This work has as its starting point the following paper [LLP22].

1 Finding induced k-chords in half graphs

The aim of this section is to prove the following proposition.

Proposition 1. For any $k \neq 1, 4$, there exists $n \in \mathbb{N}$ such that the half graph H_n contains an induced k-chord.

Before tackling the above statement, we notice the following fact.

Fact 2. For any graph G and cycle C in G, the number of chords in C is given by $e\left(G\left[V\left(C\right)\right]\right) - |C|$.

We now establish some notation. For any half graph H_n with $n \geq 3$, label its vertices as follows.

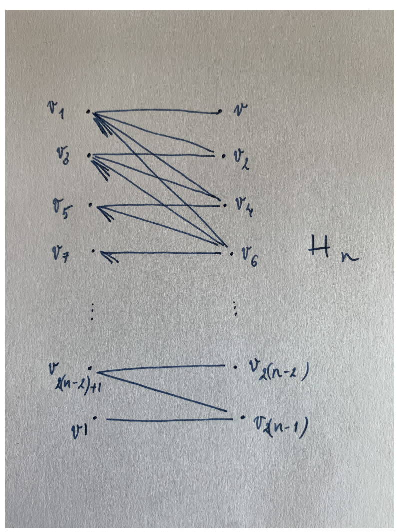

Figure 1: Vertex labeling of a half graph

Notice that the cycle $\left(v_1, v_2, \ldots, v_{2(n-1)}, v_1\right)$ induces a k-chord C_n in H_n with the number K_n of chords being (by Fact 2):

$$K_n = \left(2 + 3 + \cdots + (n-1) + (n-1)\right) - 2\left(n-1\right) = \frac{n\left(n-3\right)}{2}.$$

We now notice the following fact which will be useful in the next proofs.

Fact 3. Given a K_n-chord C_n, one can obtain a new k-chord by removing vertices $v_{2n_1}, v_{2n_2}, \ldots, v_{2n_m}, v_{2n'_1+1}, v_{2n'_2+1}, \ldots, v_{2n'_m+1}$ such that $n_1 < n_2 < \cdots < n_m < n'_1 < n'_2 < \ldots, n'_m$. This last condition is necessary to ensure that the vertices in $V(C_n) \setminus \left\{v_{2n_1}, \ldots, v_{2n_m}, v_{2n'_1+1}, \ldots, v_{2n'_m+1}\right\}$ are part of a unique cycle. The cycle (and the relative induced k-chord) we take into consideration is

$$\left(v_{2i_1+1}, v_{2j_1}, \ldots, v_{2i_{n-m}+1}, v_{2j_{n-m}}, v_{2i_1+1}\right),$$

where $\{i_1, \ldots, i_{n-m}\} = \{1, \ldots, n\} \setminus \{n'_1, \ldots, n'_m\}$ and $\{j_1, \ldots j_{n-m}\} = \{1, \ldots, n\} \setminus \{n_1, \ldots, n_m\}$ with $i_1 < \cdots < i_{n-m}$ and $j_1 < \cdots < j_{n-m}$.

Now, notice that for any $n \geq 4$, we have that $K_n - K_{n-1} = n - 2$. Our strategy is to find and induced k-chord into H_n with $K_{n-1} \leq k \leq K_n$ for al $k \neq 1, 4$. The following lemma represents a step towards this objective.

Lemma 4. For any $n \geq 4$ and any $2 \leq l \leq K_n - 2$, we have that C_n contains an induced $(K_n - l)$-chord (and thus, so does H_n).

Proof. Let $q = l - 1$. Notice that C_n has the following cycle, obtained from C_n by removing v_{2q} and $v_{2(n-2)+1}$ as in Fact 3. We have:

$$C'_n := \left(v_1, v_2, \ldots, v_{2(q-1)+1}, v_{2(q+1)}, \ldots, v_{2(n-2)}, v_{2(n-3)+1}, v_{2(n-1)}, v_1\right)$$

By Fact 3, C'_n induces a k-chord for some k. Notice that $d\left(v_{2q}\right) = q + 1 = l$ and $d\left(v_{2(n-2)+1}\right) = 2$. Since $|C'_n| = |C_n| - 2$, Fact 2 tells us that the number of induced chords of C'_n is

$$K_n - (l + 2) + 2 = K_n - l.$$

This concludes the proof. □

Now, notice that $K_4 = 2$ and $K_5 = 5$. Thus, in order to prove Proposition 1, it suffices to find n such that H_n contains an induced $(K_m - 1)$-chord for all $m \geq 6$. We claim that such n is $m + 1$.

Lemma 5. For any $m \geq 6$, we have that C_{m+1} contains an induced $(K_m - 1)$-chord (and thus, so does H_{m+1}).

Proof. Consider the K_{m+1}-chord C_{m+1}. Just as in the proof of Lemma 4, consider the k-chord C''_{m+1} obtained from C_{m+1} by removing the vertices v_2,

$v_{2(n-5)}$ $v_{2(n-3)+1}$, $v_{2(n-2)+1}$. The cycle of C''_{m+1} that we will consider is the one described in Fact 3. Notice that, since $m \geq 6$, we have that all of the above vertices are distinct. Moreover, we have $d(v_2) = 2$, $d\left(v_{2(n-5)}\right) = n-4$, $d\left(v_{2(n-3)+1}\right) = 3$, $d\left(v_{2(n-2)+1}\right) = 2$. By Fact 2, we deduce that the number of chords in C''_{m+1} is

$$K_{m+1} - (2 + (n-4) + 3 + 2) + 4 = K_{m+1} - (n-1) = K_m - 1.$$

This concludes the proof. □

As pointed out above, Lemma 4 and 5 imply Proposition 1.

2 Ramsey-type arguments

The aim of this section is that of proving the following lemma.

Lemma 6. Let $k \geq 1$ and $d \geq 1$ be some fixed constants. Let G be a graph consisting of an independent set A and three paths P_1, P_2, P_3 such that for any vertex v in any path P_i, v has at most d neighbours in A. There exists $f(k, d)$ such that, if $|A| \geq f(k, d)$, then G contains an induced k-chord.

We start by proving the following lemma.

Lemma 7. An outerplanar k'-chord contains an induced k-chord for any $k \leq k'$.

Proof. We first prove that an outerplanar k'-chord contains an induced $(k'-1)$-chord.

Let $k' \geq 1$ and let G be a k'-chord which is also outerplanar. Fix an outerplanar embedding of G. Let $(v_0, \ldots, v_{L-1})$ be the vertices making the cycle of the k-chord listed in clockwise order. By the outerplanarity, there exists a chord $\{v_i, v_j\}$ such that there is no other chord of G with one end among the vertices $\{v_i, v_{i+1} \ldots, v_j\}$, where indices are understood modulo L. Then, the graph G' induced in G by the vertices $\{v_1, \ldots, v_i, v_j, \ldots, v_{L-1}\}$, where indices are understood modulo L, is a $(k'-1)$-chord. Clearly, G' remains outerplanar.

We can reiterate this argument until we reach the desired k. □

Before proving Lemma 6, fix k and d and let G be a graph with the properties stated in Lemma 6 we make the following observation.

Observation 8. We can assume that for any vertex $v \in A$, the number of neighbours of v in any of the P_js is less than or equal to $k+1$. Or else, we are able to find a $(k+2)$-fan and thus, a k-chord.

We now prove the following lemma which significantly simplifies the proof of Lemma 6.

Lemma 9. We have that Lemma 6 holds if it holds for the case of $d = 1$.

Proof. Suppose that Lemma 6 holds for $d = 1$. Let us fix some $k \geq 1$. Let G be a graph as defined in the statement of Lemma 6. Let $N' \geq 1$ be a constant only depending on k to be determined later on. If A has $N = (N'-1) \times (k+1) \times d + 1$ vertices, then there exist N' vertices in A such that no two of these vertices share a neighbour in P_1. Let $v_1, \ldots, v_{N'}$ be such vertices. By a similar argument applied on P_2, we get the exitence of vertices $v_1, \ldots, v_{N''}$ among $v_1, \ldots, v_{N'}$ such that no two of them share a neighbour in P_2. We then repeat the same reasoning to get vertices $v_1, \ldots, v_{N'''}$ among $v_1, \ldots, v_{N''}$ such that no two share a neighbour in P_3. We set $N''' := f(k, 1)$. Then, by assumption, G has a an induced k-chord. Notice that the constants N'' and N' only depend on k and the constant N only depends on k and d. We set $f(k, d) := N$. Which proves Lemma 6. $\square$

Thus, we can prove Lemma 6 only focusing on the case $d = 1$.

Proof of Lemma 6. As noticed in Lemma 9, we can restrict ourselves to the case of $d = 1$. Let us fix some $k \geq 1$ and Also, let us call G the graph constructed in this proof, which consists of three paths P_1, P_2, P_3 and an independent set A of cardinality $|N|$ for some N depending only of k. We have the restriction that any vertex in any of the P_js has at most one neighbour in A. The aim of this proof is to determine the value of N so that G contains an induced k-chord.

Let us label the vertices of A as $u_1, \ldots, u_N$. For any path P_j, we can give a linear order "$\preceq_j$" to its vertices so that for any vertices u and v of P_j we say that $u \preceq_j v$ if "u comes after v in P_j".

For any path P_j and any vertex $u_i \in A$, denote by $p_j(u_i)$ the $\preceq_j$-smallest vertex of P_j adjacent to u_i. We remark that, since $d = 1$, if $i \neq i'$, then $p_j(u_i)$ and $p_j(u_{i'})$ are distinct.

Let $N_1 \in \mathbb{N}$ be a constant only depending on k, Erdős-Szekeres Theorem tells us that, for N big enough, there exist $v'_1, \ldots, v'_{N_1} \in A$ such that either $p_1(v_1) \preceq_1 \cdots \preceq_1 p_1\left(v'_{N_1}\right)$ or $p_1\left(v'_{N_1}\right) \preceq_1 \cdots \preceq_1 p_2(v'_1)$. We can assume without loss of generality that the former holds. Or else, we reverse the order $\preceq_1$ on P_1. Since N_1 can be arbitrarily large and only depends on k, we can assume without loss of generality that p_1 is increasing, that is, for any $i \leq j$, we have that $p_1(u_i) \preceq_1 p_1(u_j)$. We can repeat the same argument on P_2 and P_3 so that we can assume without loss of generality that the function p_j is increasing for all $j \in \{1, 2, 3\}$. Refer to Figure 2 for an illustration.

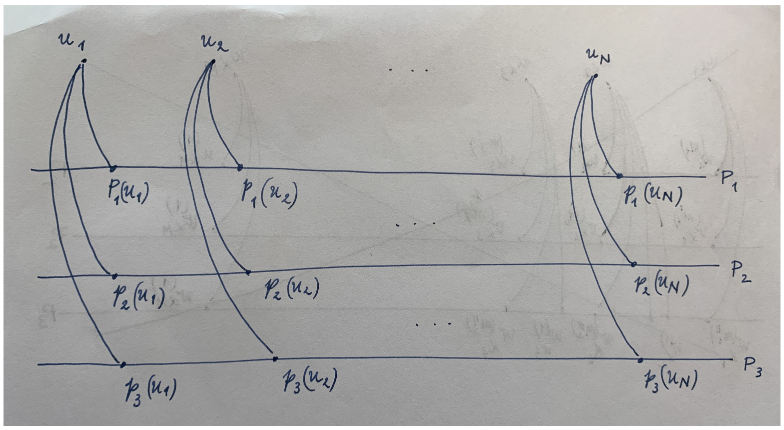

Figure 2: Graph G where the function p_j is increasing for all $j \in \{1, 2, 3\}$.

We now inductively construct a family $\left(u_{i_j}\right)_{j=1}^{M}$ for M to be determined later on, but only depending on k with the following property. For any $u_i \in A$, denote $p_1(u_i) = u_i^{(1)} \preceq_1 \cdots \preceq_1 u_i^{(k_i)}$ the neighbours of u_i in P_1. Then we have Γ: "For all $1 \leq l \leq M$, there exists m_l with $u_{i_l}^{(i)} \preceq_1 u_{i_l}^{(m_l)} \preceq_1 u_{i_l}^{(k_{i_l})}$ such that for all $1 \leq m \leq m_l$, and all $p < l$, $u_{i_p}^{(m_p)} \preceq_1 u_{i_l}^{(m)}$ and for all $m > m_l$, $u_{i_M}^{(1)} \preceq_1 u_{i_l}^{(m)}$". For an illustration refer to Figure 3.

- **Base case**. Let $q_0 := 0$ and let $A =: S_{q_0}^{(0)}$. Let also $i_1 := 1$. For any $1 \leq q \leq k_1 - 1$, let $S_q^{(1)} := \left\{u_i \middle| u_1^{(q)} \preceq_1 u_i^{(1)} \preceq_1 u_1^{(q+1)}\right\}$ and let $S_{k_1}^{(1)} := \left\{u_i \middle| u_1^{(k_1)} \preceq_1 u_i^{1}\right\}$. Notice that the $S_q^{(1)}$s form a partition of $S_{q_0}^{(0)}$ $(= A)$. So, for any constant $N^{(1)} \in \mathbb{N}$ only depending on k, we can ask N to be big enough so that for some $1 \leq q_1 \leq k_1$, we have $\left|S_{q_1}^{(1)}\right| \geq N^{(1)}$. Let $G^{(1)}$ be the subgraph of G induced by $S_{q_1}^{(1)}$, P_1, P_2, P_3.

- **Induction step**. Suppose that the set $S_{q_l}^{(l)} \subseteq A$ and the indices

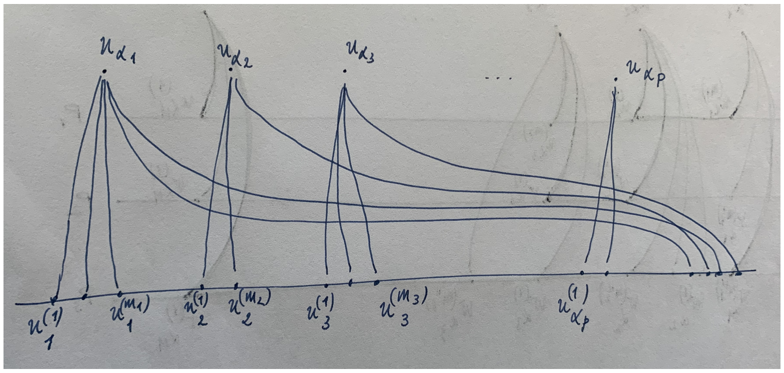

Figure 3: Outcome of the construction satisfying Γ.

$i_1, \ldots, i_l$ have been set. Define the graph $G^{(l)}$ analogously as in the base case. If $l \leq M-2$, we can use the same argument as in the base case by looking at $G^{(l)}$ and choosing $u_{i_{l+1}}$ to be the vertex in $S^{(l)}_{q_l}$ such that $u^{(1)}_i \preceq_1 u^{(1)}_j$, for all $u_j \in S^{(l)}_{q_l}$. We then find $S^{(l+1)}_{q_{l+1}} \subseteq S^{(l)}_{q_l}$ with $\left|S^{(l+1)}_{q_{l+1}}\right| \geq N^{(l+1)}$ for some $N^{(l+1)} \in \mathbb{N}$ only depending on k. If $l = M-1$, then any choice of u_{i_M} out of the vertices in $S^{(M-1)}_{q_{M-1}}$ will have the desired properties.

It is clear by the above construction and in particular by the definition of the sets $S^{(l)}_{q_l}$ and by the fact that p_1 is increasing that Γ is satisfied.

Since M is arbitrarily large and depends only on k, we can repeat this same argument relatively to P_2 by taking only the family $\left\{u_{i_j}\right\}_{j=1}^{M}$ into consideration. We reiterate this for P_3. We finally get a family $\left\{u_{\alpha_j}\right\}_{j=1}^{P}$ so that Γ holds for P_1 and analogous versions of Γ hold for P_2 and P_3.

A word on notation. Let $u_i \in \left\{u_{\alpha_j}\right\}_{j=1}^{P}$, denote by $p_2\left(u_i\right) = v^{(1)}_i \preceq_2 \cdots \preceq_2 v^{\left(k'_i\right)}_i$ (resp. $p_3\left(u_i\right) = w^{(1)}_i \preceq_3 \cdots \preceq_3 w^{\left(k'_i\right)}_i$) the neighbours of u_i in P_2 (resp. P_3). The index analogous to m_l in the construction relative to the path P_2 (resp. P_3) is denoted by m'_l (resp. m''_l) for all $1 \leq l \leq P$.

Now, consider the k'-chord induced by the following cycle:

$$\mathcal{K} = \left(u_{\alpha_1}, u^{(m_1)}_{\alpha_1}, \ldots, u^{(1)}_{\alpha_2}, u_{\alpha_2}, v^{\left(m'_2\right)}_{\alpha_2}, \ldots, v^{(1)}_{\alpha_3}, u_{\alpha_3}, \ldots, u_{\alpha_P}, w^{(1)}_{\alpha_P}, \ldots, w^{\left(m''_1\right)}_{\alpha_1}, u_{\alpha_1}\right).$$

Refer to Figure 4 for an illustration of the construction of $\mathcal{K}$.

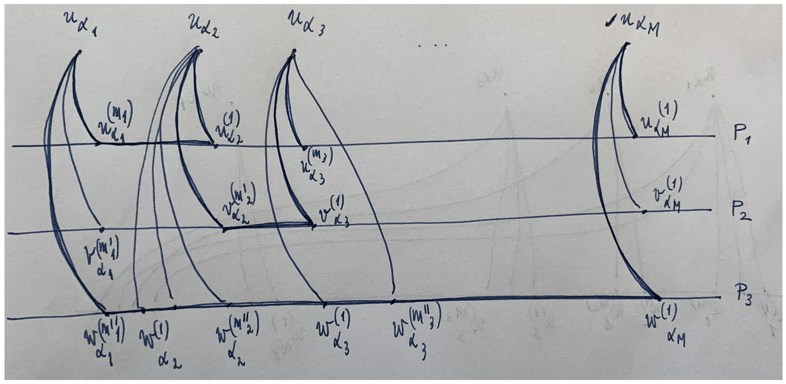

Figure 4: Construction of the k'-chord $\mathcal{K}$.

Since the P_js are induced paths, they are chordless and there are no edges connecting them. We can also assume without loss of generality that for any $1 \leq i \leq P-1$, we have $u_i^{(1)} \not\sim u_{i+1}^{(1)}$, $v_i^{(1)} \not\sim v_{i+1}^{(1)}$, $w_i^{(1)} \not\sim w_{i+1}^{(1)}$. If not, we can carry the same construction as before to obtain, instead of $\{u_{\alpha_i}\}_{i=1}^{P}$, the family $\{u_{\alpha_i}\}_{i=1}^{2P}$ and considering in the construction of $\mathcal{K}$ only the subfamily $\left\{u_{\alpha_{2i+1}}\right\}_{i=0}^{P-1}$.

Thus, by construction, the only chords of $\mathcal{K}$ are those of the form $\left\{v_{\alpha_i}, w_{\alpha_i}^{(m)}\right\}$ for $2 \leq i \leq P-1$ and $1 \leq m \leq m_i''$. Therefore, $\mathcal{K}$ has at least $P-2$ chords. Setting $P := k+2$, we get that $\mathcal{K}$ has at least k chords.

By the construction carried out trough this proof, we have that $w_{\alpha_i}^{(m_i'')} \preceq_3 w_{\alpha_{i+1}}^{(1)}$ for all $1 \leq i \leq P$. Therefore, $\mathcal{K}$ is outerplanar. We use Lemma 7 to get an induced k-chord in G.

Notice that all the constants involved in the above construction only depend on P, which, in turn, only depends on k. We thus set $f(k,1) := \mathbb{N}$, which concludes the proof. $\square$

References

[LLP22] Joonkyung Lee, Shoham Letzter, and Alexey Pokrovskiy. *Chi-boundedness of graphs containing no cycles with k chords.* 2022. arXiv: 2208.14860 [math.CO]. URL: https://arxiv.org/abs/2208.14860.