Finite-size study of the Quench Action approach in integrable spin chains

Vincenzo Alba[1], **Pasquale Calabrese**[1]

[1] International School for Advanced Studies (SISSA), Via Bonomea 265, 34136, Trieste, Italy, INFN, Sezione di Trieste

Abstract.
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1. Introduction

2. Overlap with the Neel state

Let us consider a generic n-string state

$$\lambda^a_{n;\alpha} = \lambda_{n;\alpha} + \frac{i}{2}(n+1-2a) + i\delta^a_{n;\alpha} \quad \text{with}\, a = 1, \dots, n. \tag{1}$$

We denote the generic parity invariant eigenstate as $|\{\pm\lambda_j\}_{j=1}^m, n_\infty\rangle$, where m is the number of rapidity pairs, N_∞ is the number of infinite rapidities, with $M = L/2 = N_\infty + 2m$, and $n_\infty \equiv N_\infty/L$ is the infinite rapidities density.
The overlap with the Neel state $|\Psi_0\rangle$ reads

$$\frac{\langle\Psi_0|\{\pm\lambda_j\}_{j=1}^m, n_\infty\rangle}{|||\{\lambda_j\}_{j=1}^m, n_\infty\rangle||} = \frac{\sqrt{2}N_\infty!}{\sqrt{(2N_\infty)!}}\left[\prod_{j=1}^m \frac{\sqrt{\lambda_j^2+1/4}}{4\lambda_j}\right]\sqrt{\frac{\det_m(G^+)}{\det_m(G^-)}} \tag{2}$$

where

$$G^\pm_{jk} = \delta_{jk}\left(NK_{1/2}(\lambda_j) - \sum_{l=1}^m K_1^+(\lambda_j,\lambda_l)\right) + K_1^\pm(\lambda_j,\lambda_k), \quad j,k = 1,\dots,m \tag{3}$$

and

$$K_1^\pm(\lambda,\mu) = K_1(\lambda-\mu) \pm K_1(\lambda+\mu) \tag{4}$$

and

$$K_\alpha(\lambda) \equiv \frac{2\alpha}{\lambda^2+\alpha^2} \tag{5}$$

2.1. Reduced Neel overlap

In the case of perfect strings the matrices $G^\pm_{jk}$ become ill-defined. Precisely, $K_1^+(\lambda,\mu)$ diverges if λ and μ are successive members of the same string, i.e., $|\lambda-\mu| = i$.
It is possible to rewrite (2) in terms of the string centers $\lambda_{n;\alpha}$ only. Here we restrict ourselves to rapidity configurations with no zero-momentum strings.
It is convenient to split the indices i, j of $G^\pm_{ij}$ as $i = (n,\alpha)$ $j = (m,\beta)$, with n, m being the length of the strings and α, β labelling the string centers.
The result reads

$$\frac{1}{2}G^+_{(n,\alpha)(m,\beta)} = \begin{cases} L\theta'_n(\lambda_{n;\alpha}) - \sum\limits_{(\ell,\gamma)\neq(n,\alpha)} (\Theta'_{n,\ell}(\lambda_{n;\alpha}-\lambda_{\ell;\gamma}) + \Theta'_{n,\ell}(\lambda_{n;\alpha}+\lambda_{\ell;\gamma})) & \text{if}\,(n,\alpha) = (m,\beta) \\ \Theta'_{n,m}(\lambda_{n;\alpha}-\lambda_{m;\beta}) + \Theta'_{n,m}(\lambda_{n;\alpha}+\lambda_{m;\beta}) & \text{if}\,(n,\alpha) \neq (m,\beta) \end{cases} \tag{6}$$

Here $\theta'_n(x) \equiv d/dx\theta_n(x)$ and $\Theta'(x) \equiv d/dx\Theta(x)$.
For G^-_{ij} one obtains

$$\frac{1}{2}G^-_{(n,\alpha)(m,\beta)} = \begin{cases} (L-1)\theta'_n(\lambda_{n;\alpha}) - 2\sum\limits_{k=1}^{n-1}\theta'_k(\lambda_{n;\alpha}) - \sum\limits_{(\ell,\gamma)\neq(n,\alpha)} (\Theta'_{n,\ell}(\lambda_{n;\alpha}-\lambda_{\ell;\gamma}) + \Theta'_{n,l}(\lambda_{n;\alpha}+\lambda_{\ell;\gamma})) & \text{if}\,(n, \\ \Theta'_{n,m}(\lambda_{n;\alpha}-\lambda_{m;\beta}) - \Theta'_{n,m}(\lambda_{n;\alpha}+\lambda_{m;\beta})) & \text{if}\,(n, \end{cases}$$

Finally, the multiplicative prefactor in (2) for the generic n-string can be rewritten as

$$\prod_{a=1}^{n} \frac{\sqrt{(\lambda^a_{n;\alpha})^2 + 1/4}}{4\lambda^a_{n;\alpha}} = \frac{1}{4^n} \left(\frac{\sqrt{n^2 + \lambda^2_{n;\alpha}}}{\lambda_{n;\alpha}} \prod_{k=0}^{\lceil n/2 \rceil - 1} \frac{(2k)^2 + \lambda^2_{n;\alpha}}{(2k+1)^2 + \lambda^2_{n;\alpha}} \right)^{\mathcal{P}}, \quad (8)$$

with $\mathcal{P} = +$ and $\mathcal{P} = -$ for even and odd strings, respectively.

Bethe states with nonzero Néel overlap ($N = 12$)

String content	$2I_n^+$	E	$\lvert\langle\{\lambda\}\lvert\Psi_0\rangle\rvert^2$	here
6 inf	-	0	0.002164502165	0.002164502165
2 one, 4 inf	1_1	-3.918985947229	0.096183409244	0.096183409244237
	3_1	-3.309721467891	0.011288497947	0.0112884979464673
	5_1	-2.284629676547	0.004542580506	0.0045425805061850
	7_1	-1.169169973996	0.002752622983	0.0027526229835876
	9_1	-0.317492934338	0.002116006203	0.0021160062026402
4 one, 2 inf	$1_1 3_1$	-7.070529325964	0.310133033838	0.554809782804
	$1_1 5_1$	-5.847128730477	0.129277023687	
	$1_1 7_1$	-4.570746557876	0.085992436024	
	$3_1 5_1$	-5.153853093221	0.015256395523	
	$3_1 7_1$	-3.916336243695	0.010091113504	
	$5_1 7_1$	-2.817696043731	0.004059780228	
2 two, 2 inf	1_2	-1.905667167442	0.001207238321	0.005468702625
	3_2	-1.368837200825	0.002340453815	
	5_2	-0.681173793635	0.001921010489	
1 one, 1 three, 2 inf	$0_1 0_3$	-2.668031843135	0.034959609810	0.034959609810
6 one	$1_1 3_1 5_1$	-8.387390917445	0.153412152966	0.153412152966
2 two, 2 one	$1_1 1_2$	-5.401838225870	0.040162686361	0.046134750850
	$3_1 1_2$	-4.613929948329	0.004636541934	
	$5_1 1_2$	-3.147465758841	0.001335522556	
1 three, 3 one	$0_1 2_1 0_3$	-6.340207488736	0.052743525774	0.078910020729
	$0_1 4_1 0_3$	-5.203653009936	0.015022005621	
	$0_1 6_1 0_3$	-3.788693957250	0.011144489334	
1 five, 1 one	$0_1 0_5$	-2.444293750583	0.005887902992	0.005887902992
2 three	1_3	-1.111855930538	0.001342476001	0.001342476001
1 two, 1 four	$0_2 0_4$	-1.560671012472	0.000026982174	0.000026982174

Table 1. All Bethe states for $N = 12$ with nonzero overlap with the zero-momentum Néel state. The overlap squares add up to 1 up to the precision in which the Bethe equations were solved. The $2I_n^+$ in the second column give the positive n-string quantum numbers of the parity-invariant Bethe states.

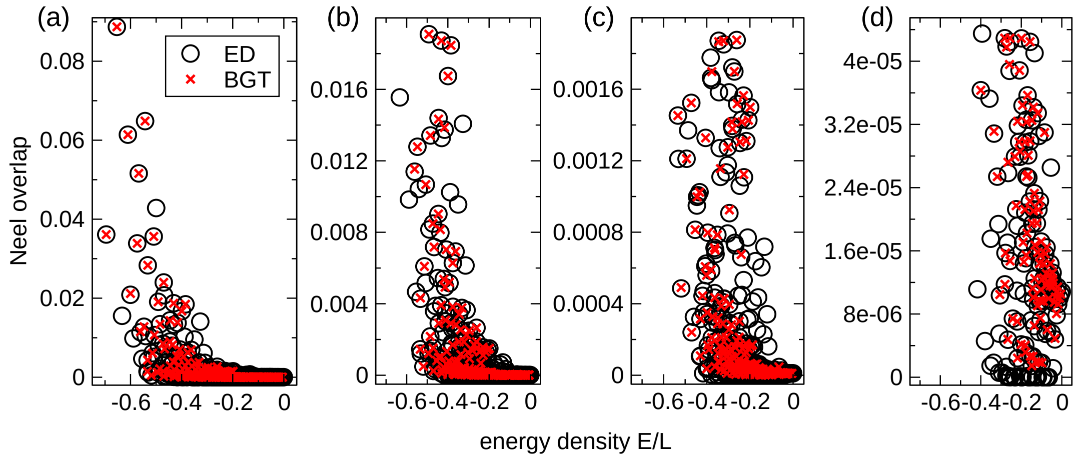

Figure 1. The squared overlap $|\langle\Psi_0|\{\lambda\}\rangle|^2$ between the the Neel state $|\Psi_0\rangle$ and the eigenstates $|\{\lambda\}\rangle$ of the XXX chain with $L = 22$ sites. Only non-zero overlaps are shown. In all the panels the x-axis shows the eigenstate energy density E/L. The circles are the exact diagonalization results for all the non-zero overlaps. The crosses are the Bethe ansatz results obtained using the Bethe-Gaudin-Takahashi equations. The missing crosses correspond to eigenstates containing zero-momentum strings. (a) Overview of all the non-zero overlaps. (b)(c)(d) The same overlaps as in (a) zooming in the regions $[0, 0.2]$, $[0, 0.020]$, and $[0, 4 \cdot 10^{-5}]$. The discrepancies between the ED and the Bethe ansatz results are attributed to the string deviations.

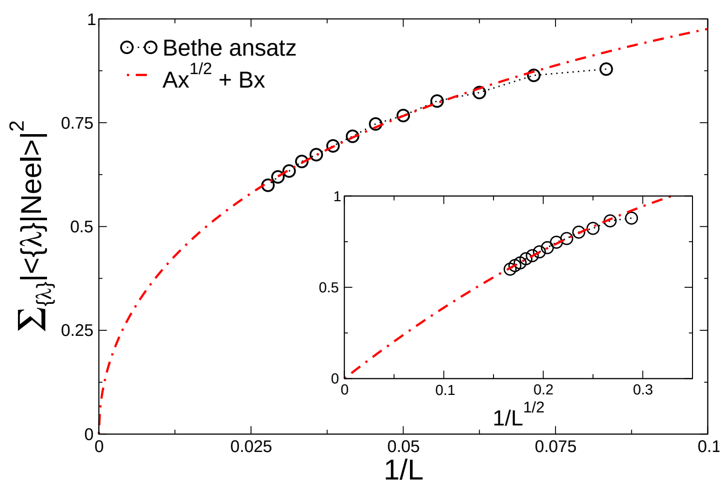

Figure 2. The overlap sum rule $\sum_{\{\lambda\}} |\langle\{\lambda\}|\Psi_0\rangle|^2$, with $|\Psi_0\rangle$ the Neel state and $|\{\lambda\}\rangle$ the eigenstates of the XXX spin chain, plotted versus $1/L$, with L the chain length. The circles are Bethe ansatz results for chains up to $L = 36$. Only the eigenstates with no zero-momentum strings are considered in the sum. The dash-dotted line is a fit to $A/L^{1/2} + B/L$, with A, B fitting parameters. Inset: The same data as in the main Figure plotted versus $1/L^{1/2}$.

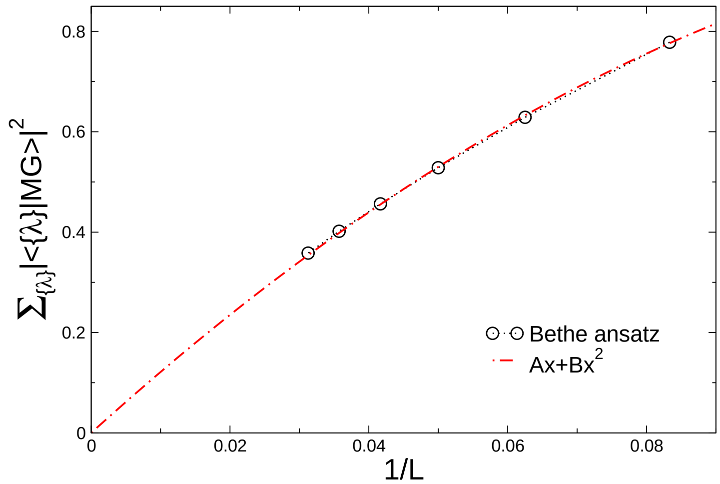

Figure 3. The overlap sum rule $\sum_{\{\lambda\}} |\langle\{\lambda\}|\Psi_0\rangle|^2$, with $|\Psi_0\rangle$ the Majumdar-Ghosh state and $|\{\lambda\}\rangle$ the eigenstates of the XXX spin chain, plotted versus $1/L$, with L the chain length. The circles are Bethe ansatz results for chains up to $L = 32$. Only the eigenstates with no zero-momentum strings are considered in the sum. The dash-dotted line is a fit to $A/L^{1/2} + B/L$, with A, B fitting parameters. Inset: The same data as in the main Figure plotted versus $1/L^{1/2}$.

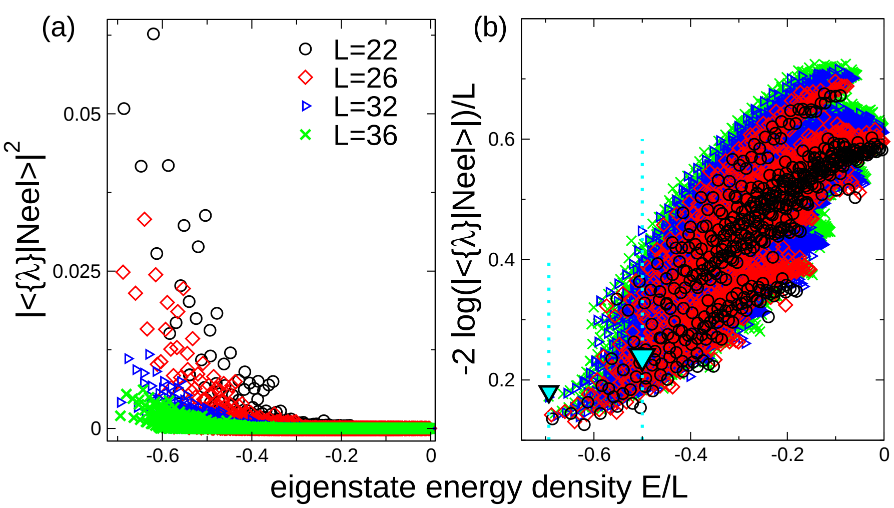

Figure 4. The overlap between the Neel state and the eigenstates of the XXX chain as a function of the eigenstate energy. (a) The squared overlap $|\langle\{\lambda\}|Neel\rangle|^2$ plotted versus the eigenstates energy density E/L. The data are Bethe ansatz results for chains of length $L = 22, 26, 32, 36$ (different symbols). (b) Same as in (a) plotting on the y-axis the combination $-2\log(|\langle\{\lambda\}|Neel\rangle|)/L$.