1 Introduction and first definitions

We define polygon-circle graphs as the intersection graphs of polygons (including segments) inscribed in a circle. The aim of this text is to prove the following.

Theorem 1. Triangle-free polygon-circle graphs have chromatic number at most 5.

Without loss of generality, we will only consider families of polygons in which no two vertices coincide.

We handle polygon-circle graphs by looking at the stereographic projection of the circle onto $\mathbb{R}$. Thus, a polygon-circle graph is represented by a family $\{h_i\}_{i=1}^{n}$ with $h_i = \left(a_1^{(i)}, \ldots, a_{k_i}^{(i)}\right)$ for some $k_i \geq 2$ with $a_j^{(i)} \in \mathbb{R}$ being the image of the stereographic projection of a vertex of the corresponding vertex of h_i.

Given a familiy $F = \{h_i\}_{i=1}^{n}$ of polygons, we denote the corresponding intersection graph $G(F)$ and we say that F is a polygon representation of $G(F)$. By abuse of notation, we will denote $\omega(G(F))$ by $\omega(F)$. Given two polygons $h_i, h_{i'}$ in F, we say that h_i and $h_{i'}$ overlap if there exists indices j and j' such that $\left(a_j^{(i)}, a_{j+1}^{(i)}\right)$ and $\left(a_{j'}^{(i')}, a_{j'+1}^{(i')}\right)$ overlap.

Given a polygon $h := (a_1, \ldots, a_k)$, a contraction of h, is a polygon h' of the form $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_k)$ where indices are unerstood modulo k. Consider a family of polygons $F = \{h_i\}_{i=1}^{n}$. Let F' be the family of polygons generated by contracting one polygon in F. We define the partial order '$\prec$' on the set of polygon-circle graphs to be the smallest order containing the relations of the form $F' \prec F$. Notice that $\prec$ is well-founded since, when $F' \prec F$, the total number of vertices of polygons in F' is strictly less than that of the polygons in F. Thus, we cannot have an infinite decreasing sequence, $\cdots \prec F'' \prec F' \prec F$.

Let $c \in \mathbb{R}$. We establish the following notation.

$$\begin{aligned}
F^0(c) &:= \left\{ h_i \in F \;\middle|\; a_1^{(i)} < c < a_{k_i}^{(i)} \right\}, \\
F^+(c) &:= \left\{ h_i \in F \;\middle|\; c < a_1^{(i)} \right\}, \\
F^-(c) &:= \left\{ h_i \in F \;\middle|\; a_{k_i}^{(i)} < c \right\}.
\end{aligned}$$

We also importantly define for $1 \leq i \leq n$ and $1 < j \leq k_i$, the following:

$$\widetilde{F}^0\left(a_j^{(i)}\right) := \left\{ h_{i'} \in F \;\middle|\; a_1^{(i)} < a_{j'}^{(i')} < a_j^{(i)} < a_{j'+1}^{(i')} \text{ for some } 1 \leq j' < k_{i'} \right\}.$$

For any $h_i \in F^0(c)$, denote $I_c(h_i) := \left(a_j^{(i)}, a_{j+1}^{(i)}\right)$ such that $c \in \left(a_j^{(i)}, a_{j+1}^{(i)}\right)$ and $O_c(h_i) := \left(a_1^{(i)}, a_{k_i}^{(i)}\right)$. For any polygon h_i, we call the segment $\left(a_1^{(i)}, a_{k_i}^{(i)}\right)$ the external segment of h_i.

Also, let h_i, $h_{i'}$ be two whose external segments overlap with $a_1^{(i)} < a_1^{(i')}$. Let $\left(a_j^{(i)}, a_{j+1}^{(i)}\right) = I_{a_1^{(i')}}(h_i)$ and let $b^{(i,i')} := a_{j+1}^{(i)}$. Let $\left(a_{j'}^{i'}, a_{j'+1}^{(i')}\right) = I_{b^{(i,i')}}(h_{i'})$ and let $a^{(i,i')} := a_{j'}^{(i')}$. Similarly, let $\left(a_{l'}^{(i')}, a_{l'+1}^{(i')}\right) = I_{a_{k_i}^{(i)}}(h_{i'})$ and let $c^{(i,i')} := a_{l'}^{(i')}$. Let $\left(a_l^{(i)}, a_{l+1}^{(i)}\right) = I_{c^{(i,i')}}(h_i)$ and let $d^{(i,i')} := a_{j+1}^{(i)}$. An illustration of this is shown in Figure 1.

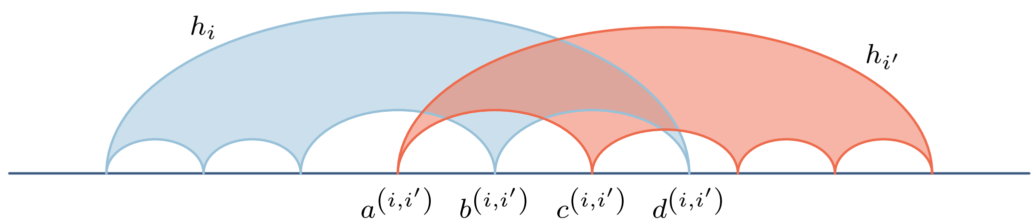

Figure 1

Remark 2. By definition, $a^{(i,i')} < b^{(i,i')}$, $c^{(i,i')} < d^{(i,i')}$ and $a^{(i,i')} \leq c^{(i,i')}$, $b^{(i,i')} \leq d^{(i,i')}$. We thus either have, $a^{(i,i')} < b^{(i,i')} < c^{(i,i')} < d^{(i,i')}$, in which case the intervals $\left(a^{(i,i')}, b^{(i,i')}\right)$ and $\left(c^{(i,i')}, d^{(i,i')}\right)$ are disjoint. Or we have $a^{(i,i')} \leq c^{(i,i')} < b^{(i,i')} \leq d^{(i,i')}$. In this case, since $c^{(i,i')}$ is a vertex of $h_{i'}$ and $c^{(i,i')} < b^{(i,i')}$, we have that $a^{(i,i')} \geq c^{(i,i')}$ and thus $a^{(i,i')} = c^{(i,i')}$. By a similar reasoning, we get that $b^{(i,i')} = d^{(i,i')}$ and so that $\left(a^{(i,i')}, b^{(i,i')}\right) = \left(c^{(i,i')}, d^{(i,i')}\right)$.

Now, consider the set of intervals of the form $\left(a^{(i,i')}, b^{(i,i')}\right)$ or $\left(c^{(i,i')}, d^{(i,i')}\right)$ and denote it $P(F)$. Take the inclusion-wise maximal subfamily of such intervals and denote it $P^0(F) \subseteq P(F)$.

Remark 3. Notice that, if $\omega(F) = 2$, the maximality of $P^0(F)$, together with Remark 2 implies that all the intervals in $P^0(F)$ are disjoint.

From now on, consider $F = \{h_i\}_{i=1}^n$ be a family of polygons with $\omega(F) = 2$. We remark the following two facts which we state without proving them.

Fact 4. For any $a_j^{(i)}$, the elements of $\widetilde{F}^0\left(a_j^{(i)}\right)$ are nested. That is, let $\widetilde{F}^0\left(a_j^{(i)}\right) =: \{h_1, \ldots, h_l\}$. Then, up to relabeling the elements of F, we have that

$$I_{a_j^{(i)}}(h_1) \subseteq I_{a_j^{(i)}}(h_2) \subseteq \cdots \subseteq I_{a_j^{(i)}}(h_l),$$
$$O_{a_j^{(i)}}(h_1) \subseteq O_{a_j^{(i)}}(h_2) \subseteq \cdots \subseteq O_{a_j^{(i)}}(h_l).$$

Fact 5. For any two polygons $h \in \widetilde{F}^0\left(a_j^{(i)}\right)$ and $h' \in \widetilde{F}^0\left(a_{j'}^{(i)}\right)$, we have that h and h' do not overlap.

2 Coloring plain polygon-circle graphs

We define a particular kind of circle graphs whose properies will be useful for the proof of Theorem 1.

Definition 6. We call a polygon-circle graph plain, if it has a polygon representation $F := \{h_i\}_{i=1}^n$ with $\omega(F) = 2$ and such that, for any $(a, b) \in P^0(F)$, there is no h_i such that $a < a_1^{(i)} < a_{k_i}^{(i)} < b$.

Plain polygon-circle graphs have the following property. We note that we will be repeatedly using Fact 4 and Fact 5 without reference.

Lemma 7. Let G be a plain polygon-circle graph and let $F := \{h_i\}_{i=1}^n$ be a polygon representation of G with the properties required by Definition 6 Then $\chi(F) \leq 3$. Moreover, there exists a 3-coloring of F such that for all $1 \leq i \leq n$ and $1 < j \leq k_i$, all polygons in $\widetilde{F}^0\left(a_j^{(i)}\right)$ have the same color. We call such coloring a 'good coloring'.

Proof. By way of contradiction, suppose that the statement is false and let $F = \{h_i\}_{i=1}^n$ be the smallest counterexample to this lemma with respect to the order $\prec$. Up to relabeling the elements of F, suppose without loss of generality that $h_1, \ldots, h_t$ are the polygons of F whose external segments (seen as itervals of the form $\left(a_1^{(i)}, a_{k_i}^{(i)}\right)$) are not contained in the external segment of any other polygon of F. We further assume that $a_1^{(1)} < a_1^{(2)} < \ldots < a_1^{(t)}$. By the minimality of F, we get that $G(F)$ is connected and $h_i \cap h_{i+1} \neq \emptyset$ for all $i \in \{1, \ldots, t-1\}$. Therefore, we have that the points $a^{(i,i+1)}$ and $b^{(i,i+1)}$ are well-defined for all $i \in \{1, \ldots, t-1\}$.

- First, suppose that $t = 1$. In this case, we have $F^0\left(a_1^{(1)}\right) = \emptyset$ and $F^0\left(a_{k_1}^{(1)}\right) = \emptyset$. Thus, contracting $a_{k_i}^{(1)}$ in h_1, either does not change the underlying graph, or implies that $G(F)$ is disconnected. In both cases, we have a contradiction.

- We are now going to prove that for all $i \in \{1, \ldots, t-1\}$, we have either $F^-\left(b^{(i,i+1)}\right) = \emptyset$ or $F^+\left(a^{(i,i+1)}\right) = \emptyset$. This is not the case for $t \geq 4$ since, in such case, we would have $h_1 \in F^-\left(b^{(2,3)}\right)$ and $h_t \in F^+\left(a^{(2,3)}\right)$. Thus, this would imply $t \leq 3$.

  By way of contradiction, suppose that there exists $i_0 \in \{1, \ldots, t-1\}$ such that $F^-\left(b^{(i_0,i_0+1)}\right) \neq \emptyset$ and $F^+\left(a^{(i_0,i_0+1)}\right) \neq \emptyset$. By the minimality of F, we have that $F_1 := F \setminus F^+\left(a^{(i_0,i_0+1)}\right)$ and $F_2 := F \setminus F^-\left(b^{i_0,i_0+1}\right)$ admit good colorings f_1 and f_2. Notice that $F_1 \cap F_2 = \{h_{i_0}, h_{i_0+1}\}$. So, we can assume without loss of generality that $f_1(h_{i_0}) = f_2(h_{i_0}) = 1$ and $f_1(h_{i_0+1}) = f_2(h_{i_0+1}) = 2$. This also allows us to define the function $f : F \longrightarrow \{1, 2, 3\}$ as follows:

$$f(h) = \begin{cases} f_1(h) & \text{if } h \in F_1, \\ f_2(h) & \text{if } h \in F_2. \end{cases}$$

  By the maximality of h_{i_0} and h_{i_0+1}, a family $\widetilde{F}^0\left(a_j^{(i)}\right)$ is either entirely contained into F_1 or F_2. Thus, if f is a coloring, it is also a good coloring.

  We now prove that f is a coloring. Let $h_i, h_{i'} \in F$ be overlapping polygons. If both $h_i, h_{i'} \in F_1$ or $h_i, h_{i'} \in F_2$, then $f(h_i) \neq f(h_{i'})$ by the fact that f_1 and f_2 are colorings. Suppose that $h_i \in F_1$ and $h_{i'} \in F_2$, then, since for no $(a, b) \in P^0(F)$ and $h_i \in F$ we have $a < a_1^{(i)} < a_{k_i}^{(i)} < b$, we deduce that

$$a_1^{(i)} < a^{(i_0,i_0+1)} < a_1^{(i')} < a_{k_i}^{(i)} < b^{(i_0,i_0+1)} < a_{k_{i'}}^{(i')}.$$

Now, since $h_{i_0+1}, h_{i'} \in \widetilde{F}^0\left(b^{(i_0,i_0+1)}\right)$, and $h_{i_0+1}, h_{i'} \in F_2$, since f_2 is a good coloring, we have that $f\left(h_{i'}\right) = f\left(h_{i_0+1}\right) = 2$. Now, since h_{i_0+1} and h_i overlap, $h_{i_0+1}, h_i \in F_1$, and f_1 is a coloring, we get that $f\left(h_{i_0+1}\right) \neq f\left(h_i\right)$. Therefore, $f\left(h_i\right) \neq f\left(h_{i'}\right) = 2$. So f is a coloring and so, it is a good coloring. This contradicts the definition of F.

We thus conclude that $F^-\left(b^{(i,i+1)}\right) = \emptyset$ or $F^+\left(a^{(i,i+1)}\right) = \emptyset$ for all $i \in \{1, \ldots, t-1\}$ and $t \leq 3$.

- Suppose now that $t = 2$. Suppose that there exists $b^{(1,2)} < a_j^{(2)} < a_{k_2}^{(2)}$. Then, since $F^0\left(a_{k_2}^{(2)}\right) = \emptyset$, we have that contracting $a_{k_2}^{(2)}$ in h_2, does not change the underlying graph. Which is not possible by the definition of F. therefore $a^{(1,2)} = a_{k_2-1}^{(2)}$. By a symmetric reasoning, we can show $b^{(1,2)} = a_2^{(1)}$. We have the setting shown in Figure 2.

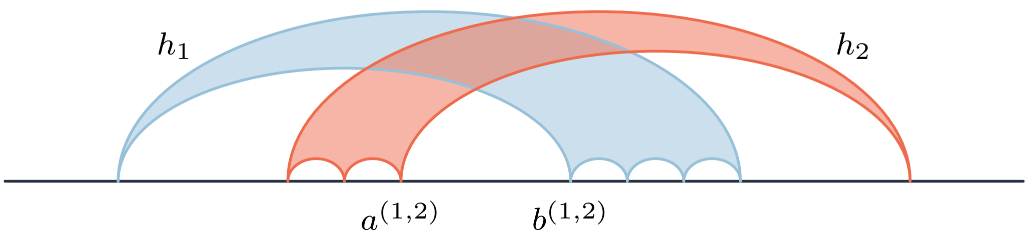

Figure 2

Consider the case of $F^+\left(a^{(1,2)}\right) = \emptyset$. Since $b^{(1,2)} = a_2^{(2)}$, h_1 only overlaps with h_2. By the minimality of F, we get a good coloring f' of $F \setminus \{h_1\}$. We can extend f' into a good coloring of F, by coloring h_1 with a color different from $f'\left(h_2\right)$.

Consider the case of $F^-\left(b^{(1,2)}\right) = \emptyset$. Since $a^{(1,2)} = a_{k_2}^{(2)}$, h_2 only overlaps with h_1. Now, take the outermost polygons overlapping with h_1:

$$\{u_j\}_{j=1}^{s} = \left\{\left(c_1^{(j)}, \ldots, c_{l_j}^{(j)}\right)\right\}_{j=1}^{(s)}$$

with

$$c_1^{(1)} < c_{l_1}^{(1)} < c_1^{(2)} < \cdots < c_{l_{s-1}}^{(s-1)} < c_1^{(s)} < c_{l_s}^{(s)}.$$

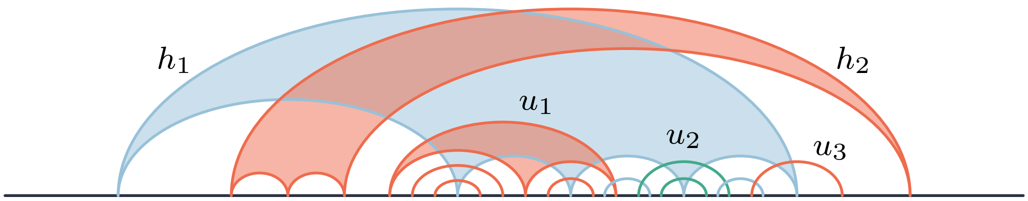

Figure 3

This setting is illustrated in Figure 3.

We can thus partition all the polygons overlapping with h_1 by looking at which u_j the are nested in. For any $1 \leq j \leq s$, denote by $N(u_j)$ the class of polygons nested in u_j. Notice that we have a good coloring f' of $F \setminus \{h_2\}$. We can extend f' into a colring f of F by coloring h_2 with the same color as the other polygons in $F^0\left(a_2^{(1)}\right)$.

For the sake of simplicity, suppose that we have $f(h_1) = 1$, $f(u_1) = 2$ and $f(h_2) = 2$. Notice that all the polygons in a given class $N(u_j)$ have the same color.

In order to obtain a good coloring of F, it suffices to color all the polygons overlapping with h_1 with color 2. If one exists, take the smallest index m $(\neq 1)$ such that the color of $N(u_m)$ is different from 2. So $f(u_m) = 3$. Define $O := F^+\left(c_{l_{m-1}}^{(m-1)}\right)$. Notice that the polygons in $F \setminus O$, that overlap with polygons in O lie in $F^0\left(c_{l_{m-1}}^{(m-1)}\right) = \widetilde{F}^0\left(c_{l_{m-1}}^{(m-1)}\right)$ by the maximality of u_{m-1}. All polygons in $\widetilde{F}^0\left(c_{l_{m-1}}^{(m-1)}\right)$ are colored with color 1. We can thus swap the colors 2 and 3 of the polygons contained in O', so that the classes $N(u_1), \ldots, N(h_{l_m})$ are colored with color 2. We can repeat this procedure until all classes $N(u_j)$ are colored with color 2.

We thus get a good coloring of F, which is a contradiction.

- We now tackle the case of $t = 3$. If $t = 3$, we have $h_3 \in F^+\left(a^{(1,2)}\right)$ and $h_1 \in F^-\left(b^{(2,3)}\right)$. Thus, we have that $F^+\left(b^{(1,2)}\right) \neq \emptyset$. By similar arguments as those used in the previous case, the minimality of F implies $a^{(1,2)} = a_1^{(2)}$, $b^{(1,2)} = a_2^{(1)}$, $a^{(2,3)} = a_{k_3-1}^{(3)}$ and $b^{(2,3)} = a_{k_2}^{(2)}$.

Also, suppose that there exists $a_2^{(1)} < a_j^{(2)} < a_{k_1}^{(1)}$ for some $j > 1$. Then, since $F^-\left(b^{(1,2)}\right) = \emptyset$, contracting $a_1^{(2)}$ in h_2 does not change the underlying graph, which contradicts the definition of F. Similarly, there is no $j < k_2$ with $a_1^{(3)} < a_j^{(2)} < a_{k_3-1}^{(3)}$. Therefore, for all $1 < j < k_2$, we have that $a_{k_1}^{(1)} < a_j^{(2)} < a_1^{(3)}$.

Suppose that there exists $1 < j < k$ such that $a_{k_1}^{(1)} < a_j^{(2)} < a_1^{(3)}$. The setting is illustrated in Figure 4.

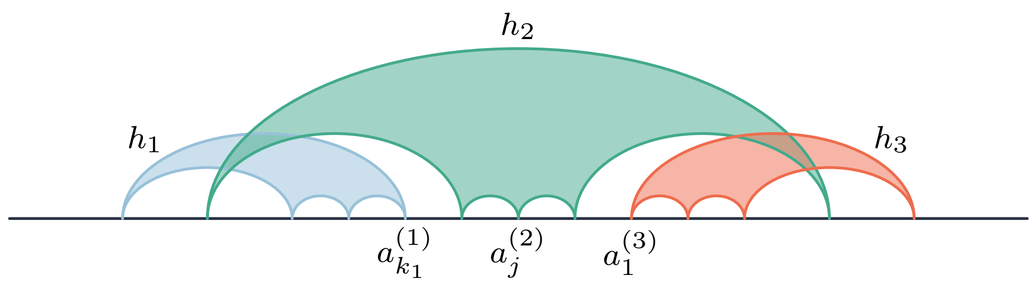

Figure 4

Contract $a_{k_2}^{(2)}$ in h_2 (call the resulting polygon h_2') and get a good coloring f' of the resulting family of polygons. Suppose without loss of generality that $f(h_1) = 1$, $f'(h_2') = 2$. If $f'(h_3) \neq 2$, then we can decontract h_2' to get a good coloring of F, which is a contradiction. Let us consider the case $f'(h_3) = 2$. Define $O' := F^+\left(a_{k_2-1}^{(2)}\right)$. Notice that all the polygons in $F \setminus O'$ that overlap with O' lie in the set $F^0\left(a_{k_2-1}^{(2)}\right) = \widetilde{F}^0\left(a_{k_2-1}^{(2)}\right)$ by the maximality of h_2. By the fact that f' is a good coloring, all polygons in $F^0\left(a_{k_2-1}^{(2)}\right)$ share the same color $\gamma \in \{1,3\}$. Let $\delta \in \{1,3\} \setminus \{\gamma\}$ We can swap the colors 2 and δ in O' by preserving the good coloring property. We can finally decontract h_2 to get a good coloring of F. A contradiction.

Thus, we deduce that h_2 is a segment. We get a good coloring f' of $F \setminus \{h_2\}$. Suppose without loss of generality that $f'(h_1) = 1$. We define $u_1, \ldots, u_s$ analogously as in the previous case. By the same argument as in the previous case, we can color all the sets $N(u_j)$ with the same color as $N(u_1)$, say 2 without loss of generality. If after this recoloring,

$f'(h_3) \neq 2$, we can extend f' into a good coloring of F by coloring h_2 with color 2. This gives a contradiction.

We thus have $f'(h_3) = 2$. Let $\gamma \in \{1,3\}$ be the color of the polygons in $F^0\left(c_{l_s}^{(s)}\right) = \widetilde{F}^0\left(c_{l_s}^{(s)}\right)$. Let $O'' := F^+\left(c_{l_s}^{(s)}\right)$. Notice that all the elements in $F \setminus O''$ that overlap elements in O'' are contained in $\widetilde{F}^0\left(c_{l_s}^{(s)}\right)$, and so have color γ. Let $\delta \in \{1,3\} \setminus \{\gamma\}$. By the above discussion, we can swap colors 2 and δ in O'' by preserving the good coloring property. We can thus extend f' into a good coloring of F by coloring h_2 with color 2 to obtain a good coloring of F. Again, a contradiction.

We have the desired result. $\square$

3 Proof of Theorem 1

We first establish some more notation. Namely, using the same notational conventions used up to this point, we denote, for $1 \leq i' \leq n$ and $1 \leq j' < k_{i'}$

$$\overline{F}^0\left(a_{j'}^{(i')}\right) := \left\{h_i \in F \;\middle|\; a_j^{(i)} < a_{j'}^{(i')} < a_{j+1}^{(i)} < a_{k_{i'}}^{(i')} \text{ for some } 1 \leq j < k_i\right\}$$

We remark in passing that, by symmetry, Lemma 7 remains true if we replace $\widetilde{F}^0\left(a_j^{(i)}\right)$ with $\overline{F}^0\left(a_{j'}^{(i')}\right)$, since this would be a mirrored version of such lemma.

We now present our main result with similar techniques as the one employed in [Kos88]. Again, in what follows, we will repeatedly use Fact 4 and Fact 5 without reference.

In particular, we prove the following statement which clearly implies Theorem 1.

Lemma 8. Let $F := \{h_i\}_{i=1}^n$ with $\omega(F) = 2$. There exist (possibly overlapping) subfamilies $F_1, \ldots, F_l$ of F with $F = \cup_{i=1}^l F_i$ and precolorings $f_1, \ldots, f_l$ with $f_k : F_k' := \cup_{i=1}^k F_i \longrightarrow \{1,2,3,4,5\}$ for all $1 \leq k \leq l$ such that the following holds for all $1 \leq k \leq l$.

1. The polygons in $F \setminus F_k'$ do not overlap with intervals in $F_{k-1}' \setminus F_k$.

2. The polygons $h_i \in F \setminus F_k'$ are such that $a < a_1^{(i)} < a_{k_i}^{(i)} < b$ for some $(a,b) \in P^0(F_k)$.

3. For each $p := (a,b) \in P^0(F_k)$, either one color is used to color all polygons in $\overline{F}^0(k)(a)$ and no more than two colors are used to color the polygons in $\widetilde{F}_k^0(b)$ or viceversa, only one color is used to color the polygons in $\widetilde{F}_k^0(b)$ and no more than two colors are used to color the polygons in $\overline{F}_k^0(a)$.

Proof. Let $F = \{h_i\}_{i=1}^n$ be a family of polygons with $\omega(F) = 2$. We construct the subfamilies $F_1, \ldots, F_k, \ldots, F_l$ and the precolorings $f_1, \ldots, f_k, \ldots, f_l$ by induction on k.

- **Base case**. Let $F_1 \subseteq F$ be the subset of polygons $h_i \in F$ such that there is no $(a,b) \in P^0(F)$ with $a < a_1^{(i)} < a_{k_i}^{(i)} < b$. By Lemma 7 there exists a coloring of F_1 with colors $\{1,2,3\}$ such that for every $h_i \in F_1$ and $1 < j \le k_i$, the polygons in $\widetilde{F}^0\left(a_j^{(i)}\right)$ have the same color. Let f_1 be such coloring. Condition 1 trivially holds. Condition 2 follows directly from the choice of F_1. Condition 3 follows from the choice of f_1 by Lemma 7.

- **Induction step**. Consider that for some $k \ge 1$, the families $F_1, \ldots, F_k$ and the precolorings $f_1, \ldots, f_k$ have been constructed. If $F \setminus F_k' = \emptyset$, then $k = l$ and the result follows. Otherwise, we construct F_{k+1} and f_{k+1} with the desired properties.

  Consider any interval $p = \left(a^{(i,i')}, b^{(i,i')}\right) \in P^0(F_k)$, such that there exists h_j with $a^{(i,i')} < a_1^{(j)} < a_{k_j}^{(j)} < b^{(i,i')}$. Denote:

$$\overline{F}^0\left(a^{(i,i')}\right) = \{u_j\}_{j=1}^t = \left\{\left(c_1^{(j)}, \ldots, c_{l_j}^{(j)}\right)\right\}_{j=1}^t,$$
$$\widetilde{F}^0\left(b^{(i,i')}\right) = \{u_j\}_{j=t+1}^s = \left\{\left(c_1^{(j)}, \ldots, c_{l_j}^{(j)}\right)\right\}_{j=t+1}^s.$$

  We note that the polygons $\{u_j\}_{j=1}^t$ and $\{u_j\}_{j=t+1}^s$ are ordered from innermost to outermost so that $p = \left(c_1^{(s)}, c_{l_s}^{(s)}\right) \cap \left(c_1^{(t)}, c_{l_t}^{(t)}\right)$. Without loss of generality, suppose that $\gamma_1 \in \{1,3\}$ is the color of the polygons $u_{t+1}, \ldots, u_s$ and let γ_2, γ_3 be the colors of the polygons $u_{t+1}, \ldots, u_s$ and let γ_2, γ_3 be the colors of the polygons $u_1, \ldots, u_t$.

  Let I_p be the set of polygons whose vertices are all in p, the polygons $u_{t+1}, \ldots, u_s$ and the segment $\left(b^{(i,i')}, c_{l_s}^{(s)} + 1\right)$. Let F_k^p be the set of polygons $h_i \in I_p$ such that, for no two polygons $h_{i'}, h_{i''} \in I_p$ we have all the vertices of h_i contained in $\left(a^{(i',i'')}, b^{(i',i'')}\right)$ or in $\left(c^{(i',i'')}, d^{(i',i'')}\right)$. By using the fact that $\omega(F) = 2$, it is easy to see that polygons in $I_p \setminus F_{k+1}^p$ do not overlap with any of the polygons of $F'_k \setminus \{u_{t+1}, \dots, u_s\}$. If p contains at least one interval, then $F_{k+1}^p \setminus \{u_{t+1}, \dots, u_s\} \cup \left\{\left(b^{(i,i')}, c_{l_s}^{(s)}\right)\right\} \neq \emptyset$. By Lemma 7, we can color $F_{k_1}^p$ with colors from $\{1,2,3,4,5\} \setminus \{\gamma_2, \gamma_3\}$ such that for each $h_i \in F_{k+1}^p$, for all $1 \leq j < k_i$, the polygons in $\overline{F}_k^{p,0}\left(a_j^{(i)}\right)$ share the same color. At the same time, since $\{u_{t+1}, \dots, u_s\} = \overline{F}_k^{p,0}\left(c_{l_s}^{(s)} + 1\right)$, we can assume that $u_{t+1}, \dots, u_s$ is colored (only) with γ_1.

Let us denote $F'^p_{k+1} := F_{k+1}^p \setminus \left\{\left(b^{(i,i')}, c_{l_s}^{(s)} + 1\right)\right\}$. Notice that the only polygons in F'^p_{k+1} which are in $F \setminus F'_k$ are $u_{t+1}, \dots, u_s$. Thus, the coloring of F'^p_{k+1} is compatible with that of F'_k. We carry out similar constructions for each $p \in P^0(F_k)$ containing at least one uncolored polygon of F. Let $F_{k+1} := \bigcup_{p \in P^0(F_k)} F'^p_k$.

By construction, Condition 2 clearly holds for $k+1$. By our induction hypothesis, polygons in $F'_{k-2} \setminus F_{k-1}$ do not overlap with polygons in $F \setminus F'_k$ and so, a fortiori, they do not overlap with polygons in $F \setminus F'_{k+1}$. Moreover, we have that polygons in $I_p \setminus F_{k+1}^p$ do not overlap with polygons in $F'_{k+1} \setminus \{u_{t+1}, \dots, u_s\}$. Together with Condition 2, this yields that polygons in $F'_{k-1} \setminus F_k$. So, Condition 1 holds for $k+1$. From the described construction, it is clear that Condition 3 holds for $k+1$.

We finally notice that the number of uncolored polygons of F strictly decreases at each step. Therefore, we will eventually have $k = l$ and our construction will be completed in a finite number of steps.

$\square$

References

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