Bethe states with nonzero M-G overlap ($L = 12$)

String content	Energy (ED)	$\lvert\langle\{\lambda\}\lvert\Psi_0\rangle\rvert^2$ (ED)	$\lvert\langle\{\lambda\}\lvert\Psi_0\rangle\rvert^2$ (B-T)
$1_1 3_1 5_1$	-8.38739	0.716616	0.7166157692239
	-6.34021	0.205891	
$5_1 1_2$	-5.40184	0.0556247	0.05403336654338
	-5.20365	0.0388322	
$3_1 1_2$	-4.61393	0.00568743	0.005582983043235
	-3.78869	0.00601941	
$1_1 1_2$	-3.14747	0.00210748	0.0021070869333835
	-2.44429	0.000129601	
	-1.56067	0.000330573	
1_3	-1.11186	0.0000117278	0.000012785579923275

Table 1. All Bethe states for $L = 12$ with nonzero overlap with the zero-momentum M-G state. The first column show the string content of the eigenstate. The second and third columns show the exact diagonalization resutls for the energy and the overlap, respectively. The last colum is the overlap obtained in the Bethe ansatz approach using the Bethe-Takahashi equations.

Bethe states with nonzero M-G overlap ($L = 16$)

String content	Energy (ED)	$\lvert\langle\{\lambda\}\lvert\Psi_0\rangle\rvert^2$ (ED)	$\lvert\langle\{\lambda\}\lvert\Psi_0\rangle\rvert^2$ (B-T)
$7_1 5_1 3_1 1_1$	-11.1423	0.517742	0.5177418283152
	-9.59129	0.244845	
$7_1 5_1 1_2$	-8.81424	0.0727096	0.07100180464371
	-8.56579	0.082852	
$7_1 3_1 1_2$	-7.96995	0.0232953	0.02300602650371
	-7.89112	0.0181713	
$7_1 1_1 1_2$	-7.51522	0.00245007	0.002427999643379
	-7.41983	0.0192443	
$5_1 3_1 1_2$	-6.80829	0.012519	0.012518660407092
	-6.78357	0.00398018	
$5_1 1_1 1_2$	-6.3398	0.00135119	0.0013511980854654
	-6.25276	0.000721952	
	-5.86683	0.00112415	
	-5.69103	0.000141052	
	-5.50802	0.00373666	
$3_1 1_2$	-5.45276	0.000466819	0.0004668223478716
	-5.06385	0.000380056	0.000166548883431
	-4.86668	0.0000395855	
	-4.64743	0.000014786	0.000016403668846624
	-4.49059	0.00155894	
	-4.25924	0.000109932	
	-3.93068	$4.12108e-6$	$4.6283550855711e-6$
	-3.92645	0.0000102752	
	-3.77995	0.0000732808	
	-3.259	0.0000348879	
	-3.12751	0.0000739906	0.00007017294942524
	-3.04176	$1.62543e-6$	
	-3.00072	$1.4684e-6$	$1.6374806913113e-6$
	-2.32465	$2.01172e-8$	
	-2.26529	$7.53377e-6$	
	-2.02465	$5.8028e-7$	$5.722094498217e-7$
	-1.38078	$2.17607e-7$	
	-1.11438	$1.825e-7$	
	-0.844856	$1.84715e-9$	$2.4957509639475e-9$

Table 2. All Bethe states for $L = 12$ with nonzero overlap with the zero-momentum M-G state. The first column show the string content of the eigenstate. The second and third columns show the exact diagonalization resutls for the energy and the overlap, respectively. The last colum is the overlap obtained in the Bethe ansatz approach using the Bethe-Takahashi equations.