

# 1 Coloring circle graphs

**WARNING:** intervals  $h_1, h_2$  overlap if  $h_1 \cap h_2 \neq \emptyset$ ,  $h_1 \not\subseteq h_2$  and  $h_2 \not\subseteq h_1$ .

We look at the class of circle graphs as the class of overlap graphs of intervals on a line. Without loss of generality, we only consider the interval models containing open intervals in which no two intervals share an endpoint.

For any such family of intervals  $F$  and any point  $c$  on a line, we set  $F^-(c) := \{(a, b) \in F \mid b < c\}$ ,  $F^0(c) := \{(a, b) \mid a < c < b\}$ ,  $F^+(c) := \{(a, b) \mid c < a\}$ . If  $\omega(F) = 2$ , for all  $(a, b) \in F$ , we have that  $F^0(a) \setminus F^0(b)$  and  $F^0(b) \setminus F^0(a)$  only contain nested intervals. Otherwise, if we had two overlapping intervals  $h_1, h_2 \in F^0(a) \setminus F^0(b)$ , we would have  $h_1, h_2, (a, b)$  forming a triangle.

**Lemma 1.** Let  $F$  be the interval model of a circle graph. Suppose that  $\omega(F) = 2$  and that for no two  $h_1, h_2 \in F$ , there exists  $h_3 \in F$  with  $h_3 \in h_1 \cap h_2$ . Then, there exists a 3-coloring of  $F$  such that for all  $(a, b) \in F$ , the intervals in  $F^0(b) \setminus F^0(a)$  have the same color.

*Proof.* By way of contradiction, suppose that the statement is incorrect and let  $F = \{h_i\}_{i=1}^n = \{(a_i, b_i)\}_{i=1}^n$  be the counterexample to this lemma with the least cardinality. Clearly,  $G(F)$  is connected. Let  $(a_1, b_1), \dots, (a_t, b_t)$ , with  $a_1 < \dots < a_t$ , be the intervals of  $F$  which are not contained in any other intervals. Notice that we have  $F^0(a_1) = F^-(a_1) = \emptyset$ . Suppose that  $|F^0(b_1)| \leq 1$ . By the minimality of  $F$ , we get that the interval model  $F' := F \setminus \{h_1\}$  is 3-colorable. It is clear that coloring  $h_1$  with a different color from  $h_2$ , we obtain a coloring for  $F$ . Which contradicts the fact that  $F$  is a counterexample to this lemma. Therefore, we conclude that  $|F^0(b_1)| \geq 2$  and  $t \geq 2$ .

By the connectedness of  $G(F)$ , we have  $h_i \cap h_{i+1} \neq \emptyset$  for all  $i \in \{1, \dots, t-1\}$ . Let us show that for all  $i \in \{1, \dots, t-1\}$  we have  $F^-(b_i) = \emptyset$  or  $F^+(a_{i+1}) = \emptyset$ . Suppose that this does not hold for some  $1 \leq i \leq t-1$ . Let  $F_1 = F \setminus F^+(a_{i+1})$  and  $F_2 = F \setminus F^-(b_i)$ .

We claim that  $F_1 \cap F_2 = \{h_i, h_{i+1}\}$ . Indeed,  $h_j \in F_1 \cap F_2$ , iff  $a_j \leq a_{i+1}$  and  $b_j \geq b_{i+1}$ . Clearly  $h_i$  and  $h_{i+1}$  have this property. Now, suppose that there exists  $h_j \in F_1 \cap F_2$  such that  $a_j < a_{i+1}$  and  $b_j > b_{i+1}$ . We cannot have  $a_j < a_i$ , because otherwise  $h_i \subseteq h_j$ . Similarly, we cannot have  $b_j > b_{i+1}$ , because otherwise, we would have  $h_{i+1} \subseteq h_j$ . Therefore we have  $a_i < a_j$  and  $b_j < b_{i+1}$ . The only remaining case is that of  $a_i < a_j < a_{i+1}$  and  $b_i < b_j < b_{i+1}$ . But if this is the case,  $\{h_i, h_j, h_{i+1}\}$  would form a triangle, which contradicts our assumptions.

In view of the minimality of  $F$ , there exist 3-colorings  $f_1, f_2$  of  $F_1$  and  $F_2$  as required by the statement of this lemma with colors 1, 2 and 3 (here we use our assumption that  $F^-(b_i)$  and  $F^+(a_{i+1})$  are non-empty and thus  $F_1$  and  $F_2$  are strictly smaller than  $F$ ). Since  $h_1$  and  $h_2$  overlap, they have different colors both in  $f_1$  and  $f_2$ , therefore, we can assume without loss of generality that  $f_1(h_i) = f_2(h_i) = 1$  and  $f_1(h_{i+1}) = f_2(h_{i+1}) = 2$ . Let

$$f(h) = \begin{cases} f_1(h) & \text{if } h \in F_1, \\ f_2(h) & \text{if } h \in F_2. \end{cases}$$

We verify that  $f$  is a coloring of  $F$ . Let  $h' = (a', b') \in F$ ,  $h'' = (a'', b'') \in F$  with  $a' < a'' < b' < b''$ . If  $\{h', h''\} \subseteq F_1$  or  $\{h', h''\} \subseteq F_2$ , then  $f(h') \neq f(h'')$ . If  $h' \in F^-(b_i)$  and  $h'' \in F^+(a_{i+1})$  then, by assumption we have  $h', h'' \not\subseteq (a_{i+1}, b_i)$ , and so we must have  $a' < a_{i+1} < a'' < b' < b_i < b''$ . Since  $f_2$  is a coloring, we have  $f(h'') = f_2(h'') = f_2(h_{i+1}) = 2$ . Since  $h'$  overlaps with  $h_{i+1}$  and  $h', h_{i+1} \in F_1$ , then  $f(h') \neq 2 = f(h'')$ . Similarly, we can verify that for all  $(a, b) \in F$  all intervals in  $F(b) \setminus F(a)$  have the same color. Thus  $f$  is a 3-coloring of  $F$ , which contradicts our choice of  $F$ . Thus, we have that  $F^-(b_i) = \emptyset$  or  $F^+(a_{i+1}) = \emptyset$  for all  $i \in \{1, \dots, t-1\}$ .

We have  $|F^+(a_2)| \geq |F^0(b_1)| \geq 2$ , and, since  $F^-(b_1) = \emptyset$ , we have  $F^0(a_2) = \{h_1\}$ .

If  $F^0(b_2) = \emptyset$ , i.e.  $t = 2$ , then  $h_2$  only overlaps with  $h_1$ . A coloring  $h_2$  with the color of the elements of  $F^0(b_1) \setminus \{h_2\}$ , we obtain a 3-coloring of  $F$  with the conditions required by the present lemma, which is a contradiction.

If  $F^0(b_2) \neq \emptyset$  and  $t \geq 3$ , we obtain (since  $h_1 \in F^-(b_2)$ ) that  $F^+(a_3) = \emptyset$  and thus  $t = 3$  and  $F^0(b_2) = \{h_3\}$ . Consider  $F \setminus \{h_2\}$ . Since  $F$  is minimal, we find that there exists a coloring  $f$  of  $F \setminus \{h_2\}$  which follows the requirement of the present lemma. Let  $h'_2 = (a'_2, b'_2)$  be the longest interval of  $F^0(b_1) \setminus \{h_2\}$  (which exists since  $|F^0(b_1)| \geq 2$ ). This way  $h'_2$  contains all the intervals in  $F^0(b_1)$ . Now, assume that  $f(h_1) = 1$  and  $f(h'_2) = 2$ . If  $f(h_3) \neq 2$ , then we have a coloring of  $F$  with the properties required by the present lemma. Which yields a contradiction. If  $f(h_3) = 2$ , then  $h_3$  and  $h'_2$  do not overlap. Thus  $h_3 \in F^+(b'_2)$ . By assumption, all intervals of  $F^0(b'_2)$  are colored with the same color  $\gamma \in \{1, 3\}$ . By recoloring all elements of  $F^+(b'_2)$  of color 2 to color  $\delta \in \{1, 3\} \setminus \gamma$  and all elements of  $F^+(b'_2)$  of color  $\delta$  with color 2, we obtain a new coloring  $f'$  of  $F \setminus \{h_2\}$ . Since  $h_3 \in F^+(b'_2)$  we get  $f'(h'_2) = 2$  and  $f'(h_3) \neq 2$ . By coloring  $h_2$  with color 2, we obtain a coloring of  $F$  with the properties required by the present lemma.  $\square$

For each pair of overlapping intervals  $h_1, h_2 \in F$ , let  $p(h_1, h_2) = h_1 \cap h_2$ . We denote by  $P(F)$  the family of such intersections. Let  $P^0(F) \subseteq P(F)$  be the inclusion-wise maximal family of  $P(F)$ .

**Lemma 2.** Let  $F$  be a family of intervals with  $\omega(F) = 2$ . Then, the intervals of  $P^0(F)$  do not intersect.

*Proof.* Let  $p_1 = h_1 \cap h_2 \in P^0(F)$  and  $p_2 = h_3 \cap h_4 \in P^0(F)$ . Therefore,  $(p_1 \cap p_2) = (a_4, b_1)$ . So  $a_1 < a_2 < a_4 < b_1 < b_3 < b_4$ . If  $b_2 < b_4$ , then  $h_1, h_2, h_4$  pairwise overlap. Thus we have a triangle, which is a contradiction. Thus  $b_4 < b_2$ . By a symmetric argument we get  $a_1 < a_3$ . But then  $p_2 = (a_4, b_3) \subsetneq (a_2, b_1) = p_1$  which contradicts the maximality of  $p_2$ .  $\square$

**Theorem 3.** Let  $F$  be a family of intervals with  $\omega(F) = 2$ .  $\chi(F) \leq 5$ .

*Proof.* We construct the desired coloring by induction on  $k$ . Let  $k = 1$ . Let  $F_1$  be the subset of intervals of  $F$  that do not lie in the intersection of overlapping intervals of  $F$ . By Lemma 1, there exists a coloring  $f_1$  of  $F_1$  with colors  $\{1, 2, 3\}$  such that for every  $(a, b) \in F_1$ , the intervals of the family  $F_1^0(b) \setminus F^0(a)$  are colored with the same color. By Lemma 2, the intervals of the family  $P^0(F_1)$  do not overlap. By definition of  $F_1$ , each interval in  $F \setminus F_1$  is contained in some (maximal) intersection  $p \in P^0(F_1)$ . Consider that the following construction has been done for  $k - 1$  with  $k \geq 2$ . We want the following properties to be true.

1. The intervals in  $F' := \bigcup_{i=1}^{k-1} F_i$  (where the  $F_i$ s might not be disjoint) are colored with colors  $\{1, \dots, 5\}$ .
2. The intervals in  $F \setminus F'$  do not overlap with intervals in  $\bigcup_{i=1}^{k-2} F_i \setminus F_{k-1}$ .
3. All intervals in  $F \setminus F'$  are contained in some interval  $p \in P^0(F^{k-1})$ .
4. For each  $p = (c, d) \in P^0(F_{k-1})$  either only one color is used to color all intervals in  $F_{k-1}^0(c) \setminus F_{k-1}^0(d)$ , and no more than two colors are used to color intervals of  $F_{k-1}^0(d) \setminus F_{k-1}^0(c)$  or viceversa.

Let us show how to carry out the  $k$ -th step of this construction. If  $F \setminus F' = \emptyset$ , then we have the desired coloring. Consider any interval  $p = (c, d) \in P^0(F_{k-1})$ , which contains at least one interval of  $F \setminus F'$ . Let

$$\begin{aligned} F_{k-1}^0(c) \setminus F_{k-1}^0(d) &= \{h_j\}_{j=1}^t = \{(c_j, d_j)\}_{j=1}^t, \\ F_{k-1}^0(d) \setminus F_{k-1}^0(c) &= \{h_j\}_{j=t+1}^s = \{(c_j, d_j)\}_{j=t+1}^s, \end{aligned}$$

and  $p = (c_s, d_s) \cap (c_t, d_t) = (c_s, d_t)$ . As already noted, each family consists of nested intervals. Without loss of generality, suppose that  $\gamma_1 \in \{1, 3\}$  is the color used to color the intervals  $h_{t+1}, \dots, h_s$  and let  $\gamma_2, \gamma_3$  be the colors used to color  $\{h_1, \dots, h_t\}$ .

Let  $I_p$  be the set of intervals in  $F$  contained in  $p$ , the intervals  $h_{t+1}, \dots, h_s$  and the interval  $(d, d_s + 1)$ . Let  $F_{k,p}$  be the set of intervals in  $I_p$  which are not contained in the overlap of any two intervals in  $I_p$ .

Let us show that the intervals from  $I_p \setminus F_{k,p}$  do not overlap with any of the intervals of  $F' \setminus \{h_{t+1}, \dots, h_s\}$ . By condition 2 of our induction hypothesis, intervals from  $F' \setminus \{h_{t+1}, \dots, h_s\}$  that can overlap with elements in  $I_p \setminus F_{k,p}$  are only intervals in  $\{h_1, \dots, h_t\}$ . Let  $(a_1, b_1) \in I_p$  be an interval contained in the overlapping of  $(a_2, b_2), (a_3, b_3) \in I_p$  and thus overlapping with some  $h_j = (c_j, d_j)$  for  $1 \leq j \leq t-1$ . Then  $a_i > c_s$  for all  $i \in \{1, 2, 3\}$ . We also get  $c_j < c_s, a_1 < d_j < b_1$ . Hence  $h_j, (a_2, b_2)$  and  $(a_3, b_3)$  pairwise overlap. Which is a contradiction.

If  $p$  contains at least one interval, then  $F_{k,p} \setminus (h_{t+1}, \dots, h_s) \cup \{(d, d_s + 1)\} \neq \emptyset$ . By Lemma 1 (replacing  $F^0(b) \setminus F^0(a)$  with  $F^0(a) \setminus F^0(b)$ ), we can color  $F_{k,p}$  with colors from  $\{1, 2, 3, 4, 5\} \setminus \{\gamma_2, \gamma_3\}$  such that for each  $(a, b) \in F_{k,p}$ , the interval from  $F_{k,p}^0(a) \setminus F_{k,p}^0(b)$  have the same color. At the same time, since  $\{h_{t+1}, \dots, h_s\} = F_{k,p}^0(d) \setminus F_{k,p}^0(d_s + 1)$  we can assume that  $h_{t+1}, \dots, h_s$  is colored with (only)  $\gamma_1$ .

Let us denote  $F'_{k,p} = F_{k,p} \setminus \{(d, d_s + 1)\}$ . It is easy to see that the coloring of  $F'_{k,p}$  is compatible with that of  $F'$ . Carrying out similar constructions for each  $p \in P^0(F_{k-1})$  containing at least one uncolored interval of  $F$ , let  $F_k := \bigcup_{p \in P^0(F_{k-1})} F'_{k,p}$ .

One can check that the induction hypotheses hold for  $k$ . Also, the number of uncolored vertices strictly decreases at each step. Therefore the coloring will be completed in a finite number of iterations.  $\square$

## 2 Coloring polygon circle graphs

We define polygon circle graphs as the set of intersection graphs of polygons (including segments) inscribed in a circle. For the sake of simplicity, we will only consider families of polygons in which no two vertices coincide. The class of circle graphs can be obtained by restricting the choice of polygons to segments.

Like circle graphs, we handle polygon circle graphs by looking at the

stereographic projection of the circle onto  $\mathbb{R}$ . Thus, a polygon circle graph is represented by a family  $\{h_i\}_{i=1}^n$  with  $(a_1^{(i)}, \dots, a_{k_i}^{(i)})$  for some  $k_i \geq 1$  with  $a_j^{(i)} \in \mathbb{R}$  being the image of the stereographic projection of a vertex of  $h_i$ .

Given a family  $F = \{h_i\}_{i=1}^n$  of polygons, we denote the corresponding intersection graph  $G(F)$ . By abuse of notation, we will denote  $\omega(G(F))$  by  $\omega(F)$ . Given two polygons  $h_i, h_{i'}$  in  $F$ , we say that  $h_i$  and  $h_{i'}$  overlap if there exists indices  $j$  and  $j'$  such that  $(a_j^{(i)}, a_{j+1}^{(i)})$  and  $(a_{j'}^{(i')}, a_{j'+1}^{(i')})$  overlap.

Given a polygon  $h := (a_1, \dots, a_k)$ , a contraction of  $h$ , is a polygon  $h'$  of the form  $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k)$  where indices are understood modulo  $k$ . Consider a family of polygons  $F = \{h_i\}_{i=1}^n$ . Let  $F'$  be the family of polygons generated by contracting one polygon in  $F$ . We define the partial order ' $\prec$ ' on the set of polygon circle graphs to be the order generated by the relations  $F' \prec F$ . Notice that  $\prec$  is well-founded since, if  $F' \prec F$ , the total number of vertices of polygons in  $F'$  is strictly less than that of the polygons in  $F$ .

Let  $c \in \mathbb{R}$ . We establish the following notation.

$$\begin{aligned} F^0(c) &:= \left\{ h_i \in F \mid a_1^{(i)} < c < a_{k_i}^{(i)} \right\}, \\ F^+(c) &:= \left\{ h_i \in F \mid c < a_1^{(i)} \right\}, \\ F^-(c) &:= \left\{ h_i \in F \mid a_{k_i}^{(i)} < c \right\}. \end{aligned}$$

We also importantly define for  $1 \leq i \leq n$  and  $1 < j \leq k_i$ , the following:

$$\widetilde{F}^0(a_j^{(i)}) := \left\{ h_{i'} \in F \mid a_1^{(i)} < a_{j'}^{(i')} < a_j^{(i)} < a_{j'+1}^{(i')} \text{ for some } 1 \leq j' < k_{i'} \right\}.$$

For any  $h_i \in F^0(c)$ , denote  $I_c(h_i) := (a_j^{(i)}, a_{j+1}^{(i)})$  such that  $c \in (a_j^{(i)}, a_{j+1}^{(i)})$  and  $O_c(h_i) := (a_1^{(i)}, a_{k_i}^{(i)})$ . For any polygon  $h_i$ , we call the segment  $(a_1^{(i)}, a_{k_i}^{(i)})$  the external segment of  $h_i$ .

Also, for any two polygons  $h_i, h_{i'}$  be two polygons whose external segments overlap with  $a_1^{(i)} < a_1^{(i')}$ . Let  $(a_j^{(i)}, a_{j+1}^{(i)}) = I_{a_1^{(i')}}(h_i)$  and let  $b^{(i,i')} := a_{j+1}^{(i)}$ . Let  $(a_{j'}^{(i')}, a_{j'+1}^{(i')}) = I_{b^{(i,i')}}(h_{i'})$  and let  $a^{(i',i)} := a_{j'}^{(i')}$ . We say that the interval  $(a^{(i',i)}, b^{(i,i')})$  is the overlap of  $h_i, h_{i'}$ .

Consider the set of intervals of the form  $(a^{(i',i)}, b^{(i,i')})$  and denote it  $P(F)$ . Take the inclusion-wise maximal subfamily of such intervals and denote it  $P^0(F) \subseteq P(F)$ .

**Remark 4.** Notice that, if  $\omega(F) = 2$ , the maximality of  $P^0(F)$  implies that all the intervals in  $P^0(F)$  are disjoint.

From now on, consider  $F = \{h_i\}_{i=1}^n$  be a family of polygons with  $\omega(F) = 2$ . We have the following two facts.

**Fact 5.** For any  $(a_j^{(i)})$ , the elements of  $\tilde{F}^0(a_j^{(i)})$  are nested. That is, let  $\tilde{F}^0(a_j^{(i)}) =: \{h_1, \dots, h_l\}$ . Then, up to relabeling the elements of  $F$ , we have that

$$\begin{aligned} I_{a_j^{(i)}}(h_1) &\subseteq I_{a_j^{(i)}}(h_2) \subseteq \dots \subseteq I_{a_j^{(i)}}(h_l), \\ O_{a_j^{(i)}}(h_1) &\subseteq O_{a_j^{(i)}}(h_2) \subseteq \dots \subseteq O_{a_j^{(i)}}(h_l). \end{aligned}$$

**Fact 6.** For any two polygons  $h \in \tilde{F}^0(a_j^{(i)})$  and  $h' \in \tilde{F}^0(a_{j'}^{(i)})$ , we have that  $h$  and  $h'$  do not overlap.

We now prove the following technical lemma.

**Lemma 7.** Let  $F := \{h_i\}_{i=1}^n$  be a family of polygons with  $\omega(F) = 2$ . Suppose that for any  $(a, b) \in P^0(F)$ , there is no  $h_i$  such that  $a < a_1^{(i)} < a_{k_i}^{(i)} < b$ . Then  $\chi(F) \leq 3$ . Moreover, there exists a 3-coloring of  $F$  such that for all  $1 \leq i \leq n$  and  $1 < j \leq k_i$ , all polygons in  $\tilde{F}^0(a_j^{(i)})$  have the same color. We call such coloring a ‘good coloring’.

*Proof.* By way of contradiction, suppose that the statement is false and let  $F = \{h_i\}_{i=1}^n$  be the smallest counterexample to this lemma with respect to the order  $\prec$ . Up to relabeling the elements of  $F$ , suppose without loss of generality that  $h_1, \dots, h_t$  are the polygons of  $F$  whose external segments (seen as intervals of the form  $(a_1^{(i)}, a_{k_i}^{(i)})$ ) are not contained in the external segment of any other polygon of  $F$ . We further assume that  $a_1^{(1)} < a_1^{(2)} < \dots < a_1^{(t)}$ . By the minimality of  $F$ , we get that  $G(F)$  is connected and  $h_i \cap h_{i+1} \neq \emptyset$  for all  $i \in \{1, \dots, t-1\}$ . Therefore, we have that the points  $a^{(i+1, i)}$  and  $b^{(i, i+1)}$  are well-defined for all  $i \in \{1, \dots, t-1\}$ .

- First, suppose that  $t = 1$ . In this case, we have  $F^0(a_1^{(1)}) = \emptyset$  and  $F^0(a_{k_1}^{(1)}) = \emptyset$ . Thus, contracting  $a_{k_1}^{(1)}$  in  $h_1$ , either does not change the underlying graph, or implies that  $G(F)$  is disconnected. In both cases, we have a contradiction.

- We are now going to prove that for all  $i \in \{1, \dots, t-1\}$ , we have either  $F^-(b^{(i,i+1)}) = \emptyset$  or  $F^+(a^{(i+1,i)}) = \emptyset$ . This is not the case for  $t \geq 4$  since, in such case, we would have  $h_1 \in F^-(b^{(2,3)})$  and  $h_t \in F^+(a^{(3,2)})$ . Thus, this would imply  $t \leq 3$ .

By way of contradiction, suppose that there exists  $i_0 \in \{1, \dots, t-1\}$  such that  $F^-(b^{(i_0,i_0+1)}) \neq \emptyset$  and  $F^+(a^{(i_0+1,i_0)}) \neq \emptyset$ . By the minimality of  $F$ , we have that  $F_1 := F \setminus F^+(a^{(i_0+1,i_0)})$  and  $F_2 := F \setminus F^-(b^{(i_0,i_0+1)})$  admit good colorings  $f_1$  and  $f_2$ . Notice that  $F_1 \cap F_2 = \{h_{i_0}, h_{i_0+1}\}$ . So, we can assume without loss of generality that  $f_1(h_{i_0}) = f_2(h_{i_0}) = 1$  and  $f_1(h_{i_0+1}) = f_2(h_{i_0+1}) = 2$ . This also allows us to define the function  $f : F \longrightarrow \{1, 2, 3\}$  as follows:

$$f(h) = \begin{cases} f_1(h) & \text{if } h \in F_1, \\ f_2(h) & \text{if } h \in F_2. \end{cases}$$

By the maximality of  $h_{i_0}$  and  $h_{i_0+1}$ , a family  $\tilde{F}^0(a_j^{(i)})$  is either entirely contained into  $F_1$  or  $F_2$ . Thus, if  $f$  is a coloring, it is also a good coloring.

We now prove that  $f$  is a coloring. Let  $h_i, h_{i'} \in F$  be overlapping polygons. If both  $h_i, h_{i'} \in F_1$  or  $h_i, h_{i'} \in F_2$ , then  $f(h_i) \neq f(h_{i'})$  by the fact that  $f_1$  and  $f_2$  are colorings. Suppose that  $h_i \in F_1$  and  $h_{i'} \in F_2$ , then, since for no  $(a, b) \in P^0(F)$  and  $h_i \in F$  we have  $a < a_1^{(i)} < a_{k_i}^{(i)} < b$ , we deduce that

$$a_1^{(i)} < a^{(i_0+1,i_0)} < a_1^{(i')} < a_{k_i}^{(i)} < b^{(i_0,i_0+1)} < a_{k_{i'}}^{(i')}.$$

Now, since  $h_{i_0+1}, h_{i'} \in \tilde{F}^0(b^{(i_0+1,i_0)})$ , and  $h_{i_0+1}, h_{i'} \in F_2$ , since  $f_2$  is a good coloring, we have that  $f(h_{i'}) = f(h_{i_0+1}) = 2$ . Now, since  $h_{i_0+1}$  and  $h_i$  overlap,  $h_{i_0+1}, h_i \in F_1$ , and  $f_1$  is a coloring, we get that  $f(h_{i_0+1}) \neq f(h_i)$ . Therefore,  $f(h_i) \neq f(h_{i'}) = 2$ . So  $f$  is a coloring and so a good coloring. This contradicts the definition of  $F$ .

We thus conclude that  $F^-(b^{(i,i+1)}) = \emptyset$  or  $F^+(a^{(i+1,i)}) = \emptyset$  for all  $i \in \{1, \dots, t-1\}$  and  $t \leq 3$ .

- Suppose now that  $t = 2$ . Suppose that there exists  $b^{(1,2)} < a_j^{(2)} < a_{k_2}^{(2)}$ . Then, since  $F^0(a_{k_2}^{(2)}) = \emptyset$ , we have that contracting  $a_{k_2}^{(2)}$  in  $h_2$ , does not

change the underlying graph. Which is not possible by the definition of  $F$ . Therefore  $a^{(2,1)} = a_{k_2}^{(2)}$ . By a symmetric reasoning, we can show  $b^{(1,2)} = a_2^{(1)}$ .

Consider the case of  $F^+(a^{(2,1)}) = \emptyset$ . Since  $b^{(1,2)} = a_2^{(2)}$ ,  $h_1$  only overlaps with  $h_2$ . By the minimality of  $F$ , we get a good coloring  $f'$  of  $F \setminus \{h_1\}$ . We can extend  $f'$  into a good coloring of  $F$ , by coloring  $h_1$  with a color different from  $f'(h_2)$ .

Consider the case of  $F^-(b^{(1,2)}) = \emptyset$ . Since  $a^{(2,1)} = a_{k_2}^{(2)}$ ,  $h_2$  only intersects with  $h_1$ . Now, take the outermost polygons overlapping with  $h_1$  and call them  $h_{i_1}, \dots, h_{i_l}$  with

$$a_1^{(i_1)} < a_{k_{i_1}}^{(i_1)} < a_1^{(i_2)} < \dots < a_{k_{i_{l-1}}}^{(i_{l-1})} < a_1^{(i_l)} < a_{k_{i_l}}^{(i_l)}.$$

We can thus partition all the polygons overlapping with  $h_1$  by looking at which  $h_{i_j}$  they are nested in. Denote such class by  $N_{h_{i_j}}$ . Notice that we have a good coloring  $f'$  of  $F \setminus \{h_2\}$ . We can extend  $f'$  into a coloring  $f$  of  $F$  by coloring  $h_2$  with the same color as the other polygons in  $F^0(a_2^{(1)})$ .

For the sake of simplicity, suppose that we have  $f(h_1) = 1$ ,  $f(h_{i_1}) = 2$  and  $f(h_2) = 2$ . Notice that all the polygons in a class  $N(h_{i_j})$  have the same color.

In order to obtain a good coloring of  $F$ , it suffices to color all the polygons overlapping with  $h_1$  with color 2. If one exists, take the smallest index  $i_k$  ( $\neq i_1$ ) such that the color of  $N(h_{i_k})$  is different from 2. So  $f(h_{i_k}) = 3$ . Define  $O := F^+(a_{k_{i_{k-1}}}^{i_{k-1}})$ . Notice that the polygons in  $F \setminus O$ , that overlap with polygons in  $O$  lie in  $F^0(a_{k_{i_{k-1}}}^{i_{k-1}}) = \tilde{F}^0(a_{k_{i_{k-1}}}^{(i_{k-1})})$  by the maximality of  $h_{i_{k-1}}$ . All polygons in  $\tilde{F}^0(a_{k_{i_{k-1}}}^{(i_{k-1})})$  are colored with color 1. We can thus swap the colors 2 and 3 of the polygons contained in  $O$ , so that the classes  $N(h_{i_1}), \dots, N(h_{i_k})$  are colored with color 2. We can repeat this procedure until all classes  $N(h_{i_j})$  are colored with color 2.

- We now tackle the case of  $t = 3$ . If  $t = 3$ , we have  $h_3 \in F^+(a^{(2,1)})$  and  $h_1 \in F^-(b^{(2,3)})$ . Thus, we have that  $F^+(b^{(2,1)}) \neq \emptyset$ . By similar



arguments as those used in the previous case, the minimality of  $F$  implies  $a^{(2,1)} = a_1^{(2)}$ ,  $b^{(1,2)} = a_2^{(1)}$ ,  $a^{(3,2)} = a_{k_3-1}^{(3)}$  and  $b^{(2,3)} = a_{k_2}^{(2)}$ .

Also, suppose that there exists  $a_2^{(1)} < a_j^{(2)} < a_{k_1}^{(1)}$  for some  $j > 1$ . Then, since  $F^-(b^{(2,1)}) = \emptyset$ , contracting  $a_1^{(2)}$  in  $h_2$  does not change the underlying graph, which contradicts the definition of  $F$ . Similarly, there is no  $j < k_2$   $a_1^{(3)} < a_j^{(2)} < a_{k_3-1}^{(3)}$ . Therefore, for all  $1 < j < k_2$ , we have that  $a_{k_1}^{(1)} < a_j^{(2)} < a_1^{(3)}$ .

Suppose that there exists  $1 < j < k$  such that  $a_{k_1}^{(1)} < a_j^{(2)} < a_1^{(3)}$ . Contract  $a_{k_2}^{(2)}$  in  $h_2$  (call the resulting polygon  $h'_2$ ) and get a good coloring  $f'$  of the resulting family of polygons. Suppose without loss of generality that  $f(h_1) = 1$ ,  $f'(h'_2) = 2$ . If  $f'(h_3) \neq 2$ , then we can decontract  $h'_2$  to get a good coloring of  $F$ , which is a contradiction. Let us consider the case  $f'(h_3) = 2$ . Define  $O' := F^+(a_{k_2-1}^{(2)})$ . Notice that all the polygons in  $F \setminus O'$  that intersect with  $O'$  lie in the set  $F^0(a_{k_2-1}^{(2)}) = \tilde{F}^0(a_{k_2-1}^{(2)})$  by the maximality of  $h_2$ . By the fact that  $f'$  is a good coloring, all polygons in  $F^0(a_{k_2-1}^{(2)})$  share the same color  $\gamma \in \{1, 3\}$ . Let  $\delta \in \{1, 3\} \setminus \{\gamma\}$ . We can swap the colors 2 and  $\delta$  in  $O'$  by preserving the good coloring property. We can finally decontract  $h_2$  to get a good coloring of  $F$ . A contradiction.

Thus, we deduce that  $h_2$  is a segment. We get a good coloring  $f'$  of  $F \setminus \{h_2\}$ . Suppose without loss of generality that  $f'(h_1) = 1$ . We define  $h_{i_1}, \dots, h_{i_l}$  analogously as in the previous case. By the same argument as in the previous case, we can color all the sets  $N(h_{i_j})$  with the same color as  $N(h_{i_1})$ , say 2 without loss of generality. If after this recoloring,  $f'(h_3) \neq 2$ , we can extend  $f'$  into a good coloring of  $F$  by coloring  $h_2$  with color 2. This gives a contradiction.

We thus have  $f'(h_3) = 2$ . Let  $\gamma \in \{1, 3\}$  be the color of the polygons in  $F^0(a_{k_{i_l}}^{(i_l)}) = \tilde{F}^0(a_{k_{i_l}}^{(i_l)})$ . Let  $O'' := F^+(a_{k_{i_l}}^{(i_l)})$ . Notice that all the elements in  $F \setminus O''$  that intersect elements in  $O''$  are contained in  $F^0(a_{k_{i_l}}^{(i_l)})$ . Let  $\delta \in \{1, 3\} \setminus \{\gamma\}$ . Therefore, we can swap colors 2 and  $\delta$  in  $O''$  by preserving the good coloring property. We can thus extend  $f'$  into a good coloring of  $F$  by coloring  $h_2$  with color 2 to obtain a good coloring of  $F$ . Again, a contradiction.

We have the desired result.  $\square$

We first establish some more notation. Namely, using the same notational conventions used up to this point, we denote, for  $1 \leq i' \leq n$  and  $1 \leq j' < k_{i'}$

$$\bar{F}^0(a_{j'}^{(i')}) := \left\{ h_i \in F \mid a_j^{(i)} < a_{j'}^{(i')} < a_{j+1}^{(i)} < a_{k_{i'}}^{(i')} \text{ for some } 1 \leq j < k_i \right\}$$

We remark in passing that, by symmetry, Lemma 7 remains true if we replace  $\tilde{F}^0(a_j^{(i)})$  with  $\bar{F}^0(a_{j'}^{(i')})$ . This is because, the case of  $t = 1$  and the proof of  $t \leq 3$  are exactly the same. Also, when dealing with the cases  $t = 2$  and  $t = 3$ , we use the minimality of  $F$  to specify some structural properties of  $F$ . After doing so, the proof of this alternative version of Lemma 7 is given by the same argument as in the proof of such lemma, only by composing the stereographic projection generating our representation of polygon circle graphs with a symmetry of  $\mathbb{R}$  with respect to the point 0.

We now present our main result with similar techniques as the one employed in the proof of Theorem 3.

**Theorem 8.** Let  $F$  be a family of polygons with  $\omega(F) = 2$ . Then,  $\chi(F) \leq 5$ .

*Proof.* Let  $F = \{h_i\}_{i=1}^n$  be a family of polygons with  $\omega(F) = 2$ . We construct the desired coloring inductively on  $k$ .

For  $k = 1$ , let  $F_1 \subseteq F$  be the subset of polygons  $h_i \in F$  such that there is no  $(a, b) \in P^0(F)$  with  $a < a_1^{(i)} < a_{k_i}^{(i)} < b$ . By Lemma 7 there exists a coloring  $f_1$  of  $F_1$  with colors  $\{1, 2, 3\}$  such that for every  $h_i \in F_1$  and  $1 < j \leq k_i$ ,  $\tilde{F}^0(a_j^{(i)})$  have the same color. By the definition of  $F_1$ , each interval in  $F \setminus F_1$  is contained into some (unique) intersection  $p \in P^0(F_1)$ . Consider that the following construction has been done for  $k - 1$  with  $k \geq 2$ . We want the following properties to be true.

1. The polygons in  $F' := \bigcup_{i=1}^{k-1} F_i$  (where the  $F_i$ s might not be disjoint) are colored with colors  $\{1, 2, 3, 4, 5\}$ .
2. The polygons in  $F \setminus F'$  do not overlap with intervals in  $\bigcup_{i=1}^{k-2} F_i \setminus F_{k-1}$ .
3. All polygons  $h_i \in F \setminus F'$  are such that  $a < a_1^{(i)} < a_{k_i}^{(i)} < b$  for some  $(a, b) \in P^0(F_{k-1})$ .

4. For each  $p \in (a^{(i',i)}, b^{(i,i')}) \in P^0(F_{k-1})$  either only one color is used to color all polygons in  $\overline{F}_{k-1}^0(a^{(i',i)})$  and no more than two colors are used to color intervals in  $\widetilde{F}_{k-1}^0(b^{(i,i')})$ , or viceversa.

Let us show how to carry out the  $k$ -th step of this construction. If  $F \setminus F' = \emptyset$ , then we are done and we have the desired coloring. Consider any interval  $p = (a^{(i',i)}, b^{(i,i')}) \in P^0(F_{k-1})$ , such that there exists  $h_j$  with  $a^{(i',i)} < a_1^{(j)} < a_{k_j}^{(j)} < b^{(i,i')}$ . Denote:

$$\begin{aligned}\overline{F}^0(a^{(i',i)}) &= \{u_j\}_{j=1}^t = \left\{ \left( c_1^{(j)}, \dots, c_{l_j}^{(j)} \right) \right\}_{j=1}^t, \\ \widetilde{F}^0(b^{(i,i')}) &= \{u_j\}_{j=t+1}^s = \left\{ \left( c_1^{(j)}, \dots, c_{l_j}^{(j)} \right) \right\}_{j=t+1}^s.\end{aligned}$$

We note that the polygons  $\{u_j\}_{j=1}^t$  and  $\{u_j\}_{j=t+1}^s$  are ordered from innermost to outermost so that  $p = (c_1^{(s)}, c_{l_s}^{(s)}) \cap (c_1^{(t)}, c_{l_t}^{(t)})$ . Without loss of generality, suppose that  $\gamma_1 \in \{1, 3\}$  is the color of the polygons  $u_{t+1}, \dots, u_s$  and let  $\gamma_2, \gamma_3$  be the colors of the polygons  $u_{t+1}, \dots, u_s$  and let  $\gamma_2, \gamma_3$  be the colors of the polygons  $u_1, \dots, u_t$ .

Let  $I_p$  be the set of polygons where vertices are all in  $p$ , the polygons  $u_{t+1}, \dots, u_s$  and the segment  $(b^{(i,i')}, c_{l_s}^{(s)} + 1)$ . Let  $F_{k,p}$  be the set of polygons in  $I_p$  whose vertices are not all in the overlap of two polygons in  $I_p$ . By the same argument as the one used in the proof of Theorem 3, polygons in  $I_p \setminus F_{k,p}$  do not overlap with any of the polygons of  $F' \setminus \{u_{t+1}, \dots, u_s\}$ . If  $p$  contains at least one interval, then  $F_{k,p} \setminus \{u_{t+1}, \dots, u_s\} \cup \left\{ (b^{(i,i')}, c_{l_s}^{(s)}) \right\} \neq \emptyset$ . By Lemma 7, we can color  $F_{k,p}$  with colors from  $\{1, 2, 3, 4, 5\} \setminus \{\gamma_2, \gamma_3\}$  such that for each  $h_i \in F_{k,p}$ , for all  $1 \leq j < k_i$ , the polygons in  $\overline{F}_{k,p}^0(a_j^{(i)})$  share the same color. At the same time, since  $\{u_{t+1}, \dots, u_s\} = \overline{F}_{k,p}^0(c_{l_s}^{(s)} + 1)$ , we can assume that  $u_{t+1}, \dots, u_s$  is colored (only) with  $\gamma_1$ .

Let us denote  $F'_{k,p} := F_{k,p} \setminus \left\{ (b^{(i,i')}, c_{l_s}^{(s)} + 1) \right\}$ . It is easy to see that the coloring of  $F'_{k,p}$  is compatible with that of  $F'$ . Carrying out similar constructions for each  $p \in P^0(F_{k-1})$  containing at least one uncolored polygons of  $F$ , let  $F_k := \bigcup_{p \in P^0(F_{k-1})} F'_{k,p}$ .

One can check that the induction hypotheses hold for  $k$ . Also, the number of uncolored polygons strictly decreases at each step. Therefore, the desired coloring will be completed in a finite number of iterations.  $\square$