1 Coloring circle graphs

WARNING: intervals h_1, h_2 overlap if $h_1 \cap h_2 \neq \emptyset$, $h_1 \not\subseteq h_2$ and $h_2 \not\subseteq h_1$.

We look at the class of circle graphs as the class of overlap graphs of intervals on a line. Without loss of generality, we only consider the interval models containing open intervals in which no two intervals share an endpoint.

For any such family of intervals F and any point c on a line, we set $F^-(c) := \{(a,b) \in F | b < c\}$, $F^0(c) := \{(a,b) | a < c < b\}$, $F^+(c) := \{(a,b) | c < a\}$. If $\omega(F) = 2$, for all $(a,b) \in F$, we have that $F^0(a) \setminus F^0(b)$ and $F^0(b) \setminus F^0(a)$ only contain nested intervals. Otherwise, if we had two overlapping intervals $h_1, h_2 \in F^0(a) \setminus F^0(b)$, we would have $h_1, h_2, (a,b)$ forming a triangle.

Lemma 1. Let F be the interval model of a circle graph. Suppose that $\omega(F) = 2$ and that for no two $h_1, h_2 \in F$, there exists $h_3 \in F$ with $h_3 \in h_1 \cap h_2$. Then, there exists a 3-coloring of F such that for all $(a,b) \in F$, the intervals in $F^0(b) \setminus F^0(a)$ have the same color.

Proof. By way of contradiction, suppose that the statement is incorrect and let $F = \{h_i\}_{i=1}^n = \{(a_i, b_i)\}_{i=1}^n$ be the counterexample to this lemma with the least cardinality. Clearly, $G(F)$ is connected. Let $(a_1, b_1), \ldots, (a_t, b_t)$, with $a_1 < \ldots < a_t$, be the intervals of F which are not contained in any other intervals. Notice that we have $F^0(a_1) = F^-(a_1) = \emptyset$. Suppose that $|F^0(b_1)| \leq 1$. By the minimality of F, we get that the interval model $F' := F \setminus \{h_1\}$ is 3-colorable. It is clear that coloring h_1 with a different color from h_2, we obtain a coloring for F. Which contradicts the fact that F is a counterexample to this lemma. Therefore, we conclude that $|F^0(b_1)| \geq 2$ and $t \geq 2$.

By the connectedness of $G(F)$, we have $h_i \cap h_{i+1} \neq \emptyset$ for all $i \in \{1, \ldots, t-1\}$. Let us show that for all $i \in \{1, \ldots, t-1\}$ we have $F^-(b_i) = \emptyset$ or $F^+(a_{i+1}) = \emptyset$. Suppose that this does not hold for some $1 \leq i \leq t-1$. Let $F_1 = F \setminus F^+(a_{i+1})$ and $F_2 = F \setminus F^-(b_i)$.

We claim that $F_1 \cap F_2 = \{h_i, h_{i+1}\}$. Indeed, $h_j \in F_1 \cap F_2$, iff $a_j \leq a_{i+1}$ and $b_j \geq b_{i+1}$. Clearly h_i and h_{i+1} have this property. Now, suppose that there exists $h_j \in F_1 \cap F_2$ such that $a_j < a_{i+1}$ and $b_j > b_{i+1}$. We cannot have $a_j < a_i$, because otherwise $h_i \subseteq h_j$. Similarly, we cannot have $b_j > b_{i+1}$, because otherwise, we would have $h_{i+1} \subseteq h_j$. Therefore we have $a_i < a_j$ and $b_j < b_{i+1}$. The only remaining case is that of $a_i < a_j < a_{i+1}$ and $b_i < b_j < b_{i+1}$. But if this is the case, $\{h_i, h_j, h_{i+1}\}$ would form a triangle, which contradicts our assumptions.

In view of the minimality of F, there exist 3-colorings f_1, f_2 of F_1 and F_2 as required by the statement of this lemma with colors 1, 2 and 3 (here we use our assumption that $F^-(b_i)$ and $F^+(a_{i+1})$ are non-empty and thus F_1 and F_2 are strictly smaller than F). Since h_1 and h_2 overlap, they have different colors both in f_1 and f_2, therefore, we can assume without loss of generality that $f_1(h_i) = f_2(h_i) = 1$ and $f_1(h_{i+1}) = f_2(h_{i+1}) = 2$. Let

$$f(h) = \begin{cases} f_1(h) & \text{if } h \in F_1, \\ f_2(h) & \text{if } h \in F_2. \end{cases}$$

We verify that f is a coloring of F. Let $h' = (a', b') \in F$, $h'' = (a'', b'') \in F$ with $a' < a'' < b' < b''$. If $\{h', h''\} \subseteq F_1$ or $\{h', h''\} \subseteq F_2$, then $f(h') \neq f(h'')$. If $h' \in F^-(b_i)$ and $h'' \in F^+(a_{i+1})$ then, by assumption we have $h', h'' \not\subseteq (a_{i+1}, b_i)$, and so we must have $a' < a_{i+1} < a'' < b' < b_i < b''$. Since f_2 is a coloring, we have $f(h'') = f_2(h'') = f_2(h_{i+1}) = 2$. Since h' overlaps with h_{i+1} and $h', h_{i+1} \in F_1$, then $f(h') \neq 2 = f(h'')$. Similarly, we can verify that for all $(a, b) \in F$ all intervals in $F(b) \setminus F(a)$ have the same color.hus f is a 3-coloring of F, which contradicts our choice of F. Thus, we have that $F^-(b_i) = \emptyset$ and $F^+(a_{i+1}) = \emptyset$ for all $i \in \{1, \ldots, t-1\}$.

We have $|F^+(a_2)| \geq |F^0(b_1)| \geq 2$, and, since $F^-(b_1) = \emptyset$, we have $F^0(a_2) = \{h_1\}$.

If $F^0(b_2) = \emptyset$, i.e. $t = 2$, then h_2 only overlaps with h_1. A coloring h_2 with the color of the elements of $F^0(b_1) \setminus \{h_2\}$, we obtain a 3-coloring of F with the conditions required by the present lemma, which is a contradiction.

If $F^0(b_2) \neq \emptyset$ and $t \geq 3$, we obtain (since $h_1 \in F^-(b_2)$) theat $F^+(a_3) = \emptyset$ and thus $t = 3$ and $F^0(b_2) = \{h_3\}$. Consider $F \setminus \{h_2\}$. Since F is minimal, we find that there exists a coloring f of $F \setminus \{h_2\}$ which follows the requirement of the present lemma. Let $h'_2 = (a'_2, b'_2)$ be the longest interval of $F^0(b_1) \setminus \{h_2\}$ (which exists since $|F^0(b_1)| \geq 2$). This way h'_2 contains all the intervals in $F^0(b_1)$. Now, assume that $f(h_1) = 1$ and $f(h'_2) = 2$. If $f(h_3) \neq 2$, then we have a coloring of F with the properties required by the present lemma. Which yields a contradiction. If $f(h_3) = 2$, then h_3 and h'_2 do not overlap. Thus $h_3 \in F^+(b'_2)$. By assumption, all intervals of $F^0(b'_2)$ are colored with the same color $\gamma \in \{1, 3\}$. By recoloring all elements of $F^+(b'_2)$ of color 2 to color $\delta \in \{1, 3\} \setminus \gamma$ and all elements of $F^+(b'_2)$ of color δ with color 2, we obtain a new coloring f' of $F \setminus \{h_2\}$. Since $h_3 \in F^+(b'_2)$, we get $f'(h'_2) = 2$ and $f'(h_3) \neq 2$. By coloring h_2 with color 2, we obtain a coloring of F with the properties required by the present lemma. $\square$

For each pair of overlapping intervals $h_1, h_2 \in F$, let $p(h_1, h_2) = h_1 \cap h_2$. We denote by $P(F)$ the family of such intersections. Let $P^0(F) \subseteq P(F)$ be the inclusion-wise maximal family of $P(F)$.

Lemma 2. Let F be a family of intervals with $\omega(F) = 2$. Then, the intervals of $P^0(F)$ do not intersect.

Proof. Let $p_1 = h_1 \cap h_2 \in P^0(F)$ and $p_2 = h_3 \cap h_4 \in P^0(F)$. Therefore, $(p_1 \cap p_2) = (a_4, b_1)$. So $a_1 < a_2 < a_4 < b_1 < b_3 < b_4$. If $b_2 < b_4$, then h_1, h_2, h_4 pairwise overlap. Thus we have a triangle, which is a contradiction. Thus $b_4 < b_2$. By a symmetric argument we get $a_1 < a_3$. But then $p_2 = (a_4, b_3) \subsetneq (a_2, b_1) = p_1$ which contradicts the maximality of p_2. $\square$

Theorem 3. Let F be a family of intervals with $\omega(F) = 2$. $\chi(F) \leq 5$.

Proof. We construct the desired coloring by induction on k. Let $k = 1$. Let F_1 be the subset of intervals of F that do not lie in the intersection of overlapping intervals of F. By Lemma 1, there exists a coloring f_1 of F_1 with colors $\{1, 2, 3\}$ such that for every $(a, b) \in F_1$, the intervals of the family $F_1^0(b) \setminus F^0(a)$ are colored with the same color. By Lemma 2, the intervals of the family $P^0(F_1)$ do not overlap. By definition of F_1, each interval in $F \setminus F_1$ is contained in some (maximal) intersection $p \in P^0(F_1)$. Consider that the following construction has been done for $k-1$ with $k \geq 2$. We want the following properties to be true.

1. The intervals in $F' := \bigcup_{i=1}^{k-1} F_i$ (where the F_is might not be disjoint) are colored with colors $\{1, \ldots, 5\}$.
2. The intervals in $F \setminus F'$ do not overlap with intervals in $\bigcup_{i=1}^{k-2} F_i \setminus F_{k-1}$.
3. All intervals in $F \setminus F'$ are contained in some interval $p \in P^0\left(F^{k-1}\right)$.
4. For each $p = (c, d) \in P^0(F_{k-1})$ either only one color is used to color all intervals in $F_{k-1}^0(c) \setminus F_{k-1}^0(d)$, and no more than two colors are used to colors intervals of $F_{k-1}^0(d) \setminus F_{k-1}^0(c)$ or viceversa.

Let us show how to carry out the k-th step of this construction. If $F \setminus F' = \emptyset$, then we are done and we have the desired coloring. Consider any interval $p = (c, d) \in P^0(F_{k-1})$, which contains at least one interval of $F \setminus F'$. Let

$$F_{k-1}^0(c) \setminus F_{k-1}^0(d) = \{h_j\}_{j=1}^{t} = \{(c_j, d_j)\}_{j=1}^{t}$$
$$F_{k-1}^0(d) \setminus F_{k-1}^0(c) = \{h_j\}_{j=t+1}^{s} = \{(c_j, d_j)\}_{j=t+1}^{s},$$

and $p = (c_s, d_s) \cap (c_t, d_t) = (c_s, d_t)$. As already noted, each family consists of nested intervals. Without loss of generality, suppose that $\gamma_1 \in \{1, 3\}$ is the color used to color the intervals $h_{t+1}, \ldots, h_s$ and let γ_2, γ_3 be the colors used to color $\{h_1, \ldots h_t\}$.

Let I_p be the set of intervals in F contained in p, the intervals $h_{t+1}, \ldots, h_s$ and the interval $(d, d_s + 1)$. Let $F_{k,p}$ be the set of intervals in I_p which are not contained in the overlap of any two intervals in I_p.

Let us show that the intervals from $I_p \setminus F_{k,p}$ do not overlap with any of the intervals of $F' \setminus \{h_{t+1}, \ldots, h_s\}$. By condition 2 of our induction hypothesis, intervals from $F' \setminus \{h_{t+1}, \ldots, h_s\}$ that can overlap with elements in $I_p \setminus F_{k,p}$ are only intervals in $\{h_1, \ldots, h_t\}$. Let $(a_1, b_1) \in I_p$ be an interval contained in the overlapping of $(a_2, b_2), (a_3, b_3) \in I_p$ and thus overlapping with some $h_j = (c_j, d_j)$ for $1 \le j \le t-1$. Then $a_i > c_s$ for all $i \in \{1, 2, 3\}$. We also get $c_j < c_s$, $a_1 < d_j < b_1$. Hence h_j, (a_2, b_2) and (a_3, b_3) pairwise overlap. Which is a contradiction.

If p contains at least one interval, then $F_{k,p} \setminus (h_{t+1}, \ldots, h_s) \cup \{(d, d_s + 1)\} \neq \emptyset$. By Lemma 1 (replacing $F^0(b) \setminus F^0(a)$ with $F^0(a) \setminus F^0(b)$), we can color $F_{k,p}$ with colors from $\{1, 2, 3, 4, 5\} \setminus \{\gamma_2, \gamma_3\}$ such that for each $(a, b) \in F_{k,p}$, the interval from $F^0_{k,p}(a) \setminus F^0_{k,p}(b)$ have the same color. At the same time, since $\{h_{t+1}, \ldots, h_s\} = F^0_{k,p}(d) \setminus F^0(d_s + 1)$ we can assume that $h_{t+1}, \ldots, h_s$ is colored with (only) γ_1.

Let us denote $F'_{k,p} = F_{k,p} \setminus \{(d, d_s + 1)\}$. It is easy to see that the coloring of $F'_{k,p}$ is compatible with that of F'. Carrying out similar constructions for each $p \in P^0(F_{k-1})$ containing at least one uncolord interval of F, let $F_k := \bigcup_{p \in P^0(F_{k-1})} F'_{k,p}$.

One can check that the induction hypotheses hold for k. Also, the number of uncolored vertices strictly decreseas at each step. Therefore the coloring will be completed in a finite number of iterations. □

2 Coloring polygon circle graphs

We define polygon circle graphs as the set of intersection graphs of polygons (including segments) inscribed in a circle. For the sake of simplicity, we will only consider families of polygons in which no two vertices coincide. The class of circle graphs can be obtained by restricting the choice of polygons to segments.

Like circle graphs, we handle polygon circle graphs by looking at the stereographic projection of the circle onto $\mathbb{R}$. Thus, a polygon circle graph is represented by a family $\{h_i\}_{i=1}^n$ with $\left(a_1^{(i)}, \dots, a_{k_i}^{(i)}\right)$ for some $k_i \geq 1$ with $a_j^{(i)} \in \mathbb{R}$ being the image of the stereographic projection of a vertex of h_i.

Given a familiy $F = \{h_i\}_{i=1}^n$ of polygons, we denote the corresponding intersection graph $G(F)$. By abuse of notation, we will denote $\omega(G(F))$ by $\omega(F)$. Given two polygons $h_i, h_{i'}$ in F, we say that h_i and $h_{i'}$ overlap if there exists indices j and j' such that $\left(a_j^{(i)}, a_{j+1}^{(i)}\right)$ and $\left(a_{j'}^{(i')}, a_{j'+1}^{(i')}\right)$ overlap.

Given a polygon $h := (a_1, \dots, a_k)$, a contraction of h, is a polygon h' of the form $(a_1, \dots, a_{i-1}, a_{i+1}, \dots a_k)$ where indices are unerstood modulo k. Consider a family of polygons $F = \{h_i\}_{i=1}^n$. Let F' be the family of polygons generated by contracting one polygon in F. We define the partial order '$\prec$' on the set of polygon circle graphs to be the order generated by the relations $F' \prec F$. Notice that $\prec$ is well-founded since, if $F' \prec F$, the total number of vertices of polygons in F' is strictly less than that of the polygons in F.

Let $c \in \mathbb{R}$. We establish the following notation.

$$
\begin{aligned}
F^0(c) &:= \left\{ h_i \in F \middle| a_1^{(i)} < c < a_{k_i}^{(i)} \right\}, \\
\widetilde{F}^0\left(a_j^{(i)}\right) &:= \left\{ h_{i'} \in F \middle| a_1^{(i)} < a_{j'}^{(i')} < a_j^{(i)} < a_{j'+1}^{(i')} \text{ for some } 1 \leq j' < k_{i'} \right\}, \\
F^+(c) &:= \left\{ h_i \in F \middle| c < a_1^{(i)} \right\}, \\
F^-(c) &:= \left\{ h_i \in F \middle| a_{k_i}^{(i)} < c \right\}.
\end{aligned}
$$

For any $h_i \in F^0(c)$, denote $I_c(h_i) := \left(a_j^{(i)}, a_{j+1}^{(i)}\right)$ such that $c \in \left(a_j^{(i)}, a_{j+1}^{(i)}\right)$ and $O_c(h_i) := \left(a_1^{(i)}, a_{k_i}^{(i)}\right)$. For any polygon h_i, we call the segment $\left(a_1^{(i)}, a_{k_i}^{(i)}\right)$ the external segment of h_i.

Also, for any two polygons h_i, $h_{i'}$ be two polygons whose external segments overlap with $a_1^{(i)} < a_1^{(i')}$. Let $\left(a_j^{(i)}, a_{j+1}^{(i)}\right) = I_{a_1^{(i')}}(h_i)$ and let $b^{(i,i')} := a_{j+1}^{(i)}$. Let $\left(a_{j'}^{i'}, a_{j'+1}^{(i')}\right) = I_{b^{(i,i')}}(h_{i'})$ and let $a^{(i',i)} := a_{j'}^{(i')}$.

Consider the set of intervals of the form $\left(a^{(i',i)}, b^{(i,i')}\right)$ and denote it $P(F)$. Take the inclusion-wise maximal subfamily of such intervals and denote it $P^0(F) \subseteq P(F)$.

Remark 4. Notice that, if $\omega(F) = 2$, the maximality of $P^0(F)$ implies that all the intervals in $P^0(F)$ are disjoint.

From now on, consider $F = \{h_i\}_{i=1}^n$ be a family of polygons with $\omega(F) = 2$. We have the following two facts.

Fact 5. For any $\left(a_j^{(i)}\right)$, the elements of $F^0\left(a_j^{(i)}\right) \setminus \left(F^0\left(a_1^{(i)}\right) \cap F\left(a_{k_i}^{(i)}\right)\right)$ are nested. That is, let

$$F^0\left(a_j^{(i)}\right) \setminus \left(F^0\left(a_1^{(i)}\right) \cap F^0\left(a_{k_i}^{(i)}\right)\right) =: \{h_1, \ldots, h_l\}.$$

Then, up to relabeling the elements of F, we have that

$$I_{a_j^{(i)}}(h_1) \subseteq I_{a_j^{(i)}}(h_2) \subseteq \cdots \subseteq I_{a_j^{(i)}}(h_l),$$
$$O_{a_j^{(i)}}(h_1) \subseteq O_{a_j^{(i)}}(h_2) \subseteq \cdots \subseteq O_{a_j^{(i)}}(h_l).$$

Fact 6. For any two polygons $h \in F^0\left(a_j^{(i)}\right) \setminus \left(F^0\left(a_1^{(i)}\right) \cap F^0\left(a_{k_i}^{(i)}\right)\right)$ and $h' \in F^0\left(a_{j'}^{(i)}\right) \setminus \left(F^0\left(a_1^{(i)}\right) \cap F^0\left(a_{k_i}^{(i)}\right)\right)$, we have that h and h' do not overlap.

We now prove the following technical lemma.

Lemma 7. Let $F := \{h_i\}_{i=1}^n$ be a family of polygons with $\omega(F) = 2$. Suppose that for any $(a, b) \in P^0(F)$, there is no h_i such that $a < a_1^{(i)} < a_{k_i}^{(i)} < b$. Then $\chi(F) \leq 3$. Moreover, there exists a 3-coloring of F such that for all i, j, all polygons in $F^0\left(a_j^{(i)}\right)$. We call such coloring a good coloring.

Proof. By way of contradiction, suppose that the statement is false and let $F = \{h_i\}_{i=1}^n$ be the smallest counterexample to this lemma with respect to the order $\prec$. Up to relabeling the elements of F, suppose without loss of generality that $h_1, \ldots, h_t$ are the polygons of F whose external segments (seen as itervals of the form $\left(a_1^{(i)}, a_{k_i}^{(i)}\right)$) are not contained in the external segment of any other polygon of F. We further assume that $a_1^{(1)} < a_1^{(2)} < \ldots < a_1^{(t)}$. By the minimality of F, we get that $G(F)$ is connected and $h_i \cap h_{i+1} \neq \emptyset$ for all $i \in \{1, \ldots, t-1\}$. Therefore, we have that the points $a^{(i+1,i)}$ and $b^{(i,i+1)}$ are well-defined for all $i \in \{1, \ldots, t-1\}$.

First, suppose that $t = 1$. In this case, we have $F^0\left(a_1^{(1)}\right) = \emptyset$ and $F^0\left(a_{k_1}^{(1)}\right) = \emptyset$. Thus, contracting $a_{k_i}^{(1)}$ in h_1, either does not change the underlying graph, or implies that $G(F)$ is disconnected. In both cases, we have a contradiction.

We are now going to prove that for all $i \in \{1, \ldots, t-1\}$, we have either $F^{-}\left(b^{(i,i+1)}\right) = \emptyset$ or $F^{+}\left(a^{(i+1,i)}\right) = \emptyset$. This is not the case for $t \geq 4$ since, in such case, we would have $h_1 \in F^{-}\left(b^{(2,3)}\right)$ and $h_t \in F^{+}\left(a^{(3,2)}\right)$. Thus, this would imply $t \leq 3$.

By way of contradiction, suppose that there exists $i_0 \in \{1, \ldots, t-1\}$ such that $F^{-}\left(b^{(i_0,i_0+1)}\right) \neq \emptyset$ and $F^{+}\left(a^{(i_0+1,i_0)}\right) \neq \emptyset$. By the minimality of F, we have that $F_1 := F \setminus F^{+}\left(a^{\{i_0+1,i_0\}}\right)$ and $F_2 := F \setminus F^{-}\left(b^{i_0,i_0+1}\right)$ admit good colorings f_1 and f_2. Notice that $F_1 \cap F_2 = \{h_{i_0}, h_{i_0+1}\}$. So, we can assume without loss of generality that $f_1(h_{i_0}) = f_2(h_{i_0}) = 1$ and $f_1(h_{i_0+1}) = f_2(h_{i_0+1}) = 2$. This also allows us to define the function $f : F \to \{1, 2, 3\}$ with $f(h) = f_1(h)$ if $h \in F_1$ and $f(h) = f_2(h)$ if $h \in F_2$. By the maximality of h_{i_0} and h_{i_0+1}, a family $\widetilde{F}^{0}\left(a_j^{(i)}\right)$ is either entirely contained into F_1 or F_2. Thus, if f is a coloring, it is also a good coloring.

We now prove that f is a coloring. Let $h_i, h_{i'} \in F$ be overlapping polygons. If both $h_i, h_{i'} \in F_1$ or $h_i, h_{i'} \in F_2$, then $f(h_i) \neq f(h_{i'})$ by the fact that f_1 and f_2 are colorings. Suppose that $h_i \in F_1$ and $h_{i'} \in F_2$, then, since for no $(a,b) \in P^{0}(F)$ and $h_i \in F$, we deduce that

$$a_1^{(i)} < a^{(i_0+1,i_0)} < a_1^{(i')} < a_{k_i}^{(i)} < b^{(i_0,i_0+1)} < a_{k_{i'}}^{(i')}.$$

Now, since $h_{i_0+1}, h_{i'} \in \widetilde{F}^{0}\left(b^{(i_0+1,i_0)}\right)$, and $h_{i_0+1}, h_{i'} \in F_2$, since f_2 is a good coloring, we have that $f(h_{i'}) = f(h_{i_0+1}) = 2$. Now, since h_{i_0+1} and h_i overlap, $h_{i_0+1}, h_i \in F_1$, and f_1 is a coloring, we get that $f(h_{i_0+1}) \neq f(h_i)$. Therefore, $f(h_i) \neq f(h_{i'}) = 2$. So f is a coloring and so a good coloring. This contradicts the definition of F.

We thus conclude that $F^{-}\left(b^{(i,i+1)}\right) = \emptyset$ or $F^{+}\left(a^{(i+1,i)}\right) = \emptyset$ for all $i \in \{1, \ldots, t-1\}$ and $t \leq 3$.

Suppose now that $t = 2$. Suppose that there exists $b^{(1,2)} < a_j^{(2)} < a_{k_2}^{(2)}$. Then, since $F^{0}\left(a_{k_2}^{(2)}\right) = \emptyset$, we have that contracting $a_{k_2}^{(2)}$ in h_2, does not change the underlying graph. Which is not possible by the definition of F. Therefore $a^{(2,1)} = a_{k_2}^{(2)}$. By a symmetric reasoning, we can show $b^{(1,2)} = a_2^{(1)}$.

Consider the case of $F^{+}\left(a^{(2,1)}\right) = \emptyset$. Since $b^{(1,2)} = a_2^{(2)}$, h_1 only overlaps with h_2. By the minimality of F, we get a good coloring f' of $F \setminus \{h_1\}$. We can extend f' into a good coloring of F, by coloring h_1 with a color different from $f'(h_2)$.

Consider the case of $F^{-}\left(b^{(1,2)}\right) = \emptyset$. Since $a^{(2,1)} = a_{k_2}^{(2)}$, h_2 only intersecs with h_1. Now, take the outermost polygons overlapping with h_1 and call them $h_{i_1}, \ldots, h_{i_l}$ with

$$a_1^{(i_1)} < a_{k_{i_1}}^{i_1} < a_1^{(i_2)} < \ldots < a_{k_{i_{l-1}}}^{(i_{l-1})} < a_1^{(i_l)} < a_{k_{i_l}}^{i_l}.$$

We can thus partition all the polygons overlapping with h_1 by looking at which h_{i_j} the are nested in. Denote such class by $N_{h_{i_j}}$. Notice that we have a good coloring f' of $F \setminus \{h_2\}$. We can extend f' into a colring f of F by coloring h_2 with the same color as the other polygons in $F^0\left(a_2^{(1)}\right)$.

For the sake of simplicity, suppose that we have $f(h_1) = 1$, $f(h_{i_1}) = 2$ and $f(h_2) = 2$. Notice that all the polygons in a class $N\left(h_{i_j}\right)$ have the same color.

In order to obtain a good coloring of F, it suffices to color all the polygons overlapping with h_1 with color 2. If one exists, take the smallest index i_k ($\neq i_1$) such that the color of $N(h_{i_k})$ is different from 2. So $f(h_{i_k}) = 3$. Define $O := F^+\left(a_{k_{i_{k-1}}}^{i_{k-1}}\right)$. Notice that the polygons in $F \setminus O$, that overlap with polygons in O lie in $O' := F^0\left(a_{k_{i_{k-1}}}^{i_{k-1}}\right) = \widetilde{F}^0\left(a_{k_{i_{k-1}}}^{(i_{k-1})}\right)$ by the maximality of $h_{i_{k-1}}$. All polygons in O' are colored with color 1. We can thus swap the colors 2 and 3 of the polygons contained in O, so that the classes $N(h_{i_1}), \ldots, N(h_{i_k})$ are colored with color 2. We can repeat this procedure until all classes $N\left(h_{i_j}\right)$ are colored with color 2.

We now tackle the case of $t = 3$. If $t = 3$, we have $h_3 \in F^+\left(a^{(2,1)}\right)$ and $h_1 \in F^-\left(b^{(2,3)}\right)$. Thus, we have that $F^+\left(b^{(2,1)}\right) \neq \emptyset$. By similar arguments as those used in the previous case, the minimality of F implies $a^{(2,1)} = a_1^{(2)}$, $b^{(2,1)} = a_2^{(1)}$, $a^{(3,2)} = a_{k_3-1}^{(3)}$ and $b^{(2,3)} = a_{k_2}^{(2)}$.

Also, suppose that there exists $a_2^{(1)} < a_j^{(2)} < a_{k_1}^{(1)}$ for some $j > 1$. Then, since $F^-\left(b^{(2,1)}\right) = \emptyset$, contracting $a_1^{(2)}$ in h_2 does not change the underlying graph, which contradicts the definition of F. Similarly, there is no $j < k_2$ $a_1^{(3)} < a_j^{(2)} < a_{k_3-1}^{(3)}$. Therefore, for all $1 < j < k_2$, we have that $a_{k_1}^{(1)} < a_j^{(2)} < a_1^{(3)}$.

Suppose that there exists $1 < j < k$ such that $a_{k_1}^{(1)} < a_j^{(2)} < a_j^{(3)}$. Contract $a_{k_2}^{(2)}$ in h_2 (call the resulting polygon h_2') and get a good coloring f' of the resulting family of polygons. Suppose without loss of generality that $f(h_1) = 1$, $f'(h_2') = 2$. If $f'(h_3) \neq 2$, then we can decontract h_2' to get a good coloring of F, which is a contradiction. Let us consider the case $f'(h_3) = 2$. Define $O' := F^+\left(a_{k_2-1}^{(2)}\right)$. Notice that all the polygons in $F \setminus O'$ that intersect with

O' lie in the set $F^0\left(a^{(2)}_{k_2-1}\right) = \widetilde{F}^0\left(a^{(2)}_{k_2-1}\right)$ by the maximality of h_2. By the fact that f' is a good coloring, all polygons in $F^0\left(a^{(2)}_{k_2-1}\right)$ share the same color $\gamma \in \{1,3\}$. Let $\delta \in \{1,3\} \setminus \{\gamma\}$ We can swap the colors 2 and δ in O' by preserving the good coloring. We can finally decontract h_2 to get a good coloring of F. A contradiction.

Thus, we deduce that h_2 is a segment. We get a good coloring f' of $F \setminus \{h_2\}$. Suppose without loss of generality that $f'(h_1) = 1$. We define $h_{i_1}, \ldots, h_{i_l}$ analogously as in the previous case. By the same argument as in the previous case, we can color all the sets $N\left(h_{i_j}\right)$ with the same color as $N\left(h_{i_1}\right)$, say 2 without loss of generality. If after this recoloring, $f'(h_3) \neq 2$, we can extend f' into a good coloring of F by coloring h_2 with color 2. This gives a contradiction. We thus have $f'(h_3) = 2$. Let $\gamma \in \{1,3\}$ be the color of the polygons in $F^0\left(a^{(i_l)}_{k_{i_l}}\right) = \widetilde{F}^0\left(a^{(i_l)}_{k_{i_l}}\right)$. Let $O'' := F^+\left(a^{(i_l)}_{k_{i_l}}\right)$. Notice that all the elements in $F \setminus O''$ that intersect elements in O'' are contained in $F^0\left(a^{(i_l)}_{k_{i_l}}\right)$. Let $\delta \in \{1,3\} \setminus \{\gamma\}$. Therefore, we can swap colors 2 and δ in O'' by preserving the good coloring property. We can thus extend f' into a good coloring of F by coloring h_2 with color 2 to obtain a good coloring of F. Again, a contradiction. $\square$