1 Introduction and first definitions

We define polygon circle graphs as the set of intersection graphs of polygons (including segments) inscribed in a circle. The aim of this text is to prove the following.

Theorem 1. Triangle-free polygon circle graphs have chromatic number at most 5.

Without loss of generality, we will only consider families of polygons in which no two vertices coincide.

We handle polygon circle graphs by looking at the stereographic projection of the circle onto $\mathbb{R}$. Thus, a polygon circle graph is represented by a family $\{h_i\}_{i=1}^n$ with $h_i = \left(a_1^{(i)}, \dots, a_{k_i}^{(i)}\right)$ for some $k_i \geq 2$ with $a_j^{(i)} \in \mathbb{R}$ being the image of the stereographic projection of a vertex of the corresponding vertex of h_i.

Given a familiy $F = \{h_i\}_{i=1}^n$ of polygons, we denote the corresponding intersection graph $G(F)$ and we say that F is a polygon representation of $G(F)$. By abuse of notation, we will denote $\omega(G(F))$ by $\omega(F)$. Given two polygons $h_i, h_{i'}$ in F, we say that h_i and $h_{i'}$ overlap if there exists indices j and j' such that $\left(a_j^{(i)}, a_{j+1}^{(i)}\right)$ and $\left(a_{j'}^{(i')}, a_{j'+1}^{(i')}\right)$ overlap.

Given a polygon $h := (a_1, \dots, a_k)$, a contraction of h, is a polygon h' of the form $(a_1, \dots, a_{i-1}, a_{i+1}, \dots a_k)$ where indices are unerstood modulo k. Consider a family of polygons $F = \{h_i\}_{i=1}^n$. Let F' be the family of polygons generated by contracting one polygon in F. We define the partial order '$\prec$' on the set of polygon circle graphs to be the smallest order containing the relations of the form $F' \prec F$. Notice that $\prec$ is well-founded since, when $F' \prec F$, the total number of vertices of polygons in F' is strictly less than that of the polygons in F. Thus, we cannot have an infinite decreasing sequence, $\dots \prec F'' \prec F' \prec F$.

Let $c \in \mathbb{R}$. We establish the following notation.

$$F^0(c) := \left\{ h_i \in F \;\middle|\; a_1^{(i)} < c < a_{k_i}^{(i)} \right\},$$
$$F^+(c) := \left\{ h_i \in F \;\middle|\; c < a_1^{(i)} \right\},$$
$$F^-(c) := \left\{ h_i \in F \;\middle|\; a_{k_i}^{(i)} < c \right\}.$$

We also importantly define for $1 \leq i \leq n$ and $1 < j \leq k_i$, the following:

$$\widetilde{F}^0\left(a_j^{(i)}\right) := \left\{ h_{i'} \in F \;\middle|\; a_1^{(i)} < a_{j'}^{(i')} < a_j^{(i)} < a_{j'+1}^{(i')} \text{ for some } 1 \leq j' < k_{i'} \right\}.$$

For any $h_i \in F^0(c)$, denote $I_c(h_i) := \left(a_j^{(i)}, a_{j+1}^{(i)}\right)$ such that $c \in \left(a_j^{(i)}, a_{j+1}^{(i)}\right)$ and $O_c(h_i) := \left(a_1^{(i)}, a_{k_i}^{(i)}\right)$. For any polygon h_i, we call the segment $\left(a_1^{(i)}, a_{k_i}^{(i)}\right)$ the external segment of h_i.

Also, let h_i, $h_{i'}$ be two whose external segments overlap with $a_1^{(i)} < a_1^{(i')}$. Let $\left(a_j^{(i)}, a_{j+1}^{(i)}\right) = I_{a_1^{(i')}}(h_i)$ and let $b^{(i,i')} := a_{j+1}^{(i)}$. Let $\left(a_{j'}^{i'}, a_{j'+1}^{(i')}\right) = I_{b^{(i,i')}}(h_{i'})$ and let $a^{(i,i')} := a_{j'}^{(i')}$. Similarly, let $\left(a_{l'}^{(i')}, a_{l'+1}^{(i')}\right) = I_{a_{k_i}^{(i)}}(h_{i'})$ and let $c^{(i,i')} := a_{l'}^{(i')}$. Let $\left(a_l^{(i)}, a_{l+1}^{(i)}\right) = I_{c^{(i,i')}}(h_i)$ and let $d^{(i,i')} := a_{j+1}^{(i)}$. An illustration of this is shown in Figure 1.

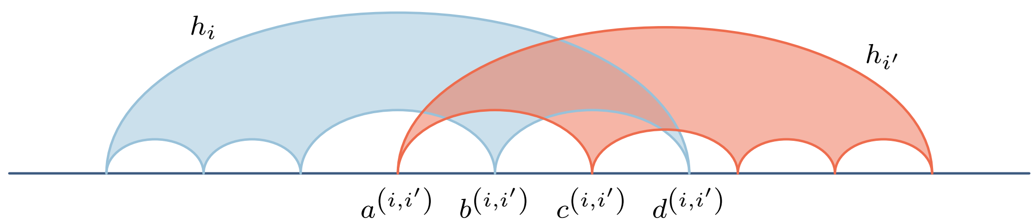

Figure 1

Remark 2. By definition, $a^{(i,i')} < b^{(i,i')}$, $c^{(i,i')} < d^{(i,i')}$ and $a^{(i,i')} \leq c^{(i,i')}$, $b^{(i,i')} \leq d^{(i,i')}$. We thus either have, $a^{(i,i')} < b^{(i,i')} < c^{(i,i')} < d^{(i,i')}$, in which case the intervals $\left(a^{(i,i')}, b^{(i,i')}\right)$ and $\left(c^{(i,i')}, d^{(i,i')}\right)$ are disjoint. Or we have $a^{(i,i')} \leq c^{(i,i')} < b^{(i,i')} \leq d^{(i,i')}$. In this case, since $c^{(i,i')}$ is a vertex of $h_{i'}$ and $c^{(i,i')} < b^{(i,i')}$, we have that $a^{(i,i')} \geq c^{(i,i')}$ and thus $a^{(i,i')} = c^{(i,i')}$. By a similar reasoning, we get that $b^{(i,i')} = d^{(i,i')}$ and so that $\left(a^{(i,i')}, b^{(i,i')}\right) = \left(c^{(i,i')}, d^{(i,i')}\right)$.

Now, consider the set of intervals of the form $\left(a^{(i,i')}, b^{(i,i')}\right)$ or $\left(c^{(i,i')}, d^{(i,i')}\right)$ and denote it $P(F)$. Take the inclusion-wise maximal subfamily of such intervals and denote it $P^0(F) \subseteq P(F)$.

Remark 3. Notice that, if $\omega(F) = 2$, the maximality of $P^0(F)$, together with Remark 2 implies that all the intervals in $P^0(F)$ are disjoint.

From now on, consider $F = \{h_i\}_{i=1}^n$ be a family of polygons with $\omega(F) = 2$. We remark the following two facts which we state without proving them.

Fact 4. For any $a_j^{(i)}$, the elements of $\widetilde{F}^0\left(a_j^{(i)}\right)$ are nested. That is, let $\widetilde{F}^0\left(a_j^{(i)}\right) =: \{h_1, \ldots, h_l\}$. Then, up to relabeling the elements of F, we have that

$$I_{a_j^{(i)}}(h_1) \subseteq I_{a_j^{(i)}}(h_2) \subseteq \cdots \subseteq I_{a_j^{(i)}}(h_l),$$
$$O_{a_j^{(i)}}(h_1) \subseteq O_{a_j^{(i)}}(h_2) \subseteq \cdots \subseteq O_{a_j^{(i)}}(h_l).$$

Fact 5. For any two polygons $h \in \widetilde{F}^0\left(a_j^{(i)}\right)$ and $h' \in \widetilde{F}^0\left(a_{j'}^{(i)}\right)$, we have that h and h' do not overlap.

2 Coloring plain polygon circle graphs

We define a particular kind of circle graphs whose properies will be useful for the proof of Theorem 1.

Definition 6. We call a polygon circle graph plain, if it has a polygon representation $F := \{h_i\}_{i=1}^n$ with $\omega(F) = 2$ and such that, for any $(a, b) \in P^0(F)$, there is no h_i such that $a < a_1^{(i)} < a_{k_i}^{(i)} < b$.

Plain polygon circle graphs have the following property. We note that we will be repeatedly using Fact 4 and Fact 5 without reference.

Lemma 7. Let G be a plain polygon circle graph and let $F := \{h_i\}_{i=1}^n$ be a polygon representation of G with the properties required by Definition 6 Then $\chi(F) \leq 3$. Moreover, there exists a 3-coloring of F such that for all $1 \leq i \leq n$ and $1 < j \leq k_i$, all polygons in $\widetilde{F}^0\left(a_j^{(i)}\right)$ have the same color. We call such coloring a 'good coloring'.

Proof. By way of contradiction, suppose that the statement is false and let $F = \{h_i\}_{i=1}^n$ be the smallest counterexample to this lemma with respect to the order $\prec$. Up to relabeling the elements of F, suppose without loss of generality that $h_1, \ldots, h_t$ are the polygons of F whose external segments (seen as itervals of the form $\left(a_1^{(i)}, a_{k_i}^{(i)}\right)$) are not contained in the external segment of any other polygon of F. We further assume that $a_1^{(1)} < a_1^{(2)} < \ldots < a_1^{(t)}$. By the minimality of F, we get that $G(F)$ is connected and $h_i \cap h_{i+1} \neq \emptyset$ for all $i \in \{1, \ldots, t-1\}$. Therefore, we have that the points $a^{(i,i+1)}$ and $b^{(i,i+1)}$ are well-defined for all $i \in \{1, \ldots, t-1\}$.

- First, suppose that $t = 1$. In this case, we have $F^0\left(a_1^{(1)}\right) = \emptyset$ and $F^0\left(a_{k_1}^{(1)}\right) = \emptyset$. Thus, contracting $a_{k_i}^{(1)}$ in h_1, either does not change the underlying graph, or implies that $G(F)$ is disconnected. In both cases, we have a contradiction.

- We are now going to prove that for all $i \in \{1, \ldots, t-1\}$, we have either $F^-\left(b^{(i,i+1)}\right) = \emptyset$ or $F^+\left(a^{(i,i+1)}\right) = \emptyset$. This is not the case for $t \geq 4$ since, in such case, we would have $h_1 \in F^-\left(b^{(2,3)}\right)$ and $h_t \in F^+\left(a^{(2,3)}\right)$. Thus, this would imply $t \leq 3$.

  By way of contradiction, suppose that there exists $i_0 \in \{1, \ldots, t-1\}$ such that $F^-\left(b^{(i_0,i_0+1)}\right) \neq \emptyset$ and $F^+\left(a^{(i_0,i_0+1)}\right) \neq \emptyset$. By the minimality of F, we have that $F_1 := F \setminus F^+\left(a^{(i_0,i_0+1)}\right)$ and $F_2 := F \setminus F^-\left(b^{i_0,i_0+1}\right)$ admit good colorings f_1 and f_2. Notice that $F_1 \cap F_2 = \{h_{i_0}, h_{i_0+1}\}$. So, we can assume without loss of generality that $f_1(h_{i_0}) = f_2(h_{i_0}) = 1$ and $f_1(h_{i_0+1}) = f_2(h_{i_0+1}) = 2$. This also allows us to define the function $f : F \longrightarrow \{1, 2, 3\}$ as follows:

$$f(h) = \begin{cases} f_1(h) & \text{if } h \in F_1, \\ f_2(h) & \text{if } h \in F_2. \end{cases}$$

  By the maximality of h_{i_0} and h_{i_0+1}, a family $\widetilde{F}^0\left(a_j^{(i)}\right)$ is either entirely contained into F_1 or F_2. Thus, if f is a coloring, it is also a good coloring.

  We now prove that f is a coloring. Let $h_i, h_{i'} \in F$ be overlapping polygons. If both $h_i, h_{i'} \in F_1$ or $h_i, h_{i'} \in F_2$, then $f(h_i) \neq f(h_{i'})$ by the fact that f_1 and f_2 are colorings. Suppose that $h_i \in F_1$ and $h_{i'} \in F_2$, then, since for no $(a, b) \in P^0(F)$ and $h_i \in F$ we have $a < a_1^{(i)} < a_{k_i}^{(i)} < b$, we deduce that

$$a_1^{(i)} < a^{(i_0,i_0+1)} < a_1^{(i')} < a_{k_i}^{(i)} < b^{(i_0,i_0+1)} < a_{k_{i'}}^{(i')}.$$

Now, since $h_{i_0+1}, h_{i'} \in \widetilde{F}^0\left(b^{(i_0,i_0+1)}\right)$, and $h_{i_0+1}, h_{i'} \in F_2$, since f_2 is a good coloring, we have that $f\left(h_{i'}\right) = f\left(h_{i_0+1}\right) = 2$. Now, since h_{i_0+1} and h_i overlap, $h_{i_0+1}, h_i \in F_1$, and f_1 is a coloring, we get that $f\left(h_{i_0+1}\right) \neq f\left(h_i\right)$. Therefore, $f\left(h_i\right) \neq f\left(h_{i'}\right) = 2$. So f is a coloring and so, it is a good coloring. This contradicts the definition of F.

We thus conclude that $F^-\left(b^{(i,i+1)}\right) = \emptyset$ or $F^+\left(a^{(i,i+1)}\right) = \emptyset$ for all $i \in \{1, \ldots, t-1\}$ and $t \leq 3$.

- Suppose now that $t = 2$. Suppose that there exists $b^{(1,2)} < a_j^{(2)} < a_{k_2}^{(2)}$. Then, since $F^0\left(a_{k_2}^{(2)}\right) = \emptyset$, we have that contracting $a_{k_2}^{(2)}$ in h_2, does not change the underlying graph. Which is not possible by the definition of F. therefore $a^{(1,2)} = a_{k_2-1}^{(2)}$. By a symmetric reasoning, we can show $b^{(1,2)} = a_2^{(1)}$. We have the setting shown in Figure 2.

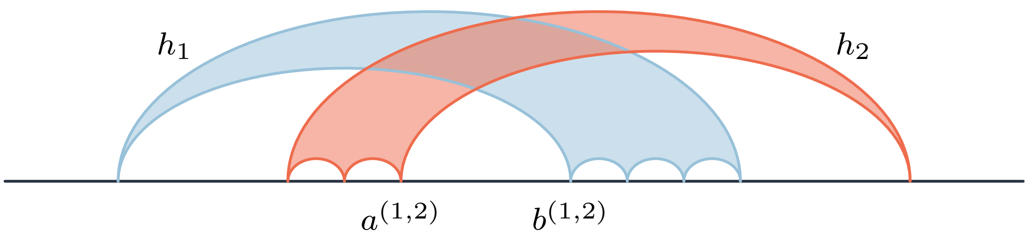

Figure 2

Consider the case of $F^+\left(a^{(1,2)}\right) = \emptyset$. Since $b^{(1,2)} = a_2^{(2)}$, h_1 only overlaps with h_2. By the minimality of F, we get a good coloring f' of $F \setminus \{h_1\}$. We can extend f' into a good coloring of F, by coloring h_1 with a color different from $f'\left(h_2\right)$.

Consider the case of $F^-\left(b^{(1,2)}\right) = \emptyset$. Since $a^{(1,2)} = a_{k_2}^{(2)}$, h_2 only overlaps with h_1. Now, take the outermost polygons overlapping with h_1:

$$\{u_j\}_{j=1}^s = \left\{\left(c_1^{(j)}, \ldots, c_{l_j}^{(j)}\right)\right\}_{j=1}^{(s)}$$

with

$$c_1^{(1)} < c_{l_1}^{(1)} < c_1^{(2)} < \cdots < c_{l_{s-1}}^{(s-1)} < c_1^{(s)} < c_{l_s}^{(s)}.$$

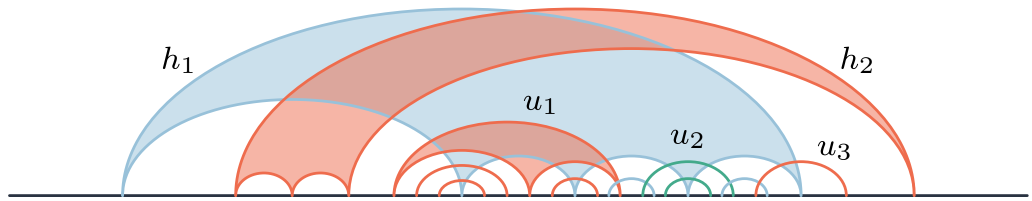

Figure 3

This setting is illustrated in Figure 3.

We can thus partition all the polygons overlapping with h_1 by looking at which u_j the are nested in. For any $1 \leq j \leq s$, denote by $N(u_j)$ the class of polygons nested in u_j. Notice that we have a good coloring f' of $F \setminus \{h_2\}$. We can extend f' into a colring f of F by coloring h_2 with the same color as the other polygons in $F^0\left(a_2^{(1)}\right)$.

For the sake of simplicity, suppose that we have $f(h_1) = 1$, $f(u_1) = 2$ and $f(h_2) = 2$. Notice that all the polygons in a given class $N(u_j)$ have the same color.

In order to obtain a good coloring of F, it suffices to color all the polygons overlapping with h_1 with color 2. If one exists, take the smallest index $m\ (\neq 1)$ such that the color of $N(u_m)$ is different from 2. So $f(u_m) = 3$. Define $O := F^+\left(c_{l_{m-1}}^{(m-1)}\right)$. Notice that the polygons in $F \setminus O$, that overlap with polygons in O lie in $F^0\left(c_{l_{m-1}}^{(m-1)}\right) = \widetilde{F}^0\left(c_{l_{m-1}}^{(m-1)}\right)$ by the maximality of u_{m-1}. All polygons in $\widetilde{F}^0\left(c_{l_{m-1}}^{(m-1)}\right)$ are colored with color 1. We can thus swap the colors 2 and 3 of the polygons contained in O', so that the classes $N(u_1), \ldots, N(h_{l_m})$ are colored with color 2. We can repeat this procedure until all classes $N(u_j)$ are colored with color 2.

We thus get a good coloring of F, which is a contradiction.

- We now tackle the case of $t = 3$. If $t = 3$, we have $h_3 \in F^+\left(a^{(1,2)}\right)$ and $h_1 \in F^-\left(b^{(2,3)}\right)$. Thus, we have that $F^+\left(b^{(1,2)}\right) \neq \emptyset$. By similar arguments as those used in the previous case, the minimality of F implies $a^{(1,2)} = a_1^{(2)}$, $b^{(1,2)} = a_2^{(1)}$, $a^{(2,3)} = a_{k_3-1}^{(3)}$ and $b^{(2,3)} = a_{k_2}^{(2)}$.

Also, suppose that there exists $a_2^{(1)} < a_j^{(2)} < a_{k_1}^{(1)}$ for some $j > 1$. Then, since $F^-\left(b^{(1,2)}\right) = \emptyset$, contracting $a_1^{(2)}$ in h_2 does not change the underlying graph, which contradicts the definition of F. Similarly, there is no $j < k_2$ with $a_1^{(3)} < a_j^{(2)} < a_{k_3-1}^{(3)}$. Therefore, for all $1 < j < k_2$, we have that $a_{k_1}^{(1)} < a_j^{(2)} < a_1^{(3)}$.

Suppose that there exists $1 < j < k$ such that $a_{k_1}^{(1)} < a_j^{(2)} < a_1^{(3)}$. The setting is illustrated in Figure 4.

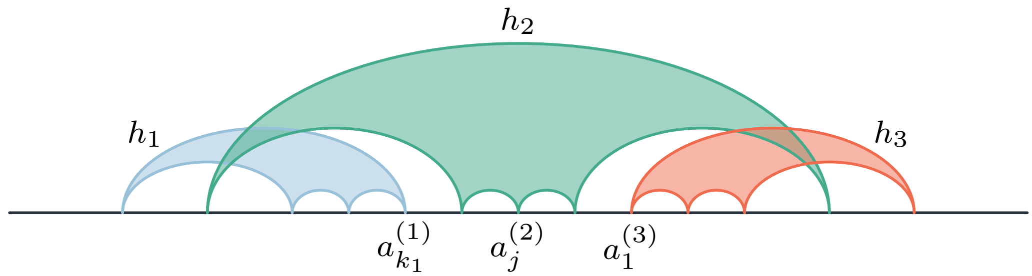

Figure 4

Contract $a_{k_2}^{(2)}$ in h_2 (call the resulting polygon h_2') and get a good coloring f' of the resulting family of polygons. Suppose without loss of generality that $f(h_1) = 1$, $f'(h_2') = 2$. If $f'(h_3) \neq 2$, then we can decontract h_2' to get a good coloring of F, which is a contradiction. Let us consider the case $f'(h_3) = 2$. Define $O' := F^+\left(a_{k_2-1}^{(2)}\right)$. Notice that all the polygons in $F \setminus O'$ that overlap with O' lie in the set $F^0\left(a_{k_2-1}^{(2)}\right) = \widetilde{F}^0\left(a_{k_2-1}^{(2)}\right)$ by the maximality of h_2. By the fact that f' is a good coloring, all polygons in $F^0\left(a_{k_2-1}^{(2)}\right)$ share the same color $\gamma \in \{1,3\}$. Let $\delta \in \{1,3\} \setminus \{\gamma\}$ We can swap the colors 2 and δ in O' by preserving the good coloring property. We can finally decontract h_2 to get a good coloring of F. A contradiction.

Thus, we deduce that h_2 is a segment. We get a good coloring f' of $F \setminus \{h_2\}$. Suppose without loss of generality that $f'(h_1) = 1$. We define $u_1, \ldots, u_s$ analogously as in the previous case. By the same argument as in the previous case, we can color all the sets $N(u_j)$ with the same color as $N(u_1)$, say 2 without loss of generality. If after this recoloring,

$f'(h_3) \neq 2$, we can extend f' into a good coloring of F by coloring h_2 with color 2. This gives a contradiction.

We thus have $f'(h_3) = 2$. Let $\gamma \in \{1,3\}$ be the color of the polygons in $F^0\left(c_{l_s}^{(s)}\right) = \widetilde{F}^0\left(c_{l_s}^{(s)}\right)$. Let $O'' := F^+\left(c_{l_s}^{(s)}\right)$. Notice that all the elements in $F \setminus O''$ that overlap elements in O'' are contained in $\widetilde{F}^0\left(c_{l_s}^{(s)}\right)$, and so have color γ. Let $\delta \in \{1,3\} \setminus \{\gamma\}$. By the above discussion, we can swap colors 2 and δ in O'' by preserving the good coloring property. We can thus extend f' into a good coloring of F by coloring h_2 with color 2 to obtain a good coloring of F. Again, a contradiction.

We have the desired result. $\square$

3 Proof of Theorem 1

We first establish some more notation. Namely, using the same notational conventions used up to this point, we denote, for $1 \leq i' \leq n$ and $1 \leq j' < k_{i'}$

$$\overline{F}^0\left(a_{j'}^{(i')}\right) := \left\{h_i \in F \;\middle|\; a_j^{(i)} < a_{j'}^{(i')} < a_{j+1}^{(i)} < a_{k_{i'}}^{(i')} \text{ for some } 1 \leq j < k_i\right\}$$

We remark in passing that, by symmetry, Lemma 7 remains true if we replace $\widetilde{F}^0\left(a_j^{(i)}\right)$ with $\overline{F}^0\left(a_{j'}^{(i')}\right)$, since this would be a mirrored version of such lemma.

We now present our main result with similar techniques as the one employed in [Kos88]. Again, in what follows, we will repeatedly use Fact 4 and Fact 5 without reference.

In particular, we prove the following statement which clearly implies Theorem 1.

Lemma 8. Let $F := \{h_i\}_{i=1}^n$ with $\omega(F) = 2$. There exist (possibly overlapping) subfamilies $F_1, \ldots, F_l$ of F with $F = \cup_{i=1}^l F_i$ and precolorings $f_1, \ldots, f_l$ with $f_k : F'_k := \cup_{i=1}^k F_i \longrightarrow \{1,2,3,4,5\}$ for all $1 \leq k \leq l$ such that the following holds for all $1 \leq k \leq l$.

1. The polygons in $F \setminus F'_k$ do not overlap with intervals in $F'_{k-1} \setminus F_k$.

2. The polygons $h_i \in F \setminus F'_k$ are such that $a < a_1^{(i)} < a_{k_i}^{(i)} < b$ for some $(a,b) \in P^0(F_k)$.

3. For each $p := (a,b) \in P^0(F_k)$, either one color is used to color all polygons in $\overline{F}^0(k)(a)$ and no more than two colors are used to color the polygons in $\widetilde{F}_k^0(b)$ or viceversa, only one color is used to color the polygons in $\widetilde{F}_k^0(b)$ and no more than two colors are used to color the polygons in $\overline{F}_k^0(a)$.

Proof. Let $F = \{h_i\}_{i=1}^n$ be a family of polygons with $\omega(F) = 2$. We construct the subfamilies $F_1, \ldots, F_k, \ldots, F_l$ and the precolorings $f_1, \ldots, f_k, \ldots, f_l$ by induction on k.

- **Base case**. Let $F_1 \subseteq F$ be the subset of polygons $h_i \in F$ such that there is no $(a,b) \in P^0(F)$ with $a < a_1^{(i)} < a_{k_i}^{(i)} < b$. By Lemma 7 there exists a coloring of F_1 with colors $\{1,2,3\}$ such that for every $h_i \in F_1$ and $1 < j \leq k_i$, the polygons in $\widetilde{F}^0\left(a_j^{(i)}\right)$ have the same color. Let f_1 be such coloring. Condition 1 trivially holds. Condition 2 follows directly from the choice of F_1. Condition 3 follows from the choice of f_1 by Lemma 7.

- **Induction step**. Consider that for some $k \geq 1$, the families $F_1, \ldots, F_k$ and the precolorings $f_1, \ldots, f_k$ have been constructed. If $F \setminus F_k' = \emptyset$, then $k = l$ and the result follows. Otherwise, we construct F_{k+1} and f_{k+1} with the desired properties.

  Consider any interval $p = \left(a^{(i,i')}, b^{(i,i')}\right) \in P^0(F_k)$, such that there exists h_j with $a^{(i,i')} < a_1^{(j)} < a_{k_j}^{(j)} < b^{(i,i')}$. Denote:

$$\overline{F}^0\left(a^{(i,i')}\right) = \{u_j\}_{j=1}^t = \left\{\left(c_1^{(j)}, \ldots, c_{l_j}^{(j)}\right)\right\}_{j=1}^t,$$
$$\widetilde{F}^0\left(b^{(i,i')}\right) = \{u_j\}_{j=t+1}^s = \left\{\left(c_1^{(j)}, \ldots, c_{l_j}^{(j)}\right)\right\}_{j=t+1}^s.$$

  We note that the polygons $\{u_j\}_{j=1}^t$ and $\{u_j\}_{j=t+1}^s$ are ordered from innermost to outermost so that $p = \left(c_1^{(s)}, c_{l_s}^{(s)}\right) \cap \left(c_1^{(t)}, c_{l_t}^{(t)}\right)$. Without loss of generality, suppose that $\gamma_1 \in \{1,3\}$ is the color of the polygons $u_{t+1}, \ldots, u_s$ and let γ_2, γ_3 be the colors of the polygons $u_{t+1}, \ldots, u_s$ and let γ_2, γ_3 be the colors of the polygons $u_1, \ldots, u_t$.

  Let I_p be the set of polygons whose vertices are all in p, the polygons $u_{t+1}, \ldots, u_s$ and the segment $\left(b^{(i,i')}, c_{l_s}^{(s)} + 1\right)$. Let F_k^p be the set of polygons $h_i \in I_p$ such that, for no two polygons $h_{i'}, h_{i''} \in I_p$ we have all the vertices of h_i contained in $\left(a^{(i',i'')}, b^{(i',i'')}\right)$ or in $\left(c^{(i',i'')}, d^{(i',i'')}\right)$. By using the fact that $\omega(F) = 2$, it is easy to see that polygons in $I_p \setminus F_{k+1}^p$ do not overlap with any of the polygons of $F'_k \setminus \{u_{t+1}, \ldots, u_s\}$. If p contains at least one interval, then $F_{k+1}^p \setminus \{u_{t+1}, \ldots, u_s\} \cup \left\{\left(b^{(i,i')}, c_{l_s}^{(s)}\right)\right\} \neq \emptyset$. By Lemma 7, we can color $F_{k_1}^p$ with colors from $\{1,2,3,4,5\} \setminus \{\gamma_2, \gamma_3\}$ such that for each $h_i \in F_{k+1}^p$, for all $1 \leq j < k_i$, the polygons in $\overline{F}_k^{p,0}\left(a_j^{(i)}\right)$ share the same color. At the same time, since $\{u_{t+1}, \ldots, u_s\} = \overline{F}_k^{p,0}\left(c_{l_s}^{(s)} + 1\right)$, we can assume that $u_{t+1}, \ldots, u_s$ is colored (only) with γ_1.

Let us denote $F'^p_{k+1} := F_{k+1}^p \setminus \left\{\left(b^{(i,i')}, c_{l_s}^{(s)} + 1\right)\right\}$. Notice that the only polygones in F'^p_{k+1} which are in $F \setminus F'_k$ are $u_{t+1}, \ldots, u_s$. Thus, the coloring of F'^p_{k+1} is compatible with that of F'_k. We carry out similar constructions for each $p \in P^0(F_k)$ containing at least one uncolored polygon of F. Let $F_{k+1} := \bigcup_{p \in P^0(F_k)} F'^p_k$.

By construction, Condition 2 clearly holds for $k+1$. By our induction hypothesis, polygons in $F'_{k-2} \setminus F_{k-1}$ do not overlap with polygons in $F \setminus F'_k$ and so, a fortiori, they do not overlap with polygons in $F \setminus F'_{k+1}$. Moreover, we have that polygons in $I_p \setminus F_{k+1}^p$ do not overlap with polygons in $F'_{k+1} \setminus \{u_{t+1}, \ldots, u_s\}$. Together with Condition 2, this yields that polygons in $F'_{k-1} \setminus F_k$. So, Condition 1 holds for $k+1$. From the described construction, it is clear that Condition 3 holds for $k+1$.

We finally notice that the number of uncolored polygons of F strictly decreases at each step. Therefore, we will eventually have $k = l$ and our construction will be completed in a finite number of steps.

□

References

[Kos88] Alexandr V Kostochka. "Upper bounds on the chromatic number of graphs". In: *Trudy Inst. Mat.(Novosibirsk)* 10.Modeli i Metody Optim. (1988), pp. 204–226.