1 Upper bounds for the chromatic number of chordal graphs and their complements

1.1 Discussion of the problem

A graph G such that $V(G) = \{v_1, \ldots, v_n\}$ is called a chord intersection graph (or simply a chord graph) if for some family $H = \{h_i, \ldots, h_n\}$ of chords of a circle the vertices v_i and v_j are adjacent if and only if h_i and h_j intersect (i.e., have a common point not belonging to the circle). Chord intersection graphs arise when considering various combinatorial problems, from sorting problems to the study of planar graphs or chain fractions [4, 24]. In particular, the problem of determining the smallest number of stacks necessary to realize a given permutation of numbers reduces to finding the chromatic number of the corresponding chord graph [4, 16].

A graph G with $V(G) = \{v_1, \ldots, v_n\}$ is called a meshing graph if for some family $F = \{h_1, \ldots, h_n\}$ of line segments the vertices v_i and v_j are adjacent if and only if the segments h_i and h_j are meshing, i.e., intersecting and $h_i \not\subseteq h_j$ and $h_j \not\subseteq h_i$. Using stereographic projection, it is easy to verify that G is a chordal graph if and only if it is a meshing graph [4].

The density and non-density of a chord graph can be found in polynomial time of the number of its vertices [4, 25]. But no such algorithms are known for finding the clique number (i.e., chromatic complement number) of chord graphs, and the problem of finding the chromatic number of chord graphs is NP-hard [26]. It is all the more important and interesting to find $\varphi(\mathcal{X}, k)$ and $\varphi(\overline{\mathcal{X}}, k)$ defined in the introduction, where $\mathcal{X}$ is the class of chord graphs and $\overline{\mathcal{X}}$ is the class of their complements. The following results are known in this direction. Karapetyan [27] proved that $4 \leq \varphi(\mathcal{X}, 2) \leq 8$ $\varphi(\mathcal{X}, k) \leq k(k+1)/2$. The second of these results was later published in [28]. Dyarfash [29] proposed a proof of the statement $\varphi(\mathcal{X}, k) \leq 2^k k^2 (k-1)$. Unfortunately, lemma 2 of [29] is incorrect. Nevertheless, an estimate follows from [29] $\varphi(\mathcal{X}, k) \leq 2^k (2^k - 2) k^2$. Below we give an asymptotically exact estimate for $k \to +\infty$ of $\varphi(\mathcal{X}, k)$. It turns out that $\varphi(\mathcal{X}, k) \sim k \log(k)$. Furthermore, using the idea of Dyarfash's proof [29], we will show that $\varphi(\mathcal{X}, k) \leq 2^k (k+2) k$, but not up to $\varphi(\mathcal{X}, 2) \leq 5$. Note that the latter result is announced but not proved in [28, 30]. In Section 4.4 we will show $\varphi(\mathcal{X}, k) \geq (k/2)(\ln k - 2)$. These results show that there are graphs in $\mathcal{X}$ that are more "complicated" than graphs from the $\mathcal{D}$-class of all intersection graphs of circle arcs. It is known [4, 9, 31] that $\varphi(\mathcal{D}, k) = [3k/2]$, $\varphi(\bar{\mathcal{D}}, k) = k + 1$ for $k \geq 1$.

It is not difficult to verify [4] that for every family H of chords of a circle there is a family H' whose chord intersection graph coincides with a similar graph for H, but whose different chords do not have common ends. Therefore, in the proof of the upperbound, we consider only families of chords or intervals whose different elements do not have common ends. In order not to clutter the notation, for a family H of chords (intervals) we will denote by $\alpha(H)$, $\chi(H)$, etc. the non-density, chromatic number, etc. for the corresponding chord intersection graph (meshing graph). Only finite families of chords or segments are considered throughout. Let us condition some more notation. The chord connecting points a and b will be denoted as an open segment $]a, b[$. If a and b are the points of the circle, we denote by $[a, b]$, the arc connecting a and b, that when we move along it from a to b we go around the circle counterclockwise. We will write $]c, d[\subseteq [a, b]$ if both ends of the chord $]c, d[$ belong to the arcs $[a, b]$. In particular $]a, b[\subseteq [a, b]$.

1.2 Upperbound for $\varphi\left(\bar{\mathcal{X}}, k\right)$.

Theorem 1. Let H be a family of chords of the circle and $\alpha(H) = k$. Then $\sigma(H) = \chi(\bar{H}) \leq \Psi(k) = \sum_{i=1}^{[k/2]} [(k+1)/i] + \varepsilon(k) - |Q(k)|$ with $Q(k) = \{m \in Z | 1 < (k+1)/4 < m < (k+1)/3\}$ and $\varepsilon(k) = 0$ if k is even or 1 otherwise.

Proof. Let us call a chord $h_0 =]a, b[\in H$ a separating chord in H if there are such chords h_1, h_2 such that $h_1 \subseteq [a, b]$, $h_2 \subseteq [b, a]$. Let H_1 be a family of non-separating chords in H and for all $i = 2, 3, \ldots, |H|$, let H_i be a family of non-separating chords in $H \setminus \bigcup_{1 \leq j \leq i-1} H_j$.

By definition of H_i, if $h =]a, b[\in H_i$, $i > 1$, then there exist chords $h', h'' \in H_{i-1}$ such that $h' \subseteq [a, b]$, $h' \subseteq [b, a]$. The chord-connecting points a and b, can be written $]a, b[$ or $]b, a[$. In

the following we denote such a chord belonging to H by $]a,b[$ if the arc $[a,b]$ does not contain chords of $\bigcup_{j\geq i} H_j \setminus \{]a,b[\}$ and by $]b,a[$ otherwise. With this notation, the graph of intersections of chords of family $H_i = \{]a_j^i, b_j^i[\}_{1\leq j\leq m_i}$ coincides with the graph of intersections of arcs of family $D_i = \{[a_j^i, b_j^i]\}_{1\leq j\leq m_i}$. But it has been noted above that, according to [31], we have $\sigma(H_i) = \sigma(D_i) \leq \alpha(D_i) + 1 = \alpha(H_i) + 1$.

Let us show that $\alpha(H_i) \leq k/i$. For $i = 1$ this follows from the assumptions of the theorem. Let $i > 1$. Let us choose $t = \alpha(H_i)$ non-intersecting chords $]a_1^i, b_1^i[, \ldots]a_i^t, b_i^t[$. By definition of H_i, for each $1 \leq j \leq t$, there exists $]a_j^{i-1}, b_j^{i-1}[\in H_{i-1}$, such that $]a_j^{i-1}, b_j^{i-1}[\subseteq]a_j^i, b_j^i[$. Similarly, if $i-1 > 1$, then for each $1 \leq j \leq k$, there exist chords $]a_j^{i-2}, b_j^{i-2}[\subseteq H_{i-2}$ such that $]a_j^{i-2}, b_j^{i-2}[\subseteq]a_j^{i-1}, a_j^{i-1}[$ etc.

Hence, there exists a family $H' = \left\{]a_j^l, b_j^l[\middle| l = 1, \ldots, i \text{ and } j = 1, \ldots, t\right\}$ of $\bigcup_{l=1}^{i} H_l$ of pairwise non-intersecting chords such that $\alpha(H_i) \leq k/i$.

Assume that $H_l \neq \emptyset$. Let us consider $]a^l, b^l[\in H_l$. If $l > 1$, then there are chords $]a_1^{l-1}, b_1^{l-1}[\in H_{l-1}$ and $]a_2^{l-1}, b_2^{l-1}[\in H_{l-1}$ such that $]a_1^{l-1}, b_1^{l-1}[\in [a^l, b^l]$ and $]a_2^{l-1}, b_2^{l-1}[\in [b^l, a^l]$. As in the previous argument, we assume the existence of a subset H'' of H of pairwise non-intersecting chords with $H'' = \left\{]a^l, b^l[\right\} \cup \left\{]a_j^i, b_j^i\middle| i = 1, \ldots, l-1 \text{ and } j = 1, 2[\right\}$. Thus $k \geq |H''| = 1 + 2(l-1)$ and $l \leq (k+1)/2$. Hence $H_i \neq \emptyset$ for all $i > (k+1)/2$ and $\square$