# IG Copula "helper function"

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This document describes in detail what the igcop_helper and igcop_helper_inv functions are in the copsupp package.

### IG Copula Family Definition

The IG copula family is defined as

$$C(u, v; \theta, k) = u + v - 1 + (1-u) H_k\left(H_k^{\leftarrow}(1-v; \theta); \theta(1-u)\right), \tag{1}$$

where the parameter space is $(\theta, k) \in [0, \infty) \times (1, \infty)$, and $H_k(\cdot; \theta) : [1, \infty) \to (0, 1]$ is defined by

$$H_k(y; \theta) = \begin{cases} \frac{1}{y} \Psi_k\left(\frac{1}{\theta \log y}\right), & y > 1; \\ 1, & y = 1, \end{cases} \tag{2}$$

where $\Psi_k : [0, \infty) \to (0, 1]$ is a concave distribution function defined by

$$\Psi_k(y) = \begin{cases} y \frac{\Gamma(k) - \Gamma^*\left(k, y^{-1}\right)}{\Gamma(k-1)} + \frac{\Gamma^*\left(k-1, y^{-1}\right)}{\Gamma(k-1)}, & y > 0; \\ 0, & y = 0, \end{cases} \tag{3}$$

where $\Gamma$ and $\Gamma^*$ are the gamma and (upper) incomplete gamma functions, respectively. Note that $H_k(\cdot; \theta)$ is strictly decreasing for all $\theta > 0$, $k > 1$, and $H_k^{\leftarrow}(\cdot; \theta)$ is the unique inverse function of $H_k(\cdot; \theta)$.

### 2|1 Distribution Function

The 2|1 distribution function of the IG copula for some $(\theta, k) \in [0, \infty) \times (1, \infty)$ and $u \in (0, 1)$ is

$$\begin{aligned} C_{2|1}(v|u; \theta, k) &= 1 - \frac{\bar{F}_{k-1}\left(\theta(1-u) \log H_k^{\leftarrow}(1-v; \theta)\right)}{H_k^{\leftarrow}(1-v; \theta)} \\ &= \varphi_{k-1}\left(H_k^{\leftarrow}(1-v; \theta); \theta(1-u)\right), \end{aligned} \tag{4}$$

$v \in (0,1)$, where $\bar{F}_{k-1}(x) = \Gamma^*(k-1,x)/\Gamma(k-1)$ for $x \geq 0$ is the Gamma survival function with shape parameter $k-1$ and unit scale parameter, and

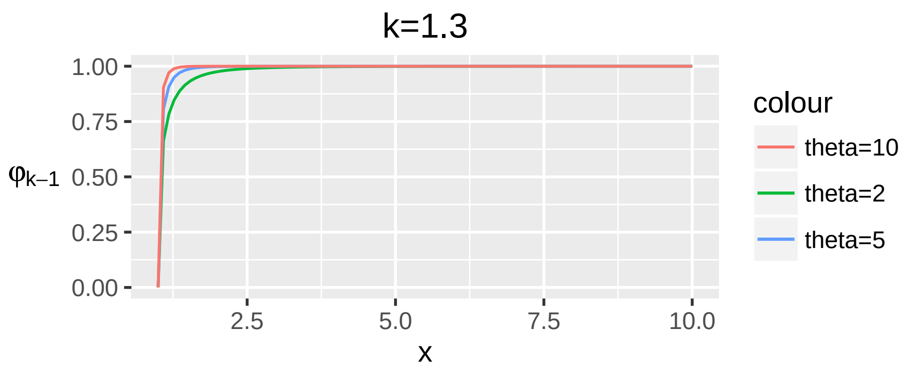
$$\varphi_{k-1}(x;\theta) = 1 - x^{-1}\bar{F}_{k-1}(\theta \log x) \tag{5}$$

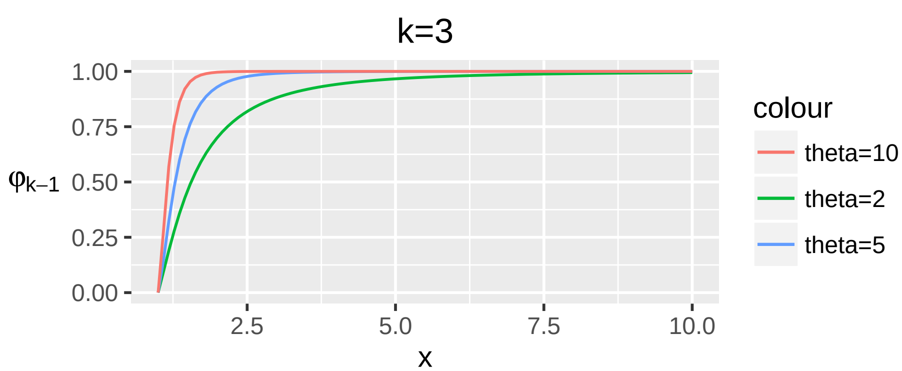
for $x > 1$. So, computing $C^{\leftarrow}_{2|1}$ amounts to computing the inverse of $\varphi_{k-1}(\cdot;\theta)$:

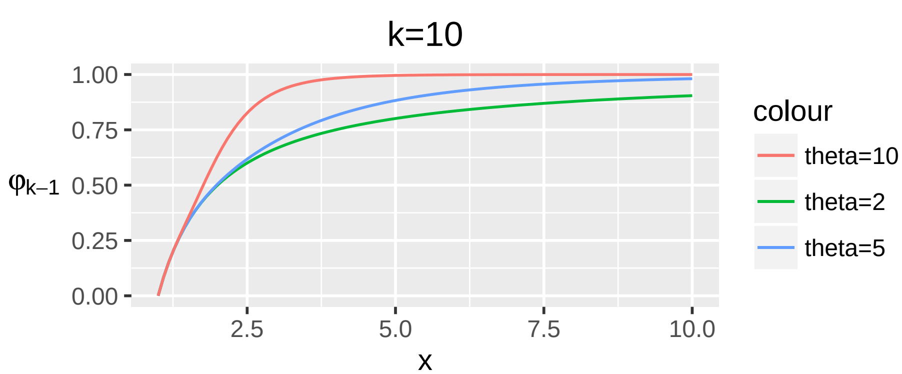
$$C^{\leftarrow}_{2|1}(\tau|u;\theta,k) = 1 - H_k\left(\varphi^{\leftarrow}_{k-1}(\tau;\theta(1-u));\theta\right), \tag{6}$$

$\tau \in (0,1)$.

However, it's not easy to compute the inverse of $\varphi_{k-1}$ – perhaps the function is too steep near 1:




Instead, try spreading out the region around 1 by taking the logarithm twice to convert the domain from $(1, \infty)$ to $\mathbb{R}$. This means looking at the function

$$\tilde{\varphi}_{k-1}(x;\theta) = \varphi_{k-1}\left(e^{e^x}, \theta\right) = 1 - e^{-e^x} \bar{F}_{k-1}(\theta \exp(x)) \tag{7}$$

for $x \in \mathbb{R}$, so that

$$C^{\leftarrow}_{2|1}(\tau|u;\theta,k) = C^{\leftarrow}_{2|1}(\tau|u;\theta,k) = 1 - H_k\left(\exp\exp\tilde{\varphi}^{\leftarrow}_{k-1}(\tau;\theta(1-u));\theta\right). \tag{8}$$

In the copsupp package, igcop_helper is $\tilde{\varphi}$, and igcop_helper_inv is $\tilde{\varphi}^{\leftarrow}$. Here are some plots of $\tilde{\varphi}$: