Lattice Sinh-Gordon model

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Chapter 1

Introduction

Our goal is to simulate a lattice regularized version of sinh-Gordon model. This model is the simplest example of an interactting field theory. It has only one particle in the spectrum and it is integrables, its S-matrix is known analytically, and show a remarkable duality property. Results obtained with thermodynamical Bethe ansatz give interesting properties, but also many dilemmas. In principle, the duality of the theory is not compatible with the mass formula. Also there is not a single UV behaviour of the theory, but there can be many values of the charge infinity. A lattice analysis can be helpful to give numerical support to many of this nonperturbative results. In particular we focus on the analytical mass formula, by doing lattice perturbation theory and then lattice spectroscopy to compare it.

In chapter 2 we give some information about of the theory in the continuum. In chapter 3 we describe the lattice setup we implemented, and in chapter 4 we show the results for lattice perturbation theory we obtained. In chapter 5 we then use lattice spectroscopy and we compare our results to the continuum results.

Chapter 2

Properties of Sinh-Gordon model in the continuum

The action that defines the model is

$$\mathcal{A} = \int d^2x \left\{ \frac{1}{2} \sum_{\nu} (\partial_\nu \phi)^2 + 2\mu \cosh(b\phi) \right\} . \tag{2.1}$$

The sinh-Gordon model has many remarkable properties. It is integrable, has a unique vacuum and the simplest symmetry Z_2: $\phi \to -\phi$. Due to integrability of the model, the S-matrix is factorizable in 2 particles S-matrix, and it is known analitically

$$S(\theta) = \frac{\sinh(\theta) - i \sin(\pi\beta)}{\sinh(\theta) + i \sin(\pi\beta)} . \tag{2.2}$$

Here θ is the rapidity, and it is related to the energy and the momentum by

$$E = m \cosh(\theta) \ ; \ P = m \sinh(\theta) . \tag{2.3}$$

It is introduced the function $\beta(b)$ defined by

$$\beta(b) = \frac{b^2}{8\pi} \frac{1}{1 + b^2/8\pi} . \tag{2.4}$$

This result is obtained by analytic continuation from an analogous calculation in the sine-Gordon model, valid in the "Coleman bound", namely $b < \sqrt{8\pi}$. It is proven that the theory doesn't develop a mass gap over this bound.

This formula reveals an interesting property, which is not manifest in the Lagrangian, a duality symmetry given by

$$b/\sqrt{8\pi} \to \sqrt{8\pi}/b \ ; \ \beta \to 1 - \beta . \tag{2.5}$$

This transformation doesn't change the position of the zeroes in the S-matrix.

2.1 Observables

Calculations of observable quantities have been made for this model. The action used in this calculations is

$$\mathcal{A} = \int d^2x \left\{ \frac{1}{16\pi} \sum_\nu (\partial_\nu \phi)^2 + 2\mu \cosh(b\phi) \right\} \tag{2.6}$$

This action is obtained by $\phi \to 1/\sqrt{8\pi}\phi$, and by rescaling the coupling accordingly. In this way the Coleman bound is at $b = 1$, simplifying the calculations.

A notable example is obtained for the quantity $e^{a\phi}$ [1], namely

$$\begin{aligned} \langle e^{a\phi} \rangle = & \left[\frac{m\Gamma(\frac{1}{2+2b^2})\Gamma(1+\frac{b^2}{2+2b^2})}{4\sqrt{\pi}} \right]^{-2a^2} \times \\ & \exp\left\{ \int_0^\infty \frac{dt}{t} \left[-\frac{\sinh^2(2abt)}{2\sinh(b^2t)\sinh(t)\cosh((1+b^2)t)} + 2a^2e^{-2t} \right] \right\}, \end{aligned} \tag{2.7}$$

where [2]

$$m = \frac{4\sqrt{\pi}}{\Gamma(\frac{1}{2+2b^2})\Gamma(1+\frac{b^2}{2+2b^2})} \left[-\frac{\mu\pi\Gamma(1+b^2)}{\Gamma(-b^2)} \right]^{\frac{1}{2+2b^2}} . \tag{2.8}$$

The last equation gives the renormalized particle mass in terms of the bare parameter μ and the coupling b. It is remarkable that this formula doesn't satisfy the duality symmetry, and that the mass goes to zero for $b = 1$ (which is equivalent to the Coleman bound, after the rescaling of $1/8\pi$ of the kinetic term). It has been obtained by analytic continuation from a mass formula for the breather in the Sin-Gordon model spectrum, valid for $b < 0.5$.

After the redefinitions $g = b\sqrt{8\pi}$ and $m_0 = 4b\sqrt{\pi\mu}$, the square of the renormalised mass can be expanded in g as

$$\begin{aligned} m^2 =& m_0^2 + \frac{m_0^2 g^2}{8\pi} \left(-\gamma_E + \psi(1/2) + 4\log 2 - \log m_0^2 \right) + \\ &+ \frac{m_0^2 g^4}{384\pi^2} \Big[3\gamma_E(2+\gamma_E) - \pi^2 - 24(1+\gamma_E - 2\log 2)\log 2 + \\ &\qquad + 3\psi(1/2)\left(-2 - 2\gamma_E + 12\log 2 + \psi(1/2)\right) + \\ &\qquad + 6\log m_0^2 \left(1 + \gamma_E - 4\log 2 - \psi(1/2)\right) + 3\log^2 m_0^2 \Big] + O(g^6) , \end{aligned} \tag{2.9}$$

where γ_E is the Euler–Mascheroni constant and the digamma function $\psi(x)$ is the logarithmic derivative of the gamma function.

We can also obtain a dimensionless quantity which is easy to estimate. It is given by

$$R = \frac{\langle e^{r\phi} \rangle \langle e^{s\phi} \rangle}{\langle e^{t\phi} \rangle \langle e^{u\phi} \rangle} . \tag{2.10}$$

The coefficients have to satisfy $r^2 + s^2 = t^2 + u^2$. When this condition is satisfied, all factors relatives to the scaling dimension cancel out, leading to the prediction

$$R = \exp \int_0^\infty \frac{dz}{z} - \left\{ \frac{\sinh(2rbz)^2 + \sinh(2sbz)^2 - \sinh(2tbz)^2 - \sinh(2ubz)^2}{2 \sinh(b^2 z) \sinh(z) \cosh((1+b^2)z)} \right\} . \tag{2.11}$$

The integral, thanks to the aforementioned condition, is convergent for small values of z, while it can diverge at infinity for some values of the vertex parameter r. TO avoid divergences, one must satisfy

$$r < \frac{b^2 + 1}{2b} . \tag{2.12}$$

This condition is obtained by observing the limit for z that goes to infinity of the integrand, and it is always satisfied for $r < 1$. For simulations we will take account of this bound, in order to avoid problems related to divergencies.

Chapter 3

Lattice setup

The lattice action (after rescaling the field as $\phi \to 1/\sqrt{8\pi}\phi$) is

$$\mathcal{A} = a^2 \sum_x \left[\frac{N_d}{8\pi}\phi(x)^2 + \frac{1}{8\pi}\sum_\nu \phi(x)\phi(x+a\nu) + 2\mu\cosh(b\phi(x)) \right] . \quad (3.1)$$

We can absorb a^2 in the dimensioned parameter μ defining $\hat{\mu} = a^2\mu$. We simulate the model using the grid code, based on a HMC algorithm. We will use $a = 1$ as usual for the rest of the notes. The forces for the HMC algorithm are given by

$$F(x) = \frac{N_d}{4\pi}\phi(x) + \frac{1}{4\pi}\sum_\nu \phi(x+\nu) + 2b\mu\sinh(b\phi(x)) . \quad (3.2)$$

For the simulations a lattice size of 256^2 has been used.

Chapter 4

Comparing lattice perturbation theory with the continuum

Simple dimensional analysis show that both the field and the parameter b are dimensionless, while μ has a dimension 2 in energy. Labelling lattice parameter with an hat, we have $\hat{\mu} = a^2\mu$. Naively the continuum is limit is obtained with $\hat{\mu} \to 0$. To compare lattice results with the continuum analytic results, we need to find the lines of constant physics.

We can do this using lattice perturbation theory. Calculating the perturbative expansion of the two point function and setting the pole position equal to the mass gives a relation between $\hat{\mu}$ and b. This can be checked non-perturbatively doing lattice spectroscopy and checking the value of the physical mass obtained.

The theory has an infinite number of interaction terms

$$\mathcal{A} = \sum_x \frac{1}{2} \sum_\nu (\delta_\nu \phi(x))^2 + \frac{\hat{m}_0^2}{g^2} \cosh(g\phi) = \tag{4.1}$$

$$= \sum_x \Bigg[N_d \phi(x)^2 + \sum_\nu \phi(x)\phi(x+\nu) + \frac{\hat{m}_0^2}{2}\phi(x)^2 + \frac{\hat{m}_0^2 g^2}{4!}\phi(x)^4 + \frac{\hat{m}_0^2 g^4}{6!}\phi(x)^6 + \dots \Bigg] \tag{4.2}$$

which leads to an infinite number of Feynman rules for n-point vertex, with n even, namely

$$V_n = -\hat{m}_0^2 g^{n-2} \,. \tag{4.3}$$

We chose this action since it doesn't have a factor of the coupling in the bare mass. There is a relation to the b and μ parameters given by

$$g = \sqrt{8\pi} b \ ; \ m_0 = 4b\sqrt{\pi\mu} \,. \tag{4.4}$$

We can expand the two point function in perturbation theory over powers of the couple g^2. In the continuum both the coupling and the field don't renormalize. There is only a multiplicative renormalization of the mass parameter, given by (2.8). Setting the two point function

$$D(p^2) = \frac{1}{\hat{p^2} + m^2} + O(\hat{p^2} + m^2) \tag{4.5}$$

for $p^2 \to -m^2$ one obtains

$$m^2 = m_0^2 + \frac{1}{2} m_0^2 g^2 T(m^2) + \frac{1}{8} m_0^2 g^4 T(m^2)^2 + O(g^6) \tag{4.6}$$

$$= m_0^2 (1 + \frac{1}{2} g^2 T(m^2) + \frac{1}{8} g^4 T(m^2)^2) + O(g^6) \tag{4.7}$$

We have the tadpole integral

$$T(m^2) = \int_{-\pi/a}^{\pi/a} \frac{d^2k}{(2\pi)^2} \frac{1}{\hat{k^2} + m^2} \tag{4.8}$$

where $\hat{k^2} = \hat{k_x}^2 + \hat{k_y}^2$ and $\hat{k} = 2/a \sin(ka/2)$. The symmetry factors 1/2 and 1/8 are obtained by multiplying the factors 1/4! and 1/6! of the action by the number of possible connection between vertices and external legs when constructing the tadpole and double tadpole integral. It immediately appears that there must be a logarithmic divergence when one substitutes $l_\mu = ak_\mu$ and takes the limit $a \to 0$.

We can do a similar operation for the coupling, by setting the full four point function with zero external momentum equal to the renormalized coupling constant. We obtain

$$g_R^2 = g^2 + \frac{\hat{m}^2 g^4}{2} T(m^2) + \frac{3\hat{m}^4 g^4}{2} V(m^2) + O(g^6) \tag{4.9}$$

We invert this relation as well, obtaining

$$g^2 = g_R^2 \left[1 - \left(\frac{3\hat{m}^2}{2} T(m^2) - \frac{3\hat{m}^4}{2} V(m^2) \right) g_R^2 \right] + O(g_R^4) \tag{4.10}$$

We can substitute $\hat{m}_0^2$ with $\hat{m}^2$ in the integrals with an error of order g^6. We invert the series

$$m_0^2 = m^2 \left[1 - \frac{1}{2} g^2 T(m^2) + \left(\frac{1}{8} T(m^2)^2 + \frac{3\hat{m}^2}{4} (T^2 + \hat{m}^2 V T) \right) g^4 \right] + O(g^6)\,. \tag{4.11}$$

This formula gives us, given a physical value of the mass m, the bare parameter m_0 for a fixed value of the lattice spacing a. We need to evaluate the tadpole and the vertex correction integrals.

4.1 Evaluation of the integrals

We substitute $l_\mu = ak_\mu$ in (4.8) obtaining

$$T(m^2) = \int_{-\pi}^{\pi} \frac{d^2 l}{(2\pi)^2} \frac{1}{\hat{l}^2 + a^2 m^2} = I_1(a^2 m^2)\,. \tag{4.12}$$

The integral has infrared divergencies for $a \to 0$. We define

$$I_n(\xi^2) = \int_{-\pi}^{\pi} \frac{d^2 l}{(2\pi)^2} \frac{1}{(\hat{l}^2 + \xi^2)^n} \tag{4.13}$$

We can evaluate numerically the value of I_1 for $a \to 0$ only dividing the integral in two parts, one within a small radius ρ, and the other outside. We obtain:

$$\begin{aligned} I_1(\xi^2) &= \int_{|l|<\rho} \frac{d^2 l}{(2\pi)^2} \frac{1}{l^2 + \xi^2} + \int_{|l|>\rho} \frac{d^2 l}{(2\pi)^2} \frac{1}{\hat{l}^2 + \xi^2} \\ &= \frac{1}{4\pi} \frac{\xi^2}{\rho^2} - \frac{1}{4\pi} \ln\left(\frac{\xi^2}{\rho^2}\right) + \int_{|l|>\rho} \frac{d^2 l}{(2\pi)^2} \frac{1}{\hat{l}^2} - \xi^2 \int_{|l|>\rho} \frac{d^2 l}{(2\pi)^2} \frac{1}{(\hat{l}^2)^2} + O(\xi^4) \\ &= Z_0 - \xi^2 C_0 - \frac{1}{4\pi} \ln(\xi^2),. \end{aligned} \tag{4.14}$$

We have expanded both parts in powers of a^2. We defined $\xi^2 = a^2 m^2$ and we introduced the quantities

$$Z_0 = \lim_{\rho \to 0} \int_{|l|>\rho} \frac{d^2 l}{(2\pi)^2} \frac{1}{\hat{l}^2} + \frac{1}{4\pi} \log(\rho^2) = 0.275794\,, \tag{4.15}$$

$$C_0 = \lim_{\rho \to 0} \int_{|l|>\rho} \frac{d^2 l}{(2\pi)^2} \frac{1}{(\hat{l}^2)^2} - \frac{1}{4\pi\rho^2}\,. \tag{4.16}$$

In the case of the tadpole integral, the quantity C_0 is ininfluent for $a \to 0$, so we have the result

$$T(a^2 m^2) = Z_0 - \frac{1}{4\pi} \ln(a^2 m^2) + O(a^2)\,. \tag{4.17}$$

For the evaluation of the vertex corrections, we note that

$$V(m^2) = \int_{-\pi/a}^{\pi/a} \frac{d^2 k}{(2\pi)^2} \frac{1}{(\hat{k}^2 + m^2)^2} = a^2 I_2(a^2 m^2)\,. \tag{4.18}$$

We use the relation

$$I_2(\xi^2) = -\frac{d}{d\xi^2} I_1(\xi^2) = C_0 + \frac{1}{4\pi\xi^2} + O(\xi^2)\,. \tag{4.19}$$

We have the final result:

$$m_0^2 = m^2 \left[1 - \frac{1}{2}g^2 I_1(a^2m^2) + \left(\frac{1+6\hat{m}^2}{8} I_1(a^2m^2) + \frac{3\hat{m}^4a^2}{4} I_1(a^2m^2) I_2(a^2m^2)\right) g^4\right] + O(g^6) \tag{4.20}$$

$$g^2 = g_R^2 \left[1 - \left(\frac{\hat{m}^2}{2} I_1(a^2m^2) - \frac{3\hat{m}^4a^2}{2} I_2(a^2m^2)\right) g_R^2\right] + O(g_R^6)\,. \tag{4.21}$$

Comments We have shown that $T(m^2) = Z_0 - \frac{1}{4\pi}\log a^2m^2 + O(a^4m^4)$. From Eq. (4.6), this implies that the coefficients of $\log a^2m^2$ at $O(g^2)$ and $(\log a^2m^2)^2$ at $O(g^4)$ match those in Eq. (2.9).

Also, we have shown that $g^2 = g_R^2 + O(a^2)$. This proves that the coupling has no divergences when taking the continuum limit, and also that it does not renormalize.

Chapter 5

Mass scaling on the lattice

We are interested in verifying the assumptions given by the mass formula (2.8) for mass values we can obtain with lattice with our setup. We want to do this motivated by the comment of the last section. Since the coefficients in front of the logarithm mathches, we want to use the result relative to the resummation of the logarithm at every order, which gives the $\mu^{1/(2+2b^2)}$ behaviour. The coefficient in front of that will be a result of the lattice regularization, and will be affected heavily by lattice artefacts.

We used as spectroscopical operator the time sliced field

$$O(t) = \sum_x \phi(t, x) \tag{5.1}$$

We chose one direction as the temporal one, and we summed over the other. Since the theory has no bound states, we expect that there shouldn't be contaminations in the first time slices, and also that the majority of the signal should be in the first time slices. Given the correlation function

$$C(t) = \sum_{t'=0}^{N_t-1} O(t')(t'+t) \tag{5.2}$$

we expect the following behaviour in this region

$$C(t) \approx e^{-m_{eff} t} \,. \tag{5.3}$$

We can extract the mass as

$$m_{eff}(t + 0.5) = \log \frac{C(t)}{C(t+1)} \,. \tag{5.4}$$

Inspired by (2.8), we want to observe a behaviour given by

$$m_{eff} = f(b)\, \mu^{\alpha} \tag{5.5}$$

with $f(b)$ an unknown function of the parameter "b", really sensible to lattice regularization artefacts, while we expect α to be close to the value obtained in the continuum, namely $1/(2+2b^2)$. We have shown in the previous section that the parameter b doesn't renormalize on the lattice, so we will assume it. We will use for this part of the analysis that for b fixed also a is fixed, so that we can check the relation between m and μ.

For various fixed beta values we proceeded to fit the mass obtained from spectroscopy for various values of μ.

The green line represent the fit with the α parameter free to vary, while the red one fixes α to be the theoretical value predicted in (2.8).

We expect the results to be close to the continuum limit for small values of the effective mass, since, given a fixed measured value of the physical mass m^* , it is related to the lattice mass simply by

$$am^* = \hat{m}_{lat} \tag{5.6}$$

with a being the lattice spacing. We used values of m_{lat} between 0.001 and 1, we believe we are reasonably close to the continuum limit in this region. Lower values of the effective mass results in a theory with a really slow growing potential, and the numerical algorhithms for these cases generates very correlated configurations. Higher values of the effective mass increase lattice artefacts considerably, and this can invalidate our assumption of the behaviour, so they will not be considered as well.

5.1 Violation of scaling

As noted in previous sections, the mass formula is valid up to $b = 1$. Nonperturbative results cease to be valid for $b > 1$. Simulations made on the lattide with $b > 1$ seems to violate the scaling laws. Interestingly, there is still a μ^α behaviour, but $\alpha = 1/(2 + 2b^2)$ is not a correct prediction anymore.

In pictures 5.7, 5.8, 5.9, 5.10, 5.11, 5.12 we show the results for $b = 1.2, 1.5, 1.7, 2.3$. We can see clearly that the red line deviates from the green line, which is in good agreement with the data.

We can show the deviation from the expected behaviour, for $b > 1$ in figure 5.13, where all the exponents from the fits are plotted alogside with the function $\alpha_{th}(b) = 1/(2 + 2b^2)$.

We show now in figure 5.14 a plot of the values obtained for the amplitudes in front of the exponential function. In principle this should be compared with the continuum equivalent described in (2.8).

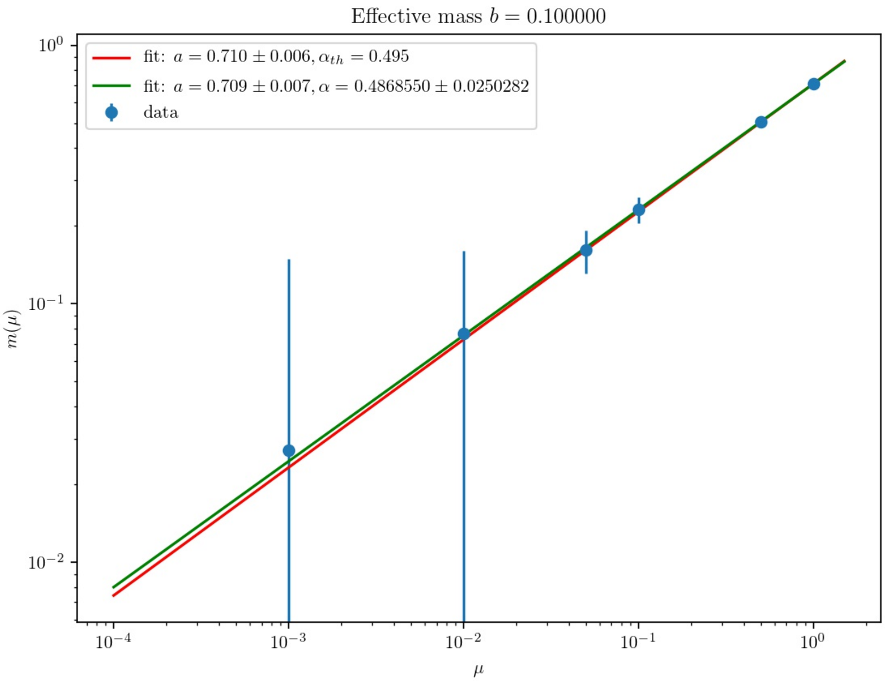

Figure 5.1: Value of the masses obtained for $b = 0.1$ and $\mu = [0.001, 0.01, 0.1, 0.5, 1]$. The χ^2 for the two fit are 0.031159556043223686 and 0.0005302957212052317, for the case with fixed $\alpha = 1/(2 + 2b^2)$ and with α free, respectively.

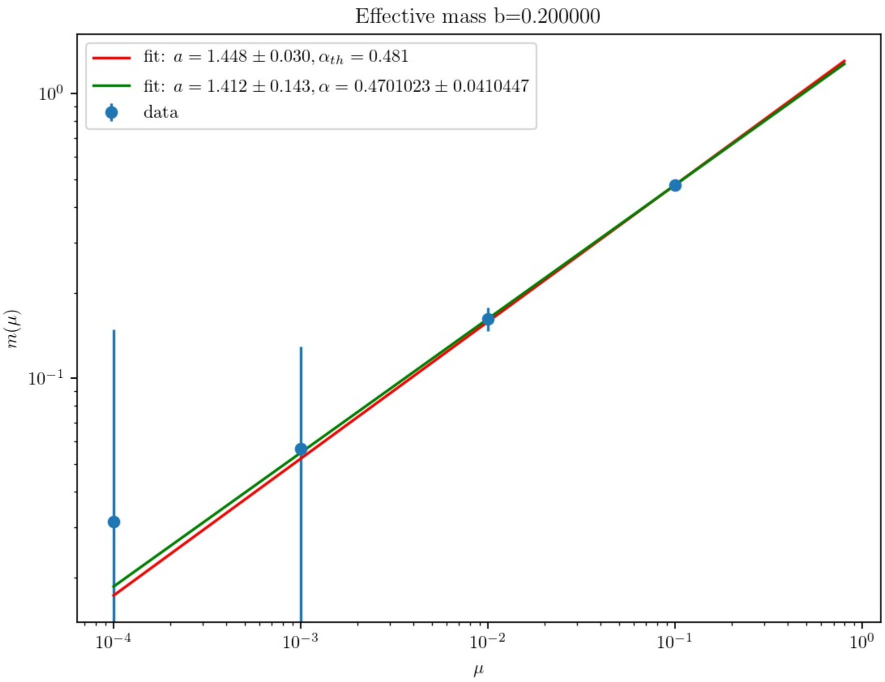

Figure 5.2: Value of the masses obtained for $b = 0.2$ and $\mu = [0.001, 0.01, 0.1, 0.5]$. The χ^2 for the two fit are 0.3488364928360326 and 0.011429390512562691, for the case with fixed $\alpha = 1/(2+2b^2)$ and with α free, respectively.

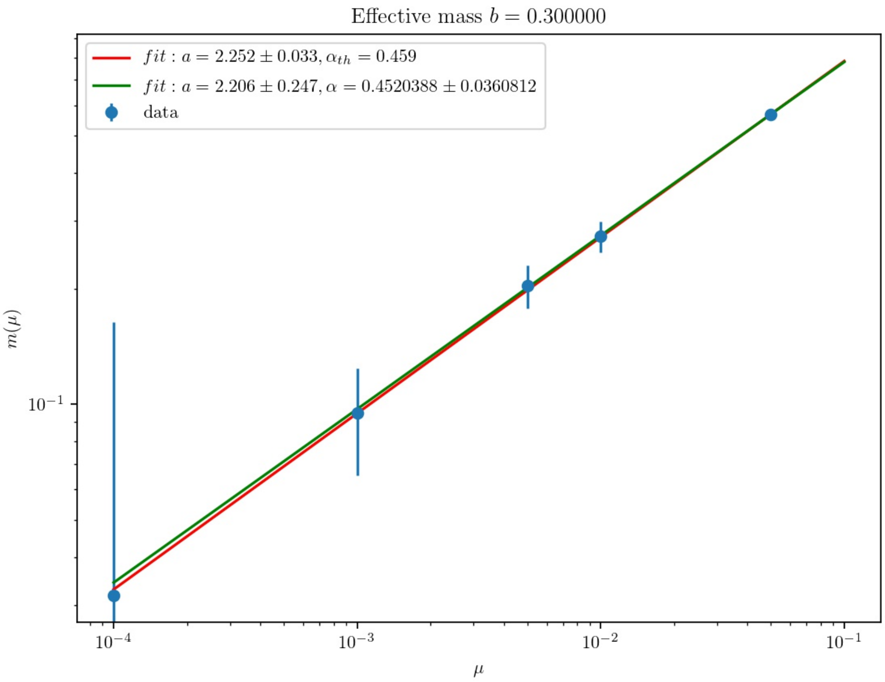

Figure 5.3: Value of the masses obtained for $b = 0.3$ and $\mu = [0.001, 0.005, 0.01, , 0.05]$. The χ^2 for the two fit are 0.017830714640868522 and 0.005743906281872949 5, for the case with fixed $\alpha = 1/(2+2b^2)$ and with α free, respectively.

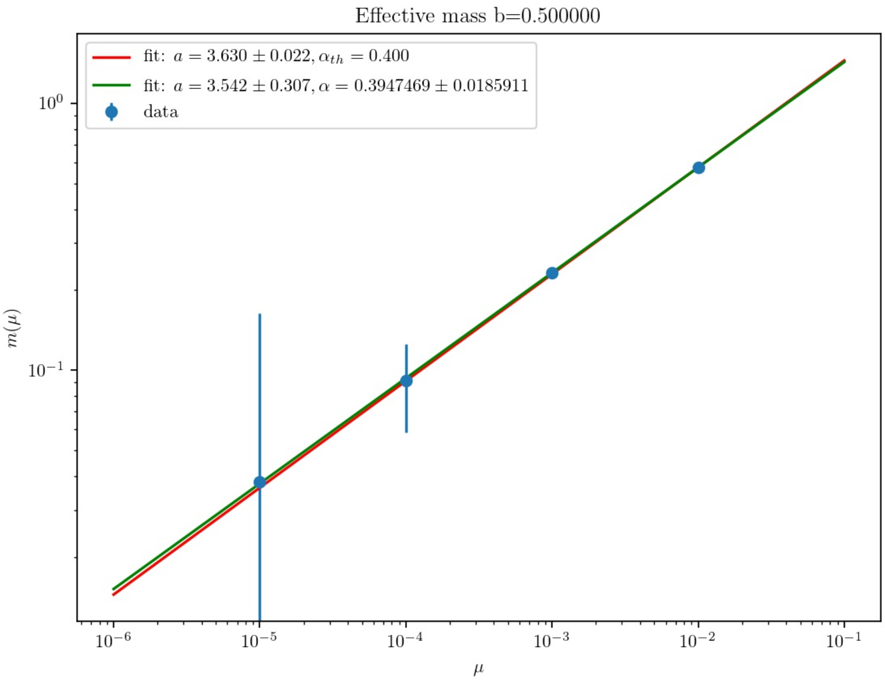

Figure 5.4: Value of the masses obtained for $b = 0.5$.

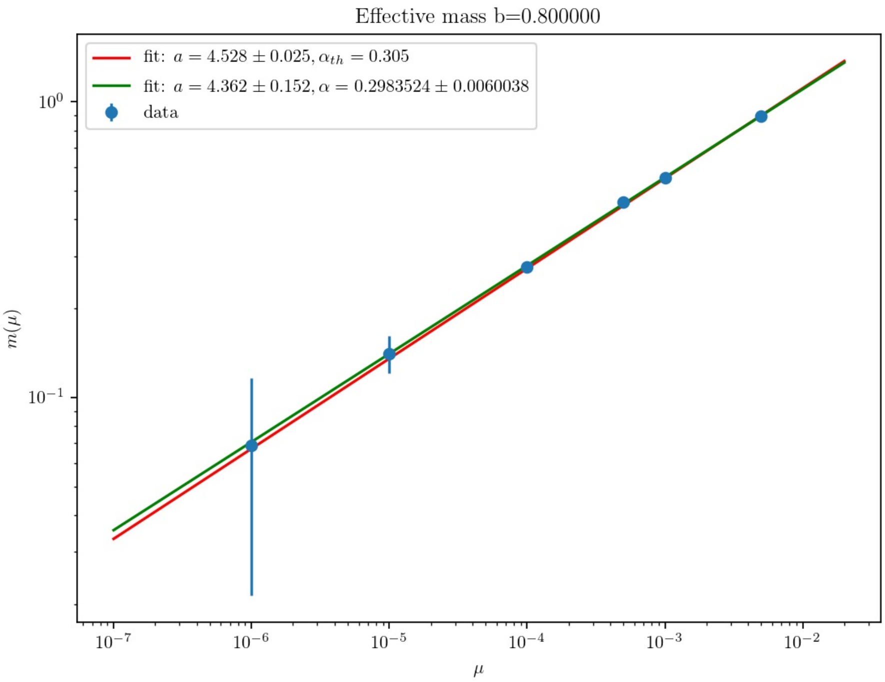

Figure 5.5: Value of the masses obtained for $b = 0.8$ and $\mu = [0.0005, 0.001, 0.005, 0.1]$. The χ^2 for the two fit are 0.7331906585415697 and 0.1862401990836134, for the case with fixed $\alpha = 1/(2+2b^2)$ and with α free, respectively.

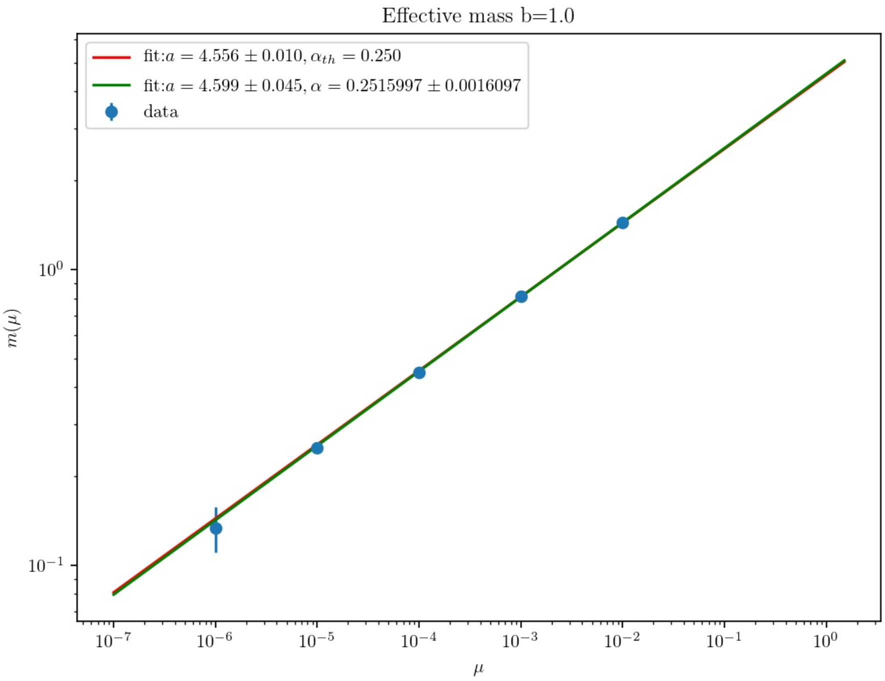

Figure 5.6: Value of the masses obtained for $b = 1.0$.

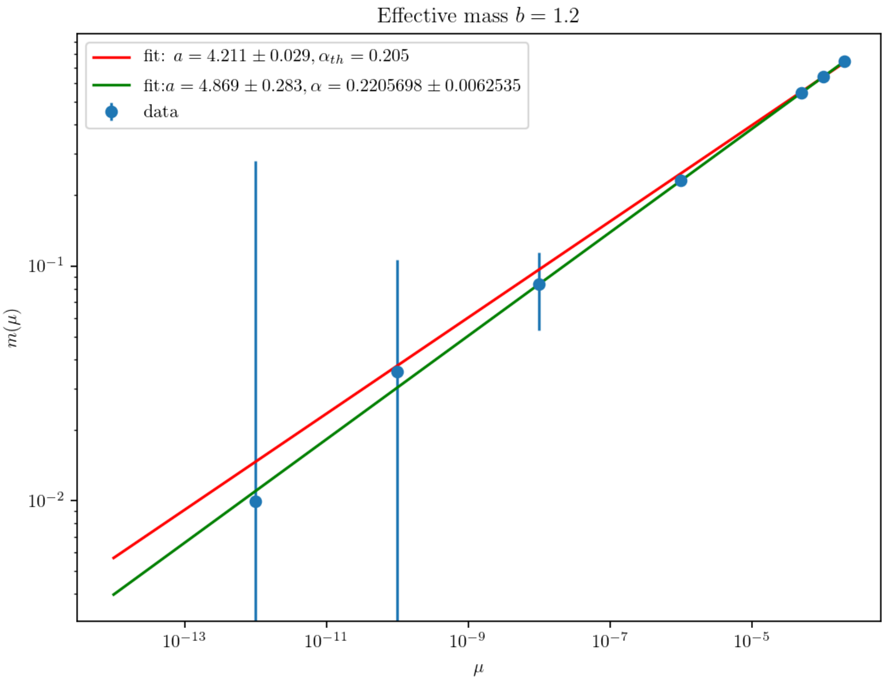

Figure 5.7: Results for b=1.2

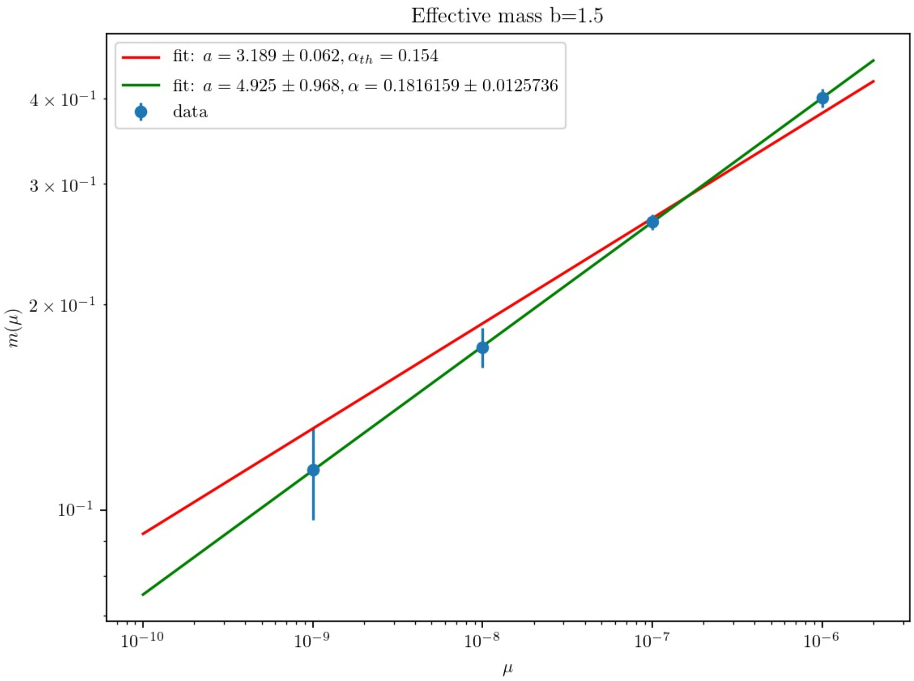

Figure 5.8: Results for b=1.5

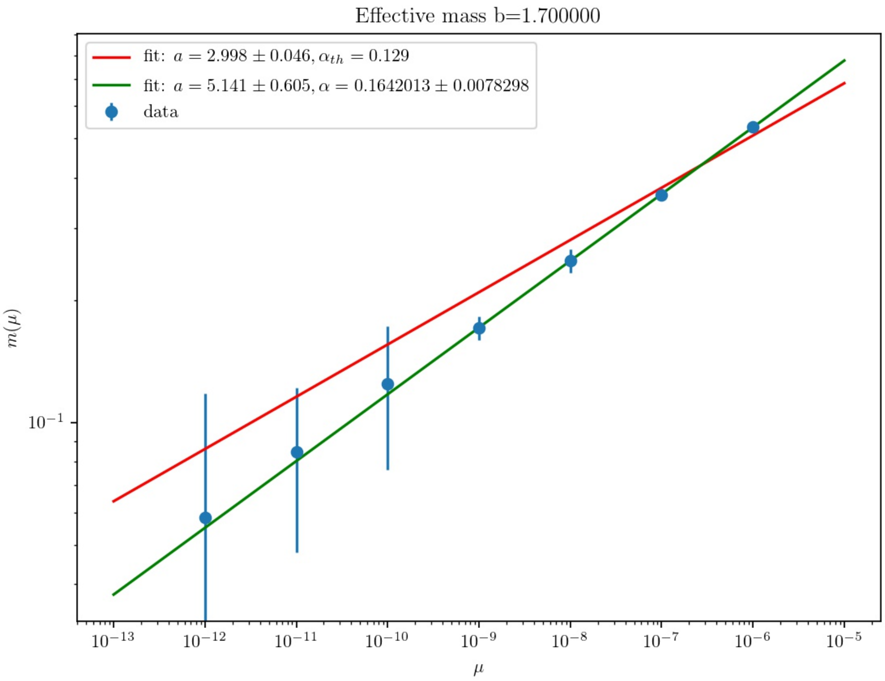

Figure 5.9: Results for b=1.7

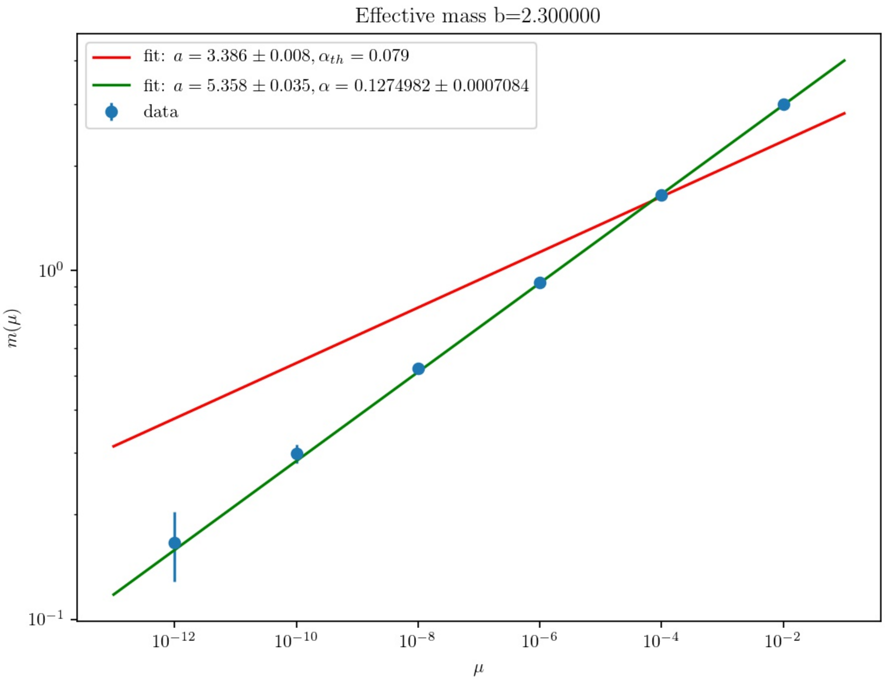

Figure 5.10: Results for b=2.3

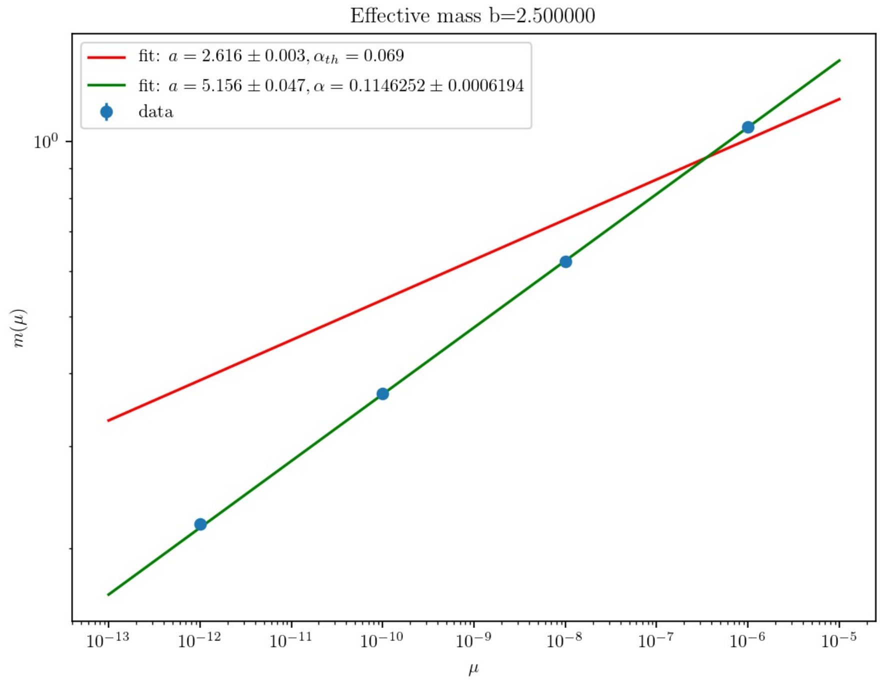

Figure 5.11: Results for b=2.5

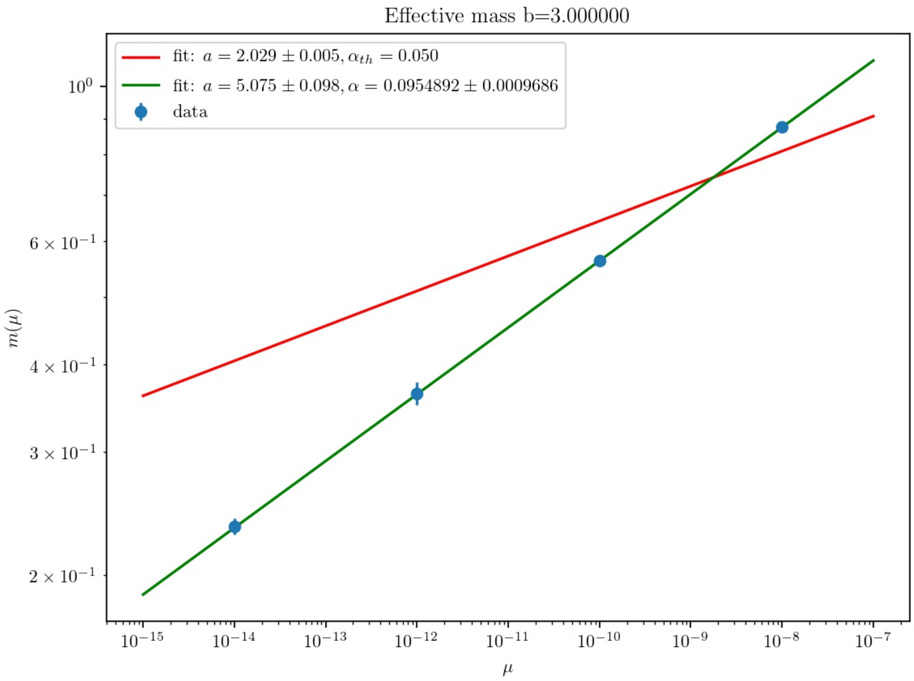

Figure 5.12: Results for b=3

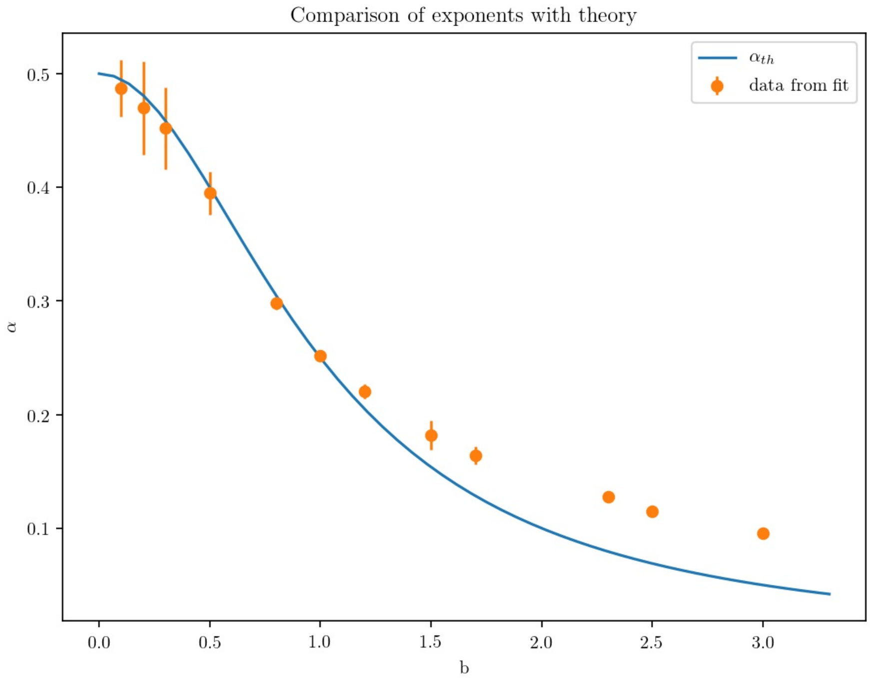

Figure 5.13: Values of the exponents obtained compared to the prediction $\alpha_{th}(b) = 1/(2 + 2b^2)$.

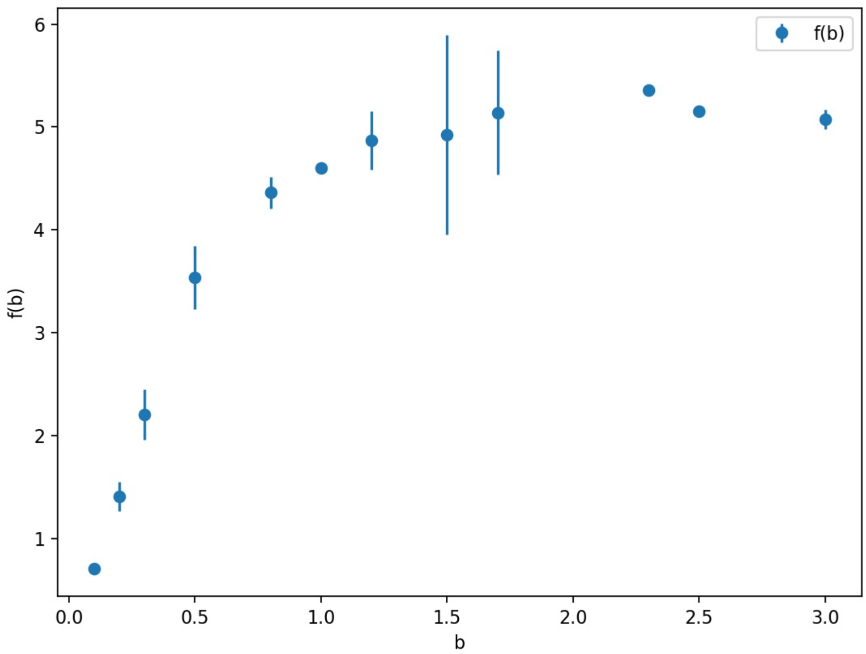

Figure 5.14: Values of the unknown function f(b) obtained.

Appendices

Appendix A

Finite volume effects

It is of most interest verifying that in our analysis finite volume effects are small and quantifiable.

For a scalar theory which has a mass gap in the non-interacting limit, effects in the variation of the mass $\Delta m = m(l) - m(\infty)$ have been quantified to decrease exponentially as $-e^{-\frac{\sqrt{3}}{2}mL}$, where m is the mass gap of the theory and L the spatial length[3].

The hypothesis has been verified for different values of b, as shown in figure A.1,A.2 and A.3.

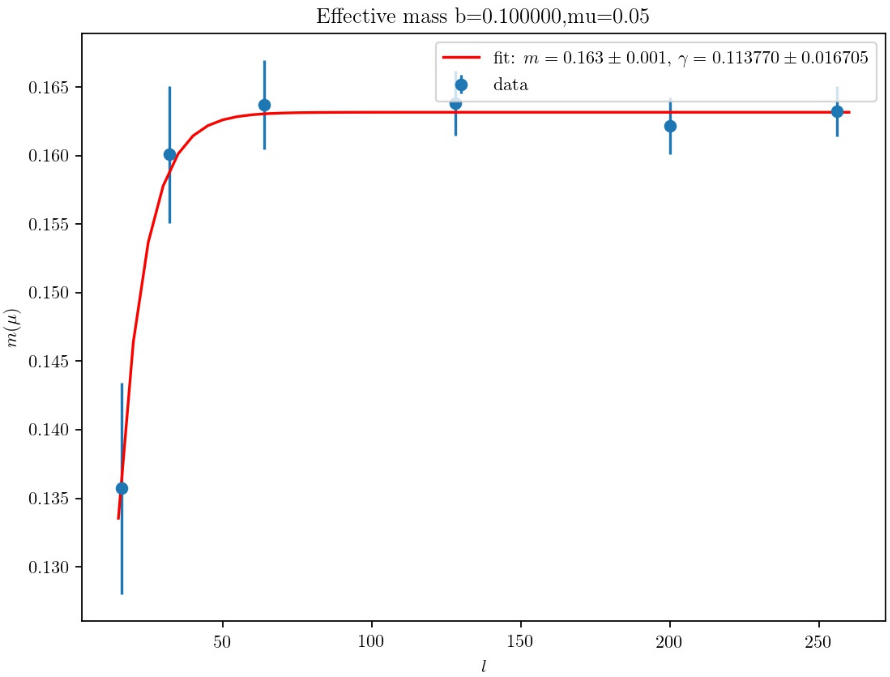

Figure A.1: Mass value dependence with respect to smaller volumes is plotted

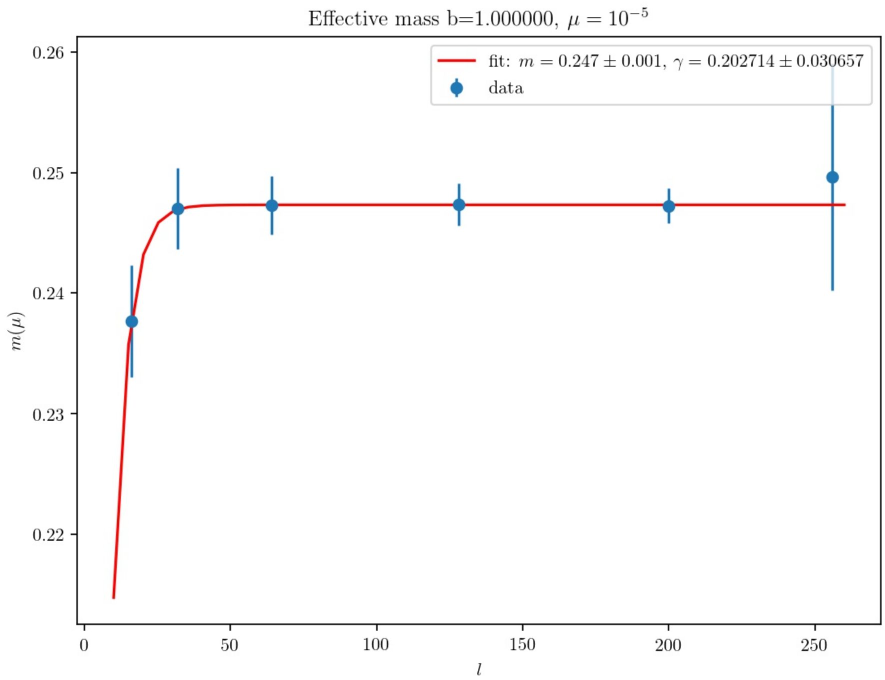

Figure A.2: Mass value dependence with respect to smaller volumes is plotted

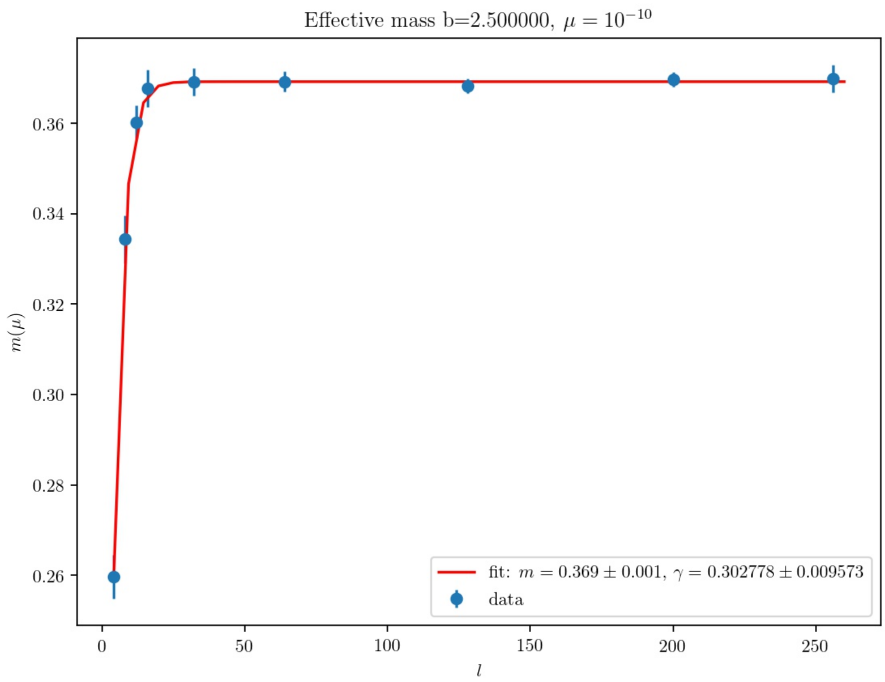

Figure A.3: Mass value dependence with respect to smaller volumes is plotted

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