

# 1 Finding $k$ -chords in half graphs

The aim of this section is to prove the following proposition.

**Proposition 1.** For any  $k \neq 1, 4$ , there exists  $n \in \mathbb{N}$  such that the half graph  $H_n$  contains an induced  $k$ -chord.

Before tackling the above statement, we notice the following fact.

**Fact 2.** For any graph  $G$  and cycle  $C$  in  $G$ , the number of chords in  $C$  is given by  $e(G[V(C)]) - |C|$ .

We now establish some notation. For any half graph  $H_n$  with  $n \geq 3$ , label its vertices as follows.

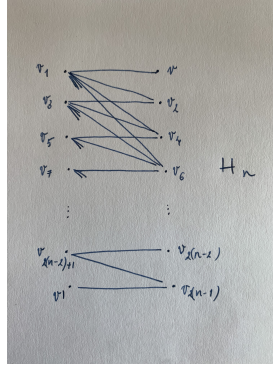


Figure 1: Vertex labeling of a half graph

Notice that the cycle  $(v_1, v_2, \dots, v_{2(n-1)}, v_1)$  induces a  $k$ -chord  $C_n$  in  $H_n$  with the number  $K_n$  of chords being (by Fact 2):

$$K_n = (2 + 3 + \dots + (n-1) + (n-1)) - 2(n-1) = \frac{n(n-3)}{2}.$$

We now notice the following fact which will be useful in the next proofs.

**Fact 3.** Given a  $K_n$ -chord  $C_n$ , one can obtain a new  $k$ -chord by removing vertices  $v_{2n_1}, v_{2n_2}, \dots, v_{2n_m}, v_{2n'_1+1}, v_{2n'_2+1}, \dots, v_{2n'_m+1}$  such that  $n_1 < n_2 < \dots < n_m < n'_1 < n'_2 < \dots, n'_m$ . This last condition is necessary to ensure that the vertices in  $v(C_n) \setminus \{v_{2n_1}, \dots, v_{2n_m}, v_{2n'_1+1}, \dots, v_{2n'_m+1}\}$  are part of

a unique cycle. The cycle (and the relative induced  $k$ -chord) we take into consideration is

$$(v_{2i_1+1}, v_{2j_1}, \dots, v_{2i_{n-m}+1}, v_{2j_{n-m}}, v_{2i_1+1}),$$

where  $\{i_1, \dots, i_{n-m}\} = \{1, \dots, n\} \setminus \{n'_1, \dots, n'_m\}$  and  $\{j_1, \dots, j_{n-m}\} = \{1, \dots, n\} \setminus \{n_1, \dots, n_m\}$  with  $i_1 < \dots < i_{n-m}$  and  $j_1 < \dots < j_{n-m}$ .

Now, notice that for any  $n \geq 4$ , we have that  $K_n - K_{n-1} = n - 2$ . Our strategy is to find an induced  $k$ -chord into  $H_n$  with  $K_{n-1} \leq k \leq K_n$  for all  $k \neq 1, 4$ . The following lemma represents a step towards this objective.

**Lemma 4.** For any  $n \geq 4$  and any  $2 \leq l \leq n - 2$ , we have that  $C_n$  contains an induced  $(K_n - l)$ -chord (and thus, so does  $H_n$ ).

*Proof.* Let  $q = l - 1$ . Notice that  $C_n$  has the following cycle, obtained from  $C_n$  by removing  $v_{2q}$  and  $v_{2(n-2)+1}$ . We have:

$$C'_n := (v_1, v_2, \dots, v_{2(q-1)+1}, v_{2(q+1)}, \dots, v_{2(n-2)}, v_{2(n-3)+1}, v_{2(n-1)}, v_1)$$

By Fact 3,  $C'_n$  induces a  $k$ -chord. Notice that  $d(v_{2q}) = q + 1 = l$  and  $d(v_{2(n-2)+1}) = 2$ . Since  $|C'_n| = |C_n| - 2$ , Fact 2 tells us that the number of chords of  $C'_n$  is  $K_n - (l - 2) - 2 = K_n - l$ .

This concludes the proof.  $\square$

Now, notice that  $K_4 = 2$  and  $K_5 = 5$ . Thus, in order to prove Proposition 1, it suffices to find  $n$  such that  $H_n$  contains an induced  $(K_m - 1)$ -chord for all  $m \geq 6$ . We claim that such  $n$  is  $m + 1$ .

**Lemma 5.** For any  $m \geq 6$ , we have that  $C_{m+1}$  contains an induced  $(K_m - 1)$ -chord (and thus, so does  $H_{m+1}$ ).

*Proof.* Consider the  $K_{m+1}$ -chord  $C_{m+1}$ . Just as in the proof of Lemma 4, consider the  $k$ -chord  $C''_{m+1}$  obtained from  $C_{m+1}$  by removing the vertices  $v_2, v_{2(n-5)}, v_{2(n-3)+1}, v_{2(n-2)+1}$ . The cycle of  $C''_{m+1}$  that we will consider is the one described in Fact 3. Notice that, since  $m \geq 6$ , we have that all of the above vertices are distinct. Moreover, we have  $d(v_2) = 2$ ,  $d(v_{2(n-5)}) = n - 4$ ,  $d(v_{2(n-3)+1}) = 3$ ,  $d(v_{2(n-2)+1}) = 2$ . By Fact 2, we deduce that the number of chords in  $C''_{m+1}$  is

$$K_{m+1} - (2 + (n - 4) + 3 + 2) - 4 = K_{m+1} - (n - 1) = K_m - 1.$$

This concludes the proof.  $\square$

As pointed out above, Lemma 4 and 5 imply Proposition 1.