# Finite-size study of the Quench Action approach in integrable spin chains

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**Abstract.**
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## 1. Introduction

We show that it is possible to numerically simulate the Quench Action approach combining Monte Carlo methods and Bethe ansatz techniques.
We focus on the situation in which the pre-quench initial state is the Neel state or the Majumdar-Ghosh state.
We investigate the importance of the zero-momentum strings in the Quench Action.
Without zero-momentum strings the overlap saturation rules are not valid, i.e., in finite size systems the vast majority of the eigenstates contain zero momentum strings.
The details on the eigenstates counting depend on the pre-quench initial state.
However, we show that one can restrict to the set of non-zero momentum strings. The fact that one neglects zero-momentum strings gives rise only to scaling corrections.
We also investigate the validity of the Bethe-Takahashi approximation for the calculation of the overlap.

## 2. The Heisenberg spin chain

Here we consdier the spin-$\frac{1}{2}$ isotropic Heisenberg chain ($XXX$ chain). The $XXX$ chain with $L$ sites is defined by the Hamiltonian

$$\mathcal{H} \equiv J\sum_{i=1}^{L}\Big[\frac{1}{2}(S_i^+S_{i+1}^- + S_i^-S_{i+1}^+) + S_i^zS_{i+1}^z\Big], \tag{1}$$

where $S_i^{\pm} \equiv (\sigma_i^x \pm i\sigma_i^y)/2$ are spin operators acting on the site $i$, $S_i^z \equiv \sigma_i^z/2$, and $\sigma_i^{x,y,z}$ the Pauli matrices. We fix $J = 1$ and use periodic boundary conditions, identifying sites $L+1$ and 1. The total magnetization $S_T^z \equiv \sum_i S_i^z = L/2 - M$, with $M$ number of down spins (particles), commutes with (1), and it is here used to label its eigenstates.

### 2.1. Bethe equations and wavefunctions

The generic eigenstate of (1) in the sector with $M$ particles can be written as

$$|\Psi_M\rangle = \sum_{1\le x_1<x_2<\ldots<x_M\le L} A_M(x_1,x_2,\ldots,x_M)|x_1,x_2,\ldots,x_M\rangle, \tag{2}$$

where the sum is over the positions $\{x_i\}$ of the particles, and $A_M(x_1,x_2,\ldots,x_M)$ is the eigenstate amplitude corresponding to particles at positions $x_1,x_2,\ldots,x_M$. $A_M(x_1,x_2,\ldots,x_M)$ is given as

$$A_M(x_1,x_2,\ldots,x_M) \equiv \sum_{\mathcal{P}\in S_M}\exp\Big[i\sum_{j=1}^{M}k_{\mathcal{P}_j}x_j + i\sum_{i<j}\theta_{\mathcal{P}_i\mathcal{P}_j}\Big]. \tag{3}$$

Here the outermost sum is over the permutations $S_M$ of the so-called quasi-momenta $\{k_1,k_2,\ldots,k_M\}$. The two-particle scattering phases $\theta_{m,n}$ are defined as

$$\theta_{m,n} \equiv \frac{1}{2i}\log\Big[-\frac{e^{ik_m+ik_n}-2e^{ik_m}+1}{e^{ik_m+ik_n}-2e^{ik_n}+1}\Big]. \tag{4}$$

The energy associated to the eigenstate (2) is

$$E = \sum_{\alpha=1}^{M} (\cos(k_\alpha) - 1). \tag{5}$$

The quasi-momenta $\{k_\alpha\}$ are obtained by solving the so-called Bethe equations

$$e^{ik_\alpha L} = \prod_{\beta \neq \alpha}^{M} \Big[ - \frac{1 - 2e^{ik_\alpha} - e^{ik_\alpha + ik_\beta}}{1 - 2e^{ik_\beta} - e^{ik_\alpha + ik_\beta}} \Big]. \tag{6}$$

It is useful to introduce the rapidities $\{\lambda_\alpha\}$ as

$$k_\alpha = \pi - 2\arctan(\lambda_\alpha) \mod 2\pi. \tag{7}$$

Taking the logarithm on both sides in (6), and using (7), one obtains the Bethe equations in logarithmic form as

$$\arctan(\lambda_\alpha) = \frac{\pi}{L} J_\alpha + \frac{1}{L} \sum_{\beta \neq \alpha} \arctan\Big(\frac{\lambda_\alpha - \lambda_\beta}{2}\Big), \tag{8}$$

where $-L/2 < J_\alpha \leq L/2$ are the so-called Bethe quantum numbers. The $J_\alpha$ are half-integers and integers for $L - M$ even and odd.

These solutions of the Bethe equations (6) form particular "string" patterns in the complex plane, in the limit of large chains $L \to \infty$(string hypothesis) [?, ?]. Specifically, rapidities forming a "string" of length $1 \leq n \leq M$ (that we defined here as $n$-string) are parametrized as

$$\lambda^j_{n;\gamma} = \lambda_{n;\gamma} - i(n - 1 - 2j) + i\delta^j_{n;\gamma}, \qquad j = 0, 1, \ldots, n-1, \tag{9}$$

where $\lambda_{n;\gamma}$ is the real part of the string (string center), and $\gamma$ labels strings with different centers, while $j$ labels the different components of the string. In (9) $\delta^j_{n;\gamma}$ are the string deviations, which typically vanish exponentially in the thermodynamic limit.

Notice that pure real rapidities are strings of unit length (1-strings).

### *2.2. Bethe-Takahashi equations.*

The string centers $\lambda_{n;\gamma}$ in (9) are obtained by solving the so-called Bethe-Takahashi equations

$$2L\theta_n(\lambda_{n;\gamma}) = 2\pi I_{n;\gamma} + \sum_{(m,\beta)\neq(n,\gamma)} \Theta_{m,n}(\lambda_{n;\gamma} - \lambda_{m;\beta}), \tag{10}$$

where the generalized scattering phases $\Theta_{m,n}$ read

$$\Theta_{m,n}(x) \equiv \begin{cases} \theta_{|n-m|}(x) + \sum\limits_{r=1}^{(n+m-|n-m|-1)/2} 2\theta_{|n-m|+2r}(x) + \theta_{n+m}(x) & \text{if} \quad n \neq m \\ \sum\limits_{r=1}^{n-1} 2\theta_{2r}(x) + \theta_{2n}(x) & \text{if} \quad n = m \end{cases}$$

and $\theta_\alpha(x) \equiv 2\arctan(x/\alpha)$. Here $I_{n;\gamma}$ are the Bethe-Takahashi quantum numbers associated with $\lambda_{n;\gamma}$.

Each $M$-particle eigenstate can be characterized by its "string content" $\mathcal{S} \equiv \{s_1, \dots, s_M\}$, with $s_n$ the number of $n$-strings.
It can be shown that $I_{n;\gamma}$ are integers or half-integers for $L - s_n$ odd and even, respectively.
Clearly, the constraint $\sum_{\alpha=1}^{M} \alpha s_\alpha = M$ has to be satisfied. The upper bound for the Bethe-Takahashi quantum numbers can be derived as

$$|I_{n;\gamma}| \leq I_n^{(MAX)} \equiv \frac{1}{2}(L - 1 - \sum_{m=1}^{M} t_{m,n} s_m), \tag{11}$$

where $t_{m,n} \equiv 2\text{min}(n, m) - \delta_{m,n}$.

## 3. Overlap with the Neel state

Here we restrict to the parity-invariant eigenstates. These are the only eigenstates with non-zero overlap with the Neel state. Parity-invariant eigenstates contain only pairs of rapidities with opposite sign.
We denote the generic parity invariant eigenstate as $|\{\pm\lambda_j\}_{j=1}^{m}, n_\infty\rangle$, where $m$ is the number of rapidity pairs, $N_\infty$ is the number of infinite rapidities, with $M = L/2 = N_\infty + 2m$, and $n_\infty \equiv N_\infty/L$ is the density of infinite rapidities.
The overlap with the Neel state $|N\rangle$ reads

$$\frac{\langle N|\{\pm\lambda_j\}_{j=1}^{m}, n_\infty\rangle}{||\{\lambda_j\}_{j=1}^{m}, n_\infty\rangle||} = \frac{\sqrt{2}N_\infty!}{\sqrt{(2N_\infty)!}} \left[\prod_{j=1}^{m} \frac{\sqrt{\lambda_j^2 + 1}}{4\lambda_j}\right] \sqrt{\frac{\det_m(G^+)}{\det_m(G^-)}} \tag{12}$$

where

$$G_{jk}^{\pm} = \delta_{jk}\Big(N K_{1/2}(\lambda_j) - \sum_{l=1}^{m} K_1^+(\lambda_j, \lambda_l)\Big) + K_1^{\pm}(\lambda_j, \lambda_k), \qquad j, k = 1, \dots, m \tag{13}$$

and

$$K_1^{\pm}(\lambda, \mu) = K_1(\lambda - \mu) \pm K_1(\lambda + \mu) \tag{14}$$

and

$$K_\alpha(\lambda) \equiv \frac{8\alpha}{\lambda^2 + 4\alpha^2} \tag{15}$$

### *3.1. Reduced Neel overlap*

Here we consider the overlap formula for the Neel state (12) in the limit $L \to \infty$, assuming that the rapidities form perfect strings.
In the case of perfect strings the matrices $G_{jk}^{\pm}$ become ill-defined. Precisely, $K_1^{\pm}(\lambda, \mu)$ diverges if $\lambda$ and $\mu$ are successive members of the same string, i.e., $|\lambda - \mu| = 2i$.
It is possible to rewrite (12) in terms of the string centers $\lambda_{n;\alpha}$ only. Here we restrict ourselves to rapidity configurations with no zero-momentum strings. Our results are not valid for zero-rapidity strings. These would require the knowledge of the precise form of

the string deviations, i.e., the dependence of the string deviations on $L$, as it has been pointed out in Ref. [1].
It is convenient to split the indices $i, j$ of $G^{\pm}_{ij}$ as $i = (n, \alpha)$ $j = (m, \beta)$, with $n, m$ being the length of the strings and $\alpha, \beta$ labelling the string centers.
The result reads

$$\frac{1}{2}G^{+}_{(n,\alpha)(m,\beta)} = \begin{cases} L\theta'_n(\lambda_{n;\alpha}) - \sum\limits_{(\ell,\gamma)\neq(n,\alpha)} (\Theta'_{n,\ell}(\lambda_{n;\alpha} - \lambda_{\ell;\gamma}) + \Theta'_{n,\ell}(\lambda_{n;\alpha} + \lambda_{\ell;\gamma})) & \text{if } (n,\alpha) = (m,\beta) \\ \Theta'_{n,m}(\lambda_{n;\alpha} - \lambda_{m;\beta}) + \Theta'_{n,m}(\lambda_{n;\alpha} + \lambda_{m;\beta}) & \text{if } (n,\alpha) \neq (m,\beta) \end{cases} \tag{16}$$

Here $\theta'_n(x) \equiv d/dx\theta_n(x) = 2n/(n^2 + x^2)$ and $\Theta'(x) \equiv d/dx\Theta(x)$.
For $G^{-}_{ij}$ one obtains

$$\frac{1}{2}G^{-}_{(n,\alpha)(m,\beta)} = \begin{cases} (L-1)\theta'_n(\lambda_{n;\alpha}) - 2\sum\limits_{k=1}^{n-1}\theta'_k(\lambda_{n;\alpha}) - \sum\limits_{(\ell,\gamma)\neq(n,\alpha)} (\Theta'_{n,\ell}(\lambda_{n;\alpha} - \lambda_{\ell;\gamma}) + \Theta'_{n,l}(\lambda_{n;\alpha} + \lambda_{\ell;\gamma})) & \text{if } (n, \\ \Theta'_{n,m}(\lambda_{n;\alpha} - \lambda_{m;\beta}) - \Theta'_{n,m}(\lambda_{n;\alpha} + \lambda_{m;\beta})) & \text{if } (n, \end{cases}$$

Finally, the multiplicative prefactor in (12) for the generic $n$-string can be rewritten as

$$\prod_{a=1}^{n} \frac{\sqrt{(\lambda^a_{n;\alpha})^2 + 1}}{4\lambda^a_{n;\alpha}} = \frac{1}{4^n}\left(\frac{\sqrt{n^2 + \lambda^2_{n;\alpha}}}{\lambda_{n;\alpha}} \prod_{k=0}^{\lceil n/2\rceil - 1} \frac{(2k)^2 + \lambda^2_{n;\alpha}}{(2k+1)^2 + \lambda^2_{n;\alpha}}\right)^{\mathcal{P}}, \tag{18}$$

with $\mathcal{P} = +$ and $\mathcal{P} = -$ for even and odd strings, respectively.

## 4. Overlap with the Majumdar-Ghosh state

The overlap between the generic eigenstate of the $XXX$ chain and the Majumdar-Ghosh state is obtained fro the overlap with the Neel state as

$$\langle MG|\{\lambda_j\}^M_{j=1}\rangle = \prod_{j=1}^{M} \frac{1}{\sqrt{2}}\Big(1 - \frac{\lambda_j - i}{\lambda_j + i}\Big)\langle N|\{\lambda_j\}^M_{j=1}\rangle \tag{19}$$

The mutliplicative factor in (19), using the string hypothesis for the generic $n$-string is rewritten as

$$\prod_{j=1}^{M} \frac{1}{\sqrt{2}}\Big(1 - \frac{\lambda_j - i}{\lambda_j + i}\Big) = 2^n \prod_{k=0}^{\lfloor n/2\rfloor} \frac{1}{[\lambda^2_{n;\gamma} + (2k + (1 - (-1)^n)/2)^2]^2} \tag{20}$$

### *4.1. Counting of eigenstates with non-zero Neel overlap*

We numerically checked that the number of states with non-zero overlap with the Neel state is given as

$$Z_N = 2^{\frac{L}{2}-1} + \frac{1}{2}B\left(\frac{L}{2}, \frac{L}{4}\right) + 1, \tag{21}$$

with $B(x, y)$ denoting the binomial coefficient. The contribution 1 accounts for the ferromagnetic state. Here we assumed that $L$ is divisible by four. Here $Z_N$ is be obtained as the total number of parity-invariant Bethe-Gaudin-Takahashi quantum numbers.

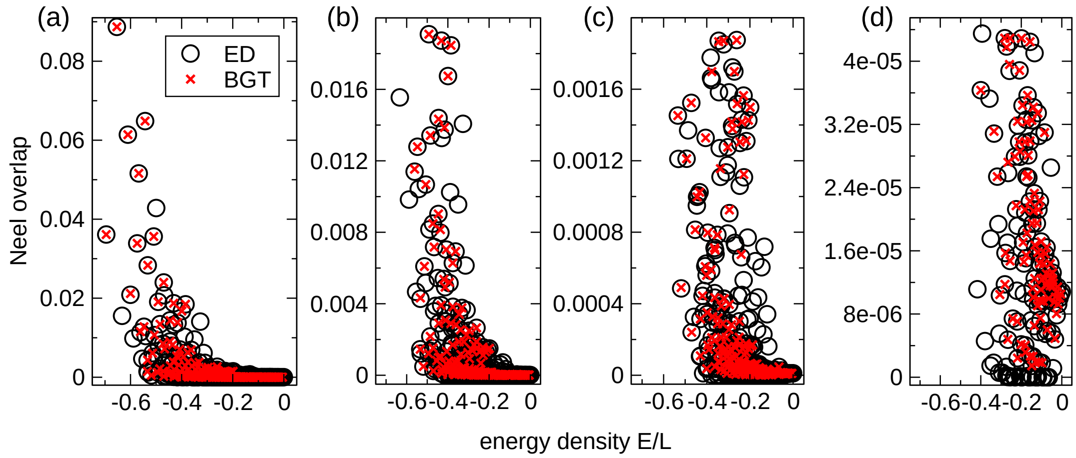


**Figure 1.** The squared overlap $|\langle\Psi_0|\{\lambda\}\rangle|^2$ between the the Neel state $|\Psi_0\rangle$ and the eigenstates $|\{\lambda\}\rangle$ of the $XXX$ chain with $L = 22$ sites. Only non-zero overlaps are shown. In all the panels the $x$-axis shows the eigenstate energy density $E/L$. The circles are the exact diagonalization results for all the non-zero overlaps. The crosses are the Bethe ansatz results obtained using the Bethe-Gaudin-Takahashi equations. The missing crosses correspond to eigenstates containing zero-momentum strings. (a) Overview of all the non-zero overlaps. (b)(c)(d) The same overlaps as in (a) zooming in the regions $[0, 0.2]$, $[0, 0.020]$, and $[0, 4 \cdot 10^{-5}]$. The discrepancies between the ED and the Bethe ansatz results are attributed to the string deviations.

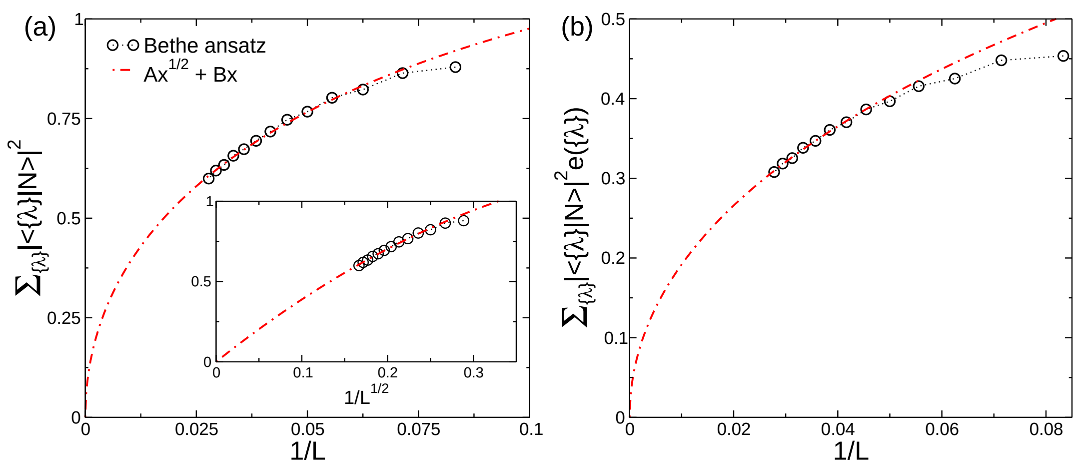


**Figure 2.** Overlap sum rules for the Neel state. (a) The overlap sum rule $\sum_{\{\lambda\}} |\langle\{\lambda\}|N\rangle|^2 = 1$, with $|N\rangle$ the Neel state and $|\{\lambda\}\rangle$ the eigenstates of the $XXX$ spin chain. The $x$-axis shows $1/L$, with $L$ the chain length. The circles are Bethe ansatz results for chains up to $L = 36$. The results are obtained via a full scanning of the chain Hilbert space. Only the eigenstates with no zero-momentum strings are considered. The dash-dotted line is a fit to $A/L^{1/2}+B/L$, with $A, B$ fitting parameters. Inset: The same data as in the main Figure plotted versus $1/L^{1/2}$. (b) The same as in (a) for the sum rule $\sum_{\{\lambda\}} |\langle\{\lambda\}|N\rangle|^2 e(\{\lambda\}) = 1/2$, with $e(\{\lambda\})$ the eigenstates energy density.

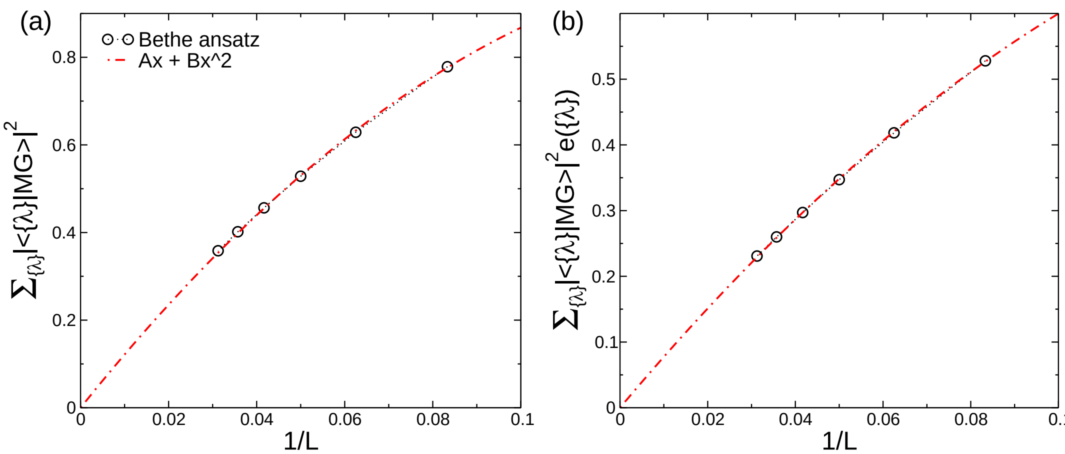


**Figure 3.** Overlap sum rules for the Majumdar-Ghosh (MG) state. (a) The overlap sum rule $\sum_{\{\lambda\}} |\langle\{\lambda\}|MG\rangle|^2 = 1$, with $|MG\rangle$ the Majumdar-Ghosh state and $|\{\lambda\}\rangle$ the eigenstates of the $XXX$ spin chain. The $x$-axis shows $1/L$, with $L$ the chain length. The circles are Bethe ansatz results for chains up to $L = 36$. The results are obtained via a full scanning of the chain Hilbert space. Only the eigenstates with no zero-momentum strings are considered. The dash-dotted line is a fit to $A/L + B/L^2$, with $A, B$ fitting parameters. (b) The same as in (a) for the sum rule $\sum_{\{\lambda\}} |\langle\{\lambda\}|MG\rangle|^2 e(\{\lambda\}) = 2/3$, with $e(\{\lambda\})$ the eigenstates energy density.

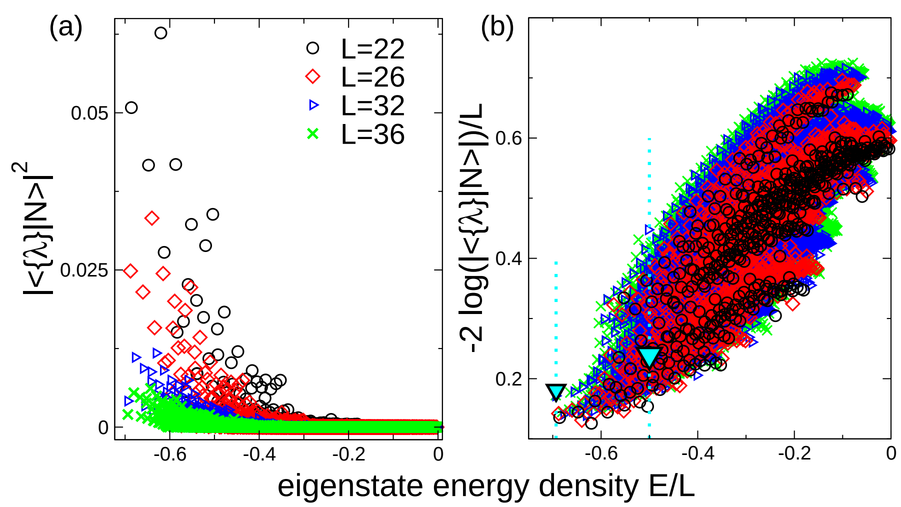


**Figure 4.** The overlap between the Neel state and the eigenstates of the $XXX$ chain as a function of the eigenstate energy. (a) The squared overlap $|\langle\{\lambda\}|Neel\rangle|^2$ plotted versus the eigenstates energy density $E/L$. The data are Bethe ansatz results for chains of length $L = 22, 26, 32, 36$ (different symbols). (b) Same as in (a) plotting on the $y$-axis the combination $-2\log(|\langle\{\lambda\}|Neel\rangle|)/L$.

After excluding the zero-momentum strings the total number of states with non-zero overlap with the Neel state is

$$Z'_N = B\left(\frac{L}{2}, \frac{L}{4}\right) \tag{22}$$

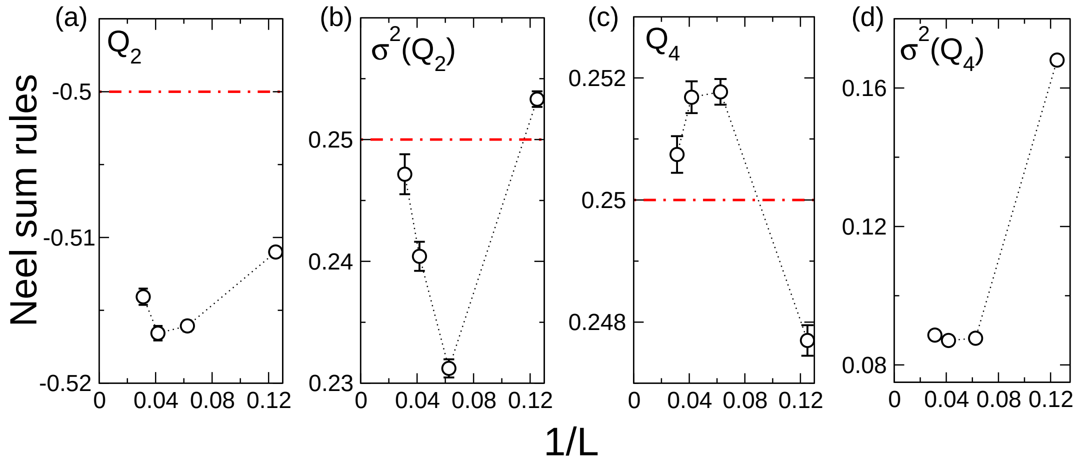


**Figure 5.** The overlap between the Neel state and the eigenstates of the $XXX$ chain as a function of the eigenstate energy. (a) The squared overlap $|\langle\{\lambda\}|Neel\rangle|^2$ plotted versus the eigenstates energy density $E/L$. The data are Bethe ansatz results for chains of length $L = 22, 26, 32, 36$ (different symbols). (b) Same as in (a) plotting on the $y$-axis the combination $-2\log(|\langle\{\lambda\}|Neel\rangle|)/L$.

Importantly, the fraction of eigenstates corresponding to non-zero momentum strings is vanishing in the thermodynamic limit as

$$\frac{Z'_N}{Z_N} \propto \frac{4}{\sqrt{\pi L}}. \tag{23}$$

### *4.2. Proof of (22)*

Here we prove (22). The proof follows very closely the one of the completeness of the Bethe ansatz eigenstates **cita qualcosa**. The proof is based on the counting of all the possible parity-invariant quantum number configurations.

Given a generic $M$-particle eigenstate of the $XXX$ chain, due to parity invariance, if one excludes the zero-momentum strings only the $n$-strings with length $n \leq M/2$ can contribute.

Thus, we denote the string content $\mathcal{S}_{PI}$ of a generic parity-invariant with no zero-momentum string eigenstate as $\mathcal{S}_{PI} \equiv \{s_1, \dots, s_{M/2}\}$. Clearly one has the constraint $\sum_k k s_k = M/2$. We also restrict ourselves for simplicity to $L$ divisible by four.

The number of available parity-invariant quantum numbers $\mathcal{N}_n$ in the $n$-string sector is given as

$$\mathcal{N}_n(L, \mathcal{S}_{PI}) = \frac{L}{2} - \sum_{m=1}^{M/2} t_{nm} s_m. \tag{24}$$

Thus, the number of parity-invariant quantum number configurations $\mathcal{N}(L, \mathcal{S}_{PI})$ compatible with string content $\mathcal{S}_{PI}$ is obtained by choosing in all the possible ways the quantum numbers as

$$\mathcal{N}(L, \mathcal{S}_{PI}) = \prod_{k=1}^{M/2} B(\mathcal{N}_k, s_k). \tag{25}$$

It is useful to focus on the situation with fixed number of particles $0 \leq \ell \leq M/2$ and number $1 \leq q \leq M/2$ of independent strings. The total number of eigenstates $\mathcal{N}''(L, \ell, q)$ is obtained as

$$\mathcal{N}''(L, \ell, q) = \sum_{\{s_k\}\; s.t \sum k s_k = \ell, \sum s_k = q} \mathcal{N}(L, \mathcal{S}_{PI}). \tag{26}$$

where the sum is over all possible string contents $\{s_k\}$ compatible with the constraints $\sum_k s_k = q$ and $\sum_k k s_k = \ell$.
The idea is now to derive a recursive relation for $\mathcal{N}''(L, \ell, q)$. It is useful to consider the shifted string content $\mathcal{S}'_{PI}$ as

$$\mathcal{S}'_{PI} \equiv \{s_{m+1}\} \quad \text{with}\, s_m \in \mathcal{S}_{PI}. \tag{27}$$

Observing that

$$t_{ij} = t_{i-1,j-1} + 2, \tag{28}$$

one can write for $n > 1$ that

$$\mathcal{N}_n(L, \mathcal{S}_{PI}) = \mathcal{N}_{n-1}(L - 4q, \mathcal{S}'_{PI}), \tag{29}$$

which implies

$$Z(L, \mathcal{S}) = B(Z_1(L, \mathcal{S}), s_1) Z(L - 4q, \mathcal{S}'). \tag{30}$$

Substituting in (26) one obtains

$$Z(L, \ell, q) = \sum_{s=0}^{q-1} B\Big(\Big\lceil \frac{L}{2} - \frac{1}{2} \Big\rceil - 2q + s, s\Big) Z(L - 4q, \ell - q, q - s), \tag{31}$$

i.e.,

$$Z(L, \ell, q) = \sum_{s=0}^{q-1} B\Big(\frac{L}{2} - 2q + s, s\Big) Z(L - 4q, \ell - q, q - s). \tag{32}$$

The initial condition $Z(L, 1, 1)$ reads

$$Z(L, 1, 1) = \frac{L}{2} - 1 \tag{33}$$

One can show that this leads to

$$Z(L, \ell, q) = \frac{L/2 - 2\ell + 1}{L/2 - \ell + 1} B(L/2 - \ell + 1, q) B(\ell - 1, q - 1). \tag{34}$$

By summing over $q = 1, \ldots, \ell$ one obtains

$$\sum_{q=1}^{\ell} Z(L, \ell, q) = B(L/2, \ell) - B(L/2, \ell - 1). \tag{35}$$

Summing over $\ell$ one obtains (22).

### *4.3. Proof of non-zero Neel overlap counting*

Here we prove (21).

Let us consider the string configurations as before in which all the particle are distributed in at most $M/2$-strings, as $\mathcal{S}_{PI} = \{s_1, s_2, \dots, s_{M/2}\}$. Now we have $\sum_k k s_k = M$.

The number of available parity-invariant quantum numbers $\mathcal{N}_n$ in the $n$-string sector is now given as

$$\mathcal{N}_n(L, \mathcal{S}_{PI}) = \left\lfloor \frac{L}{2} - \frac{1}{2} \sum_{m=1}^{M/2} t_{nm} s_m \right\rfloor . \tag{36}$$

Thus, the number of parity-invariant quantum number configurations $\mathcal{N}(L, \mathcal{S}_{PI})$ compatible with string content $\mathcal{S}_{PI}$ is obtained by choosing in all the possible ways the quantum numbers as

$$\mathcal{N}(L, \mathcal{S}_{PI}) = \prod_{k=1}^{M/2} B\left(\mathcal{N}_k, \left\lfloor \frac{s_k}{2} \right\rfloor\right) . \tag{37}$$

We focus now to the situation with fixed number of particles $0 \le \ell \le M/2$ and number $1 \le q \le M/2$ of independent strings.

The total number of eigenstates is given as

$$\mathcal{N}''(L, \ell, q) = \sum_{\{s_k\}\, s.t \sum k s_k = \ell, \sum s_k = q} \mathcal{N}(L, \mathcal{S}_{PI}). \tag{38}$$

where the sum is over all possible string contents $\{s_k\}$ compatible with the constraints $\sum_k s_k = q$ and $\sum_k k s_k = \ell$.

The idea is now to derive a recursive relation for $\mathcal{N}''(L, \ell, q)$. It is useful to consider the shifted string content $\mathcal{S}'_{PI}$ as

$$\mathcal{S}'_{PI} \equiv \{s_{m+1}\} \quad \text{with}\, s_m \in \mathcal{S}_{PI}. \tag{39}$$

Observing that

$$t_{ij} = t_{i-1,j-1} + 2, \tag{40}$$

one can write for $n > 1$ that

$$\mathcal{N}_n(L, \mathcal{S}_{PI}) = \mathcal{N}_{n-1}(L - 2q, \mathcal{S}'_{PI}), \tag{41}$$

which implies that

$$Z(L, \mathcal{S}) = B(Z_1(L, \mathcal{S}), \lfloor s_1/2 \rfloor) Z(L - 2q, \mathcal{S}') \tag{42}$$

One then has

$$Z(L, \ell, q) = \sum_{s=0}^{q-1} B\Big(\frac{L}{2} - q + \left\lfloor \frac{s}{2} \right\rfloor, \left\lfloor \frac{s}{2} \right\rfloor\Big) Z\left(L - 2q, \ell - q, q - s\right). \tag{43}$$

The initial condition $Z(L, 1, 1)$ reads

$$Z(L, 1, 1) = \frac{L}{2} - 1 \tag{44}$$

[1] P. Calabrese and P. Le Doussal, J. Stat. Mech. (2014) P05004.

Bethe states with nonzero Néel overlap ($N = 12$)

| String content | $2I_n^+$ | E | $\lvert\langle\{\lambda\}\lvert\Psi_0\rangle\rvert^2$ | here |
|---|---|---|---|---|
| 6 inf | - | 0 | 0.002164502165 | 0.002164502165 |
| 2 one, 4 inf | $1_1$ | $-3.918985947229$ | 0.096183409244 | 0.096183409244237 |
| | $3_1$ | $-3.309721467891$ | 0.011288497947 | 0.0112884979464673 |
| | $5_1$ | $-2.284629676547$ | 0.004542580506 | 0.0045425805061850 |
| | $7_1$ | $-1.169169973996$ | 0.002752622983 | 0.0027526229835876 |
| | $9_1$ | $-0.317492934338$ | 0.002116006203 | 0.0021160062026402 |
| 4 one, 2 inf | $1_1 3_1$ | $-7.070529325964$ | 0.310133033838 | 0.554809782804 |
| | $1_1 5_1$ | $-5.847128730477$ | 0.129277023687 | |
| | $1_1 7_1$ | $-4.570746557876$ | 0.085992436024 | |
| | $3_1 5_1$ | $-5.153853093221$ | 0.015256395523 | |
| | $3_1 7_1$ | $-3.916336243695$ | 0.010091113504 | |
| | $5_1 7_1$ | $-2.817696043731$ | 0.004059780228 | |
| 2 two, 2 inf | $1_2$ | $-1.905667167442$ | 0.001207238321 | 0.005468702625 |
| | $3_2$ | $-1.368837200825$ | 0.002340453815 | |
| | $5_2$ | $-0.681173793635$ | 0.001921010489 | |
| 1 one, 1 three, 2 inf | $0_1 0_3$ | $-2.668031843135$ | 0.034959609810 | 0.034959609810 |
| 6 one | $1_1 3_1 5_1$ | $-8.387390917445$ | 0.153412152966 | 0.153412152966 |
| 2 two, 2 one | $1_1 1_2$ | $-5.401838225870$ | 0.040162686361 | 0.046134750850 |
| | $3_1 1_2$ | $-4.613929948329$ | 0.004636541934 | |
| | $5_1 1_2$ | $-3.147465758841$ | 0.001335522556 | |
| 1 three, 3 one | $0_1 2_1 0_3$ | $-6.340207488736$ | 0.052743525774 | 0.078910020729 |
| | $0_1 4_1 0_3$ | $-5.203653009936$ | 0.015022005621 | |
| | $0_1 6_1 0_3$ | $-3.788693957250$ | 0.011144489334 | |
| 1 five, 1 one | $0_1 0_5$ | $-2.444293750583$ | 0.005887902992 | 0.005887902992 |
| 2 three | $1_3$ | $-1.111855930538$ | 0.001342476001 | 0.001342476001 |
| 1 two, 1 four | $0_2 0_4$ | $-1.560671012472$ | 0.000026982174 | 0.000026982174 |

**Table 1.** All Bethe states for $N = 12$ with nonzero overlap with the zero-momentum Néel state. The overlap squares add up to 1 up to the precision in which the Bethe equations were solved. The $2I_n^+$ in the second column give the positive $n$-string quantum numbers of the parity-invariant Bethe states.

Bethe states with nonzero M-G overlap ($L = 12$)

| String content | Energy (ED) | $\lvert\langle\{\lambda\}\lvert\Psi_0\rangle\rvert^2$ (ED) | $\lvert\langle\{\lambda\}\lvert\Psi_0\rangle\rvert^2$ (B-T) |
|---|---|---|---|
| $1_1 3_1 5_1$ | $-8.38739$ | 0.716616 | 0.7166157692239 |
| | $-6.34021$ | 0.205891 | |
| $5_1 1_2$ | $-5.40184$ | 0.0556247 | 0.05403336654338 |
| | $-5.20365$ | 0.0388322 | |
| $3_1 1_2$ | $-4.61393$ | 0.00568743 | 0.005582983043235 |
| | $-3.78869$ | 0.00601941 | |
| $1_1 1_2$ | $-3.14747$ | 0.00210748 | 0.0021070869333835 |
| | $-2.44429$ | 0.000129601 | |
| | $-1.56067$ | 0.000330573 | |
| $1_3$ | $-1.11186$ | 0.0000117278 | 0.000012785579923275 |

**Table 2.** All Bethe states for $L = 12$ with nonzero overlap with the zero-momentum M-G state. The first column show the string content of the eigenstate. The second and third columns show the exact diagonalization resutls for the energy and the overlap, respectively. The last colum is the overlap obtained in the Bethe ansatz approach using the Bethe-Takahashi equations.

Bethe states with nonzero M-G overlap ($L = 16$)

| String content | Energy (ED) | $\vert\langle\{\lambda\}\vert\Psi_0\rangle\vert^2$ (ED) | $\vert\langle\{\lambda\}\vert\Psi_0\rangle\vert^2$ (B-T) |
|---|---|---|---|
| $7_1 5_1 3_1 1_1$ | $-11.1423$ | 0.517742 | 0.5177418283152 |
| | $-9.59129$ | 0.244845 | |
| $7_1 5_1 1_2$ | $-8.81424$ | 0.0727096 | 0.07100180464371 |
| | $-8.56579$ | 0.082852 | |
| $7_1 3_1 1_2$ | $-7.96995$ | 0.0232953 | 0.02300602650371 |
| | $-7.89112$ | 0.0181713 | |
| $7_1 1_1 1_2$ | $-7.51522$ | 0.00245007 | 0.002427999643379 |
| | $-7.41983$ | 0.0192443 | |
| $5_1 3_1 1_2$ | $-6.80829$ | 0.012519 | 0.012518660407092 |
| | $-6.78357$ | 0.00398018 | |
| $5_1 1_1 1_2$ | $-6.3398$ | 0.00135119 | 0.0013511980854654 |
| | $-6.25276$ | 0.000721952 | |
| | $-5.86683$ | 0.00112415 | |
| | $-5.69103$ | 0.000141052 | |
| | $-5.50802$ | 0.00373666 | |
| $3_1 1_2$ | $-5.45276$ | 0.000466819 | 0.0004668223478716 |
| | $-5.06385$ | 0.000380056 | 0.000166548883431 |
| | $-4.86668$ | 0.0000395855 | |
| | $-4.64743$ | 0.000014786 | 0.000016403668846624 |
| | $-4.49059$ | 0.00155894 | |
| | $-4.25924$ | 0.000109932 | |
| | $-3.93068$ | $4.12108e-6$ | $4.6283550855711e-6$ |
| | $-3.92645$ | 0.0000102752 | |
| | $-3.77995$ | 0.0000732808 | |
| | $-3.259$ | 0.0000348879 | |
| | $-3.12751$ | 0.0000739906 | 0.00007017294942524 |
| | $-3.04176$ | $1.62543e-6$ | |
| | $-3.00072$ | $1.4684e-6$ | $1.6374806913113e-6$ |
| | $-2.32465$ | $2.01172e-8$ | |
| | $-2.26529$ | $7.53377e-6$ | |
| | $-2.02465$ | $5.8028e-7$ | $5.722094498217e-7$ |
| | $-1.38078$ | $2.17607e-7$ | |
| | $-1.11438$ | $1.825e-7$ | |
| | $-0.844856$ | $1.84715e-9$ | $2.4957509639475e-9$ |

**Table 3.** All Bethe states for $L = 12$ with nonzero overlap with the zero-momentum M-G state. The first column show the string content of the eigenstate. The second and third columns show the exact diagonalization resutls for the energy and the overlap, respectively. The last colum is the overlap obtained in the Bethe ansatz approach using the Bethe-Takahashi equations.