

The logarithmic negativity in random spin chains

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I. INTRODUCTION

II. THE DISORDERED XX CHAIN

The disordered XX chain is defined by the Hamiltonian

$$\mathcal{H}_{XX} = \sum_{i=1}^{L-1} J_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + h \sum_{i=1}^L S_i^z, \quad (1)$$

with $S_i^{x,y,z} \equiv \sigma_i^{x,y,z}/2$, σ_i^α being the Pauli matrices acting on site i . For periodic boundary conditions one has an extra term in Eq. (1) connecting site L with site 1. Hereafter we fix $h = 0$ and choose J_i uniformly distributed in $[0, 1]$. After the Jordan-Wigner transformation

$$c_i = \left(\prod_{m=1}^{i-1} \sigma_m^z \right) \frac{\sigma_i^x - i\sigma_i^y}{2}, \quad (2)$$

(1) is recast in the free-fermionic form

$$\mathcal{H}_{XX} = \frac{1}{2} \sum_{i=1}^{L-1} J_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \frac{h}{2} \sum_{i=1}^{L-1} c_i^\dagger c_i, \quad (3)$$

with c_i spinless fermionic operators satisfying the canonical anticommutation relations $\{c_m, c_n^\dagger\} = \delta_{m,n}$. The mapping between Eq. (1) and Eq. (3) is exact apart from boundary terms (that we neglect here) giving a vanishing contribution (as $1/L$) to physical quantities in the large chain limit.

By imposing that the single-particle eigenstates $|\Psi_q\rangle$ (with q an integer labelling the different eigenstates) of (3) are of the form

$$\eta_q^\dagger |0\rangle \equiv |\Psi_q\rangle = \sum_i \Phi_q(i) c_i^\dagger |0\rangle, \quad (4)$$

with $|0\rangle$ the vacuum and $\Phi_q(i)$ the eigenstate amplitudes. The Schrödinger equation gives the equation for $\Phi_q(i)$ as

$$(J_i \Phi_q(i+1) + J_{i-1} \Phi_q(i-1))/2 = \epsilon_q \Phi_q(i), \quad i = 1, \dots, L, \quad (5)$$

and $J_L = 0$. Eq. (5) corresponds to finding the eigenvectors of the banded matrix $T = (J_j \delta_{i,j+1} + J_{j-1} \delta_{i,j-1})/2$. The eigenvalues corresponds to the single-particle eigenenergies ϵ_q .

The ground state $|GS\rangle$ of (1) is obtained by filling all the negative modes ϵ_q as

$$|GS\rangle = \eta_{q_M}^\dagger \eta_{q_{M-1}}^\dagger \cdots \eta_{q_1}^\dagger |0\rangle. \quad (6)$$

One has the anticommutation relations

$$\{\eta_q^\dagger, c_j^\dagger\} = \{\eta_q, c_j\} = 0 \quad (7)$$

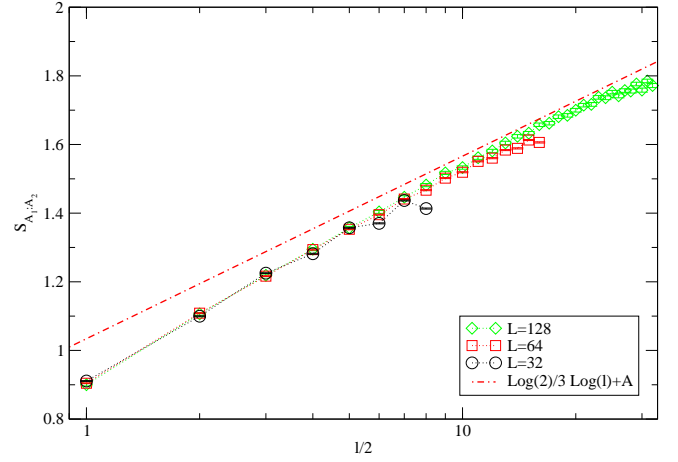


FIG. 1.

and

$$\{\eta_q^\dagger, c_j\} = \Phi_q(j) \delta_{q,j}, \quad \{\eta_q, c_j^\dagger\} = \Phi_q^*(j) \delta_{q,j} \quad (8)$$

These imply that the expectation value of the two-point function in a generic eigenstate of (1) is given as

$$\langle c_i^\dagger c_j \rangle = \sum_q \Phi_q^*(i) \Phi_q(j), \quad (9)$$

where the sum is over the q single-particle excitations forming the eigenstate.

One can show that the eigenvalues of the matrix T are organized in pairs with opposite sign. Given the components $\Phi_1(i)$ of the eigenvector with eigenvalue ϵ_q , the components of the eigenvector with $-\epsilon_q$ are given as $(-1)^{i+1} \Phi_q(i)$. This also implies that the ground state of (1) is in the sector with $M = L/2$ fermions.

The reduced density matrix for a subsystem A is determined by the correlation matrix restricted to A as

$$C_{ij} \equiv \langle c_i^\dagger c_j \rangle, \quad (10)$$

where $i, j \in A$. In particular, given the eigenvalues λ_k of C , the entanglement entropy S_A is given as

$$S_A = - \sum_k (\lambda_k \log \lambda_k + (1 - \lambda_k) \log (1 - \lambda_k)). \quad (11)$$

III. THE ENTANGLEMENT NEGATIVITY IN DMRG

Here we show how to calculate the logarithmic negativity in DMRG simulations.

We consider a generic 1D lattice of L sites. We restrict ourselves to the situation with open boundary conditions. On each site i of the lattice we consider a local Hilbert space \mathbb{H}_i of finite dimension d (e.g., for spin-1/2 systems one has $d = 2$). A generic wavefunction $|\Psi\rangle$ corresponding to a pure state living in $\mathbb{H}^{\otimes L}$ can be written in Matrix-Product-State (MPS) language as

$$|\Psi\rangle = \sum_{\sigma_1, \sigma_2, \dots, \sigma_L} A_{\alpha_1}^{\sigma_1} A_{\alpha_1 \alpha_2}^{\sigma_2} \cdots A_{\alpha_L}^{\sigma_L} |\sigma_1, \dots, \sigma_L\rangle. \quad (12)$$

Here $1 \leq \sigma_i \leq d$ is the so-called physical index that labels the states in the local Hilbert space, whereas $1 \leq \alpha_i \leq \chi_i$ are the virtual indices, and χ_i is the bond dimension of the MPS.

In (12) the sum over the repeated indices α_i is assumed and $|\sigma_1, \sigma_2, \dots, \sigma_L\rangle \equiv |\sigma_1\rangle \otimes |\sigma_2\rangle \cdots |\sigma_L\rangle$, with $|\sigma_i\rangle$ an element of the base of the Hilbert space at site i .

At each site $1 < i < L$, for each fixed σ_i , $A_{\alpha_{i-1} \alpha_i}^{\sigma_i}$ is a $\chi_i \times \chi_i$ matrix, while $A_{\alpha_1}^{\sigma_1}$ and $A_{\alpha_L}^{\sigma_L}$ are vectors.

The MPS in (12) can be pictorially represented as in Fig. From (12) one obtains that $\langle \Psi |$ is given as

$$\langle \Psi | = \langle \sigma_1, \dots, \sigma_L | \sum_{\sigma_1, \sigma_2, \dots, \sigma_L} \bar{A}_{\alpha_1}^{\sigma_1} \bar{A}_{\alpha_1 \alpha_2}^{\sigma_2} \cdots \bar{A}_{\alpha_L}^{\sigma_L} \quad (13)$$

where the bar in $\bar{A}_{\alpha_{i-1} \alpha_i}^{\sigma_i}$ represents the complex conjugation.

The full system density matrix $\rho \equiv |\Psi\rangle\langle\Psi|$ is obtained as

$$\rho = \sum_{\substack{\sigma_1, \dots, \sigma_L \\ \sigma'_1, \dots, \sigma'_L}} (A_{\alpha_1}^{\sigma_1} \bar{A}_{\alpha'_1}^{\sigma'_1}) (A_{\alpha_1 \alpha_2}^{\sigma_2} \bar{A}_{\alpha'_1 \alpha'_2}^{\sigma'_2}) \cdots (A_{\alpha_L}^{\sigma_L} \bar{A}_{\alpha'_L}^{\sigma'_L}) |\sigma_1\rangle\langle\sigma'_1| \otimes \cdots \otimes |\sigma_L\rangle\langle\sigma'_L|. \quad (14)$$

The pictorial representation of ρ is shown in Fig.

We now introduce a tripartition of the system as $S = A_1 \cup A_1 \cup B$, where A_1 and A_2 are the subintervals of interest and B the remaining part of the system. To be specific we consider the case of two intervals of length ℓ_1 and ℓ_2 at mutual distance

d with $A_1 = [s+1, \dots, s+\ell_1]$ $A_2 = [s+\ell_1+d+1, \dots, s+\ell_1+d+\ell_2]$ with A_1 shifted from the left boundary of the chain by s sites. The reduced density matrix $\rho_{A_1 \cup A_2}$ is obtained by tracing over the degrees of freedom of B as

$$\rho_{A_1 \cup A_2} = \sum_{\{\sigma_i, \sigma'_i: i \in A_1 \cup A_2\}} \delta_{\nu_1, \nu'_1} \underbrace{\left(\prod_{n \in A_1} A_{\nu_n, \nu_{n+1}}^{\sigma_n} \right)}_{F_1} \left(\prod_{n \in A_1} \bar{A}_{\nu'_n, \nu'_{n+1}}^{\sigma'_n} \right) [E_{12}]_{\nu'_{\ell_1+1}, \nu'_{\ell_1+d+1}}^{\nu_{\ell_1+1}, \nu_{\ell_1+d+1}} \underbrace{\left(\prod_{m \in A_2} A_{\nu_m, \nu_{m+1}}^{\sigma_m} \right)}_{F_2} \left(\prod_{m \in A_2} \bar{A}_{\nu'_m, \nu'_{m+1}}^{\sigma'_m} \right) \delta_{\nu_{\ell_1+\ell_2+d+1}, \nu'_{\ell_1+\ell_2+d+1}} \prod_{i \in A_1 \cup A_2} |\sigma_i\rangle\langle\sigma'_i|. \quad (15)$$

Here the index $n \in [1, \dots, \ell_1]$ ($m \in [d+\ell_1+1, \dots, d+\ell_1+\ell_2]$) counts only the spins in the interval A_1 (A_2). Notice that the two Kronecker delta in (15) are because the virtual indices ν_1, ν'_1 and $\nu_{\ell_1+\ell_2+d+1}, \nu'_{\ell_1+\ell_2+d+1}$ are contracted. Moreover, repeated indices are summed over, and the term in the first square bracket in (15) depends on ν_{ℓ_1+1} and ν'_{ℓ_1+1} . Finally, the tensor F_1 depends on the physical indices $\sigma_1, \dots, \sigma_{\ell_1}$ and on ν_1, ν_{ℓ_1+1} . In (15) we defined the “transfer matrix” between the two intervals E as

$$[E_{12}]_{\nu'_{\ell_1+1}, \nu'_{\ell_1+d+1}}^{\nu_{\ell_1+1}, \nu_{\ell_1+d+1}} \equiv \prod_{j \in B_1} A_{\nu_j, \nu_{j+1}}^{\sigma_j} \bar{A}_{\nu'_j, \nu'_{j+1}}^{\sigma'_j}. \quad (16)$$

Notice that all the spin indices are contracted in (16). Also, the computational cost for constructing E is $\mathcal{O}(\chi_{max}^6)$, with χ_{max} the largest bond dimension in the chain. It is also convenient for later to define the analogous tensors E_1 and E_2 for the intervals A_1 and A_2 as

$$[E_1]_{\nu'_1, \nu'_{\ell_1+1}}^{\nu_1, \nu_{\ell_1+1}} \equiv \prod_{j \in A_1} A_{\nu_j, \nu_{j+1}}^{\sigma_j} \bar{A}_{\nu'_j, \nu'_{j+1}}^{\sigma'_j} \quad (17)$$

$$[E_2]_{\nu'_{\ell_1+d+1}, \nu'_{\ell_1+\ell_2+d+1}}^{\nu_{\ell_1+d+1}, \nu_{\ell_1+\ell_2+d+1}} \equiv \prod_{j \in A_2} A_{\nu_j, \nu_{j+1}}^{\sigma_j} \bar{A}_{\nu'_j, \nu'_{j+1}}^{\sigma'_j}. \quad (18)$$

It is convenient to recast (15) as

$$\rho_{A_1 \cup A_2} = \sum_{\sigma_i: i \in A_1} [F_1]_{\nu_1, \nu_{\ell_1+1}}^{\sigma_1, \dots, \sigma_{\ell_1}} \prod_{j \in A_1} |\sigma_j\rangle \langle \sigma'_j| \sum_{\sigma'_i: i \in A_1} [\bar{F}_1]_{\nu'_1, \nu'_{\ell_1+1}}^{\sigma'_1, \dots, \sigma'_{\ell_1}} [E_{12}]_{\nu'_{\ell_1+1}, \nu'_{\ell_1+d+1}}^{\nu_{\ell_1+1}, \nu_{\ell_1+d+1}} \delta_{\nu_{\ell_1+\ell_2+d+1}, \nu'_{\ell_1+\ell_2+d+1}} \delta_{\nu_1, \nu'_1} \\ \otimes \sum_{\sigma_i: i \in A_2} [F_2]_{\nu_{\ell_1+d+1}, \nu_{\ell_1+\ell_2+d+1}}^{\sigma_{\ell_1+d+1}, \dots, \sigma_{\ell_1+\ell_2+d}} \prod_{j \in A_2} |\sigma_j\rangle \langle \sigma'_j| \sum_{\sigma'_i: i \in A_2} [\bar{F}_2]_{\nu'_{\ell_1+d+1}, \nu'_{\ell_1+\ell_2+d+1}}^{\sigma'_{\ell_1+d+1}, \dots, \sigma'_{\ell_1+\ell_2+d}}. \quad (19)$$

This can be rewritten in a more convenient form as

$$\rho_{A_1 \cup A_2} = \sum_{\sigma_i: i \in A_1} [F_1]_{\nu_{1L}, \nu_{1R}}^{\{\sigma_i\}} \prod_{j \in A_1} |\sigma_j\rangle \langle \sigma'_j| \sum_{\sigma'_i: i \in A_1} [\bar{F}_1]_{\nu'_{1L}, \nu'_{1R}}^{\{\sigma'_i\}} [E_{12}]_{\nu'_{1R}, \nu'_{2L}}^{\nu_{1R}, \nu_{2L}} \delta_{\nu_{2R}, \nu'_{2R}} \delta_{\nu_{1L}, \nu'_{1L}} \\ \otimes \sum_{\sigma_i: i \in A_2} [F_2]_{\nu_{2L}, \nu_{2R}}^{\{\sigma_i\}} \prod_{j \in A_2} |\sigma_j\rangle \langle \sigma'_j| \sum_{\sigma'_i: i \in A_2} [\bar{F}_2]_{\nu'_{2L}, \nu'_{2R}}^{\{\sigma'_i\}}, \quad (20)$$

where we defined the indices $\nu_{1L} \equiv \nu_1, \nu_{1R} \equiv \nu_{\ell_1+1}$ and $\nu_{2L} \equiv \nu_{\ell_1+1}, \nu_{2R} \equiv \nu_{\ell_1+\ell_2+d+1}$.

After introducing the states

$$|w_{\nu_{1L}, \nu_{1R}}^{(1)}\rangle \equiv \sum_{\sigma_i: i \in A_1} [F_1]_{\nu_{1L}, \nu_{1R}}^{\{\sigma_i\}} \prod_{j \in A_1} |\sigma_j\rangle, \quad (21)$$

$$|w_{\nu_{2L}, \nu_{2R}}^{(2)}\rangle \equiv \sum_{\sigma_i: i \in A_2} [F_2]_{\nu_{2L}, \nu_{2R}}^{\{\sigma_i\}} \prod_{j \in A_2} |\sigma_j\rangle. \quad (22)$$

Eq. (20) can be rewritten in the compact form

$$\rho_{A_1 \cup A_2} = \delta_{\nu_{1L}, \nu'_{1L}} \delta_{\nu_{2R}, \nu'_{2R}} [E_{12}]_{\nu'_{1R}, \nu'_{2L}}^{\nu_{1R}, \nu_{2L}} |w_{\nu_{1L}, \nu_{1R}}^{(1)}\rangle \langle w_{\nu'_{1L}, \nu'_{1R}}^{(1)}| \quad (23)$$

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