The logarithmic negativity in random spin chains

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I. INTRODUCTION

II. THE DISORDERED XX CHAIN

The disordered XX chain is defined by the Hamiltonian

$$\mathcal{H}_{XX} = \sum_{i=1}^{L-1} J_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + h \sum_{i=1}^{L} S_i^z, \quad (1)$$

with $S_i^{x,y,z} \equiv \sigma_i^{x,y,z}/2$, σ_i^α being the Pauli matrices acting on site i. For periodic boundary conditions one has an extra term in Eq. (1) connecting site L with site 1. Hereafter we fix $h = 0$ and choose J_i uniformily distributed in $[0, 1]$. After the Jordan-Wigner transformation

$$c_i = \Big(\prod_{m=1}^{i-1} \sigma_m^z \Big) \frac{\sigma_i^x - i\sigma_i^y}{2}, \quad (2)$$

(1) is recast in the free-fermionic form

$$\mathcal{H}_{XX} = \frac{1}{2} \sum_{i=1}^{L-1} J_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \frac{h}{2} \sum_{i=1}^{L-1} c_i^\dagger c_i, \quad (3)$$

with c_i spinless fermionic operators satisfying the canonical anticommutation relations $\{c_m, c_n^\dagger\} = \delta_{m,n}$. The mapping between Eq. (1) and Eq. (3) is exact apart from boundary terms (that we neglect here) giving a vanishing contribution (as $1/L$) to physical quantities in the large chain limit.

By imposing that the single-particle eigenstates $|\Psi_q\rangle$ (with q an integer labelling the different eigenstates) of (3) are of the form

$$\eta_q^\dagger |0\rangle \equiv |\Psi_q\rangle = \sum_i \Phi_q(i) c_i^\dagger |0\rangle, \quad (4)$$

with $|0\rangle$ the vacuum and $\Phi_q(i)$ the eigenstate amplitudes. The Schrödinger equation gives the equation for $\Phi_q(i)$ as

$$(J_i \Phi_q(i+1) + J_{i-1} \Phi_q(i-1))/2 = \epsilon_q \Phi_q(i), \quad i = 1, \dots, L, \quad (5)$$

and $J_L = 0$. Eq. (5) corresponds to finding the eigenvectors of the banded matrix $T = (J_j \delta_{i,j+1} + J_{j-1} \delta_{i,j-1})/2$. The eigenvalues corresponds to the single-particle eigenergies ϵ_q.

The ground state $|GS\rangle$ of (1) is obtained by filling all the negative modes ϵ_q as

$$|GS\rangle = \eta_{q_M}^\dagger \eta_{q_{M-1}}^\dagger \cdots \eta_{q_1}^\dagger |0\rangle. \quad (6)$$

One has the anticommutation relations

$$\{\eta_q^\dagger, c_j^\dagger\} = \{\eta_q, c_j\} = 0 \quad (7)$$

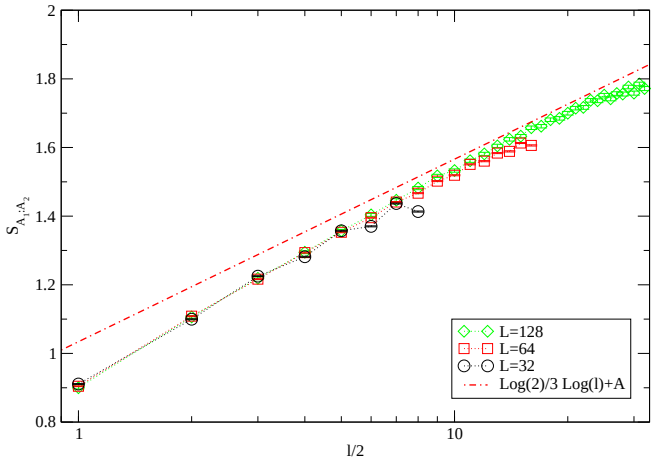

FIG. 1.

and

$$\{\eta_q^\dagger, c_j\} = \Phi_q(j) \delta_{k,j}, \quad \{\eta_q, c_j^\dagger\} = \Phi^*(j) \delta_{k,j} \quad (8)$$

These imply that the expectation value of the two-point function in a generic eigenstate of (1) is given as

$$\langle c_i^\dagger c_j \rangle = \sum_q \Phi_q^*(i) \Phi_q(j), \quad (9)$$

where the sum if over the q single-particle excitations forming the eigenstate.

One can show that the eigenvalus of the matrix T are organized in pairs with opposite sign. Given the components $\Phi_1(i)$ of the eigenvector with eigenvalue ϵ_q, the components of the eigenvector with $-\epsilon_q$ are given as $(-1)^{i+1} \Phi_q(i)$. This also implies that the ground state of (1) is in the sector with $M = L/2$ fermions.

The reduced density matrix for a subsystem A is determined by the correlation matrix restricted to A as

$$\mathcal{C}_{ij} \equiv \langle c_i^\dagger c_j \rangle, \quad (10)$$

where $i, j \in A$. In particular, given the eigenvalues λ_k of $\mathcal{C}$, the entanglement entropy S_A is given as

$$S_A = -\sum_k (\lambda_k \log \lambda_k + (1 - \lambda_k) \log(1 - \lambda_k)). \quad (11)$$

III. THE ENTANGLEMENT NEGATIVITY IN DMRG

Here we show how to calculate the logarithmic negativity in DMRG simulations.

We consider a generic $1D$ lattice of L sites. We restrict ourselves to the situation with open boundary conditions. On each site i of the lattice we consider a local Hilbert space $\mathbb{H}_i$ of finite dimension d (e.g., for spin-$1/2$ systems one has $d = 2$). A generic wavefunction $|\Psi\rangle$ corresponding to a pure stae living in $\mathbb{H}^{\otimes L}$ can be written in Matrix-Product-State (MPS) language as

$$|\Psi\rangle = \sum_{\sigma_1,\sigma_2,\ldots,\sigma_L} A^{\sigma_1}_{\alpha_1} A^{\sigma_2}_{\alpha_1\alpha_2} \cdots A^{\sigma_L}_{\alpha_L} |\sigma_1,\ldots,\sigma_L\rangle. \quad (12)$$

Here $1 \leq \sigma_i \leq d$ is the so-called physical index that labels the states in the local Hilbert space, whereas $1 \leq \alpha_i \leq \chi_i$ are the virtual indices, and χ_i is the bond dimension of the MPS. In (12) we the sum over the repeated indices α_i is assumed and $|\sigma_1,\sigma_2,\ldots,\sigma_L\rangle \equiv |\sigma_1\rangle \otimes |\sigma_2\rangle \cdots |\sigma_L\rangle$, with $|\sigma_i\rangle$ an element of the base of the Hilbert space at site i.

At each site $1 < i < L$, for each fixed σ_i, $A^{\sigma_i}_{\alpha_{i-1}\alpha_i}$ is a $\chi_i \times \chi_i$ matrix, while $A^{\sigma_1}_{\alpha_1}$ and $A^{\sigma_L}_{\alpha_L}$ are vectors.

The MPS in (12) can be pictorially represented as in Fig. .... From (12) one obtains that $\langle\Psi|$ is given as

$$\langle\Psi| = \langle\sigma_1,\ldots,\sigma_L| \sum_{\sigma_1,\sigma_2,\ldots,\sigma_L} \bar{A}^{\sigma_1}_{\alpha_1} \bar{A}^{\sigma_2}_{\alpha_1\alpha_2} \cdots \bar{A}^{\sigma_L}_{\alpha_L} \quad (13)$$

where the bar in $\bar{A}^{\sigma_i}_{\alpha_{i-1}\alpha_i}$ represents the complex conjugation.

The full system density matrix $\rho \equiv |\Psi\rangle\langle\Psi|$ is obtained as

$$\rho = \sum_{\substack{\sigma_1,\ldots,\sigma_L \\ \sigma'_1,\ldots,\sigma'_L}} (A^{\sigma_1}_{\alpha_1} \bar{A}^{\sigma'_1}_{\alpha'_1})(A^{\sigma_2}_{\alpha_1,\alpha_2} \bar{A}^{\sigma'_2}_{\alpha'_1,\alpha'_2}) \cdots (A^{\sigma_L}_{\alpha_L} \bar{A}^{\sigma'_L}_{\alpha'_L}) |\sigma_1\rangle\langle\sigma'_1| \otimes \cdots \otimes |\sigma_L\rangle\langle\sigma'_L|. \quad (14)$$

The pictorial representation of ρ is shown in Fig. ... .

We now introduce a tripartition of the system as $S = A_1 \cup A_1 \cup B$, where A_1 and A_2 are the subintervals of interest and B the remaining part of the system. To be specific we consider the case of two intervals of length ℓ_1 and ℓ_2 at mutual distance d with $A_1 = [s+1,\ldots,s+\ell_1]$ $A_2 = [s+\ell_1+d+1,\ldots,s+\ell_1+d+\ell_2]$ with A_1 shifted from the left boundary of the chain by s sites. The reduced density matrix $\rho_{A1\cup A_2}$ is obtained by tracing over the degrees of freedom of B as

$$\rho_{A_1\cup A_2} = \sum_{\{\sigma_i,\sigma'_i:\, i\in A_1\cup A_2\}} \delta_{\nu_1,\nu'_1} \underbrace{\left(\prod_{n\in A_1} A^{\sigma_n}_{\nu_n,\nu_{n+1}}\right)}_{F_1} \left(\prod_{n\in A_1} \bar{A}^{\sigma'_n}_{\nu'_n,\nu'_{n+1}}\right) [E_{12}]^{\nu_{\ell_1+1},\nu_{\ell_1+d+1}}_{\nu'_{\ell_1+1},\nu'_{\ell_1+d+1}}$$
$$\underbrace{\left(\prod_{m\in A_2} A^{\sigma_m}_{\nu_m,\nu_{m+1}}\right)}_{F_2} \left(\prod_{m\in A_2} \bar{A}^{\sigma'_m}_{\nu'_m,\nu'_{m+1}}\right) \delta_{\nu_{\ell_1+\ell_2+d+1},\nu'_{\ell_1+\ell_2+d+1}} \prod_{i\in A_1\cup A_2} |\sigma_i\rangle\langle\sigma'_i|. \quad (15)$$

Here the index $n \in [1,\ldots,\ell_1]$ ($m \in [d+\ell_1+1,\ldots,d+\ell_1+\ell_2]$) counts only the spins in the interval A_1 (A_2). Notice that the two Kronecker delta in (15) are because the virtual indices ν_1,ν'_1 and $\nu_{\ell_1+\ell_2+d+1},\nu'_{\ell_1+\ell_2+d+1}$ are contracted. They arise from contracting the matrices A lying outside the interval of interest. Moreover, repeated indices are summed over, and the term in the first square bracket in (15) depends on ν_{ℓ_1+1} and ν'_{ℓ_1+1}. Finally, the tensor F_1 depends on the physical indices $\sigma_1,\ldots,\sigma_{\ell_1}$ and on ν_1,ν_{ℓ_1+1}. In (15) we defined the "transfer matrix" between the two intervals E as

$$[E_{12}]^{\nu_{\ell_1+1},\nu_{\ell_1+d+1}}_{\nu'_{\ell_1+1},\nu'_{\ell_1+d+1}} \equiv \prod_{j\in B_1} A^{\sigma_j}_{\nu_j,\nu_{j+1}} \bar{A}^{\sigma_j}_{\nu'_j,\nu'_{j+1}}. \quad (16)$$

Notice that all the spin indices are contracted in (16). Also, the computational cost for constructing E is $\mathcal{O}(\chi^6_{max})$, with χ_{max} the largest bond dimension in the chain. It is also convenient for later to define the analogous tensors E_1 and E_2 for the intervals A_1 and A_2 as

$$[E_1]^{\nu_1,\nu_{\ell_1+1}}_{\nu'_1,\nu'_{\ell_1+1}} \equiv \prod_{j\in A_1} A^{\sigma_j}_{\nu_j,\nu_{j+1}} \bar{A}^{\sigma_j}_{\nu'_j,\nu'_{j+1}} \quad (17)$$

$$[E_2]^{\nu_{\ell_1+d+1},\nu_{\ell_1+\ell_2+d+1}}_{\nu'_{\ell_1+d+1},\nu'_{\ell_1+\ell_2+d+1}} \equiv \prod_{j\in A_2} A^{\sigma_j}_{\nu_j,\nu_{j+1}} \bar{A}^{\sigma_j}_{\nu'_j,\nu'_{j+1}}. \quad (18)$$

It is convenient to recast (15) as

$$\rho_{A_1\cup A_2} = \sum_{\sigma_i:\, i\in A_1} [F_1]^{\sigma_1,\ldots,\sigma_{\ell_1}}_{\nu_1,\nu_{\ell_1+1}} \prod_{j\in A_1} |\sigma_j\rangle\langle\sigma'_j| \sum_{\sigma'_i:\, i\in A_1} [\bar{F}_1]^{\sigma'_1,\ldots,\sigma'_{\ell_1}}_{\nu'_1,\nu'_{\ell_1+1}} [E_{12}]^{\nu_{\ell_1+1},\nu_{\ell_1+d+1}}_{\nu'_{\ell_1+1},\nu'_{\ell_1+d+1}} \delta_{\nu_{\ell_1+\ell_2+d+1},\nu'_{\ell_1+\ell_2+d+1}} \delta_{\nu_1,\nu'_1}$$
$$\otimes \sum_{\sigma_i:\, i\in A_2} [F_2]^{\sigma_{\ell_1+d+1},\ldots,\sigma_{\ell_1+\ell_2+d}}_{\nu_{\ell_1+d+1},\nu_{\ell_1+\ell_2+d+1}} \prod_{j\in A_2} |\sigma_j\rangle\langle\sigma'_j| \sum_{\sigma'_i:\, i\in A_2} [\bar{F}_2]^{\sigma'_{\ell_1+d+1},\ldots,\sigma'_{\ell_1+\ell_2+d}}_{\nu'_{\ell_1+d+1},\nu'_{\ell_1+\ell_2+d+1}}. \quad (19)$$

This can be rewritten in a more convenient form as

$$\rho_{A_1\cup A_2} = \sum_{\sigma_i:\, i\in A_1} [F_1]^{\{\sigma_i\}}_{\nu_{1L},\nu_{1R}} \prod_{j\in A_1} |\sigma_j\rangle\langle\sigma'_j| \sum_{\sigma'_i:\, i\in A_1} [\bar{F}_1]^{\{\sigma'_i\}}_{\nu'_{1L},\nu'_{1R}} [E_{12}]^{\nu_{1R},\nu_{2L}}_{\nu'_{1R},\nu'_{2L}} \delta_{\nu_{2R},\nu'_{2R}} \delta_{\nu_{1L},\nu'_{1L}}$$
$$\otimes \sum_{\sigma_i:\, i\in A_2} [F_2]^{\{\sigma_i\}}_{\nu_{2L},\nu_{2R}} \prod_{j\in A_2} |\sigma_j\rangle\langle\sigma'_j| \sum_{\sigma'_i:\, i\in A_2} [\bar{F}_2]^{\{\sigma'_i\}}_{\nu'_{2L},\nu'_{2R}}, \quad (20)$$

where we defined the indices $\nu_{1L} \equiv \nu_1, \nu_{1R} \equiv \nu_{\ell_1+1}$ and $\nu_{2L} \equiv \nu_{\ell_1+1}, \nu_{2R} \equiv \nu_{\ell_1+\ell_2+d+1}$.

After introducing the states

$$|w^{(1)}_{\nu_{1L},\nu_{1R}}\rangle \equiv \sum_{\sigma_i:\, i\in A_1} [F_1]^{\{\sigma_i\}}_{\nu_{1L},\nu_{1R}} \prod_{j\in A_1} |\sigma_j\rangle, \quad (21)$$
$$|w^{(2)}_{\nu_{2L},\nu_{2R}}\rangle \equiv \sum_{\sigma_i:\, i\in A_2} [F_2]^{\{\sigma_i\}}_{\nu_{2L},\nu_{2R}} \prod_{j\in A_2} |\sigma_j\rangle. \quad (22)$$

Eq. (20) can be rewritten in the compact form

$$\rho_{A_1\cup A_2} = \delta_{\nu_{1L},\nu'_{1L}} \delta_{\nu_{2R},\nu'_{2R}} [E_{12}]^{\nu_{1R},\nu_{2L}}_{\nu'_{1R},\nu'_{2L}} |w^{(1)}_{\nu_{1L},\nu_{1R}}\rangle\langle w^{(1)}_{\nu'_{1L},\nu'_{1R}}| \otimes |w^{(2)}_{\nu_{2L},\nu_{2R}}\rangle\langle w^{(2)}_{\nu'_{2L},\nu'_{2R}}|. \quad (23)$$

One should now observe that, although the vectors $|w^{(1)}_{\nu_{1L},\nu_{1R}}\rangle$ provide a basis for the interval A_1 (similarly for A_2) they are not orthonormal (not even orthogonal). Clearly, one has that

$$\langle w^{(1)}_{\nu_{1L},\nu_{1R}} | w^{(1)}_{\nu'_{1L},\nu'_{1R}}\rangle = [E_1]^{\nu_{1L},\nu_{1R}}_{\nu'_{1L},\nu'_{1R}} \quad (24)$$
$$\langle w^{(2)}_{\nu_{2L},\nu_{2R}} | w^{(2)}_{\nu'_{2L},\nu'_{2R}}\rangle = [E_2]^{\nu_{2L},\nu_{2R}}_{\nu'_{2L},\nu'_{2R}}. \quad (25)$$

A notable exception when the basis given by $|w^{(1)}_{\nu_{1L},\nu_{1R}}\rangle$ and $|w^{(2)}_{\nu_{2L},\nu_{2R}}\rangle$ is orthogonal is when the two intervals A_1 and A_2 are attached at the left and right edges of the chain, respectively. In this situation the reduced density matrix is given as

$$[\rho_{A_1\cup A_2}]^{(\nu_{1R},\nu_{2L})}_{(\nu'_{1R},\nu'_{2L})} = [E_{12}]^{\nu_{1R},\nu_{2L}}_{\nu'_{1R},\nu'_{2L}} \quad (26)$$

and it is a $\chi^2_{max} \times \chi^2_{max}$ matrix. In the more general case, one way to promote the basis to an orthonormal one is by performing a singular value decomposition (SVD) of the matrices E_1 and E_2. Given the SVD of E_1 as $E_1 = UDU^\dagger$ with U orthogonal and D diagonal, then the vectors $|v^{(1)}_m\rangle$

$$|v^{(1)}_m\rangle \equiv [N_1]^m_{\nu_{1L},\nu_{1R}} |w^{(1)}_{\nu_{1L},\nu_{1R}}\rangle \quad (27)$$

with $N_1 \equiv D^{-\frac{1}{2}}U$ are orthonormal. This leads to

$$\rho_{A_1\cup A_2} = \quad (28)$$

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