igcond and igcondinv functions

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This document describes in detail what the igcond and igcondinv functions are in the copsupp package.

IG Copula Family Definition

The IG copula family is defined as

$$C\left(u, v; \theta, k\right) = u + v - 1 + \left(1 - u\right) H_k\left(H_k^{\leftarrow}\left(1 - v; \theta\right); \theta\left(1 - u\right)\right), \tag{1}$$

where the parameter space is $(\theta, k) \in [0, \infty) \times (1, \infty)$, and $H_k(\cdot; \theta) : [1, \infty) \to (0, 1]$ is defined by

$$H_k\left(y; \theta\right) = \begin{cases} \frac{1}{y} \Psi_k\left(\frac{1}{\theta \log y}\right), & y > 1; \\ 1, & y = 1, \end{cases} \tag{2}$$

where $\Psi_k : [0, \infty) \to (0, 1]$ is a concave distribution function defined by

$$\Psi_k\left(y\right) = \begin{cases} y \frac{\Gamma(k) - \Gamma^*\left(k, y^{-1}\right)}{\Gamma(k-1)} + \frac{\Gamma^*\left(k-1, y^{-1}\right)}{\Gamma(k-1)}, & y > 0; \\ 0, & y = 0, \end{cases} \tag{3}$$

where Γ and Γ^* are the gamma and (upper) incomplete gamma functions, respectively. Note that $H_k(\cdot; \theta)$ is strictly decreasing for all $\theta > 0$, $k > 1$, and $H_k^{\leftarrow}(\cdot; \theta)$ is the unique inverse function of $H_k(\cdot; \theta)$.

2|1 Distribution Function

The 2|1 distribution function of the IG copula for some $(\theta, k) \in [0, \infty) \times (1, \infty)$ and $u \in (0, 1)$ is

$$\begin{aligned} C_{2|1}\left(v | u; \theta, k\right) &= 1 - \frac{\bar{F}_{k-1}\left(\theta\left(1 - u\right) \log H_k^{\leftarrow}\left(1 - v; \theta\right)\right)}{H_k^{\leftarrow}\left(1 - v; \theta\right)} \\ &= 1 - \varphi_k\left(H_k^{\leftarrow}\left(1 - v; \theta\right); \theta\left(1 - u\right)\right), \end{aligned} \tag{4}$$

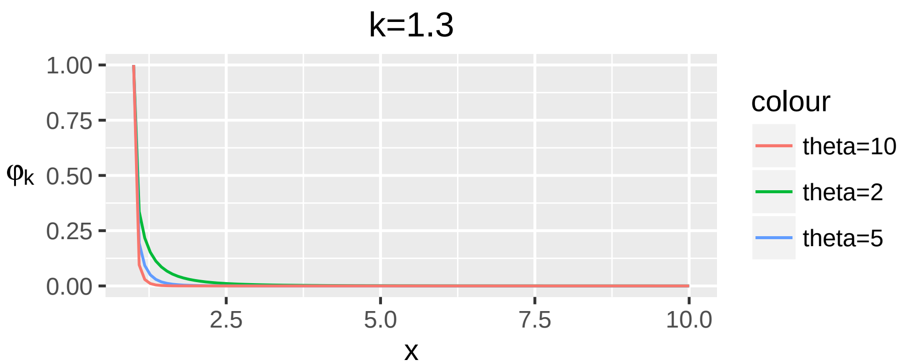
$v \in (0,1)$, where $\bar{F}_{k-1}(x) = \Gamma^*(k-1,x)/\Gamma(k-1)$ for $x \geq 0$ is the Gamma survival function with shape parameter $k-1$ and unit scale parameter, and

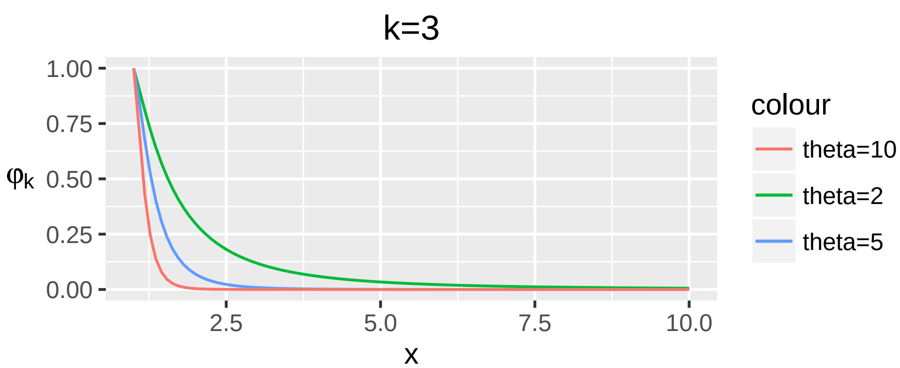
$$\varphi_k(x;\eta) = x^{-1}\bar{F}_{k-1}(\eta \log x) \tag{5}$$

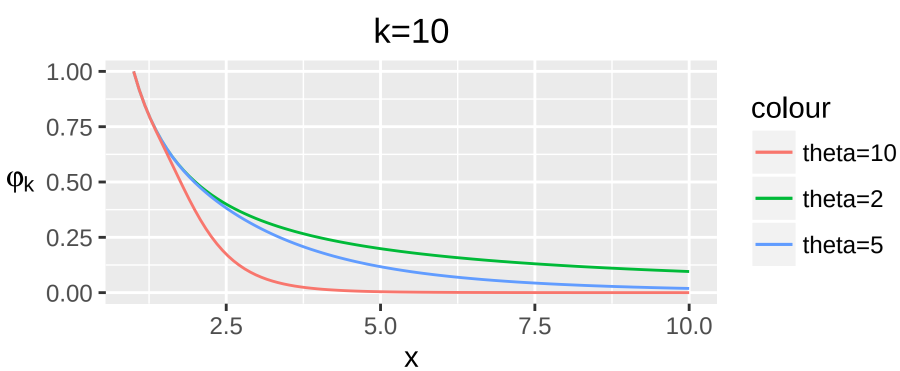
for $x > 1$. So, computing $C_{2|1}^{\leftarrow}$ amounts to computing the inverse of $\varphi_k(\cdot;\theta)$:

$$C_{2|1}^{\leftarrow}(\tau|u;\theta,k) = 1 - H_k\left(\varphi_k^{\leftarrow}(1-\tau;\theta(1-u));\theta\right), \tag{6}$$

$\tau \in (0,1)$. Here are some plots of φ_k:







In the copsupp package, igcond is φ_k, and igcondinv is $\varphi_k^{\leftarrow}$.

To compute igcondinv by solving $\varphi_k(x;\eta) = p$ for x, the Newton-Raphson algorithm is used to solve $g(x) = 0$ for x, where

$$g(x) = xp - \bar{F}_{k-1}(\eta \log x),$$

with derivative

$$g'(x) = p + \frac{\eta}{x} f_{k-1}(\eta \log x).$$

A starting value is used by noting that φ_k is the product of two invertible survival functions, and is therefore smaller than those two survival functions. The smaller of the two *roots* of these survival functions is therefore an upper bound for the root of φ_k. The starting point is taken to be immediately to the left of this upper bound.