1 Algorithms for coloring circle graphs

WARNING: intervals h_1, h_2 overlap if $h_1 \cap h_2 \neq \emptyset$ $h_1 \not\subseteq h_2$ and $h_2 \not\subseteq h_1$.

We look at the class of circle graphs as the class of overlap graphs of intervals on a line. Without loss of generality, we only consider the interval models containing open intervals in which no two intervals share an endpoint.

I am assuming opennes

For any such family of intervals F and any point c on a line, we set $F^-(c) := \{(a,b) \in F | b < c\}$, $F^0(c) := \{(a,b) | a < c < b\}$, $F^+(c) := \{(a,b) | c < a\}$. If $\omega(F) = 2$, for all $(a,b) \in F$, we have that $F^0(a) \setminus F^0(b)$ and $F^0(b) \setminus F^0(a)$ only contain nested intervals. Otherwise, if we had two overlapping intervals $h_1, h_2 \in F^0(a) \setminus F^0(b)$, we would have $h_1, h_2, (a,b)$ forming a triangle.

Lemma 1. Let F be the interval model of a circle graph. Suppose that $\omega(F) = 2$ and that for no two $h_1, h_2 \in F$, there exists $h_3 \in F$ with $h_3 \in h_1 \cap h_2$. Then, there exists a 3-coloring of F such that for all $(a,b) \in F$, the intervals in $F^0(a) \setminus F^0(b)$ have the same color.

I guess that the remarkable result is the 3-coloring part

Proof. By way of contradiction, suppose that the statement is incorrect and let $F = \{h_i\}_{i=1}^n = \{(a_i, b_i)\}_{i=1}^n$ be the counterexample to this lemma with the least cardinality. Clearly, $G(F)$ is connected. Let $(a_1, b_1), \ldots, (a_t, b_t)$, with $a_1 < \ldots < a_t$, be the intervals of F which are not contained in any other intervals. Notice that we have $F^0(a_1) = F^-(a_1) = \emptyset$. Suppose that $|F^0(b_1)| \leq 1$. By the minimality of F, we get that the interval model $F' := F \setminus \{h_1\}$ is 3-colorable. It is clear that coloring h_1 with a different color from h_2, we obtain a coloring for F. Which contradicts the fact that F is a counterexample to this lemma. Therefore, we conclude that $|F^0(b_1)| \geq 2$ and $t \geq 2$.

if $G(F)$ was not connected, at least one of its connected components would countervene to the lemma and we would have a smaller counterexample

By the connectedness of $G(F)$, we have $h_i \cap h_{i+1} \neq \emptyset$ for all $i \in \{1, \ldots, t-1\}$. Let us show that for all $i \in \{1, \ldots, t-1\}$ we have $F^-(b_i) = \emptyset$ or $F^+(a_{i+1}) = \emptyset$. Suppose that this does not hold for some $1 \leq i \leq t-1$. Let $F_1 = F \setminus F^+(a_{i+1})$ and $F_2 = F \setminus F^-(b_i)$.

That would imply that $i = 1$ or $i = t$ for all $i \in \{1, \ldots, t\}$ and so $t \leq 2$

We claim that $F_1 \cap F_2 = \{h_i, h_{i+1}\}$. Indeed, $h_j \in F_1 \cap F_2$, iff $a_j \leq a_{i+1}$ and $b_j \geq b_{i+1}$. Clearly h_i and h_{i+1} have this property. Now, suppose that there exists $h_j \in F_1 \cap F_2$ such that $a_j < a_{i+1}$ and $b_j > b_{i+1}$. We cannot have $a_j < a_i$, because otherwise $h_i \subseteq h_j$. Similarly, we cannot have $b_j > b_{i+1}$, because otherwise, we would have $h_{i+1} \subseteq h_j$. Therefore we have $a_i < a_j$ and $b_j < b_{i+1}$. The only remaining case is that of $a_i < a_j < a_{i+1}$ and $b_i < b_j < b_{i+1}$. But if this is the case, $\{h_i, h_j, h_{i+1}\}$ would form a triangle, which contradicts our assumptions.

In view of the minimality of F, there exist 3-colorings f_1, f_2 of F_1 and F_2 as required by the statement of this lemma with colors 1, 2 and 3 (here we use our assumption that $F^-(b_i)$ and $F^+(a_{i+1})$ are non-empty and thus F_1 and F_2 are strictly smaller than F). Since h_1 and h_2 overlap, they have different colors both in f_1 and f_2, therefore, we can assume without loss of generality that $f_1(h_i) = f_2(h_i) = 1$ and $f_1(h_{i+1}) = f_2(h_{i+1}) = 2$. Let

$$f(h) = \begin{cases} f_1(h) & \text{if } h \in F_1, \\ f_2(h) & \text{if } h \in F_2. \end{cases}$$

We verify that f is a coloring of F. Let $h' = (a', b') \in F$, $h'' = (a'', b'') \in F$ with $a' < a'' < b' < b''$. If $\{h', h''\} \subseteq F_1$ or $\{h', h''\} \subseteq F_2$, then $f(h') \neq f(h'')$. If $h' \in F^-(b_i)$ and $h'' \in F^+(a_{i+1})$ then, by assumption we have $h', h'' \not\subseteq (a_{i+1}, b_i)$, and so we must have $a' < a_{i+1} < a'' < b' < b_i < b''$. Since f_2 is a coloring, we have $f(h'') = f_2(h'') = f_2(h_{i+1}) = 2$. Since h' overlaps with h_{i+1} and $h', h_{i+1} \in F_1$, then $f(h') \neq 2 = f(h'')$. Similarly, we can verify that for all $(a, b) \in F$ all intervals in $F^0(b) \setminus F^0(a)$ have the same color. Thus f is a 3-coloring of F, which contradicts our choice of F. Thus, we have that $F^-(b_i) = \emptyset$ and $F^+(a_{i+1}) = \emptyset$ for all $i \in \{1, \ldots, t-1\}$.

So far we proved that the colorings f_1 and f_2 can be fused into one coloring f.

We can do this by introducing a trick like $(d, d_s + 1)$. This time it should be just h_i added to F_2.

We have $|F^+(a_2)| \geq \left|F^0(b_1)\right| \geq 2$, and, since $F^-(b_1) = \emptyset$, we have $F^0(a_2) = \{h_1\}$.

If $F^0(b_2) = \emptyset$, i.e. $t = 2$, then h_2 only overlaps with h_1. A coloring h_2 with the color of the elements of $F^0(b_1) \setminus \{h_2\}$, we obtain a 3-coloring of F with the conditions required by the present lemma, which is a contradiction.

If $F^0(b_2) \neq \emptyset$ and $t \geq 3$, we obtain (since $h_1 \in F^-(b_2)$) theat $F^+(a_3) = \emptyset$ and thus $t = 3$ and $F^0(b_2) = \{h_3\}$. Consider $F \setminus h_2$. Since F is minimal, we find that there exists a coloring f of $F \setminus \{h_2\}$ which follows the requirement of the present lemma. Let $h_2' = (a_2', b_2')$ be the longest interval of $F^0(b_1) \setminus \{h_2\}$ (which exists since $\left|F^0(b_1)\right| \geq 2$). This way h_2' contains all the intervals in $F^0(b_1)$. Now, assume that $f(h_1) = 1$ and $f(h_2') = 2$. If $f(h_3) \neq 2$, then we have a coloring of F with the properties required by the present lemma. Which yields a contradiction. If $f(h_3) = 2$, then h_3 and h_2' do not overlap. Thus $h_3 \in F^+(b_2')$. By assumption, all intervals of $F^0(b_2')$ are colored with the same color $\gamma \in \{1, 3\}$. By recoloring all elements of $F^+(b_2')$ of color 2 to color $\delta \in \{1, 3\} \setminus \gamma$ and all elements of $F^+(b_2')$ of color δ with color 2, we obtain a new coloring f' of $F \setminus \{h_2\}$. Since $h_3 \in F^+(b_2')$, we get $f'(h_2') = 2$ and $f'(h_3) \neq 2$. By coloring h_2 with color 2, we obtain a coloring of F with the properties required by the present lemma. □

Now that we have this why do we go on?

We essentially flip 2 and δ, if there was no conflict before, there is no conflict now

since h_2' and h_3 do not overlap

this is because $F^0(a_2) = \{h_1\}$ and $F^0(b_2) = \{h_3\}$

Possibly $h_3 = h_2$

For each pair of overlapping intervals $h_1, h_2 \in F$, let $p(h_1, h_2) = h_1 \cap h_2$. We denote by $P(F)$ the family of such intersections. Let $P^0(F) \subseteq P(F)$ be the inclusion-wise maximal family of $P(F)$.

Lemma 2. Let F be a family of intervals with $\omega(F) = 2$. Then, the intervals of $P^0(F)$ do not intersect.

Proof. Let $p_1 = h_1 \cap h_2 \in P^0(F)$ and $p_2 = h_3 \cap h_4 \in P^0(F)$. Therefore, $(p_1 \cap p_2) = (a_4, b_1)$. So $a_1 < a_2 < a_4 < b_1 < b_3 < b_4$. If $b_2 < b_4$, then h_1, h_2, h_4 pairwise overlap. Thus we have a triangle, which is a contradiction. Thus $b_4 < b_2$. By a symmetric argument we get $a_1 < a_3$. But then $p_2 = (a_4, b_3) \subsetneq (a_2, b_1) = p_1$ which contradicts the maximality of p_2. □

Theorem 3. Let F be a family of intervals with $\omega(F) = 2$. $\chi(F) \leq 5$.

Proof. We construct the desired coloring by induction on k. Let $k = 1$. Let F_1 be the subset of intervals of F that do not lie in the intersection of overlapping intervals of F. By Lemma 1, there exists a coloring f_1 of F_1 with colors $\{1, 2, 3\}$ such that for every $(a, b) \in F_1$, the intervals of the family $F_1^0(b) \setminus F^0(a)$ are

k is the number of "layers"

colored with the same color. By Lemma 2, the intervals of the family $P^0(F_1)$ do not overlap. By definition of F_1, each interval in $F \setminus F_1$ is contained in some (maximal) intersection $p \in P^0(F_1)$. Consider that the following construction has been done for $k-1$ with $k \geq 2$. We want the following properties to be true.

1. The intervals in $F' := \bigcup_{i=1}^{k-1} F_i$ (where the F_is might not be disjoint) are colored with colors $\{1, \ldots, 5\}$.

2. The intervals in $F \setminus F'$ do not overlap with intervals in $\bigcup_{i=1}^{k-2} F_i \setminus F_{k-1}$.

3. All intervals in $F \setminus F'$ are contained in some interval $p \in P^0\left(F^{k-1}\right)$.

4. For each $p = (c,d) \in P^0(F_{k-1})$ either only one color is used to color all intervals in $F^0_{k-1}(c) \setminus F^0_{k-1}(d)$, and no more than two colors are used to colors intervals of $F^0_{k-1}(d) \setminus F^0_{k-1}(d)$ or viceversa.

Aren't they? No. As we will see later we will have the intervals $h_{t+1}, \ldots, h_s$ which are in F' but also in $F_{p,k}$.

Otherwise $\omega(F) \geq 3$

Let us show how to carry out the k-th step of this construction. If $F \setminus F' = \emptyset$, then we are done and we have the desired coloring. Consider any interval $p = (c,d) \in P^0(F_{k-1})$, which contains at least one interval of $F \setminus F'$. Let

For $k = 1$, this is Lemma 2

$$F^0_{k-1}(c) \setminus F^0_{k-1}(d) = \{h_j\}_{j=1}^{t} = \{(c_j, d_j)\}_{j=1}^{t}$$
$$F^0_{k-1}(d) \setminus F^0_{k-1}(c) = \{h_j\}_{j=t+1}^{s} = \{(c_j, d_j)\}_{j=t+1}^{s},$$

by condition 3

and $p = (c_s, d_s) \cap (c_t, d_t) = (c_s, d_t)$. As already noted, each family consists of nested intervals. Without loss of generality, suppose that $\gamma_1 \in \{1,3\}$ is the color used to color the intervals $h_{t+1}, \ldots, h_s$ and let γ_2, γ_3 be the colors used to color $\{h_1, \ldots h_t\}$.

otherwise $\omega(F) \geq 3$

Could also be only γ_2

Let I_p be the set of intervals in F contained in p, the intervals $h_{t+1}, \ldots, h_s$ and the interval $(d, d_s + 1)$. Let $F_{k,p}$ be the set of intervals in I_p which are not contained in the overlap of any two intervals in I_p.

Let us show that the intervals from $I_p \setminus F_{k,p}$ do not overlap with any of the intervals of $F' \setminus \{h_{t+1}, \ldots, h_s\}$. . By condition 2 of our induction hypothesis, intervals from $F' \setminus \{h_{t+1}, \ldots, h_s\}$ that can overlap with elements in $I_p \setminus F_{k,p}$ are only intervals in $\{h_1, \ldots, h_t\}$. Let $(a_1, b_1) \in I_p$ be an interval contained in the overlapping of $(a_2, b_2), (a_3, b_3) \in I_p$ and thus overlapping with some $h_j = (c_j, d_j)$ for $1 \leq j \leq t-1$. Then $a_i > c_s$ for all $i \in \{1,2,3\}$. We also get $c_j < c_s$, $a_1 < d_j < b_1$. Hence h_j, (a_2, b_2) and (a_3, b_3) pairwise overlap. Which is a contradiction.

The following levels can only overlap with $\{h_{t+1}, \ldots, h_s\}$

This is true for intervals overlapping with p, but what about intervals overlapping with d? The only intervals in I_p overlapping with $(d, d_s + 1)$ are $h_{t+1}, \ldots, h_s$ which are also in $F_{k,p}$.

If p contains at least one interval, then $F_{k,p} \setminus (h_{t+1}, \ldots, h_s) \cup \{(d, d_s + 1)\} \neq \emptyset$. By Lemma 1 (replacing $F^0(b) \setminus F^0(a)$ with $F^0(a) \setminus F^0(b)$), we can color $F_{k,p}$ with colors from $\{1,2,3,4,5\} \setminus \{\gamma_2, \gamma_3\}$ such that for each $(a,b) \in F_{k,p}$, the interval from $F^0_{k,p}(a) \setminus F^0_{k,p}(b)$ have the same color. At the same time, since $\{h_{t+1}, \ldots, h_s\} = F^0_{k,p}(d) \setminus F^0(d_s + 1)$ we can assume that $h_{t+1}, \ldots, h_s$ is colored with (only) γ_1.

Actually $F^0(d_s + 1)$ is empty, but we want to apply the statement of Lemma 1 rigorously. This is why we introduce the segment $(d, d_s + 1)$ in the first place.

Let us denote $F'_{k,p} = F_{k,p} \setminus \{(d, d_s + 1)\}$. It is easy to see that the coloring of $F'_{k,p}$ is compatible with that of F'. Carrying out similar constructions for each $p \in P^0(F_{k-1})$ containing at least one uncolord interval of F, let $F_k := \bigcup_{p \in P^0(F_{k-1})} F'_{k,p}$.

One can check that the induction hypotheses hold for k. Also, the number of uncolored vertices strictly decreseas at each step. Therefore the coloring will be completed in a finite number of iterations. $\square$