

# Lecture 04: Domains

---

Vincenzo Ciancia

April 04, 2025

## Section 1: Introduction to Semantic Domains

In programming language semantics, **semantic domains** are mathematical structures used to give meaning to syntactic constructs. They provide the foundation for defining the behavior of programs in a precise, mathematical way.

## Key Semantic Domains in Programming Languages

- **Syntactic Domains:** Represent the structure of programs (tokens, parse trees, syntax trees).
- **Semantic Domains:** Represent the meaning of programs (values, environments, states).

The relationship between these domains is at the heart of language semantics:

Syntax -> Semantics

Program -> Meaning

## Core Semantic Domains

In our mini-interpreter, we'll work with several fundamental semantic domains:

- **Expressible Values:** Values that can be produced by evaluating expressions
- **Denotable Values (DVal):** Values that can be bound to identifiers in the environment
- **Memorable Values (MVal):** Values that can be stored in memory/state
- **Identifiers:** Names of constants, variables, functions, modules, etc.
- **Locations:** Memory addresses of variables
- **Environment:** Maps identifiers to denotable values, representing the binding context
- **State:** Maps memory locations to memorable values, representing the program's memory

While these domains often overlap, they aren't necessarily identical. Understanding the differences is crucial for language design.

## Side Effects and Pure Functions

A fundamental distinction in programming language semantics is between **pure** computations and those that produce **side effects**.

## Pure Functions

A **pure function** is a computation that: 1. Always produces the same output for the same input 2. Has no observable effects beyond computing its result

In mathematical terms, a pure function is simply a mapping from inputs to outputs, like mathematical functions (e.g.,  $\sin(x)$ ,  $\log(x)$ ).

```
def add(x: int, y: int) -> int:  
    return x + y # Pure: same inputs always give same output
```

## Side Effects

A **side effect** is any observable change to the system state that occurs during computation, beyond returning a value. Common side effects include:

1. **Memory updates:** Modifying variables or data structures

```
x = 10  # Changes the program's state
```

Q: in this context, what is a function that does **not** return always the same value for the same inputs?

2. **Input/Output operations:**

```
print("Hello")  # Affects the external world (terminal)
```

### 3. File operations:

```
with open("data.txt", "w") as f:  
    f.write("data")  # Changes the file system
```

### 4. Network operations:

```
requests.get("https://example.com")  # Interacts with external systems
```



## Memory Updates as Side Effects

In our interpreter design, memory (state) updates are a primary form of side effect. When we update state:

```
def state_update(state: State, location: int, value: MVal) -> State:
    new_state = state.copy()
    new_state[location] = value
    return new_state
```

We're representing a change to the program's memory. In a real computer, this would modify memory cells directly. Our functional implementation returns a new state rather than modifying the existing one, but conceptually it represents the same side effect.

## Side Effects in Programming Languages

Languages differ in how they handle side effects:

- **Purely functional languages** (like Haskell) isolate side effects using type systems and monads
- **Imperative languages** (like C, Python) embrace side effects as their primary mechanism for computation
- **Hybrid languages** (like Scala, OCaml) support both styles

Understanding side effects is crucial for language design because they impact: - Program correctness (pure functions are easier to reason about) - Parallelization (side effects complicate parallel execution) - Optimization (pure functions allow more aggressive optimizations)

In our interpreter, we'll model side effects using explicit state passing, maintaining the mathematical clarity of our semantics while accurately representing the behavior of stateful programs.

## Section 2: Denotable vs. Memorizable Values

## Denotable Values (DVal)

Denotable values are those that can be bound to identifiers in an environment. In our implementation, they include:

```
type Num = int    # A type alias for integers
type DenOperator = Callable[[int, int], int]
type DVal = int | DenOperator  # Denotable values
```

DVal includes: - **Numbers**: Simple integer values - **Operators**: Functions that take two integers and return an integer

## Memorable Values (MVal)

Memorable values are those that can be stored in memory (the state). In our implementation:

```
type MVal = int # Memorable values
```

MVal only includes integers, not functions. This highlights an important distinction:

*Not everything that can be bound to a name can be stored in memory.*

This distinction is crucial for understanding: - Why some languages don't support first-class functions - Why some types require special treatment in memory management - How languages with different type systems handle values differently

### **Section 3: Environment and State as Functions**

In our treatment of semantic domains, we adopt a purely functional approach where environments are represented as functions, and states are represented as dataclasses containing store functions, rather than mutable data structures. This approach aligns with the mathematical view of semantic domains and provides a clean conceptual model for understanding program behavior.

## Functional Programming: A Brief Digression

Before diving into our function-based implementation of environments and state, it's worth taking a brief detour to discuss functional programming concepts, as they form the foundation of our approach.

Functional programming is a paradigm where computations are treated as evaluations of mathematical functions, emphasizing immutable data and avoiding side effects. Python, while not a pure functional language, supports many functional programming techniques.

## Functions as First-Class Citizens

In functional programming, functions are “first-class citizens” — they can be:

- Assigned to variables
- Passed as arguments to other functions
- Returned from functions
- Stored in data structures

For example:

```
# Function assigned to a variable
```

```
increment = lambda x: x + 1
```

```
# Function passed as an argument
```

```
def apply_twice(f, x):
```

```
    return f(f(x))
```

```
result = apply_twice(increment, 3) # Returns 5
```



## Higher-Order Functions: Map

A common pattern in functional programming is applying a function to each element in a collection. Python's `map` function does exactly this:

```
numbers = [1, 2, 3, 4, 5]

# Apply a function to each element
squared = list(map(lambda x: x**2, numbers)) # [1, 4, 9, 16, 25]

# Equivalent to a list comprehension
squared_alt = [x**2 for x in numbers] # [1, 4, 9, 16, 25]
```

## Function Composition

Functional programming emphasizes building complex behaviors by composing simpler functions:

```
def compose(f, g):  
    return lambda x: f(g(x))  
  
# Compose two functions  
negate_and_square = compose(lambda x: -x, lambda x: x**2)  
result = negate_and_square(5) # -25
```

## Pure Functions and Immutability

Pure functions always produce the same output for the same input and have no side effects. This property makes them predictable and easier to reason about:

```
# Pure function
```

```
def add(a, b):  
    return a + b
```

```
# Impure function (has side effects)
```

```
def add_and_print(a, b):  
    result = a + b  
    print(f"The result is {result}") # Side effect: printing  
    return result
```

## Relevance to Semantic Domains

These functional programming concepts directly inform our approach to implementing semantic domains:

1. We represent environments and stores as functions, not data structures
2. We use higher-order functions to create updated environments and stores
3. We maintain immutability through functional updates rather than mutations
4. We compose simple operations to build complex behaviors

With this foundation in mind, let's explore how we represent environments and states as functions.

## Environment as a Function

An environment is mathematically a function that maps identifiers to denotable values:

```
type Environment = Callable[[str], DVal]
```

This means an environment is a function that:

- Takes an identifier (string) as input
- Returns a denotable value (DVal)
- Raises an error if the identifier is not defined

While many practical implementations use dictionaries or hash tables for efficiency, conceptually an environment is simply a function:

```
Environment: Identifier → DVal
```

## State as a Dataclass

Unlike environment, which is purely a function, a state encapsulates both a store function and allocation information:

```
@dataclass
class State:
    store: Callable[[Location], MVal] # The store function
    next_loc: int                     # Next available location
```

This represents a state as: - A dataclass containing a store function and next location counter - The store function takes a location as input and returns the value at that location - The next\_loc field tracks the next available memory location for allocation

Conceptually, the store component maintains the mapping:

Store: Location  $\rightarrow$  MVal

While the next\_loc component tracks allocation state.

## Section 4: Functional Updates

In a purely functional approach, we don't mutate existing environments or states. Instead, we create new functions or dataclasses that encapsulate the updated behavior.

## Environment Updates

Instead of modifying a dictionary, we define a new function that returns the new value for the updated identifier and delegates to the original environment for all other identifiers:

```
def env_extend(env: Environment, name: str, value: DVal) -> Environment:

    """Create new environment with an added binding"""

    def new_env(n: str) -> DVal:

        if n == name:

            return value

        return env(n)

    return new_env
```

This function returns a new environment that: - Returns `value` when asked for `name` - Delegates to the original environment for all other identifiers

This approach: - Preserves referential transparency - Enables easy implementation of lexical scoping - Facilitates reasoning about program behavior - Models the mathematical concept of function extension



## State Updates

Similarly, state updates create new State objects with updated store functions:

```
def state_update(state: State, location: Location, value: MVal) -> State:
    """Create new state with an updated value at given location"""

    def new_store(loc: Location) -> MVal:
        if loc == location:
            return value

        return state.store(loc)

    return State(store=new_store, next_loc=state.next_loc)
```

This function creates a new State object that:

- Contains a new store function that returns the new value when asked for the specified location
- Delegates to the original state's store function for all other locations
- Preserves the next\_loc value from the original state

## Empty Environment and State

The primitives for creating empty environments and states define initial values:

```
def empty_environment() -> Environment:
    """Create an empty environment function"""
    def env(name: str) -> DVal:
        raise ValueError(f"Undefined identifier: {name}")
    return env

def empty_memory() -> State:
    """Create an empty memory state"""
    def store(location: Location) -> MVal:
        raise ValueError(f"Undefined memory location: {location}")
    return State(store=store, next_loc=0)
```

Note that `empty_memory` returns a `State` dataclass initialized with an empty store function and `next_loc` set to 0.

## Memory Allocation

In a complete interpreter, we use the State dataclass to implement memory allocation elegantly:

```
def allocate(state: State, value: MVal) -> tuple[State, Location]:  
  
    """Allocate a new memory location and store a value there.  
    Returns the updated state and the new location."""  
  
    location = state.next_loc  
  
    new_state = state_update(state, location, value)  
  
    # Return state with incremented next_loc and the allocated location  
  
    return State(store=new_state.store, next_loc=location + 1), location
```

This function: 1. Gets the next available location from the state 2. Updates the store function to map this location to the provided value 3. Returns a new state with the incremented next\_loc and the allocated location

By bundling the store function with the next\_loc counter in our State dataclass, we maintain a purely functional approach while elegantly handling the allocation challenge. This design demonstrates how functional programming can manage state without side effects by making state changes explicit in the return values of functions.

## Initial Environment Setup

In our functional implementation, the initial environment is built by starting with an empty environment and extending it with each operator:

```
def create_initial_env() -> Environment:

    """Create an environment populated with standard operators"""

    env = empty_environment()

    env = env_extend(env, "+", add)

    env = env_extend(env, "-", subtract)

    env = env_extend(env, "*", multiply)

    env = env_extend(env, "/", divide)

    env = env_extend(env, "%", modulo)

    return env
```

This builds up the environment incrementally, adding each binding through functional extension.

## **Section 5: Environment-Based Interpretation**

Traditional interpreters often use pattern matching on operators directly in the evaluation function. An environment-based approach takes a more abstract view, treating operators as first-class values in the environment.

## Traditional Approach (from Chapter 3)

```
def evaluate(ast: Expression) -> int:

    match ast:

        case Number(value):

            return value

        case BinaryExpression(op, left, right):

            left_value = evaluate(left)

            right_value = evaluate(right)

            match op:

                case "+":

                    return left_value + right_value

                case "-":

                    return left_value - right_value

            # ... other operations
```

# Environment-Based Approach

```
def evaluate(ast: Expression, env: Environment) -> MVal:

  match ast:

    case Number(value):

      return value

    case BinaryExpression(op, left, right):

      try:

        # Get operator from environment

        operator = env_lookup(env, op)

        # Ensure it's a DenOperator

        if not isinstance(operator, Callable):

          raise ValueError(f"{op} is not a function")

        # Evaluate operands and apply operator

        left_value = evaluate(left, env)

        right_value = evaluate(right, env)
```

## Benefits of the Environment-Based Approach

- **Extensibility:** New operators can be added to the environment without modifying the evaluator
- **First-class operations:** Operators are values that can be passed, returned, and manipulated
- **Consistent treatment:** Operators and other identifiers are handled uniformly
- **Semantic clarity:** The environment explicitly represents the mapping from names to meanings



## Section 6: Implementing an Environment-Based Interpreter

Our `domains.py` file implements a complete environment-based interpreter:

1. **Define semantic domains:** Denotable and memorizable values
2. **Implement operators:** Define functions for arithmetic operations
3. **Create the environment:** Populate with standard operators
4. **Evaluate expressions:** Using the environment to look up operators
5. **REPL:** Interactive environment for testing the interpreter

## **Section 7: Extending the Interpreter**

This approach makes it easy to extend the language with new features:

## Adding New Operators

To add a new operator, simply define its function and add it to the environment:

```
def power(x: int, y: int) -> int:
    return x ** y

# Extend environment
env = env_extend(create_initial_env(), "**", power)
```

```
### Adding Variables
```

To support variables, extend the AST **with** a variable node **and** update the evaluator:

```
```python
```

```
@dataclass
```

```
class Variable:
```

```
    name: str
```

```
# Update Expression type
```

```
type Expression = Number | BinaryExpression | Variable
```

# Update evaluate function

```
def evaluate(ast: Expression, env: Environment) -> MVal:

  match ast:

    case Variable(name):

      value = env_lookup(env, name)

      # Additional check might be needed if variables can only be Numbers

      if not isinstance(value, int):

        raise ValueError(f"{name} is not a number")

      return value

    # ... existing cases
```

## Additional Resources

- Denotational Semantics (Wikipedia)
- Programming Language Semantics (Stanford)
- Functional Programming and Lambda Calculus
- Environment and Store in Programming Languages

## Section 8: Primitives for Environment and Memory

In the formal semantics of programming languages, we work with primitives that define how environments and memory operate. These primitives capture the essential operations needed to model variable bindings and memory allocation.

## Locations as a Semantic Domain

Before discussing memory operations, we must recognize **Locations** as a fundamental semantic domain:

```
type Location = int  # In our implementation, locations are integers
```

Locations (sometimes called addresses) are abstract entities that serve as references to memory cells. They are a semantic domain distinct from integers used in calculation, even though we might represent them as integers in an implementation. In a language's formal semantics, locations are opaque values that only make sense in the context of memory operations.



## Memory (State) Primitives

Memory, also called the store or state, is represented by the State dataclass which contains both a store function mapping locations to memorizable values and a next\_loc field. The core operations include:

1. **empty\_memory**: Creates an initial, empty memory state

```
def empty_memory() -> State:
    """Create an empty memory state"""
    def store(location: Location) -> MVal:
        raise ValueError(f"Undefined memory location: {location}")
    return State(store=store, next_loc=0)
```

2. **update**: Modifies the memory at a specific location

```
def state_update(state: State, location: Location, value: MVal) -> State:
    """Create new state with an updated value at given location"""
    def new_store(loc: Location) -> MVal:
        if loc == location:
            return value
        return state.store(loc)
```

Memory in programming languages has a maximum size, determined by hardware or system configuration. When a program needs more memory than available:

- In simple interpreters: An “out of memory” error occurs
- In modern operating systems: Memory expansion mechanisms activate, such as:
  - Virtual memory paging to disk
  - Heap expansion
  - Garbage collection (freeing unused memory)

# Environment Primitives

The environment maps identifiers to denotable values. Its essential operations include:

1. **empty\_environment**: Creates an initial, empty environment

```
def empty_environment() -> Environment:
    """Create an empty environment function"""

    def env(name: str) -> DVal:
        raise ValueError(f"Undefined identifier: {name}")

    return env
```

2. **bind (or extend)**: Adds a new binding to an environment

```
def env_extend(env: Environment, name: str, value: DVal) -> Environment:
    """Create new environment with an added binding"""

    def new_env(n: str) -> DVal:
        if n == name:
            return value

        return env(n)

    return new_env
```

## Memory and Environment in Language Semantics

The interaction between environments and memory is central to understanding language features:

- **Variables:** Bind identifiers to locations (in the environment) which then hold values (in memory)
- **Assignment:** Changes memory but not the environment (the variable still refers to the same location)
- **Scoping:** Creates nested environments with different bindings for the same identifier
- **Parameter passing:** Creates bindings between formal parameters and actual arguments

This conceptual separation allows language designers to reason clearly about the effects of operations and ensure language features interact correctly.

## **Section 9: Practical Implementation and Testing**

Our environment-based interpreter includes practical components for execution and testing, demonstrating how theoretical concepts translate to code.

## Parsing and AST Construction

In our implementation, we use the Lark parser to convert text expressions into parse trees:

```
def parse_ast(expression: str) -> Expression:
    """Parse a string expression into an AST"""
    parse_tree = parser.parse(expression)
    return transform_parse_tree(parse_tree)
```

The `transform_parse_tree` function then converts these parse trees into our AST representation, ready for evaluation.

# Interactive REPL

A Read-Eval-Print Loop allows interactive testing of the interpreter:

```
def REPL():  
  
    """Read-Evaluate-Print Loop with environment"""  
  
    env = create_initial_env()  
  
    print("Mini-interpreter with environment (type 'exit' to quit)")  
    print("Available operators: +, -, *, /, %")  
  
    while True:  
        expression = input("Enter an expression (exit to quit): ")  
  
        if expression == "exit":  
            break  
  
        try:  
            ast = parse_ast(expression)  
            result = evaluate(ast, env)  
            print(result)
```

## Automated Tests

Testing ensures the interpreter behaves as expected across various expressions:

```
def run_tests():  
    """Run test expressions to verify the parser and evaluator"""  
  
    test_expressions = [  
        "1+2",  
        "3*4",  
        "5-3",  
        "10/2",  
        "10%3",  
        "(1+2)*3",  
        "1+(2*3)",  
        "10/(2+3)",  
        "10%(2+3)",  
    ]  
  
    env = create_initial_env()
```



## **Section 10: Conclusion and Next Steps**

## Summary of Key Concepts

In this chapter, we've explored:

1. **Semantic domains** as mathematical structures that give meaning to programs
2. **Environment-based interpretation** as a flexible approach to language implementation
3. **Functional representations** of environments and state
4. **Primitive operations** for manipulating environments and memory
5. **The distinction** between denotable and memorizable values

These concepts form the foundation for understanding more complex language features in subsequent chapters.

## Looking Forward

The environment-based approach introduced here will be extended in future chapters to support:

1. **Variables and assignment:** Using environments to bind identifiers to memory locations
2. **Scoping mechanisms:** Creating nested environments for block-structured code
3. **Functions and closures:** Capturing environments for later execution
4. **Typing systems:** Adding constraints on what values expressions can produce

By building on these semantic foundations, we can construct a rich understanding of programming language design and implementation.

## Exercises for the Reader

1. Extend the interpreter to support variables using the Variable AST node described in Section 7
2. Add support for multi-character operators (e.g., “\*\*” for exponentiation)
3. Implement a memory system with the primitives described in Section 8
4. Add support for conditional expressions (if-then-else)
5. Implement a simple block structure with local variables

These exercises will deepen your understanding of language semantics and interpreter design while preparing you for more advanced topics in the next chapters.