

1 Finding induced k -chords in half graphs

The aim of this section is to prove the following proposition.

Proposition 1. For any $k \neq 1, 4$, there exists $n \in \mathbb{N}$ such that the half graph H_n contains an induced k -chord.

Before tackling the above statement, we notice the following fact.

Fact 2. For any graph G and cycle C in G , the number of chords in C is given by $e(G[V(C)]) - |C|$.

We now establish some notation. For any half graph H_n with $n \geq 3$, label its vertices as follows.

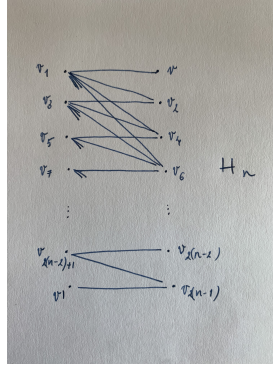


Figure 1: Vertex labeling of a half graph

Notice that the cycle $(v_1, v_2, \dots, v_{2(n-1)}, v_1)$ induces a k -chord C_n in H_n with the number K_n of chords being (by Fact 2):

$$K_n = (2 + 3 + \dots + (n-1) + (n-1)) - 2(n-1) = \frac{n(n-3)}{2}.$$

We now notice the following fact which will be useful in the next proofs.

Fact 3. Given a K_n -chord C_n , one can obtain a new k -chord by removing vertices $v_{2n_1}, v_{2n_2}, \dots, v_{2n_m}, v_{2n'_1+1}, v_{2n'_2+1}, \dots, v_{2n'_m+1}$ such that $n_1 < n_2 < \dots < n_m < n'_1 < n'_2 < \dots, n'_m$. This last condition is necessary to ensure that the vertices in $V(C_n) \setminus \{v_{2n_1}, \dots, v_{2n_m}, v_{2n'_1+1}, \dots, v_{2n'_m+1}\}$ are part of

a unique cycle. The cycle (and the relative induced k -chord) we take into consideration is

$$(v_{2i_1+1}, v_{2j_1}, \dots, v_{2i_{n-m}+1}, v_{2j_{n-m}}, v_{2i_1+1}),$$

where $\{i_1, \dots, i_{n-m}\} = \{1, \dots, n\} \setminus \{n'_1, \dots, n'_m\}$ and $\{j_1, \dots, j_{n-m}\} = \{1, \dots, n\} \setminus \{n_1, \dots, n_m\}$ with $i_1 < \dots < i_{n-m}$ and $j_1 < \dots < j_{n-m}$.

Now, notice that for any $n \geq 4$, we have that $K_n - K_{n-1} = n - 2$. Our strategy is to find an induced k -chord into H_n with $K_{n-1} \leq k \leq K_n$ for all $k \neq 1, 4$. The following lemma represents a step towards this objective.

Lemma 4. For any $n \geq 4$ and any $2 \leq l \leq K_n - 2$, we have that C_n contains an induced $(K_n - l)$ -chord (and thus, so does H_n).

Proof. Let $q = l - 1$. Notice that C_n has the following cycle, obtained from C_n by removing v_{2q} and $v_{2(n-2)+1}$ as in Fact 3. We have:

$$C'_n := (v_1, v_2, \dots, v_{2(q-1)+1}, v_{2(q+1)}, \dots, v_{2(n-2)}, v_{2(n-3)+1}, v_{2(n-1)}, v_1)$$

By Fact 3, C'_n induces a k -chord for some k . Notice that $d(v_{2q}) = q + 1 = l$ and $d(v_{2(n-2)+1}) = 2$. Since $|C'_n| = |C_n| - 2$, Fact 2 tells us that the number of induced chords of C'_n is

$$K_n - (l + 2) + 2 = K_n - l.$$

This concludes the proof. \square

Now, notice that $K_4 = 2$ and $K_5 = 5$. Thus, in order to prove Proposition 1, it suffices to find n such that H_n contains an induced $(K_m - 1)$ -chord for all $m \geq 6$. We claim that such n is $m + 1$.

Lemma 5. For any $m \geq 6$, we have that C_{m+1} contains an induced $(K_m - 1)$ -chord (and thus, so does H_{m+1}).

Proof. Consider the K_{m+1} -chord C_{m+1} . Just as in the proof of Lemma 4, consider the k -chord C''_{m+1} obtained from C_{m+1} by removing the vertices $v_2, v_{2(n-5)}, v_{2(n-3)+1}, v_{2(n-2)+1}$. The cycle of C''_{m+1} that we will consider is the one described in Fact 3. Notice that, since $m \geq 6$, we have that all of the above vertices are distinct. Moreover, we have $d(v_2) = 2$, $d(v_{2(n-5)}) = n - 4$, $d(v_{2(n-3)+1}) = 3$, $d(v_{2(n-2)+1}) = 2$. By Fact 2, we deduce that the number of chords in C''_{m+1} is

$$K_{m+1} - (2 + (n - 4) + 3 + 2) + 4 = K_{m+1} - (n - 1) = K_m - 1.$$

This concludes the proof. \square

As pointed out above, Lemma 4 and 5 imply Proposition 1.

2 Ramsey-type arguments

The aim of this section is that of proving the following lemma.

Lemma 6. For any fixed k, d , there exists some $f(k, d)$ such that every graph G consisting of three induced paths P_1, P_2, P_3 and an independent set of at least $f(k, d)$ vertices A where each vertex in A has at least one neighbour in each P_i and no vertex of G has more than d neighbours in A , we have that G contains an induced k -chord.

We start by proving the following lemma.

Lemma 7. An outerplanar k' -chord contains an induced k -chord for any $k' \leq k$.

Proof. We first prove that an outerplanar k' -chord contains an induced $(k' - 1)$ -chord.

Let $k' \geq 1$ and let G be a k' -chord which is also outerplanar. Fix an outerplanar embedding of G . Let (v_0, \dots, v_{L-1}) be the vertices making the cycle of the k' -chord listed in clockwise order. By the outerplanarity, there exists a chord $\{v_i, v_j\}$ such that there is no other chord of G with one end among the vertices $\{v_i, v_{i+1}, \dots, v_j\}$, where indices are understood modulo L . Then, the graph G' induced in G by the vertices $\{v_1, \dots, v_i, v_j, \dots, v_{L-1}\}$, where indices are understood modulo L , is a $(k' - 1)$ -chord. Clearly, G' remains outerplanar.

We can reiterate this argument until we reach the desired k . \square