

Jamming transition in a cellular automaton model for traffic flow

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Abstract

The cellular automaton model for traffic flow exhibits a jamming transition from a free-flow phase to a congested phase. In the deterministic case this transition corresponds to a critical point with diverging correlation length. In the presence of noise, however, no consistent picture has emerged up to now. We present data from numerical simulations which suggest the absence of critical behavior. The transition of the deterministic case is smeared out and one only observes the remnants of the critical point.

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I. INTRODUCTION

The cellular automaton (CA) approach to traffic flow theory [1] has attracted much interest in recent years (see e.g. [2]). Compared to the earlier attempts in modeling traffic flow (see e.g. [2–5] and references therein) CA models can be used very efficiently for computer simulations. This makes it possible to perform real time simulations even for very large networks [6,7]. Due to the relevance of these models for applications it is important to understand the underlying physics thoroughly.

The cellular automaton model of Nagel and Schreckenberg (NaSch model) [1] provides a simple but quite realistic description of traffic flow. The road is divided into L cells so that the model is discrete in space and time. Each cell can either be empty or occupied by one of N cars $j = 1, \dots, N$ with velocities $v_j = 0, \dots, v_{max}$. v_{max} is assumed to be the same for all the cars. The update is divided into four steps which are applied *in parallel* to all cars. The first step (R1) is an acceleration step. The velocities v_j of each car j not already propagating with the maximum velocity v_{max} are increased by one. The second step (R2) is designed to avoid accidents. If a car has d_j empty cells in front of it and its velocity (after step (R1)) exceeds d_j the velocity is reduced to d_j . Up to now the dynamics is completely deterministic. Noise is introduced via the randomisation step (R3). Here the velocities of moving cars ($v_j \geq 1$) are decreased by one with probability p . The steps (R1)–(R3) give the new velocity v_j for each car j . In the last step of the update procedure the positions x_j of the cars are shifted by v_j cells (R4) to $x_j + v_j$. We consider here only periodic boundary conditions. Thus the model contains three parameters: the maximum velocity v_{max} , the probability for braking p , and the average density $\rho = N/L$.

A basic feature of traffic models is the relation between density ρ and the average flow $J = \rho \bar{v}$ (fundamental diagram) where $\bar{v} = \frac{1}{N} \sum_{j=1}^N v_j$ is the average velocity. Fig. 1 shows the fundamental diagram for different values of p . One observes two effects as the noise p is increased, namely a decrease of the flow and a shift of the maximum towards smaller densities. In the low-density limit $\rho \ll 1$ one always finds free flow behavior with $J(\rho) \simeq$

$(v_{max} - p)\rho$ whereas for high densities $1 - \rho \ll 1$ one has $J(\rho) \simeq (1 - p)(1 - \rho)$.

Although the model cannot be solved exactly for arbitrary parameter values, two limits of the model can be treated analytically. Firstly the case $v_{max} = 1$ which is solved exactly with improved mean field methods [8,9]. Here the fundamental diagram is symmetric due to particle-hole symmetry. Considering larger maximum velocities $v_{max} \geq 2$ one can obtain solutions only in the deterministic limit $p = 0$ where the flow is given by

$$J(\rho) = \min(\rho v_{max}, 1 - \rho). \quad (1)$$

In the free flow regime where $J(\rho) = \rho v_{max}$ all cars propagate with maximum velocity whereas in the jammed phase the flow is limited by the number of empty cells. These limits will be discussed further in the later sections.

Such a transition from a free flow regime at low densities to a congested flow regime where start and stop waves dominate the dynamics are quite typically for traffic flow. Several attempts have been made to explain the nature of this transition in the CA model [10]-[15]. It seems, however, that no consensus has been reached yet. Here we present results of an extensive numerical investigation of the parameter dependence of the transition in the NaSch model. We examine several quantities which give information about the location ρ_c and the nature of the phase transition.

The outline of the paper is as follows: In the next section we discuss the relaxation into the steady state. Section 3 is devoted to measurements of an order-parameter. Section 4 shows the behavior of the spatial correlation function. Our results are discussed in the final section.

II. RELAXATION

A characteristic feature of a second order phase transition is the divergence of the relaxation time at the transition point. For technical reasons Csányi and Kertész [13] made no direct measurements of the relaxation time, but used the following approach: Starting from

a random configuration of cars with velocity $v_j = 0$ the average velocity $\bar{v}(t)$ is measured at each time step t . For $t \rightarrow \infty$ the system reaches a stationary state with average velocity $\langle \bar{v}_\infty \rangle$. The relaxation time is characterised by the parameter [13]

$$\tau = \int_0^\infty \left[\min\{v^*(t), \langle \bar{v}_\infty \rangle\} - \langle \bar{v}(t) \rangle \right] dt . \quad (2)$$

$v^*(t)$ denotes the average velocity in the acceleration phase $t \rightarrow 0$ for low vehicle density $\rho \rightarrow 0$. Because the vehicles do not interact with each other, $v^*(t) = (1 - p)t$ holds in this regime. Thus the relaxation time is obtained by summing up the deviations of the average velocity $\langle \bar{v}(t) \rangle$ from the values of a system with one single vehicle which can move without interactions with other cars ($\rho \rightarrow 0$). One finds a maximum of the relaxation parameter near, but *below*, the density of maximum flow for $p = 0.25$ and $v_{max} = 5$ (see [13]).

Within this investigation we extended the set of braking parameters ($p = 0 \dots 0.75$) in order to study the parameter dependence of the maximum of τ . We took into account system sizes up to $L = 30000$ where the position ρ_c of the maximum of τ becomes size independent. The transition density is given by ρ_c of the largest system we took into account.

The results for $p = 0$ and $p = 0.25$ are shown in Figs. 2 and 3. A comparison of ρ_c with the density of maximum flow $\rho(q_{max})$ shows smaller values of the transition densities for all values of p taken into account. Taking the magnitude of τ as a characteristic value for the relaxation time one can estimate the dynamical exponent. Furthermore the scaling behavior of the width $\sigma(L)$ and height $\tau_m(L)$ of the peak has been taken into account,

$$\tau_m(L) \propto L^z, \quad \sigma(L) \propto L^{-1/\nu}. \quad (3)$$

We find $z = 0.28$ and $\nu = 5.7$ for $p = 0.5$ and $z = 0.36$ and $\nu = 6.8$ for $p = 0.25$. Note that the peak is not symmetric so that it is difficult to determine its width. Comparing our data with [13] two facts have to be mentioned. First, our results for $p = 0.25$ are completely different from the exponents obtained in [13]. The second remarkable point is the occurrence of negative values for the relaxation times (see Fig. 3). This effect is not shown in [13]. One can think that it emerges from inaccurate measurements or finite size effects, but the

negative values are a consequence of the definition (2). If we look at the time evolution of $\langle \bar{v}(t) \rangle / \langle \bar{v}_\infty \rangle$ we see the reason for this unpleasant feature (Fig. 4): For $\rho > \rho_c$ the system gets temporarily into states which have a higher average velocity than the stationary state such that $\langle \bar{v}(t) \rangle > \langle \bar{v}_\infty \rangle$ holds within this time interval. This over-reaction is a consequence of the relaxation mechanism which can be divided into two phases for $p > 0$. Within the first few time steps small clusters which occur in the initial configuration vanish. The second phase is characterized by the growth of surviving jams. More and more cars get trapped into large jams and therefore the average flow decreases to its stationary value. This decrease causes negative values of τ at large densities.

Finally one should note that (2) can only be interpreted as a relaxation time for a purely exponential decay, $\langle \bar{v}_\infty \rangle - \langle \bar{v}(t) \rangle \propto e^{-t/\tau}$. Figure 4 shows, however, that this is not the case for $\rho > \rho_c$, where one even finds a non-monotonic relaxational behavior. In order to get a clear picture of the nature of the transition one should therefore examine various quantities.

III. ORDER PARAMETER

For a proper description of the transition one should introduce an order parameter which has a qualitative different behavior within the two phases. A first candidate would be the analogue of the magnetisation in the Ising model, i.e. the number of cars. However, since this quantity is conserved in the NaSch model it can not serve as an order parameter. Therefore the density of nearest neighbour pairs

$$m = \frac{1}{L} \sum_{i=1}^L n_i n_{i+1}, \quad (4)$$

with $n_i = 0$ for an empty cell and $n_i = 1$ for a cell occupied by a car (irrespective of its velocity), is the simplest choice of a local quantity with a nontrivial behavior at the transition density. Taking into account the braking rule (R2) m gives the density of those cars with velocity 0 which had to brake due to the next car ahead. Although the order parameter introduced in [10] is defined as a time average it shows a quite similar behavior.

For large time periods it measures the densities of cars with velocity 0 [16]. Vilar et al. [10] only investigated the deterministic case $p = 0$ for which their order parameter is identical to ours, but also in the presence of noise the values differ only slightly. First we will discuss the behavior of the order parameter in the case $p = 0$. Below the transition density,

$$\rho_c = \frac{1}{v_{max} + 1}, \quad (5)$$

the order parameter vanishes because every car has at least v_{max} empty sites in front and propagates with v_{max} . Within the jammed phase the flow is limited by the number of empty cells and also stopped cars occur. In the presence of noise the behavior of the order parameter qualitatively changes in the vicinity of the transition density. Within this region m decays exponentially. Assuming m is a possible choice for the order parameter this implies the absence of criticality in the nondeterministic case. Figure 6 shows that the order parameter does not exhibit a sharp transition. Although it becomes rather small for small densities it is still different from zero. The situation is quite similar to the behavior of the order parameter in finite systems [17]. The transition is smeared out by the noise and the transition density is shifted towards smaller values. In order to have a suitable criterion for the determination of the transition density, we analysed the scaling behavior of the order parameter near the transition density ρ_c . Fig. 7 shows that one gets a quite reasonable data collapse using the scaling form

$$\overline{m}(\rho) = \Pi(p)m(\rho + \Delta\rho_c). \quad (6)$$

$\Pi(p)$ is a scaling factor and $\Delta\rho_c$ is the shift of the transition density compared to the deterministic value (5). The values of the transition densities are shown in Fig. 8. This results are in good agreement with the results obtained from the measurement of τ .

IV. SPATIAL CORRELATIONS

A striking feature of second order phase transitions is the occurrence of a diverging length scale at criticality and a corresponding algebraic decay of the correlation function. Using

lattice gas variables the density-density correlation function is defined by

$$G(r) = \frac{1}{L} \sum_{i=1}^L n_i n_{i+r} - \rho^2. \quad (7)$$

Again it is very instructive to consider the deterministic case ($p = 0$) first. In the vicinity of the transition density one observes a decay of the amplitude of $G(r)$ for larger values of the distance between the sites. Precisely at ρ_c , however, the correlation function is given by

$$G(r) = \begin{cases} \rho_c - \rho_c^2 & \text{for } r \equiv 0 \pmod{v_{max} + 1} \\ -\rho_c^2 & \text{else} \end{cases} \quad (8)$$

because there are exactly v_{max} empty sites in front of each car. Considering small, but finite, values of p the correlation function has the same structure as in the deterministic case, but the amplitude decays exponentially for all values of ρ . The decay of the amplitude determines the correlation length for a given pair of (p, ρ) , which is finite for all densities in the presence of noise. The maximal value of the correlation length ξ_{max} determines the transition density for small values of p . Numerically we find

$$\xi_{max} \sim p^{-\frac{1}{2}}. \quad (9)$$

In fact, this picture can be confirmed analytically for $v_{max} = 1$ [18]. Using the results of [8] one obtains $\xi_{max}^{-1} = \ln\left(\frac{1-p}{1-p-\sqrt{p}}\right)$ for the correlation length ξ_{max} at $\rho = 1/2$. Therefore $\xi_{max} \propto 1/\sqrt{p}$ for small p . This exponent seems to be independent of v_{max} although the particle-hole symmetry is broken for $v_{max} > 1$. If one considers larger values of p the correlation length gives not the relevant length scale which is then determined by the size distribution of jams. A numerical analysis of this limit is quite difficult and has to be referred to future work.

V. DISCUSSION AND SUMMARY

Our results suggest a consistent picture of the jamming transition in the NaSch CA. Measurements of the order parameter and the correlation function show that critical behavior only occurs in the deterministic limit where the transition density is given by $\rho_c = (v_{max} +$

$1)^{-1}$ (see also [10,19]). The presence of any noise destroys long-range correlations. The behavior is analogous to a second order phase transition in finite systems [17]. We have, however, checked carefully that our results are not affected by finite-size effects and are solely due to the presence of noise.

Analogous behavior is also found in the Ising chain in a transverse field [20]. The transverse field Γ is the control parameter and corresponds to the density ρ in the NaSch model whereas the temperature T corresponds to the noise parameter p . This correspondance can be used to predict scaling laws. These predictions are currently under investigation and results will be published elsewhere.

We found qualitatively the same behavior of the relaxation parameter as shown in [13], but some important new features have been observed. An important result is the occurrence of negative values of τ which is a consequence of the relaxation mechanism beyond the transition density: Within the first few time steps small jams which are present in the initial condition die out. The second phase is dominated by the formation of large jams. Thus at a certain time intervall the average flow is systematically larger than the stationary value, which causes negative contributions at that time. Consequently one has to question whether τ gives meaningful results concerning the relaxation time or not.

The order parameter does not vanish exactly, but the transition density could be determined from the scaling behavior. This suggests that the system is not critical in a strict sense. Measurements of the density-density correlation function and the correlation length confirm this picture. We find a finite correlation length in the presence of noise ($p > 0$). The maximum correlation length diverges in the deterministic limit like $\frac{1}{\sqrt{p}}$ for all values of v_{max} we investigated ($v_{max} = 2, 3, 5$). For the case $v_{max} = 1$ this result can also be confirmed by analytical calculations.

Our conclusions have to be compared with those of other investigations where signals of a second order transition also in nondeterministic cases have been found. From our point of view these results are either a consequence of a special limit considered or the methods chosen. Nagel and Paczuski [12] showed the existence of self-organized critical behavior for

the outflow region of a large jam in the cruise-control limit. They found a scale-invariant size distribution of jams from measurements far downstream of the megajam. In this region most of the cars propagate without any fluctuations such that this limit is also an example for scale-invariance in deterministic flow. Very recently an investigation of the probabilistic version of the NaSch model has been performed [15] and it has been argued, that at the jamming transition critical behavior occurs also for the nondeterministic cases. However, the order parameter introduced in [15] does also not vanish exactly below the transition density. All the data presented in [15] are consistent with our interpretation of the nature of the transition. In contrast to the view of [15] we expect true phase separation only in the limit $p \rightarrow 1$.

Another indication for the absence of critical behavior is the well-established fact (see e.g. [13] and Fig. 8) that the density of maximum flow $\rho(q_{max})$ and the transition density ρ_c are different. It would be rather surprising if the system exhibits a genuine second order phase transition with diverging correlation length. Correlations obviously favor states with higher flow (see e.g. Figs. 9 and 10, which show that occupancies of cells in front of a car are suppressed which is the generalization of the particle-hole attraction observed in [8] for $v_{max} = 1$). Therefore one should expect that the state with the strongest correlations is also the state with the highest flow, as in the deterministic case.

In conclusion, we found the absence of criticality in the NaSch model in the presence of noise. For finite p the second order transition of the deterministic case is smeared out, similar to the situation of a second order transition in a finite system. Here this effect is caused by the presence of noise, $p > 0$. For small values of p one finds an ordering transition close to $\rho_c = 1/(v_{max} + 1)$. Larger values of the noise favour the formation of jams and a tendency to phase separation occurs (see also [14,15,21]). We therefore are currently investigating the limit $p \rightarrow 1$ more carefully. The results will be presented in a future publication.

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FIGURES

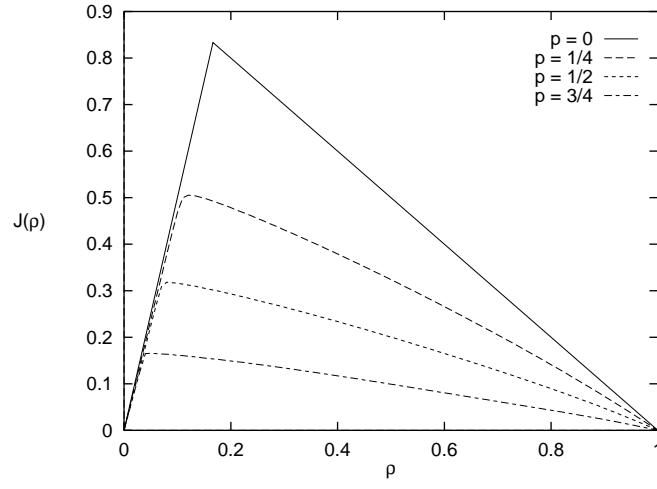


FIG. 1. Fundamental diagram for different values of p ($v_{\max} = 5$).

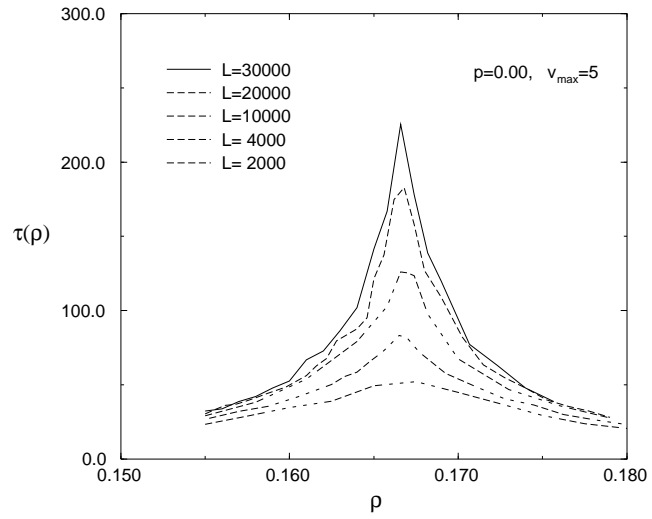


FIG. 2. Relaxation parameter near the transition density for different system sizes ($v_{\max} = 5, p = 0$).

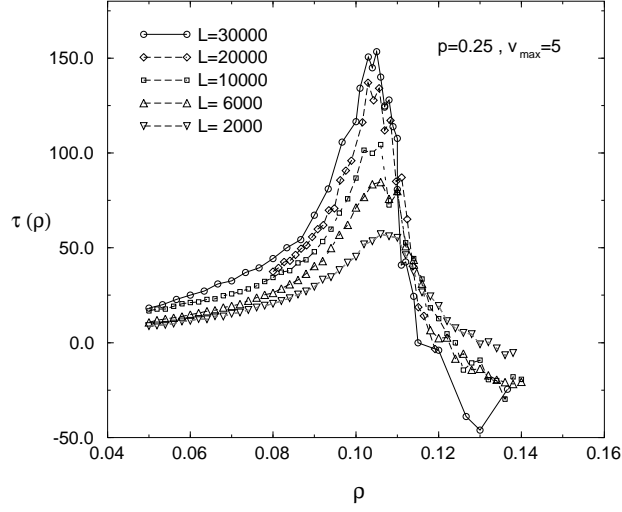


FIG. 3. Relaxation parameter near the transition density for a higher value of the braking probability ($v_{max} = 5, p = 0.25$).

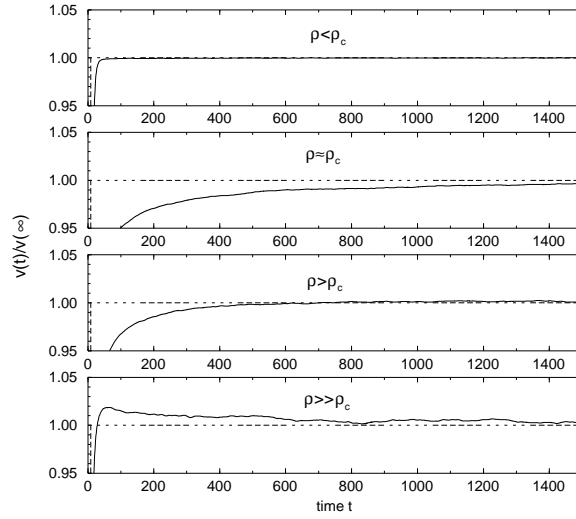


FIG. 4. Time dependence of the average velocity. After a few time steps the average velocity reaches its absolut maximum.

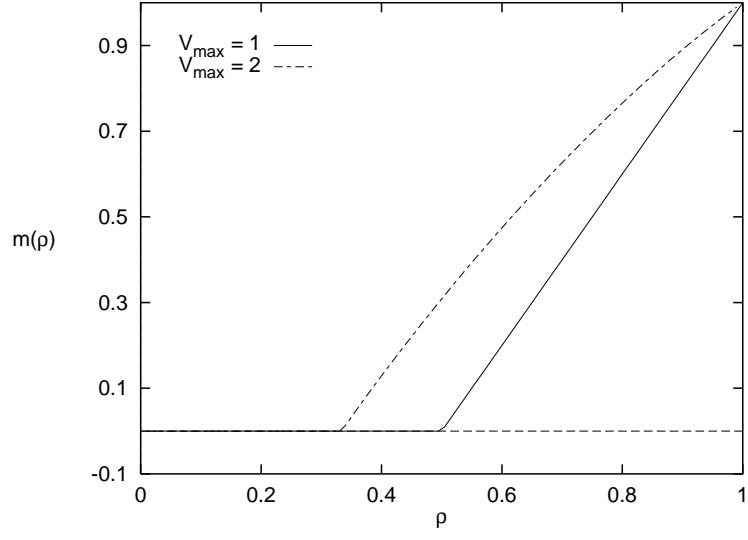


FIG. 5. Order parameter for the deterministic model ($v_{max} = 1, 2$). Below the transition density m vanishes exactly.

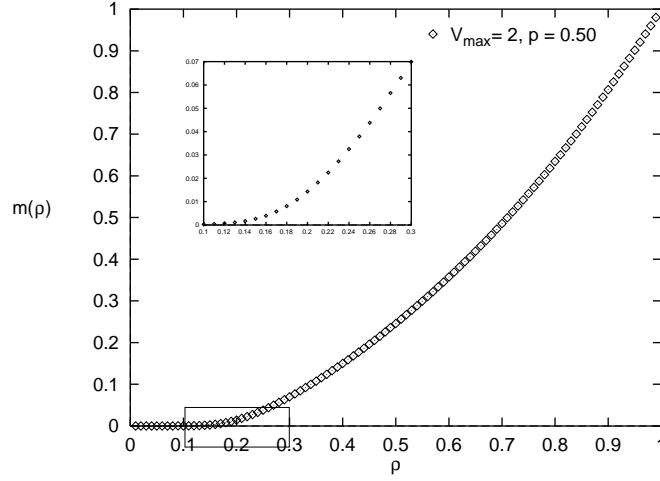


FIG. 6. Behavior of the order parameter for a finite braking probability. It does not vanish exactly for $\rho < \rho_c$ but converges smoothly to zero even for small values of the braking probability p .

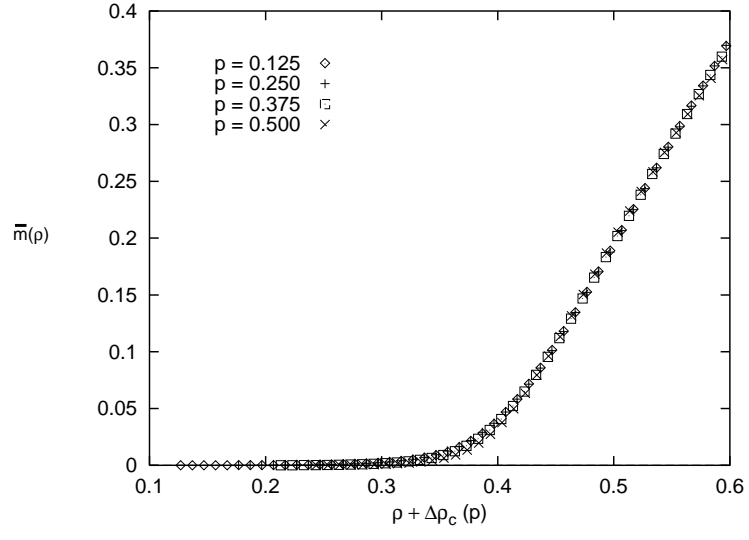


FIG. 7. Scaling-plot of the order parameter. In the vicinity of the transition density one gets a reasonable data collapse. The density shift determines the transition density for a given p .

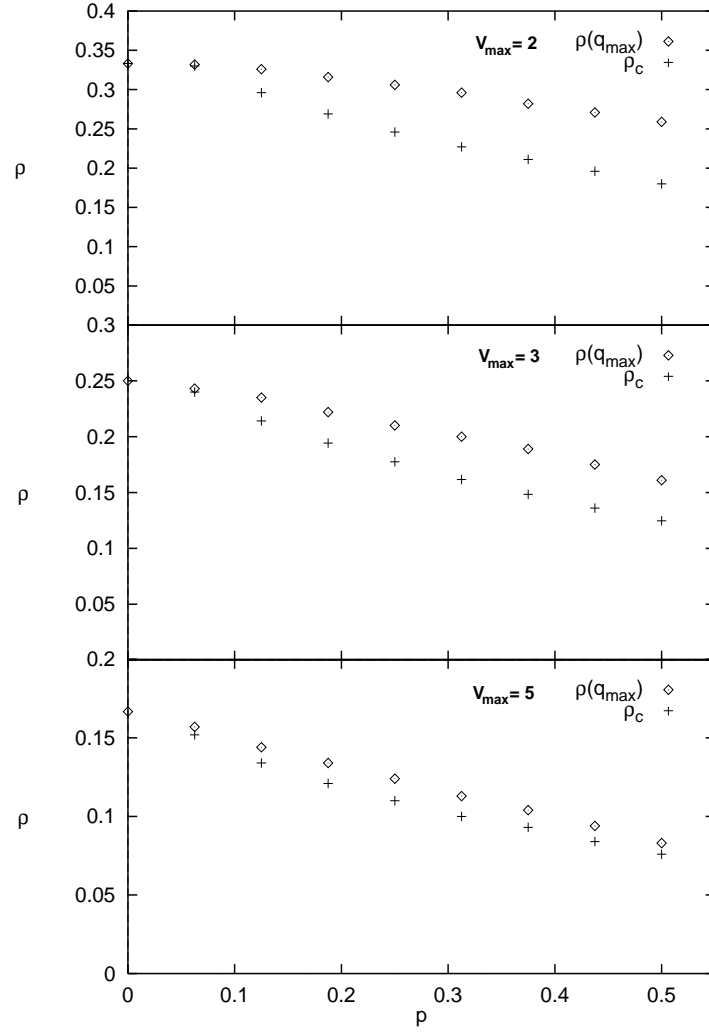


FIG. 8. Comparison between transition density and density of maximum flow.

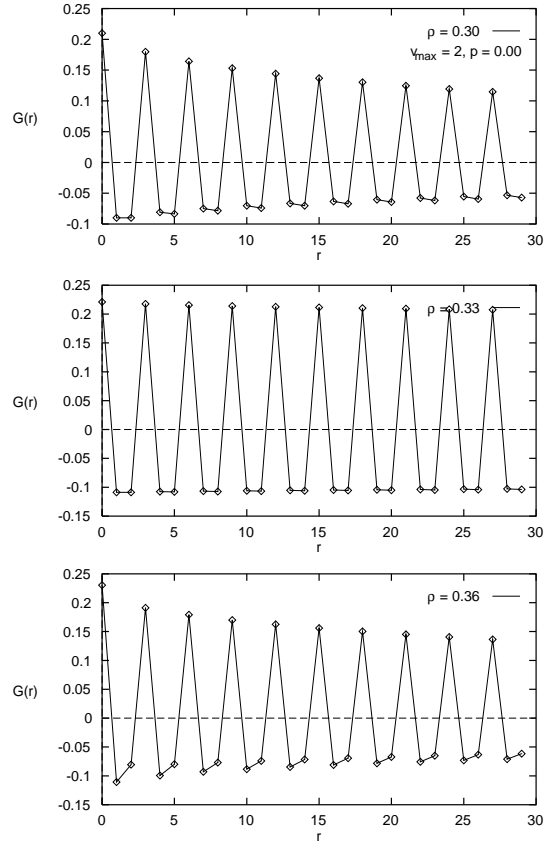


FIG. 9. Correlation function in the vicinity of the phase transition for the deterministic limit.

At $\rho = \rho_c$ the amplitude is independent of the distance r .

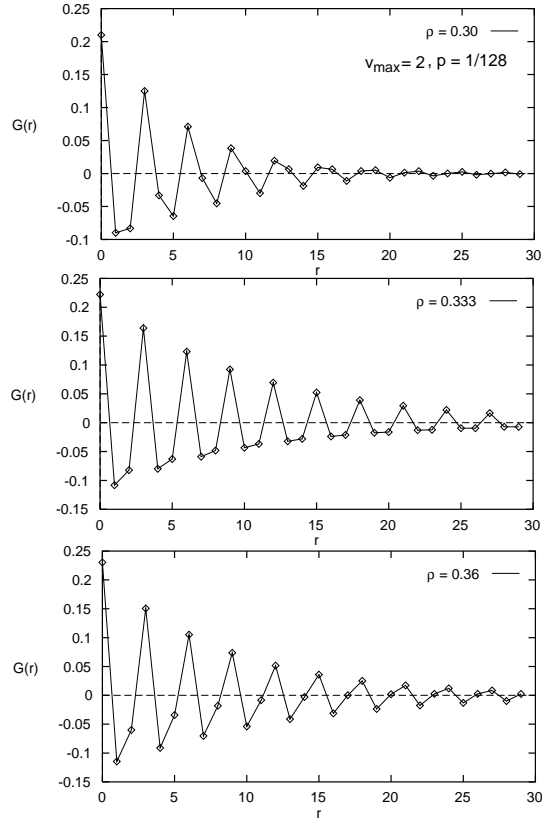


FIG. 10. Correlation function in the presence of noise. The amplitude of the correlation function decays exponentially for all values of ρ .

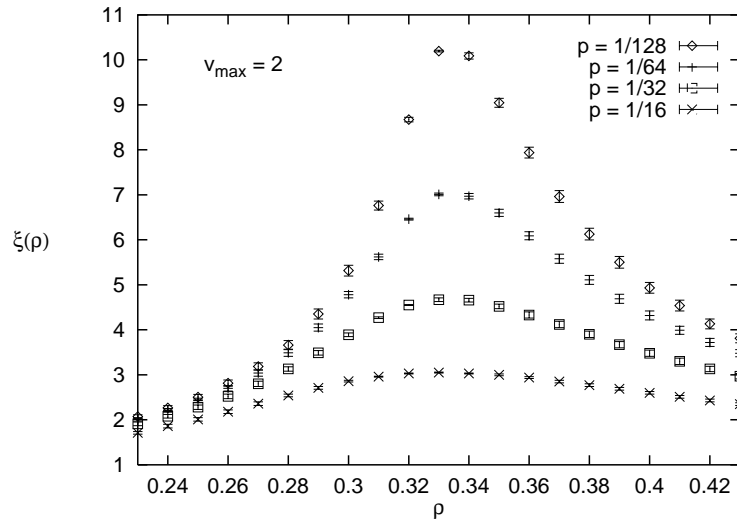


FIG. 11. Density dependence of the correlation length in the vicinity of the transition density.

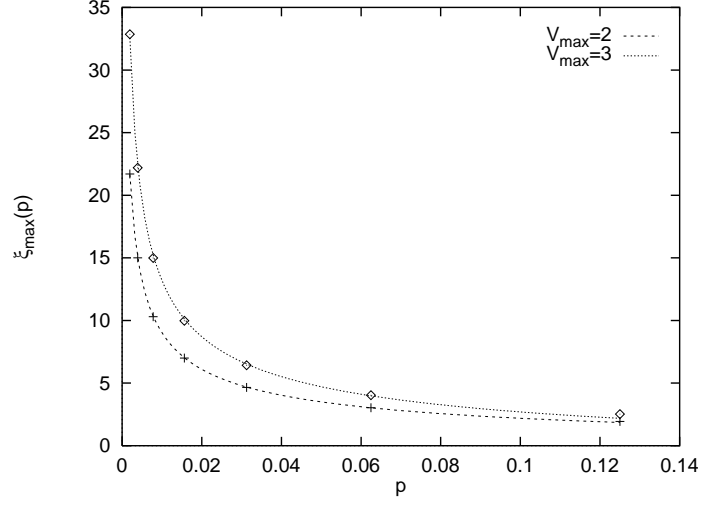


FIG. 12. Noise dependence of ξ_{max} for different maximum velocities. Independent of the maximum velocity $\xi_{max}(p) \sim 1/\sqrt{p}$ holds in the limit $p \rightarrow 0$.