1 Finding k-chords in half graphs

The aim of this section is to prove the following proposition.

Proposition 1. For any $k \neq 1, 4$, there exists $n \in \mathbb{N}$ such that the half graph H_n contains an induced k-chord.

Before tackling the above statement, we notice the following fact.

Fact 2. For any graph G and cycle C in G, the number of chords in C is given by $|e(G[V(C)])| - |C|$.

We now establish some notation. For any half graph H_n with $n \geq 3$, label its vertices as follows.

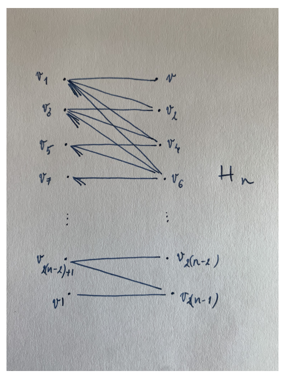

Figure 1: Vertex labeling of a half graph

Notice that the cycle $\left(v_1, v_2, \ldots, v_{2(n-1)}, v_1\right)$ induces a k-chord C_n in H_n with the number K_n of chords being (by Fact 2):

$$K_n = (2 + 3 + \cdots + (n-1) + (n-1)) - 2(n-1) = \frac{n(n-3)}{2}.$$

We now notice the following fact which will be useful in the next proofs.

Fact 3. Given a K_n-chord C_n, one can obtain a new k-chord by removing vertices $v_{2n_1}, v_{2n_2}, \ldots, v_{2n_m}, v_{2n'_1+1}, v_{2n'_2+1}, \ldots, v_{2n'_m+1}$ such that $n_1 < n_2 < \cdots < n_m < n'_1 < n'_2 < \ldots, n'_m$. This last condition is necessary to ensure that the vertices in $v(C_n) \setminus \left\{v_{2n_1}, \ldots, v_{2n_m}, v_{2n'_1+1}, \ldots, v_{2n'_m+1}\right\}$ are part of a unique cycle. The cycle (and the relative induced k-chord) we take into consideration is

$$\left(v_{2i_1+1}, v_{2j_1}, \ldots, v_{2i_{n-m}+1}, v_{2j_{n-m}}, v_{2i_1+1}\right),$$

where $\{i_1, \ldots, i_{n-m}\} = \{1, \ldots, n\} \backslash \{n'_1, \ldots, n'_m\}$ and $\{j_1, \ldots j_{n-m}\} = \{1, \ldots, n\} \backslash \{n_1, \ldots, n_m\}$ with $i_1 < \cdots < i_{n-m}$ and $j_1 < \cdots < j_{n-m}$.

Now, notice that for any $n \geq 4$, we have that $K_n - K_{n-1} = n - 2$. Our strategy is to find and induced k-chord into H_n with $K_{n-1} \leq k \leq K_n$ for al $k \neq 1, 4$. The following lemma represents a step towards this objective.

Lemma 4. For any $n \geq 4$ and any $2 \leq l \leq n-2$, we have that C_n contains an induced $(K_n - l)$-chord (and thus, so does H_n).

Proof. Let $q = l - 1$. Notice that C_n has the following cycle, obtained from C_n by removing v_{2q} and $v_{2(n-2)+1}$. We have:

$$C'_n := \left(v_1, v_2, \ldots, v_{2(q-1)+1}, v_{2(q+1)}, \ldots, v_{2(n-2)}, v_{2(n-3)+1}, v_{2(n-1)}, v_1\right)$$

By Fact 3, C'_n induces a k-chord. Notice that $d\left(v_{2q}\right) = q + 1 = l$ and $d\left(v_{2(n-2)+1}\right) = 2$. Since $|C'_n| = |C_n| - 2$, Fact 2 tells us that the number of chords of C'_n is $K_n - (l-2) - 2 = K_n - l$.

This concludes the proof. □

Now, notice that $K_4 = 2$ and $K_5 = 5$. Thus, in order to prove Proposition 1, it suffices to find n such that H_n contains an induced $(K_m - 1)$-chord for all $m \geq 6$. We claim that such n is $m + 1$.

Lemma 5. For any $m \geq 6$, we have that C_{m+1} contains an induced $(K_m - 1)$-chord (and thus, so does H_{m+1}).

Proof. Consider the K_{m+1}-chord C_{m+1}. Just as in the proof of Lemma 4, consider the k-chord C'''_{m+1} obtained from C_{m+1} by removing the vertices v_2, $v_{2(n-5)}$ $v_{2(n-3)+1}$, $v_{2(n-2)+1}$. The cycle of C'''_{m+1} that we will consider is the one described in Fact 3. Notice that, since $m \geq 6$, we have that all of the above vertices are distinct. Moreover, we have $d\left(v_2\right) = 2$, $d\left(v_{2(n-5)}\right) = n - 4$, $d\left(v_{2(n-3)+1}\right) = 3$, $d\left(v_{2(n-2)+1}\right) = 2$. By Fact 2, we deduce that the number of chords in C'''_{m+1} is

$$K_{m+1} - (2 + (n-4) + 3 + 2) - 4 = K_{m+1} - (n-1) = K_m - 1.$$

This concludes the proof. □

As pointed out above, Lemma 4 and 5 imply Proposition 1.