

This work has as its starting point the following paper [LLP22].

1 Finding induced k -chords in half graphs

The aim of this section is to prove the following proposition.

Proposition 1. For any $k \neq 1, 4$, there exists $n \in \mathbb{N}$ such that the half graph H_n contains an induced k -chord.

Before tackling the above statement, we notice the following fact.

Fact 2. For any graph G and cycle C in G , the number of chords in C is given by $e(G[V(C)]) - |C|$.

We now establish some notation. For any half graph H_n with $n \geq 3$, label its vertices as follows.

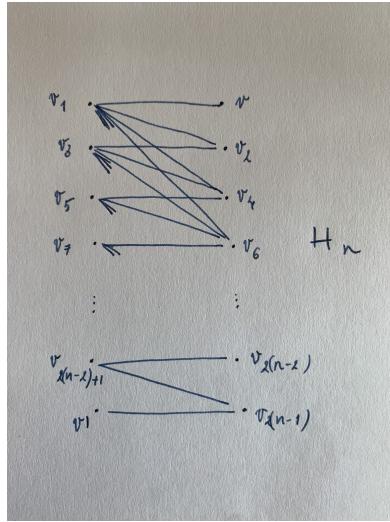


Figure 1: Vertex labeling of a half graph

Notice that the cycle $(v_1, v_2, \dots, v_{2(n-1)}, v_1)$ induces a k -chord C_n in H_n with the number K_n of chords being (by Fact 2):

$$K_n = (2 + 3 + \dots + (n-1) + (n-1)) - 2(n-1) = \frac{n(n-3)}{2}.$$

We now notice the following fact which will be useful in the next proofs.

Fact 3. Given a K_n -chord C_n , one can obtain a new k -chord by removing vertices $v_{2n_1}, v_{2n_2}, \dots, v_{2n_m}, v_{2n'_1+1}, v_{2n'_2+1}, \dots, v_{2n'_m+1}$ such that $n_1 < n_2 < \dots < n_m < n'_1 < n'_2 < \dots < n'_m$. This last condition is necessary to ensure that the vertices in $V(C_n) \setminus \{v_{2n_1}, \dots, v_{2n_m}, v_{2n'_1+1}, \dots, v_{2n'_m+1}\}$ are part of a unique cycle. The cycle (and the relative induced k -chord) we take into consideration is

$$(v_{2i_1+1}, v_{2j_1}, \dots, v_{2i_{n-m}+1}, v_{2j_{n-m}}, v_{2i_1+1}),$$

where $\{i_1, \dots, i_{n-m}\} = \{1, \dots, n\} \setminus \{n'_1, \dots, n'_m\}$ and $\{j_1, \dots, j_{n-m}\} = \{1, \dots, n\} \setminus \{n_1, \dots, n_m\}$ with $i_1 < \dots < i_{n-m}$ and $j_1 < \dots < j_{n-m}$.

Now, notice that for any $n \geq 4$, we have that $K_n - K_{n-1} = n - 2$. Our strategy is to find an induced k -chord into H_n with $K_{n-1} \leq k \leq K_n$ for all $k \neq 1, 4$. The following lemma represents a step towards this objective.

Lemma 4. For any $n \geq 4$ and any $2 \leq l \leq K_n - 2$, we have that C_n contains an induced $(K_n - l)$ -chord (and thus, so does H_n).

Proof. Let $q = l - 1$. Notice that C_n has the following cycle, obtained from C_n by removing v_{2q} and $v_{2(n-2)+1}$ as in Fact 3. We have:

$$C'_n := (v_1, v_2, \dots, v_{2(q-1)+1}, v_{2(q+1)}, \dots, v_{2(n-2)}, v_{2(n-3)+1}, v_{2(n-1)}, v_1)$$

By Fact 3, C'_n induces a k -chord for some k . Notice that $d(v_{2q}) = q + 1 = l$ and $d(v_{2(n-2)+1}) = 2$. Since $|C'_n| = |C_n| - 2$, Fact 2 tells us that the number of induced chords of C'_n is

$$K_n - (l + 2) + 2 = K_n - l.$$

This concludes the proof. \square

Now, notice that $K_4 = 2$ and $K_5 = 5$. Thus, in order to prove Proposition 1, it suffices to find n such that H_n contains an induced $(K_m - 1)$ -chord for all $m \geq 6$. We claim that such n is $m + 1$.

Lemma 5. For any $m \geq 6$, we have that C_{m+1} contains an induced $(K_m - 1)$ -chord (and thus, so does H_{m+1}).

Proof. Consider the K_{m+1} -chord C_{m+1} . Just as in the proof of Lemma 4, consider the k -chord C''_{m+1} obtained from C_{m+1} by removing the vertices v_2 ,

$v_{2(n-5)}, v_{2(n-3)+1}, v_{2(n-2)+1}$. The cycle of C''_{m+1} that we will consider is the one described in Fact 3. Notice that, since $m \geq 6$, we have that all of the above vertices are distinct. Moreover, we have $d(v_2) = 2$, $d(v_{2(n-5)}) = n-4$, $d(v_{2(n-3)+1}) = 3$, $d(v_{2(n-2)+1}) = 2$. By Fact 2, we deduce that the number of chords in C''_{m+1} is

$$K_{m+1} - (2 + (n-4) + 3 + 2) + 4 = K_{m+1} - (n-1) = K_m - 1.$$

This concludes the proof. \square

As pointed out above, Lemma 4 and 5 imply Proposition 1.

2 Ramsey-type arguments

The aim of this section is that of proving the following lemma.

Lemma 6. For any fixed k, d , there exists some $f(k, d)$ such that every graph G consisting of three induced paths P_1, P_2, P_3 and an independent set of at least $f(k, d)$ vertices A where each vertex in A has at least one neighbour in each P_i and no vertex of G has more than d neighbours in A , we have that G contains an induced k -chord.

We start by proving the following lemma.

Lemma 7. An outerplanar k' -chord contains an induced k -chord for any $k' \leq k$.

Proof. We first prove that an outerplanar k' -chord contains an induced $(k' - 1)$ -chord.

Let $k' \geq 1$ and let G be a k' -chord which is also outerplanar. Fix an outerplanar embedding of G . Let (v_0, \dots, v_{L-1}) be the vertices making the cycle of the k -chord listed in clockwise order. By the outerplanarity, there exists a chord $\{v_i, v_j\}$ such that there is no other chord of G with one end among the vertices $\{v_i, v_{i+1}, \dots, v_j\}$, where indices are understood modulo L . Then, the graph G' induced in G by the vertices $\{v_1, \dots, v_i, v_j, \dots, v_{L-1}\}$, where indices are understood modulo L , is a $(k' - 1)$ -chord. Clearly, G' remains outerplanar.

We can reiterate this argument until we reach the desired k . \square

Before proving Lemma 6, fix k and d and let G be a graph with the properties stated in Lemma 6 we make the following observation.

Observation 8. We can assume that for any vertex $v \in A$, the number of neighbours of v in any of the P_j s is less than or equal to $k + 1$. Or else, we are able to find a $(k + 2)$ -fan and thus, a k -chord.

We now prove the following lemma which significantly simplifies the proof of Lemma 6.

Lemma 9. We have that Lemma 6 holds if it holds for the case of $d = 1$.

Proof. We observe that the number of neighbours of each $v \in A$ in P_1 is less than or equal to $k + 1$ and the number of neighbours in A of each vertex of P_1 is bounded above by d . So, let $N \geq 1$ be a constant only depending on k and d , if A has $N' = (N - 1) \times (k + 1) \times d + 1$ vertices, then there exist N vertices in A such that no two of these vertices share a neighbour in P_1 . Since we can ask N to be arbitrarily large and N' only depends on k and d , we can assume without loss of generality that no two vertices of A share a neighbour in P_1 . We can then repeat the same argument for P_2 and P_3 . All in all, we can assume without loss of generality that no two vertices in A share a common neighbour in any of the P_j s.

Therefore, we can restrict the proof of Lemma 6 to the case of $d = 1$. \square

We are now ready to prove Lemma 6.

Proof of Lemma 6. As noticed in Lemma 9, we can restrict ourselves to the case of $d = 1$. Let us fix some $k \geq 1$ and Also, let us call G the graph constructed in this proof, which consists of three paths P_1 , P_2 , P_3 and an independent set A of cardinality $|N|$ for some N depending only of k . We have the restriction that any vertex in any of the P_j s has at most one neighbour in A . The aim of this proof is to determine the value of N so that G contains an induced k -chord.

Let us label the vertices of A as u_1, \dots, u_N . For any path P_j , we can give a linear order “ \preceq_j ” to its vertices so that for any vertices u and v of P_j we say that $u \preceq_j v$ if “ u comes after v in P_j ”.

For any path P_j and any vertex $u_i \in A$, denote by $p_j(u_i)$ the \preceq_j -smallest vertex of P_j adjacent to u_i . We remark that, since $d = 1$, if $i \neq i'$, then $p_j(u_i)$ and $p_j(u_{i'})$ are distinct.

Let $N_1 \in \mathbb{N}$ be a constant only depending on k , Erdős-Szekeres Theorem tells us that, for N big enough, there exist $v'_1, \dots, v'_{N_1} \in A$ such that either $p_1(v_1) \preceq_1 \dots \preceq_1 p_1(v'_{N_1})$ or $p_1(v'_{N_1}) \preceq_1 \dots \preceq_1 p_2(v'_1)$. We can assume without loss of generality that the former holds. Or else, we reverse the

order \preceq_1 on P_1 . Since N_1 can be arbitrarily large and only depends on k , we can assume without loss of generality that p_1 is increasing, that is, for any $i \leq j$, we have that $p_1(u_i) \preceq_1 p_1(u_j)$. We can repeat the same argument on P_2 and P_3 so that we can assume without loss of generality that the function p_j is increasing for all $j \in \{1, 2, 3\}$. Refer to Figure 2 for an illustration.

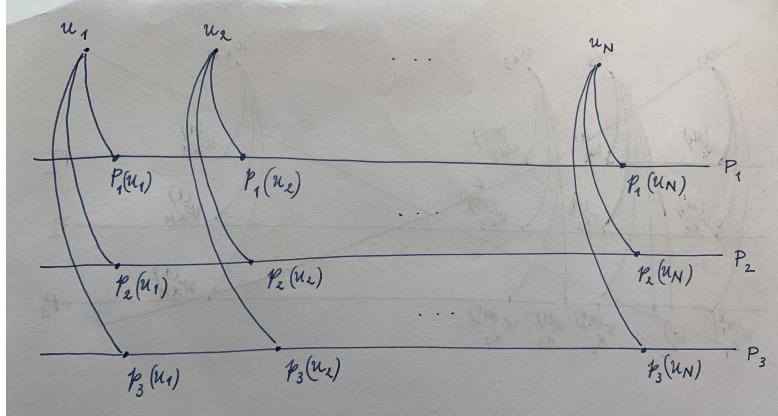


Figure 2: Graph G where the function p_j is increasing for all $j \in \{1, 2, 3\}$.

We now inductively construct a family $(u_{i_j})_{j=1}^M$ for M to be determined later on, but only depending on k with the following property. For any $u_i \in A$, denote $p_1(u_i) = u_i^{(1)} \preceq_1 \dots \preceq_1 u_i^{(k_i)}$ the neighbours of u_i in P_1 . Then we have Γ : “For all $1 \leq l \leq M$, there exists m_l with $u_{i_l}^{(i_l)} \preceq_1 u_{i_l}^{(m_l)} \preceq_1 u_{i_l}^{(k_{i_l})}$ such that for all $1 \leq m \leq m_l$, and all $p < l$, $u_{i_p}^{(m_p)} \preceq_1 u_{i_l}^{(m)}$ and for all $m > m_l$, $u_{i_M}^{(1)} \preceq_1 u_{i_l}^{(m)}$ ”. For an illustration refer to Figure 3.

- **Base case.** Let $q_0 := 0$ and let $A =: S_{q_0}^{(0)}$. Let also $i_1 := 1$. For any $1 \leq q \leq k_1 - 1$, let $S_q^{(1)} := \left\{ u_i \mid u_1^{(q)} \preceq_1 u_i^{(1)} \preceq_1 u_1^{(q+1)} \right\}$ and let $S_{k_1}^{(1)} := \left\{ u_i \mid u_1^{(k_1)} \preceq_1 u_i^{(1)} \right\}$. Notice that the $S_q^{(1)}$ s form a partition of $S_{q_0}^{(0)}$ ($= A$). So, for any constant $N^{(1)} \in \mathbb{N}$ only depending on k , we can ask N to be big enough so that for some $1 \leq q_1 \leq k_1$, we have $|S_{q_1}^{(1)}| \geq N^{(1)}$. Let $G^{(1)}$ be the subgraph of G induced by $S_{q_1}^{(1)}$, P_1 , P_2 , P_3 .

- **Induction step.** Suppose that the set $S_{q_l}^{(l)} \subseteq A$ and the indices i_1, \dots, i_l have been set. Define the graph $G^{(l)}$ analogously as in the

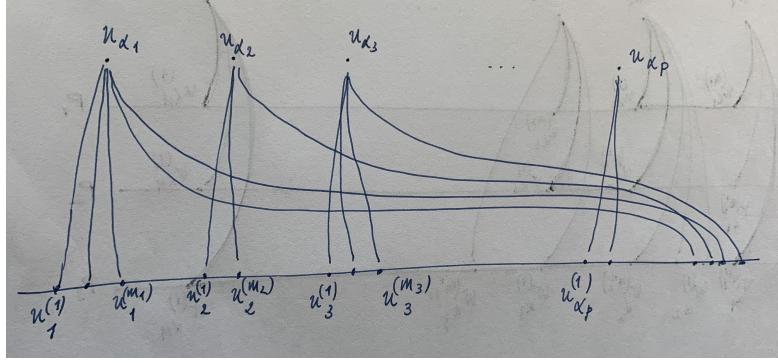


Figure 3: Outcome of the construction satisfying Γ .

base case. If $l \leq M - 2$, we can use the same argument as in the base case by looking at $G^{(l)}$ and choosing $u_{i_{l+1}}$ to be the vertex in $S_{q_l}^{(l)}$ such that $u_i^{(1)} \preceq_1 u_j^{(1)}$, for all $u_j \in S_{q_l}^{(l)}$. We then find $S_{q_{l+1}}^{(l+1)} \subseteq S_{q_l}^{(l)}$ with $|S_{q_{l+1}}^{(l+1)}| \geq N^{(l+1)}$ for some $N^{(l+1)} \in \mathbb{N}$ only depending on k . If $l = M - 1$, then any choice of u_{i_M} out of the vertices in $S_{q_{M-1}}^{(M-1)}$ will have the desired properties.

It is clear by the above construction and in particular by the definition of the sets $S_{q_l}^{(l)}$ and by the fact that p_1 is increasing that Γ is satisfied.

Since M is arbitrarily large and depends only on k , we can repeat this same argument relatively to P_2 by taking only the family $\{u_{i_j}\}_{j=1}^M$ into consideration. We reiterate this for P_3 . We finally get a family $\{u_{\alpha_j}\}_{j=1}^P$ so that Γ holds for P_1 and analogous versions of Γ hold for P_2 and P_3 .

A word on notation. Let $u_i \in \{u_{\alpha_j}\}_{j=1}^P$, denote by $p_2(u_i) = v_i^{(1)} \preceq_2 \dots \preceq_2 v_i^{(k'_i)}$ (resp. $p_3(u_i) = w_i^{(1)} \preceq_3 \dots \preceq_3 w_i^{(k'_i)}$) the neighbours of u_i in P_2 (resp. P_3). The index analogous to m_l in the construction relative to the path P_2 (resp. P_3) is denoted by m'_l (resp. m''_l) for all $1 \leq l \leq P$.

Now, consider the k' -chord induced by the following cycle:

$$\mathcal{K} = \left(u_{\alpha_1}, u_{\alpha_1}^{(m_1)}, \dots, u_{\alpha_2}^{(1)}, u_{\alpha_2}, v_{\alpha_2}^{(m'_2)}, \dots, v_{\alpha_3}^{(1)}, u_{\alpha_3}, \dots, u_{\alpha_P}, w_{\alpha_P}^{(1)}, \dots, w_{\alpha_1}^{(m''_1)}, u_{\alpha_1} \right).$$

Refer to Figure 4 for an illustration of the construction of \mathcal{K} .

Since the P_j 's are induced paths, they are chordless and there are no edges connecting them. We can also assume without loss of generality that for any

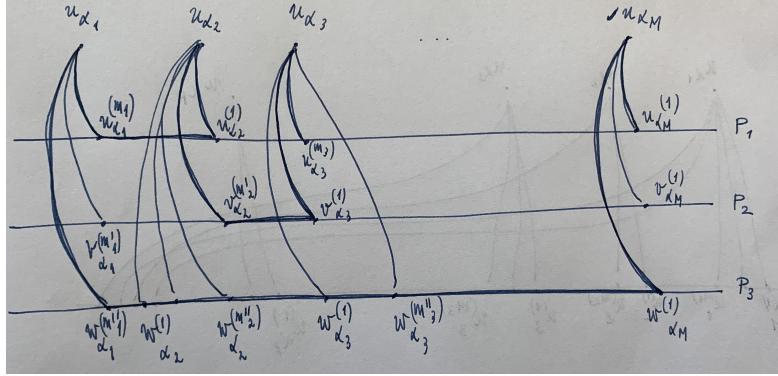


Figure 4: Construction of the k' -chord \mathcal{K} .

$1 \leq i \leq P - 1$, we have $u_i^{(1)} \not\sim u_{i+1}^{(1)}$, $v_i^{(1)} \not\sim v_{i+1}^{(1)}$, $w_i^{(1)} \not\sim w_{i+1}^{(1)}$. If not, we can carry the same construction as before to obtain, instead of $\{u_{\alpha_i}\}_{i=1}^P$, the family $\{u_{\alpha_i}\}_{i=1}^{2P}$ and considering in the construction of \mathcal{K} only the subfamily $\{u_{\alpha_{2i+1}}\}_{i=0}^{P-1}$.

Thus, by construction, the only chords of \mathcal{K} are those of the form $\{v_{\alpha_i}, w_{\alpha_i}^{(m)}\}$ for $2 \leq i \leq P - 1$ and $1 \leq m \leq m''_i$. Therefore, \mathcal{K} has at least $P - 2$ chords. Setting $P := k + 2$, we get that \mathcal{K} has at least k chords.

By the construction carried out through this proof, we have that $w_{\alpha_i}^{(m''_i)} \preceq_3 w_{\alpha_{i+1}}^{(1)}$ for all $1 \leq i \leq P$. Therefore, \mathcal{K} is outerplanar. We use Lemma 7 to get an induced k -chord in G .

Notice that all the constants involved in the above construction only depend on P , which, in turn, only depends on k . We thus set $f(k, 1) := \mathbb{N}$, which concludes the proof. \square

References

- [LLP22] Joonkyung Lee, Shoham Letzter, and Alexey Pokrovskiy. *Chiboundedness of graphs containing no cycles with k chords*. 2022. arXiv: 2208.14860 [math.CO]. URL: <https://arxiv.org/abs/2208.14860>.