IG and IGL Copula Families: Functions in copsupp

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This document provides details of the functions for the IG and IGL copula family, found in the copsupp package.

Section 1 describes the copula families and their functions in copsupp; Section 2 describes a useful way to compute the IG 2|1 quantile function; Section 3 describes computing the inverse of the IG generating function.

1 IG and IGL Copula Families

The IGL copula family can be defined as

$$C_{IGL}\left(u, v; k\right) = u\Psi_k\left(u^{-1}\Psi_k^{\leftarrow}\left(v\right)\right),$$

where $\Psi_k : [0, \infty) \to (0, 1]$ is a concave distribution function defined by

$$\Psi_k\left(y\right) = \begin{cases} y\frac{\Gamma(k)-\Gamma^*\left(k, y^{-1}\right)}{\Gamma(k-1)} + \frac{\Gamma^*\left(k-1, y^{-1}\right)}{\Gamma(k-1)}, & y > 0; \\ 0, & y = 0, \end{cases} \tag{1}$$

and Γ and Γ^* are the gamma and (upper) incomplete gamma functions, respectively. The parameter space is $k > 1$. The copula family is denoted iglcop in copsupp. The function Ψ_k is called the IGL copula *generating function*, and its name in copsupp is igl_gen.

The IG copula family can be defined as

$$C_{IG}\left(u, v; \theta, k\right) = u + v - 1 + \left(1 - u\right) H_k\left(H_k^{\leftarrow}\left(1 - v; \theta\right); \theta\left(1 - u\right)\right), \tag{2}$$

where the parameter space is $(\theta, k) \in [0, \infty) \times (1, \infty)$, and $H_k\left(\cdot; \theta\right) : [1, \infty) \to (0, 1]$ is defined by

$$H_k\left(y; \theta\right) = \begin{cases} \frac{1}{y}\Psi_k\left(\frac{1}{\theta \log y}\right), & y > 1; \\ 1, & y = 1. \end{cases} \tag{3}$$

Note that $H_k(\cdot;\theta)$ is strictly decreasing for all $\theta > 0$, $k > 1$, and $H_k^{\leftarrow}(\cdot;\theta)$ is the unique inverse function of $H_k(\cdot;\theta)$. The copula family is denoted `igcop` in `copsupp`. The function H_k is called the IG copula *generating function*, and its name in `copsupp` is `ig_gen`.

2 igcond and igcondinv functions

The computation of the IGL 2|1 quantile function is simple. But, computing the IGL 2|1 quantile function in a "direct" way first requires computing an inverse. This section describes a way to avoid this "nested inverse" with functions in `copsupp` called `igcond` and `igcondinv`.

The 2|1 distribution function of the IG copula for some $(\theta, k) \in [0,\infty) \times (1,\infty)$ and $u \in (0,1)$ is

$$\begin{aligned} C_{IGL,2|1}(v|u;\theta,k) &= 1 - \frac{\bar{F}_{k-1}\left(\theta(1-u)\log H_k^{\leftarrow}(1-v;\theta)\right)}{H_k^{\leftarrow}(1-v;\theta)} \\ &= 1 - \varphi_k\left(H_k^{\leftarrow}(1-v;\theta);\theta(1-u)\right), \end{aligned} \tag{4}$$

$v \in (0,1)$, where $\bar{F}_{k-1}(x) = \Gamma^*(k-1,x)/\Gamma(k-1)$ for $x \geq 0$ is the Gamma survival function with shape parameter $k-1$ and unit scale parameter, and

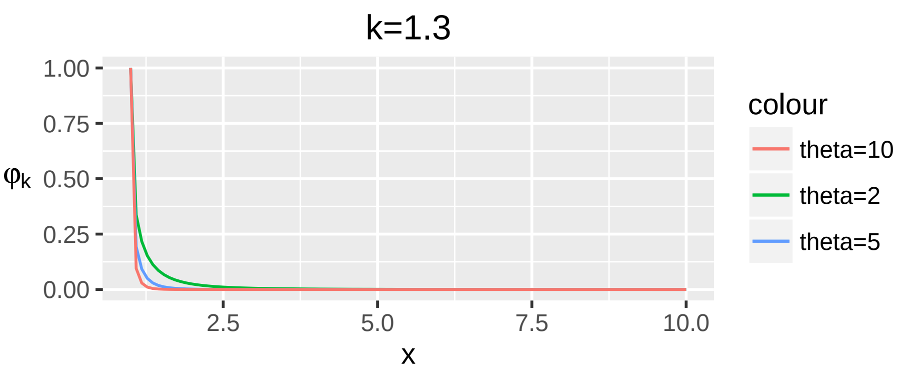
$$\varphi_k(x;\eta) = x^{-1}\bar{F}_{k-1}(\eta\log x) \tag{5}$$

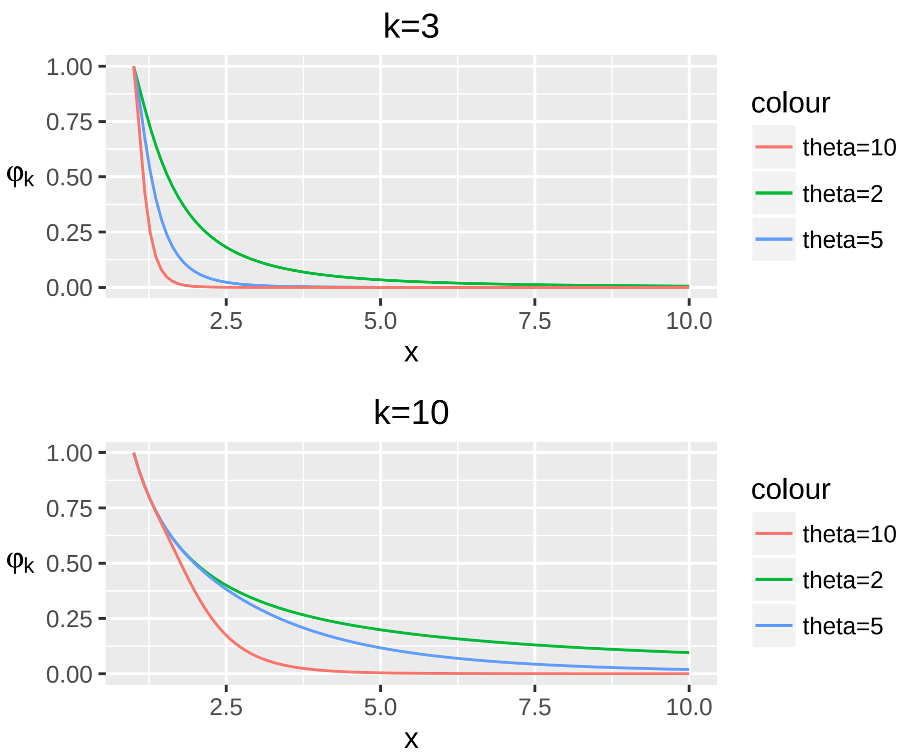
for $x > 1$. So, computing $C_{2|1}^{\leftarrow}$ amounts to computing the inverse of $\varphi_k(\cdot;\theta)$:

$$C_{IGL,2|1}^{\leftarrow}(\tau|u;\theta,k) = 1 - H_k\left(\varphi_k^{\leftarrow}(1-\tau;\theta(1-u));\theta\right), \tag{6}$$

$\tau \in (0,1)$. Without this φ_k function, one would first need to compute $H_k^{\leftarrow}(1-v;\theta)$ to compute $C_{IGL,2|1}^{\leftarrow}(v|u;\theta,k)$, and this may lead to inaccuracies due to error amplification.

In the `copsupp` package, `igcond` is φ_k, and `igcondinv` is $\varphi_k^{\leftarrow}$. Here are some plots of φ_k:





To compute igcondinv by solving $\varphi_k(x;\eta) = p$ for x, the Newton-Raphson algorithm is used to solve $g(x) = 0$ for x, where

$$g(x) = xp - \bar{F}_{k-1}(\eta \log x),$$

with derivative

$$g'(x) = p + \frac{\eta}{x} f_{k-1}(\eta \log x).$$

A starting value is used by noting that φ_k is the product of two invertible survival functions, and is therefore smaller than those two survival functions. The smaller of the two *roots* of these survival functions is therefore an upper bound for the root of φ_k. The starting point is taken to be immediately to the left of this upper bound.

3 ig_geninv function

Although computing $H_k^{\leftarrow}$ can be bypassed when computing the IG 2|1 quantile function, it is needed when computing other copula quantities, such as the copula distribution function. The inverse $H_k^{\leftarrow}$ is called ig_geninv in copsupp.

To compute $H_k^{\leftarrow}(p, \theta)$ for $p \in (0, 1)$ and some $\theta > 0$, Newton-Raphson can be used to solve $h(x) = 0$ for x, where

$$h(x) = xp - \Psi_k\left(\frac{1}{\theta \log x}\right).$$

This function has derivative

$$h'(x) = p + \frac{1}{\theta x (\log x)^2} \Psi_k' \left(\frac{1}{\theta \log x} \right).$$

To get a good starting value for the algorithm, note that the function H_k is a product of two survival functions, just like φ_k is (as described in Section 2). And like inverting φ_k, the inverse $H_k^{\leftarrow}(p;\theta)$ for some $\theta > 0$ has an upper bound that's the minimum of the roots of the individual survival functions:

$$\min \left\{ \frac{1}{p}, \exp\left(\frac{1}{\theta \Psi_k^{\leftarrow}(p)} \right) \right\},$$

where Ψ_k is defined in Equation (1). Computing the inverse $\Psi_k^{\leftarrow}$ should not be problematic in terms of accuracy of the final result, since this computation is only used to determine a starting value.