRR \sim 24.32\%$$

Similarly,

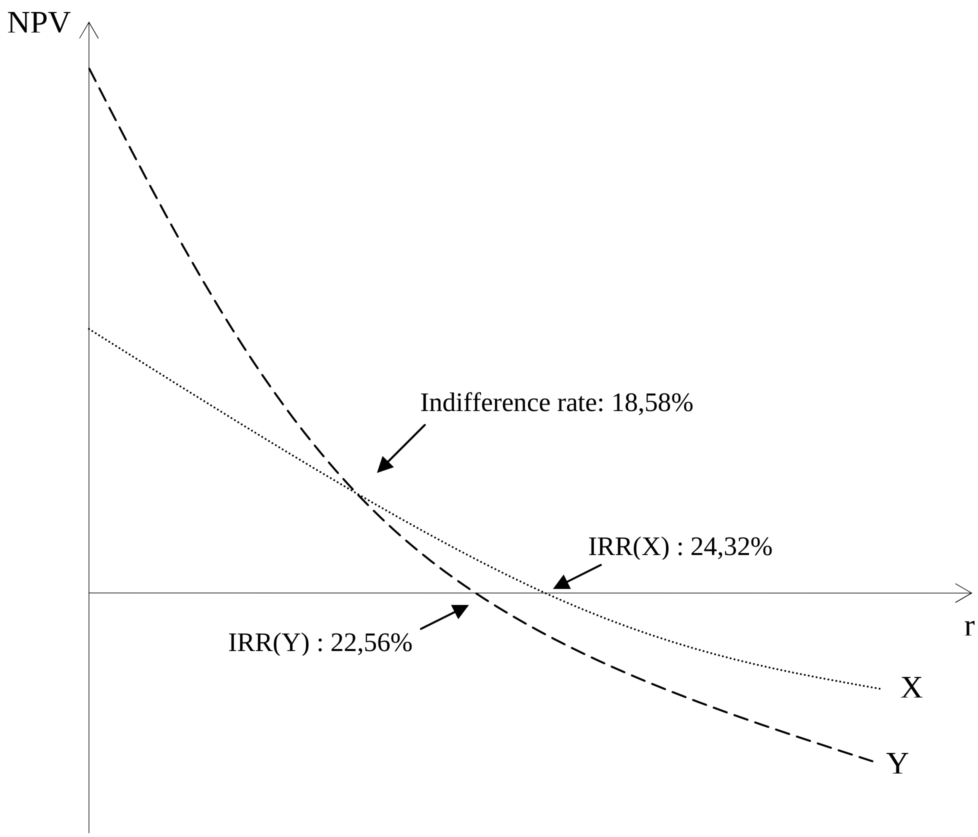
$$\text{IRR(Y)}: \quad -3\ 000 + \frac{300}{(1+r)} + \frac{500}{(1+r)^2} + \frac{800}{(1+r)^3} + \frac{2\ 200}{(1+r)^4} + \frac{2\ 800}{(1+r)^5} = 0$$

If $r_1 = 0.20 \implies NPV_1 = 246.40$
If $r_2 = 0.25 \implies NPV_2 = -211.8$

We can write:

$$\frac{246.40}{246.40 + 211.8} = \frac{IRR - 0.20}{0.25 - 0.20} \qquad \implies \qquad IRR \sim 22,56\%$$

$\implies \quad X \succ Y$


NPV
Indifference rate: 18,58%
IRR(X) : 24,32%
r
IRR(Y) : 22,56%
X
Y

d) The interest rate which sets equal both NPV can be determined by calculting the IRR of the difference in cash-flows of the two projects. The difference in cash-flows are:

	0	1	2	3	4	5
X	-3 000	1 100	1 100	1 100	1 100	1 100
Y	-3 000	300	500	800	2 200	2 800
X-Y	0	800	600	300	-1 100	-1 700

$$\text{IRR(X-Y)}: \quad \frac{800}{(1+r)} + \frac{600}{(1+r)^2} + \frac{300}{(1+r)^3} - \frac{1\ 100}{(1+r)^4} - \frac{1\ 700}{(1+r)^5} = 0$$

If $r_1 = 0.15 \implies NPV_1 = -127.54$
If $r_2 = 0.25 \implies NPV_2 = 169.98$

We can write:

$$\frac{169,98}{169,98 + 127,54} = \frac{0,25 - IRR}{0,25 - 0,15} \qquad \Longrightarrow \qquad IRR(X - Y) \sim 18,6\% \quad \text{(see Graph)}$$

Question 3

Machine 1(million):

- Cost: - 120
- Resale after 5 years: 25 (t=5)
- Tax on capital gain: - 0,25 $\times$ 25 = - 6.25 (t=5)
- Increase in EBITDA: 35 $\times$ 0.75 = 26.25 / year for 5 years
- Annual depreciation: 120/5 = 24
- Tax savings on DEP: 24 $\times$ 0.25 = 6

$$\begin{aligned}\text{NPV(1)} &= -120 + \frac{26.25}{0.1}\left[1 - \frac{1}{(1.1)^5}\right] + \frac{6}{0.1}\left[1 - \frac{1}{(1.1)^5}\right] + \frac{(25 - 6.25)}{(1.1)^5} \\ &= 13.90\end{aligned}$$

Machine 2(million):

- Cost: - 75

- Resale after 4 years: 10 (t=4)
- Tax on capital gain: - 0.25 $\times$ 10 = - 2.5 (t=4)
- Increase in EBITDA: 25 $\times$ 0.75 = 18.75 / year for 4 years
- Annual depreciation: 75/4 = 18.75
- Tax savings on DEP: 18.75 $\times$ 0.25 = 4.69

$$\begin{aligned}\text{NPV}(2) &= -75 + \frac{18.75}{0.1}\left[1 - \frac{1}{(1.1)^4}\right] + \frac{4.69}{0.1}\left[1 - \frac{1}{(1.1)^4}\right] + \frac{(10-2.5)}{(1.1)^4} \\ &= 4.42\end{aligned}$$

Conclusion: Machine 1 $\succ$ Machine 2

Problem: The two projects have a different time horizon.

Method 1: Equivalent annuity (EA)

Machine 1:

$$13.90 = \frac{EA(1)}{0.1}\left[1 - \frac{1}{(1.1)^5}\right]$$

$\Longrightarrow$ EA(1) = 3.666

Machine 2:

$$4.60 = \frac{EA(2)}{0.1}\left[1 - \frac{1}{(1.1)^4}\right]$$

$\Longrightarrow$ EA(2) = 1.393

Conclusion: Machine 1 $\succ$ Machine 2

Method 2: Smallest common denominator of expected life: 20 years

Machine 1: Done 4 times

$$\begin{aligned}\text{NPV} &= \text{VAN}(1) + \frac{\text{VAN}(1)}{(1.1)^5} + \frac{\text{VAN}(1)}{(1.1)^{10}} + \frac{\text{VAN}(1)}{(1.1)^{15}} \\ &= 13.90 + \frac{13.90}{(1.1)^5} + \frac{13.90}{(1.1)^{10}} + \frac{13.90}{(1.1)^{15}} \\ &= 31.21\end{aligned}$$

Machine 2: Done 5 times

$$
\begin{aligned}
\text{NPV} &= \text{VAN(2)} + \frac{\text{VAN(2)}}{(1.1)^4} + \frac{\text{VAN(2)}}{(1.1)^8} + \frac{\text{VAN(2)}}{(1.1)^{12}} + \frac{\text{VAN(2)}}{(1.1)^{16}} \\
&= 4.42 + \frac{4.42}{(1.1)^4} + \frac{4.42}{(1.1)^8} + \frac{4.42}{(1.1)^{12}} + \frac{4.42}{(1.1)^{16}} \\
&= 11.86
\end{aligned}
$$

Conclusion: Machine 1 $\succ$ Machine 2.

Question 4

Cash-flow calculations:

Flux	t=0	t=1...9	t=10
Investment	- 1 000		
EBITDA after tax (sales × margin)		100	100
Tax savings on Depreciation		50	50
Δ WCR	- 300		300
Sales of machine			200
Tax on capital gain			-100
Total	- 1 300	150	550

a) NPV calculation

$$
\text{NPV} = -1\ 300 + \frac{150}{0,04}\left[1 - \frac{1}{(1,04)^9}\right] + \frac{550}{(1,04)^{10}} = 186,86
$$

IRR calculation

$$
\text{NPV} = -1\ 300 + \frac{150}{r}\left[1 - \frac{1}{(1+r)^9}\right] + \frac{550}{(1+r)^{10}} = 0
$$

If $r = 0,04 \quad \Rightarrow \quad NPV = 186,86$
If $r = 0,07 \quad \Rightarrow \quad NPV = -43,12$

By linear interpolation, we get: $r = 0,06375$

b) NPV calculation in nominal terms with π, the expected inflation.

t=0

- Investment: - 1 000

- Δ WCR: - 300

$$\text{Annual Cash Flow} = -1\ 300$$

t=1...9

- EBITDA after tax: $200\ (0,5)\ (1+\pi)^t = 100\ (1+\pi)^t$
- Tax savings on depreciation : $100 \times 0,5 = 50$
- Δ WCR: $-300\ \pi\ (1+\pi)^{t-1}$

$$\text{Annual Cash Flows} = 50 + 100\ (1+\pi)^t - 300\ \pi\ (1+\pi)^{t-1}$$

t=10

- EBITDA after tax: $100\ (1+\pi)^{10}$
- Tax savings on depreciation: $100 \times 0,5 = 50$
- Δ WCR: $-300\ \pi\ (1+\pi)^9$
- Recovery of the WCR: $300\ (1+\pi)^{10}$
- Sale of machine (after tax): $100\ (1+\pi)^{10}$

$$\begin{aligned} \text{Annual Cash Flows} &= 50 + 100\ (1+\pi)^{10} - 300\ \pi\ (1+\pi)^9 + 300\ (1+\pi)^{10} + 100\ (1+\pi)^{10} \\ &= 50 + 500\ (1+\pi)^{10} - 300\ \pi\ (1+\pi)^9 \end{aligned}$$

Nominal interest rate: $r_n = r_r + \pi = 0,10$

Tax rate	0,5
WACC (real)	0,04

	0	1	2	3	4	5	6	7	8	9	10
CAPEX	-1000										
Depreciation		100	100	100	100	100	100	100	100	100	100
Units produced		10000	10000	10000	10000	10000	10000	10000	10000	10000	10000
EBITDA margin		0,02	0,02	0,02	0,02	0,02	0,02	0,02	0,02	0,02	0,02
Delta WCR	-300										300
Resale of machine											200

NPV Real

	0	1	2	3	4	5	6	7	8	9	10
CAPEX	-1000										
EBITDA (after tax)		100	100	100	100	100	100	100	100	100	100
Tax shield (DEP)		50	50	50	50	50	50	50	50	50	50
Delta WCR	-300										
Recovery of WCR											300
Resale of machine											200
Tax on CG											-100
CF	-1300	150	150	150	150	150	150	150	150	150	550
PV(CF)	-1300	144,2308	138,6834	133,3495	128,2206	123,2891	118,5472	113,9877	109,6035	105,388	371,5603

NPV	186,86

NPV Nominal

Inflation rate	0,06
WACC (nominal)	0,1

	0	1	2	3	4	5	6	7	8	9	10
CAPEX	-1000										
EBITDA (after tax)		106	112,36	119,1016	126,2477	133,8226	141,8519	150,363	159,3848	168,9479	179,0848
Tax shield (DEP)		50	50	50	50	50	50	50	50	50	50
Delta WCR	-300	-18	-19,08	-20,2248	-21,4383	-22,7246	-24,0881	-25,5333	-27,0653	-28,6893	-30,4106
Recovery of WCR											537,2543
Resale of machine											358,1695
Tax on CG											-179,085
CF	-1300	138	143,28	148,8768	154,8094	161,098	167,7639	174,8297	182,3195	190,2586	915,0132
PV(CF)	-1300	125,4545	118,4132	111,8533	105,7369	100,0292	94,69832	89,71527	85,05337	80,68823	352,7772

NPV	-35,5804

c) The NPV with inflation is lower since depreciation is not indexed on inflation. The use of a higher discount rate decreases even more the NPV.

Assistantship II

3 Uncertainty Theory

Question 1

An agent prefers gamble 1 to gamble 2 iff:

$$E[u(\cdot)_1] > E[u(\cdot)_2]$$

<u>Gamble 1</u>

$$E\left[u\left(\cdot\right)\right] = 0,15u\left(-4\right) + 0,20u\left(6\right) + 0,35u\left(10\right) + 0,30u\left(15\right)$$

<u>Gamble 2</u>

$$E\left[u\left(\cdot\right)\right] = 0,15u\left(-4\right) + 0,10u\left(3\right) + 0,10u\left(9\right) + 0,35u\left(10\right) + 0,10u\left(11\right) + 0,20u\left(17\right)$$

If gamble 1 is preferred to gamble 2, then we can write:

$$\begin{aligned} 0,15u\left(-4\right) + 0,20u\left(6\right) + 0,35u\left(10\right) + 0,30u\left(15\right) \quad > \quad & 0,15u\left(-4\right) + 0,10u\left(3\right) + 0,10u\left(9\right) \\ & + 0,35u\left(10\right) + 0,10u\left(11\right) + 0,20u\left(17\right) \end{aligned}$$

or

$$0,20u\left(6\right) + 0,30u\left(15\right) > 0,10u\left(3\right) + 0,10u\left(9\right) + 0,10u\left(11\right) + 0,20u\left(17\right)$$

We know that for a risk averse agent: $u[E(W)] > E[u(W)]$. Hence,

$$u\left(6\right) > 0,5u\left(3\right) + 0,5u\left(9\right) \Longrightarrow 0,20u\left(6\right) > 0,10u\left(3\right) + 0,10u\left(9\right)$$

and

$$u\left(15\right) > 0,33u\left(11\right) + 0,67u\left(17\right) \Longrightarrow 0,30u\left(15\right) > 0,10u\left(11\right) + 0,20u\left(17\right)$$

Conclusion: $E\left[u\left(\cdot\right)_1\right] > E\left[u\left(\cdot\right)_2\right]$

Question 2

a) Since they both have a 50% of winning the treasure and they agree on these probabilities, there is no need to call for an arbitrator who is going to decide on the winner with a 50% chance. Indeed, they would get the same outcome but they would need to pay a fee. Another possibility is to toss a coin in which case they both have 50% of winning.

Given the uncertainty of this game, the decision must be based on the expected utility criteria.

$$E[U(w_f)] = \sum_s p_s \, U(w_{f,s})$$

Let's compare the utilities of John and Jack:

<u>John</u>

Arbitrator: $E[U(w_f)] = 0,5U\,(300\ 000) + 0,5U\,(700\ 000) = 692,2$

Toss a coin: $E[U(w_f)] = 0,5U\,(300\ 000) + 0,5U\,(800\ 000) = 721,1$

<u>Jack</u>

Arbitrator: $E[U(w_f)] = 0,5U\,(0) + 0,5U\,(400\ 000) = 316,2$

Toss a coin: $E[U(w_f)] = 0,5U\,(0) + 0,5U\,(500\ 000) = 353,6$

Conclusion: A simple game such as tossing a coin is preferred to the arbitrator.

There exists another solution: $50\% - 50\%$ split where each get 250 000 euros

John: $U(550\ 000) = 741,62$

Jack: $U(250\ 000) = 500$

Conclusion: The $50\% - 50\%$ split is preferred to toss a coin. This stems from the agent's risk aversion where they preferred to receive the expected gain with certainty than playing the gamble itself.

b) Let's compare the utilities between using the arbitrator and tossing a coin::

<u>John</u>

Arbitrator: $E[U(w_f)] = 0,1U\,(300\ 000) + 0,9U\,(700\ 000) = 807,8$

Toss a coin: $E[U(w_f)] = 0,5U\,(300\ 000) + 0,5U\,(800\ 000) = 721,1$

<u>Jack</u>

Arbitrator: $E[U(w_f)] = 0,1U\,(0) + 0,9U\,(400\ 000) = 569,2$
Toss a coin: $E[U(w_f)] = 0,5U\,(0) + 0,5U\,(400\ 000) = 353,6$

Conclusion: Both John and Jack prefer the arbitrator to tossing a coin.

Question 3

a) The attitude towards risk is defined by the second derivative of the utility function:

$$\frac{\partial U(w_f)}{\partial w_f} = \frac{1}{w_f} > 0$$

$$\frac{\partial^2 U(w_f)}{\partial w_f^2} = -\frac{1}{w_f^2} < 0$$

Hence, the agent is risk averse.

b) First, let's calculate the final wealth in each state of nature:

w_f	P(w_f)
100	0,05
140	0,15
190	0,10
200	0,70

We can the calculate the expected utility:

$$\begin{aligned} E[U(w_f)] &= 0,05U(100) + 0,15U(140) + 0,10U(190) + 0,70U(200) \\ &= 0,05\ ln(100) + 0,15\ ln(140) + 0,10\ ln(190) + 0,70\ ln(200) \\ &= 5,205029 \end{aligned}$$

The certainty equivalent is given by the inverse of the expected utility function. The inverse of the logarithmic function is the exponential:

$$\begin{aligned} w^* &= U^{-1}E[U(w_f)] \\ &= e^{5,205029} \\ &= 182,2 \end{aligned}$$

The insurance premium (P^*) is given by the condition :

$$U(w_0 + P^*) = E[U(w_f)]$$

This implies that:

$$w_0 + P^* = w^* \qquad \Longrightarrow \qquad P^* = w^* - w_0$$

Hence,

$$P^* = 182,2 - 200 = -17,8$$

c) Let $U(w_f) = (w_f)^{1/2}$

We recalculate the expected utility :

$$\begin{aligned} E[U(w_f)] &= 0,05\ (100)^{1/2} + 0,15\ (140)^{1/2} + 0,10\ (190)^{1/2} + 0,70\ (200)^{1/2} \\ &= 13,55272 \end{aligned}$$

The certainty equivalent is equal to:

$$\begin{aligned} w^* &= U^{-1}E[U(w_f)] \\ &= (13,55272)^2 \\ &= 183,7 \end{aligned}$$

The insurance premium (P^*) is equal to:

$$\begin{aligned} P^* &= w^* - w_0 \\ &= 183,7 - 200 \\ &= -16,3 \end{aligned}$$

d) Using the definition of absolute risk aversion, we can compare the degree of risk aversion based on the two utility function.

Agent 1: $U(w_f) = ln(w_f)$

$$\begin{aligned} ARA(W) = -\frac{U''(W)}{U'(W)} \\ = \frac{1/W}{1/W^2} \\ = 1/W \end{aligned}$$

Agent 2: $U(w_f) = w_f^{1/2}$

$$\begin{aligned} ARA(W) &= -\frac{U''(W)}{U'(W)} \\ &= \frac{\frac{1}{4}W^{-3/2}}{\frac{1}{2}W^{-1/2}} \\ &= \frac{1}{2W^2} \end{aligned}$$

Since ARA(W) is larger for agent 1, he is more risk averse and is willing to pay a higher insurance premium to get rid of risk.

Question 4

a)

i) Method 1:

Calculate the expected utility:

$$\begin{aligned} E[U(W_f)] &= 0.5\ U(6\ 000) + 0.5\ U(4\ 000) \\ &= 0.5\ \ln(6\ 000) + 0.5\ \ln(4\ 000) \\ &= 8.4967825 \end{aligned}$$

The certainty equivalent is equal to:

$$\begin{aligned} W^* &= U^{-1}E[U(W_f)] \\ &= e^{8.4967825} \\ W^* &= 4\ 898.98 \end{aligned}$$

The agent is indifferent between the wealth associated with the gamble and receiving 4 898.98 with certainty.

The maximum price that he would be willing to pay to get rid of the risk is defined by P^*:

$$P^* = W^* - W_0 = 4\ 898.98 - 5\ 000 = -101.02$$

Given that the cost of insurance is 125 euros, the agent prefers not to get insured.

b) The initial wealth is now 4 000.

$$\begin{aligned} E[U(W_f)] &= 0.5\ U(3\ 000) + 0.5\ U(5\ 000) \\ &= 0.5\ \ln(3\ 000) + 0.5\ \ln(5\ 000) \\ &= 8.26178 \end{aligned}$$

The certainty equivalent is equal to:

$$\begin{aligned} W^* &= U^{-1}E[U(W_f)] \\ &= e^{8.26178} \\ W^* &= 3\ 872.98 \end{aligned}$$

The maximum price that he would be willing to pay to get rid of the risk is again defined by P^*:

$$P^* = W^* - W_0 = 3\ 872.98 - 4\ 000 = -127.02$$

Given that the cost of insurance is 125 euros, the agent buys the insurance.

ii) Method 2:

We know that P^* is such that:

$$U(W_0 + P^*) = E[U(W_f)]$$

Let $P^* = -125$ and $W_0 = 5\ 000$

$$U(W_0 + P^*) = U(5\ 000 - 125) = \ln(4\ 875) = 8.491875$$

$$\begin{aligned} E[U(w_f)] &= 0.5\ U(6\ 000) + 0.5\ U(4\ 000) \\ &= 0.5\ \ln(6\ 000) + 0.5\ \ln(4\ 000) \\ &= 8.4967825 \end{aligned}$$

Conclusion: $U(w_0 + P^*) < E[U(w_f)]$. The agent does not buy the insurance.

b)

Let $P^* = 125$ and $W_0 = 4\ 000$

$$U(W_0 + P^*) = U(4\ 000 - 125) = \ln(3\ 875) = 8.2623$$

$$\begin{aligned} E[U(w_f)] &= 0.5\ U(3\ 000) + 0.5\ U(5\ 000) \\ &= 0.5\ \ln(3\ 000) + 0.5\ \ln(5\ 000) \\ &= 8.26178 \end{aligned}$$

Conclusion: $U(w_0 + P^*) > E[U(w_f)]$. The agent buys the insurance.

Question 5

a) Expected gain

$$E\left(x\right) = 0,5\left(40\ 000 - 19\ 000\right) + 0,5\left(0 - 19\ 000\right) = 1\ 000$$

b) Certainty equivalent certain, risk premium and insurance premium

i) Calculuation of the expected utility:

$$\begin{aligned} E\left[U\left(w_0 + \tilde{x}\right)\right] &= 0,5U\left(121\ 000\right) + 0,5U\left(81\ 000\right) \\ &= 0,5\left(121\ 000\right)^{1/2} + 0,5\left(81\ 000\right)^{1/2} \\ &= 316,23 \end{aligned}$$

ii) The certainty equivalent is given by the inverse function of the expected utility. Given that the utility function is equal to $w^{\frac{1}{2}}$, its inverse is equal to w^2:

$$\begin{aligned} w^* &= U^{-1}E[U(w_f)] \\ &= (316,23)^2 \\ &= 100\ 000 \end{aligned}$$

iii) The risk premium is given by $\Pi \ni U\left(w_0 - \Pi + E\left(x\right)\right) = E\left[U\left(w\right)\right] = U(w^*)$. We can then write:

$$\begin{aligned} w_0 + E\left(x\right) - \Pi &= w^* \\ \Pi &= w_0 + E\left(x\right) - w^* \\ \Pi &= 100\ 000 + 1\ 000 - 100\ 000 \\ \Pi &= 1\ 000 \end{aligned}$$

since $E(x) = 1\ 000$

iv) The insurance premium (P^*) is given by the condition: $U(w_0 + P^*) = E[U(w_f)]$. This implies that:

$$w_0 + P^* = w^* \qquad \Longrightarrow \qquad P^* = w^* - w_0$$

Thus,

$$P^* = 100\ 000 - 100\ 000 = 0$$

c) The agent is indifferent between investing and not investing.
d) Calculation of the expected utility :

$$\begin{aligned} E\left[U\left(w_0 + \tilde{x}\right)\right] &= 0,5\left(81\ 000 - 19\ 000 + 40\ 000\right)^{1/2} + 0,5\left(81\ 000 - 19\ 000 + 0\right)^{1/2} \\ &= 284,19 \end{aligned}$$

$\Rightarrow \; w^* = (284,19)^2 = 80\ 762$

If the agent does not invest in the second round, he gets:

$$U\,(81\ 000) = 284,60$$

$\Rightarrow U\,(81\ 000) > E\,[U\,(W_0 + \tilde{x})] = 284,19$

Thus, he will not invest.

e) Yes he would invest. This is the inverse logic of d).

Question 6:

The base case scenario is the following:

Event	Prob	House	Savings	Final Wealth	$U(W_f)$
No earthquake	0.98	1 000 000	500 000	1 500 000	14.221
Earthquake	0.02	500 000	500 000	1 000 000	13.816

a) Expected utility associated with facing the risk (no insurance):

- Expected utility: $E(U(W_f)) = (0.98 \times 14.221) + (0.02 \times 13.816) = 14.213$

b) Certainty equivalent and the insurance premium that your parents would be willing to pay.

- Certainty equivalent: $W^* = e^{14.213} = 1\ 487\ 885.23$
- Insurance premium: $I = W^* - W_0 = 1\ 487\ 885.23 - 1\ 500\ 000 = -12\ 114.77$

Conclusion: Your parents would be willing to pay up to 12 114.77 to get insured. An insurance that sells for 10 000 is then a good deal. They should buy the insurance.

Alternative solution. Suppose that your parents decide to take the insurance at a cost of 10 000. Their wealth will be the following:

Event	Prob	House	Savings	Insurance compensation	Final Wealth	$U(W_f)$
No earthquake	0.98	1 000 000	490 000	0	1 490 000	14.214
Earthquake	0.02	500 000	490 000	500 000	1 490 000	14.214

Note that the savings go down by 10 000 since your parents paid that amount to buy the insurance.

- Expected utility: $E(U(W_f)) = 14.214$
- Certainty equivalent: $W^* = e^{14.214} = 1\ 490\ 000$

The expected utility (14.214) is higher with the insurance than without it (14.213). Alternatively, the certainty equivalent with the insurance (1 490 000) is higher than the certainty equivalent without insurance (1 487 885.23), they should buy the insurance.

c) If your parents buy stocks of the construction company, we have the following:

Event	Prob	House	Savings	Stock	Final Wealth	$U(W_f)$
No earthquake	0.98	1 000 000	450 000	40 000	1 490 000	14.214
Earthquake	0.02	500 000	450 000	75 000	1 025 000	13.840

Note that the savings go down by 50 000 since your parents paid that amount to buy the stocks. In this case:

- Expected utility is $E(U(W_f)) = (0.98 \times 14.214) + (0.02 \times 13.840) = 14.207$

d) Since the expected utlity with the stocks (14.207) is lower than the expected utility without them (14.213), your parents should not buy the stocks.
The same calculation can be done with the certainty equivalent with the stocks.

- Certainty equivalent: $W^* = e^{14.207} = 1\ 478\ 893.91$

Since the certainty equivalent with the stocks (1 478 893.91) is lower than the certainty equivalent without the stocks (1 487 885.23), your parents should not buy the stocks.

4 Portofolio Theory: The Mean-Variance Model

Question 1

Construction of a Portfolio Frontier with 2 Assets

State of nature	Probability	Return Asset 1	Return Asset 2	Return Asset 3
A	0,18	-0,05	0,18	-0,12
B	0,22	0,12	-0,08	0,12
C	0,25	0,30	0,16	-0,06
D	0,35	0,22	0,16	0,04

Expect. Return		0,1694	0,1108	0,0038
Variance		0,0144	0,0103	0,0072
Standard Deviation		0,1198	0,1016	0,0849

Cov(1,2)	0,0018
Corr(1,2)	0,1493
Cov(1,3)	0,0022
Corr(1,3)	0,2147
Cov(2,3)	-0,0066
Corr(2,3)	-0,7630

Portfolio 1: Assets 1 et 2

W1	W2	E(r)	Stand. Dev.	Variance
2	-1	0,2280	0,2460	0,0605
1,5	-0,5	0,1987	0,1794	0,0322
1,25	-0,25	0,1841	0,1482	0,0219
1	0	0,1694	0,1198	0,0144
0,9	0,1	0,1635	0,1098	0,0121
0,8	0,2	0,1577	0,1009	0,0102
0,7	0,3	0,1518	0,0934	0,0087
0,6	0,4	0,1460	0,0877	0,0077
0,5	0,5	0,1401	0,0841	0,0071
0,4	0,6	0,1342	0,0830	0,0069
0,3	0,7	0,1284	0,0843	0,0071
0,2	0,8	0,1225	0,0881	0,0078
0,1	0,9	0,1167	0,0940	0,0088
0	1	0,1108	0,1016	0,0103
-0,25	1,25	0,0962	0,1261	0,0159
-0,5	1,5	0,0815	0,1552	0,0241
-1	2	0,0522	0,2200	0,0484

Global Min. variance portfolio

W1*	0,4041
W2*	0,5959
E(Return)	0,1345
Variance	0,0069
Standard Deviation	0,0830

Portfolio 2: Assets 2 et 3

W2	W3	E(r)	Stand. Dev.	Variance
2	-1	0,2178	0,2735	0,0748
1,5	-0,5	0,1643	0,1868	0,0349
1,25	-0,25	0,1376	0,1439	0,0207
1	0	0,1108	0,1016	0,0103
0,9	0,1	0,1001	0,0851	0,0072
0,8	0,2	0,0894	0,0692	0,0048
0,7	0,3	0,0787	0,0542	0,0029
0,6	0,4	0,0680	0,0414	0,0017
0,5	0,5	0,0573	0,0330	0,0011
0,4	0,6	0,0466	0,0330	0,0011
0,3	0,7	0,0359	0,0412	0,0017
0,2	0,8	0,0252	0,0540	0,0029
0,1	0,9	0,0145	0,0690	0,0048
0	1	0,0038	0,0849	0,0072
-0,25	1,25	-0,0230	0,1266	0,0160
-0,5	1,5	-0,0497	0,1693	0,0287
-1	2	-0,1032	0,2559	0,0655

Global Min. variance portfolio

W2*	0,4492
W3*	0,5508
E(Return)	0,0519
Variance	0,0010
Standard Deviation	0,0318

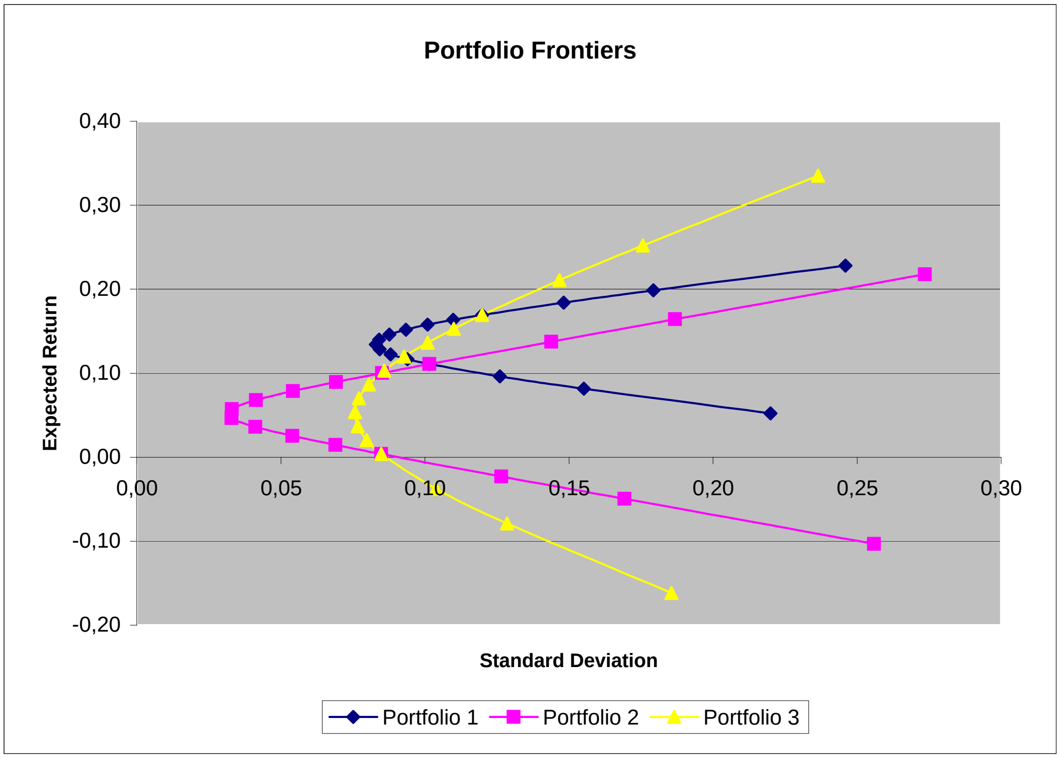
Portfolio 3: Assets 1 et 3

W1	W3	E(r)	Stand. Dev.	Variance
2	-1	0,3350	0,2365	0,0559
1,5	-0,5	0,2522	0,1756	0,0308
1,25	-0,25	0,2108	0,1467	0,0215
1	0	0,1694	0,1198	0,0144
0,9	0,1	0,1528	0,1100	0,0121
0,8	0,2	0,1363	0,1009	0,0102
0,7	0,3	0,1197	0,0928	0,0086
0,6	0,4	0,1032	0,0859	0,0074
0,5	0,5	0,0866	0,0805	0,0065
0,4	0,6	0,0700	0,0771	0,0059
0,3	0,7	0,0535	0,0758	0,0057
0,2	0,8	0,0369	0,0767	0,0059
0,1	0,9	0,0204	0,0798	0,0064
0	1	0,0038	0,0849	0,0072
-0,25	1,25	-0,0376	0,1039	0,0108
-0,5	1,5	-0,0790	0,1286	0,0165
-1	2	-0,1618	0,1856	0,0344

Global Min. variance portfolio

W1*	0,2920
W3*	0,7080
E(Return)	0,0521
Variance	0,0057
Standard Deviation	0,0758

Construction of a Portfolio Frontier with 2 Assets



Question 2

a)

$$\begin{aligned}
\sigma^2(r_p) &= W_a^2\sigma_a^2 + (1-W_a)^2\sigma_b^2 + 2W_a(1-W_a)\sigma_a\sigma_b\rho_{a.b} \\
&= 0.000225W_a^2 + 0.0007(1-W_a)^2 + 0.0003W_a(1-W_a) \\
&= 0.000625W_a^2 - 0.0011W_a + 0.0007
\end{aligned}$$

The global miminum variance portfolio is given by:

$$\frac{\partial\sigma^2(r_p)}{\partial Wa} = 0.00125W_a - 0.0011 = 0$$

$$\Longrightarrow \quad (W_a^*.W_b^*) = (0.88; 0.12)$$

or alternatively:

$$\begin{aligned}
W_a^* &= \frac{\sigma_b^2 - \sigma_a\sigma_b\rho_{a.b}}{\sigma_a^2 + \sigma_b^2 - 2\sigma_a\sigma_b\rho_{a.b}} \\
&= \frac{0.0007 - 0.00015}{0.000225 + 0.0007 - 0.0003} \\
&= 0.88
\end{aligned}$$

b) We solve for W_a in the $E(r_p)$ equation and then replace in $\sigma^2(r_p)$.

$$\begin{aligned}
E(r_p) &= W_aE(r_a) + W_bE(r_b) \\
&= 0.03W_a + 0.06(1-W_a)
\end{aligned}$$

$$\Longrightarrow \quad W_a = 2 - \frac{E(r_p)}{0.03}$$

In the variance equation:

$$\begin{aligned}
\sigma^2(r_p) &= 0.000625\left(2 - \frac{E(r_p)}{0.03}\right)^2 - 0.0011\left(2 - \frac{E(r_p)}{0.03}\right) + 0.0007 \\
&= 0.000625\left(4 - \frac{4}{0.03}E(r_p) + \frac{E(r_p)^2}{0.0009}\right) - 0.0022 + 0.0367E(r_p) + 0.0007 \\
&= 0.694E(r_p)^2 - 0.0467E(r_p) + 0.001
\end{aligned}$$

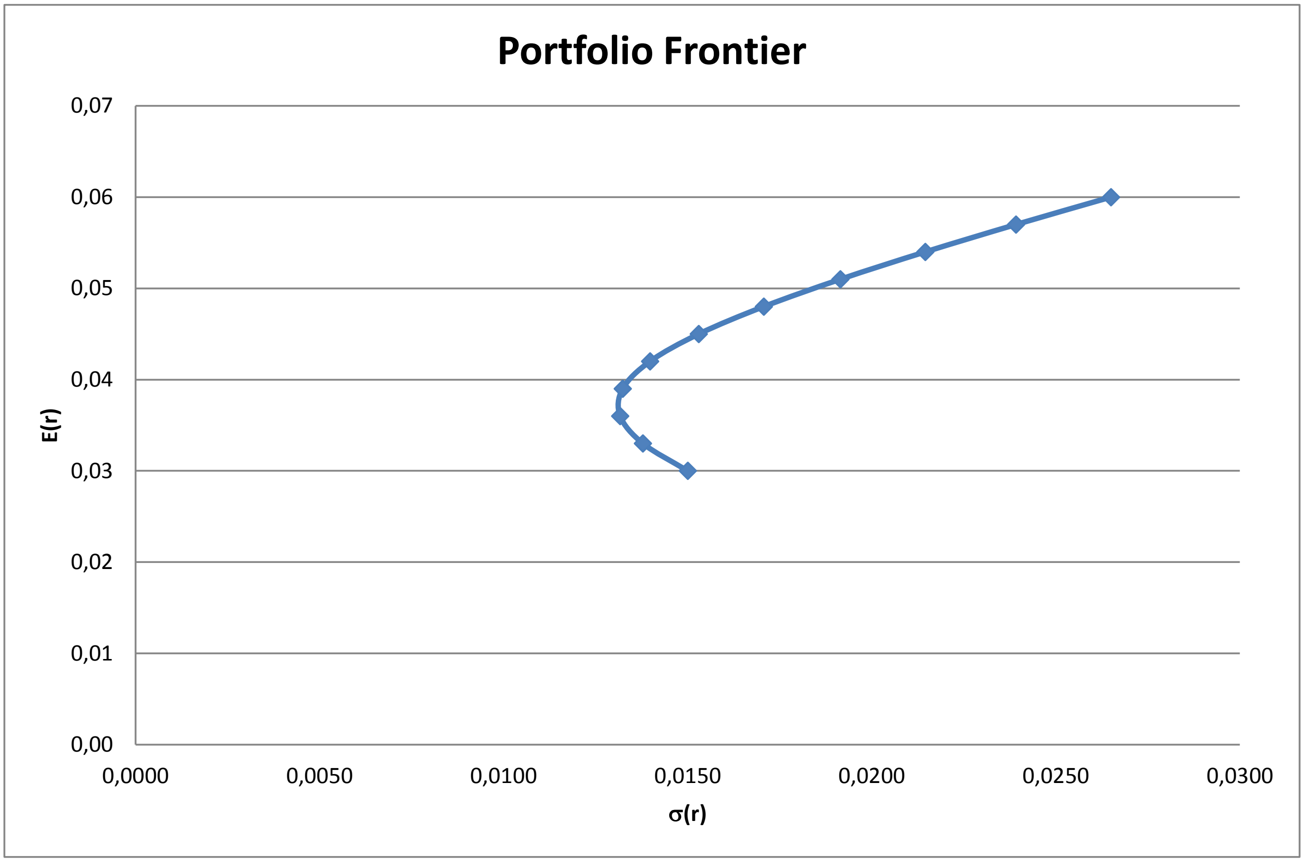
$$\Longrightarrow \quad \sigma(r_p) = \left[0.694E(r_p)^2 - 0.0467E(r_p) + 0.001\right]^{1/2}$$

c) We can trace the portfolio with fictive portfolios:

W_a	W_b	$E(r_p)$	$\sigma(r_p)$
0	1	0.060	0.02650
0.10	0.90	0.057	0.02392
0.20	0.80	0.054	0.02146
0.30	0.70	0.051	0.01915
0.40	0.60	0.048	0.01707
0.50	0.50	0.045	0.01530
0.60	0.40	0.042	0.01398
0.70	0.30	0.039	0.01323
0.80	0.20	0.036	0.01317
0.90	0.10	0.033	0.01378
1	0	0.030	0.01500

The global minimum variance portfolio is $(W_a^*, W_b^*) = (0.88; 0.12)$ and has the following characteristics:

$E(r_p) = 0.0336$
$\sigma(r_p) = 0.0147$


Portfolio Frontier
0,07
0,06
0,05
0,04
0,03
0,02
0,01
0,00
E(r)
0,0000
0,0050
0,0100
0,0150
0,0200
0,0250
0,0300
σ(r)

Question 3

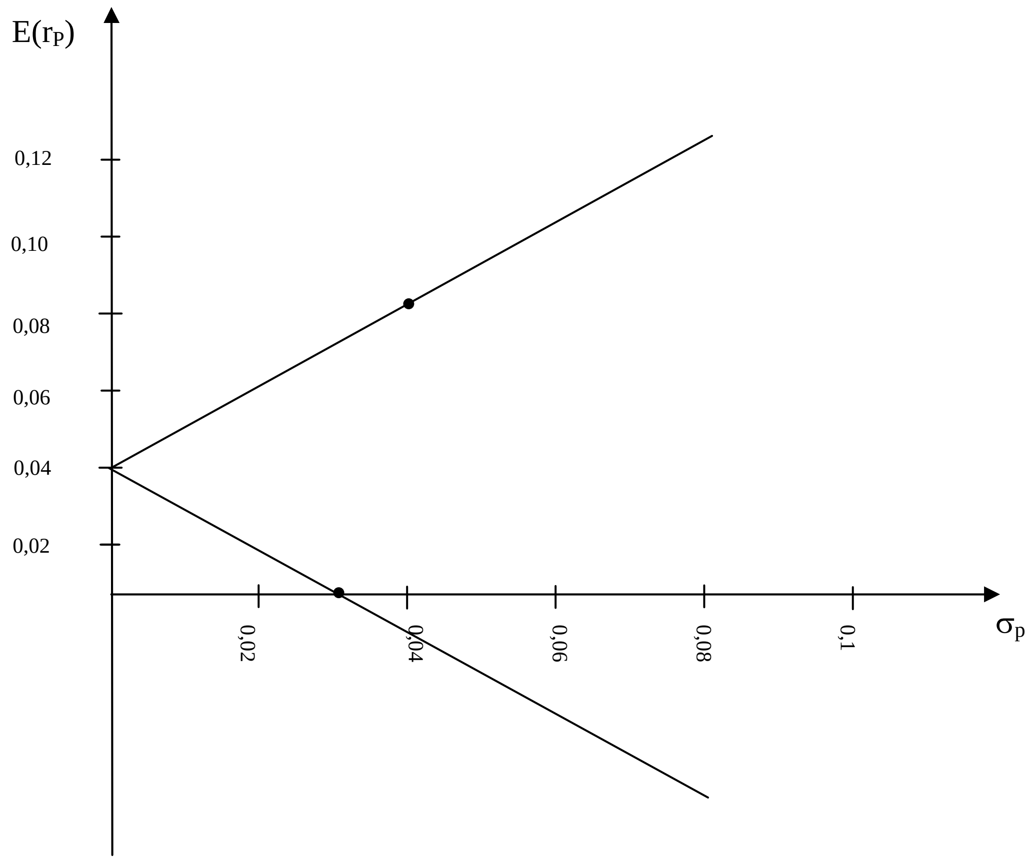
a) The equation of the capital market line is given by:

$$\begin{aligned} E(r_p) &= r_f + \left(\frac{E(r_m) - r_f}{\sigma_m}\right)\sigma_p \\ &= 0,04 + \left(\frac{0,12 - 0,04}{0,06}\right)\sigma_p \\ &= 0,04 + 1,333\sigma_p \end{aligned}$$

This is the equation of a straight line with an intercept equal to 0.04 and a slope of 1.33.

b) To draw the CML, we can either use the above equation or construct portfolios.

w_m	w_f	$E(r_p)$	$\sigma(r_p)$
-1,5	2,5	-0,080	0,09
-1,0	2,0	-0,040	0,06
-0,5	1,5	0,000	0,03
0,0	1,0	0,040	0,00
0,5	0,5	0,080	0,03
1,0	0,0	0,120	0,06
1,5	-0,5	0,160	0,09


E(rP)
0,12
0,10
0,08
0,06
0,04
0,02
0,02
0,04
0,06
0,08
0,1
σP

c) Replacing the values in the equation of the CML, we get:

$$0,16 = 0,04 + 1,333\sigma_p$$

$$\implies \quad \sigma_p = 0,09$$

In an economy with a risk-free asset, we know that $\sigma_p = w_m \sigma_m$. hence, we get:

$$0,09 = 0,06 w_m \qquad \implies \qquad w_m = 1,5 \ \text{ et } \ w_f = -0,5 \ \text{ (Borrow)}$$

The final portfolio is equal to:

- Equity: $50\ 000 \times 1,5 = 75\ 000$
- Loan: $-25\ 000$

Question 4

a)

$$E(r_p) = \sum_{i=1}^{N} w_i E(r_i) = \sum_{i=1}^{N} \frac{0,15}{25} = 0,15$$

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

Let $w_i = w_j = 1/N$. Alors,

$$\begin{aligned} \sigma_p^2 &= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} \\ &= \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2 + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \sigma_{ij} \end{aligned}$$

We can rewrite this as:

$$\begin{aligned}
\sigma_p^2 &= \frac{1}{N}\sum_{i=1}^{N}\frac{\sigma^2}{N} + \frac{(N-1)}{N}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N}\frac{\sigma_{ij}}{N(N-1)} \\
&= \frac{1}{N}\,\bar{\sigma}^2 + \left(1-\frac{1}{N}\right)\bar{\sigma}_{ij}
\end{aligned}$$

Ler $w_i = w_j = 1/25$. Then,

$$\sigma_p^2 = \frac{1}{25}\,\bar{\sigma}^2 + \left(1-\frac{1}{25}\right)\bar{\sigma}_{ij}$$

If $\rho_{ij} = 0 \implies \sigma_{ij} = 0$ et $\bar{\sigma}_{ij} = 0$. Hence,

$$\begin{aligned}
\sigma_p^2 &= \frac{1}{25}\,\bar{\sigma}^2 \\
&= \frac{0,07}{25} \\
\sigma_p^2 &= 0,0028 \\
\sigma_p &= 0,053
\end{aligned}$$

If $\rho_{ij} = 0,2 \implies \sigma_{ij} = \rho_{i,j}\sigma_i\sigma_j = 0,2(0,07)^{1/2}(0,07)^{1/2} = 0,014.$

This implies that $\bar{\sigma}_{ij} = 0,014$

$$\begin{aligned}
\sigma_p^2 &= \frac{1}{25}\,\bar{\sigma}^2 + \left(1-\frac{1}{25}\right)\bar{\sigma}_{ij} \\
&= \frac{0,07}{25} + \left(1-\frac{1}{25}\right)0,014 \\
\sigma_p^2 &= 0,01624 \\
\sigma_p &= 0,1274
\end{aligned}$$

b) Consider a portfolio E composed of the risk free asset and the portfolio of risky assets P. The equation of the Capital Market Line is given by:

$$\begin{aligned}
E(r_e) &= r_f + \left(\frac{E(r_p)-r_f}{\sigma_p}\right)\sigma_e \\
&= 0,05 + \left(\frac{0,10}{0,053}\right)\sigma_e \\
&= 0,05 + 1,89\ \sigma_e
\end{aligned}$$

If we want a portfolio offering a 20% expected return, then,

$$\begin{aligned}
0,20 &= 0,05 + 1,89\ \sigma_e \\
0,15 &= 1,89\ \sigma_e
\end{aligned}$$

Hence $\sigma_e = 0,079$

We know that for an efficient portfolio,

$$\begin{aligned} \sigma_e &= w_p \, \sigma_p \\ 0,079 &= 0,053 \, w_p \\ w_p &= 1,5 \end{aligned}$$

Hence, $w_p = 1,5$ et $w_f = -0,5$.

The investor borrows 50% of his initial wealth and buys 150% of portfolio P.

Other method:

$$\begin{aligned} E(r_e) &= w_f r_f + (1 - w_f)E(r_p) \\ 0,2 &= 0,05 w_f + 0,15(1 - w_f) \\ 0,05 &= -0,10 w_f \\ w_f &= -0,5 \qquad \Longrightarrow \qquad w_p = 1,5 \end{aligned}$$

Question 5

In an economy with a risk free asset, all efficient portfolios are on the Capital Market Line and a are combination of the risk free asset and the tangent portfolio (M). The equation of the CML is:

$$\begin{aligned} E(r_p) &= r_f + \left[\frac{E(r_M) - r_f}{\sigma_M}\right] \sigma_p \\ &= 0.05 + 1.03846\sigma_p \end{aligned}$$

In addition,

$$\sigma_p = w_M \sigma_M$$

i) If $E(r_p) = 0.10$, we replace in the CML equation:

$$\begin{aligned} E(r_p) &= 0.05 + 1.03846\sigma_p \\ 0.1 &= 0.05 + 1.03846\sigma_p \\ 0.05 &= 1.03846\sigma_p \\ 0.04815 &= \sigma_p \end{aligned}$$

We then replace in the standard deviation equation,

$$\begin{aligned} w_M &= \frac{\sigma_p}{\sigma_M} \\ &= \frac{0.04815}{0.13} \\ &= 0.3704 \end{aligned}$$

The optimal portfolio contains 37.04% of the tangent portfolio (M) and 62.86% of the risk-free asset.

ii) If $\sigma_p = 0.08$, we replace in the standard deviation equation,

$$\begin{aligned} w_M &= \frac{\sigma_p}{\sigma_M} \\ &= \frac{0.08}{0.13} \\ &= 0.6154 \end{aligned}$$

The optimal portfolio contains 61.54% of the tangent portfolio (M) and 38.46% of the risk-free asset.

The expected return of this portfolio is: :

$$\begin{aligned} E(r_p) &= 0.05 + 1.03846\sigma_p \\ &= 0.05 + 1.03846(0.08) \\ &= 0.1331 \end{aligned}$$

or alternatively,

$$\begin{aligned} E(r_p) &= w_f r_f + w_M E(r_M) \\ &= (0.3846)(0.05) + (0.6154)(0.185) \\ &= 0.1331 \end{aligned}$$

Question 6

1. We know that:

$$E(R_P) = w_A \cdot E(R_A) + (1 - w_A) \cdot E(R_B),$$

where w_A is the proportion of wealth invested in asset A. Hence, to achieve an expected return of 5%, you want a portfolio such that:

$$w_A = \frac{E(R_P) - E(r_B)}{E(R_A) - E(R_B)} = \frac{0.05 - 0.10}{0.07 - 0.10} = 166.67\%,$$

and $w_B = 1 - w_A = -66.67\%$. The covariance between the two assets is

$$\sigma_{A,B} = \sigma_A \sigma_B = 0.02$$

The risk of the portfolio is given by:

$$\sigma_P = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_{A,B}} = 3.33\%.$$

You need to short-sell asset B in order to achieve a return lower that asset A alone.

2. The global minimum variance has the following composition:

$$w_A^g = \frac{\sigma_B^2 - \sigma_{A,B}}{\sigma_A^2 + \sigma_B^2 + 2\sigma_{A,B}} = 200\%,$$

and $w_B^g = 1 - w_A^g = -100\%$. Since both assets have perfect positive correlation, the global minimum variance portfolio has zero variance. In effect, you can verify that

$$\sigma_{GMVP} = \sqrt{(w_A^g)^2\sigma_A^2 + (w_B^g)^2\sigma_B^2 + 2w_A^g w_B^g \sigma_{A,B}} = 0,$$

and that the expected return is equal to

$$E(R_{GMPV}) = w_A^g \cdot E(r_A) + w_B^g \cdot E(r_B) = 4\%.$$

An asset with no variance is a risk-free asset, so in this case the expected return of the global minimum variance portfolio is the risk-free rate. Therefore, the investment opportunity set is just a triangle. The upper portion (the efficient frontier) goes through the GMV portfolio, and assets A, B and the portfolio computed in b). This is shown in Figure 1.

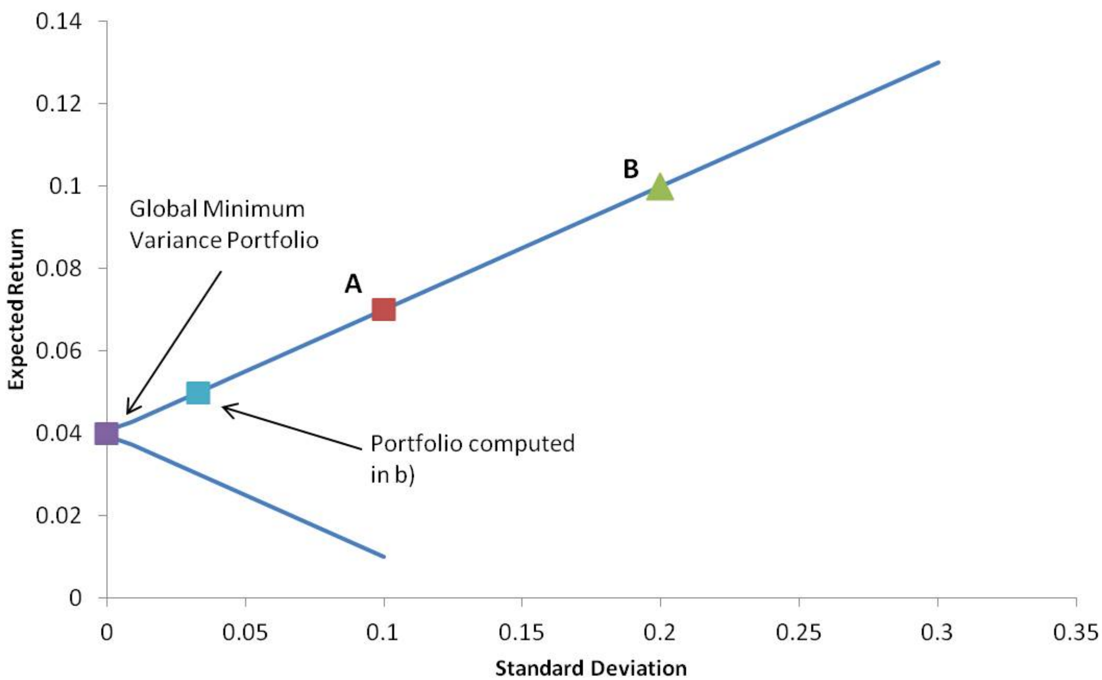


Figure 1: Investment Opportunity Set

3. As explained before, assets A, B and the portfolio computed in b) are efficient since they are in the upper portion of the investment opportunity set.

4. If there is another riskless asset yielding a return of 3%, then there is an arbitrage opportunity. It means that you can borrow risk-free at 3% and reinvest the proceeds at 4%, also risk-free.

   For example, you could borrow \$10 000 at 3% (meaning that you short-sell the equivalent of \$10 000 of this riskless asset) and invest that amount in the GMV portfolio (you buy \$20 000 of asset A and sell short \$10 000 of asset B). After one year, you collect (for sure!) $10000 \cdot 1.04 = 10\,400$ but only have to pay back $10\,000 \cdot 1.0300 = 10\,300$ for your loan. Hence you make a **riskless** profit of \$100, or 1.00% per dollar borrowed.

Assistantship III

5 The Capital Asset Pricing Model (CAPM)

Question 1

a)

$$E\left(r_m\right) = 0,1\left(-0,2\right) + 0,2\left(-0,1\right) + 0,3\left(0,1\right) + 0,3\left(0,25\right) + 0,1\left(0,35\right) = 0,10$$

$$E\left(r_4\right) = 0,1\left(-0,15\right) + 0,2\left(-0,3\right) + 0,3\left(0,15\right) + 0,3\left(0,3\right) + 0,1\left(0,4\right) = 0,10$$

$$\sigma_m^2 = \sum_s p_s[r_{m,s} - E(r_m)]^2 = 0,03 \quad \Longrightarrow \quad \sigma_m = 0,1732$$

$$\sigma_4^2 = \sum_s p_s[r_{4,s} - E(r_4)]^2 = 0,06 \quad \Longrightarrow \quad \sigma_4 = 0,2449$$

We know that $\rho_{m,m} = 1$ since it is the market portfolio. In addition,

$$\rho_{4,m} = \frac{cov\left(r_4,r_m\right)}{\sigma_4\ \sigma_m}$$

with

$$cov\left(r_4, r_m\right) = \sum_s p_s[r_{4,s} - E\left(r_4\right)][r_{m,s} - E\left(r_m\right)] = 0,04 \Longrightarrow \quad \rho_{4,m} = 0,94$$

Hence,

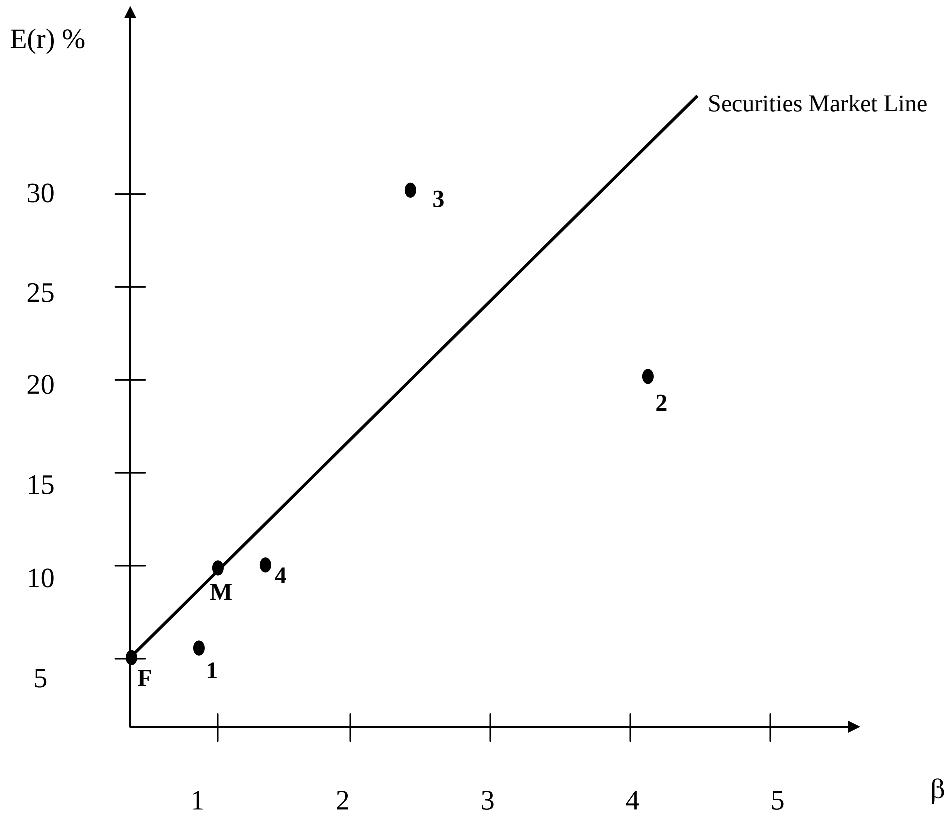
$$\beta_1 = \frac{cov\left(r_1,r_m\right)}{\sigma_m^2} = \frac{\sigma_1\ \sigma_m\ \rho_{1,m}}{\sigma_m^2} = 0,83$$

$$\beta_2 = \frac{\sigma_2\ \sigma_m\ \rho_{2,m}}{\sigma_m^2} = 4,04$$

$$\beta_3 = \frac{\sigma_3\ \sigma_m\ \rho_{3,m}}{\sigma_m^2} = 2,31$$

$$\beta_4 = \frac{\sigma_4\ \sigma_m\ \rho_{4,m}}{\sigma_m^2} = 1,33$$

b)


E(r) %
Securities Market Line
30
25
20
15
10
5
3
2
4
M
1
F
1
2
3
4
5
β

c) The equation of the Securities Market Line (SML) is given by:

$$E\left(r_i\right) = r_f + \beta_i[E\left(r_m\right) - r_f]$$

Hence, in equilibrium:

$$E(r_1) = 0,05 + 0,83[0,10 - 0,05] = 9,15\%$$

$$E(r_2) = 0,05 + 4,04[0,10 - 0,05] = 25,2\%$$

$$E(r_3) = 0,05 + 2,31[0,10 - 0,05] = 16,55\%$$

$$E(r_4) = 0,05 + 1,33[0,10 - 0,05] = 11,66\%$$

d) The ranking of assets is:

- $\beta \quad : F > 1 > M > 4 > 3 > 2$
- $\sigma \quad : F > M > 4 > 2 > 3 > 1$

Reason: Beta measures only systematic risk while σ measures total risk (systematic + non systematic)

Question 2

a)

$$E\left(r_m\right) = \sum_s p_s \; r_{m,s} = 0.13$$

$$E\left(r_{jm}\right) = \sum_s p_s \; r_{jm,s} = 0.15$$

$$\sigma_m^2 = \sum_s p_s[r_{m,s} - E\left(r_m\right)]^2 = 0.0091 \quad \Longrightarrow \quad \sigma_m = 0,095$$

$$\sigma_{jm}^2 = \sum_s p_s[r_{jm,s} - E\left(r_{jm}\right)]^2 = 0.0525 \quad \Longrightarrow \quad \sigma_{jm} = 0,229$$

$$cov\,(r_m, r_{jm}) = \sum_s p_s[r_{m,s} - E\,(r_m)](r_{jm,s} - E(r_{jm})) = 0.017$$

$$\Longrightarrow \quad \beta_{jm} = \frac{cov(r_m, r_{jm})}{\sigma_m^2} = 1.87$$

$$\rho_{jm,m} = \frac{cov(r_m, r_{jm})}{\sigma_m \sigma_{jm}} = 0.781$$

b) If $r_f = 0.08$, according to the CAPM:

$$\begin{aligned} E\,(r_{jm}) &= r_f + \beta_{jm}[E\,(r_m) - r_f] \\ &= 0.08 + 1.87[0.13 - 0.08] \\ E\,(r_{jm}) &= 0.1735 \end{aligned}$$

Based on the study, the expected return of JM (15%) is lower than its equilibrium expected return as predicted by the CAPM.

$\Longrightarrow$ Should not buy the stocks of JM.

Question 3

a)

Method 1

$$\begin{aligned} E\,(r_e) &= r_f + \beta_e[E\,(r_m) - r_f] \\ 0,10 &= 0,04 + \beta_e[0,14 - 0,04] \end{aligned}$$

$$\Longrightarrow \quad \beta_e = 0,6$$

Method 2

$$\beta_e = w_m\ \beta_m + w_f\ \beta_f = w_m \qquad \text{car} \quad \beta_m = 1 \quad \text{et} \quad \beta_f = 0$$

Since

$$\begin{aligned} E(r_e) &= w_m E(r_m) + (1 - w_m) r_f \\ 0,10 &= 0,14 w_m + 0,04(1 - w_m) \\ 0,06 &= 0,10 w_m \\ w_m &= 0,6 \qquad \Longrightarrow \quad \beta_e = 0,6 \qquad \text{because} \quad w_m = \beta_e \end{aligned}$$

b)

Method 1

Efficient portfolios have no specific risk. Using the equation for $E\left(r_e\right)$, we can write:

$$\sigma_e^2 = \beta_e^2 \sigma_m^2 + \epsilon_e^2$$

Since $\epsilon_e^2 = 0$ (zero specific risk), we have:

$$\sigma_e^2 = \beta_e^2 \sigma_m^2$$

$$\sigma_e = \beta_e \sigma_m = 0,6\,(0,12) = 0,072$$

Method 2

Using the Capital Market Line, we can write,

$$E(r_e) = r_f + \left[\frac{E(r_m) - r_f}{\sigma_m}\right] \sigma_e$$

$$\sigma_e = \left[\frac{E(r_e) - r_f}{E(r_m) - r_f}\right] \sigma_m$$

$$\sigma_e = \left[\frac{0,10 - 0,04}{0,14 - 0,04}\right] 0,12 = 0,072$$

Method 3

$$\begin{aligned}
\sigma_e^2 &= w_m^2 \sigma_m^2 + w_f^2 \sigma_f^2 + 2 w_m w_f \sigma_m \sigma_f \;\rho_{f,m} \\
&= w_m^2 \sigma_m^2 \\
\sigma_e^2 &= w_m \sigma_m \\
&= 0,6 \times 0,12 \\
&= 0,072
\end{aligned}$$

c)

$$\rho_{e,m} = \frac{cov\,(r_e, r_m)}{\sigma_e \sigma_m}$$

Method 1

We know that:

$$\beta_e = \frac{cov\,(r_e, r_m)}{\sigma_{m^2}}$$

$$\implies \quad cov\,(r_e r_m) = \beta_e \sigma_m^2 = 0,6(0,0144) = 0,00864$$

Hence,

$$\rho_{e,m} = \frac{0,00864}{(0,072)(0,12)} = 1$$

Method 2

$$\rho_{e,m} = \frac{\sigma_{e,m}}{\sigma_e \sigma_m}$$

But,

$$\beta_e = \frac{\sigma_{e,m}}{\sigma_m^2} \qquad \implies \quad \sigma_{e,m} = \beta_e\ \sigma_m^2$$

Thus,

$$\rho_{e,m} = \frac{\beta_e\ \sigma_m^2}{\sigma_e \sigma_m} = \beta_e \frac{\sigma_m}{\sigma_e}$$

We know that $\sigma_e = \beta_e \sigma_m$. Thus,

$$\rho_{e,m} = \frac{\beta_e\ \sigma_m}{\beta_e\ \sigma_m} = 1$$

Concusion: An efficient portfolio is perfectly correlated with the market portfolio.

Question 4

We know that,

$$\rho_{j,m} = \frac{\sigma_{j,m}}{\sigma_j \sigma_m}$$

On the basis of the data, we can write:

$$0,8 = \frac{\sigma_{j,m}}{(0,25)(0,2)}$$

Thus,

$$\sigma_{j,m} = 0,8(0,25)(0,2) = 0,04$$

We also know that:

$$\beta_j = \frac{\sigma_{j,m}}{\sigma_m^2} = \frac{0,04}{(0,2)^2} = 1$$

The systematic risque (beta) of a portfolio is equal to the weighted average of the systematic risk (beta) of each individual asset in the portfolio.

$$\begin{aligned} \beta_p &= w_f \beta_f + w_j \beta_j \\ 1,6 &= (1-w_j)(0) + w_j(1) \end{aligned}$$

$$\implies \quad w_j = 1,6 \ \text{ and } \ w_f = -0,6$$

The investor must borrow 60% of his wealth and buy 160% of JOJO's stocks in order to get a portfolio with a beta of 1.6.

Question 5

Information:

- $E(r_Q) = 0,25$
- $E(r_m) = 0,15$
- $\sigma_m) = 0,10$
- $r_f = 0,05$

a) An efficient portefolio, Q, is composed of the risk-free asset and the market portfolio, (w_f, w_m). We can then write:

$$\begin{aligned} E(r_Q) &= w_f\ r_f + w_m\ E(r_m) \\ &= w_f\ r_f + (1 - w_f)\ E(r_m) \\ 0,25 &= 0,05w_f + 0,15(1 - w_f) \\ 0,1 &= -0,1w_f \\ w_f &= -1 \qquad \Longrightarrow \qquad w_m = 2 \end{aligned}$$

In addition, we know that the beta of a portfolio is equal to the weighted average of the beta of each asset in the portfolio:

$$\begin{aligned} \beta_Q &= w_f\ \beta_f + w_m\ \beta_m \\ &= w_m \qquad \text{since } \beta_f = 0 \text{ and } \beta_m = 1 \\ &= 2 \end{aligned}$$

b) For an efficient portfolio, the variance equation is given by:

$$\begin{aligned} \sigma_Q^2 &= w_f^2\sigma_f^2 + w_m^2\sigma_m^2 + 2w_f w_m \sigma_{f,m} \\ &= w_m^2\sigma_m^2 \end{aligned}$$

Thus,

$$\sigma_Q = w_m\sigma_m = 2(0,1) = 0,2$$

In addition, we can verify the expected return of Q:

$$E(r_Q) = w_f\ r_f + w_m\ E(r_m) = -1(0,05) + 2(0,15) = 0,25$$

or

$$E(r_Q) = r_f + \beta_Q[E(r_m) - f_f] = 0,05 + 2(0,1) = 0,25$$

c) We know that an efficient portfolio with the risk-free asset and the market portfolio has no specific (non-systematic) risk. Hence, the only source of risk is the systematic (market) risk. Indeed, we can write:

$$\begin{aligned}
\sigma_Q^2 &= \beta_Q^2\ \sigma_m^2 + \sigma_\epsilon^2 \\
&= \beta_Q^2\ \sigma_m^2 \qquad \text{since } \sigma_\epsilon^2 = 0 \\
\sigma_Q &= \beta_Q\ \sigma_m \\
&= 2(0,1) \\
&= 0,2
\end{aligned}$$

Thus, the risk as measured by the standard deviation reflects only systematic risk as captured by the beta.

d) In the case of an individual asset, the total risk mesaured by the variance captures systematic and non-systematic risk.

$$\sigma_i^2 = \beta_i^2\ \sigma_m^2 + \sigma_\epsilon^2$$

But, according to the CAPM, in equilibrium, the expected return of any asset is a function of the systematic risk only. Thus, the appropriate measure of risk is the beta.

Question 6

1. We can compute the expected return of A, B, C from the CAPM:

$$\begin{aligned}
E(R_A) &= 0.04 + 1.2 \cdot (0.10 - 0.04) = 0.112 \\
E(R_B) &= 0.04 + 0.7 \cdot (0.10 - 0.04) = 0.082 \\
E(R_C) &= 0.04 + 2.0 \cdot (0.10 - 0.04) = 0.16
\end{aligned}$$

2. The maximum Sharpe ratio of the economy is equal to the Sharpe ratio of M

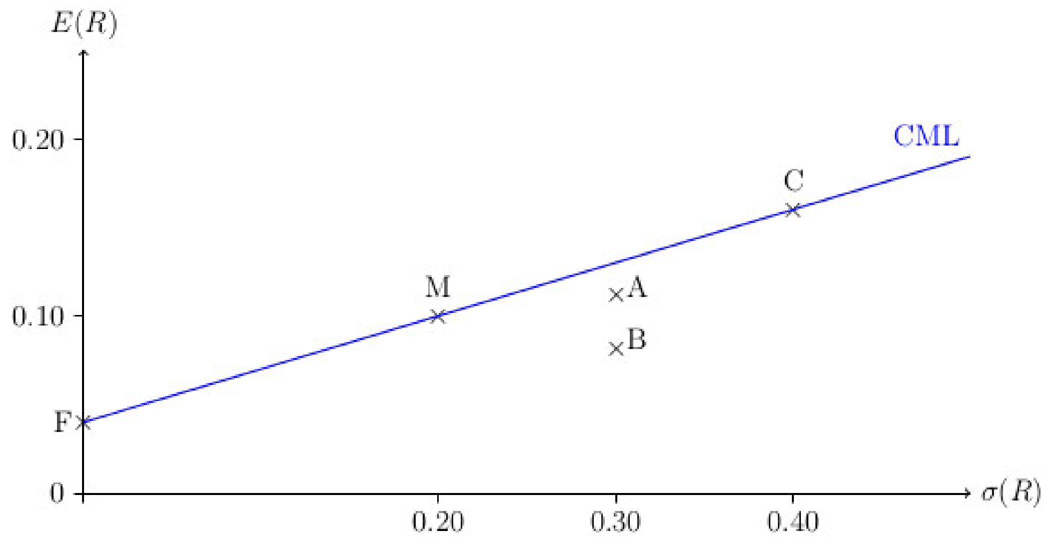
$$SR_M = \frac{0.10 - 0.04}{0.20} = 0.30$$

since M is efficient. To determine whether A, B, or C are efficient, we compute their Sharpe ratios and compare them with the Sharpe ratio of M:

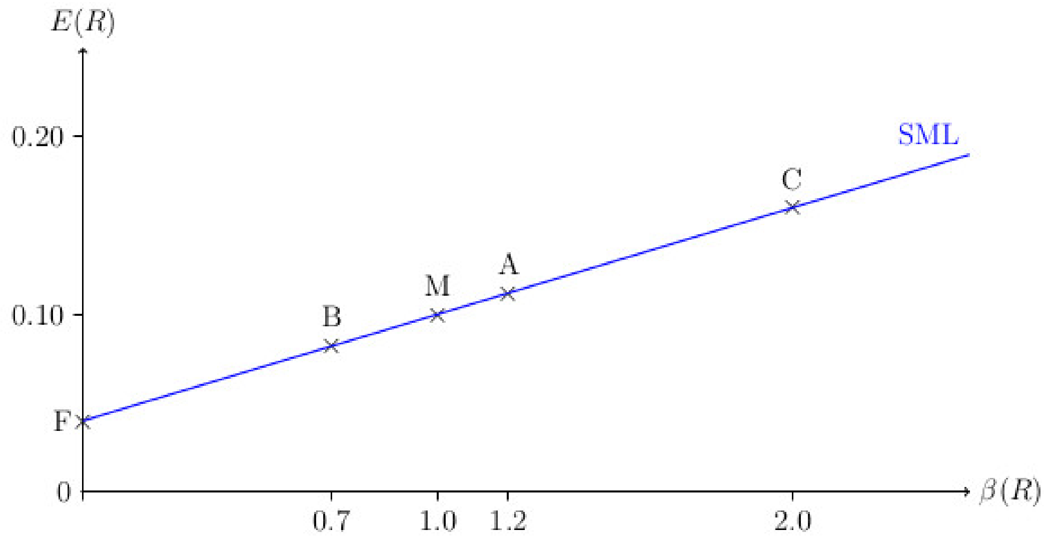
$$\begin{aligned}
SR_A &= \frac{0.112 - 0.04}{0.3} = 0.24 < SR_M \\
SR_B &= \frac{0.082 - 0.04}{0.3} = 0.14 < SR_M \\
SR_C &= \frac{0.16 - 0.04}{0.4} = 0.30 = SR_M
\end{aligned}$$

Hence, only C is efficient whereas A and B are inefficient.

3. The CML is as follows:



4. The SML is as follows:



5. In this problem there are 3 efficient portfolios: F, M, and C. Therefore, we can obtain any efficient portfolio by combining two of them.

   Method 1: Using F and M.

   If we denote by w the weight in M, then we should have:

   $$(1-w)\beta_F + w\beta_M = 1.5$$

   Since $\beta_F = 0$ and $\beta_M = 1$, we find that $w = 1.5$. Hence, to obtain an efficient portfolio with a beta of 1.5 we need to borrow 50 % and invest 150% in M.

   Method 2: Using F and C.

   If we denote by w the weight in C, then we should have:

   $$(1-w)\beta_F + w\beta_C = 1.5$$

   Since $\beta_F = 0$ and $\beta_C = 2$, we find that $w = 0.75$. Hence, to obtain an efficient portfolio with a beta of 1.5 we need to invest 25 % in F and 75% in C.

   Method 3: Using M and C.

   If we denote by w the weight in C, then we should have:

   $$(1-w)\beta_M + w\beta_C = 1.5$$

   Since $\beta_M = 1$ and $\beta_C = 2$, we find that $w = 0.5$. Hence, to obtain an efficient portfolio with a beta of 1.5 we need to invest 50 % in M and 50% in C.

Expected return and standard deviation: Regardless of the method used to build the portfolio, the expected return of this portfolio is

$$E(R_P) = 0.04 + 1.5 \cdot (0.10 - 0.04) = 0.13$$

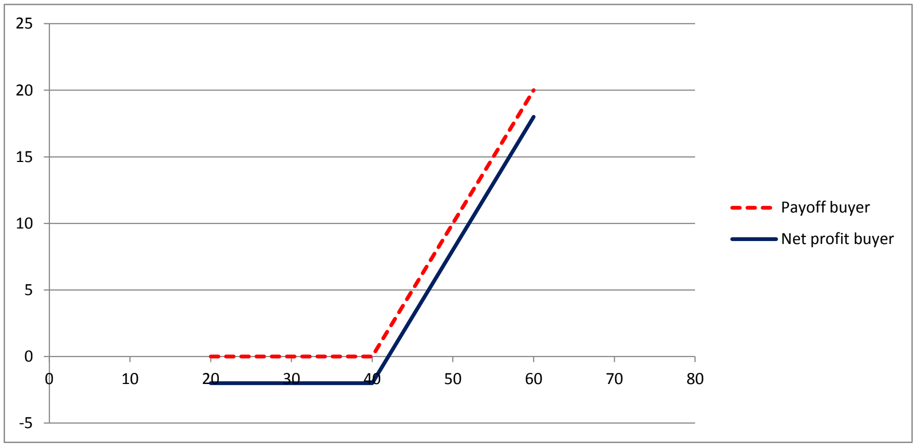
and its standard deviation

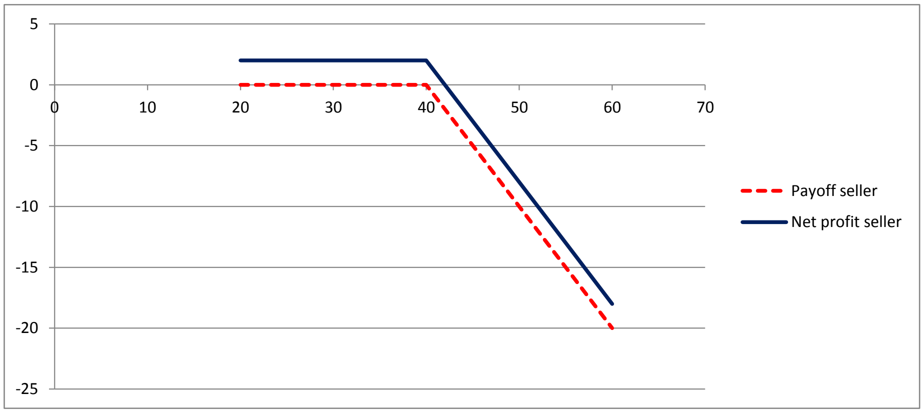
$$\sigma(R_P) = 1.5 \cdot 0.20 = 0.30$$

6 Options Valuation

Question 1

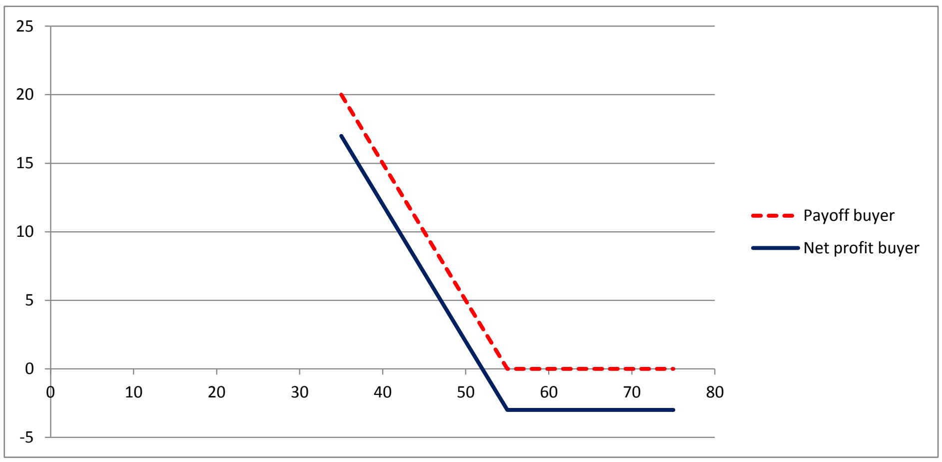
	Call	Buyer Price	Buyer Payoff	Buyer Net Profit	Seller Price	Seller Payoff	Seller Net profit
Strike	40	20	0	-2	20	0	2
Premium	2	25	0	-2	25	0	2
		30	0	-2	30	0	2
		35	0	-2	35	0	2
		40	0	-2	40	0	2
		45	5	3	45	-5	-3
		50	10	8	50	-10	-8
		55	15	13	55	-15	-13
		60	20	18	60	-20	-18

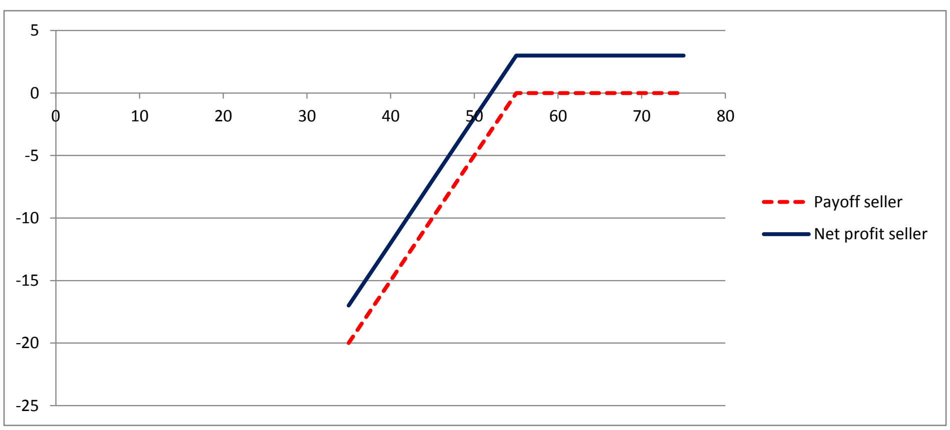




Question 2

	Call	Buyer			Seller		
		Price	Payoff	Net Profit	Price	Payoff	Net Profit
Strike	55	35	20	17	35	-20	-17
Premium	3	40	15	12	40	-15	-12
		45	10	7	45	-10	-7
		50	5	2	50	-5	-2
		55	0	-3	55	0	3
		60	0	-3	60	0	3
		65	0	-3	65	0	3
		70	0	-3	70	0	3
		75	0	-3	75	0	3

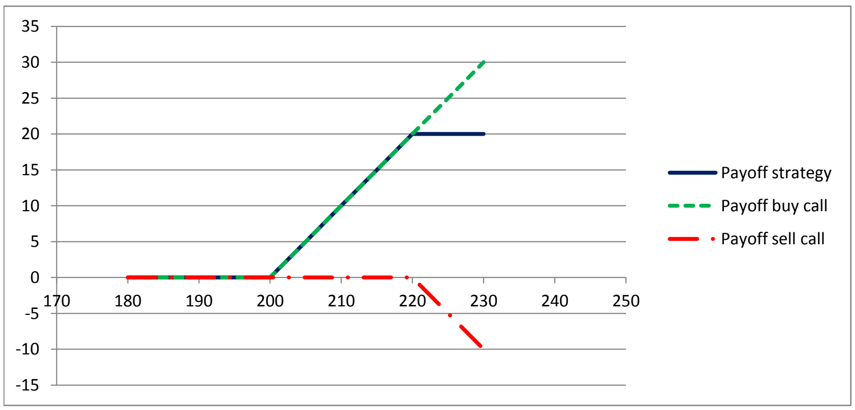


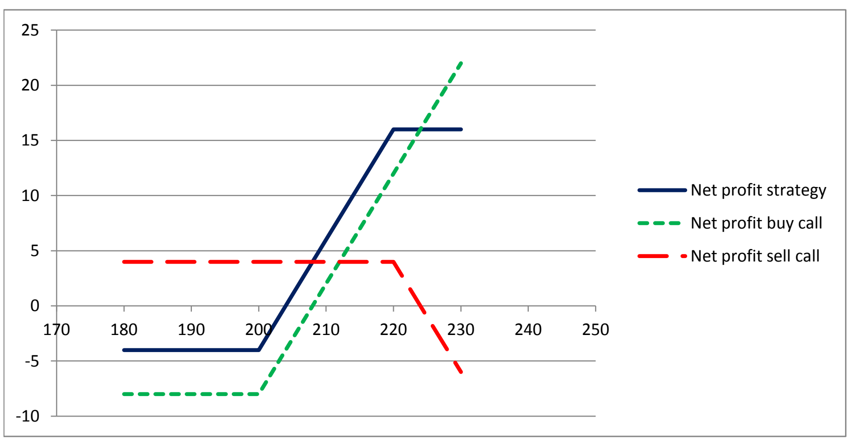


Question 3

	Buy call	Sell call
Prix d'exercice	200	220
Prime	8	4

Payoff and net profit						
	Buy call		Sell call		Strategy	
Price	Payoff	Net profit	Payoff	Net profit	Payoff	Net profit
180	0	-8	0	4	0	-4
185	0	-8	0	4	0	-4
190	0	-8	0	4	0	-4
195	0	-8	0	4	0	-4
200	0	-8	0	4	0	-4
205	5	-3	0	4	5	1
210	10	2	0	4	10	6
215	15	7	0	4	15	11
220	20	12	0	4	20	16
225	25	17	-5	-1	20	16
230	30	22	-10	-6	20	16





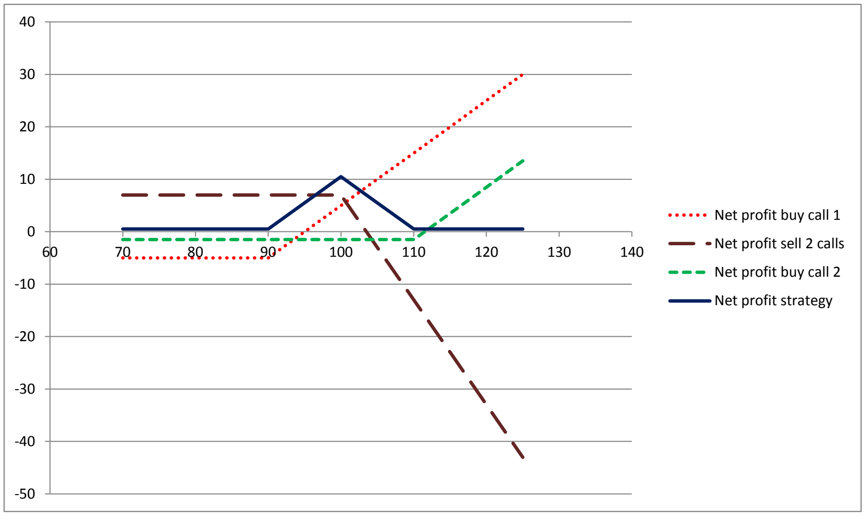
Assistantship IV

6. Options Valuation (Cont')

Question 4

	Buy call	Sell 2 calls	Buy call
Strike	90	100	110
Premium	5	3,5	1,5

Payof and net profit								
	Buy call		Sell 2 calls		Buy call		Strategy	
Price	Payoff	Net profit	Payoff	Net profit	Payoff	Net profit	Payoff	Net profit
70	0	-5	0	7	0	-1,5	0	0,5
75	0	-5	0	7	0	-1,5	0	0,5
80	0	-5	0	7	0	-1,5	0	0,5
85	0	-5	0	7	0	-1,5	0	0,5
90	0	-5	0	7	0	-1,5	0	0,5
95	5	0	0	7	0	-1,5	5	5,5
100	10	5	0	7	0	-1,5	10	10,5
105	15	10	-10	-3	0	-1,5	5	5,5
110	20	15	-20	-13	0	-1,5	0	0,5
115	25	20	-30	-23	5	3,5	0	0,5
120	30	25	-40	-33	10	8,5	0	0,5
125	35	30	-50	-43	15	13,5	0	0,5

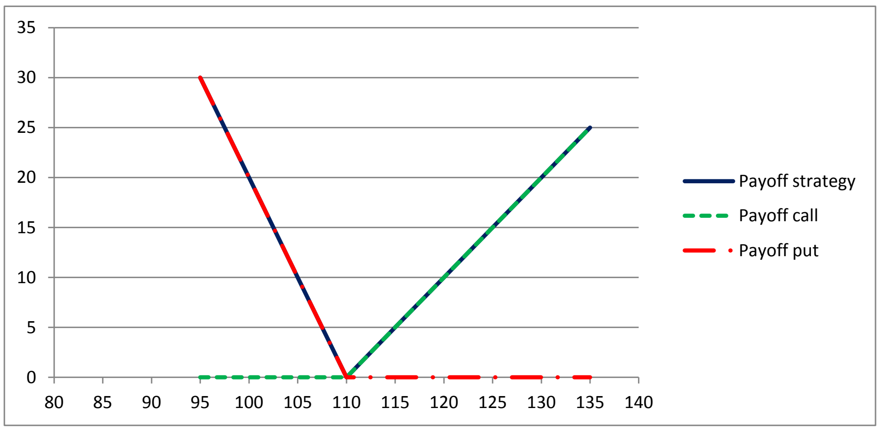


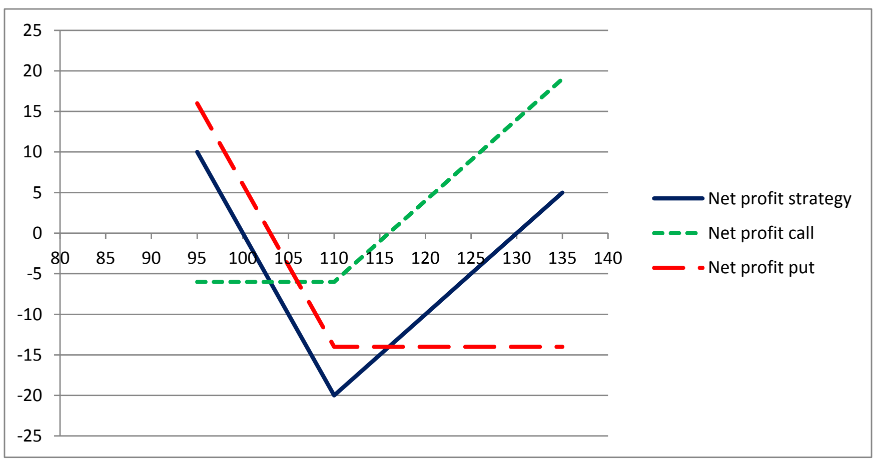
Question 5

	Call	Put
Strike	110	110
Premium	6	7

Payoff and Net profit

	Call		2 Puts		Strategy	
Price	Payoff	Net profit	Payoff	Net profit	Payoff	Net profit
95	0	-6	30	16	30	10
100	0	-6	20	6	20	0
105	0	-6	10	-4	10	-10
110	0	-6	0	-14	0	-20
115	5	-1	0	-14	5	-15
120	10	4	0	-14	10	-10
125	15	9	0	-14	15	-5
130	20	14	0	-14	20	0
135	25	19	0	-14	25	5





Question 6

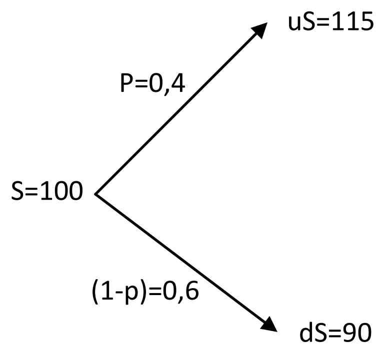
a) et b)

Elements of answer.

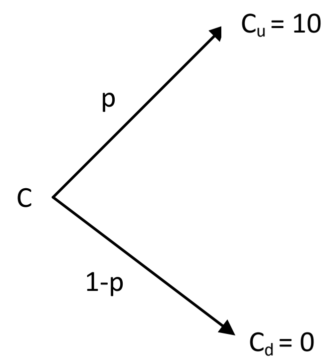
- $uS = (1 + U)S = 1.15 \times 100 = 115 \quad \text{with} \quad p = 0.4$
- $dS = (1 + d)S = 0.90 \times 100 = 90 \quad \text{with} \quad p = 0.6$
- $C_u = Max[0; uS - K] = Max[0; 10] = 10$
- $C_d = Max[0; dS - K] = Max[0; -15] = 0$

Binomial Tree: 1 period

Stock



Call



c) Replication portfolio

$$\Delta^* = \frac{C_u - C_d}{uS - dS} = \frac{10}{25} = 0.4$$

$$B^* = \frac{C_u - \Delta^* uS}{1 + r_f} = \frac{10 - (0.4)115}{1.05} = -34.29$$

The replication portfolio consists in:

- buying 0.4 shares of the underlying stock.
- borrowing 34.29.

d) "Risk-neutral" probabilities

$$q = \frac{(1 + r_f) - d}{u - d} = \frac{1.05 - 0.9}{1.15 - 0.9} = 0.6 \qquad \rightarrow \qquad (1 - q) = 0.4$$

e) Call premium today. 2 equivalent methods

Method 1: Present value of the expected payoffs of the call discounted at the risk-free rate and calculated on the basis of the risk-neutral probabilities.

$$C = \frac{qC_u + (1 - q)C_d}{1 + r_f} = \frac{0.6(10) + 0.4(0)}{1.05} = 5.71$$

Method 2: Value of the replication portfolio.

$$C = \Delta^* S + B^* = 0.4(100) - 34.29 = 5.71$$

7 Capital Structure Theory

Question 1

a) $\tau = 0$

According to Modigliani & Miller with perfect capital markets : $V^u = V^l$. Thus,

$V^u = V_e^u = 900$

$V^l = V_e^l + V_d^l \quad \Longrightarrow \quad V_e^l = 900 - 400 = 500$

We know that $E(r_e)^u = 0,18$. In addition,

$$V^u = \frac{E(F)^u}{E(r_e)^u} \quad \Longrightarrow \quad E(F)^u = 0,18 \times 900 = 162$$

Hence, if U and L have the same value, they must generate the same cash flow F

$\Longrightarrow E(F)^l = 162$

On the basis of M& M Proposition II:

$$\begin{aligned} E(r_e)^l &= E(r_0) + \frac{V_d}{V_e}\left(E(r_0) - r_d\right) \qquad \text{with} \qquad E(r_0) = E(r_e)^u \\ &= 0,18 + \frac{400}{500}\left(0,18 - 0,12\right) \\ E(r_e)^l &= 0,228 \end{aligned}$$

In addition,

$WACC^u = E(r_e)^u = 0,18$

$$WACC^l = \frac{400}{900}\ (0,12) + \frac{500}{900}\ (0,228) = 0,18$$

$\Longrightarrow \quad WACC^u = WACC^l$

b)

$V^l = V^u + \tau V_d^l = 900 + 0,5\,(400)$

$\Longrightarrow \quad V^l = 1\ 100$

$\Longrightarrow \quad V_e^l = V^l - V_d^l = 700$

According to M&M Proposition II:

$$\begin{aligned} E(r_e)^l &= E(r_0) + \frac{V_d}{V_e}\left(E(r_0) - r_d\right)(1-\tau) \\ &= 0,18 + \frac{400}{700}(0,18 - 0,12)(0,5) \\ E(r_e)^l &= 0,1971 \end{aligned}$$

$$\begin{aligned} WACC^l &= E(r_0)\left(1 - \tau \frac{V_d}{V_d + V_e}\right) \quad \text{with} \quad E(r_0) = E(r_e)^u \\ &= 0,18\left(1 - 0,5 \times \frac{400}{1\ 100}\right) \\ WACC^l &= 0,1472 \end{aligned}$$

Alternatively,

$$\begin{aligned} WACC^l &= \frac{400}{1\ 100}(0,12)(0,5) + \frac{700}{1\ 100}(0,1971) \\ &= 0,1472 \end{aligned}$$

Question 2

The operational cash flows of Starlight are equal to: $EBIT(1-\tau) = 1\ 000\ 000 \times 0,55 = 550\ 000$.

a) The value of equity of Starlight with no debt is equal to the number of shares times the price per share

$$V_e^u = 100\ 000 \times 343,75 = 3\ 437\ 500$$

Hence, we can write,

$$E(r_e)^u = \frac{EBIT(1-\tau)}{V_e^u} = \frac{550\ 000}{3\ 437\ 500} = 0,16$$

Since the firm has no debt, the $WACC = E(r_e)^u = 0,16$

b) According to M&M (with perpetuity),

$$\begin{aligned}
V^l &= V^u + \text{VA(tax savings on interests)} \\
&= V^u + \frac{\tau\ r\ V_d}{r} \\
&= V^u + \tau\ V_d \\
&= 3\ 437\ 500 + (0,45 \times 2\ 000\ 000) \\
V^l &= 4\ 337\ 500
\end{aligned}$$

Thus,

$$V_e^l = V^l - V_d = 2\ 337\ 500$$

We can calculate the cost of equity using the M&M definition (Prop. II):

$$\begin{aligned}
E(r_e)^l &= E(r_0) + \frac{V_d}{V_e}[E(r_0) - r](1-\tau) \\
&= 0,16 + \frac{2\ 000\ 000}{2\ 337\ 500}[0,16 - 0,10](0,55) \\
&= 0,1882
\end{aligned}$$

or on the basis of the cash flows to shareholders:

$$\begin{aligned}
E(r_e)^l &= \frac{EBIT(1-\tau) + \tau\ rV_d - rV_d}{V_e^u} \\
&= \frac{550\ 000 + (0,45 \times 2\ 000\ 000) - (0,1 \times 2\ 000\ 000)}{2\ 337\ 500} \\
&= 0,1882
\end{aligned}$$

The WACC can be calculated using the M&M definition:

$$WACC = E(r_0)\left[1 - \tau\frac{V_d}{V}\right] = 0,16\left[1 - 0,45\ \frac{2\ 000\ 000}{4\ 337\ 500}\right] = 0,1268$$

or alternatively by using the classic definition

$$\begin{aligned}
WACC &= \frac{V_d}{V}r(1-\tau) + \frac{V_e}{V}E(r_e) \\
&= \frac{2\ 000\ 000}{4\ 337\ 500}\ (0,1)(0,55) + \frac{2\ 337\ 500}{4\ 337\ 500}\ (0,1882) \\
&= 0,1268
\end{aligned}$$

c) In order to calculate the new price per share, we can use the fact leverage generates benefits to shareholders in the form of tax savings on interests. Therefore, the benefit per share is equal to the total value of the tax shield divided by the number of shares: $900\ 000/10\ 000 = 90$.

The value of a share increases to: $343,75 + 90 = 433,75$.

The number of shares repurchased is equal to 2 000 000/433, 75 = 4 611

There are then 5 389 shares on the market at a price of 433.75 for a total market value of equity of 2 337 500.

d) Let $E(r_m) = 0,15$

i) Before the change in capital structure, Starlight had no debt. Hence,

$$\begin{aligned} E(r_e)^u &= r_f + \beta^u[E(r_m) - r_f] \\ 0,16 &= 0,10 + \beta^u[0,05] \\ \beta^u &= 1,2 \end{aligned}$$

ii) With the new capital strucutre;

$$\begin{aligned} E(r_e)^l &= r_f + \beta^l[E(r_m) - r_f] \\ 0,1882 &= 0,10 + \beta^l[0,05] \\ \beta^l &= 1,764 \end{aligned}$$

or alternatively

$$\begin{aligned} \beta^l &= \left[1 + (1-\tau)\frac{V_d}{V_e}\right]\beta^u \\ &= \left[1 + (0,55)\frac{2}{2,3375}\right]1,2 \\ \beta^l &= 1,764 \end{aligned}$$

Question 3

Information

- E(EBITDA) = 960 for X and Y at t=1
- $WACC^x = 0,2$
- $\tau = 0$
- $V_d^x/V^x = 0,5$
- $V_d^y/V^y = 0,6$
- $V^y = 1,05V^x$

a) The value of a firm is equal to the present value of the casf flows to the firm. We can then write:

$$V^x = \frac{E(EBITDA)}{1 + WACC^x} = \frac{960}{1,2} = 800$$

We know that $V_d^x = 0,5V^x = 400$.

$$V_d^x = 0,5V^x = 400$$

This implies that:

$$V_e^x = V^x - V_d^x = 400$$

According to the information, $V^y = 1,05V^x = 840$. In addition,

$$V_d^y = 0,6V^y = 504$$

This implies that:

$$V_e^y = V^y - V_d^y = 336$$

b) Based on M&M without taxes, we should have the following relationship: $V^x = V^y$. As we can see, the value of firm Y is higher than the value of firm X for the same cash flows. Hence, firm Y is overvalued by the market. Therefore, we can construct an arbitrage strategy which allows us to exploit this error in valuation.

Strategy	CF (M) $t = 0$	CF (M) $t = 1$
Buy shares of X	- 400	$\tilde{F} - V_d^x(1 + r)$ $= 960 - 400(1 + r)$
Sell shares of Y	336	$-[\tilde{F} - V_d^y(1 + r)]$ $= -960 + 504(1 + r)$
Borrow amount equal to 104	104	$104(1 + r)$
Net CF	40	0

c) If X is correctly valued, that is $V^x = 800$, we should have $V^y = 800$. Given that we assume that the debt of Y is correctly valued, that is $V_d^y = 504$, then $V_e^y = V^y - V_d^y = 296$.

Question 4

a) Let $\tilde{R}$ be the net operational cash flow to RGB in one year. The certainty equivalent (CE) is equal to:

$$E(\tilde{R}) - \frac{cov(\tilde{R}, r_m)}{\sigma_m^2}[E(r_m) - r_f]$$

We know that,

$$\begin{aligned} E\left(\tilde{R}\right) &= \sum_s p_s\ R_s \\ &= 0,2(1M) + 0,2(5M) + 0,4(8M) + 0,2(10M) \\ &= 6\ 400\ 000 \end{aligned}$$

$$cov(\tilde{R}, r_m) = \sum_s\ p_s(R_s - E(\tilde{R}))(r_{m,s} - E(r_m)) = 884\ 000$$

$$E(r_m) = \sum_s p_s\ r_{m,s} = 0,2(-0,3) + 0,2(0) + 0,4(0,2) + 0,2(0,6) = 0,14$$

$$\sigma_m^2 = \sum_s p_s\ (r_{m,s} - E(r_m))^2 = 0,0864$$

Thus,

$$CE = 6\ 400\ 000 - \frac{884\ 000}{0,0864}(0,14 - 0,06) = 5\ 581\ 482$$

The present value of $\tilde{R}$ is equal to its CE discounted at the risk free rate onver one year.

$$V = \frac{5\ 581\ 482}{1,06} = 5\ 265\ 549 = V^u$$

The price of one share is then equal to V divided by the number of shares:

$$P = \frac{5\ 265\ 549}{2\ 000\ 000} = 2,633$$

Note: the cost of capital of RGB is equal to the cost of equity since the firm has no debt.

On an annuity basis, we can write:

$$E(r_e) = \frac{E(\tilde{R}) - V}{V} = \frac{6\ 400\ 000 - 5\ 265\ 549}{5\ 265\ 549} = 21,53\%$$

Thus,

$$\beta_e^u = \frac{E(r_e) - r_f}{E(r_m) - r_f} = \frac{0,2153 - 0,06}{0,14 - 0,06} = 1,94$$

b) Loan = 3 000 000

Note: One should note that some of the classic M&M formulas do not apply in this one year period setting since they rely on the perpetuity assumption.

The tax savings on interests are equal to:

$$\tau\ r_f\ V_d = 0,4 \times 0,06 \times 3\ 000\ 000 = 72\ 000$$

The present value of tax savings over one year is equal to:

$$\text{PV(Tax savings)} = \frac{72\ 000}{1,06} = 67\ 925$$

We can write,

$$V^l = V^u + \text{PV(Tax savings)} = 5\ 265\ 549 + 67\ 925 = 5\ 333\ 473$$

Thus,

$$V_e^l = V^l - V_d^l = 5\ 333\ 473 - 3\ 000\ 000 = 2\ 333\ 473$$

The increase in the price of one share following the loan is equal to the PV(Tax savings) per share:

$$\Delta P = \frac{67\ 925}{2\ 000\ 000} = 0,034$$

The new share price before the repurchase of the shares is:

$$P' = P_0 + \Delta P = 2,633 + 0,034 = 2,667$$

The number of shares repurchased is equal to the amount of the loan divided by the new share price:

$$\text{Nbr shares repurchased} = \frac{3\ 000\ 000}{2,667} = 1\ 124\ 971$$

The number of shares left on the market is: $2\ 000\ 000 - 1\ 124\ 971 = 875\ 029$

The value of equity after the share repurchase is: $875\ 029 \times 2,667 = 2\ 333\ 473$

As we have seen, the relationship defined in class between β^l_{fp} et β^u does not anymore since they rely on the perpetuity assumption. However, we have an alternative relationship in the one period case.

$$\begin{aligned}
\beta^l_e &= \left[1 + \left(1 - \frac{\tau\ r_f}{(1+r_f)}\right)\frac{V_d}{V_e}\right]\beta^u \\
&= \left[1 + \left(1 - \frac{0,4(0,06)}{(1,06)}\right)\frac{3\ 000\ 000}{2\ 333\ 473}\right]1,94 \\
\beta^l_e &= 4,38
\end{aligned}$$

Cost of equity: 2 methods

i) CAPM

$$\begin{aligned}
E(r_e) &= r_f + \beta_e\left[E(r_m) - r_f\right] \\
&= 0,06 + 4,38(0,08) \\
&= 0,410
\end{aligned}$$

ii) Cash flows to equity

We can write that the value of equity is equal to the present value of cash flows to equity-holders:

$$V_e = \frac{E(CF_e)}{(1+E(r_e))} \qquad \Longrightarrow \qquad E(r_e) = \frac{E(FM_e)}{V_e} - 1$$

The cash flows to equityholders in one year are equal to the net operational cash flows plus les tax savings on interests associated to the loan minus the repayment of the debt and the interests.

$$\begin{aligned} E(CF_e) &= E(\tilde{R}) + \text{Tax savings} - V_d(1 + r_f) \\ &= 6\ 400\ 000 + 72\ 000 - 3\ 000\ 000(1,06) \\ &= 3\ 292\ 000 \end{aligned}$$

Thus,

$$E(r_e) = \frac{3\ 292\ 000}{2\ 333\ 473} - 1 = 41,08\%$$

Using this result and the CAPM, we can verify the calculation of the β_e^l

$$\beta_e^l = \frac{E(r_e) - r_f}{E(r_m) - r_f} = \frac{0,4108 - 0,06}{0,14 - 0,06} = 4,38$$

Cost of capital: 2 methods:

i) CAPM

$$\begin{aligned} WACC &= \frac{V_d}{V}\, r_f(1 - \tau) + \frac{V_e}{V}\, E(r_e) \\ &= \frac{3\ 000\ 000}{5\ 333\ 473}(0,06)(0,6) + \frac{2\ 333\ 473}{5\ 333\ 473}(0,4108) \\ WACC &= 20,0\% \end{aligned}$$

ii) Cash flows to the firm

We know that the value of a firm is equal to the present value of cash flows to the firm:

$$V = \frac{E(CF)}{(1 + WACC)} \quad \Longrightarrow \quad WACC = \frac{E(CF)}{V} - 1$$

But the cash flows to the firm in one year is equal to the net operational cash flows: $E(\tilde{R}) = 6\ 400\ 000$.

Thus,

$$WACC = \frac{6\ 400\ 000}{5\ 333\ 473} - 1 = 20,0\%$$

Question 5

a) The Free Cash Flows for UFO are:

Flows	t = 0	t = 1 - ∞
Acquisition machines + WCR + plant	- 100	
Annual maintenance		- 10
EBITDA after tax		20
Tax savings on depreciation		5
Free Cash Flows	- 100	15

The IRR of this investment is given by:

$$0 = -100 + \frac{15}{IRR} \qquad \Longrightarrow \qquad IRR = 15\%$$

b) The cash flows to equityholders can be calculated using the fundamental indentity in financial theory. Indeed, we know that

$$A_t + C_t = F_t + Ec_t$$

where

A_t: cash flows to equityholders
C_t: cash flows to creditors
F_t: real cash flows from investments
Ec_t: tax savings from debt financing (interests tax shield)

We can then write that,

$$A_t = F_t + Ec_t - C_t$$

Calculation of cash flows:

Flows	t = 0	t = 1 - ∞
F_t	- 100	15
Ec_t	0	$1,5^{*}$
C_t	75	-3^{**}
A_t	- 25	13,5

$*$: Ec_t for $t > 0 = 0,5 \times 3 =$ 1,5
$**$: C_t for $t > 0 = +75$ (repayment of debt) + 3 (interests on debt) - 75 (new debt)

c) The market value of the shares is equal to the present value of the equity cash flows:

$$V = \sum_{t=1}^{\infty} \frac{13,5}{(1,135)^t} = \frac{13,5}{0,135} = 100$$

The increase in wealth for equityholders is equal to the difference between their initial investment and the value of equity: -25 + 100 = 75

The maximum rate of return for equityholders over which the project is not profitable is such that they recover only their initial investment:

$$25 = \frac{13,5}{k} \qquad \Longrightarrow \qquad k = 54\%$$

Question 6

Valuation
Company J

	N-1	N	N+1	N+2	N+3	N+4	TV
Sales growth rate (%)							4%
Sales	6 000,00	10 000,00	11 500,00	13 000,00	17 995,00	23 393,00	
EBIT margin	0,04	0,08	0,10	0,10	0,11	0,11	
EBIT	240,00	800,00	1 150,00	1 300,00	1 889,48	2 573,23	
Taxes	60,00	200,00	287,50	325,00	472,37	643,31	
EBIT after tax	180,00	600,00	862,50	975,00	1 417,11	1 929,92	
+ Depreciation	67,00	101,00	131,00	165,00	198,00	229,00	
- CAPEX	100,00	120,00	125,00	160,00	210,00	275,00	
- Variation WCR	-309,00	59,00	97,00	154,00	100,00	162,00	
Firm Cash Flows (FCF)	456,00	522,00	771,50	826,00	1 305,11	1 721,92	
Tax rate (%)	0,25	0,25	0,25	0,25	0,25	0,25	0,25
Debt in capital structure (%)	20%	20%	20%	20%	20%	20%	25%
Debt / Equity	0,2500	0,2500	0,2500	0,2500	0,2500	0,2500	0,3333
Equity Beta (no debt)	1,26	1,26	1,26	1,26	1,26	1,26	1,26
Equity Beta (with dette)	1,50	1,50	1,50	1,50	1,50	1,50	1,58
Treasury Bill Rate	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%	5,0%
Market premium	6,0%	6,0%	6,0%	6,0%	6,0%	6,0%	6,0%
Cost of equity	14,00%	14,00%	14,00%	14,00%	14,00%	14,00%	14,47%
Net cost of debt	3,8%	3,8%	3,8%	3,8%	3,8%	3,8%	3,8%
Cost of capital	11,95%	11,95%	11,95%	11,95%	11,95%	11,95%	11,79%
Present value of FCF		466,28	615,58	588,72	830,90	979,25	
Terminal value							22 980,29
PV (terminal value)							13 160,94

Enterprise value	16 641,68