Systems, Measurements, and Controls

a coursepack for ME 365 and ME 375

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January 3, 2025

About this Coursepack

 It is a work in progress and may contain many mistakes and personal notes regarding future improvements. Be sure to consult additional sources and follow all relevant standards for engineering analysis and design. If you find mistakes or have recommendations for improvements, please let him know. Previous versions of this coursepack were intended to be used in conjunction with the textbooks [1, 2]. The current version was developed to parallel slides presented in many sections of ME365 and ME375 at Purdue. This coursepack may contain examples paralleling those from these and other sources or modifications thereof. This coursepack is intended to be used as a resource by ME 365 and ME 375 students at Purdue.

To the greatest extent possible, the topics covered by this coursepack are organized into a coherent sequence. At the highest level, this coursepack is organized into parts (i.e., groups of chapters) and chapters. This high-level organization is best leveraged by using the *Bookmarks* or *Table of Contents* pane in a pdf viewer but is also presented in Contents. Also, it's possible to navigate this coursepack by lecture number using List of ME 365 Lectures, List of ME 375 Lectures, and the lecture numbers identified in the footer. There are, however, two significant constraints on how the topics are organized. First, the approved course descriptions, outcomes, and lists of topics for ME 365 and ME 375 dictate the scope of this coursepack and whether topics appear in the first or second *half.* Second, content relevant to a biweekly lab experiment is ideally presented prior to (or approximately concurrent with) the experiment.

Contents

List of ME 365 Lectures

List of ME 375 Lectures

Chapter 1

Introduction

- Use the ME375 robot as a motivating example to tie together the seemingly disparate topics of ME 365 and ME 375.
- Systems -
  - System -
    * "... a set of connected things or parts forming a complex whole ..." [The Oxford Pocket Dictionary of Current English. Encyclopedia.com. 16 Aug. 2021 https://www.encyclopedia.com.]
    * "... a combination of elements intended to act together to accomplish an objective." [2]
    * "A system is an arrangment, set, or collection of things connected or related in such a manner as to form an entirety or whole [3]."
  - Dynamic System -
    * "A dynamic system is one whose present output depends on past inputs." [2]
    * Dynamic - often we use this adjective to describe systems. "(of a process or system) characterized by constant change, activity, or progress ..." [The Oxford Pocket Dictionary of Current English. Encyclopedia.com. 16 Aug. 2021 https://www.encyclopedia.com.]
  - System structure
    * Element - a term we use to refer to the "things or parts" that form a system [The Oxford Pocket Dictionary of Current English. Encyclopedia.com. 16 Aug. 2021 https://www.encyclopedia.com.].
    * Subsystems
  - System inputs and outputs
    * Input - "That which is put in or taken in, or which is operated on or utilized by any process or system (either material or abstract)." ["input, n." OED Online, Oxford

University Press, December 2021, www.oed.com/view/Entry/96482. Accessed 22 February 2022.].

    * Output - a result of an input.
    * Cartoon of a generic system

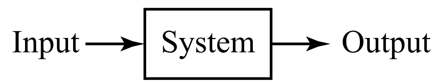


Figure 1.1: Cartoon of a generic system.

  - Examples of dynamic systems:
    * Bicycle: rider, crank, chain, wheels, shifter, handle bar
    * Vehicle suspension: linkages, springs, dampers
    * Air conditioner: compressor, fan, ducts

- Measurements
  - Measurements - "A measurement assigns a specific value to a physical variable [4]."
  - Purpose - To quantify system performance/bahavior (and changes thereof)
  - Error - "Difference between the value indicated by a measurment system and the actual value ... [4]"
  - The design of measurement systems and the interpretation of the output of measurement systems can be complicated.
  - Example
    * GPS location on a map with blue bubble indicating the uncertainty in the location

- Control
  - Control -
    * "... we define control to be the use of algorithms and feedback in engineered systems [5]."
    * "A control system consists of subsystems and processes (or plants) assembled for the purpose of obtaining a desired output with desired performance, given a specified input [6]."
    * "A control system is an arrangement of physical components connected or related in such a manner as to form and/or act as an entire unit [3]."
    * "... we can define the term "control" as the alteration of a dynamic system such that it behaves as if it were a different dynamic system [7]."
  - Feedback -

* “The term feedback refers to a situation in which two (or more) dynamical systems are connected together such that each system influences the other and their dynamics are thus strongly coupled [5].”
* “The central idea [of feedback control] is that a system’s output can be measured and fed back to a controller of some kind and used to effect the control [8].”

– Example: Cruise control

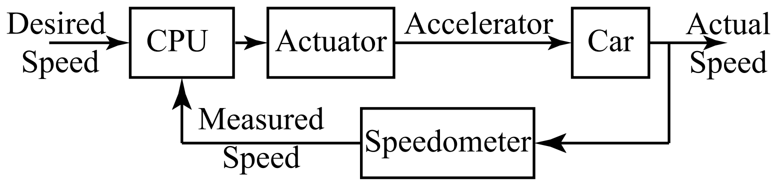


Figure 1.2: Cartoon of a cruise control feedback loop.

* Desired speed (set by driver)
* Computer
* Actuator and accelerator
* Car
* Actual speed
* Speedometer
* Measured Speed
* Feedback (computer compares desired and measured speeds and makes adjustments to the signal it sends to the actuator that pushes the accelerator)

Part I

Static Measurements

Chapter 2

Measurement System Structure and Static Calibration

2.1 Measurement System Structure

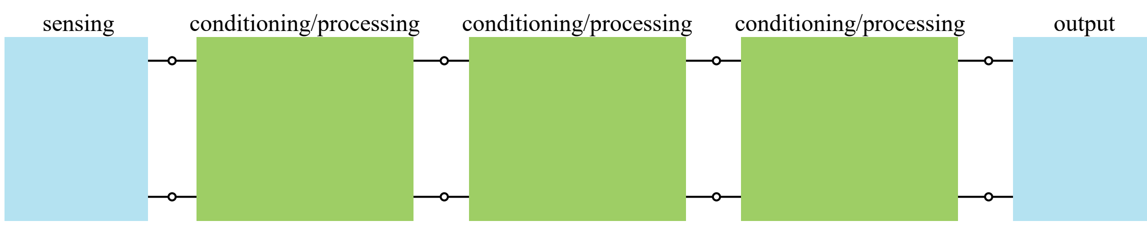


Figure 2.1: Generic measurement system structure

- Measurement systems typically consist of multiple stages [4] or elements [9, 10]; see Figure 2.1.
- Stages or elements often fall into the following categories
  - Sensing [4, 9, 10]
    - Sensing elements include: thermocouples, piezoelectric sensors, strain gauge bridge circuit, etc.
    - (Note, some authors distinguish sensors from transducers; see, for example, [4]. However, it may not be easy or even very useful to distinguish them.)
    - Sensor -
      - "A sensor is often defined as a "device that receives and responds to a signal or stimulus" [11]."

        - "the element of a measuring instrument or a measuring chain to which a measurand is directly applied [12]."
    - Transducer -
        - "The transducer converts this sensed information into a detectable signal form ... [4]."
        - "A transducer is a sensor that in some way obeys reciprocity ... [13]."
  - Signal conditioning [4, 9] and processing [9]:
    - Signal conditioner - "Signal conditioning equipment takes the transducer signal and modifies it to a desired form [4]."
        - may consist of amplifiers, filters, and demodulators [14]
    - Examples:
        - Amplifier
        - Filter
        - Modulator and Demodulator
        - Analog to digital converter
  - Output [4]; data presentation [9]; or indicator or recorder [10]

- Example
  - Temperature measurement with a thermocouple
    - sensor and/or transducer - thermocouple
    - signal conditioner - amplifier and/or filter
    - data acquisition and display - analog to digital converter, digital display
    - Note temperature is the input and voltage is the output

2.2 Static Calibration and Measurement System Characteristics

- Static calibration
  - Standard - "The known value or basis of a calibration [4]."
    - Standards are maintained at various levels (i.e., lab, organization, ... worldwide)
    - Primary standards - "A primary standard defines the value of a unit. It provides a means to describe the unit with a unique number that can be understood throughout the world [4]."
  - Calibration - "A calibration is the act of applying a known value of input to a measurement system for the purpose of observing the system output [4]."

* Static calibration - "The term 'static' refers to a calibration procedure in which the values of the variables involved remain constant ... [4]."
* When a measurement system is used, its calibration facilitates mapping a reading from the system (i.e., output) to a measurement of the measurand (i.e., input).

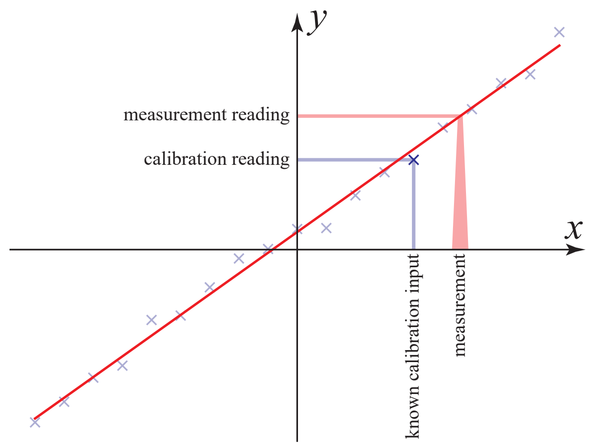


Figure 2.2: An example calibration curve in which each data point (x) indicates a known input and corresponding output reading. A curve is fit to the calibration data; and its inverse may be used to convert an output reading to an expected input value (i.e., measurements). Measurements systems have errors as suggested by the width of the red shaded region at the x-axis.

- General method for static calibration
  * Apply a set of known and unchanging (static) measurands (i.e., standards), $\{x_1, x_2, ..., x_i, ..., x_n\}$ to the measurement system to obtain the corresponding outputs, $\{y_1, y_2, ..., y_i, ..., y_n\}$. See Figure 2.2.
    · The number (n), spacing, and range of the inputs must be sufficient to characterize the instrument appropriately.
    · In addition, the order in which the inputs are applied and the number of repetitions of each input must be sufficient.
    · Note, determining the number, spacing, range, order, and repetitions of the inputs is not trivial. Authors/manufacturers may differ substantially in their approach. The presentation in [15] includes the following recommendations: use no less than 5 points, evenly distribute the points over the range, and complete three cycles (i.e., sequential applications of the known inputs in ascending and then descending order).
  * Construct a calibration curve as a scatter plot of all input-output pairs. Inputs (i.e., known standards) are plotted along x and outputs are plotted along y.

    * Fit a curve to the data (In ME 365, typically a line fit is used.)
        · In ME 365, typically we use a least-squares linear fit to all of the calibration data within the intended range. Sometimes the intended range of a system is restricted (thereby excluding some calibration data) to obtain the best fit.
        · See `polyfit` functions described in:
          Python - see Appendix D.2 Common Mathematical Functions
          MATLAB® - see Appendix E.3 Common Mathematical Functions
  - Measurement vs. calibration; see Figure 2.2.
    * In the calibration process, known inputs are applied to the system and output readings are used to construct a curve fit.
    * In the measurement process, the curve fit is used to convert output readings to expected input values.
  - Example
    * Thermocouple
        · Might use a ice water ($\sim 0°$C) and boiling water ($\sim 100°$C) as the standard

- Measurement system characteristics
  - Range; see, for example, [4, 9]
    * input range - the range of input values for which the system is intended, $[\min_i(x_i),\ \max_i(x_i)]$
    * output range - the range of output values for which the system is intended, $[\min_i(y_i),\ \max_i(y_i)]$
    * Note, the intended range of a system may differ from the range of its calibration when calibration points depart from the intended curve fit.
  - Span; see, for example, [4, 9]
    * input span - the difference between the maximum and minimum values of the input range, $\max_i(x_i) - \min_i(x_i)$
    * output span - the difference between the maximum and minimum values of the output range, $\max_i(y_i) - \min_i(y_i)$
  - Full scale deflection (f.s.d.) [9] or full scale operating range (FSO) [4] - equivalent to the output span
  - Linear measurement system characteristics
    * Many measurement systems have an approximately linear relationship between their input and output.

$$\begin{aligned} y(x) &\approx y_{\text{linear}}(x) \qquad (2.1) \\ &\approx K_0 x + b \qquad (2.2) \end{aligned}$$

        · We take $y_{\text{linear}}(x)$ to be the best (i.e., least-squares) linear approximation to the calibration data.

    · Linear models are sufficient for many measurement systems and are preferred for their simplicity. However, when linear models are insufficient, nonlinear functions (e.g., higher order polynomials) may be used to fit the data.

  * Sensitivity (K_0) - The slope of the static calibration curve

$$K_0 = \frac{dy_{\text{linear}}}{dx}; \tag{2.3}$$

  see, for example, [9, 4, 10]

    · Also known as static sensitivity or static gain [4]
    · For nonlinear systems, sensitivity is a function of the input, $K_0(x)$.

  * Bias (b) - y-intercept to the best linear approximation to the static calibration curve [14]

– Resolution

  * (Input) Resolution -
    · We use the terms 'input resolution' and 'resolution' interchangeably
    · "The smallest detectable change in measured value as indicated by the measuring system [4]."
    · "Resolution is defined as the largest change in [input] that can occur without any corresponding change in [output] [9]."

  * 'output device resolution' [14] - Measurements are read from an output element or stage in a measurement system. Output elements/stages often include, for example, one or more of the following features that impact resolution:
    · Dial indicator - The resolution of a dial indicator depends upon the spacing between divisions marked around the dial.
    · Digital display - The resolution of a digital display depends upon the least significant change that can be detected using the display.
    · Binary signal - The resolution of a binary signal depends upon the significance of the least significant bit.

  * Relationship between (input) resolution and output device resolution
    · Assuming only the output element/stage of a measurement system degrades resolution
    · Assuming a linear approximation to the calibration curve is used
    · Relationship

$$\text{Input Resolution} = \frac{\text{Output Device Resolution}}{\text{Sensitivity}}; \tag{2.4}$$

    see [14]

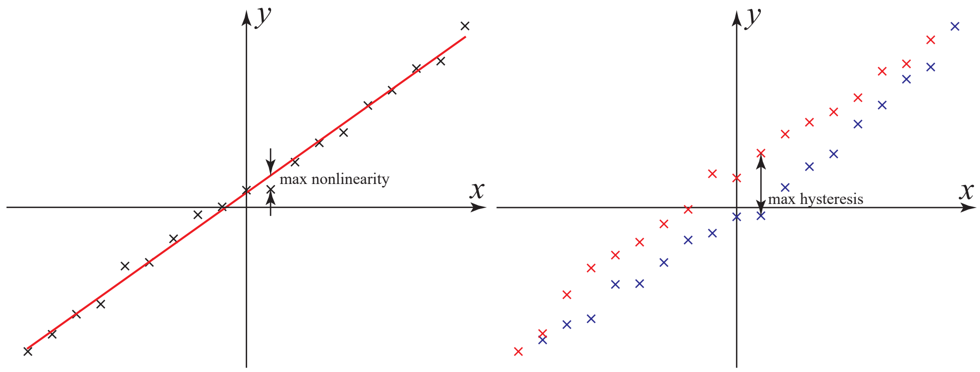


Figure 2.3: (Left) Maximum nonlinearity. (Right) Maximum hysteresis.

- Calibration errors
  - There are several terms used to describe measurement system errors including nonlinearity, hysteresis, and repeatability. Because there are multiple ways to define these errors, definitions must be provided when they are quantified for a system. Unfortunately, this doesn't seem to be common practice. Here we provide our definition of these terms.
  - Absolute error - "Absolute error, ϵ, is defined as the difference between the true value applied to a measurement system and the indicated value of the system ...[4]."
    * Generally, it's impossible to determine the absolute error exactly because the exact value of the measurand is almost always uncertain.
  - Nonlinearity error
    * Nonlinearity is a measure of the difference between a linear approximation to the calibration curve, $y_{\text{linear}}(x_i)$, and the calibration data, y_i
    * There are multiple notions/definitions of nonlinearity
      · Some authors/manufacturers use the term 'linearity' rather than 'nonlinearity'; see, for example, [11].
      · Primarily, definitions differ in how a line is constructed to fit the calibration data; see, for example, [16].
      · Further differences result from how nonlinearity is computed given multiple calibration points with the same measurand due to, for example, hysteresis and/or repeatability.
    * For simplicity, in ME 365 we evaluate nonlinearity with respect to a least-squares fit to all of the calibration data.
      · If there are multiple points with the same measurand, we evaluate nonlinearity using the point with the largest deviation.

- As a consequence of our definition, nonlinearity error may include contributions from hysteresis and repeatability errors.
- Note, the method presented in [15, 10] uses the average calibration curve (i.e., the average of all points, including upscale and downscale readings, with the same value of the measurand) to both construct the line fit and calculate nonlinearity.

* Nonlinearity -

$$\text{nonlinearity}(x_i) = y_i - y_{\text{linear}}(x_i); \tag{2.5}$$

see, for example, [4, 9].

* Maximum Nonlinearity - maximum nonlinearity is the largest difference between the linear fit and the calibration data. Sometimes maximum nonlinearity is expressed as a percentage of the output span (i.e., full scale deflection or f.s.d.). [9, 14]

$$\text{max nonlinearity (\%fsd)} = 100\% \frac{\max_i |\text{nonlinearity}(x_i)|}{\text{output span}} \tag{2.6}$$

* See Figure 2.3(Left).

– Hysteresis error

* "Hysteresis error refers to differences in the values found between going upscale and downscale in a sequential test [4]."
* To quantify hysteresis, generally the calibration inputs are applied in increasing and then decreasing order. According to [10, 15], for example, this process is repeated multiple times.
* Hysteresis is a form of nonlinearity.
* Although there are multiple definitions of hysteresis, here we compute maximum hysteresis considering all calibration cycles. In other words, the maximum hysteresis may result from a large output (while decreasing the input) in one cycle and a small output (while increasing the input) in another cycle.
* Maximum hysteresis as a percentage of the output span is given by

$$\text{max hysteresis (\%fsd)} = 100\% \frac{\max_i |y_{i,\text{increasing}} - y_{i,\text{decreasing}}|}{\text{output span}}; \tag{2.7}$$

see, for example, [9, 4, 14].

* See Figure 2.3(Right).
* As a consequence of our definition, hysteresis error may include contributions from deadband error (see, for example, [15] for a definition) and repeatability error.
* Example - a mass balance with friction at the pivot

– Repeatability error

* "The repeatability error is the maximum variability of successive measurements of the same value of input approached in the same direction [10]."

    * Repeatability is best characterized using statistical techniques including those described in Chapters 4 Probability and 5 Statistics and Uncertainty Propagation
  - Accuracy
    * It's not clear how accuracy is best defined in the context of a measurement system in which the input and output have totally different units. We propose the following approach in which we distinguish between input and output accuracy.
    * (Input) Accuracy
      · Maximum difference (over the intended intput range) between an exact value of a measurand and the value obtained when properly using the measurement system. (This implies the use of a calibration to convert from the output of the measurement system.)
      · This definition of accuracy seems consistent with, for example, [11, 17]
    * Output Accuracy
      · We define output accuracy as the maximum deviation in the output from the least-squares fit (i.e., our definition of maximum nonlinearity).
      · "Measured accuracy is determined ... by selecting the greatest positive and negative deviations of any measured value from the ideal value. Measured accuracy is reported as a maximum plus and maximum minus deviations in percent ideal output span [15]."
      · This definition of accuracy seems consistent with, for example, [10, 15]
    * For simplicity,
      · This definition is similar to how the term "accuracy" is used in [10].
      · Here, we assume that calibration data includes sufficient random deviations (i.e., repeatability errors) from the linear fit so as to bound the expected measurement error with a high level of confidence.
  - Depending on the definitions used and the calibration data, the errors measured by maximum nonlinearity and maximum hysteresis may be correlated. For example, the effects of hysteresis may contribute to both the maximum hysteresis and maximum nonlinearity of a system.

Example 2.1: Thermocouple

The following series of 12 calibration points are to be used to construct a linear calibration for a temperature measurement system with input T (° C) and output v (V). Assume the measurements were taken in the order provided.

#	T (° C)	v (V)
1	10	0.01957
2	18	0.04943
3	26	0.05348
4	34	0.08584
5	42	0.09710
6	50	0.11992
7	58	0.13735
8	66	0.14016
9	74	0.16162
10	82	0.18348
11	90	0.19920
12	98	0.21714

a. Find the input range and span.

- Input range: $[10,\ 98]$ °C
- Input span: $98 - 10$ °C$=88$ °C

b. Find the output range and span.

- Output range: $[0.01957,\ 0.21714]$ V
- Output span: $0.21714 - 0.01957$ V$= 0.19757$ V

c. Plot the calibration curve and find the least-squares linear fit.

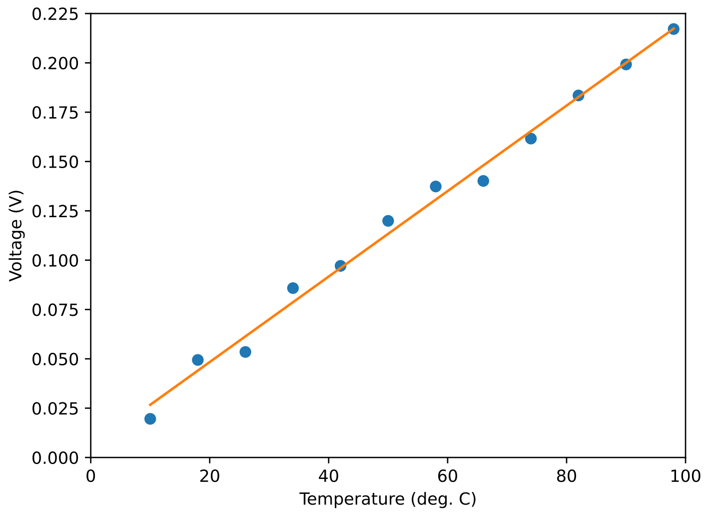


Figure 2.4:

- Linear fit: $v = \left(0.00216639\ \frac{\text{V}}{^\circ\text{C}}\right) T + (0.00503888\ \text{V})$

d. Find the sensitivity and bias.

- Sensitivity: $K_0 = 0.00216639\ \frac{\text{V}}{^\circ\text{C}}$
- Bias: $b = 0.00503888$ V

e. If the output is measured using a voltmeter with 0.00001 V resolution, find the input resolution.

$$\text{Input Resolution} = \frac{\text{Output Device Resolution}}{\text{Sensitivity}} \tag{2.8}$$

$$= \frac{0.00001\ \text{V}}{0.00216639\ \frac{\text{V}}{^\circ\text{C}}} \tag{2.9}$$

$$= 0.00461597404\ ^\circ\text{C} \tag{2.10}$$

Note, too many significant digits have been provided here.

f. Find the maximum nonlinearity in %fsd

$$\text{max nonlinearity (\%fsd)} = 3.991\% \tag{2.11}$$

g. Find the maximum hysteresis in %fsd

- The calibration data doesn't include upscale and downscale measurements so we cannot evaluate hysteresis

update with a script.

Example 2.2: Example with hysteresis and multiple calibration cycles

Example 2.3: example in which calibration points are excluded in the linear fit

Chapter 3

Analog to Digital Conversion

3.1 Analog and Digital Signals

- Analog signals
  - "... an analog signal is an electrical signal whose value varies in analogy with a physical quantity ... [18]"
  - "Analog describes a signal that is continuous in time [4]."
- Digital signals
  - "A digital signal ... can take only a finite number of values [18]."
  - "... a digital signal provides a quantized magnitude at discrete times [4]."
  - Attributes
    - Quantization - "Quantization assigns a single number to represent a range of magnitudes of a continuous signal [4]."
    - Discrete time - "... information about the magnitude of the signal is available only at discrete points in time. A discrete time signal usually results from measuring a continuous variable at finite time intervals [4]."
  - Binary signal - "The most common digital signals are binary signals. A binary signal is a signal that can take only one of two discrete values and is therefore characterized by transitions between two states [18]."
- Advantages and disadvantages
  -

3.2 Binary Signals and Representations

- Binary Signals
  - One convention, of many, is for nominally 0 V and 5 V to represent binary 0 and 1 respectively
  - "... the most widely used choice of levels are those used in TTL (transistor-transistor logic) and, in which positive true, or 1, corresponds to a minimum output level of +2.4 V ... and false, or 0, corresponds to a maximum output level of +0.4 V ... [19]"
- Binary information
  - Bit - "Binary systems use the binary digit or bit as the smallest unit of information [4]."
  - Groups of bits
    - Nibble - group of four bits [18]
    - Byte - group of eight bits [18]
    - Word - group of 16 bits [18]
    - Word - "... a collection of bits is used to express numerical information in the entity called a word [4]."
- Unipolar (see, for example, [19]; or non-negative) binary representations of integers; see, for example [20]
  - Straight binary
    - Straight binary numbers are represented with digits (i.e., bits) arranged from most to least significant bit
    - Range of an $n-$bit straight binary number $[0, ..., 2^n - 1]$
    - Conversion from decimal integer to straight binary integer; see, for example, [18]
      - Apply successive divisions by 2
      - The whole part (quotient) is used in subsequent divisions while the remainder is kept for the binary representation.

Example 3.1: Convert decimal integer to straight binary

division	quotient	remainder	significance
42/2	21	0	$0 \cdot 2^0$
21/2	10	1	$1 \cdot 2^1$
10/2	5	0	$0 \cdot 2^2$
5/2	2	1	$1 \cdot 2^3$
2/2	1	0	$0 \cdot 2^4$
1/2	0	1	$1 \cdot 2^5$

$(42)_{10} \to (101010)_2$

* Conversion from straight binary to decimainteger

Example 3.2: Convert straight binary to decimal integer

$$(101010)_2 \to (1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0)_{10} \quad (3.1)$$
$$\to (1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0)_{10} \quad (3.2)$$
$$\to (1 \cdot 32 + 0 \cdot 16 + 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 0 \cdot 1)_{10} \quad (3.3)$$
$$\to (42)_{10} \quad (3.4)$$

* Arithmatic
  · Addition works similar to decimal addition.
  · Subtraction works similar to decimal subtraction.
  · Overflow -

– Binary coded decimal (BCD)

* Typically, in binary coded decimal represention s, each digit of a decimal number is represented by a 4 bit binary number.
* Applications for binary coded decimals include 7 segment displays seen on, for example, many calculators or digital multimeters. It can be easier to map a binary coded decimal representation into the inputs of numeric displays of decimal numbers; see, for example, [19, 9].
* Because 4 bits can represent more than just the 10 digits from 0 to 9, binary coded decimal repersentations use more than the minimum number of bits necessary to represent a desimal number; see, for example, [18]

• Bipolar (see, for example, [19]) binary representations of integers; see, for example, [20]

– Sign-magnitude convention [18]

* Structure

    · Sign bit (most significant bit, or leading bit) - indicates the sign of the number: 0 for positive and 1 for negative; see, for example, [18]
    · Sign magnitude representations are inefficient in that they include two codes for zero: +0 and -0.
- Two's complement
  * Popularity
    · Most computers use two's complement to represent numbers and perform integer arithmatic; see, for example, [18, 10]
    · "The popularity of twos complement coding lies in the ease with which mathematical operations can be performed ... [19]."
  * Structure
    · Sign bit (most significant bit, or leading bit) indicates the sign of the number: 0 for non-negative and 1 for negative; see, for example, [18, 10]
    · Two's complement represents one more negative integer than positive integers: $[-2^{n-1}, ..., 2^{n-1} - 1]$. (According to [19], the -2^{n-1} is "not normally used in computations ...")
  * Non-negative integers are represented following the approach of straight binary, provided there are sufficient bits to represent the number (i.e., there are sufficient bits such that the leading bit is 0).
  * Sign change (from positive to negative or negative to positive) - sign changes are made using the following two's complement operation; see, for example, [10, 19]
    · Invert each bit (0 to 1 and 1 to 0)
    · Add 1 to the result
  * Addition - two's complement representations may be added following rules for addition of straight binary (with some exceptions)
    · carrout
    · overflow - check out: `http://sandbox.mc.edu/~bennet/cs110/tc/orules.html#:~:text=Two's%20Complement%20Overflow%20Rules&text=The%20rules%20for%20detecting%20overflow,result%2C%20the%20sum%20has%20overflowed.` and `http://sandbox.mc.edu/~bennet/cs110/textbook/module3_2.html`
  * Subtraction - subtraction is replaced by a sign change and addition; see, for example, [18]

- Decimal interpretation of 3-bit binary representations (following the notation of [20])

binary representation	straight binary →decimal	sign magnitude →decimal	two's complement →decimal
$(000)_2$	$(0)_{10}$	$(+0)_{10}$	$(0)_{10}$
$(001)_2$	$(1)_{10}$	$(+1)_{10}$	$(+1)_{10}$
$(010)_2$	$(2)_{10}$	$(+2)_{10}$	$(+2)_{10}$
$(011)_2$	$(3)_{10}$	$(+3)_{10}$	$(+3)_{10}$
$(100)_2$	$(4)_{10}$	$(-0)_{10}$	$(-4)_{10}$
$(101)_2$	$(5)_{10}$	$(-1)_{10}$	$(-3)_{10}$
$(110)_2$	$(6)_{10}$	$(-2)_{10}$	$(-2)_{10}$
$(111)_2$	$(7)_{10}$	$(-3)_{10}$	$(-1)_{10}$

Example 3.3:

Represent the decimal number $(5)_{10}$ using a four bit two's complement binary number

- Convert magnitude to binary:

division	quotient	remainder	significance
5/2	2	1	$1 \cdot 2^0$
2/2	1	0	$0 \cdot 2^1$
1/2	0	1	$1 \cdot 2^2$

  – Magnitude: $(5)_{10} = (0101)_2$

- Account for the sign using two's complement:
  – No change needed because the number is positive.
  – Result: $(5)_{10} = (0101)_2$

Example 3.4:

Represent the decimal number $(-5)_{10}$ using a four bit two's complement binary number

- Convert magnitude to binary:

division	quotient	remainder	significance
5/2	2	1	$1 \cdot 2^0$
2/2	1	0	$0 \cdot 2^1$
1/2	0	1	$1 \cdot 2^2$

  – Magnitude: $(5)_{10} = (0101)_2$

- Account for the sign using two's complement:
  - Invert each bit: $(1010)_2$
  - Add one: $(-5)_{10} = (1011)_2$

Example 3.5:

Represent the four bit two's complement binary number $(0101)_2$ using a decimal number

- Account for the sign using two's complement:
  - No change needed because the number is positive.
- Convert to decimal magnitude

$$(0101)_2 = \left(0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0\right)_{10} \tag{3.5}$$
$$= (5)_{10} \tag{3.6}$$

Example 3.6:

Represent the four bit two's complement binary number $(1011)_2$ using a decimal number

- Account for the sign using two's complement:
  - Invert each bit: $(0100)_2$
  - Add one: $(1011)_2 = -(0101)_2$
- Convert to decimal magnitude

$$-(0101)_2 = -\left(0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0\right)_{10} \tag{3.7}$$
$$= (-5)_{10} \tag{3.8}$$

Example 3.7:

Find the decimal equivalent to the sum of two's complement binary numbers: $(0010)_2 + (1011)_2$

- Method 1:

– Add binary numbers

$$\begin{array}{r} (0010)_2 \\ \underline{+(1011)_2} \\ (1101)_2 \end{array} \tag{3.9}$$

– Convert to decimal

$$\begin{aligned}
(1101)_2 &= -(0010 + 0001)_2 && (3.10)\\
&= -(0011)_2 && (3.11)\\
&= -(0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0)_{10} && (3.12)\\
&= (-3)_{10} && (3.13)
\end{aligned}$$

- Method 2:

– Convert to decimal

$$\begin{aligned}
(0010)_2 &= (0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0)_{10} && (3.14)\\
&= (2)_{10} && (3.15)\\
(1011)_2 &= -(0100 + 0001)_2 && (3.16)\\
&= -(0101)_2 && (3.17)\\
&= -(0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0)_{10} && (3.18)\\
&= (-5)_{10} && (3.19)
\end{aligned}$$

– Add decimal numbers

$$\begin{aligned}
(0010)_2 + (1011)_2 &= (2)_{10} + (-5)_{10} && (3.20)\\
&= (-3)_{10} && (3.21)
\end{aligned}$$

Example 3.8:

Find the decimal equivalent to the difference of two's complement binary numbers: $(0010)_2 - (1011)_2$

- Method 1:

- Convert to an addition problem

$$
\begin{aligned}
(0010)_2 - (1011)_2 &= (0010)_2 + (-(1011)_2) && (3.22)\\
&= (0010)_2 + (0100 + 0001)_2 && (3.23)\\
&= (0010)_2 + (0101)_2 && (3.24)
\end{aligned}
$$

- Add binary numbers

$$
\begin{array}{r}
(0010)_2 \\
\underline{+(0101)_2} \\
(0111)_2
\end{array}
\qquad (3.25)
$$

- Convert to decimal

$$
\begin{aligned}
(0111)_2 &= (0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0)_{10} && (3.26)\\
&= (7)_{10} && (3.27)
\end{aligned}
$$

- Method 2:
  - Convert to decimal

$$
\begin{aligned}
(0010)_2 &= (0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0)_{10} && (3.28)\\
&= (2)_{10} && (3.29)\\
(1011)_2 &= -(0100 + 0001)_2 && (3.30)\\
&= -(0101)_2 && (3.31)\\
&= -(0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0)_{10} && (3.32)\\
&= (-5)_{10} && (3.33)
\end{aligned}
$$

  - Subtract decimal numbers

$$
\begin{aligned}
(0010)_2 - (1011)_2 &= (2)_{10} - (-5)_{10} && (3.34)\\
&= (7)_{10} && (3.35)
\end{aligned}
$$

3.3 Analog to Digital Conversion

- Range and resolution; see, for example, [20]
  - Range

* Minimum input voltage
  · Minimum input voltage $V_{\min}$
  · Nominal minimum input voltage $V_{\min,\text{nom}}$
  · Relationship
$$V_{\min} = V_{\min,\text{nom}} \tag{3.36}$$
* Maximum input voltage
  · Maximum input voltage $V_{\max}$
  · Nominal maximum input voltage $V_{\max,\text{nom}}$
  · Relationship
$$V_{\max} = V_{\max,\text{nom}} - Q \tag{3.37}$$
* Note, reference [20] emphasizes the distinction between 'true span' and 'nominal span'
  · "In many practical ADC's the input range is quoted as being ± R Volts but in fact the true input range is –R to R-Q Volts [20]."
  · The myRIO spec sheet, [21], uses the terms 'Nominal Range,' 'Maximum Positive Reading' and 'Maximum Negative Reading'
  · "The resolution of an ADC is usually expressed as the number of bits in its digital output code. For example, an ADC with an n-bit resolution has 2^n possible digital codes which define 2^n step levels. However, since the first (zero) step and the last step are only one half of a full width, the full-scale range (FSR) is divided into 2^n–1 step widths [22]."
  · "It is important to note that the analog value represented by the all-ones code is not full-scale (abbreviated FS), but FS – 1 LSB. This is a common convention in data conversion notation and applies to both ADCs and DACs [19]."

– Resolution (or 'quantization interval' [9, 20]) of an n-bit analog to digital converter
$$Q = \frac{V_{\max,\text{nom}} - V_{\min,\text{nom}}}{2^n} \tag{3.38}$$
$$= \frac{V_{\max} - V_{\min}}{2^n - 1} \tag{3.39}$$
* [9], for example, provides a formula using the true range
* [13], for example, provides a formula using the nominal span.
* If we think of the curves in Figure 3.1 as calibration curves, the resolution (i.e., input resolution) of the ADC is given by Q and measured in volts while the output resolution is 1. Often the output integer of an ADC is mapped into a quantity with dimensions such that the input and output resolutions are both effectively Q.

– Quantization error [4] or input resolution error [10]
$$\text{Quantization error} = \frac{Q}{2} \tag{3.40}$$

- Mapping between input voltage V_{in} and output decimal integer I_{out}
  - Input (voltage) to output (decimal integer)
    * Only non-negative numbers coded (unsigned); see, for example, [10, 20]

$$I_{\text{out}} = \text{round}\left\{\frac{V_{\text{in}} - V_{\text{min,nom}}}{Q}\right\} \tag{3.41}$$

      assuming $V_{\text{min}} \leq V_{\text{in}} \leq V_{\text{max}}$
    * Positive and negative numbers coded (signed); see, for example, [20]

$$I_{\text{out}} = \text{round}\left\{\frac{V_{\text{in}}}{Q}\right\} \tag{3.42}$$

      assuming $V_{\text{min}} \leq V_{\text{in}} \leq V_{\text{max}}$
      Note, [10] presents a similar formula which differs in that it incorporates a two's complement conversion.
  - Output (decimal integer) to input (voltage)
    * Only non-negative numbers coded (unsigned); see, for example, [20, 10]

$$V_{\text{in}} = I_{\text{out}}Q + V_{\text{min,nom}} \pm \frac{Q}{2} \tag{3.43}$$

      assuming $V_{\text{min}} \leq V_{\text{in}} \leq V_{\text{max}}$
    * Positive and negative numbers coded (signed); see, for example, [20]

$$V_{\text{in}} = I_{\text{out}}Q \pm \frac{Q}{2} \tag{3.44}$$

      assuming $V_{\text{min}} \leq V_{\text{in}} \leq V_{\text{max}}$
      Note, [10] presents a similar formula which differs in that it incorporates a two's complement conversion.
  - Saturation
    * Saturation (or clipping) occurs when an input voltage is above or below the range of an analog to digital converter; see, for example, [4, 10, 20].
  - Warning, be careful with your calculations particularly if you are rounding Q. Rounding errors can build up to significant levels in these problems.

- To get the most out of an analog to digital converter, typically we want the signal to use nearly all of the input range leaving a little cushion to avoid saturation/clipping. Amplifiers may be used to fit the signal to the input range of an analog to digital converter.

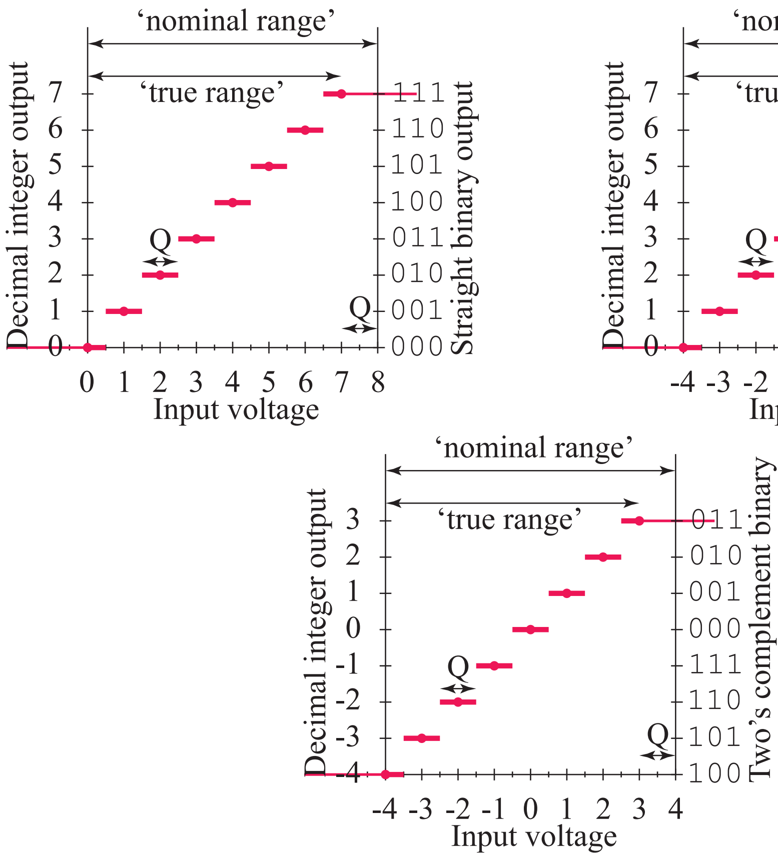


Figure 3.1: (To-left) Input-output relationship for an analog to digital converter with nominal input range of 0 to 8 V and only non-negative numbers coded. (Top-right) Input-output relationship for an analog to digital converter with nominal input range of -4 to 4 V and only non-negative numbers coded. (Bottom) Input-output relationship for an analog to digital converter with nominal input range of -4 to 4 V and positive and negative numbers coded. See, for example, [13]

Chapter 4

Probability

This chapter serves as a brief review and is a work in progress that may contain significant mistakes. Be sure to consult appropriate textbooks and follow required standards when performing critical statistical/uncertainty analyses.

4.1 Experiments, Random Variabless and Probability Functions

- Experiment
  - An experiment is a procedure that yields an outcome (see, for example, [23]) that is, in general, random.
- Random variable
  - Random variable -
    - A random variable is a function that maps the outcome of an experiment (or procedure) into a numerical value; see, for example, [24, 10].
  - Discrete vs. continuous random variables
    - Continuous random variables can output "... any real value in an interval ... [24]."
    - Discrete random variables can output any of a discrete set of real values
  - Generally, we will represent random variables with upper-case symbols (e.g., X); see, for example, [24]. (However, not all upper-case symbols necessarily represent random variables.)
- Probability functions and corresponding notation
  - Probability $P[\cdot]$ -

  * Probability of the experimental outcome described within the bracket
  * If an experiment were to be repeated many times (approaching infinity), the probability of a specific output is the ratio of the occurrences of that outcome to the number of repetitions of the experiment.
- Probability mass function: (pertains to discrete random variables)
  * The probability mass function, $p_X(x)$, is a function that maps each of the outcomes of a discrete random variable to its associated probability.

$$p_X(x) = P[X = x] \tag{4.1}$$

- Probability density function: $f_X(x)$
  * The probability density function, $f_X(x)$, maps an outcome of a random variable, X, to the ratio of the probability associated with getting the outcome within a small increment of the random variable to the increment.

$$f_X(x) = \lim_{\Delta x \to 0} \frac{P\left[x - \frac{\Delta x}{2} < X < x + \frac{\Delta x}{2}\right]}{\Delta x} \tag{4.2}$$

  * The probability that the outcome of a random process is among the available outcomes must be 1.

$$\int_{-\infty}^{\infty} f_X(x)dx = P[-\infty < X < \infty] \tag{4.3}$$

$$= 1 \tag{4.4}$$

- Cumulative distribution function:

$$F_X(x) = P[X \leq x] \tag{4.5}$$

$$= \int_{-\infty}^{x} f_X(\xi)d\xi \tag{4.6}$$

- Examples:
  * Example: Probability that a continuous random variable, X, is between x_1 and x_2:

$$P[x_1 < X < x_2] = \int_{x_1}^{x_2} f_X(\xi)d\xi \tag{4.7}$$

$$= F_X(x_2) - F_X(x_1) \tag{4.8}$$

  * Example: Probability that a continuous random variable, X, is less than x_1:

$$P[X < x_1] = \int_{-\infty}^{x_1} f_X(\xi)d\xi \tag{4.9}$$

$$= F_X(x_1) - \cancelto{0}{F_X(-\infty)} \tag{4.10}$$

$$= F_X(x_1) \tag{4.11}$$

* Example: Probability that a continuous random variable, X, is equal to x_1:

$$\begin{aligned} P[X = x_1] &= \int_{x_1}^{x_1} f_X(\xi) d\xi & (4.12) \\ &= F_X(x_1) - F_X(x_1) & (4.13) \\ &= 0 & (4.14) \end{aligned}$$

  · An exception may occur when the probability density function is the impulse function

- Common probability distributions
  - Uniform distribution
    * A uniform distribution is completely described by its lower and upper bounds, a and b respectively.
    * A random variable that follows a uniform distribution is equally likely to take on any value in the range from a to b. (The probability density function is a constant in the range from a to b.)
    * Probability density function

$$f_X(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \leq x < b \\ 0 & b \leq x \end{cases} \tag{4.15}$$

    * Cumulative distribution function

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a}(x-a) & a \leq x < b \\ 1 & b \leq x \end{cases} \tag{4.16}$$

  - Normal or Gaussian distribution
    * A normal or Gaussian distribution is bell shaped and completely described by its mean, μ, and standard deviation, σ.
    * Probability density function

$$f_X(x) = \frac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}} \tag{4.17}$$

    * Cumulative distribution function

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} p(\xi) d\xi & (4.18) \\ &= \int_{-\infty}^{x} \frac{\exp\left(-\frac{(\xi-\mu)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}} d\xi & (4.19) \end{aligned}$$

- Standard normal distribution ($\mu = 0$ and $\sigma = 1$)
  - Probability density function

$$f_Z(z) = \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} \tag{4.20}$$

  - A normally distributed random variable, X, with arbitrary mean, μ and standard deviation, σ, can be mapped into a random variable, Z, with standard normal distrubution using the change of coordinates:

$$Z = \frac{X - \mu}{\sigma} \tag{4.21}$$

  - Used as the basis for normal distribution tables
    - The two-sided area under the standard normal distribution (see Figure 4.1) is tabulated in Table 4.1
    - Note, statistical tables come in a variety of forms.
    - Not all standard normal distribution tables are two sided like this. now how to use the table you have. $z = 1.00 \rightarrow 68.27\%$,
      $z = 1.96 \rightarrow 95.00\%$,
      $z = 2.00 \rightarrow 95.45\%$,
      $z = 2.58 \rightarrow 99\%$,
      $z = 3.00 \rightarrow 99.73\%$,
      $z = 4.00 \rightarrow 99.99\%$

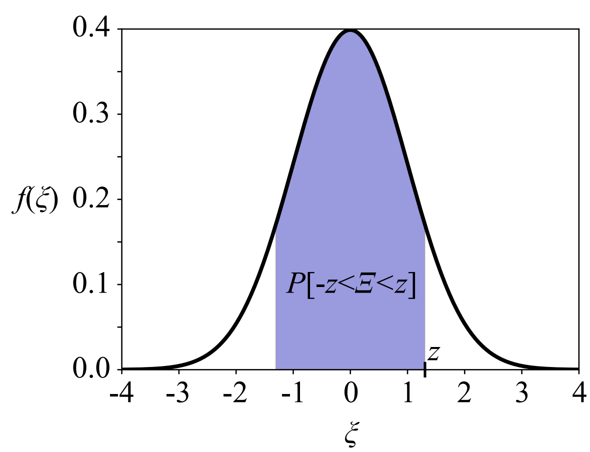


Figure 4.1: Probability density function for the standard normal distribution with two-sided area shaded.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0080	0.0160	0.0239	0.0319	0.0399	0.0478	0.0558	0.0638	0.0717
0.1	0.0797	0.0876	0.0955	0.1034	0.1113	0.1192	0.1271	0.1350	0.1428	0.1507
0.2	0.1585	0.1663	0.1741	0.1819	0.1897	0.1974	0.2051	0.2128	0.2205	0.2282
0.3	0.2358	0.2434	0.2510	0.2586	0.2661	0.2737	0.2812	0.2886	0.2961	0.3035
0.4	0.3108	0.3182	0.3255	0.3328	0.3401	0.3473	0.3545	0.3616	0.3688	0.3759
0.5	0.3829	0.3899	0.3969	0.4039	0.4108	0.4177	0.4245	0.4313	0.4381	0.4448
0.6	0.4515	0.4581	0.4647	0.4713	0.4778	0.4843	0.4907	0.4971	0.5035	0.5098
0.7	0.5161	0.5223	0.5285	0.5346	0.5407	0.5467	0.5527	0.5587	0.5646	0.5705
0.8	0.5763	0.5821	0.5878	0.5935	0.5991	0.6047	0.6102	0.6157	0.6211	0.6265
0.9	0.6319	0.6372	0.6424	0.6476	0.6528	0.6579	0.6629	0.6680	0.6729	0.6778
1.0	0.6827	0.6875	0.6923	0.6970	0.7017	0.7063	0.7109	0.7154	0.7199	0.7243
1.1	0.7287	0.7330	0.7373	0.7415	0.7457	0.7499	0.7540	0.7580	0.7620	0.7660
1.2	0.7699	0.7737	0.7775	0.7813	0.7850	0.7887	0.7923	0.7959	0.7995	0.8029
1.3	0.8064	0.8098	0.8132	0.8165	0.8198	0.8230	0.8262	0.8293	0.8324	0.8355
1.4	0.8385	0.8415	0.8444	0.8473	0.8501	0.8529	0.8557	0.8584	0.8611	0.8638
1.5	0.8664	0.8690	0.8715	0.8740	0.8764	0.8789	0.8812	0.8836	0.8859	0.8882
1.6	0.8904	0.8926	0.8948	0.8969	0.8990	0.9011	0.9031	0.9051	0.9070	0.9090
1.7	0.9109	0.9127	0.9146	0.9164	0.9181	0.9199	0.9216	0.9233	0.9249	0.9265
1.8	0.9281	0.9297	0.9312	0.9328	0.9342	0.9357	0.9371	0.9385	0.9399	0.9412
1.9	0.9426	0.9439	0.9451	0.9464	0.9476	0.9488	0.9500	0.9512	0.9523	0.9534
2.0	0.9545	0.9556	0.9566	0.9576	0.9586	0.9596	0.9606	0.9615	0.9625	0.9634
2.1	0.9643	0.9651	0.9660	0.9668	0.9676	0.9684	0.9692	0.9700	0.9707	0.9715
2.2	0.9722	0.9729	0.9736	0.9743	0.9749	0.9756	0.9762	0.9768	0.9774	0.9780
2.3	0.9786	0.9791	0.9797	0.9802	0.9807	0.9812	0.9817	0.9822	0.9827	0.9832
2.4	0.9836	0.9840	0.9845	0.9849	0.9853	0.9857	0.9861	0.9865	0.9869	0.9872
2.5	0.9876	0.9879	0.9883	0.9886	0.9889	0.9892	0.9895	0.9898	0.9901	0.9904
2.6	0.9907	0.9909	0.9912	0.9915	0.9917	0.9920	0.9922	0.9924	0.9926	0.9929
2.7	0.9931	0.9933	0.9935	0.9937	0.9939	0.9940	0.9942	0.9944	0.9946	0.9947
2.8	0.9949	0.9950	0.9952	0.9953	0.9955	0.9956	0.9958	0.9959	0.9960	0.9961
2.9	0.9963	0.9964	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972
3.0	0.9973	0.9974	0.9975	0.9976	0.9976	0.9977	0.9978	0.9979	0.9979	0.9980
3.1	0.9981	0.9981	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986
3.2	0.9986	0.9987	0.9987	0.9988	0.9988	0.9988	0.9989	0.9989	0.9990	0.9990
3.3	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.4	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0.9995
3.5	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0.9997
3.6	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998	0.9998
3.7	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

Table 4.1: Probability within $\pm z$; $P[-z < Z < z]$. Table generated using Scipy and Python.

Example 4.1:

Given a uniformly distributed random variable, X, from $a = 1$ to $b = 3$, find:

a. an expression for the probability density function,

- The probability density function will be 0 outside 1 to 3 and constant within 1 to 3

$$f_X(x) = \begin{cases} 0 & x < 1 \\ c & 1 \leq x < 3 \\ 0 & 3 \leq x \end{cases} \tag{4.22}$$

- The total area under the curve must be 1

$$1 = \int_{-\infty}^{\infty} f_X(x) \tag{4.23}$$
$$= \int_{1}^{3} c dx \tag{4.24}$$
$$= cx|_1^3 \tag{4.25}$$
$$1 = 2c \tag{4.26}$$
$$c = \frac{1}{2} \tag{4.27}$$
$$f_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \leq x < 3 \\ 0 & 3 \leq x \end{cases} \tag{4.28}$$

b. an expression for the cumulative distribution function,

- the cumulative distribution function is the integral of the probability density function

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi) d\xi \tag{4.29}$$
$$= \int_{-\infty}^{x} \left\{ \begin{array}{ll} 0 & \xi < 1 \\ \frac{1}{2} & 1 \leq \xi < 3 \\ 0 & 3 \leq \xi \end{array} \right\} d\xi \tag{4.30}$$
$$= \begin{cases} 0 & x < 1 \\ \frac{1}{2}(x-1) & 1 \leq x < 3 \\ 1 & 3 \leq x \end{cases} \tag{4.31}$$

c. $P[X=2]$ (i.e., the probability that $X=2$), and

$$\begin{aligned} P[X=2] &= \int_2^2 f_X(x)dx && (4.32)\\ &= F_X(2)-F_X(2) && (4.33)\\ &= 0 && (4.34) \end{aligned}$$

d. $P[1.5<X<2]$ (i.e., the probability that $1.5<X<2$).

$$\begin{aligned} P[1.5<X<2] &= \int_{1.5}^2 f_X(x)dx && (4.35)\\ &= F_X(2)-F_X(1.5) && (4.36)\\ &= \frac{1}{2}(2-1)-\frac{1}{2}(1.5-1) && (4.37)\\ &= \frac{1}{2}-\frac{1}{4} && (4.38)\\ &= \frac{1}{4} && (4.39) \end{aligned}$$

Example 4.2:

Given a normally distributed random variable, X, with mean $\mu=2$ and standard deviation $\sigma=\frac{1}{2}$, find:

a. $P[X=2]$ (i.e., the probability that $X=2$), and

$$\begin{aligned} P[X=2] &= \int_2^2 f_X(x)dx && (4.40)\\ &= F_X(2)-F_X(2) && (4.41)\\ &= 0 && (4.42) \end{aligned}$$

b. $P[1.0<X<1.5]$ (i.e., the probability that $1.0<X<1.5$).

- map the values $x_1 = 1.0$ into $x_2 = 1.5$ into standard normal distribution

$$
\begin{aligned}
z &= \frac{x-\mu}{\sigma} && (4.43)\\
z_1 &= \frac{x_1-\mu}{\sigma} && (4.44)\\
&= \frac{1-2}{0.5} && (4.45)\\
&= -2 && (4.46)\\
z_2 &= \frac{x_2-\mu}{\sigma} && (4.47)\\
&= \frac{1.5-2}{0.5} && (4.48)\\
&= -1 && (4.49)
\end{aligned}
$$

- calculate probability and manipulate expression into terms that can be read from Table 4.1

$$
\begin{aligned}
P[1.0 < X < 1.5] &= P[-2 < Z < -1] && (4.50)\\
&= \int_{-2}^{-1} f_Z(z)dz && (4.51)\\
&= \int_{-2}^{0} f_Z(z)dz - \int_{-1}^{0} f_Z(z)dz && (4.52)\\
&= \frac{2}{2}\left(\int_{-2}^{0} f_Z(z)dz - \int_{-1}^{0} f_Z(z)dz\right) && (4.53)\\
&= \frac{1}{2}\left(\int_{-2}^{2} f_Z(z)dz - \int_{-1}^{1} f_Z(z)dz\right) && (4.54)
\end{aligned}
$$

- Read values from Table 4.1

$$
\begin{aligned}
P[1.0 < X < 1.5] &= \frac{1}{2}(0.9545 - 0.6827) && (4.55)\\
&= 0.1359 && (4.56)
\end{aligned}
$$

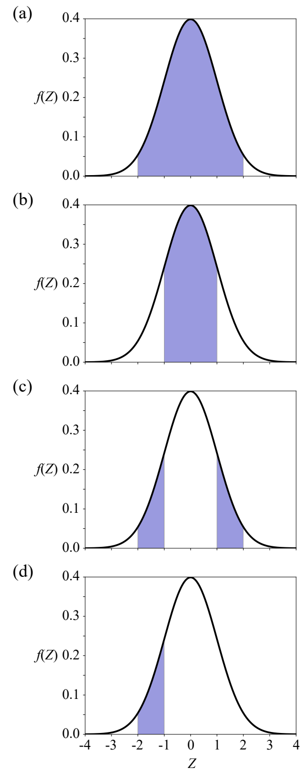


Figure 4.2: (a) $P[-2 < Z < 2]$. (b) $P[-1 < Z < 1]$. (c) $P[-2 < Z < 2] - P[-1 < Z < 1]$. (d) $\frac{P[-2<Z<2]-P[-1<Z<1]}{2}$

c. $P[1.0 < X < 2.5]$ (i.e., the probability that $1.0 < X < 2.5$).

- map the values $x_1 = 1.0$ into $x_2 = 2.5$ into standard normal distribution

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} & (4.57) \\ z_1 &= \frac{x_1-\mu}{\sigma} & (4.58) \\ &= \frac{1-2}{0.5} & (4.59) \\ &= -2 & (4.60) \\ z_2 &= \frac{x_2-\mu}{\sigma} & (4.61) \\ &= \frac{2.5-2}{0.5} & (4.62) \\ &= 1 & (4.63) \end{aligned}$$

- calculate probability and manipulate expression into terms that can be read from Table 4.1

$$\begin{aligned} P[1.0 < X < 2.5] &= P[-2 < Z < 1] & (4.64) \\ &= \int_{-2}^{1} f_Z(z)dz & (4.65) \\ &= \int_{-2}^{0} f_Z(z)dz + \int_{0}^{1} f_Z(z)dz & (4.66) \\ &= \frac{2}{2}\left(\int_{-2}^{0} f_Z(z)dz + \int_{0}^{1} f_Z(z)dz\right) & (4.67) \\ &= \frac{1}{2}\left(\int_{-2}^{2} f_Z(z)dz + \int_{-1}^{1} f_Z(z)dz\right) & (4.68) \end{aligned}$$

- Read values from Table 4.1

$$\begin{aligned} P[1.0 < X < 2.5] &= \frac{1}{2}(0.9545 + 0.6827) & (4.69) \\ &= 0.8186 & (4.70) \end{aligned}$$

4.2 Expectation

- The expected value of a random variable is its mean; see, for example, [24].
- Discrete random variables
  - Expectation of a random variable
    * Mean (or population mean) of a random variable X with probability mass function $p_X(x_i) = P[X = x_i]$

$$\begin{aligned} \mu_X &= E[X] && (4.71)\\ &= \sum_{i=1}^{n} x_i p_X(x_i) && (4.72)\\ &= \sum_{i=1}^{n} \frac{x_i}{n} && (4.73) \end{aligned}$$

      · (expected value of the random variable, 'center of mass of a distribution' [24])
    * Variance σ_X^2 and standard deviation σ_X (or population variance and standard deviation)

$$\begin{aligned} \sigma_X^2 &= E\left[(X - \mu_X)^2\right] && (4.74)\\ &= \sum_{i=1}^{n} (x_i - \mu_X)^2 p_X(x_i) && (4.75)\\ &= \sum_{i=1}^{n} \frac{(x_i - \mu_X)^2}{n} && (4.76) \end{aligned}$$

      · ("... variance tells us how spread out a distribution is [24].")
  - Expectation of a function of a single random variable
    * Given some function, $g(X)$, of a discrete random variable, X, with probability mass function, $p_X(x_i) = P[X = x_i]$, the expected value is

$$\begin{aligned} \mu_{g(X)} &= E[g(X)] && (4.77)\\ &= \sum_{i=1}^{n} g(x_i) p_X(x_i); && (4.78) \end{aligned}$$

      see, for example, [24]. If the population is infinite, then the upper limit of the sum is replaced with ∞.
    * Special case: small σ_X

· Expected value of $g(X)$ (using a Taylor series expansion for $g(X)$)

$$\begin{aligned}
\mu_{g(X)} &= \sum_{i=1}^{n} g(x_i) p_X(x_i) && (4.79) \\
&= \sum_{i=1}^{n} \left(g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} (x_i - \mu_X) + ... \right) p_X(x_i) && (4.80)
\end{aligned}$$

for $\sigma_X^2 \ll \left| \frac{g(\mu_X)}{\frac{d^2 g}{dX^2}} \right|$ (Do we need to assume that g is a continuous function?)

$$\begin{aligned}
\mu_{g(X)} &\approx \sum_{i=1}^{n} \left(g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} (x_i - \mu_X) \right) p_X(x_i) && (4.81) \\
&\approx g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} \sum_{i=1}^{n} (x_i - \mu_X) p_X(x_i) && (4.82) \\
&\approx g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} \left(\cancel{\sum_{i=1}^{n} x_i p_X(x_i) - \mu_X} \right) && (4.83) \\
&\approx g(\mu_X) && (4.84)
\end{aligned}$$

· variance of $g(X)$ (using a Taylor series expansion for $g(X)$)

$$\begin{aligned}
\sigma_{g(X)}^2 &= E\left[(g(X) - E[g(x)])^2 \right] && (4.85) \\
&= \sum_{i=1}^{n} (g(x_i) - E[g(x)])^2 \, p_X(x_i) && (4.86) \\
&= \sum_{i=1}^{n} \left(g(x_i) - \mu_{g(X)} \right)^2 p_X(x_i) && (4.87) \\
&= \sum_{i=1}^{n} \left(g^2(x_i) - 2g(x_i)\mu_{g(X)} + \mu_{g(x)}^2 \right) p_X(x_i) && (4.88) \\
&= -2\mu_{g(X)}^2 + \mu_{g(X)}^2 + \sum_{i=1}^{n} g^2(x_i) p_X(x_i) && (4.89) \\
&= -\mu_{g(X)}^2 + \sum_{i=1}^{n} g^2(x_i) p_X(x_i) && (4.90) \\
&= -\mu_{g(X)}^2 + \sum_{i=1}^{n} \left(g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} (x_i - \mu_X) + ... \right)^2 p_X(x_i) && (4.91)
\end{aligned}$$

for $\sigma_X^2 \ll \left|\frac{g(\mu_X)}{\frac{d^2 g}{dX^2}}\right|$ (Do we need to assume that g is a continuous function?)

$$\sigma^2_{g(X)} \approx -\mu^2_{g(X)} + \sum_{i=1}^{n}\left(g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X}(x_i - \mu_X)\right)^2 p_X(x_i) \tag{4.92}$$

$$\approx -\mu^2_{g(X)} + \sum_{i=1}^{n}\left(g^2(\mu_X) + 2g(\mu_X)\left.\frac{dg}{dX}\right|_{\mu_X}\cancel{(x_i - \mu_X)}\right.$$

$$\left. + \left.\frac{dg}{dX}\right|^2_{\mu_X}(x_i - \mu_X)^2\right)p_X(x_i)] \tag{4.93}$$

$$\approx -\mu^2_{g(X)} + \sum_{i=1}^{n}\left(g^2(\mu_X) + \left.\frac{dg}{dX}\right|^2_{\mu_X}(x_i - \mu_X)^2\right)p_X(x_i) \tag{4.94}$$

$$\approx \cancel{-\mu^2_{g(X)}} + \cancel{g^2(\mu_X)} + \left.\frac{dg}{dX}\right|^2_{\mu_X}\sigma_X^2 \tag{4.95}$$

$$\approx \left.\frac{dg}{dX}\right|^2_{\mu_X}\sigma_X^2 \tag{4.96}$$

– Expectation of a function of multiple random variables

  * Given some function, $g(X, Y)$, of two discrete uncorrelated random variables, X and Y, with probability mass functions, $p_X(x_i) = P[X = x_i]$ and $p_Y(y_j) = P[Y = y_j]$, the expected value is

$$\mu_{g(X,Y)} = E[g(X, Y)] \tag{4.97}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} g(x_i, y_j)p_X(x_i)p_Y(y_j) \tag{4.98}$$

  * Special case: small σ_X and σ_Y
    · Expected value of $g(X, Y)$ (using Taylor series expansion for $g(X, Y)$)

$$\mu_{g(X,Y)} = \sum_{i=1}^{n}\sum_{j=1}^{n} g(x_i, y_j)p_X(x_i)p_Y(y_j) \tag{4.99}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(g(\mu_X, \mu_Y) + \left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}(x_i - \mu_X)\right.$$

$$\left. + \left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}(y_j - \mu_Y) + ...\right)p_X(x_i)p_Y(y_j) \tag{4.100}$$

for small σ_X and σ_Y (Do we need to assume that g is a continuous function?)

$$\begin{aligned}
\mu_{g(X,Y)} &\approx \sum_{i=1}^{n}\sum_{j=1}^{n}\left(g(\mu_X,\mu_Y) + \left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}\cancel{(x_i-\mu_X)}\right. \\
&\quad \left. + \left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}\cancel{(y_j-\mu_Y)}\right)p_X(x_i)p_Y(y_j) &(4.101)\\
&\approx g(\mu_X,\mu_Y) &(4.102)
\end{aligned}$$

· variance of $g(X,Y)$ (using a Taylor series expansion for $g(X,Y)$)

$$\begin{aligned}
\sigma^2_{g(X,Y)} &= E\left[(g(X,Y)-E[g(X,Y)])^2\right] &(4.103)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(g(x_i,y_j)-E[g(X,Y)])^2\,p_X(x_i)p_Y(y_j) &(4.104)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(g(x_i,y_j)-\mu_{g(X,Y)}\right)^2 p_X(x_i)p_Y(y_j) &(4.105)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(g^2(x_i,y_j)-2g(x_i,y_j)\mu_{g(X,Y)}+\mu^2_{g(X,Y)}\right)p_X(x_i)p_Y(y_j) &(4.106)\\
&= -2\mu^2_{g(X,Y)}+\mu^2_{g(X,Y)}+\sum_{i=1}^{n}\sum_{j=1}^{n}g^2(x_i,y_j)p_X(x_i)p_Y(y_j) &(4.107)\\
&= -\mu^2_{g(X,Y)}+\sum_{i=1}^{n}\sum_{j=1}^{n}g^2(x_i,y_j)p_X(x_i)p_Y(y_j) &(4.108)\\
&= -\mu^2_{g(X,Y)}+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(g(\mu_X,\mu_Y)+\left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}(x_i-\mu_X)\right.\\
&\quad \left.+\left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}(y_i-\mu_Y)+\ldots\right)^2 p_X(x_i)p_Y(y_j) &(4.109)
\end{aligned}$$

for $\sigma_X^2 \ll \left|\frac{g(\mu_X)}{\frac{\partial^2 g}{\partial X^2}}\right|$ (Do we need to assume that g is a continuous function?)

$$\begin{aligned}
\sigma^2_{g(X)} &\approx -\mu^2_{g(X,Y)} + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(g(\mu_X,\mu_Y) + \left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}(x_i-\mu_X)\right. \\
&\quad \left. + \left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}(y_i-\mu_Y)\right)^2 p_X(x_i)p_Y(y_j) && (4.110)\\
&\approx -\mu^2_{g(X,Y)} + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(g^2(\mu_X,\mu_Y) + \left.\frac{\partial g}{\partial X}\right|^2_{\mu_X,\mu_Y}(x_i-\mu_X)^2\right. \\
&\quad \left. + \left.\frac{\partial g}{\partial Y}\right|^2_{\mu_X,\mu_Y}(y_j-\mu_Y)^2\right) p_X(x_i)p_Y(y_j) && (4.111)\\
&\approx \cancel{-\mu^2_{g(X,Y)}} + \cancel{g^2(\mu_X,\mu_Y)} + \left.\frac{\partial g}{\partial X}\right|^2_{\mu_X,\mu_Y}\sigma_X^2 + \left.\frac{\partial g}{\partial Y}\right|^2_{\mu_X,\mu_Y}\sigma_Y^2 && (4.112)\\
&\approx \left.\frac{\partial g}{\partial X}\right|^2_{\mu_X,\mu_Y}\sigma_X^2 + \left.\frac{\partial g}{\partial Y}\right|^2_{\mu_X,\mu_Y}\sigma_Y^2 && (4.113)
\end{aligned}$$

- Continuous random variables
  - Expectation of a random variable
    * Mean (expected value of the random variable, 'center of mass of a distribution' [24])

$$\begin{aligned}
\mu_X &= E[X] && (4.114)\\
&= \int_{-\infty}^{\infty} x f_X(x)dx && (4.115)
\end{aligned}$$

    * Variance σ_X^2 and standard deviation σ_X (variance is a measure of the spread of a distribution; see, for example, [24].")

$$\begin{aligned}
\sigma_X^2 &= E\left[(X-\mu_X)^2\right] && (4.116)\\
&= \int_{-\infty}^{\infty}(x-\mu_X)^2 f_X(x)dx && (4.117)
\end{aligned}$$

  - Expectation of a function of a single random variable
    * Given some function, $g(X)$, of a continuous random variable, X, with probability density function, $f_X(x)$, the expected value is

$$\begin{aligned}
\mu_X &= E[g(X)] && (4.118)\\
&= \int_{-\infty}^{\infty} g(x)f_X(x)dx; && (4.119)
\end{aligned}$$

see, for example, [24]

* Special case: small σ_X
  - Expected value of $g(X)$ (using a Taylor series expansion for $g(X)$)

$$
\begin{aligned}
\mu_{g(X)} &= E\left[g(X)\right] && (4.120)\\
&= \int_{-\infty}^{\infty} g(x) f_X(x) dx && (4.121)\\
&= \int_{-\infty}^{\infty} \left(g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} (x-\mu_X) + ... \right) f_X(x) dx && (4.122)
\end{aligned}
$$

for $\sigma_X^2 \ll \left| \frac{g(\mu_X)}{\frac{d^2g}{dX^2}} \right|$ (Do we need to assume that g is a continuous function?)

$$
\begin{aligned}
\mu_{g(X)} &\approx \int_{-\infty}^{\infty} \left(g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} (x-\mu_X) \right) f_X(x) dx && (4.123)\\
&\approx g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} \int_{-\infty}^{\infty} (x-\mu_X) f_X(x) dx && (4.124)\\
&\approx g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} \left(\cancel{\int_{-\infty}^{\infty} x f_X(x) dx - \mu_X} \right) && (4.125)\\
&\approx g(\mu_X) && (4.126)
\end{aligned}
$$

  - variance of $g(X)$ (using a Taylor series expansion for $g(X)$)

$$
\begin{aligned}
\sigma^2_{g(X)} &= E\left[\left(g(X) - E[g(x)] \right)^2 \right] && (4.127)\\
&= \int_{-\infty}^{\infty} \left(g(x) - E[g(x)] \right)^2 f_X(x) dx && (4.128)\\
&= \int_{-\infty}^{\infty} \left(g(x) - \mu_{g(X)} \right)^2 f_X(x) dx && (4.129)\\
&= \int_{-\infty}^{\infty} \left(g^2(x) - 2g(x)\mu_{g(X)} + \mu^2_{g(x)} \right) f_X(x) dx && (4.130)\\
&= -2\mu^2_{g(X)} + \mu^2_{g(X)} + \int_{-\infty}^{\infty} g^2(x) f_X(x) dx && (4.131)\\
&= -\mu^2_{g(X)} + \int_{-\infty}^{\infty} g^2(x) f_X(x) dx && (4.132)\\
&= -\mu^2_{g(X)} + \int_{-\infty}^{\infty} \left(g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} (x-\mu_X) + ... \right)^2 f_X(x) dx && (4.133)
\end{aligned}
$$

for $\sigma_X^2 \ll \left|\frac{g(\mu_X)}{\frac{d^2g}{dX^2}}\right|$ (Do we need to assume that g is a continuous function?)

$$
\begin{aligned}
\sigma_{g(X)}^2 &\approx -\mu_{g(X)}^2 + \int_{-\infty}^{\infty} \left(g(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X} (x-\mu_X) \right)^2 f_X(x)dx && (4.134) \\
&\approx -\mu_{g(X)}^2 + \int_{-\infty}^{\infty} \left(g^2(\mu_X) + 2g(\mu_X) \left.\frac{dg}{dX}\right|_{\mu_X} \cancel{(x-\mu_X)} \right. \\
&\quad \left. + \left.\frac{dg}{dX}\right|_{\mu_X}^2 (x-\mu_X)^2 \right) f_X(x)dx && (4.135) \\
&\approx -\mu_{g(X)}^2 + \int_{-\infty}^{\infty} \left(g^2(\mu_X) + \left.\frac{dg}{dX}\right|_{\mu_X}^2 (x-\mu_X)^2 \right) f_X(x)dx && (4.136) \\
&\approx \cancel{-\mu_{g(X)}^2} + \cancel{g^2(\mu_X)} + \left.\frac{dg}{dX}\right|_{\mu_X}^2 \sigma_X^2 && (4.137) \\
&\approx \left.\frac{dg}{dX}\right|_{\mu_X}^2 \sigma_X^2 && (4.138)
\end{aligned}
$$

- Expectation of a function of multiple random variables
  - Given some function, $g(X,Y)$, of two continuous uncorrelated random variables, X and Y, with probability density functions, $f_X(x)$ and $f_Y(y)$, the expected value is

$$
\begin{aligned}
\mu_{g(X,Y)} &= E[g(X,Y)] && (4.139) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_X(x) f_Y(y) dxdy && (4.140)
\end{aligned}
$$

  - Special case: small σ_X and σ_Y
    - Expected value of $g(X,Y)$ (using Taylor series expansion for $g(X,Y)$)

$$
\begin{aligned}
\mu_{g(X,Y)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_X(x) f_Y(y) dxdy && (4.141) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(g(\mu_X,\mu_Y) + \left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y} (x-\mu_X) \right. \\
&\quad \left. + \left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y} (y-\mu_Y) + ... \right) f_X(x) f_Y(y) dxdy && (4.142)
\end{aligned}
$$

for small σ_X and σ_Y (Do we need to assume that g is a continuous function?)

$$\begin{aligned}
\mu_{g(X,Y)} &\approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(g(\mu_X,\mu_Y)+\left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}\cancel{(x-\mu_X)}\right. \\
&\quad \left.+\left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}\cancel{(y-\mu_Y)}\right)f_X(x)f_Y(y)dxdy && (4.143)\\
&\approx g(\mu_X,\mu_Y) && (4.144)
\end{aligned}$$

· variance of $g(X,Y)$ (using a Taylor series expansion for $g(X,Y)$)

$$\begin{aligned}
\sigma^2_{g(X,Y)} &= E\left[\left(g(X,Y)-E[g(X,Y)]\right)^2\right] && (4.145)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(g(x,y)-E[g(X,Y)]\right)^2 f_X(x)f_Y(y)dxdy && (4.146)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(g(x,y)-\mu_{g(X,Y)}\right)^2 f_X(x)f_Y(y)dxdy && (4.147)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(g^2(x,y)-2g(x,y)\mu_{g(X,Y)}+\mu^2_{g(X,Y)}\right) f_X(x)f_Y(y)dxdy && (4.148)\\
&= -2\mu^2_{g(X,Y)}+\mu^2_{g(X,Y)}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2(x,y)f_X(x)f_Y(y)dxdy && (4.149)\\
&= -\mu^2_{g(X,Y)}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2(x,y)f_X(x)f_Y(y)dxdy && (4.150)\\
&= -\mu^2_{g(X,Y)}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(g(\mu_X,\mu_Y)+\left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}(x-\mu_X)\right. \\
&\quad \left.+\left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}(y-\mu_Y)+...\right)^2 f_X(x)f_Y(y)dxdy && (4.151)
\end{aligned}$$

for $\sigma_X^2 \ll \left|\frac{g(\mu_X)}{\frac{\partial^2 g}{\partial X^2}}\right|$ (Do we need to assume that g is a continuous function?)

$$\begin{aligned}
\sigma_{g(X)}^2 &\approx -\mu_{g(X,Y)}^2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(g(\mu_X,\mu_Y) + \left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}(x-\mu_X)\right. \\
&\quad \left. + \left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}(y-\mu_Y)\right)^2 f_X(x)f_Y(y)dxdy \qquad (4.152) \\
&\approx -\mu_{g(X,Y)}^2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(g^2(\mu_X,\mu_Y) + \left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}^2(x-\mu_X)^2\right. \\
&\quad \left. + \left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}^2(y-\mu_Y)^2\right) f_X(x)f_Y(y)dxdy \qquad (4.153) \\
&\approx \cancel{-\mu_{g(X,Y)}^2} + \cancel{g^2(\mu_X,\mu_Y)} + \left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}^2 \sigma_X^2 + \left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}^2 \sigma_Y^2 \qquad (4.154) \\
&\approx \left.\frac{\partial g}{\partial X}\right|_{\mu_X,\mu_Y}^2 \sigma_X^2 + \left.\frac{\partial g}{\partial Y}\right|_{\mu_X,\mu_Y}^2 \sigma_Y^2 \qquad (4.155)
\end{aligned}$$

- Expectation is a linear operator [24]
  - Expectation is a linear operator

$$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y] \qquad (4.156)$$

  - Proof (considering continuous random variables without loss of generatlity)

$$\begin{aligned}
E[\alpha X + \beta Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\alpha x + \beta y)\, f_X(x)f_Y(y)dxdy \qquad (4.157) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha x f_X(x)f_Y(y)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta y f_X(x)f_Y(y)dxdy \qquad (4.158) \\
&= \alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_X(x)f_Y(y)dxdy + \beta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_X(x)f_Y(y)dxdy \qquad (4.159) \\
&= \alpha\int_{-\infty}^{\infty} x f_X(x)dx\int_{-\infty}^{\infty} f_Y(y)dy + \beta\int_{-\infty}^{\infty} y f_Y(y)dy\int_{-\infty}^{\infty} f_X(x)dx \qquad (4.160) \\
&= \alpha\int_{-\infty}^{\infty} x f_X(x)dx 1 + \beta\int_{-\infty}^{\infty} y f_Y(y)dy 1 \qquad (4.161) \\
&= \alpha E[X] + \beta E[Y] \qquad (4.162)
\end{aligned}$$

Example 4.3:

Considering a random variable, X, with uniform distribution from $a = 1$ to $b = 3$, find expressions for the

a. expected value, $\mu_X = E[X]$, and

b. variance, σ_X^2.

Example 4.4:

Considering the mean, $\bar{X}_{10}$, of 10 random variables, $\{X_1, X_2, ..., X_{10}\}$, each with uniform distribution from $a = 1$ to $b = 3$,

$$\bar{X}_{10} = \frac{\sum_{i=1}^{10} X_i}{10}, \tag{4.163}$$

find:

a. an expression for the expected value of $\bar{X}_{10}$,

$$\mu_{\bar{X}_{10}} = E(\bar{X}_{10}), \tag{4.164}$$

and

b. an expression the variance of $\bar{X}_{10}$ (considering all possible outcomes for $\bar{X}_{10}$),

$$\sigma_{\bar{X}_{10}}^2 = E\left(\left(\bar{X}_{10} - E(\bar{X}_{10})\right)^2\right). \tag{4.165}$$

Example 4.5:

Considering two random variables, X and Y, with means μ_X and μ_Y and variances σ_X^2 and σ_Y^2, find expressions for

a. expected value of the sum, $\mu_{X+Y} = E[X + Y]$, and

b. variance of the sum, $\sigma_{X+Y}^2 = E[((X + Y) - \mu_{X+Y})^2]$.

4.3 Monte Carlo Methods and Data Visualization

- Monte Carlo Simulations
  - Monte Carlo simulations

    * Monte Carlo simulations are simply simulations that employ random numbers; see, for example, [24].
    * Monte Carlo simulations are named after a casino in Monaco; see, for example, [24]
    * Monte Carlo simulations are used, for example, when hand calculations are cumbersome, impractical, or simply not possible.
  - Relevant python and MATLAB® commands

- Data Visualization
  - Histogram
    * Frequency vs. x
    * Sometimes normalized to relative frequency vs. x
    * Requires selecting the bin size
      · "It is customary to have from 5 to 15 bins [10]."
      · Typically, bins are of equal width.
      · Quantitiative guidelines for selecting bins are available in the literature. However, it seems that selecting bin size and origin can be somewhat of an art form.
  - Cumulative distribution function
    * No bin size selection required
  - others
    * box plot
    * error bars
  - Software may be used to visualize data using, for example, the following commands:
    * Python - see Appendix D.4 Representing and Solving Linear Time Invariant Systems
      · `plot`
      · `hist`
      · `cdfplot`
      · `boxplot`
      · `errorbar`
    * MATLAB® - see Appendix E.5 Representing and Solving Linear Time Invariant Systems
      · `plot`
      · `histogram`
      · `cdfplot`
      · `boxplot`
      · `boxchart`
      · `errorbar`

Example 4.6: Monte Carlo

Consider 10 random variables, $\{X_1, X_2, ..., X_{10}\}$, each with uniform distribution from $a = 0$ to $b = 1$, the function

$$g(x) = \frac{2}{\pi} \tan^{-1}(10x - 5) + 2, \tag{4.166}$$

and the average

$$\bar{G}_{10} = \frac{1}{10} \sum_{i=1}^{10} g(X_i). \tag{4.167}$$

Use Monte Carlo methods to do the following:

a. construct plots to visualize the distribution for X_i,

b. determine the expected value for X_i,

c. find the variance of X_i,

d. construct plots to visualize the distribution for Y_i,

e. determine the expected value for Y_i,

f. find the variance of Y_i,

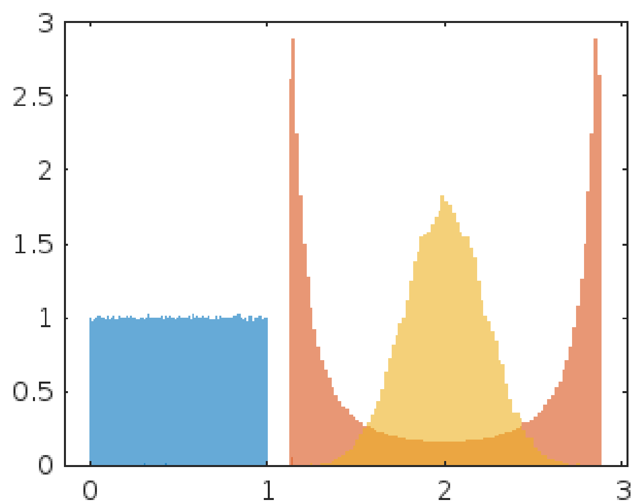
g. construct plots to visualize the distribution for $\bar{G}_{10}$,

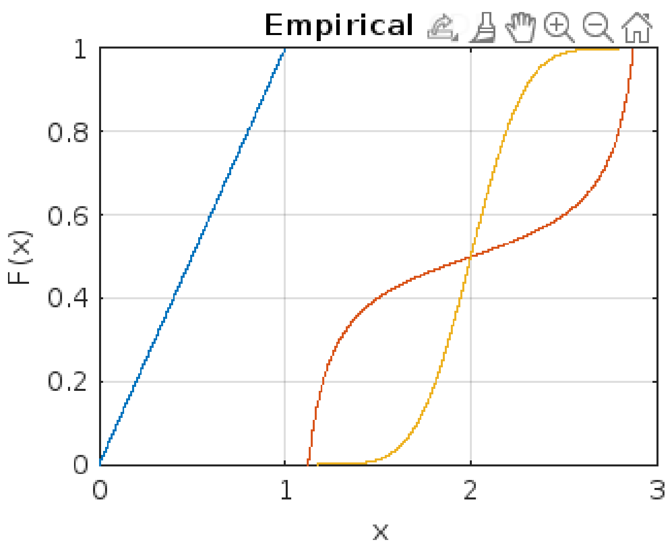
h. determine the expected value for $\bar{G}_{10}$, and

i. determine the variance for $\bar{G}_{10}$.


```
clear all
clc
close all
N=100000;%Number of runs
Nsamples=10;%number of samples per run
X=rand(N,Nsamples);%
Y=2/pi*atan(10*X-5)+2;
G10=mean(Y,2);

histogram(X(:),'Normalization','pdf','LineStyle','none')
hold on
histogram(Y(:),'Normalization','pdf','LineStyle','none')
histogram(G10,'Normalization','pdf','LineStyle','none')

figure()
cdfplot(X(:))
hold on
cdfplot(Y(:))
cdfplot(G10)

display(['E[X]= ',num2str(mean(X(:)))])
display(['E[(X-E[X])^2]= ',num2str(var(X(:)))])
display(['E[Y]= ',num2str(mean(Y(:)))])
display(['E[(Y-E[Y])^2]= ',num2str(var(Y(:)))])
display(['E[G_10]= ',num2str(mean(G10))])
display(['E[(G_10-E[G_10])^2]= ',num2str(var(G10))])
```

```
E[X]= 0.50036
E[(X-E[X])^2]= 0.083276
E[Y]= 2.0009
E[(Y-E[Y])^2]= 0.49406
E[G_10]= 2.0009
E[(G_10-E[G_10])^2]= 0.049279
```





Published with MATLAB® R2022a

Chapter 5

Statistics and Uncertainty Propagation

This chapter serves as a brief review and is a work in progress that may contain significant mistakes. Be sure to consult appropriate textbooks and follow required standards when performing critical statistical/uncertainty analyses.

5.1 Sample Statistics and Parameter Estimation

- Population vs. Sample
  - Population -
    - In statistics, population refers to all of the items we are trying to characterize. Because populations are often very large, it can be impractical to measure every item in a population.
    - "The population comprises the entire collection of objects, measurements, observations, and so on whose properties are under consideration and about which some generalizations are to be made [25]."
    - Note, population mean and standard deviation were presented in the previous chapter.
  - Sample -
    - Sample - A sample is the part of the population that is actually measured in order to characterize the population.
    - "A sample is a representative subset of a population on which an experiment is performed and numerical data are obtained [25]."
- Sample statistics

- Sample of size n (independent random variables, $\{X_1,\ X_2,\ X_3,\ ...,\ X_i,\ ...,\ X_n\}$, drawn from the same distribution).
- Sample mean (for sample size of n)

$$\bar{X}_n = \sum_{i=1}^{n} \frac{X_i}{n} \tag{5.1}$$

  (Here the subscript of $\bar{X}_n$ indicates sample size rather than sample number.)
- Sample variance (for sample size n)
  * Case 1: The sample mean is available and the population mean is not (common)

$$S_X^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X}_n)^2}{n-1} \tag{5.2}$$

    · Interestingly, S_X^2 is an unbiased estimate of the population variance while S_n is a biased estimate for the population standard deviation [24].
    · Note, the sample variance is a random variable; see, for example, [24].
  * Case 2: The population mean is available (uncommon)

$$S_X^2 = \sum_{i=1}^{n} \frac{(X_i - \mu_X)^2}{n} \tag{5.3}$$

- Outliers
  * Outliers are sample measurements that are not representative of the population.
  * Sometimes outliers are rejected (i.e., not included in sample statistics) if they meet specific criteria.
    · There are no universally accepted criteria for outlier rejection.
    · The three sigma method is one criterion for outlier detection; see, for example, [4].

- Law of large numbers - "The law of large numbers (LLN) says that as n grows, the sample mean converges to the true mean μ ... [24]"

- Standard deviation of the means

  - Consider n independent random variables, $\{X_1,\ X_2,\ ...,\ X_n\}$ drawn from an arbitrary (but the same distribution) with mean μ_X and variance σ_X^2.
  - We define $\bar{X}_n$ as a random variable representing the mean of this sample.
  - We can compute the mean of $\bar{X}_n$ by applying the linearity property to the expectation of $\bar{X}_n$:

$$\begin{aligned}
\mu_{\bar{X}_n} &= E[\bar{X}_n] && (5.4)\\
&= E\left[\sum_{i=1}^{n}\frac{X_i}{n}\right] && (5.5)\\
&= \frac{1}{n}E\left[\sum_{i=1}^{n}X_i\right] && (5.6)\\
&= \frac{1}{n}n\sum_{i=1}^{n}E\left[X_i\right] && (5.7)\\
&= \mu_X && (5.8)
\end{aligned}$$

  - Variance of $\bar{X}_n$

$$\begin{aligned}
\sigma_{\bar{X}_n}^2 &= E[(\bar{X}_n-\mu_X)^2] && (5.9)\\
&= E\left[\left(\sum_{i=1}^{n}\frac{X_i}{n}-\mu_X\right)^2\right] && (5.10)\\
&= E\left[\left(\sum_{i=1}^{n}\frac{X_i}{n}-\frac{n}{n}\mu_X\right)^2\right] && (5.11)\\
&= \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n}(X_i-\mu_X)\right)^2\right] && (5.12)
\end{aligned}$$

because X_i are independent and uncorrelated we can swap the order of the sum and the square (see proofs in previous lecture?)

$$\sigma^2_{\bar{X}_n} = \frac{1}{n^2} E\left[\sum_{i=1}^{n}(X_i - \mu_X)^2\right] \quad (5.13)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} E\left[(X_i - \mu_X)^2\right] \quad (5.14)$$

$$= \frac{1}{n^2} n E\left[(X_i - \mu_X)^2\right] \quad (5.15)$$

$$= \frac{1}{n}\sigma^2_X \quad (5.16)$$

$$= \frac{\sigma^2_X}{n} \quad (5.17)$$

– Regardless of sample size (n), we can relate the population standard deviation to the standard deviation of the means:

$$\sigma_{\bar{X}_n} = \frac{\sigma_X}{\sqrt{n}}, \quad (5.18)$$

- Central limit theorem
  - "The [central limit theorem] states that for large n, the distribution of $\bar{X}_n$ after standardization approaches a standard normal distribution [24]."
    * "The distribution of the individual X_j can be anything in the world, as long as the mean and variance are finite [24]."
  - The standard deviation of this normal distribution is given by the standard deviation of the means.
- Student-t distribution
  - Consider n independent random variables, $\{X_1,\ X_2,\ \ldots,\ X_n\}$ drawn from a same normal distribution with mean μ_X and variance σ^2_X.
    * The normality of X_i and the central limit theorem dictate that the mean of the n random variables, $\bar{X}_n$, follows the normal distribution
    * The standard deviation of the means, Equation 5.18, and Equation 4.21 yield a map from the normally distributed $\bar{X}_n$ to the standard normal distribution

$$Z = \frac{\bar{X}_n - \mu_X}{\frac{\sigma_X}{\sqrt{n}}} \quad (5.19)$$

  - A random variable T, which follows the t-distribution, is obtained by substituting the sample standard deviation, S_X, in place of the population standard deviation, σ_X,

$$T = \frac{\bar{X}_n - \mu_X}{\frac{S_X}{\sqrt{n}}} \quad (5.20)$$

- The t-distribution is parameterized by it's degrees of freedom, ν
  * The sample size, n, plays a major role in Equation 5.20, which defines the random variable T and thereby parameterizes the t-distribution.
    · n is explicitly present (as $\sqrt{n}$) in Equation 5.20
    · n is also implicitly present in Equation 5.20 because the distribution for the random variable S_X (i.e., the sample standard deviation) depends upon the sample size.
  * We use the degrees of freedom, ν, rather than the sample size, n, to parameterize the t-distribution
    · Degrees of freedom - "It is given by the number of independent measurements minus the minimum number of measurements that are theoretically necessary to estimate a statistical parameter [10]."
    · If the population mean is unknown, $\nu = n - 1$ (most common case).
    · If the population mean is known, $\nu = n$ (uncommon case).
- Probability density function.

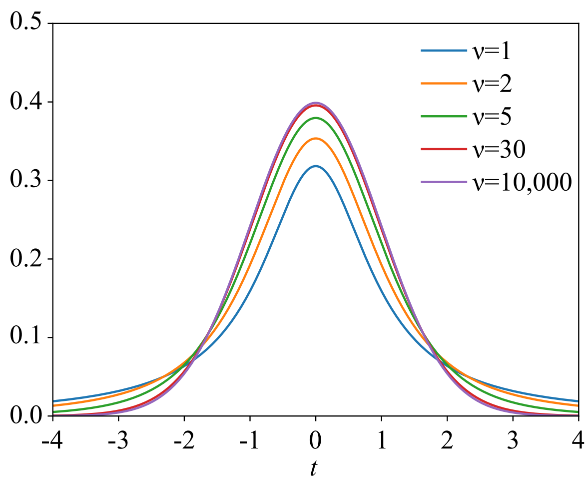


Figure 5.1: Probability density function for the t-distribution with several values for the degrees of freedom, ν.

- Notes
  * As the sample size increases, the sample variance (or standard deviation) approaches the population variance (or standard deviation)

$$\lim_{n\to\infty} S_X^2 \to \sigma_X^2 \tag{5.21}$$

    * Therefore, the t-distribution approaches the standard normal distribution in the limit as $n \to \infty$

$$Z = \lim_{n\to\infty} \frac{\bar{X}_n - \mu_X}{\frac{S_X}{\sqrt{n}}} \tag{5.22}$$

    * Because S_X is based on a finite sample size, the t-distribution is broader than the standard normal distribution.
    * (Note, $\frac{S_X^2}{\sigma_X^2}(n-1)$ follows a chi squared distribution.)

- t-value
    * For a given confidence level, $1-\alpha$, the t-value describes the most likely range for the standardized sample mean; see, for example, [10]

$$P\left[-t_{\nu,1-\alpha} < \frac{\bar{X}_n - \mu_X}{\frac{S_X}{\sqrt{n}}} < t_{\nu,1-\alpha}\right] = 1-\alpha \tag{5.23}$$

    * Often we rearrange the argument of the propability function, $P[]$, to estimate the population mean, μ_X, from $\bar{X}_n$, S_X, and $t_{\nu,1-\alpha}$ (with a desired confidence level, $1-\alpha$).

$$\mu_X = \bar{X}_n \pm t_{\nu,1-\alpha}\frac{S_X}{\sqrt{n}} \tag{5.24}$$

    * Confidence level $1-\alpha$:
        · In this case, "The confidence level is the probability that the population mean will fall within the specified interval ... [10]."
    * Note, here our t-values describes a two-sided interval. Other texts may use different notation.
    * t-values are obtained from Table 5.1 for the appropriate degrees of freedom (usually $\nu = n-1$) and desired confidence level $(1-\alpha)$.

ν	$1-\alpha$ 0.50	0.90	0.95	0.99
1	1.000	6.314	12.706	63.657
2	0.816	2.920	4.303	9.925
3	0.765	2.353	3.182	5.841
4	0.741	2.132	2.776	4.604
5	0.727	2.015	2.571	4.032
6	0.718	1.943	2.447	3.707
7	0.711	1.895	2.365	3.499
8	0.706	1.860	2.306	3.355
9	0.703	1.833	2.262	3.250
10	0.700	1.812	2.228	3.169
11	0.697	1.796	2.201	3.106
12	0.695	1.782	2.179	3.055
13	0.694	1.771	2.160	3.012
14	0.692	1.761	2.145	2.977
15	0.691	1.753	2.131	2.947
16	0.690	1.746	2.120	2.921
17	0.689	1.740	2.110	2.898
18	0.688	1.734	2.101	2.878
19	0.688	1.729	2.093	2.861
20	0.687	1.725	2.086	2.845
21	0.686	1.721	2.080	2.831
22	0.686	1.717	2.074	2.819
23	0.685	1.714	2.069	2.807
24	0.685	1.711	2.064	2.797
25	0.684	1.708	2.060	2.787
26	0.684	1.706	2.056	2.779
27	0.684	1.703	2.052	2.771
28	0.683	1.701	2.048	2.763
29	0.683	1.699	2.045	2.756
30	0.683	1.697	2.042	2.750
∞	0.674	1.645	1.960	2.576

Table 5.1: t-values for probability $P[-t_{\nu,1-\alpha} < T < t_{\nu,1-\alpha}] = 1-\alpha$ with ν being the number of degrees of freedom. Table generated using Scipy and Python.

- Parameter estimation (see, for example, [25])
  - Estimate of the population mean with a $1-\alpha$ confidence level
    - Given large sample size (often $n > 30$; see, for example, [10])

$$\mu_X = \bar{X}_n \pm z_{1-\alpha}\frac{S_X}{\sqrt{n}} \tag{5.25}$$

      - $z_{1-\alpha}$ is obtained from the standard normal distribution; see Table 4.1.
      - The population need not follow a normal distribution [25]
    - Given small sample size

$$\mu_X = \bar{X}_n \pm t_{\nu,1-\alpha}\frac{S_X}{\sqrt{n}} \tag{5.26}$$

      - $t_{\nu,1-\alpha}$ is obtained from the Student's t distribution; see Table 5.1.
      - Technically, the random variable, X, should follow a normal distribution for this approach to be used. In many applications, however, X may be sufficiently close to normal that we can use this approach.
  - Estimate of the population standard deviation with confidence level $1-\alpha$ (Need to update the notation here to make it consistent. Also, this isn't traditionally covered in ME365.)

$$\frac{S_X\sqrt{n-1}}{\chi_{\nu,\alpha/2}} \leq \sigma \leq \frac{S_X\sqrt{n-1}}{\chi_{\nu,1-\alpha/2}} \tag{5.27}$$

    - $\chi_{\nu,\alpha/2}$ and $\chi_{\nu,1-\alpha/2}$ are obtained from the χ^2 distribution
    - Intended for populations that follow a normal distribution but often used for populations with different distributions [25]

- Correlation and statistical independence

- Software may be used to analyze sample data and generate random numbers using, for example, the following commands:
  - Python: - see Appendix D.5 Probability, Statistics, and Uncertainty
    - `mean`
    - `var`
    - `std`
    - `default_rng`
    - `standard_normal`
    - `uniform`
    - `integers`
  - MATLAB®: see Appendix E.6 Probability, Statistics, and Uncertainty

* `mean`
* `var`
* `std`
* `rng`
* `randn`
* `rand`
* `randi`

Example 5.1: Estimate population mean given sample

Four ($n = 4$) independent measurements of the diameter of a ball bearing yield an average of $\bar{D}_4 = 5.010$ mm and sample standard deviation of $S_D = 0.020$ mm. Neglecting any bias uncertainty in the system, find the best estimate for the 95% confidence interval for the average diameter of the ball bearing, μ_D. Assume the ball bearings are normally distributed.

- Sample

$$n = 4 \tag{5.28}$$
$$\bar{D}_4 = 5.010 \text{ mm} \tag{5.29}$$
$$S_D = 0.020 \text{ mm} \tag{5.30}$$

- Estimate of population mean

$$\mu_D = \bar{D}_n \pm t_{\nu,1-\alpha} \frac{S_D}{\sqrt{n}} \tag{5.31}$$

  – degrees of freedom

$$\nu = n - 1 \tag{5.32}$$
$$= 4 - 1 \tag{5.33}$$
$$= 3 \tag{5.34}$$

  because we don't know the population mean

  – 95% confidence level

$$1 - \alpha = 0.95 \tag{5.35}$$

  – The t-value may be obtained from the row in which $\nu = 3$ and the column labeled $1 - \alpha = 0.95$ of the t-table, Table 5.1.

$$t_{3,95\%} = 3.182 \tag{5.36}$$

– Estimate of population mean

$$\mu_D = \bar{D}_4 \pm t_{3,95\%} \frac{S_D}{\sqrt{4}} \tag{5.37}$$

$$= 5.010 \pm 3.182 \frac{0.020}{\sqrt{4}} \text{ mm} \tag{5.38}$$

$$= 5.010 \pm 3.182 \frac{0.020}{2} \text{ mm} \tag{5.39}$$

$$= 5.010 \pm 0.032 \text{ mm} \tag{5.40}$$

5.2 Uncertainty, Error, and Accuracy

- Error - Measurement error is the difference between the measurement/reading and the exact value.
- Uncertainty - "Since the magnitude of the error in any measurement can only be estimated, one refers to an estimate of the error in the measurement as uncertainty present in the measured value [4]."
- Types of error/uncertainty; see, for example, [10]
  - R_X: Random (or precision) error/uncertainty describes, for example, the variability among many measurements of the same quantity
    - Random uncertainty of a **single measurement** in which S_X is an estimate for the expected standard deviation of measurements $\{X_1,\ X_2,\ ...,\ X_n\}$

      $$R_X = t_{\nu,1-\alpha} S_X \tag{5.41}$$

      Here $t_{\nu,1-\alpha}$ is the t values with the appropriate degrees of freedom and confidence level corresponding to the estimate of S_X.
    - Random uncertainty of the **sample mean of n measurements** for which S_X is the sample standard deviation of measurements $\{X_1,\ X_2,\ ...,\ X_n\}$

      $$R_{\bar{X}_n} = t_{\nu,1-\alpha} \frac{S_X}{\sqrt{n}} \tag{5.42}$$

      Here $t_{\nu,1-\alpha}$ is the t value with the appropriate degrees of freedom and confidence level.
  - B_X: Bias (or systematic) error/uncertainty describes, for example, the difference between the exact value of a quantity and the average of many repeated measurements of the same quantity.

- U_X: Uncertainty in the measurement X is the combination of random and bias uncertainties (with the same confidence level). Random and bias uncertainties are combined using the root sum of squares (RSS) method

$$U_X = \sqrt{R_X^2 + B_X^2} \tag{5.43}$$

  * This assumes R_X and B_X are uncorrelated (which they should be).
  * For the uncertainty when using the average of serval measurements to estimate the population mean value, substitute $R_{\bar{X}_n}$ for R_X

- "When an experiment is repeated using different equipment or by different experimenters, the bias errors of successive experiments are unrelated. If enough different experiments are performed, the bias errors are effectively randomized, and they become another form of precision error in the set of all experiments [26]."

• Accuracy specification

- "It is common for manufacturers to supply the uncertainty descriptor called accuracy. As defined in ANSI/ISA (1979), accuracy is a number that usually includes errors due to hysteresis, linearity, and repeatability, but assumes that the calibration is still valid and that zero errors (offsets) have been adjusted to close to zero. When accuracy is reported, it is treated as systematic uncertainty. This approach is conservative ... (pp. 203 [10])"
- Proposed assumptions (unless the manufacturers documentation states otherwise):
  * Assume the manufacturers stated accuracy "... bounds errors determined during the calibration process ... [10]."
  * Assume the manufacturer's stated accuracy corresponds to a confidence level higher than about 95% [10].
  * Treat the manufacturer's stated accuracy as a measure of bias uncertainty (even though it likely accounts for bias and random uncertainties); see, for example, [10].
- Spec sheets often report accuracy as:

$$\begin{aligned} \text{accuracy} &= \pm(\text{percent} \times \text{reading} + \text{counts} \times \text{resolution} && (5.44) \\ &= \pm(\%\text{reading} + \text{counts}) && (5.45) \end{aligned}$$

  * Percent is a measure of the 'gain error' of the measurement [27]
  * Reading is the reading from the equipment
  * Counts (also called digits) is a measure of the 'offset error' of the measurement and depends upon the range selected [27]
  * Resolution is the smallest change that can be detected in the reading.

• Significant digits and rounding

- The following approach is based on that presented in [28]
- Measurement uncertainty, U_X, is often expressed with a confidence interval of about 95%
- Do not round intermediate calculations
- Maintain two significant digits in the measurement uncertainty, U_X
- Round the measured value to the same significant digit as the measurement uncertainty
- (Use the even/odd method of rounding to avoid rounding asymmetry.)

Example 5.2: Estimate the population mean given a sample mean and standard deviation as well as the instrument bias.

An inexpensive plastic caliper is used to make five ($n = 5$) independent measurements of the diameter of a ball bearing. The average reading is $\bar{D}_5 = 5.010$ mm and the sample standard deviation of the readings is $S_D = 0.020$ mm. For all of these readings, the caliper has a bias uncertainty of ± 0.050 mm at the 95% confidence level. Find the 95% confidence interval for the average diameter of the ball bearing, μ_D.

- Random uncertainty (assuming all possible caliper readings of the ball bearing are normally distributed)

$$R_{\bar{D}_5} = t_{4,95\%}\frac{S_D}{\sqrt{5}} \tag{5.46}$$

- Bias uncertainty

$$B_{\bar{D}_5} = 0.050 \text{ mm} \tag{5.47}$$

- Uncertainty

$$\begin{aligned} U_{\bar{D}_5} &= \sqrt{R_{\bar{D}_5}^2 + B_{\bar{D}_5}^2} && (5.48) \\ &= \sqrt{\left(t_{4,95\%}\frac{S_D}{\sqrt{5}}\right)^2 + 0.050^2} && (5.49) \\ &= \sqrt{\left(2.776\frac{0.020}{\sqrt{5}}\right)^2 + 0.050^2} && (5.50) \\ &= 0.0558 \text{ mm} && (5.51) \end{aligned}$$

Note, the t-value ($t_{4,95\%} = 2.776$) may be obtained from the row in which $\nu = n - 1 = 4$ and the column labeled $1 - \alpha = 0.95$ of the t-table, Table 5.1.

- 95% confidence interval

$$\mu_D = 5.010 \pm 0.056 \text{ mm} \tag{5.52}$$

5.3 Uncertainty Propagation

- Consider a quantity g that is a function of multiple quantities x, y, ...

$$g = g(x, y, ...). \tag{5.53}$$

  – We can express a linear approximation to the function for x, y, ... near x_0, y_0, ... as

$$g(x, y, ...) \approx g(x_0, y_0, ...) + \left.\frac{\partial g}{\partial x}\right|_{x_0, y_0, ...} (x - x_0) + \left.\frac{\partial g}{\partial y}\right|_{x_0, y_0, ...} (y - y_0) + ... \tag{5.54}$$

  – Let's represent measurements of x, y, ... with the random variables X, Y, which have means and variances given by μ_X, μ_Y, ... and σ_X^2, σ_Y^2. Using these random variables, we can represent the result of g as the random variable

$$G = g(X, Y, ...). \tag{5.55}$$

  – We can compute the expected value for G using the linear approximation for g and linearizing about μ_X, μ_Y, ...

$$\begin{aligned}
\mu_G &= E[G] && (5.56)\\
&= E\left[g(X, Y, ...)\right] && (5.57)\\
&\approx E\left[g(\mu_X, \mu_Y, ...) + \left.\frac{\partial g}{\partial x}\right|_{\mu_X, \mu_Y, ...} (X - \mu_X) + \left.\frac{\partial g}{\partial y}\right|_{\mu_X, \mu_Y, ...} (Y - \mu_Y) + ...\right] && (5.58)\\
&\approx E\left[g(\mu_X, \mu_Y, ...)\right] + E\left[\left.\frac{\partial g}{\partial x}\right|_{\mu_X, \mu_Y, ...} (X - \mu_X)\right] + E\left[\left.\frac{\partial g}{\partial y}\right|_{\mu_X, \mu_Y, ...} (Y - \mu_Y)\right] + ... && (5.59)\\
&\approx E\left[g(\mu_X, \mu_Y, ...)\right] + \left.\frac{\partial g}{\partial x}\right|_{\mu_X, \mu_Y, ...} \cancelto{0}{E[(X - \mu_X)]} + \left.\frac{\partial g}{\partial y}\right|_{\mu_X, \mu_Y, ...} \cancelto{0}{E[(Y - \mu_Y)]} + ... && (5.60)\\
&\approx g(\mu_X, \mu_Y, ...) && (5.61)
\end{aligned}$$

– Similarly, we can compute the variance of G as

$$\sigma_G^2 = E\left[(G-\mu_G)^2\right] \tag{5.62}$$

$$\approx E\left[\left(\cancel{g(\mu_X,\mu_Y,...)} + \left.\frac{\partial g}{\partial x}\right|_{\mu_X,\mu_Y,...}(X-\mu_X) + \left.\frac{\partial g}{\partial y}\right|_{\mu_X,\mu_Y,...}(Y-\mu_Y) + ... - \cancel{\mu_G}\right)^2\right] \tag{5.63}$$

$$\approx E\left[\left(\left.\frac{\partial g}{\partial x}\right|_{\mu_X,\mu_Y,...}(X-\mu_X) + \left.\frac{\partial g}{\partial y}\right|_{\mu_X,\mu_Y,...}(Y-\mu_Y) + ...\right)^2\right] \tag{5.64}$$

$$\approx E\left[\left(\left.\frac{\partial g}{\partial x}\right|_{\mu_X,\mu_Y,...}(X-\mu_X)\right)^2\right] + E\left[\left(\left.\frac{\partial g}{\partial x}\right|_{\mu_X,\mu_Y,...}(X-\mu_X)\right)\left(\left.\frac{\partial g}{\partial y}\right|_{\mu_X,\mu_Y,...}(Y-\mu_Y)\right)\right]$$
$$+E\left[\left(\left.\frac{\partial g}{\partial y}\right|_{\mu_X,\mu_Y,...}(Y-\mu_Y)\right)^2 + ...\right] \tag{5.65}$$

Here the cancelation occured because we assume that X, Y, ... are uncorrelated.

$$\sigma_G^2 \approx \left(\left.\frac{\partial g}{\partial x}\right|_{\mu_X,\mu_Y,...}\right)^2 E\left[(X-\mu_X)^2\right] + \left(\left.\frac{\partial g}{\partial y}\right|_{\mu_X,\mu_Y,...}\right)^2 E\left[(Y-\mu_Y)^2\right] + ... \tag{5.66}$$

$$\approx \left(\left.\frac{\partial g}{\partial x}\right|_{\mu_X,\mu_Y,...}\right)^2 \sigma_X^2 + \left(\left.\frac{\partial g}{\partial y}\right|_{\mu_X,\mu_Y,...}\right)^2 \sigma_Y^2 + ... \tag{5.67}$$

$$\sigma_G \approx \left[\left(\left.\frac{\partial g}{\partial x}\right|_{\mu_X,\mu_Y,...}\sigma_X\right)^2 + \left(\left.\frac{\partial g}{\partial y}\right|_{\mu_X,\mu_Y,...}\sigma_Y\right)^2 + ...\right]^{\frac{1}{2}} \tag{5.68}$$

- Case 1: Uncertainty propagation for a single-run; see, for example, [25]
  – Considering X_1, Y_1, and Z_1 to be random variables representing one measurement each of three separate quantities, an estimate for the result, μ_G, may be obtained as

$$G_1 = g(X_1, Y_1, Z_1). \tag{5.69}$$

  – Supposing the measurements (X_1, Y_1, Z_1) have uncorrelated uncertainties of U_X, U_Y, and U_Z respectively (corresponding to consistent confidence levels), the uncertainty in the result, U_G, may be obtained via the root sum of squares (RSS) method

$$U_G = \sqrt{\left(\left.\frac{\partial r}{\mu_X}\right|_{X_1,Y_1,Z_1} U_X\right)^2 + \left(\left.\frac{\partial r}{\mu_Y}\right|_{X_1,Y_1,Z_1} U_Y\right)^2 + \left(\left.\frac{\partial r}{\mu_Z}\right|_{X_1,Y_1,Z_1} U_Z\right)^2}. \tag{5.70}$$

    * Assumptions and limitations
        · The uncertainties U_X, U_Y, and U_Z account for both bias and precision errors
        · The uncertainties U_X, U_Y, and U_Z are uncorrelated/independent [25]
        · Uncertainties U_X, U_Y, and U_Z must be relatively small to apply this linear approach
        · Uncertainties U_X, U_Y, and U_Z have the same confidence level
        · See Equations 4.138, 4.155, and 5.68 and their corresponding derivations.

- Case 2: Uncertainty propagation for multiple-runs; see, for example, [25] (Update the notation to make it consistent with everything else.)
  - Consider n measurements of each quantity
    * measurements: $\{X_1,\ X_2,\ ...,\ X_n\}$, $\{Y_1,\ Y_2,\ ...,\ Y_n\}$, $\{Z_1,\ Z_2,\ ...,\ Z_n\}$
    * means: $\bar{X}_n$, $\bar{Y}_n$, and $\bar{Z}_n$
  - Bias uncertainty of the result by RSS

$$B_G = \sqrt{\left(\left.\frac{\partial g}{\partial \mu_X}\right|_{\bar{X}_n,\bar{Y}_n,\bar{Z}_n} B_X\right)^2 + \left(\left.\frac{\partial g}{\partial \mu_Y}\right|_{\bar{X}_n,\bar{Y}_n,\bar{Z}_n} B_Y\right)^2 + \left(\left.\frac{\partial g}{\partial \mu_Z}\right|_{\bar{X}_n,\bar{Y}_n,\bar{Z}_n} B_Z\right)^2} \tag{5.71}$$

    * B_X, B_Y, and B_Z are the bias uncertainties in the measurements of the quantities
    * Note, partial derivatives are evaluated at the average measured values for the quantities
  - Random uncertainty of the result

$$R_G = t_{\nu,\alpha/2}\frac{S_G}{\sqrt{n}} \tag{5.72}$$

    * S_G is the sample standard deviation of the results from the n runs (i.e., sample standard deviation of $\{g(X_1, Y_1, Z_1),\ g(X_2, Y_2, Z_2),\ ...,\ g(X_n, Y_n, Z_n)\} = \{G_1,\ G_2,\ ...,\ G_n\}$)
    * n is the number of runs (number of times the experiment was run with all quantities measured one time per experiemental run)
    * $t_{\nu,\alpha/2}$ is the t value with the appropriate degrees of freedom and and confidence level
  - uncertainty in the result

$$U_G = \sqrt{B_G^2 + R_G^2} \tag{5.73}$$

  - Confidence interval for the result

$$\bar{G}_n \pm U_G \tag{5.74}$$

    * $\bar{G}_n$ is the average of the results of the n runs

Example 5.3: Uncertainty of a result given parameter uncertainties

The instantaneous power dissipated by a resistor, p, can be expressed as

$$p = \frac{v^2}{R}$$

where v is the voltage across the resistor and R is the resistance of the resistor. Find a symbolic expression for the uncertainty in the calculation of instantaneous power at the 95% confidence level, u_p, given the following variables representing measured values and uncertainties (also at a 95% confidence level):

- $v = v_0 \pm u_v$
- $R = R_0 \pm u_R$.

You may assume the uncertainties are relatively small.

- Uncertainty propagation

$$u_p = \sqrt{\left(\left.\frac{\partial p}{\partial v}\right|_{v_0,R_0} u_v \right)^2 + \left(\left.\frac{\partial p}{\partial R}\right|_{v_0,R_0} u_R \right)^2} \tag{5.75}$$

$$= \sqrt{\left(\frac{2v_0}{R_0} u_v \right)^2 + \left(-\frac{v_0^2}{R_0^2} u_R \right)^2} \tag{5.76}$$

Part II

Mechanical Systems Modeling

Chapter 6

Kinematics and Newton's Laws

- Introduction to dynamic system modeling
  - In this course we will develop models for mechanical, electrical, thermal, and hydraulic systems.
  - We start by deriving governing equations for mechanical systems of rigid bodies which have the form of ordinary differential equations. Although electrical, thermal, and hydraulic systems are modeled by differential equations, discussion of these systems is reserved for after we study the response of systems modeled by lineary ordinary differential equations. (This sequence facilitates better synchronization with the lab.)
  - For simplicity, we focus on models which correspond to linear time invariant systems of ordinary differential equations

6.1 Kinematics

- Kinematics - "The study of motion without regard to forces [29]."

6.1.1 Degrees of Freedom

- Degrees of freedom - definitions
  - A system's number of degrees of freedom "is equal to the number of independent parameters (measurements) which are needed to uniquely define its position in space at any instant of time [29]."
  - The number of degrees of freedom is "the number of inputs which need to be provided in order to create a predictable output [29]"
  - "... the minimum number of independent coordinates necessary to describe the configuration of a system [30]."

- Rigid body degrees of freedom
  - An unconstrained rigid body in 3-dimensional space has 6 degrees of freedom. (It can translate in 3 dimensions and rotate about 3 axes.)
  - A rigid body constrained to planar motion has 3 degrees of freedom. (It can translate in two dimensions and rotate about one axis.) In this class, we restrict our discussion to planar systems.
- Kinematic pairs (i.e., joints) eliminate degrees of freedom from a system
  - Gruebler's equation can be used to account for the number of degrees of freedom of a planar system

$$\text{degrees of freedom} = 3n - \text{joints} \tag{6.1}$$

    n: is the number of rigid bodies
    joints: is the number degrees of freedom removed from the system by kinematic pairs
- Common kinematic pairs (joints) in planar systems
  - Pin joints: typically eliminate 2 degrees of freedom (the positions of the bodies must be the same at the pin joint)
  - Inextensible cables: typically eliminate 1 degree of freedom (length of the cable)
  - Prismatic (or sliding) joints: typically eliminate 2 degrees of freedom (no relative motion perpendicular to the joint and no relative rotation)
  - Rolling without slip: typically eliminate 2 degrees of freedom (no slip at the point of contact and surfaces remain in contact)
  - Ground: eliminate 3 degrees of freedom
  - Note, there are cases when Eq. 6.1 doesn't hold. This can occur when, for example, kinematic pairs are redundant.

Example 6.1:

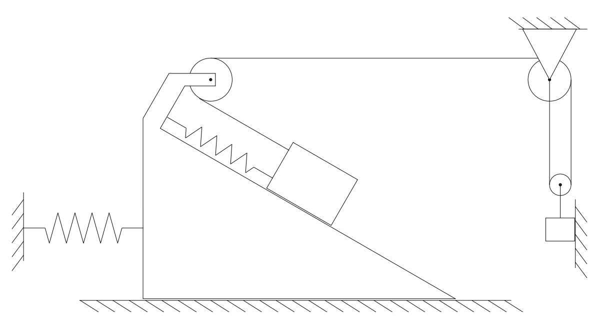


Figure 6.1:

$$
\begin{aligned}
& 3\ \frac{\text{dof}}{\text{rigid body}} \times 3 \text{ rigid bodies} \\
-2\ \frac{\text{dof}}{\text{prismatic joint}} \times 3 \text{ prismatic joints} - 1\ \frac{\text{dof}}{\text{cable}} \times 1 \text{ cable} \quad &= \quad 2 \text{ dof} \qquad (6.2)
\end{aligned}
$$

Example 6.2:

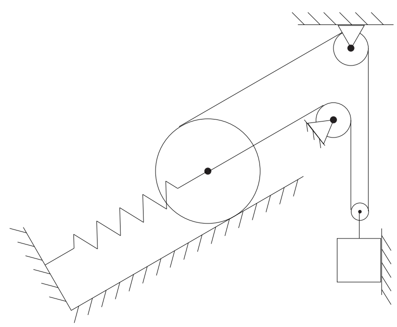


Figure 6.2:

$$
\begin{aligned}
3\ \frac{\text{dof}}{\text{rigid body}} \times 2 \text{ rigid bodies} - 2\ \frac{\text{dof}}{\text{pure rolling}} \times 1 \text{ pure rolling} & \\
-2\ \frac{\text{dof}}{\text{prismatic joint}} \times 1 \text{ prismatic joint} - 1\ \frac{\text{dof}}{\text{cable}} \times 1 \text{ cable} \quad &= \quad 1 \text{ dof} \qquad (6.3)
\end{aligned}
$$

Example 6.3: Double pendulum

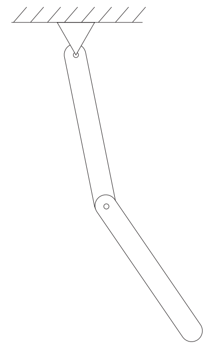


Figure 6.3:

$$3\ \frac{\text{dof}}{\text{rigid body}} \times 2 \text{ rigid bodies} - 2\ \frac{\text{dof}}{\text{pin joint}} \times 2 \text{ pin joints} \quad = \quad 2 \text{ dof} \qquad (6.4)$$

6.1.2 Coordinates, reference frames, and vectors

- The terms 'coordinate system' and 'reference frame' are related, often difficult to distinguish, and sometimes used interchangeably.
- Coordinates
  - (Generalized) Coordinate - Here we use the term 'coordinate' to refer to any variable representing the state of a system. Frequently, coordinates have some meaningful physical significance such as the distance between two points of interest or the relative angle between the sides of two rigid bodies of interest.
  - Coordinate system -
    * A coordinate system is a collection of coordinates used "... to uniquely determine the position of the points or other geometric elements on a manifold such as Euclidean space [wikipedia, Coordinate System, accessed 6/20/2022]."
    * Coordinate systems require reference points and/or directions from which distances and angles can be measured. We call these reference points and/or directions 'reference frames.'
    * "The study of dynamics uses two basic types of coordinates: rectilinear and curvilinear. Rectilinear coordinates describe the components of the motion in fixed di-

rections. Curvilinear coordinate systems incorporate the properties of the path that the particle follows [30]."

· Examples of rectilinear coordinate systems are: Cartesian

· Examples of curvilinear coordinate systems are: Normal-Tangential, Cylindrical, Spherical

∗ Here we are not concerned about using a single standard coordinate system for an entire dynamic system. Instead we select a collection (or system) of coordinates to describe the states of a system.

- Reference frame -
  - A reference frame is a collection of points or directions from which distances and angles can be measured and or observations can be made.
  - Reference frames often serve as the origin of a coordinate system.
  - Multiple reference frames may be used to describe the dynamics of a single system.
  - For Newton's "... laws to have meaning, the motion must be measured relative to some reference frame [31]." An inertial reference frame is one in which Newton's laws apply (a body subject to no force maintains a constant velocity in an inertial reference frame.) Inertial reference frames are non-accelerating, they may translate with constant velocity but they may not rotate.
- Vectors
  - Geometric vector - for lack of a better definition, a geometric vector is a quantity that can be depicted as an arrow with magnitude and direction.
  - Algebraic vector -
    - ∗ An algebraic vector is a list/array of scalars/numbers that conveys magnitude and direction with the value of its components.
    - ∗ Often, an algebraic vector is represented by a single column matrix.
    - ∗ Algebraic vectors require an associated coordinate system, reference frame, and/or set of basis vectors in order to be mapped to an equivalent geometric vector.
  - The time derivative of a vector depends upon the reference frame from which the vector is to be observed in the differentiation.

6.1.3 Kinematic constraint equations

- Constraint equations account for relationships between coordinates due to mechanical joints when more coordinates are used than degrees of freedom.

- Relationship between constraints, coordinates, and degrees of freedom

$$\text{(number of constraint equations)} = \text{(number of coordinates)} - \text{(number of degrees of freedom)} \tag{6.5}$$

  - (number of constraint equations): is the number of scalar kinematic constraint equations needed to relate the coordinates
  - (number of coordinates): is the number of coordinates used to completely describe the state (e.g., positions and orientations) of bodies in the system (sometimes more coordinates are used than are necessary)
  - (number of degrees of freedom): is the number of degrees of freedom of the system

- Vector loop
  - Vector loop - A collection of vectors that sum to 0
  - Can be written based on, for example, position and/or velocity
    - position equation
$$0 = \vec{r}_B - \vec{r}_A - \vec{r}_{B/A} \tag{6.6}$$
    - velocity equation
$$0 = \vec{v}_B - \vec{v}_A - \vec{v}_{B/A} \tag{6.7}$$
  - Vector loops serve as kinematic constraint equations

Example 6.4: Double pendulum

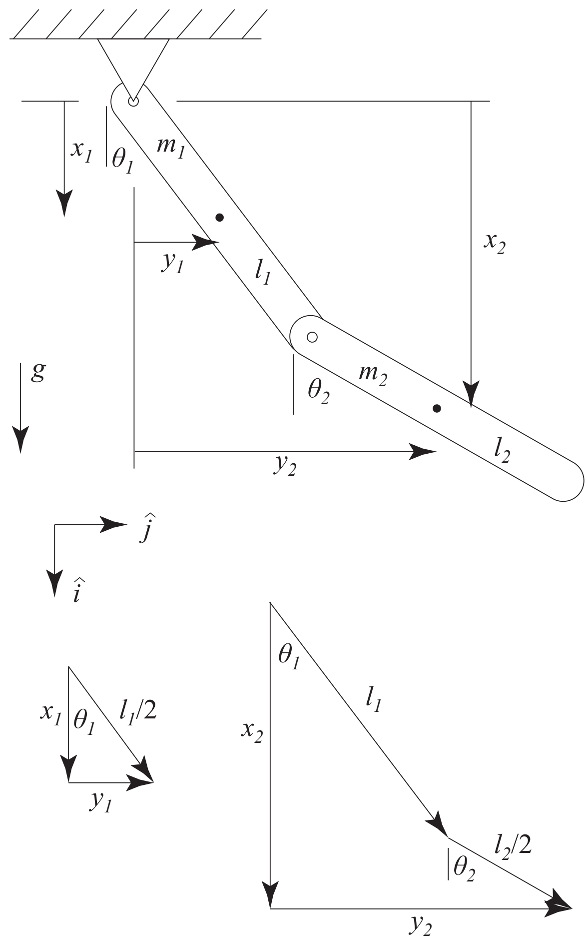


Figure 6.4:

- Coordinates: x_1, y_1, θ_1, x_2, y_2, θ_2
- Vector loops (based on position)

$$0 = x_1\hat{i} + y_1\hat{j} - \frac{l_1}{2}\left(\cos(\theta_1)\hat{i} + \sin(\theta_1)\hat{j}\right) \tag{6.8}$$

$$\begin{aligned} 0 &= x_2\hat{i} + y_2\hat{j} - \frac{l_2}{2}\left(\cos(\theta_2)\hat{i} + \sin(\theta_2)\hat{j}\right) \\ &\quad -l_1\left(\cos(\theta_1)\hat{i} + \sin(\theta_1)\hat{j}\right) \end{aligned} \tag{6.9}$$

or in scalar form

$$
\begin{aligned}
0 &= x_1 - \frac{l_1}{2}\cos(\theta_1) && (6.10)\\
0 &= y_1 - \frac{l_1}{2}\sin(\theta_1) && (6.11)\\
0 &= x_2 - \frac{l_2}{2}\cos(\theta_2) - l_1\cos(\theta_1) && (6.12)\\
0 &= y_2 - \frac{l_2}{2}\sin(\theta_2) - l_1\sin(\theta_1) && (6.13)
\end{aligned}
$$

- The first and second time derivatives of these equations may be useful to relate position, velocity and acceleration (here I show derivatives for only the first loop)

$$
\begin{aligned}
0 &= \dot{x}_1 + \frac{l}{2}\dot{\theta}_1\sin(\theta_1) && (6.14)\\
0 &= \ddot{x}_1 + \frac{l}{2}\ddot{\theta}_1\sin(\theta_1) + \frac{l}{2}\dot{\theta}_1^2\cos(\theta_1) && (6.15)\\
0 &= \dot{y}_1 - \frac{l}{2}\dot{\theta}_1\cos(\theta_1) && (6.16)\\
0 &= \ddot{y}_1 - \frac{l}{2}\ddot{\theta}_1\cos(\theta_1) + \frac{l}{2}\dot{\theta}_1^2\sin(\theta_1) && (6.17)\\
& && (6.18)
\end{aligned}
$$

Example 6.5:

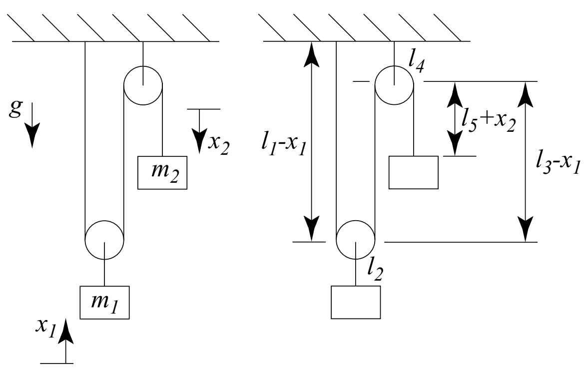


Figure 6.5:

Cables and pulleys (Is there a reference for this figure?)

- Assuming an inextensible and taut cable
- Coordinates: x_1 and x_2
  In addition we introduce some constants so that we can write expressions for the length of each segment of the cable
- Vector loop (based on position, one dimensional along the contour of the cable)

$$0 = l_1 - x_1 + l_2 + l_3 - x_1 + l_4 + l_5 + x_2 - L_{tot} \tag{6.19}$$

differentiate with respect to time to eliminate all the constants

$$0 = \frac{d}{dt}(l_1 - x_1 + l_2 + l_3 - x_1 + l_4 + l_5 + x_2 - L_{tot}) \tag{6.20}$$
$$0 = -2\dot{x}_1 + \dot{x}_2 \tag{6.21}$$

Example 6.6: Gears

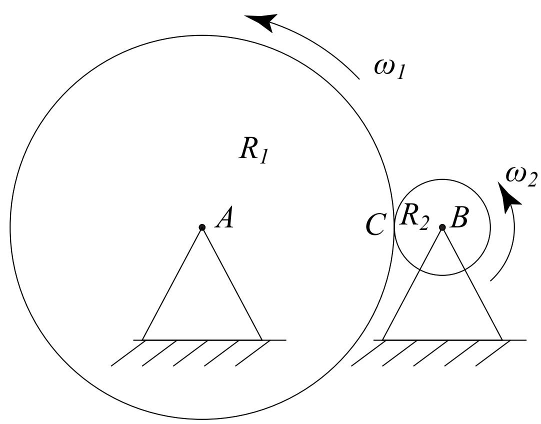


Figure 6.6:

- Vector loop (based on velocity)

$$0 = \vec{v}_{B/A} - (\vec{v}_{C/A} + \vec{v}_{B/C}) \tag{6.22}$$
$$= 0 - (R_1\omega_1\hat{j} + R_2\omega_2\hat{j}) \tag{6.23}$$
$$= R_1\omega_1 + R_2\omega_2 \tag{6.24}$$

Example 6.7: Rolling without slip

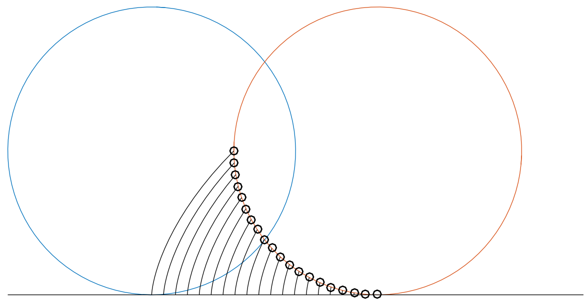


Figure 6.7:

- Pure rolling: no slip (no instantaneous velocity at point of contact
- Vector loop (based on velocity)

$$\begin{aligned} 0 &= \vec{v}_B - \vec{v}_A - \vec{v}_{B/A} &\quad (6.25)\\ &= \dot{x}\hat{\imath} - 0 - R\omega\hat{\imath} &\quad (6.26)\\ &= \dot{x} - R\omega &\quad (6.27) \end{aligned}$$

- Note, this derivation doesn't hold for a disk experiencing any sliding (example: bowling ball)

Example 6.8: Balance Beam

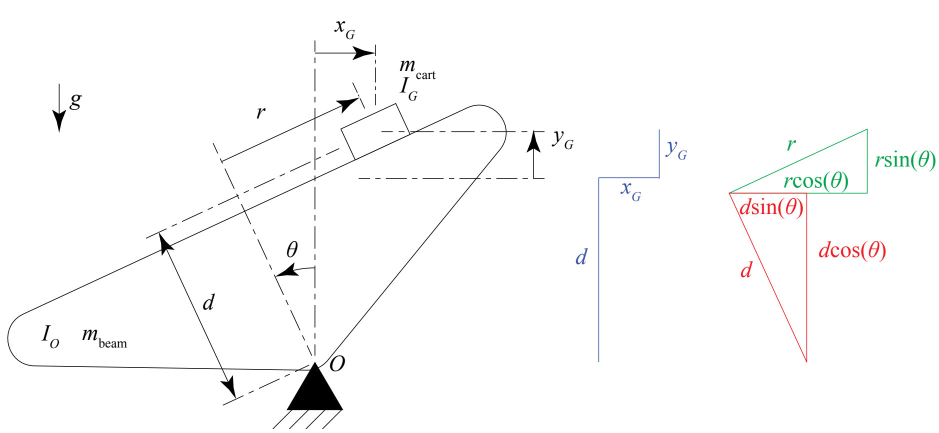


Figure 6.8:

- Vector loop

$$d\hat{j} + x_G\hat{i} + y_G\hat{j} = d(-\sin(\theta)\hat{i} + \cos(\theta)\hat{j}) + r(\cos(\theta)\hat{i} + \sin(\theta)\hat{j}) \tag{6.28}$$

$$x_G = -d\sin(\theta) + r\cos(\theta) \tag{6.29}$$

$$y_G = -d + d\cos(\theta) + r\sin(\theta) \tag{6.30}$$

6.2 Newton's Laws and Moment Equations

- Newton's laws and their implications
  - Summary of Newton's laws
    1. In the absence of external forces, the linear momentum of a particle remains constant
    2. The time rate of change of linear momentum of a particle is equal to the net force ($\vec{F}$) acting on it

$$\frac{d}{dt}(m\vec{v}) = \vec{F} \tag{6.31}$$
$$m\vec{a} = \vec{F} \tag{6.32}$$

       (assuming the mass of the particle is constant)
    3. Forces exerted due to interaction between particles are equal and opposite

$$f_{ij} = f_{ji} \tag{6.33}$$

    * Notes
      · Newton's laws are valid when particle velocities/accelerations are measured relative to an inertial reference frame (i.e., the reference frame is not rotating and not accelerating).
      · Typically, we will write vectors with respect to ground/earth. Though earth moves, it approximates an inertial reference frame when short distances, small velocities, and short time periods are considered [30].
  - Application to systems of particles in translation
    * Consider a system consisting of N particles
    * Define the center of gravity of the system as the mass-weighted average position of the particles in the system

$$\vec{r}_G = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} \tag{6.34}$$
$$= \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{m} \tag{6.35}$$

      where we define the total mass of the system $m = \sum_{i=1}^{N} m_i$
    * Newton's 2nd Law for the system

$$\sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_i \tag{6.36}$$
$$m\vec{a}_G = \tag{6.37}$$

Note, all internal forces cancel in the summation on the right hand side following Newton's 3rd Law such that $\vec{F}_i$ need only consist of external forces.

- Application to rigid bodies in translation
  * Consider a rigid body of mass m consisting of an infinite number of point particles of mass dm
  * Define the center of gravity of the system as the mass-weighted average position of the system

$$\vec{r}_G = \frac{\int_{\text{body}} \vec{r} dm}{\int_{\text{body}} dm} \tag{6.38}$$

$$= \frac{\int_{\text{body}} \vec{r} dm}{m} \tag{6.39}$$

where $m = \int_{\text{body}} dm$ is the total mass of the rigid body

  * Newton's 2nd Law for the system

$$m\vec{a}_G = \int_{\text{body}} d\vec{F} \tag{6.40}$$

Note all internal forces cancel in the integral on the right hand side following Newton's 3rd Law such that $d\vec{F}$need only consist of external forces.

- Moment equations (See [32])
  - Moment about an arbitrary point P of Newton's second law for a particle
    * Take the moment of Newton's 2nd Law about an arbitrary point P

$$\vec{F} = (m\ddot{\vec{r}}) \tag{6.41}$$

$$(\vec{r} - \vec{r}_P) \times \vec{F} = (\vec{r} - \vec{r}_P) \times (m\ddot{\vec{r}}) \tag{6.42}$$

$$\vec{M}_P = (\vec{r} - \vec{r}_P) \times (m\ddot{\vec{r}}) \tag{6.43}$$

Here $\vec{M}_P$ is defined as $(\vec{r} - \vec{r}_P) \times \vec{F}$

  - Moment about an arbitrary point P of Newton's second law for a system of N particles

$$\vec{M}_P = \sum_{i=1}^{N} (\vec{r}_i - \vec{r}_P) \times m_i \ddot{\vec{r}}_i \tag{6.44}$$

where $\vec{M}_P$ is defined as $\sum_{i=1}^{N} (\vec{r}_i - \vec{r}_P) \times \vec{F}_i$.

  - Moment about an arbitrary point P of Newton's second law for a rigid body

$$\vec{M}_P = m(\vec{r}_G - \vec{r}_P) \times \ddot{\vec{r}}_G + I_G\vec{\alpha} + \vec{\omega} \times I_G\vec{\omega} \tag{6.45}$$

  * Here, $\vec{M}_P = \int_{\text{body}} (\vec{r} - \vec{r}_P) \times d\vec{F}$.
  * Note, the last term above, $\vec{\omega} \times I_G\vec{\omega}$, is 0 when the system is planar.
  * Special cases:
    · Moment about the center of mass (i.e., $P = G$ and $\vec{r}_G = \vec{r}_P$), $\vec{M}_G = I_G\vec{\alpha}$
    · Moment about a fixed pivot point P on the body ($\vec{\dot{r}}_P = 0$) and about which the body rotates, $\vec{M}_P = I_P\vec{\alpha}$ where $I_P = I_G + m||\vec{r}_G - \vec{r}_P||^2$ by the parallel axis theorem.
  * Generally, writing correct moment equations (about an arbitrary point P) for rigid bodies is challenging.
    · Students are encouraged to master writing moment equations about the center of mass for rigid bodies before attempting to write moment equations about other points (even if doing so complicates the problem algebraically).
    · Use of "kinetic diagrams" as described in [33] (see pages 101 and 387-388), for example, might facilitate writing moment equations about arbitrary points.

- Free-body diagram
  - See chapter on free-body diagrams in [34]
  - A free-body diagram is analogous to the closed systems discussed in thermodynamics and fluid dynamics; see, for example, [34].
  - In part, free-body diagrams serve to distinguish between internal and externa
    * Internal forces are forces that associated with interactions between elements within the systems
    * External forces are forces that associated with interactions between elements of the system and those external to the system
  - The term 'body' in the context of free-body diagrams should be interpreted as system [34]
  - Method to construct free-body diagrams
    * Sketch the system (e.g., rigid body) as if it were entirely separated from all other systems
    * If coordinates are included in the diagram, they should be drawn in such a way that they cannot be misinterpreted as forces or moments.
    * Identify all the connections that were severed by separating the system from all other systems. Replace these connections with arrows representing forces and/or moments and define the associated variables by labeling the arrows.
    * Often, the direction to draw arrows representing forces and moments is arbitrary. (Avoid solving the equations of motion in your head to determine which direction to draw an arrow. Instead, all equations must be consistent with the variables defined by the associated free-body diagrams.)

Chapter 7

Equations of Motion

7.1 Force-motion Relationships

- Mechanical systems elements
  - Translation
    - Mass (rigid body)
      - Newton's second law
$$\vec{F} = m\vec{a}_G \tag{7.1}$$
        Note, $\vec{a}_G$ is the acceleration of the center of gravity
      - Kinetic energy for translation
$$KE = \frac{1}{2}mv_G^2 \tag{7.2}$$
    - Springs
      - Linear spring - the restoring force is proportional to the displacement from unstretched positions (x_1 and x_2)
      - Force displacement relation
$$F = k(x_1 - x_2) \tag{7.3}$$
      - Potential energy
$$PE = \frac{1}{2}k(x_1 - x_2)^2 \tag{7.4}$$
      - Many objects may be approximated as a spring: coil spring, cantilever beam, extension of rod or cable, etc.
      - General spring element - the restoring force could be any general function of displacement
    - Damping

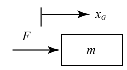
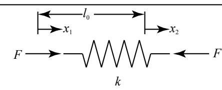
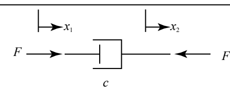
Element	Symbol/FBD	Force-Motion Relationship	Energy and Power
Mass		$F = m\ddot{x}_G$	$KE = \frac{1}{2}m\dot{x}_G^2$
Spring		$F = k(x_1 - x_2)$	$PE = \frac{1}{2}k(x_1 - x_2)^2$
Damper		$F = k(\dot{x}_1 - \dot{x}_2)$	$P = -c(\dot{x}_1 - \dot{x}_2)^2$

Figure 7.1:

- Proportional damping force opposes relative velocity

$$F = c(\dot{x}_1 - \dot{x}_2) \tag{7.5}$$

- Damping dissipates energy. The rate of mechanical energy dissipation (heat generation)

$$P = c(\dot{x}_1 - \dot{x}_2)^2 \tag{7.6}$$

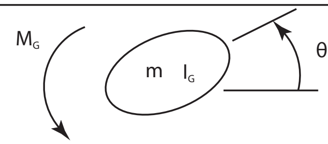
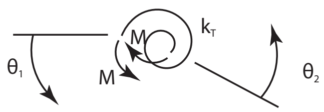
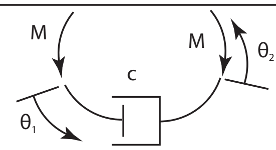
Element	FBD	Input/output Relationship
Inertia		$M_G = I_G\ddot{\theta}$
Spring		$M = k_T(\theta_1 - \theta_2)$
Damper		$M = c(\dot{\theta}_1 - \dot{\theta}_2)$

Figure 7.2:

- Rotation
  * Rotational inertia (planar, about an arbitrary point P)
    · Planar moment balance about an arbitrary point P

$$\vec{M}_P = m(\vec{r}_G - \vec{r}_P) \times \vec{a}_G + I_G\vec{\alpha} \tag{7.7}$$

here we shall restrict ourselves to planar motion.

    · The mass moment of inertia is obtained from

$$I_G = \int_{\text{body}} (r - r_G)^2 dm \tag{7.8}$$

    · Common mass moments of inertia [2]: sphere $I_G = \frac{2}{5}mR^2$, disk (symmetry axis) $I_G = \frac{1}{2}mR^2$, slender rod $I_G = \frac{ml^2}{12}$
    · Kinetic energy for planar rotation

$$KE = \frac{1}{2}I_G\omega^2 \tag{7.9}$$

  * Spring
    · Moment rotation relation

$$T = k_T(\theta_1 - \theta_2) \tag{7.10}$$

        - Potential energy

$$PE = \frac{1}{2}k_T(\theta_1 - \theta_2)^2 \tag{7.11}$$

    - Damper
        - Moment angular velocity relation

$$T = c_T(\dot{\theta}_1 - \dot{\theta}_2) \tag{7.12}$$

        - Rate of energy dissipation (power)

$$P = c_T(\dot{\theta}_1 - \dot{\theta}_2)^2 \tag{7.13}$$

- Weight

$$w = mg \tag{7.14}$$

- Dry friction
  - Dry friction opposes relative motion or impending relative motion; see, for example, [30].
  - Static friction - Friction force developed when there is no relative motion. Its magnitude is limited by

$$F_f \leq \mu_s N \tag{7.15}$$

  - Dynamic friction (or Coulomb friction) - Friction force developed when there is relative motion. Its magnitude is

$$F_f = \mu_d N \tag{7.16}$$

  - Often, it is assumed that more force is required to overcome static friction than is required to maintain motion ($\mu_s > \mu_d$)
  - See the discussion in [34] to justify that it makes more sense to consider $\mu_s = \mu_d$

7.2 Newtonian Approach to Deriving Equations of Motion

- Method
  - Establish coordinates for displacements and rotations and indicate the positive direction.
  - Draw free-body diagrams for each independent rigid body.
  - Write force-motion relationships (input/output equations) for the system, including
    - Rigid bodies
      - Ttranslation: $\vec{F} = m\vec{a}$

- · Rotation: $\vec{M}_G = I_G\vec{\alpha}$ about the mass center (preferred), $\vec{M}_P = I_P\vec{\alpha}$ about a fixed pivot point P, or more generally $\vec{M}_P = I_G\vec{\alpha} + m\vec{r}_{G/P} \times \vec{a}_G$ about arbitrary point P. (Here we assume the body is symmetric about the plane.)
- ∗ Springs
- ∗ Dampers
- ∗ Dry friction
- ∗ Weight
- ∗ Other

– Write kinematic constraint equations relating the coordinates.

– Use substitution (and possibly equilibrium) to simplify the system of equations.

- ∗ Note, simpler equations of motion often result when coordinates are measured from an equilibrium state; see examples below.

7.3 Examples

7.3.1 Basic Planar Dynamics with Kinematics

Example 7.1: Inverted Pendulum

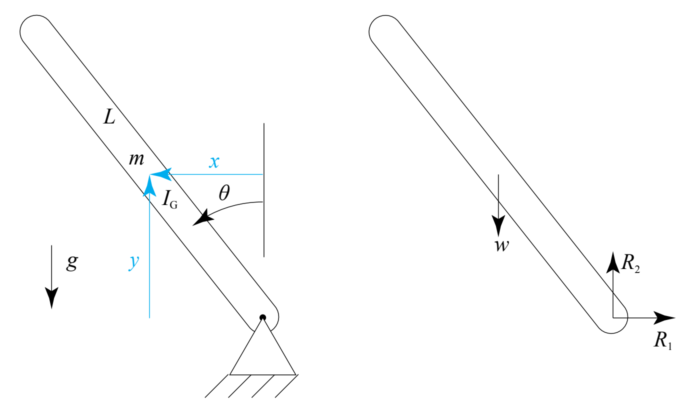


Figure 7.3: A uniform bar of length L, mass m, and mass moment of inertia I_G is supported by a pin joint at one end.

Derive the equation of motion for the system in **Figure 7.3** in terms of the coordinate θ.

- Establish coordinates: θ, x, y
- Free-body diagram
  - w weight
  - R_1 and R_2 reactions forces at the pin
- Write force-motion relationships
  - Weight
$$w = mg \tag{7.17}$$
  - Translation in x direction
$$-R_1 = m\ddot{x} \tag{7.18}$$
  - Translation in y direction
$$R_2 - w = m\ddot{y} \tag{7.19}$$

– Rotation about mass center

$$R_1 y + R_2 x = I_G \ddot{\theta} \tag{7.20}$$

- Kinematics

$$\begin{aligned}
x &= \frac{L}{2}\sin(\theta) && (7.21)\\
\dot{x} &= \frac{L}{2}\cos(\theta)\dot{\theta} && (7.22)\\
\ddot{x} &= -\frac{L}{2}\sin(\theta)\dot{\theta}^2 + \frac{L}{2}\cos(\theta)\ddot{\theta} && (7.23)\\
y &= \frac{L}{2}\cos(\theta) && (7.24)\\
\dot{y} &= -\frac{L}{2}\sin(\theta)\dot{\theta} && (7.25)\\
\ddot{y} &= -\frac{L}{2}\cos(\theta)\dot{\theta}^2 - \frac{L}{2}\sin(\theta)\ddot{\theta} && (7.26)
\end{aligned}$$

- Substitution

– Substitute Equations 7.21, 7.24, 7.18, and 7.19 into Equation 7.20

$$-m\ddot{x}\frac{L}{2}\cos(\theta) + (w + m\ddot{y})\frac{L}{2}\sin(\theta) = I_G\ddot{\theta} \tag{7.27}$$

– Substitute Equations 7.17, 7.23, and 7.26 into the above and simplify

$$\begin{aligned}
-m\left(\cancel{-\frac{L}{2}\sin(\theta)\dot{\theta}^2} + \frac{L}{2}\cos(\theta)\ddot{\theta}\right)\frac{L}{2}\cos(\theta) & \\
+\left(mg + m\left(\cancel{-\frac{L}{2}\cos(\theta)\dot{\theta}^2} - \frac{L}{2}\sin(\theta)\ddot{\theta}\right)\right)\frac{L}{2}\sin(\theta) &= I_G\ddot{\theta} && (7.28)\\
\left(I_G + m\frac{L^2}{4}\right)\ddot{\theta} - mg\frac{L}{2}\sin(\theta) &= 0 && (7.29)
\end{aligned}$$

Example 7.2:

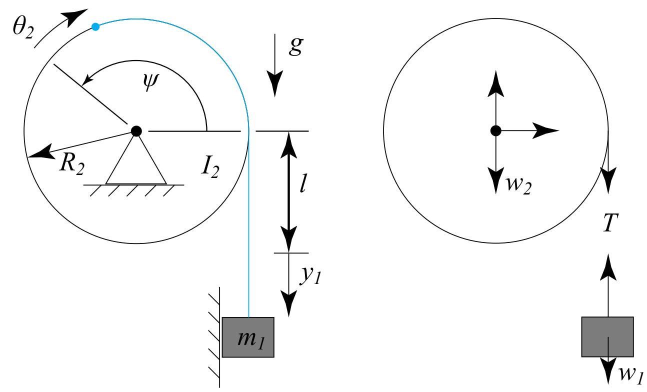


Figure 7.4: Assume the cable is inextensible and remains taut. Also, assume that the state in which $\theta_2 = 0$ corresponds to $y_1 = 0$.

Derive the equation of motion for the in Figure 7.4 in terms of the coordinate y_1.

- Establish coordinates: y_1, θ_2
- Free-body diagrams
  - w_1 and w_2 weights
  - T tension in cable
  - other reaction forces from pin joint not labeled
- Write force-motion relationships
  - Weight
$$w_1 = m_1 g \tag{7.30}$$
  - Translation along y_1 of m_1
$$w_1 - T = m_1 \ddot{y}_1 \tag{7.31}$$
  - Rotation about mass center of I_2 in the θ_2 direction
$$TR_2 = I_2 \ddot{\theta}_2 \tag{7.32}$$
- Kinematics
  - ψ an angle from the horizontal to the attachment point when $\theta_2 = 0$

- l: constant length of cable from the tangent point on the disk to the zero position of y_1
- L: total length of cable

$$\begin{aligned} L &= R_2(\psi - \theta_2) + l + y_1 \\ \frac{d}{dt}(L) &= \frac{d}{dt}(R_2(\psi - \theta_2) + l + y_1) \\ 0 &= -R_2\dot{\theta}_2 + \dot{y}_1 \end{aligned} \tag{7.33}$$

- Substitution
  - Substitute Equation 7.30 into 7.31 into 7.32

$$m_1(g - \ddot{y}_1)R_2 = I_2\ddot{\theta}_2 \tag{7.34}$$

  - Substitute Equation 7.33 into the above and simplify

$$m_1(g - R_2\ddot{\theta}_2)R_2 = I_2\ddot{\theta}_2 \tag{7.35}$$

$$m_1 g R_2 = (m_1 R_2^2 + I_2)\ddot{\theta}_2 \tag{7.36}$$

Example 7.3:

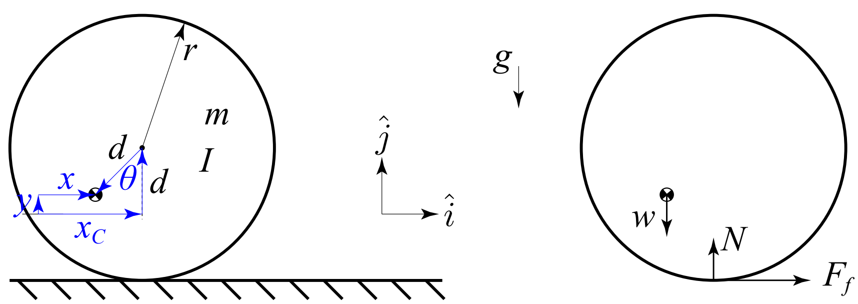


Figure 7.5: The center of mass of a disk is at a distance d from its geometric center. The radius of the disk is r and its mass moment of inertia about the center of mass is I. Assume all the coordinates (x, y, θ) are measured from equilibrium and the disk remains in contact with the ground and rolls without slip.

Derive the equation of motion for the mechanical system in Figure 7.5 in terms of the coordinate θ.

- Establish coordinates: x_C, x, y, θ
- Free-body diagram
  - w weight
  - F_f friction force
  - N normal force
- Write force-motion relationships
  - Weight
$$w = mg \tag{7.37}$$
  - Translation in x direction
$$F_f = m\ddot{x} \tag{7.38}$$
  - Translation in y direction
$$N - w = m\ddot{y} \tag{7.39}$$
  - Rotation about mass center in θ direction
$$-F_f(r - d + y) - N(x_C - x) = I\ddot{\theta} \tag{7.40}$$
- Kinematics
  - This is a 1 degree of freedom system. 3 coordinates are provided and 1 coordinate is introduced, totaling 4 coordinates. $4-1=3$ scalar kinematic constraint equations are needed.
  - Introduce the coordinate x_C. An additional scalar kinematic constraint equation will be necessary to eliminate x_C.
  - Roll back disk to a state when $0 < \theta < \pi/2$ (because it's easier trig is easier with angles between 0° and 90°).
  - Draw the vector loop and write the corresponding equations
$$y\hat{\jmath} + x\hat{\imath} = x_C\hat{\imath} + d\hat{\jmath} - d\sin(\theta)\hat{\imath} - d\cos(\theta)\hat{\jmath} \tag{7.41}$$
  - Write two independent scalar kinematic constraint equations, one for each direction $\hat{\imath}$ and $\hat{\jmath}$. And differentiate twice.

$\hat{i}$ direction

$$\begin{aligned} x &= x_C - d\sin(\theta) && (7.42) \\ \dot{x} &= \dot{x}_C - d\dot{\theta}\cos(\theta) && (7.43) \\ \ddot{x} &= \ddot{x}_C - d\ddot{\theta}\cos(\theta) + d\dot{\theta}^2\sin(\theta) && (7.44) \end{aligned}$$

$\hat{j}$ direction

$$\begin{aligned} y &= d - d\cos(\theta) && (7.45) \\ \dot{y} &= d\dot{\theta}\sin(\theta) && (7.46) \\ \ddot{y} &= d\ddot{\theta}\sin(\theta) + d\dot{\theta}^2\cos(\theta) && (7.47) \end{aligned}$$

– Roll without slip

$$\begin{aligned} x_C &= r\theta && (7.48) \\ \dot{x}_C &= r\dot{\theta} && (7.49) \\ \ddot{x}_C &= r\ddot{\theta} && (7.50) \end{aligned}$$

- Substitution
  - Substitute Eqs. 7.38, 7.39, 7.42, and 7.45 into 7.40

$$-m\ddot{x}(r - \cancel{d} + \cancel{d} - d\cos(\theta)) - m(\ddot{y} + g)(\cancel{x_C} - \cancel{x_C} + d\sin(\theta)) = I\ddot{\theta} \qquad (7.51)$$

  - Substitute Eqs. 7.44 and 7.47 into the above

$$\begin{aligned} -m(\ddot{x}_C - d\ddot{\theta}\cos(\theta) + d\dot{\theta}^2\sin(\theta))(r - d\cos(\theta)) & \\ -m(d\ddot{\theta}\sin(\theta) + d\dot{\theta}^2\cos(\theta) + g)d\sin(\theta) &= I\ddot{\theta} && (7.52) \end{aligned}$$

  - Substitute Eq. 7.50 into the above and simplify to obtain the equation of motion

$$\begin{aligned} I\ddot{\theta} &= -m(r\ddot{\theta} - d\ddot{\theta}\cos(\theta) + d\dot{\theta}^2\sin(\theta))r \\ &\quad + m(r\ddot{\theta} - d\ddot{\theta}\cos(\theta) + d\dot{\theta}^2\sin(\theta))d\cos(\theta) \\ &\quad - m(d\ddot{\theta}\sin(\theta) + d\dot{\theta}^2\cos(\theta) + g)d\sin(\theta) && (7.53) \\ &= -mr^2\ddot{\theta} + 2mrd\ddot{\theta}\cos(\theta) - mrd\dot{\theta}^2\sin(\theta) - md^2\ddot{\theta} - mgd\sin(\theta) && (7.54) \end{aligned}$$

$$\boxed{\left(I + mr^2 - 2mrd\cos(\theta) + md^2\right)\ddot{\theta} + mrd\dot{\theta}^2\sin(\theta) = -mgd\sin(\theta)} \qquad (7.55)$$

Example 7.4:

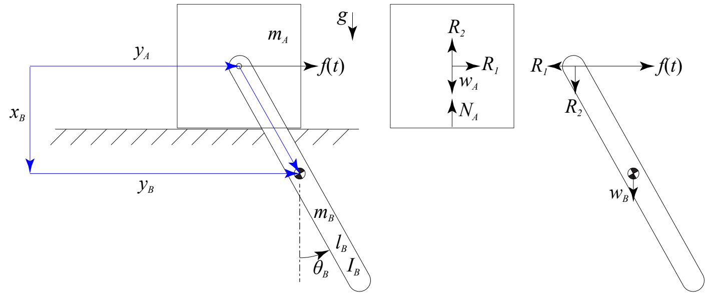


Figure 7.6: A force $f(t)$ is applied to a mechanical system consisting of mass and bar pinned together. Assume mass m_A slides on a frictionless surface and the pin joint is frictionless. Note, the mass moment of inertia I_B is with respect to the respective mass centers.

Derive the equations of motion for the mechanical system in Figure 7.6 in terms of the coordinates y_A and θ_B.

- Establish coordinates: x_B, y_A, y_B, and θ_B
- Free-body diagrams
  - w_A weight
  - w_B weight
  - N_A normal force
  - R_1 reaction force
  - R_2 reaction force
- Write force-motion relationships
  - Weight

$$w_A = m_A g \tag{7.56}$$

$$w_B = m_B g \tag{7.57}$$

  - Translation in y_A direction for m_A

$$R_1 = m_A \ddot{y}_A \tag{7.58}$$

- Translation in vertical direction for m_A (assume remains in contact with surface - no acceleration in the vertical direction)

$$-R_2 + w_A - N_A = 0 \tag{7.59}$$

- Translation in y_B direction for m_B

$$f - R_1 = m_B \ddot{y}_B \tag{7.60}$$

- Translation in x_B direction for m_B

$$R_2 + w_B = m_B \ddot{x}_B \tag{7.61}$$

- Rotation about mass center in θ_B direction

$$R_2 \frac{l_B}{2} \sin(\theta_B) + (R_1 - f) \frac{l_B}{2} \cos(\theta_B) = I_B \ddot{\theta}_B \tag{7.62}$$

- Kinematics
  - This is a 2 degree of freedom system. 4 coordinates are provided. $4 - 2 = 2$ scalar kinematic constraint equations are needed.
  - Draw the vector loop and write the corresponding equations

$$x_B \hat{i} + y_B \hat{j} = y_A \hat{j} + \frac{l_B}{2} \cos(\theta_B) \hat{i} + \frac{l_B}{2} \sin(\theta_B) \hat{j} \tag{7.63}$$

  - Write two independent scalar kinematic constraint equations, one for each direction $\hat{i}$ and $\hat{j}$. And differentiate twice.

$\hat{i}$ direction

$$x_B = \frac{l_B}{2} \cos(\theta_B) \tag{7.64}$$

$$\dot{x}_B = -\frac{l_B}{2} \dot{\theta}_B \sin(\theta_B) \tag{7.65}$$

$$\ddot{x}_B = -\frac{l_B}{2} \ddot{\theta}_B \sin(\theta_B) - \frac{l_B}{2} \dot{\theta}_B^2 \cos(\theta_B) \tag{7.66}$$

$\hat{j}$ direction

$$y_B = y_A + \frac{l_B}{2} \sin(\theta_B) \tag{7.67}$$

$$\dot{y}_B = \dot{y}_A + \frac{l_B}{2} \dot{\theta}_B \cos(\theta_B) \tag{7.68}$$

$$\ddot{y}_B = \ddot{y}_A + \frac{l_B}{2} \ddot{\theta}_B \cos(\theta_B) - \frac{l_B}{2} \dot{\theta}_B^2 \sin(\theta_B) \tag{7.69}$$

- Substitution
  - Add Eqns. 7.58 and 7.60

$$\cancel{R_1} + f - \cancel{R_1} = m_A\ddot{y}_A + m_B\ddot{y}_B \tag{7.70}$$

  - Substitute Eq. 7.69 into the above to obtain the first equation of motion

$$\boxed{f = m_A\ddot{y}_A + m_B\left(\ddot{y}_A + \frac{l_B}{2}\ddot{\theta}_B\cos(\theta_B) - \frac{l_B}{2}\dot{\theta}_B^2\sin(\theta_B)\right)} \tag{7.71}$$

  - Eqns. 7.61 and 7.60 into Eq. 7.62

$$(m_B\ddot{x}_B - m_Bg)\,\frac{l_B}{2}\sin(\theta_B) - m_B\ddot{y}_B\frac{l_B}{2}\cos(\theta_B) = I_B\ddot{\theta}_B \tag{7.72}$$

  - Substitute Eqns. 7.66 and 7.69 into the above and simplify to obtain the second equation of motion

$$\begin{aligned}\left(m_B\left(-\frac{l_B}{2}\ddot{\theta}_B\sin(\theta_B) - \cancel{\frac{l_B}{2}\dot{\theta}^2\cos(\theta_B)}\right) - m_Bg\right)\frac{l_B}{2}\sin(\theta_B)&\\ -m_B\left(\ddot{y}_A + \frac{l_B}{2}\ddot{\theta}_B\cos(\theta_B) - \cancel{\frac{l_B}{2}\dot{\theta}_B^2\sin(\theta_B)}\right)\frac{l_B}{2}\cos(\theta_B) &= I_B\ddot{\theta}_B\end{aligned} \tag{7.73}$$

$$\boxed{\left(I_B + m_B\frac{l_B^2}{4}\right)\ddot{\theta}_B + m_B\ddot{y}_A\frac{l_B}{2}\cos(\theta_B) + m_Bg\frac{l_B}{2}\sin(\theta_B) = 0} \tag{7.74}$$

Example 7.5: Balance beam

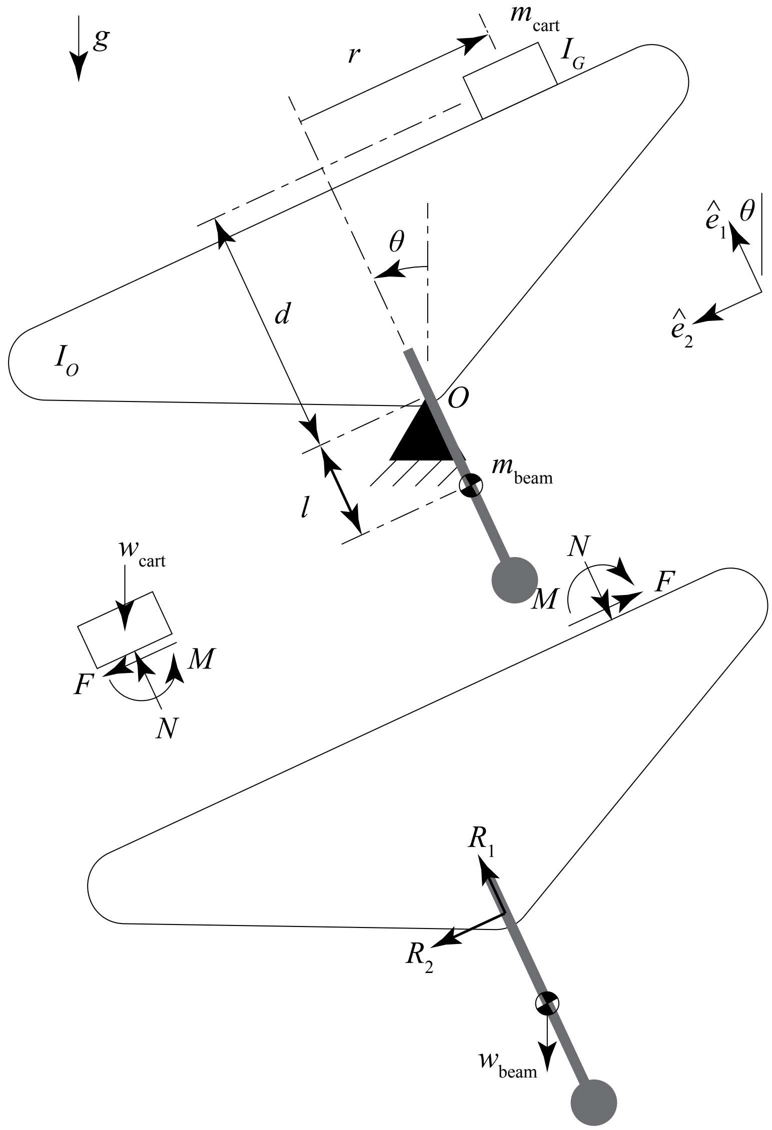


Figure 7.7: The actuator that drives cart with prescribed displacement, r, applies an unknown force, F.

Note, this diagram differs from what is in the pre-lab - update this presentation.

Derive the equation of motion for the balance beam system in terms of the unknown angle θ considering the distance r to be a known input.

- coordinate: θ, $\vec{p}$ (position of the cart), and r is a known input
- FBD
- Force motion
  - Weight

$$\begin{aligned} w_{\text{cart}} &= m_{\text{cart}} g && (7.75) \\ w_{\text{beam}} &= m_{\text{beam}} g && (7.76) \end{aligned}$$

  - $F = ma$ of cart in $\hat{e}_1$ direction

$$N - w_{\text{cart}} \cos(\theta) = m_{\text{cart}} \ddot{\vec{p}} \cdot \hat{e}_1 \tag{7.77}$$

  - $F = ma$ of cart in $\hat{e}_2$ direction

$$F + w_{\text{cart}} \sin(\theta) = m_{\text{cart}} \ddot{\vec{p}} \cdot \hat{e}_2 \tag{7.78}$$

  - $M = I\alpha$ of cart in θ direction (assuming N and F act through center of mass)

$$M = I_G \ddot{\theta} \tag{7.79}$$

  - $M = I\alpha$ of beam in θ direction

$$-Nr - Fd - w_{\text{beam}} l \sin(\theta) - M = I_O \ddot{\theta} \tag{7.80}$$

- Kinematics

$$\begin{aligned} \vec{p}_{\text{cart}} &= d\hat{e}_1 - r\hat{e}_2 && (7.81) \\ \dot{\vec{p}}_{\text{cart}} &= r\dot{\theta}\hat{e}_1 + (d\dot{\theta} - \dot{r})\hat{e}_2 && (7.82) \\ \ddot{\vec{p}}_{\text{cart}} &= \left(r\ddot{\theta} - d\dot{\theta}^2 + 2\dot{r}\dot{\theta}\right)\hat{e}_1 + \left(d\ddot{\theta} + r\dot{\theta}^2 - \ddot{r}\right)\hat{e}_2 && (7.83) \end{aligned}$$

- Substitution (double check the following answer and show work)
  - Sub Eqns. 7.77-7.79 into Eq. 7.80

$$\begin{aligned} &- \left(w_{\text{cart}} \cos(\theta) + m_{\text{cart}} \ddot{\vec{p}} \cdot \hat{e}_1\right) r \\ &- \left(-w_{\text{cart}} \sin(\theta) + m_{\text{cart}} \ddot{\vec{p}} \cdot \hat{e}_2\right) d \\ &- w_{\text{beam}} l \sin(\theta) - I_G \ddot{\theta} = I_O \ddot{\theta} && (7.84) \end{aligned}$$

– Sub Eqns. 7.75, 7.76, and 7.83 intp Eq. 7.84

$$\begin{aligned} -\left(m_{\text{cart}} g\cos(\theta) + m_{\text{cart}}\left(r\ddot{\theta} - d\dot{\theta}^2 + 2\dot{r}\dot{\theta}\right)\right) r & \\ -\left(-m_{\text{cart}} g\sin(\theta) + m_{\text{cart}}\left(d\ddot{\theta} + r\dot{\theta}^2 - \ddot{r}\right)\right) d & \\ -m_{\text{beam}} g l\sin(\theta) - I_G\ddot{\theta} &= I_O\ddot{\theta} \end{aligned} \tag{7.85}$$

– Rearranging

$$\begin{aligned} \left(I_G + I_O + m_{\text{cart}}\left(d^2 + r^2\right)\right)\ddot{\theta} + m_{\text{cart}}\left(2r\dot{r}\dot{\theta} - \ddot{r}d\right) & \\ +\left(m_{\text{beam}} l - m_{\text{cart}} d\right) g\sin(\theta) + m_{\text{cart}} g r\cos(\theta) &= 0 \end{aligned} \tag{7.86}$$

Remember, we consider r to be a known input motion corresponding to the internal force F acting between the cart and beam. Consequently, the system has one degree of freedom and one equation of motion.

- To develop a control algorithm, it might be worth approximating the system for small angle θ and small r (unless we add a balance weight). add linearization to the linearization appendix

7.3.2 Systems With Springs and Dampers

Example 7.6:

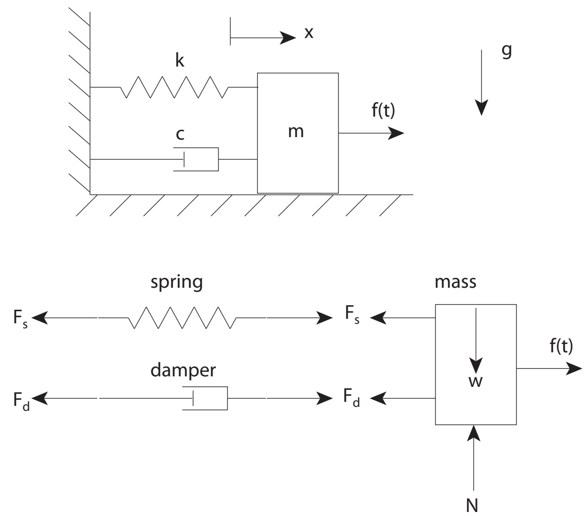


Figure 7.8: Assume the surface is frictionless.

Example - Mass spring damper system.

- Establish coordinates for displacements and rotations and indicate the positive direction.
  Coordinate: x measured from the position at which the spring is stress-free
- Draw free-body diagrams for each independent rigid body (We usually only draw a free-body diagram of the masses for this type of system. However, here we draw free-body diagrams for each element in the system.)
- Write force-motion relationships (input/output equations) for the system
  Spring
  $$F_s = kx \tag{7.87}$$
  Damper
  $$F_d = c\dot{x} \tag{7.88}$$
  weight
  $$w = mg \tag{7.89}$$

$F = ma$ of the mass in the x direction

$$-F_s - F_d + f(t) = m\ddot{x} \tag{7.90}$$

$F = ma$ of the mass in the vertical direction (unnecessary)

$$-w + N = 0 \tag{7.91}$$

- Write kinematic constraint equations relating the coordinates.
  None.

- Use substitution (and possibly equilibrium) to simplify the system of equations.
  Substitute Eqns. 7.87, 7.88, and 7.89 into 7.90

$$-kx - c\dot{x} + f(t) = m\ddot{x} \tag{7.92}$$

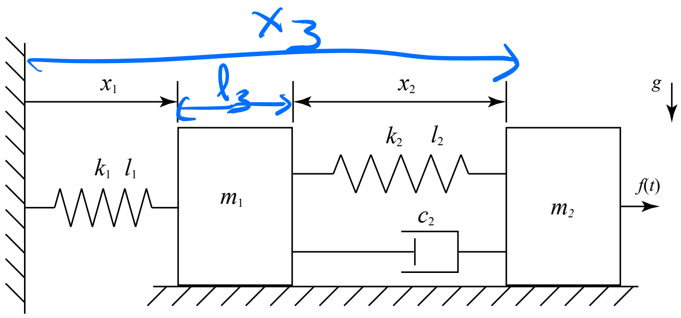


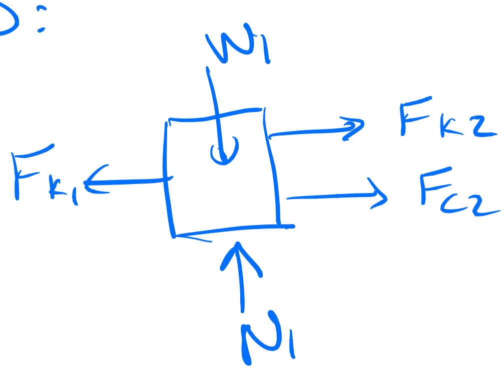
Figure 2: A force $f(t)$ is applied to a system with two masses and springs. Assume the surface is frictionless. Lengths l_1 and l_2 are the free lengths of the two springs. Use this Figure for **Problem 2**.

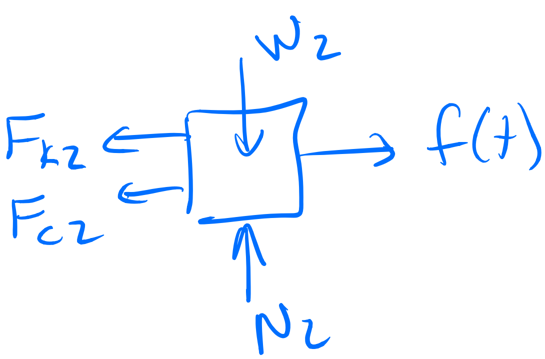
2. Derive the equations of motion for the mechanical system in **Figure 2** in terms of the coordinates x_1 and x_2.

---

coordinates: x_1 x_2 x_3

FBD:





Input output

$\Sigma F = ma$ m_1 x_1-dir

$$-F_{k1} + F_{k2} + F_{c2} = m_1 \ddot{x}_1 \quad ①$$

$\Sigma F = ma$ m_2 x_3-dir

$$-F_{k2} - F_{c2} + f(t) = m_2 \ddot{x}_3 \quad ②$$

k_1

$$F_{k1} = k_1(x_1 - l_1) \quad ③$$

k_2

$$F_{k2} = k_2(x_2 - l_2) \quad ④$$

c_2

$$F_{c2} = c_2 \dot{x}_2 \quad ⑤$$

kinematics

$$x_3 = x_1 + l_3 + x_2$$

$$\dot{x}_3 = \dot{x}_1 + \dot{x}_2$$

$$\ddot{x}_3 = \ddot{x}_1 + \ddot{x}_2 \quad ⑥$$

substitution

equations ③ - ⑤ into ①

$$-k_1(x_1-l_1)+k_2(x_2-l_2)+c_2\dot{x}_2 = m_1\ddot{x}_1$$

rearranging

$$\boxed{m_1\ddot{x}_1 - c_2\dot{x}_2 + k_1x_1 - k_2x_2 = k_1l_1 - k_2l_2}$$

equations ④ - ⑥ into ②

$$-k_2(x_2-l_2) - c_2\dot{x}_2 + f(t) = m_2(\ddot{x}_1+\ddot{x}_2)$$

rearranging

$$\boxed{m_2\ddot{x}_1 + m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 = k_2l_2 + f(t)}$$

Example 7.7:

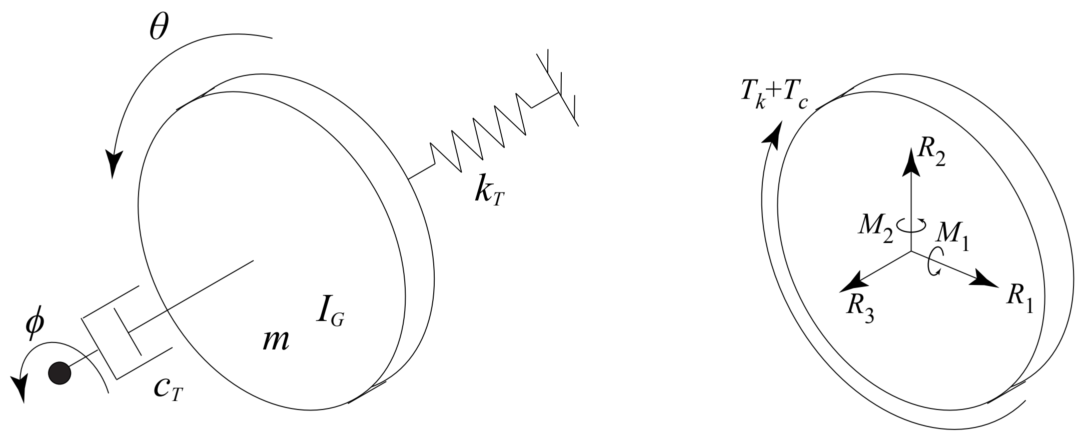


Figure 7.9: A known angular rotation $\phi(t)$ is applied to one end of a torsional damper. Assume the disk is supported by a frictionless bearing. Note, the damper is a torsional damper and the spring is a torsional spring. The coordinate θ measures the rotation of the disk from the state in which the spring is stress-free.

Derive the equation(s) of motion for the mechanical system in **Figure 7.9**.

- Establish coordinates: θ, ϕ
- Free-body diagram
  - R_1, R_2, and R_3 reactions forces at shaft
  - M_1 and M_2 reactions moments at shaft
  - T_k torque from spring
  - T_c torque from damper
- Write force-motion relationships
  - Spring

$$T_k = k_T\theta \tag{7.93}$$

  - Damper

$$T_c = c_T(\dot{\theta} - \dot{\phi}) \tag{7.94}$$

– Moment equation about shaft in θ direction

$$-T_k - T_c = I_G\ddot{\theta} \tag{7.95}$$

- Kinematics - NA
- Substitution
  – Substitute Equations 7.93 and 7.94 into Equation 7.95

$$-k_T\theta - c_T(\dot{\theta} - \dot{\phi}) = I_G\ddot{\theta} \tag{7.96}$$

7.3.3 Coordinates Measured From an Equilibrium State

Example 7.8:

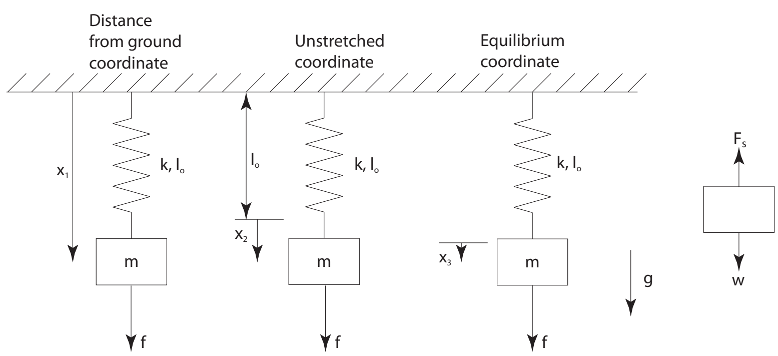


Figure 7.10: Here, l_o is the free length of the spring.

Example - possible coordinate choices for mass-spring system

- Establish coordinates for displacements and rotations and indicate the positive direction.
  Here we might consider three choices of coordinates: displacement from ground (x_1), displacement from unstretched position (x_2), and displacement from equilibrium position (x_3) with $f = 0$
- Draw free-body diagrams for each independent rigid body.
- Write force-motion relationships (input/output equations) for the system
  Spring (three expressions for the spring force depending on the choice of coordi-

nate)

$$F_s = k(x_1 - l_o) \qquad F_s = kx_2 \qquad F_s = k(\delta + x_3) \tag{7.97}$$

Here δ is the equilibrium stretch of the spring when $f = 0$. At this point in our derivation, δ is unknown.
Gravity

$$w = mg \tag{7.98}$$

Force balance ($F = ma$) in the downward direction. Three equations are provided depending on your choice of coordinate.

$$m\ddot{x}_1 = w - F_s + f \qquad m\ddot{x}_2 = w - F_s + f \qquad m\ddot{x}_3 = w - F_s + f \tag{7.99}$$

- Write kinematic constraint equations relating the coordinates.
  One coordinate and one degree of freedom so no need for constraints. However, since we are deriving the equations of motion 3 times all at once, we can relate the coordinates that we are using.

$$x_1 = l_o + x_2 = l_o + \delta + x_3 \tag{7.100}$$

- Use substitution (and possibly equilibrium) to simplify the system of equations.

$$m\ddot{x}_1 = mg - k(x_1 - l_o) + f \qquad m\ddot{x}_2 = mg - kx_2 + f \qquad m\ddot{x}_3 = mg - k(\delta + x_3) + f \tag{7.101}$$

We can simplify the third equation using equilibrium (At equilibrium with $f = 0$, $\dot{x}_3 = 0$, $\ddot{x}_3 = 0$, and because x_3 is measured from equilibrium $x_3 = 0$)

$$0 = mg - k\delta \tag{7.102}$$

Therefore

$$m\ddot{x}_3 = -kx_3 + f \tag{7.103}$$

- Note: Using the coordinate x_3 yields the simplest equation of motion. For this reason, typically coordinates measured from an equilibrium state (like x_3 in this case) are preferred.

Example 7.9:

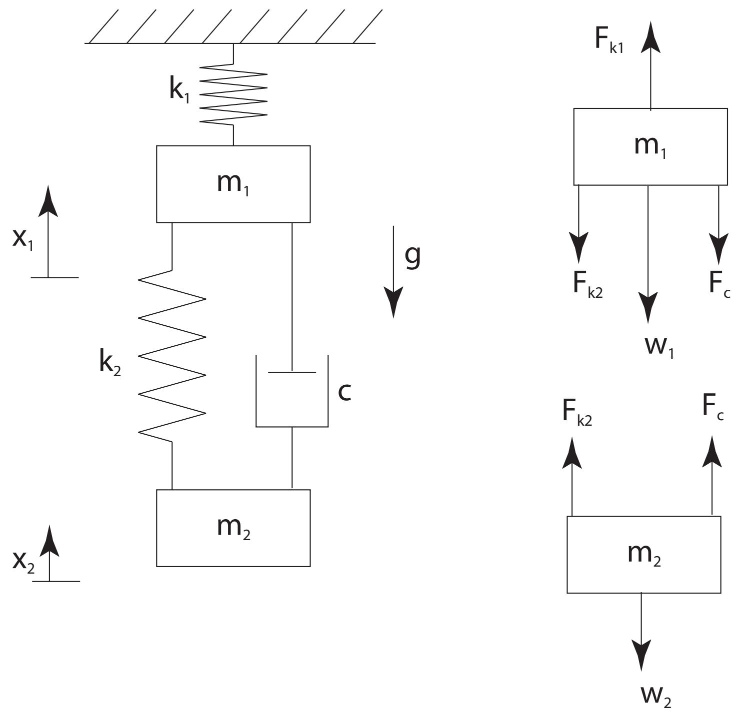


Figure 7.11:

A 2 degree of freedom system

- Establish coordinates for displacements and rotations and indicate the positive direction.
  Measure x_1 and x_2 as displacements from equilibrium position.
- Draw free-body diagrams for each independent rigid body.
- Write force-motion relationships (input/output equations) for the system
  Springs

$$F_{k1} = k_1(-x_1 + \delta_1) \tag{7.104}$$

$$F_{k2} = k_2(x_1 - x_2 + \delta_2) \tag{7.105}$$

  Here we define δ_1 and δ_2 as the equilibrium stretch in the springs
  Damper

$$F_c = c(\dot{x}_1 - \dot{x}_2) \tag{7.106}$$

Gravity

$$w_1 = m_1 g \tag{7.107}$$
$$w_2 = m_2 g \tag{7.108}$$

Force balance ($F = ma$) for mass m_1 in the positive x_1 direction

$$F_{k1} - w_1 - F_{k2} - F_c \quad = \quad m_1 \ddot{x}_1 \tag{7.109}$$

Force balance ($F = ma$) for mass m_2 in the positive x_2 direction

$$F_{k2} + F_c - w_2 \quad = \quad m_2 \ddot{x}_2 \tag{7.110}$$

- Write kinematic constraint equations relating the coordinates.
  Two coordinates and two degrees of freedom, so no need for constraint equations.
- Use substitution (and possibly equilibrium) to simplify the system of equations.
  Substitute Eqns. 7.104, 7.105, 7.106, and 7.107 into Eq. 7.109.

$$k_1(-x_1 + \delta_1) - m_1 g - k_2(x_1 - x_2 + \delta_2) - c(\dot{x}_1 - \dot{x}_2) \quad = \quad m_1 \ddot{x}_1 \tag{7.111}$$

Substitute Eqns. 7.105, 7.106, and 7.108 into Eq. 7.110.

$$k_2(x_1 - x_2 + \delta_2) + c(\dot{x}_1 - \dot{x}_2) - m_2 g \quad = \quad m_2 \ddot{x}_2 \tag{7.112}$$

Note, at equilibrium $\ddot{x}_1 = \dot{x}_1 = \ddot{x}_2 = \dot{x}_2 = 0$. Furthermore, because our coordinates are measured from equilibrium $x_1 = x_2 = 0$ at equilibrium

$$k_1\delta_1 - m_1 g - k_2\delta_2 = 0 \tag{7.113}$$

$$k_2\delta_2 - m_2 g = 0 \tag{7.114}$$

Therefore, we can cancel terms from our equations of motion:

$$m_1 \ddot{x}_1 = -c(\dot{x}_1 - \dot{x}_2) - k_1 x_1 - k_2(x_1 - x_2) + \cancelto{0}{k_1\delta_1 - m_1 g - k_2\delta_2} \tag{7.115}$$

$$m_2 \ddot{x}_2 = c(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) + \cancelto{0}{k_2\delta_2 - m_2 g} \tag{7.116}$$

and finally we obtain

$$m_1 \ddot{x}_1 \quad = \quad -c(\dot{x}_1 - \dot{x}_2) - k_1 x_1 - k_2(x_1 - x_2) \tag{7.117}$$
$$m_2 \ddot{x}_2 \quad = \quad c(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) \tag{7.118}$$

Example 7.10:

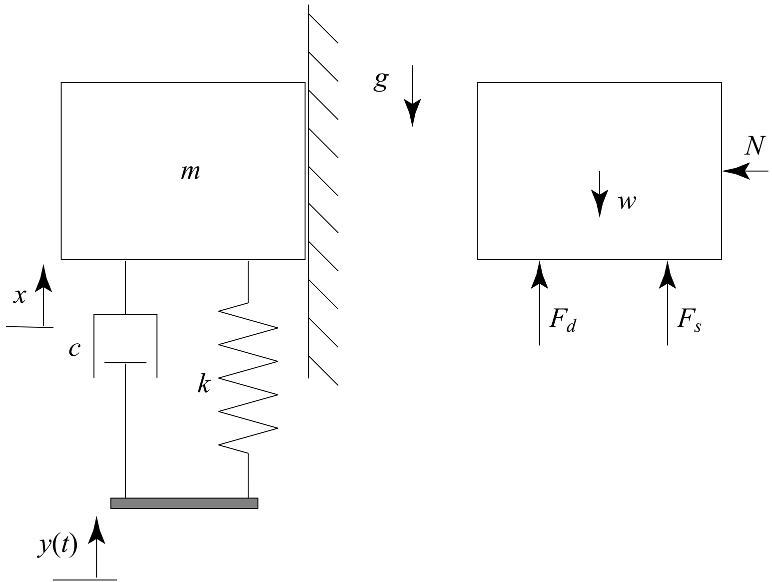


Figure 7.12: A known input displacement $y(t)$ is applied to a massless rigid bar which is connected to a mass spring damper system. Assume the mass remains in contact with and slides along a frictionless surface. The coordinate x is measured from the equilibrium with $y = 0$.

Derive the equation of motion for the mechanical system in Figure 7.12 in terms of the coordinate x.

- Establish coordinates: x, y, and δ the equilibrium compression in the spring
- Free-body diagrams
  - w weight
  - N normal force
  - F_d compressive force supported by damper
  - F_s compressive force supported by spring
- Write force-motion relationships
  - Weight

$$w = mg \tag{7.119}$$

- Translation in x direction

$$F_d + F_s - w = m\ddot{x} \tag{7.120}$$

- Spring

$$F_s = k(y - x + \delta) \tag{7.121}$$

- Damper

$$F_d = c(\dot{y} - \dot{x}) \tag{7.122}$$

- Kinematics - NA
- Substitution
  - Substitute Eqns. 7.119, 7.121, and 7.122 into Eq. 7.120

$$c(\dot{y} - \dot{x}) + k(y - x + \delta) - mg = m\ddot{x} \tag{7.123}$$

  - The equilibrium with $y = 0$ corresponds to $x = 0$ such that

$$k\delta - mg = 0 \tag{7.124}$$

  - Therefore

$$c(\dot{y} - \dot{x}) + k(y - x) = m\ddot{x} \tag{7.125}$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky} \tag{7.126}$$

Part III

Dynamic System Response and Representation

Chapter 8

Laplace Transform

8.1 Introduction to Differential Equations

- See, for example, [35]
- Independent vs. dependent variables
  - Independent variable - a variable that changes independent from all other variables. Usually in this class, time is the independent variable.
  - Dependent variable - variable that depends on other variables. In this class, the dependent variable is often called the 'output' of the system.
- Ordinary vs. partial differential equations
  - Ordinary differential equations (ODEs) - ordinary differential equations consist of dependent variables and their derivatives with respect to a single independent variable. For this class, the independent variable is usually time.
  - Partial differential equations (PDEs)- partial differential equations consist of dependent variables and their derivatives with respect to more than one independent variable. We will focus on ODEs in this class.
- Initial conditions - ODEs typically have a family of solutions. Known values of the dependent variable and its derivatives at a point in time allow us to find a specific solution. Usually the point in time is $t = 0$ and therefor the values are known as initial conditions. An ODE with specified ICs is known as an initial value problem.
- Order of ODEs - The order of an ODE is the order its highest derivative.
- System of differential equations. When two or more differential equations must be solved simultaneously, they form a system of differential equations.

- Linear vs. nonlinear dynamic systems
  - The response of a dynamic system at time t represented by $y(t)$ depends upon the system's initial conditions $(y(0), \dot{y}(0), ...)$ and the time history of the input $u(t)$ from $t = 0$ to t.
  - Linear dynamic system properties
    * If y_{h1} and y_{h2} are solutions to the homogeneous ode

$$a_2(t)\ddot{y} + a_1(t)\dot{y} + a_0(t)y = 0 \tag{8.1}$$

      then αy_{h1} and $\alpha y_{h1} + \beta y_{h2}$ are solutions to the ode. Note no ICs have been provided - we are not considering an IVP at this point.
    * If y_{p1} is a particular solution to the ode

$$a_2(t)\ddot{y} + a_1(t)\dot{y} + a_0(t)y = u_1(t) \tag{8.2}$$

      and x_{p2} is a particular solution to the ode

$$a_2(t)\ddot{y} + a_1(t)\dot{y} + a_0(t)y = u_2(t) \tag{8.3}$$

      then:
      · αy_{p1} is a particular solution to the ode

$$a_2(t)\ddot{y} + a_1(t)\dot{y} + a_0(t)y = \alpha u_1(t) \tag{8.4}$$

      (i.e., this property is termed 'homogeneity' in, for example, [7])
      · $y_{p1} + y_{p2}$ is a particular solution to the ode

$$a_2(t)\ddot{y} + a_1(t)\dot{y} + a_0(t)y = u_1(t) + u_2(t) \tag{8.5}$$

      (i.e., this property is termed 'additivity' in, for example, [7])
      · $\alpha \dot{y}_{p1}$ is a particular solution to the ode

$$a_2(t)\ddot{y} + a_1(t)\dot{y} + a_0(t)y = \alpha \dot{u}_1(t) \tag{8.6}$$

  - Identification
    * "... a linear n^{th} order [dynamic system] is one whose n^{th} derivative depends in a linear way on each of the lower derivatives, and also in a linear way on the forcing function, if any [7]."
    * Linear examples:
      · $\dot{y} + 2y = 3t^4$
      · $\dot{y} + 2t^3 y = 4$

- $\ddot{y} + 2\dot{y} + 3y = 4\sin(5t)$
- $\dot{y} + 2t^3 y = u$
- $\ddot{y} + 2\dot{y} + 3y = 4\sin(5t)u$

* Nonlinear examples:
  - $\ddot{y} + 2\sin(y) = 0$
  - $\ddot{y} + \dot{y}^2 + 3y = 4$
  - $\ddot{y} + 2y\dot{y} + 3y = 4$
  - $\ddot{y} + 2yu = 0$
  - $\ddot{y} + \dot{y}^2 + 3y = u$
  - $\ddot{y} + 2y + 3y = \dot{u}u$

- Linear differential equation vs. linear dynamic system
  * It appears that in the context of differential equations, arbitrary inputs (u) are not always considered and linearity with respect to the input isn't always discussed.
- Standard form for linear ODEs
  * Lefthand side - functions of the dependent variable.
  * Righthand side - everything else. Sometimes called the input or forcing function.
- Homogeneous vs non-homogeneous Linear ODE
  * Homogeneous linear ODE - All terms in the linear ODE include a dependent variable. (In standard form, the right hand side is zero.)
  * Non-homogeneous linear ODE - At least some terms in the linear ODE do not include a dependent variable. (In standard form, the right hand side is nonzero.)
- Linear Time Invariant (LTI) systems -
  * A linear time invariant system is a system of ordinary differential equations in which the coefficients to the dependent variables do not depend on time. (However, the input or forcing terms may depend on time.)
  * Examples: $\dot{y} + 2y = 3$, $\dot{y} + 2y = 3t^4$
  * Not LTI: $\dot{y} + 2t^3 y = 4$, $\ddot{y} + 2\sin(3t)\dot{y} + 4y = 5$
  * In general, LTI systems are easier to solve than linear time varying ODEs and nonlinear ODEs.

8.2 Laplace Transform Definition and Properties

- Introduction to Laplace transform
  - Method of undetermined coefficients (or trial solution method).
    * Review the method of undetermined coefficients in Appendix C.

  * The method of undetermined coefficients leverages the unique relationship between the exponential function, e^{st}, and it's derivative

$$\frac{d}{dt}\left(e^{st}\right) = s\left(e^{st}\right) \tag{8.7}$$

    for constant s.
  * The homogeneous solution of an LTI system of differential equations is the superposition of exponential terms

$$y_h = D_1 e^{s_1 t} + D_2 e^{s_2 t} + ... \tag{8.8}$$

- A major advantage of Laplace transform methods (over the method of undetermined coefficients, for example) is that it can convert differential equations into algebraic equations.
- Review complex numbers
  * complex number

$$j = \sqrt{-1} \tag{8.9}$$

  * Euler's equation and trigonometric relationships

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{8.10}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{8.11}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{8.12}$$

• Definition of Laplace Transform

- Definition and notation:

$$L[y(t)] = \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-st} y(t) dt\right] \tag{8.13}$$

$$Y(s) = L[y(t)] \tag{8.14}$$

  * In order to evaluate this integral, we restrict the domain of the complex number s to include only those values that annihilate the integrand as $t \to \infty$.
  * In this class we will use a one-sided Laplace transform (with the lower limit of integration set to 0^-).
  * See [36] for a discussion on the lower limit.
  * Inner product interpretation of the Laplace transform
    · The inner product of two vectors, $\vec{a} = \{a_1, a_2, a_3\}^T$ and $\vec{b} = \{b_1, b_2, b_3\}^T$, (also known as as the dot product) can be written as

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{8.15}$$

- The inner product of two vectors is a measure of their alignment. For example, the inner product of orthogonal/perpendicular vectors is zero and the inner product of parallel unit vectors is 1. Also, the inner product of vectors may be used to resolve the components of a vector into a basis of orthogonal unit vectors.
- We can define the inner product of two functions, $a(t)$ and $b(t)$, over an interval of time, 0 to T, as
$$\langle a(t), b(t)\rangle = \int_0^T a(t)b(t)dt \tag{8.16}$$
- The inner product of two functions is a measure of how much the functions look alike and may be used to resolve a function into a basis of orthonormal functions.
- The Laplace transform may be thought of as a measure of how much a function, $y(t)$, looks like the exponential function, e^{-st}, considering a range of values of s. This is significant because De^{st} is the assumed form of the homogeneous solution in the method of undetermined coefficients. (Note, because s in the method of undetermined coefficients is treated as a constant, we don't need to be too concerned about the distinction between s and $-s$.)

– Inverse Laplace transform notation
$$y(t) = L^{-1}[Y(s)] \tag{8.17}$$

- Table of common Laplace transforms [See, for example, [35, 1, 6]]

Table 8.1: Common Laplace transforms [See, for example, [35, 1, 6]]

$y(t),\ t \geq 0$	$Y(s) = \int_{0^-}^{\infty} y(t)e^{-st}dt$
$\delta(t)$	1
$u_s(t)$	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$

- Derivations of common Laplace transforms [See, for example, [2, 1]]
  - Impulse
    - Definition of unit impulse function (see, for example, [8, 35])

$$\delta(t) = \lim_{D\to 0}\begin{cases} 0 & t<0 \\ \frac{1}{D} & 0 \le t \le D \\ 0 & D<t \end{cases} \tag{8.18}$$

      - Infinite at $t = 0$, 0 elsewhere
      - Unit area, $\int_{-\infty}^{\infty}\delta(t)dt = 1$
      - Could be used to represent a bat hitting a baseball, for example.
    - Laplace transform

$$\begin{aligned} L[\delta(t)] &= \lim_{T\to\infty}\left[\int_{0^-}^{T}\delta(t)e^{-st}dt\right] && (8.19)\\ &= \lim_{D\to 0}\int_{0^-}^{D}\delta(t)e^{-st}dt && (8.20)\\ &= e^{-s0}\lim_{D\to 0}\int_{0^-}^{D}\frac{1}{D}dt && (8.21)\\ &= \lim_{D\to 0}\frac{D}{D} && (8.22)\\ &= 1 && (8.23)\end{aligned}$$

  - Constant $y(t) = 1$

$$\begin{aligned} L[x(t)] &= \lim_{T\to\infty}\left[\int_{0^-}^{T}e^{-st}1dt\right] && (8.24)\\ &= \lim_{T\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^T && (8.25)\\ &= \lim_{T\to\infty}\left[-\frac{1}{s}e^{-sT}+\frac{1}{s}e^{-s0}\right] && (8.26)\\ &= \frac{1}{s} && (8.27)\end{aligned}$$

    - Note, here we require $\mathbb{R}\{s\} > 0$ for the limit to be obtained.
  - Step function $y(t) = u_s(t)$. (see, for example, [35])
    - Unit step function

$$u_s(t) = \begin{cases} 0 & t<0 \\ \text{undefined} & t=0 \\ 1 & 0<t \end{cases} \tag{8.28}$$

- · The unit step function is undefined at $t = 0$.
- ∗ Laplace transform

$$L[x(t)] = \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-st}u_s(t)dt\right] \tag{8.29}$$

$$= \lim_{T\to\infty}\left[\cancelto{0}{\int_{0^-}^{0} e^{-st}u_s(t)dt} + \int_{0^+}^{T} e^{-st}dt\right] \tag{8.30}$$

$$= \lim_{T\to\infty}\left[-\frac{1}{s}e^{-st}\right]_{0^+}^{T} \tag{8.31}$$

$$= \lim_{T\to\infty}\left[-\frac{1}{s}e^{-sT} + \frac{1}{s}e^{-s0^+}\right] \tag{8.32}$$

$$= \frac{1}{s} \tag{8.33}$$

- ∗ Note, here we require $\mathbb{R}\{s\} > 0$ for the limit to be obtained and we assume $u_s(0)$ is not infinite.
- ∗ The constant and step functions have the same value for all $t > 0$ and differ only for $t \leq 0$. Consequently, they have the same Laplace transform.

- Ramp function $y(t) = tu_s(t)$
  * Ramp function
$$y(t) = tu_s(t) \tag{8.34}$$
  * Laplace transform
$$\begin{aligned}
L[x(t)] &= \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-st}tu_s(t)dt\right] && (8.35)\\
&= \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-st}tdt\right] && (8.36)\\
&= \lim_{T\to\infty}\left[-\frac{t}{s}e^{-st}\Big|_{0^-}^{T} + \int_{0^-}^{T}\frac{1}{s}e^{-st}dt\right] && (8.37)\\
&= \lim_{T\to\infty}\left[\frac{T}{s}e^{-sT} - \frac{1}{s^2}e^{-st}\Big|_{0^-}^{T}\right] && (8.38)\\
&= \lim_{T\to\infty}\left[\frac{T}{s}e^{-sT} - \frac{1}{s^2}e^{-sT} + \frac{1}{s^2}\right] && (8.39)\\
&= \cancelto{0}{\lim_{T\to\infty}\left[\frac{T}{s}e^{-sT} - \frac{1}{s^2}e^{-sT}\right]} + \frac{1}{s^2} && (8.40)\\
&= \frac{1}{s^2} && (8.41)
\end{aligned}$$
    · Note, here used integration by parts.
  * Note, here we require $\mathbb{R}\{s\} > 0$ for the limit to be obtained.
- Exponential function $y(t) = e^{-at}$
$$\begin{aligned}
L[y(t)] &= \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-st}e^{-at}dt\right] && (8.42)\\
&= \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-(s+a)t}dt\right] && (8.43)\\
&= \lim_{T\to\infty}\left[-\frac{1}{s+a}e^{-(s+a)T} + \frac{1}{s+a}e^{-(s+a)0}\right] && (8.44)\\
&= \frac{1}{s+a} && (8.45)
\end{aligned}$$
  * Note, here we require $\mathbb{R}\{s+a\} > 0$ for the limit to be obtained.

- Sine and cosine functions (See, for example, [2])
  - Derive the Laplace transform for

$$y(t) = e^{(-a+jb)t} \tag{8.46}$$

  - Now consider the Laplace transform of

$$\begin{aligned}
L[y(t)] &= \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-st}e^{(-a+jb)t}dt\right] && (8.47)\\
&= \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-(s+a-jb)t}dt\right] && (8.48)\\
&= \lim_{T\to\infty}\left[-\frac{1}{s+a-jb}e^{-(s+a-jb)t}\right]_{0^-}^{T} && (8.49)\\
&= \frac{1}{s+a-jb} && (8.50)\\
& && (8.51)
\end{aligned}$$

  - Note, here we require $\mathbb{R}\{s+a\} > 0$ for the limit to be obtained.
  - Multiply numerator and denominator by the complex conjugate of denominator to see complex and real part

$$\begin{aligned}
L[y(t)] &= \frac{1(s+a+jb)}{(s+a-jb)(s+a+jb)} && (8.52)\\
&= \frac{s+a+jb}{(s+a)^2+b^2} && (8.53)\\
&= \frac{s+a}{(s+a)^2+b^2} + j\frac{b}{(s+a)^2+b^2} && (8.54)\\
& && (8.55)
\end{aligned}$$

  - Employ the Euler relation $e^{j\theta} = \cos\theta + j\sin(\theta)$

$$\begin{aligned}
e^{-at}\cos(bt) &= \mathbb{R}\left\{e^{(-a+jb)t}\right\} && (8.56)\\
L[e^{-at}\cos(\omega t)] &= L\left[\mathbb{R}\left\{e^{(-a+jb)t}\right\}\right] && (8.57)\\
&= \mathbb{R}\left\{L\left[e^{(-a+jb)t}\right]\right\} && (8.58)\\
&= \frac{s+a}{(s+a)^2+b^2} && (8.59)
\end{aligned}$$

By linearity we can change the order of $L[]$ and $\mathbb{R}\{\}$. Similarly,

$$\begin{aligned}
e^{-at}\sin(bt) &= \mathbb{I}\left\{e^{(-a+jb)t}\right\} && (8.60)\\
L[e^{-at}\sin(\omega t)] &= L\left[\mathbb{I}\left\{e^{(-a+jb)t}\right\}\right] && (8.61)\\
&= \mathbb{I}\left\{L\left[e^{(-a+jb)t}\right]\right\} && (8.62)\\
&= \frac{b}{(s+a)^2+b^2} && (8.63)
\end{aligned}$$

* In addition, we can set $a = 0$ to obtain Laplace transforms for $\sin(\omega t)$ and $\cos(\omega t)$

- Table of Laplace transform properties (see, for example, [35, 1, 6])

Table 8.2: Laplace transforms properties [See, for example, [35, 1, 6]]

$y(t)$	$Y(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$e^{-at}f(t)$	$F(s+a)$
$tf(t)$	$-\frac{dF(s)}{ds}$
$u_s(t-D)f(t-D)$	$e^{-sD}F(s)$
$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1}f(t)}{dt^{k-1}}\Big\vert_{t=0^-}$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
$\lim_{t\to 0^+} y(t) = \lim_{s\to\infty} sY(s)$	
$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$	

- Derivation of Laplace transform properties (see, for example, [2, 1])
  - Linearity

$$L[af(t)+bg(t)] = aL[f(t)] + bL[g(t)] \tag{8.64}$$

$$L^{-1}[aF(s)+bG(s)] = af(t)+bg(t) \tag{8.65}$$

    * Proof

$$\begin{aligned} L[af(t)+bg(t)] &= \lim_{T\to\infty}\int_{0^-}^{T} e^{-st}\left(af(t)+bg(t)\right)dt && (8.66)\\ &= a\lim_{T\to\infty}\int_{0^-}^{T} e^{-st}f(t)dt \\ &\quad +b\lim_{T\to\infty}\int_{0^-}^{T} e^{-st}g(t)dt && (8.67)\\ &= aF(s)+bG(s) && (8.68)\end{aligned}$$

- "Shifting along s-axis," "shift in frequency," or "multiplication (modulation) of $f(t)$ by an exponential expression" (see, [8, 35])

$$L[e^{-at}y(t)] = Y(s+a) \tag{8.69}$$

  * Proof

$$\begin{aligned} L[e^{-at}y(t)] &= \lim_{T\to\infty}\int_{0^-}^{T} e^{-st}e^{-at}y(t)dt && (8.70)\\ &= \lim_{T\to\infty}\int_{0^-}^{T} e^{-(s+a)t}y(t)dt && (8.71)\\ &= Y(s+a) && (8.72)\end{aligned}$$

- Multiplication by time (t)

$$L[ty(t)] = -\frac{dY(s)}{ds} \tag{8.73}$$

  * Proof

$$\begin{aligned} -\frac{d}{ds}L[y(t)] &= -\frac{d}{ds}\lim_{T\to\infty}\int_{0^-}^{T} e^{-st}y(t)dt && (8.74)\\ &= \lim_{T\to\infty}\int_{0^-}^{T} te^{-st}y(t)dt && (8.75)\\ &= L[ty(t)] && (8.76)\end{aligned}$$

- Shifting along t-axis (time delay)

$$\begin{aligned} y(t) &= \begin{cases} 0 & t < T_0 \\ f(t-T_0) & t > T_0. \end{cases} && (8.77)\\ &= u_s(t-T_0)f(t-T_0) && (8.78)\end{aligned}$$

$$Y(s) = e^{-sT_0}F(s) \tag{8.79}$$

  * Proof (see, for example, [8])

$$\begin{aligned} L[y(t)] &= \lim_{T\to\infty}\int_{0^-}^{T} e^{-st}u_s(t-T_0)f(t-T_0)dt && (8.80)\\ &= \lim_{T\to\infty}\int_{-T_0}^{T} e^{-s(\tau+T_0)}u_s(\tau)f(\tau)d\tau && (8.81)\\ &= e^{-sT_0}\lim_{T\to\infty}\int_{0^-}^{T} e^{-s\tau}f(\tau)d\tau && (8.82)\\ &= e^{-sT_0}F(s) && (8.83)\end{aligned}$$

Here we defined $\tau = t - T_0$. Note, $u_s(\tau)f(\tau)$ is 0 for $\tau < T_0$.

- Derivative property (differentiation) (see [2])

$$L\left[\frac{d^n y}{dt^n}\right] = s^n Y(s) - \sum_{k=1}^{n} s^{n-k} \left.\frac{d^{k-1}y}{dt^{k-1}}\right|_{t=0^-} \tag{8.84}$$

  * Proof for $n = 1$ (see derivation in [2])

$$L\left[\frac{dy}{dt}\right] = \lim_{T\to\infty} \int_{0^-}^{T} \frac{dy}{dt} e^{-st} dt \tag{8.85}$$

  employ integration by parts ($\int_a^b u dv = uv|_a^b - \int_a^b v du$) with $u = e^{-st}$ and $dv = \frac{dy}{dt}$

$$= \lim_{T\to\infty} \left. y e^{-st}\right|_{0^-}^{T} - \lim_{T\to\infty} \int_{0^-}^{T} (-s) y e^{-st} dt \tag{8.86}$$

$$= -y(0) + s \lim_{T\to\infty} \int_{0^-}^{T} y e^{-st} dt \tag{8.87}$$

$$= -y(0) + sL[y(t)] \tag{8.88}$$

$$= -y(0) + sY(s) \tag{8.89}$$

- Integration
- Initial value theorem
  * Result

$$y(0^+) = \lim_{t\to 0^+} y(t) \tag{8.90}$$

$$= \lim_{s\to\infty} [sY(s)] \tag{8.91}$$

  * Limitations (see, for example, [35])
    · Theorem is valid if the limit exists (degree of the numerator of $X(s)$ is less than that of the denominator).
    · Laplace transforms of $y(t)$ and dy/dt must exist
    · $y(t)$ must be either continuous or have a step discontinuity at $t = 0$ [1]
    · "In contrast with the Final Value Theorem, the Initial Value Theorem can be applied to any function $F(s)$ [8]."
  * Proof (see wikipedia and [8])
    · From the derivative property we have

$$L\left[\frac{dy}{dt}\right] = sY(s) - x(0^-) \tag{8.92}$$

$$sY(s) = y(0^-) + L\left[\frac{dy}{dt}\right] \tag{8.93}$$

· Now we take the limit as $s \to \infty$

$$\begin{aligned}
\lim_{s\to\infty} sY(s) &= x(0^-) + \lim_{s\to\infty} L\left[\frac{dy}{dt}\right] && (8.94)\\
&= y(0^-) + \lim_{s\to\infty} \int_{0^-}^{\infty} \frac{dy}{dt} e^{-st} dt && (8.95)\\
&= y(0^-) + \lim_{s\to\infty} \int_{0^-}^{0^+} \frac{dy}{dt} e^{-st} dt + \cancelto{0}{\lim_{s\to\infty} \int_{0^+}^{\infty} \frac{dy}{dt} e^{-st} dt} && (8.96)\\
&= y(0^-) + y(0^+) - y(0^-) && (8.97)\\
&= y(0^+) && (8.98)
\end{aligned}$$

– Final value theorem

  ∗ Equation

$$\begin{aligned}
y(\infty) &= \lim_{t\to\infty} y(t) && (8.99)\\
&= \lim_{s\to 0} sY(s) && (8.100)
\end{aligned}$$

  ∗ Limitations [35]

    · $y(t)$ and $\frac{dy}{dt}$ must posses Laplace transforms

    · "If all poles of $sY(s)$ are in the left half of the s-plane, then $\lim_{t\to\infty} = \lim_{s\to 0} sY(s)$ [8]." (This is equivalent to stating that $\lim_{t\to\infty} y(t)$ must be a constant [35].)

  ∗ Proof (see wikipedia and [8])

    · From the derivative property we have

$$\begin{aligned}
\int_{0^-}^{\infty} \frac{dy(t)}{dt} e^{-st} dt &= sY(s) - y(0^-) && (8.101)\\
\lim_{s\to 0} \int_{0^-}^{\infty} \frac{dy(t)}{dt} e^{-st} dt &= \lim_{s\to 0} \left(sY(s) - y(0^-)\right) && (8.102)\\
\int_{0^-}^{\infty} \frac{dy(t)}{dt} dt &= \lim_{s\to 0} \left(sY(s) - y(0^-)\right) && (8.103)\\
y(\infty) - \cancel{y(0^-)} &= \lim_{s\to 0} \left(sY(s) - \cancel{y(0^-)}\right) && (8.104)\\
y(\infty) &= \lim_{s\to 0} sY(s) && (8.105)
\end{aligned}$$

• Inverse Laplace transform by partial fraction expansion (see [2, 8])

– Transforms usually result in rational function something like

$$Y(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0} \tag{8.106}$$

– We will focus on the case when $m \leq n$ (proper rational functions)
– To simplify the inverse Laplace transform we factor the denominator
– For distinct poles (roots of the denominator) using the "cover-up method" [8])
  * For poles $s = p_1,\ p_2,\ ...,\ p_n$ we can write as:

$$Y(s) = \frac{N(s)}{(s-p_1)(s-p_2)...(s-p_n)} \tag{8.107}$$
$$= \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + ... + \frac{C_n}{s-p_n} \tag{8.108}$$

  * It can be shown that

$$C_i = \lim_{s \to p_i} [Y(s)(s-p_i)] \tag{8.109}$$

  * The result for $x(t)$ is then

$$y(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} + ... + C_n e^{p_n t} \tag{8.110}$$

– For repeated poles (roots of the denominator)
  * For poles $s = p_1,\ p_2,\ ...,\ p_n$ and suppose k roots are identical, we can write as:

$$Y(s) = \frac{N(s)}{(s-p_1)^k (s-p_{k+1})(s-p_{k+2})...(s-p_n)} \tag{8.111}$$
$$= \frac{C_1}{(s-p_1)^k} + \frac{C_2}{(s-p_1)^{k-1}} + ... + \frac{C_p}{s-p_1} \tag{8.112}$$
$$+ \frac{C_{k+1}}{s-p_{k+1}} + ... + \frac{C_n}{s-p_n} \tag{8.113}$$

  * For non-repeated poles, the coefficients are the same as before

$$C_i = \lim_{s \to p_i} [Y(s)(s-p_i)] \tag{8.114}$$

  * However, for repeated poles, the coefficients are

$$C_1 = \lim_{s \to p_1} \left[Y(s)(s-p_1)^k \right] \tag{8.115}$$
$$C_2 = \lim_{s \to p_1} \left\{ \frac{d}{ds} \left[Y(s)(s-p_1)^k \right] \right\} \tag{8.116}$$
$$C_i = \lim_{s \to p_1} \left\{ \frac{1}{(i-1)!} \frac{d^{i-1}}{ds^{i-1}} \left[Y(s)(s-p_1)^k \right] \right\} \tag{8.117}$$

  * The result for $y(t)$ is then

$$y(t) = C_1 \frac{t^{k-1}}{(k-1)!} e^{p_1 t} + C_2 \frac{t^{k-2}}{(k-2)!} e^{p_1 t} + ... + C_k e^{p_1 t} \tag{8.118}$$
$$+ C_{k+1} e^{p_{k+1} t} + ... + C_n e^{p_n t} \tag{8.119}$$

– Proof

  * Distinct poles

$$
\begin{align}
Y(s) &= \frac{N(s)}{D(s)} \tag{8.120}\\
&= \frac{N(s)}{(s-p_1)(s-p_2)...(s-p_n)} \tag{8.121}\\
&= \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + ... + \frac{C_n}{s-p_n} \tag{8.122}
\end{align}
$$

Now let's write

$$
\begin{align}
Y(s)(s-p_1) &= \frac{C_1(s-p_1)}{s-p_1} + \frac{C_2(s-p_1)}{s-p_2} + ... + \frac{C_n(s-p_1)}{s-p_n} \tag{8.123}\\
\lim_{s\to p_1} Y(s)(s-p_1) &= \lim_{s\to p_1}\left(\frac{C_1}{s-p_1} + \frac{C_2}{s+r_2} + ... + \frac{C_n}{s-p_n}\right)(s-p_1) \tag{8.124}\\
&= C_1 + \lim_{s\to p_1}\left(\frac{C_2}{s-p_2} + ... + \frac{C_n}{s-p_n}\right)\cancelto{0}{(s-p_1)} \tag{8.125}\\
C_1 &= \lim_{s\to p_1} Y(s)(s-p_1) \tag{8.126}
\end{align}
$$

  * Repeated poles - need to fix the following proof

$$
\begin{align}
Y(s) &= \frac{N(s)}{D(s)} \tag{8.127}\\
&= \frac{N(s)}{(s-p_1)^k(s-p_{k+1})(s-p_{k+2})...(s-p_n)} \tag{8.128}\\
&= \frac{C_1}{(s-p_1)^k} + \frac{C_2}{(s-p_1)^{k-1}} + ... + \frac{C_p}{s-p_1} \tag{8.129}\\
&\quad + \frac{C_{k+1}}{s-p_{k+1}} + ... + \frac{C_n}{s-p_n} \tag{8.130}\\
Y(s)(s-p_1)^k &= \frac{C_1}{(s-p_1)^k} + \frac{C_2}{(s-p_1)^{k-1}} + ... + \frac{C_p}{s-p_1} \tag{8.131}\\
&\quad + \frac{C_{k+1}}{s-p_{k+1}} + ... + \frac{C_n}{s-p_n} \tag{8.132}
\end{align}
$$

– What if $m \geq n$

  * Requires long division
  * "For a transform $F(s)$ representing the response of any physical system, $m \leq n$ [8]."

    * "If the order m of the numerator of $X(s)$ is greater than the order n of the denominator, the transform can be represented by a polynomial ... plus a ratio of two polynomials where the denominator degree is greater than the numerator degree [35]."

- Software may be used to complete a partial fraction expansion using, for example, the following commands:
    * Python: - see Appendix D.4 Representing and Solving Linear Time Invariant Systems
        · `residue`
    * MATLAB® - see Appendix E.5 Representing and Solving Linear Time Invariant Systems
        · `residue`

Example 8.1:

Find the Laplace transform of the function $y(t) = 4\sin(2t) + 6e^{-3t}$.

- Laplace transform

$$Y(s) = L[4\sin(2t) + 6e^{-3t}] \tag{8.133}$$

- Linearity property

$$Y(s) = L[4\sin(2t)] + L[6e^{-3t}] \tag{8.134}$$
$$= 4L[\sin(2t)] + 6L[e^{-3t}] \tag{8.135}$$

- Laplace transform table

$$Y(s) = 4\frac{2}{s^2+2^2} + 6\frac{1}{s+3} \tag{8.136}$$
$$= \frac{8}{s^2+2^2} + \frac{6}{s+3} \tag{8.137}$$

Example 8.2:

Find the Laplace transform of the function $y(t) = 4e^{-2t}\cos(3t)$.

- Laplace transform

$$Y(s) = L[4e^{-2t}\cos(3t)] \tag{8.138}$$

- Linearity property

$$Y(s) = 4L[e^{-2t}\cos(3t)] \tag{8.139}$$

- Laplace transform table

$$Y(s) = 4\frac{s+2}{(s+2)^2+3^2} \tag{8.140}$$

Example 8.3:

Find the Laplace transform of the function

$$y(t) = \begin{cases} 0 & t < \pi \\ \sin(t-\pi) & \pi < t \end{cases}. \tag{8.141}$$

- Incorporate unit step function

$$y(t) = \sin(t-\pi)u_s(t-\pi) \tag{8.142}$$

- Shifting along time property

$$Y(s) = e^{-s\pi}L[\sin(t)] \tag{8.143}$$

- Laplace transform table

$$Y(s) = e^{-s\pi}\frac{1}{s^2+1^2} \tag{8.144}$$

Example 8.4:

Find the inverse Laplace transform of the function

$$Y(s) = \frac{1}{2s+1}. \tag{8.145}$$

- Rearrange to match an entry on the table

$$Y(s) = \frac{\frac{1}{2}}{s+\frac{1}{2}} \tag{8.146}$$

- Laplace transform table

$$y(t) = \frac{1}{2}e^{-\frac{1}{2}t} \tag{8.147}$$

Example 8.5:

Find the inverse Laplace transform of the function

$$Y(s) = \frac{s+3}{s^2+3s+2}. \tag{8.148}$$

- Factor the denominator

$$\begin{aligned} s &= \frac{-3 \pm \sqrt{3^2 - 4\cdot 2}}{2} && (8.149)\\ &= \{-1, -2\} && (8.150)\\ Y(s) &= \frac{s+3}{(s+1)(s+2)} && (8.151) \end{aligned}$$

- Partial fraction expansion

$$\begin{aligned} Y(s) &= \frac{A}{s+1} + \frac{B}{s+2} && (8.152)\\ A &= \lim_{s\to -1} \frac{s+3}{\cancel{(s+1)}(s+2)}\cancel{(s+1)} && (8.153)\\ &= \frac{-1+3}{-1+2} && (8.154)\\ &= 2 && (8.155)\\ B &= \lim_{s\to -2} \frac{s+3}{(s+1)\cancel{(s+2)}}\cancel{(s+2)} && (8.156)\\ &= \frac{-2+3}{-2+1} && (8.157)\\ &= -1 && (8.158)\\ Y(s) &= \frac{2}{s+1} - \frac{1}{s+2} && (8.159) \end{aligned}$$

- Laplace transform table

$$y(t) = 2e^{-t} - e^{-2t} \tag{8.160}$$

Example 8.6:

Find the inverse Laplace transform of the function

$$Y(s) = \frac{s+3}{s^2}. \tag{8.161}$$

- Partial fraction expansion

$$\begin{aligned} Y(s) &= \frac{A}{s^2} + \frac{B}{s} && (8.162) \\ A &= \lim_{s \to 0} \frac{s+3}{\cancel{s^2}} \cancel{s^2} && (8.163) \\ &= 3 && (8.164) \\ B &= \lim_{s \to 0} \frac{d}{ds} \left(\frac{s+3}{\cancel{s^2}} \cancel{s^2} \right) && (8.165) \\ &= 1 && (8.166) \\ Y(s) &= \frac{3}{s^2} + \frac{1}{s} && (8.167) \end{aligned}$$

- Laplace transform table

$$y(t) = 3t + 1 \quad (8.168)$$

Example 8.7:

Find the inverse Laplace transform of the function

$$Y(s) = \frac{2s+1}{(s+3)^2}. \quad (8.169)$$

- Partial fraction expansion

$$\begin{aligned} Y(s) &= \frac{A}{(s+3)^2} + \frac{B}{s+3} && (8.170) \\ A &= \lim_{s \to -3} \frac{2s+1}{\cancel{(s+3)^2}} \cancel{(s+3)^2} && (8.171) \\ &= -5 && (8.172) \\ B &= \lim_{s \to -3} \frac{d}{ds} \left(\frac{2s+1}{\cancel{(s+3)^2}} \cancel{(s+3)^2} \right) && (8.173) \\ &= 2 && (8.174) \\ Y(s) &= \frac{-5}{(s+3)^2} + \frac{2}{s+3} && (8.175) \end{aligned}$$

- Rearrange to use multiply by time property

$$Y(s) = \frac{d}{ds}\left(\frac{5}{s+3}\right) + \frac{2}{s+3} \tag{8.176}$$

$$y(t) = -tL^{-1}\left[\frac{5}{s+3}\right] + L^{-1}\left[\frac{2}{s+3}\right] \tag{8.177}$$

- Laplace transform table

$$y(t) = -5te^{-3t} + 2e^{-3t} \tag{8.178}$$

- Examples work out some more of the following examples
  - Inverse Laplace transform
    - Multiple poles
      1. $Y(s) = \frac{3s+5}{3s^2+4s+1}$
         $y(t) = 2e^{-\frac{1}{3}t} - e^{-t}$
      2. $Y(s) = \frac{(4s+6)}{(s+1)(s+2)(s+3)}$
         $y(t) = e^{-t} + 2e^{-2t} - 3e^{-3t}$
      3. $Y(s) = \frac{(s+3)}{(s+1)(s+2)(s+2)}$
         $y(t) = -te^{-2t} + 2e^{-t} - 2e^{-2t}$
      4. $Y(s) = \frac{2}{s^2-2^2}$
         $y(t) = -\frac{1}{2}te^{-2t} + \frac{1}{2}e^{-2t}$
    - Complex poles
      1. $Y(s) = \frac{3}{s^2+4s+13}$
      2. $Y(s) = \frac{s+2}{s^2+4s+13}$
      3. $Y(s) = \frac{1}{3(s^2+4s+13)}$
      4. $Y(s) = \frac{3s+4}{s^2+4s+13}$
      5. $Y(s) = \frac{s^2+3s+26}{s(s^2+4s+13)}$
         $y(t) = 2 - e^{-2t}(\cos(3t) + \sin(3t))$
    - Other
      1. $Y(s) = \left(\frac{1}{s+3}\right)e^{-4s}$

8.3 Solving ODEs with Laplace Transforms

- Solving ODEs with the Laplace transform
  - Overview
    - The application of mainstream Laplace transform approaches is restricted to linear time invariant systems; see, for example, [37]
    - Converts linear time invariant systems of differential equations into algebraic equations.
  - Method
    - Take Laplace transform of the ODE.
      - Apply the linearity property to consider individual terms
      - Use derivative property to account for derivatives and initial conditions (and, if the input is not 0 fo r$t < 0$, initial values of the input)
    - Algebraically isolate the Laplace transform of the dependent variable ($X(s)$)
    - Take the inverse Laplace transform
      - Often use partial fraction expansion to divide a complicated expression into simple transforms
      - Use linearity property to consider terms individually

Example 8.8: first order ODE with no input

Use Laplace transform to solve the initial value problem

$$\tau\dot{x} + x = 0,\ x(0) = x_0. \tag{8.179}$$

- Laplace transform

$$\tau(sX(s) - x_0) + X(s) = 0 \tag{8.180}$$

- Isolate $X(s)$

$$X(s) = \frac{\tau x_0}{\tau s + 1} \tag{8.181}$$

- Inverse Laplace transform
  - Partial fraction expansion

$$X(s) = \frac{\tau x_0}{\tau s + 1} \tag{8.182}$$

$$= \frac{x_0}{s + \frac{1}{\tau}} \tag{8.183}$$

– Inverse Laplace transform

$$x(t) = L^{-1}\left[\frac{x_0}{s+\frac{1}{\tau}}\right] \tag{8.184}$$

$$= x_0 e^{-\frac{t}{\tau}} \tag{8.185}$$

Example 8.9: first order ODE with impulse input

$$\tau\dot{x} + x = A\delta(t),\ x(0) = x_0 \tag{8.186}$$

- Laplace transform

$$\tau(sX(s) - x_0) + X(s) = A \tag{8.187}$$

- Isolate $X(s)$

$$X(s) = \frac{A + \tau x_0}{\tau s + 1} \tag{8.188}$$

- Inverse Laplace transform
  - Partial fraction expansion

$$X(s) = \frac{A + \tau x_0}{\tau s + 1} \tag{8.189}$$

$$= \frac{\frac{A}{\tau} + x_0}{s + \frac{1}{\tau}} \tag{8.190}$$

  - Inverse Laplace transform

$$x(t) = L^{-1}\left[\frac{\frac{A}{\tau} + x_0}{s + \frac{1}{\tau}}\right] \tag{8.191}$$

$$= \left(\frac{A}{\tau} + x_0\right) e^{-\frac{t}{\tau}} \tag{8.192}$$

Example 8.10: first order ODE with step input

Use Laplace transform to solve the initial value problem

$$\tau\dot{x} + x = Au_s(t),\ x(0) = x_0. \tag{8.193}$$

- Laplace transform
$$\tau(sX(s) - x_0) + X(s) = \frac{A}{s} \tag{8.194}$$
- Isolate $X(s)$
$$X(s) = \frac{\frac{A}{s} + \tau x_0}{\tau s + 1} \tag{8.195}$$
- Inverse Laplace transform
  - Partial fraction expansion

$$\begin{aligned} X(s) &= \frac{\frac{A}{s} + \tau x_0}{\tau s + 1} && (8.196) \\ &= \frac{\frac{A}{\tau} + x_0 s}{s(s + \frac{1}{\tau})} && (8.197) \\ &= \frac{C_1}{s} + \frac{C_2}{s + \frac{1}{\tau}} && (8.198) \end{aligned}$$

$$\begin{aligned} C_1 &= \lim_{s \to 0} \frac{\frac{A}{\tau} + x_0 s}{\cancel{s}(s + \frac{1}{\tau})} \cancel{s} && (8.199) \\ &= \frac{\frac{A}{\tau}}{\frac{1}{\tau}} && (8.200) \\ &= A && (8.201) \\ C_2 &= \lim_{s \to -\frac{1}{\tau}} \frac{\frac{A}{\tau} + x_0 s}{s\cancel{(s + \frac{1}{\tau})}} \cancel{(s + \frac{1}{\tau})} && (8.202) \\ &= \frac{\frac{A}{\tau} + x_0 \left(-\frac{1}{\tau}\right)}{-\frac{1}{\tau}} && (8.203) \\ &= -A + x_0 && (8.204) \end{aligned}$$

$$X(s) = \frac{A}{s} + \frac{-A + x_0}{s + \frac{1}{\tau}} \tag{8.205}$$

  - Inverse Laplace transform

$$\begin{aligned} x(t) &= L^{-1}\left[\frac{A}{s} + \frac{-A + x_0}{s + \frac{1}{\tau}}\right] && (8.206) \\ &= A + (-A + x_0)\, e^{-\frac{t}{\tau}} && (8.207) \end{aligned}$$

Example 8.11: Undamped second order ODE with no input

Use Laplace transform to solve the initial value problem

$$m\ddot{x} + kx = 0,\ x(0) = x_0,\ v(0) = v_0. \tag{8.208}$$

- Laplace transform
$$m(s^2X(s) - sx_0 - v_0) + kX(s) = 0 \tag{8.209}$$

- Isolate $X(s)$
$$X(s) = \frac{msx_0 + mv_0}{ms^2 + k} \tag{8.210}$$

- Option 1: Inverse Laplace transform with complex numbers
  - Partial fraction expansion using complex numbers

$$\begin{aligned}
X(s) &= \frac{sx_0 + v_0}{s^2 + \frac{k}{m}} && (8.211)\\
&= \frac{sx_0 + v_0}{\left(s + j\sqrt{\frac{k}{m}}\right)\left(s - j\sqrt{\frac{k}{m}}\right)} && (8.212)\\
&= \frac{C_1}{s + j\sqrt{\frac{k}{m}}} + \frac{C_2}{s - j\sqrt{\frac{k}{m}}} && (8.213)\\
& && (8.214)
\end{aligned}$$

$$\begin{aligned}
C_1 &= \lim_{s \to -j\sqrt{\frac{k}{m}}} \frac{sx_0 + v_0}{\cancel{\left(s + j\sqrt{\frac{k}{m}}\right)}\left(s - j\sqrt{\frac{k}{m}}\right)}\cancel{\left(s + j\sqrt{\frac{k}{m}}\right)} && (8.215)\\
&= \frac{-j\sqrt{\frac{k}{m}}x_0 + v_0}{-j\sqrt{\frac{k}{m}} - j\sqrt{\frac{k}{m}}} && (8.216)\\
&= \frac{\sqrt{\frac{k}{m}}x_0 + jv_0}{2\sqrt{\frac{k}{m}}} && (8.217)
\end{aligned}$$

$$
\begin{aligned}
C_2 &= \lim_{s \to j\sqrt{\frac{k}{m}}} \frac{sx_0 + v_0}{\left(s + j\sqrt{\frac{k}{m}}\right)\cancel{\left(s - j\sqrt{\frac{k}{m}}\right)}} \cancel{\left(s - j\sqrt{\frac{k}{m}}\right)} && (8.218) \\
&= \frac{j\sqrt{\frac{k}{m}}x_0 + v_0}{j\sqrt{\frac{k}{m}} + j\sqrt{\frac{k}{m}}} && (8.219) \\
&= \frac{\sqrt{\frac{k}{m}}x_0 - jv_0}{2\sqrt{\frac{k}{m}}} && (8.220)
\end{aligned}
$$

$$
X(s) = \frac{\frac{\sqrt{\frac{k}{m}}x_0 + jv_0}{2\sqrt{\frac{k}{m}}}}{s + j\sqrt{\frac{k}{m}}} + \frac{\frac{\sqrt{\frac{k}{m}}x_0 - jv_0}{2\sqrt{\frac{k}{m}}}}{s - j\sqrt{\frac{k}{m}}} \qquad (8.221)
$$

– Inverse Laplace transform

$$
\begin{aligned}
x(t) &= L^{-1}\left[\frac{\frac{\sqrt{\frac{k}{m}}x_0 + jv_0}{2\sqrt{\frac{k}{m}}}}{s + j\sqrt{\frac{k}{m}}} + \frac{\frac{\sqrt{\frac{k}{m}}x_0 - jv_0}{2\sqrt{\frac{k}{m}}}}{s - j\sqrt{\frac{k}{m}}}\right] && (8.222) \\
&= \frac{\sqrt{\frac{k}{m}}x_0 + jv_0}{2\sqrt{\frac{k}{m}}} e^{-j\sqrt{\frac{k}{m}}t} + \frac{\sqrt{\frac{k}{m}}x_0 - jv_0}{2\sqrt{\frac{k}{m}}} e^{j\sqrt{\frac{k}{m}}t} && (8.223) \\
&= \frac{\sqrt{\frac{k}{m}}x_0 + jv_0}{2\sqrt{\frac{k}{m}}} \left(\cos\left(\sqrt{\frac{k}{m}}t\right) - j\sin\left(\sqrt{\frac{k}{m}}t\right)\right) \\
&\quad + \frac{\sqrt{\frac{k}{m}}x_0 - jv_0}{2\sqrt{\frac{k}{m}}} \left(\cos\left(\sqrt{\frac{k}{m}}t\right) + j\sin\left(\sqrt{\frac{k}{m}}t\right)\right) && (8.224) \\
&= x_0\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{v_0}{\sqrt{\frac{k}{m}}}\sin\left(\sqrt{\frac{k}{m}}t\right) && (8.225)
\end{aligned}
$$

- Option 2: Inverse Laplace transform using Laplace transform table entries for sin and cos

$$X(s) = \frac{sx_0 + v_0}{s^2 + \frac{k}{m}} \tag{8.226}$$

$$= \frac{sx_0 + v_0}{s^2 + \sqrt{\frac{k}{m}}^2} \tag{8.227}$$

$$= \frac{sx_0}{s^2 + \sqrt{\frac{k}{m}}^2} + \frac{v_0}{s^2 + \sqrt{\frac{k}{m}}^2} \tag{8.228}$$

$$= x_0 \frac{s}{s^2 + \sqrt{\frac{k}{m}}^2} + \frac{v_0}{\sqrt{\frac{k}{m}}} \frac{\sqrt{\frac{k}{m}}}{s^2 + \sqrt{\frac{k}{m}}^2} \tag{8.229}$$

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}} t\right) + \frac{v_0}{\sqrt{\frac{k}{m}}} \sin\left(\sqrt{\frac{k}{m}} t\right) \tag{8.230}$$

Example 8.12: Underdamped ($0 < \zeta < 1$) second order ODE with no input

Use Laplace transform to solve the initial value problem

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = 0,\ y(0) = y_0,\ \dot{y}(0) = v_0. \tag{8.231}$$

- Laplace transform

$$1(s^2 Y(s) - sy_0 - v_0) + 2\zeta\omega_n(sY(s) - y_0) + \omega_n^2 Y(s) = 0 \tag{8.232}$$

- Isolate $Y(s)$

$$Y(s) = \frac{y_0 s + v_0 + 2\zeta\omega_n y_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8.233}$$

- Option 1: Inverse Laplace transform with complex numbers
  - Partial fraction expansion using complex numbers

$$Y(s) = \frac{y_0 s + v_0 + 2\zeta\omega_n y_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8.234}$$

$$= \frac{y_0 s + v_0 + 2\zeta\omega_n y_0}{\left(s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)\left(s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)} \tag{8.235}$$

$$= \frac{C_1}{s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}} + \frac{C_2}{s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}} \tag{8.236}$$

$$
\begin{aligned}
C_1 &= \lim_{s \to -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}} \frac{sy_0 + v_0 + 2\zeta\omega_n y_0}{\cancel{\left(s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)}\left(s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)} \\
&\quad \cdot \cancel{\left(s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)} && (8.237) \\
&= \lim_{s \to -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}} \frac{sy_0 + v_0 + 2\zeta\omega_n y_0}{\left(s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)} && (8.238) \\
&= \frac{\left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right) y_0 + v_0 + 2\zeta\omega_n y_0}{-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}} && (8.239) \\
&= \frac{\left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right) y_0 + v_0 + 2\zeta\omega_n y_0}{2j\omega_n\sqrt{1-\zeta^2}} && (8.240) \\
&= \frac{\omega_n\sqrt{1-\zeta^2}x_0 - j(v_0 + \zeta\omega_n y_0)}{2\omega_n\sqrt{1-\zeta^2}} && (8.241) \\
&= \frac{1}{2}y_0 - j\frac{v_0 + \zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}} && (8.242)
\end{aligned}
$$

Similarly, we can show

$$
C_2 = \frac{1}{2}y_0 + j\frac{v_0 + \zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}} \qquad (8.243)
$$

$$
Y(s) = \frac{\frac{1}{2}y_0 - j\frac{v_0+\zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}}}{s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}} + \frac{\frac{1}{2}y_0 + j\frac{v_0+\zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}}}{s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}} \qquad (8.244)
$$

– Inverse Laplace transform

$$y(t) = L^{-1}\left[\frac{\frac{1}{2}y_0 - j\frac{v_0+\zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}}}{s+\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}} + \frac{\frac{1}{2}y_0 + j\frac{v_0+\zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}}}{s+\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}\right] \tag{8.245}$$

$$= \left(\frac{1}{2}y_0 - j\frac{v_0+\zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}}\right)e^{\left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)t} + \left(\frac{1}{2}y_0 + j\frac{v_0+\zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}}\right)e^{\left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)t} \tag{8.246}$$

$$= \left(\frac{1}{2}y_0 - j\frac{v_0+\zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}}\right)e^{-\zeta\omega_n t}\left(\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + j\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right) + \left(\frac{1}{2}y_0 + j\frac{v_0+\zeta\omega_n y_0}{2\omega_n\sqrt{1-\zeta^2}}\right)e^{-\zeta\omega_n t}\left(\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) - j\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right) \tag{8.247}$$

$$= y_0\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \frac{v_0+\zeta\omega_n y_0}{\omega_n\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) \tag{8.248}$$

- Option 2: Inverse Laplace transform using Laplace transform table entries for sin and cos

$$Y(s) = \frac{y_0 s + v_0 + 2\zeta\omega_n y_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8.249}$$

$$= \frac{y_0 s + v_0 + 2\zeta\omega_n y_0}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \tag{8.250}$$

$$= y_0\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} + \frac{v_0+\zeta\omega_n x_0}{\omega_n\sqrt{1-\zeta^2}}\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \tag{8.251}$$

$$y(t) = y_0\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \frac{v_0+\zeta\omega_n y_0}{\omega_n\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) \tag{8.252}$$

Example 8.13:

Use Laplace transform to solve the initial value problem

$$\ddot{y} + 2\dot{y} + 5y = 5u_s(t),\ y(0) = -1,\ \dot{y}(0) = 0. \tag{8.253}$$

- Laplace transform

$$1(s^2Y(s) - sy_0 - v_0) + 2(sY(s) - y_0)5Y(s) = 5\frac{1}{s} \tag{8.254}$$

$$(s^2Y(s) - s(-1) - 0) + 2(sY(s) - (-1)) + 5Y(s) = 5\frac{1}{s} \tag{8.255}$$

- Isolate $Y(s)$

$$Y(s) = \frac{-s - 2 + \frac{5}{s}}{s^2 + 2s + 5} \tag{8.256}$$

$$= \frac{-s^2 - 2s + 5}{s(s^2 + 2s + 5)} \tag{8.257}$$

- Option 1: Inverse Laplace transform with complex numbers
  - Partial fraction expansion using complex numbers

$$Y(s) = \frac{-s^2 - 2s + 5}{s(s^2 + 2s + 5)} \tag{8.258}$$

$$= \frac{-s^2 - 2s + 5}{s(s + 1 - 2j)(s + 1 + 2j)} \tag{8.259}$$

$$= C_1\frac{1}{s} + \frac{C_2}{s + 1 - 2j} + \frac{C_3}{s + 1 + 2j} \tag{8.260}$$

$$C_1 = \lim_{s\to 0} \frac{-s^2 - 2s + 5}{\cancel{s}(s + 1 - 2j)(s + 1 + 2j)}\cancel{s} \tag{8.261}$$

$$= \frac{5}{(1 - 2j)(1 + 2j)} \tag{8.262}$$

$$= \frac{5}{1 + 4} \tag{8.263}$$

$$= 1 \tag{8.264}$$

$$
\begin{aligned}
C_2 &= \lim_{s\to -1+2j} \frac{-s^2-2s+5}{s\cancel{(s+1-2j)}(s+1+2j)}\cancel{(s+1-2j)} && (8.265)\\
&= \lim_{s\to -1+2j} \frac{-s^2-2s+5}{s(s+1+2j)} && (8.266)\\
&= \frac{-(-1+2j)^2-2(-1+2j)+5}{(-1+2j)(-1+2j+1+2j)} && (8.267)\\
&= \frac{-(1-4j-4)+2-4j+5}{(-1+2j)(4j)} && (8.268)\\
&= \frac{10}{-8-4j} && (8.269)\\
&= \frac{5}{-4-2j} && (8.270)\\
&= \frac{5(-4+2j)}{(-4-2j)(-4+2j)} && (8.271)\\
&= \frac{5(-4+2j)}{20} && (8.272)\\
&= -1+\frac{1}{2}j && (8.273)
\end{aligned}
$$

Similarly, we can show

$$
C_3 = -1-\frac{1}{2}j \quad (8.274)
$$

$$
\begin{aligned}
Y(s) &= C_1\frac{1}{s}+\frac{C_2}{s+1-2j}+\frac{C_3}{s+1+2j} && (8.275)\\
&= 1\frac{1}{s}+\frac{-1+\frac{1}{2}j}{s+1-2j}+\frac{-1-\frac{1}{2}j}{s+1+2j} && (8.276)
\end{aligned}
$$

– Inverse Laplace transform

$$
\begin{aligned}
y(t) &= 1+\left(-1+\frac{1}{2}j\right)e^{(-1+2j)t}+\left(-1-\frac{1}{2}j\right)e^{(-1-2j)t} && (8.277)\\
&= 1+e^{-t}\left(\left(-1+\frac{1}{2}j\right)e^{2jt}+\left(-1-\frac{1}{2}j\right)e^{-2jt}\right) && (8.278)\\
&= 1+e^{-t}\left(-1\left(e^{2jt}+e^{-2jt}\right)-\frac{1}{2j}\left(e^{2jt}-e^{-2jt}\right)\right) && (8.279)\\
&= 1+e^{-t}\left(-2\left(\frac{e^{2jt}+e^{-2jt}}{2}\right)-\left(\frac{e^{2jt}-e^{-2jt}}{2j}\right)\right) && (8.280)\\
&= 1+e^{-t}\left(-2\cos(2t)-\sin(2t)\right) && (8.281)
\end{aligned}
$$

- Option 2: Inverse Laplace transform using Laplace transform table entries for sin and cos
  - Partial fraction expansion

$$
\begin{aligned}
Y(s) &= \frac{-s^2-2s+5}{s(s^2+2s+5)} && (8.282)\\
&= \frac{-s^2-2s+5}{s\left((s+1)^2+2^2\right)} && (8.283)\\
&= \frac{C_1}{s}+C_2\frac{s+1}{(s+1)^2+2^2}+C_3\frac{2}{(s+1)^2+2^2} && (8.284)\\
& && (8.285)
\end{aligned}
$$

  - common denominator

$$
Y(s) = \frac{C_1(s^2+2s+5)+C_2 s(s+1)+C_3 s2}{s\left((s+1)^2+2^2\right)} \tag{8.286}
$$

  - Equate coefficients of the numerator

$$
\begin{aligned}
s^2 :&\quad C_1+C_2=-1 && (8.287)\\
s^1 :&\quad 2C_1+C_2+2C_3=-2 && (8.288)\\
s^0 :&\quad 5C_1=5 && (8.289)
\end{aligned}
$$

  - Solve for coefficients

$$
\begin{aligned}
C_1 &= 1 && (8.290)\\
C_2 &= -2 && (8.291)\\
C_3 &= -1 && (8.292)
\end{aligned}
$$

  - Inverse laplace transform

$$
\begin{aligned}
Y(s) &= \frac{1}{s}-2\frac{s+1}{(s+1)^2+2^2}-1\frac{2}{(s+1)^2+2^2} && (8.293)\\
y(t) &= 1-2e^{-t}\cos(2t)-e^{-t}\sin(2t) && (8.294)
\end{aligned}
$$

Example 8.14:

- $\ddot{y} + 3\dot{y} + 2y =$
- $\ddot{y} + 2\dot{y} + 5y =$
- Example - two degree of freedom system
- Example - state space model

8.4 Parts of Solution

- Free response vs. forced response
  - Free response - "The part of the response that depends on the initial conditions [2]."
    - The free-response may be obtained by setting the forcing terms to 0 and solving.
  - Forced response - "The part of the response that depends on the forcing function [2]."
    - The forced response may be obtained by setting the initial conditions to 0 and solving.
  - Note, the sum of free response and forced response is the total response
  - Note, some authors use 'Forced response' to refer to the particular solution and 'natural response' to refer to the homogeneous solution [6]. These definitions are slightly different in that what we call forced response includes contributions from both homogeneous and particular solutions.
- Steady State vs Transient response -
  - Transient response -
    - "The part of the response that disappears with time [35]." Similar definitions appear in [3, 38, 4], for example.
    - "... that which goes from the initial state to the final state [39]."
    - [40, 41], for example, suggest that the transient response is equivalent to the homogeneous solution.
  - Steady State response -
    - "The part of the response that remains with time [35]." Similar definitions appear in [3, 38], for example .
    - "... the manner in which the system output behaves as t approaches infinity [39]."
    - "**steady response** A portion of the time-dependent system response, which either remains constant or repeats its waveform with time [4]."

    * [40, 41], for example, suggest that the steady-state response is equivalent to the particular solution.
  - Together the steady state and transient responses combine to form the total response
    * Some authors suggest this is a mathematical sum or superposition of two terms
    * Other authors seem to suggest that the transient and steady states combine to form the total response but span distinct regions of time; see, for example, [40, 42, 6].
  - Special cases
    * Unstable systems - Some authors include as part of the transient response any part that grows unbounded due to instability of the system; see, for example, `https://www.ece.rutgers.edu/~gajic/psfiles/chap6.pdf`
    * Nonperiodic steady state - Some authors restrict the steady state response to cases in which it is constant or periodic; see, for example, [42, 40]

Chapter 9

Dynamic System Representation

9.1 Transfer Functions

- The total response of a system is the sum of contributions from the initial conditions (free response) and input (forced response).
- Transfer function
  - Definition
    - For single input single output systems, the transfer function is the Laplace transform of the **forced response** divided by the Laplace transform of the input

$$G(s) = \frac{Y(s)}{U(s)} \tag{9.1}$$

    - All initial conditions of the output and initial values of the input are to be set to 0.
    - The transfer function considers the input to be zero for $t < 0$ and turn on for $t > 0$, like step function.
    - For systems with multiple inputs and/or outputs
      - Each input-output pair of a system yields a transfer function.
      - The total output of a system, is the superposition of all contributions.
      - To obtain a specific transfer function, consider the corresponding input and temporily set others to zero.
- Transfer functions are expressed in two common forms
  - Ratio of polynomials

* Transfer function as a ratio of polynomials

$$G(s) = \frac{Y(s)}{U(s)} \tag{9.2}$$

$$= \frac{N(s)}{D(s)} \tag{9.3}$$

$$= \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0} \tag{9.4}$$

* Rational function - ratio of polynomials
  · Proper - A rational function is termed proper when the order of the numerator is less than or equal to that of the denominator, $m \leq n$.
  · Strictly proper - A rational function is termed strictly proper when the order of the numerator is less than that of the denominator, $m < n$.
  · Biproper (or semi-proper) - A transfer function is biproper when the orders of the numerator and denominator are equivalent, $m = n$; see, for example, [43]
  · Improper - A rational function is termed improper when the order of the denominator is less than that of the numerator, $n < m$.
* Relative degree (or pole excess) - the difference between the degree of the denominator and numerator, $n - m$, is called the relative degree of the system [37, 5, 43].
* Normalization
  · Often, we divide numerator and denominator of the transfer function by the leading coefficient of the denominator forcing the leading coefficient of the denominator to become 1; see, for example, [38, 43].
  · In some circumstances, we will force the last coefficient of the denominator to 1 instead.
  · One benefit of these normalizations is that they reveal the minimum the number of parameters (i.e., coefficients) required to represent the system: $n + m + 1$.
* Static sensitivity (K_0) [10, 4] (aka, static gain, DC gain [8], or zero frequency gain [5], steady-state gain)
  · The static sensitivity can be obtained from

$$K_0 = G(0) \tag{9.5}$$

$$= \frac{b_0}{a_0} \tag{9.6}$$

$$= K\frac{(-z_1)(-z_2)...(-z_m)}{(-p_1)(-p_2)...(-p_n)} \tag{9.7}$$

  · Describes the steady state response for stable systems given a unit step input $f(t) = u_s(t)$

$$\lim_{t\to\infty} y(t) = K_0 u_s(t) \tag{9.8}$$

  - The static sensitivity is 0 when there is a zero at 0 and infinity when there is a pole at 0.
  - Note, static sensitivity, K_0, differs from gain, K

- Ratio in factored zero pole form [8] or pole zero gain model (see, for example, `https://www.mathworks.com/help/control/ref/zpk.html`)
  - Factored zero pole form

$$G(s) = K\frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)} \tag{9.9}$$

  - Pole (p_i) - root of the denominator of a transfer function
    - "The *poles* of a transfer function are (1) the values of the Laplace transform variable, s, that cause the transfer function to become infinite or (2) any roots of the denominator of the transfer function that are common to roots of the numerator [44]."
  - Zero (z_i) - root of the numerator of a transfer function
    - "The *zeros* of a transfer function are (1) the values of the Laplace transform variable, s, that cause the transfer function to become zero or (2) any roots of the numerator of the transfer function that are common to roots of the denominator [44]."
  - Gain, transfer function gain [8], or gain factor [3] (K)

$$K = \frac{b_m}{1} \tag{9.10}$$

  - Knowledge of the poles and zeros of a system tell us much about system dynamics (e.g., system stability).

- Realizable systems
  - Not all systems that can be constructed mathematically can be realized physically
  - Physical systems have limitations
    - Causal -
      - "A system is said to be causal if the value of the output at time t_0 depends on the values of the input and output for all t up to time t_0 but no further, i.e., only for $t \leq t_0$ [7]."
      - A noncausal system would be capable of determining the future; see, for example, [7].
    - Physical laws - physical systems must obey physical laws (e.g., the laws of thermodynamics dictate that perpetual motion machines cannot exist)

    - Finite inputs and outputs - the inputs and outputs of physical systems must be finite
  - Improper systems are not realizable.
    - The output of a system with improper transfer function includes a contribution from the derivative(s) of the input
      - is the definition of derivative two sided?
    - Finite outputs
      - Abrupt changes in the input would be differentiated by an improper system yielding large spikes in the output (e.g., the derivative of the step function is the impulse function).
      - Improper systems are not realizable because physical systems have limitations on the magnitude of their outputs.
    - Causal
      - "It is known that for a system to be causal, its transfer function (if it has one) must be proper [7]."

- Software may be used to represent transfer functions and numerically solve for the response of the systems they represent using, for example, the following commands:
  - Python: - see Appendix D.4 Representing and Solving Linear Time Invariant Systems
    - `tf`
    - `step_response`
    - `impulse_response`
    - `forced_response`
  - MATLAB® - see Appendix E.5 Representing and Solving Linear Time Invariant Systems
    - `tf`
    - `zpk`
    - `step`
    - `impulse`
    - `lsim`

Example 9.1: first order ordinary differential equation

Find the transfer function for the following system with input $u(t)$ and output $y(t)$,

$$a_1\dot{y} + a_0 y = b_0 u(t). \tag{9.11}$$

- Laplace transform (take the initial conditions as 0 when calculating the transfer function)

$$a_1 sY(s) + a_0 Y(s) = b_0 U(s) \tag{9.12}$$

- Therefore the transfer function is

$$G(s) = \frac{Y(s)}{U(s)} \tag{9.13}$$

$$= \frac{b_0}{a_1 s + a_0} \tag{9.14}$$

$$= \frac{\frac{b_0}{a_1}}{s + \frac{a_0}{a_1}} \tag{9.15}$$

- Note, the denominator is the characteristic polynomial of the system
- Note, with experience, you could obtain the differential equation from the transfer function

Example 9.2: second order ordinary differential equation

Find the transfer function for the following system with input $u(t)$ and output $y(t)$,

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_0 u(t). \tag{9.16}$$

- Laplace transform (take the initial conditions as 0 when calculating the transfer function)

$$a_2 s^2 Y(s) + a_1 sY(s) + a_0 Y(s) = b_0 U(s) \tag{9.17}$$

- Therefore the transfer function is

$$G(s) = \frac{Y(s)}{U(s)} \tag{9.18}$$

$$= \frac{b_0}{a_2 s^2 + a_1 s + a_0} \tag{9.19}$$

$$= \frac{\frac{b_0}{a_2}}{s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}} \tag{9.20}$$

- Again, the denominator is the characteristic polynomial of the system

Example 9.3: system with a derivative of the input

Find the transfer function for the following system with input $u(t)$ and output $y(t)$,

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = b_1\dot{u}(t) + b_0 u(t). \tag{9.21}$$

- Laplace transform (take the initial conditions as 0 when calculating the transfer function)
$$a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_1 s U(s) + b_0 U(s) \tag{9.22}$$

- Therefore the transfer function is

$$G(s) = \frac{Y(s)}{U(s)} \tag{9.23}$$
$$= \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} \tag{9.24}$$
$$= \frac{\frac{b_1}{a_2}s + \frac{b_0}{a_2}}{s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}} \tag{9.25}$$

- Again, the denominator is the characteristic polynomial of the system

Example 9.4: system with two inputs

Find the transfer function for the following system with inputs $u_1(t)$ and $u_2(t)$ and output $y(t)$,
$$a_2\ddot{y} + a_1\dot{y} + a_0 y = b_1 u_1(t) + b_2 u_2(t). \tag{9.26}$$

- With two inputs and one output, the system has two transfer functions
- First consider the first input (and ignore the second input)
  - Laplace transform (take the initial conditions as 0 when calculating the transfer function)
$$a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_1 U_1(s) \tag{9.27}$$
  - Therefore the first transfer function is

$$G_1(s) = \frac{Y(s)}{U_1(s)} \tag{9.28}$$
$$= \frac{b_1}{a_2 s^2 + a_1 s + a_0} \tag{9.29}$$
$$= \frac{\frac{b_1}{a_2}}{s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}} \tag{9.30}$$

- Next consider the second input (and ignore the first input)
  - Laplace transform (take the initial conditions as 0 when calculating the transfer function)
$$a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_2 U_2(s) \tag{9.31}$$
  - Therefore the first transfer function is
$$G_2(s) = \frac{Y(s)}{U_2(s)} \tag{9.32}$$
$$= \frac{b_2}{a_2 s^2 + a_1 s + a_0} \tag{9.33}$$
$$= \frac{\frac{b_2}{a_2}}{s^2 + \frac{a_1}{a_2} s + \frac{a_0}{a_2}} \tag{9.34}$$

Example 9.5: system with two outputs and one input

Find the transfer function for the following system with input $u(t)$ and outputs $x_1(t)$ and $x_2(t)$,

$$\dot{x}_1 = -2x_1 + x_2 + u(t) \tag{9.35}$$
$$\dot{x}_2 = x_1 - 2x_2. \tag{9.36}$$

- With two outputs and one input, the system has two transfer functions
- Laplace transform yields a system of algebraic equations

$$sX_1(s) = -2X_1(s) + X_2(s) + U(s) \tag{9.37}$$
$$sX_2(s) = X_1(s) - 2X_2(s) \tag{9.38}$$

- Now using substitution to solve for $X_1(s)$ and $X_2(s)$ in terms of $U(s)$ yields
  - From the second equation

$$X_2(s) = \frac{1}{s+2} X_1(s) \tag{9.39}$$

– substituting into the first equation yields

$$\begin{aligned}
sX_1(s) &= -2X_1(s) + \frac{1}{s+2}X_1(s) + U(s) && (9.40)\\
\left(s+2-\frac{1}{s+2}\right)X_1(s) &= U(s) && (9.41)\\
\frac{X_1(s)}{U(s)} &= \frac{1}{s+2-\frac{1}{s+2}} && (9.42)\\
G_1(s) &= \frac{s+2}{s^2+4s+3} && (9.43)
\end{aligned}$$

– Substituting further

$$\begin{aligned}
\frac{X_2(s)}{U(s)} &= \frac{1}{s+2}\frac{X_1(s)}{U(s)} && (9.44)\\
G_2(s) &= \frac{1}{s^2+4s+3} && (9.45)
\end{aligned}$$

9.2 Block Diagrams

- Introduction to block diagrams
  - Block diagrams take on a lot of different forms
    - Laplace transform domain representations using transfer functions
    - Time domain representations
    - Software:
      - LabVIEW™ - LabVIEW is a trademark of National Instruments. Neither Dr. Lillian, nor any software programs or other goods or services offered by Dr. Lillian, are affiliated with, endorsed by, or sponsored by National Instruments. See `https://www.ni.com/en/about-ni/legal/trademarks-and-logo-guidelines.html`.
      - Simulink® - MATLAB and Simulink are registered trademarks of The MathWorks, Inc. See mathworks.com/trademarks for a list of additional trademarks.
      - .
    - Flow charts
  - For our purposes, block diagrams are depictions of dynamic systems that feature input output relationships (e.g., transfer functions) of various elements and subsystems comprising a complete system.
  - Here we focus on block diagrams in the Laplace transform domain such that they depict the linear algebraic relationships between signals/variables within a system. (Functions should be of s rather than t.)
  - Generally, our block diagrams do not look like the physical system they represent.
  - A block diagram representation of a system is not unique; see, for example, [35].

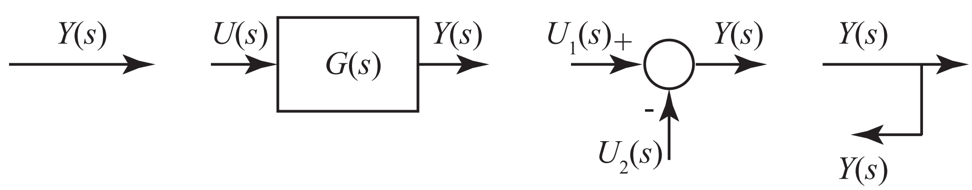


Figure 9.1: See, for example, [35]

- Block diagram symbols (see, for example, [35])
  - Arrow - Arrows represent system variables and indicate a direction into or out of blocks and summers; see, for example [35].

- Block - Blocks represent the transfer function relationship between their input and output variables/arrows. Algebraically, a block corresponds to multiplication

$$Y(s) = G(s)U(s). \tag{9.46}$$

- Summer or summing junction - Algebraically, the output of a summer is the sum/difference of inputs

$$Y(s) = U_1(s) - U_2(s); \tag{9.47}$$

  see, for example, [35, 6]

- Takeoff or pickoff point - A takeoff or pickoff point simply splits an arrow such that its corresponding variable can go to multiple locations on the diagram; see, for example, [35, 6].

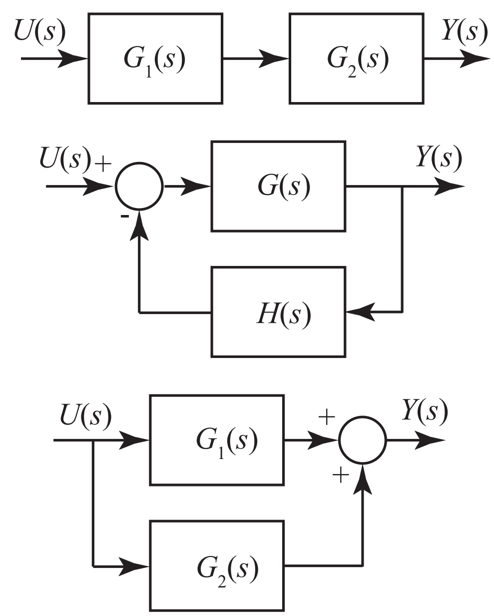


Figure 9.2: Common block diagram structures include (top) series or cascade, (middle) feedback, and (bottom) parallel.

- Common structures
  - Series or cascade; see, for example, [35, 6, 8].

$$\frac{Y(s)}{U(s)} = G_1(s)G_2(s) \tag{9.48}$$

- (negative) Feedback; see, for example, [35, 6, 8].

$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{9.49}$$

- Parallel; see, for example, [8, 6]

$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s) \tag{9.50}$$

- Note, software can be used to find equivalent transfer functions for common block diagram structures using, for example, the following commands:
  * Python: - see Appendix D.4 Representing and Solving Linear Time Invariant Systems
    · `series`
    · `parallel`
    · `feedback`
  * MATLAB® - see Appendix E.5 Representing and Solving Linear Time Invariant Systems
    · `series`
    · `parallel`
    · `feedback`

• Block diagram simplification
- Complicated single input single output block diagrams can be simplified into a single block or transfer function representing the input and output relationship of the system
- For systems with multiple inputs and/or outputs
  * Each input-output pair of a system yields a transfer function.
  * Superposition - The total output of a system, is the superposition of all contributions.
  * To obtain a specific transfer function, consider the corresponding input and temporily set others to zero. (Do not temporarily set outputs to zero.)
- Algebraic simplification methods
  * Method 1:
    · Define variables representing all unlabeled arrows
    · Write equations relating the inputs and outputs for each block and summer
    · Use substitution to find the relationship between input and output of the system by eliminating variables algebraically

* Method 2:
  * Write the output on the left hand side of an equation
  * Begin writting the right hand side with the input (leave space to the right and left of the input).
  * Follow the path from input to output on the block diagram and add sums and products when summers and blocks are reached respectively.
  * Algebraically simplify the resulting equation
- Graphical simplification methods
  * Substitute common structures for blocks with equivalent transfer functions
  * Rearrange blocks, summers, and/or arrows.
  * Experience with block diagram algebra can make simplification intuitive.
- Limitations to block diagram and transfer function algebra. This section needs to be checked and completed.
  * Physical limitations
    * Perfect pole-zero cancellation is not practical. With physical systems there will always be at least minor differences between pole and zero.
    * Improper transfer functions are not realizable. (therefore, strictly proper transfer functions may not be inverted).
    * Limitations of digital computers have finite precision and discretize time. (However, these computation limitations are somewhat out of the scope of ME 365 where we focus on continuous time LTI systems.)
  * Mathematical limitations
    * Transfer functions and block diagrams focus on the forced response. Operations that are not necessarily problematic with respect to the forced response, may yield problems with respect to the free response.
    * Much like we cannot divide by 0, some algebraic operations are invalid with transfer functions and block diagrams.

Example 9.6: Blocks in series (cascade)

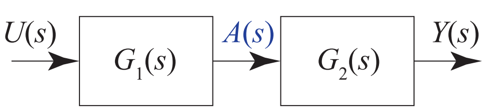


Figure 9.3:

- Algebraic Method 1

- Label arrows: $A(s)$
- Write equations for blocks and summers

$$
\begin{aligned}
A(s) &= G_1(s)U(s) && (9.51)\\
Y(s) &= G_2(s)A(s) && (9.52)
\end{aligned}
$$

- Substitution

$$
\begin{aligned}
Y(s) &= G_2(s)G_1(s)U(s) && (9.53)\\
\frac{Y(s)}{U(s)} &= G_2(s)G_1(s) && (9.54)
\end{aligned}
$$

- Algebraic Method 2
  - Output on left

$$Y(s) = \underline{\hspace{6cm}} \qquad (9.55)$$

  - Input on right

$$Y(s) = \underline{\hspace{2.5cm}}U(s)\underline{\hspace{2.5cm}} \qquad (9.56)$$

  - Blocks and summers

$$
\begin{aligned}
Y(s) &= \underline{\hspace{1.2cm}}G_1(s)U(s)\underline{\hspace{2.5cm}} && (9.57)\\
&= G_1(s)G_2(s)U(s)\underline{\hspace{2.5cm}} && (9.58)
\end{aligned}
$$

  - Simplification

$$\frac{Y(s)}{U(s)} = G_2(s)G_1(s) \qquad (9.59)$$

Example 9.7: Negative feedback

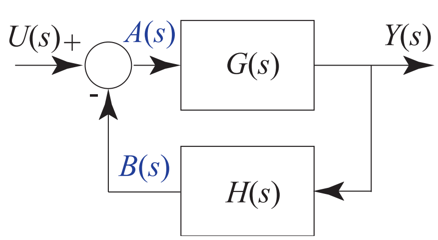


Figure 9.4:

- Algebraic Method 1
  - Label arrows: $A(s)$ and $B(s)$
  - Write equations for blocks and summers

$$\begin{aligned} A(s) &= U(s) - B(s) && (9.60) \\ Y(s) &= G(s)A(s) && (9.61) \\ B(s) &= H(s)Y(s) && (9.62) \end{aligned}$$

  - Substitution

$$\begin{aligned} \frac{Y(s)}{G(s)} &= U(s) - H(s)Y(s) && (9.63) \\ \left(\frac{1}{G(s)} + H(s)\right)Y(s) &= U(s) && (9.64) \\ \frac{Y(s)}{U(s)} &= \frac{1}{\frac{1}{G(s)} + H(s)} && (9.65) \\ &= \frac{G(s)}{1 + G(s)H(s)} && (9.66) \end{aligned}$$

- Algebraic Method 2
  - Output on left

$$Y(s) = \underline{\hspace{6cm}} \qquad (9.67)$$

  - Input on right

$$Y(s) = \underline{\hspace{2.5cm}}U(s)\underline{\hspace{2.5cm}} \qquad (9.68)$$

  - Blocks and summers

$$\begin{aligned} Y(s) &= \underline{\hspace{2.5cm}}U(s) - \underline{\hspace{2.5cm}} && (9.69) \\ &= \underline{\hspace{2.5cm}}U(s) - H(s)Y(s)\underline{\hspace{0.6cm}} && (9.70) \\ &= \underline{\hspace{1.2cm}}G(s)(U(s) - H(s)Y(s))\underline{\hspace{0.6cm}} && (9.71) \end{aligned}$$

  - Simplification

$$\begin{aligned} \frac{Y(s)}{U(s)} &= G(s)(U(s) - H(s)Y(s)) && (9.72) \\ \frac{Y(s)}{U(s)} &= \frac{G(s)}{1 + G(s)H(s)} && (9.73) \end{aligned}$$

Example 9.8: Parallel

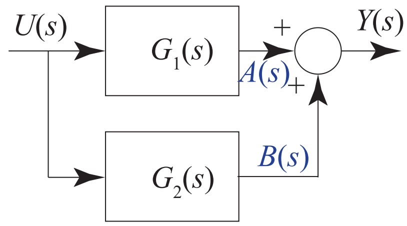


Figure 9.5:

- Algebraic Method 1
  - Label arrows: $A(s)$ and $B(s)$
  - Write equations for blocks and summers

$$\begin{aligned} A(s) &= G_1 U(s) \quad (9.74)\\ B(s) &= G_2(s)U(s) \quad (9.75)\\ Y(s) &= A(s) + B(s) \quad (9.76) \end{aligned}$$

  - Substitution

$$\begin{aligned} Y(s) &= G_1 U(s) + G_2(s)U(s) \quad (9.77)\\ &= (G_1(s) + G_2(s))U(s) \quad (9.78)\\ \frac{Y(s)}{U(s)} &= G_1(s) + G_2(s) \quad (9.79) \end{aligned}$$

- Algebraic Method 2
  - Output on left

$$Y(s) = \underline{\qquad\qquad\qquad\qquad\qquad} \quad (9.80)$$

  - Input on right

$$Y(s) = \underline{\qquad\qquad}U(s)\underline{\qquad\qquad} \quad (9.81)$$

  - Blocks and summers

$$\begin{aligned} Y(s) &= \underline{\qquad}G_1(s)U(s)\underline{\qquad\qquad} \quad (9.82)\\ &= \underline{\qquad}G_1(s)U(s) + \underline{\qquad\qquad} \quad (9.83)\\ &= \underline{\qquad}G_1(s)U(s) + G_2(s)U(s) \quad (9.84) \end{aligned}$$

– Simplification

$$Y(s) = G_1(s)U(s) + G_2(s)U(s) \quad (9.85)$$

$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s) \quad (9.86)$$

Example 9.9:

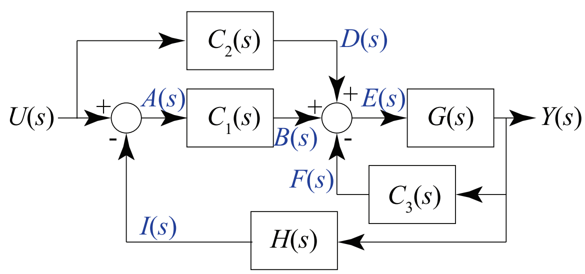


Figure 9.6:

- Algebraic Method 1
  - Label arrows: $A(s)$, $B(s)$, $D(s)$, $E(s)$, $F(s)$, and $I(s)$
  - Write equations for blocks and summers

$$A(s) = U(s) - I(s) \quad (9.87)$$

$$B(s) = C_1(s)A(s) \quad (9.88)$$

$$D(s) = C_2(s)U(s) \quad (9.89)$$

$$E(s) = B(s) + D(s) - F(s) \quad (9.90)$$

$$Y(s) = G(s)E(s) \quad (9.91)$$

$$F(s) = C_3(s)Y(s) \quad (9.92)$$

$$I(s) = H(s)Y(s) \quad (9.93)$$

– Substitution

$$
\begin{aligned}
E(s) &= B(s) + D(s) - F(s) && (9.94)\\
\frac{Y(s)}{G(s)} &= C_1(s)(U(s) - H(s)Y(s)) + C_2(s)U(s) - C_3(s)Y(s) && (9.95)\\
\left(\frac{1}{G(s)} + C_1(s)H(s) + C_3(s)\right)Y(s) &= (C_1(s) + C_2(s))U(s) && (9.96)\\
\frac{Y(s)}{U(s)} &= \frac{C_1(s) + C_2(s)}{\frac{1}{G(s)} + C_1(s)H(s) + C_3(s)} && (9.97)\\
&= \frac{G(s)(C_1(s) + C_2(s))}{1 + G(s)(C_1(s)H(s) + C_3(s))} && (9.98)
\end{aligned}
$$

- Algebraic Method 2

– Output on left

$$Y(s) = \underline{\hspace{6cm}} \tag{9.99}$$

– Input on right

$$Y(s) = \underline{\hspace{3cm}}U(s)\underline{\hspace{7cm}} \tag{9.100}$$

– Blocks and summers

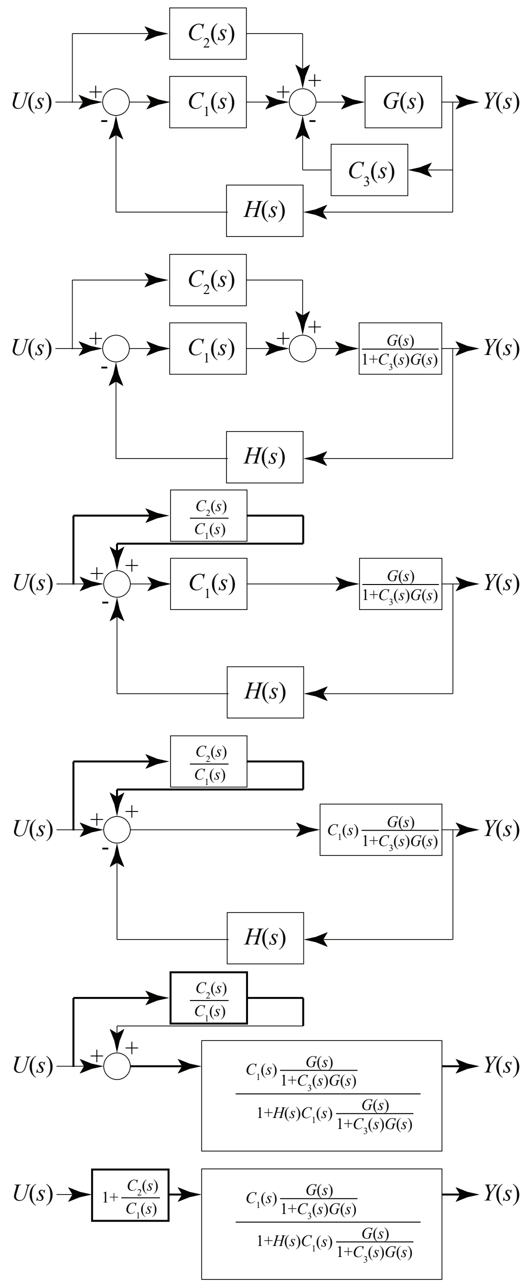
$$
\begin{aligned}
Y(s) &= \underline{\hspace{3cm}}U(s) - \underline{\hspace{6.5cm}} && (9.101)\\
&= \underline{\hspace{3cm}}U(s) - H(s)Y(s)\underline{\hspace{5cm}} && (9.102)\\
&= \underline{\hspace{1.5cm}}C_1(s)(U(s) - H(s)Y(s))\underline{\hspace{5cm}} && (9.103)\\
&= \underline{\hspace{1.5cm}}C_1(s)(U(s) - H(s)Y(s)) + C_2(s)U(s) - C_3(s)Y(s) && (9.104)\\
&= \underline{\hspace{0.3cm}}G(s)[C_1(s)(U(s) - H(s)Y(s)) + C_2(s)U(s) - C_3(s)Y(s)] && (9.105)
\end{aligned}
$$

– Simplification

$$Y(s) = G(s)[C_1(s)(U(s) - H(s)Y(s)) + C_2(s)U(s) - C_3(s)Y(s)] \tag{9.106}$$

$$
\begin{aligned}
(1 + G(s)C_1(s)H(s) + G(s)C_3(s))Y(s) &= (G(s)C_1(s) + G(s)C_2(s))U(s) && (9.107)\\
\frac{Y(s)}{U(s)} &= \frac{G(s)(C_1(s) + C_2(s))}{1 + G(s)(C_1(s)H(s) + C_3(s))} && (9.108)
\end{aligned}
$$

- Graphical simplification



$C_2(s)$
$U(s)$
$C_1(s)$
$G(s)$
$Y(s)$
$C_3(s)$
$H(s)$
$C_2(s)$
$U(s)$
$C_1(s)$
$\frac{G(s)}{1+C_3(s)G(s)}$
$Y(s)$
$H(s)$
$\frac{C_2(s)}{C_1(s)}$
$U(s)$
$C_1(s)$
$\frac{G(s)}{1+C_3(s)G(s)}$
$Y(s)$
$H(s)$
$\frac{C_2(s)}{C_1(s)}$
$U(s)$
$C_1(s)\frac{G(s)}{1+C_3(s)G(s)}$
$Y(s)$
$H(s)$
$\frac{C_2(s)}{C_1(s)}$
$U(s)$
$\frac{C_1(s)\frac{G(s)}{1+C_3(s)G(s)}}{1+H(s)C_1(s)\frac{G(s)}{1+C_3(s)G(s)}}$
$Y(s)$
$U(s)$
$1+\frac{C_2(s)}{C_1(s)}$
$\frac{C_1(s)\frac{G(s)}{1+C_3(s)G(s)}}{1+H(s)C_1(s)\frac{G(s)}{1+C_3(s)G(s)}}$
$Y(s)$

Figure 9.7:

- Block diagram construction - see 9.4 System Representations and Equivalence

9.3 State Space

- Introduction to state space (or state variable) models
  - Many higher-order differential equations may be mapped into systems of first-order differential equations (i.e., state space or state variable form).
    * "... every linear nth-order differential equation can be reduced to ... [45]" a state space model.
    * Not all systems of ordinary differential equations may be expressed in state space form; see, for example, [45].
  - State space (or state variable) form is a "standard and compact" means of representing dynamic systems and therefore it's convenient for use with computer solvers [2].

- Construction of state space models
  - Define state variables.
    * The number of state variables necessary to represent a system is equal to the number of initial conditions required to find the response of the system. (The sum of the orders of the highest derivative of each of the dependent variables. Often, the number of state variables necessary to represent a system is the sum of the order of each ordinary differential equation in the system.)
    * Traditionally, x_i is used to denote the i^{th} state variable. (In most sections of ME365, q_i is used to denote the i^{th} state variable.)
    * There are many valid definitions of state variables for any one system.
      · Common approach: If the p^{th} derivative of a dependent variable (e.g., $\frac{d^p y}{dt^y}$) appears in a system we use p state variables to represent the dependent variable (e.g., y) and its first $p-1$ derivatives.
      · Other approaches:
  - Write state equations
    * State equations are written with the time derivative of a state variable on the left and functions of the state variables and time on the right.

$$\dot{x}_1 = f(t, x_1, x_2, ...) \tag{9.109}$$
$$\dot{x}_2 = f(t, x_1, x_2, ...) \tag{9.110}$$
$$\vdots \tag{9.111}$$

    * Some state equations will arise directly from the definition of the state variables.
      · Example: if two state variables are defined as $x_1 = \theta$ and $x_2 = \dot{\theta}$, then one state equation will be $\dot{x}_1 = x_2$
    * Others come from substituting state variables into the system's ordinary differential equations.

Example 9.10: mass-spring-damper

Construct a state space model for the equation of motion for a mass-spring-damper system,

$$m\ddot{y} + c\dot{y} + ky = f. \tag{9.112}$$

- Define state variables

$$\begin{aligned} x_1 &= y & (9.113) \\ x_2 &= \dot{y} & (9.114) \end{aligned}$$

- Write state equations
  - from the definitions above we can write

$$\dot{x}_1 = x_2 \tag{9.115}$$

  - Substituting x_1 and x_2 into the original equation

$$m\dot{x}_2 + cx_2 + kx_1 = f \tag{9.116}$$

$$\dot{x}_2 = \frac{1}{m}\left(-kx_1 - cx_2 + f\right) \tag{9.117}$$

  - Writing the state equations together we have

$$\begin{aligned} \dot{x}_1 &= x_2 & (9.118) \\ \dot{x}_2 &= \frac{1}{m}\left(-kx_1 - cx_2 + f\right) & (9.119) \end{aligned}$$

Example 9.11: Two degree of freedom mass spring damper system

Construct a state variable model given the following equations of motion

$$\begin{aligned} m_1\ddot{y}_1 &= -(k_1+k_2)y_1 + k_2y_2 - (c_1+c_2)\dot{y}_1 + c_2\dot{y}_2 + f_1 & (9.120) \\ m_2\ddot{y}_2 &= k_2y_1 - (k_2+k_3)y_2 + c_2\dot{y}_1 - (c_2+c_3)\dot{y}_2 + f_2 & (9.121) \end{aligned}$$

where $m_1,\ m_2,\ c_1,\ c_2,\ c_3,\ k_1,\ k_2$,and k_3 are constant and the inputs f_1 and f_2 are known functions of time.

- Define state variables

$$x_1 = y_1 \tag{9.122}$$
$$x_2 = \dot{y}_1 \tag{9.123}$$
$$x_3 = y_2 \tag{9.124}$$
$$x_4 = \dot{y}_2 \tag{9.125}$$

- Write state equations
  - From the definition of the state variables we can write

$$\dot{x}_1 = x_2 \tag{9.126}$$
$$\dot{x}_3 = x_4 \tag{9.127}$$

  - The remaining equations come from substituting the state variables into the governing equations

$$m_1\dot{x}_2 = -(k_1+k_2)x_1 + k_2x_3 - (c_1+c_2)x_2 + c_2x_4 + f_1 \tag{9.128}$$
$$\dot{x}_2 = \frac{1}{m_1}\left(-(k_1+k_2)x_1 + k_2x_3 - (c_1+c_2)x_2 + c_2x_4 + f_1\right) \tag{9.129}$$
$$m_2\dot{x}_4 = k_2x_1 - (k_2+k_3)x_3 + c_2x_2 - (c_2+c_3)x_4 + f_2 \tag{9.130}$$
$$\dot{x}_4 = \frac{1}{m_2}\left(k_2x_1 - (k_2+k_3)x_3 + c_2x_2 - (c_2+c_3)x_4 + f_2\right) \tag{9.131}$$

  - Rewriting all the state equations together yields

$$\dot{x}_1 = x_2 \tag{9.132}$$
$$\dot{x}_2 = \frac{1}{m_1}\left(-(k_1+k_2)x_1 + k_2x_3 - (c_1+c_2)x_2 + c_2x_4 + f_1\right) \tag{9.133}$$
$$\dot{x}_3 = x_4 \tag{9.134}$$
$$\dot{x}_4 = \frac{1}{m_2}\left(k_2x_1 - (k_2+k_3)x_3 + c_2x_2 - (c_2+c_3)x_4 + f_2\right) \tag{9.135}$$

Example 9.12: transverse deflection of a beam

Construct a state space model for the deflection, $y(x)$, of a beam with distributed load $w(x)$,

$$EI\frac{d^4y}{dx^4} = w(x). \tag{9.136}$$

(Note, the independent variable in this system is position, x, rather than time, t.)

- Define state variables

$$z_1 = y \tag{9.137}$$
$$z_2 = \frac{dy}{dx} \tag{9.138}$$
$$z_3 = \frac{d^2y}{dx^2} \tag{9.139}$$
$$z_4 = \frac{d^3y}{dx^3} \tag{9.140}$$

- Write state equations

$$\frac{dz_1}{dx} = z_2 \tag{9.141}$$
$$\frac{dz_2}{dx} = z_3 \tag{9.142}$$
$$\frac{dz_3}{dx} = z_4 \tag{9.143}$$
$$\frac{dz_4}{dx} = \frac{1}{EI}w(x) \tag{9.144}$$

Example 9.13: inverted pendulum

Construct a state variable model for a pendulum with equation of motion,

$$ml^2\ddot{\theta} + K_T\theta - mgl\sin(\theta) = 0 \tag{9.145}$$

- Define state variables

$$x_1 = \theta \tag{9.146}$$
$$x_2 = \dot{\theta} \tag{9.147}$$

- Write state equations
  - From definition of state variables

$$\dot{x}_1 = x_2 \tag{9.148}$$

  - Substituting state variables into the governing equation

$$ml^2\dot{x}_2 + K_Tx_1 - mgl\sin(x_1) = 0 \tag{9.149}$$
$$\dot{x}_2 = -\frac{1}{ml^2}\left(K_Tx_1 - mgl\sin(x_1)\right) \tag{9.150}$$

- Writing state equations together yields

$$\dot{x}_1 = x_2 \tag{9.151}$$
$$\dot{x}_2 = \frac{1}{ml^2}(-K_T x_1 + mgl\sin(x_1)) \tag{9.152}$$

- Note this system is nonlinear
- This example is completed as part of the appendices on MATLAB® and Python - need to link

Example 9.14: Phase portrait

Solve for and plot the trajectory of the following mass spring system through state space,

$$\ddot{y} + 4y = 0,\ y(0) = 1,\ \dot{y}(0) = 0. \tag{9.153}$$

- Define state variables:

$$x_1 = y \tag{9.154}$$
$$x_2 = \dot{y} \tag{9.155}$$

- Write state equations
  - From definition of state variables

$$\dot{x}_1 = x_2 \tag{9.156}$$

  - Substituting state variables into the governing equation

$$\dot{x}_2 + 4x_1 = 0 \tag{9.157}$$
$$\dot{x}_2 = -4x_1 \tag{9.158}$$

  - Writing state equations together yields

$$\dot{x}_1 = x_2 \tag{9.159}$$
$$\dot{x}_2 = -4x_1 \tag{9.160}$$

- Interestingly, we know this mass-spring system is conservative. Therefore we can write

$$\mathrm{KE}(t)+\mathrm{PE}(t) = \mathrm{KE}(0)+\mathrm{PE}(0) \tag{9.161}$$

$$\frac{1}{2}m\dot{y}^2(t)+\frac{1}{2}ky^2(t) = \frac{1}{2}m\dot{y}^2(0)+\frac{1}{2}ky^2(0) \tag{9.162}$$

$$\frac{1}{2}mx_2^2(t)+\frac{1}{2}kx_1^2(t) = \frac{1}{2}\cdot 1\cdot 0^2+\frac{1}{2}\cdot 4\cdot 1 \tag{9.163}$$

$$\frac{1}{2}x_2^2(t)+\frac{1}{2}4x_1^2(t) = \frac{1}{2}4 \tag{9.164}$$

$$\frac{x_2^2}{4}+\frac{x_1^2}{1} = 1 \tag{9.165}$$

Note, this is an equation for an ellipse. The ellipse is ploted in Figure 9.8. Such plots (known as phase portraits) can be useful in understanding dynamic systems.

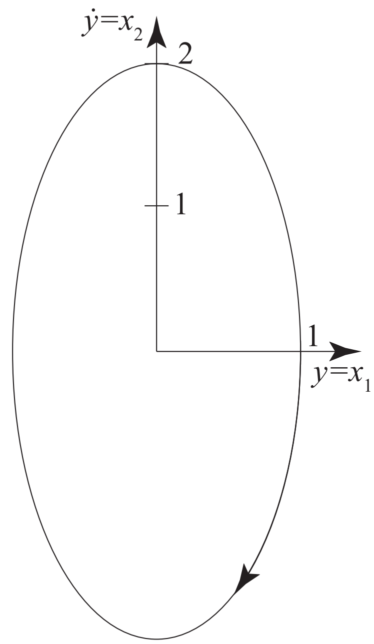


Figure 9.8:

Example 9.15: pendulum phase portrait

note pendulum is nonlinear

- Software may be used to numerically solve for the response of (linear and nonlinear) systems represented in state space.
  - Python - see Appendix D.3 Numeric Solution of Initial Value Problems Via State Space
  - MATLAB® - see Appendix E.4 Numeric Solution of Initial Value Problems Via State Space

- Standard form of linear state space models (See, for example, [35, 6, 8, 7])
  - This form is exclusively for linear systems of ordinary differential equations
  - Standard form of state equation

$$\dot{\vec{x}} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u} \tag{9.166}$$

    * State vector: $\vec{x}$. Column vector with n rows
    * System matrix: $\mathbf{A}$. Matrix with n rows and n columns
    * Input vector: $\vec{u}$. Column vector with m rows
    * Input matrix: $\mathbf{B}$. Matrix with n rows and m columns
  - Standard form of output equation

$$\vec{y} = \mathbf{C}\vec{x} + \mathbf{D}\vec{u} \tag{9.167}$$

    * Output vector: $\vec{y}$. Column vector with p rows
    * Output matrix: $\mathbf{C}$. Matrix with p rows and n columns
    * 'Feedforward matrix' [6], 'direct transmission term' [8], or 'feedthrough matrix' [7]: $\mathbf{D}$. Matrix with p rows and m columns
    * Often, the relevant outputs of a system are functions of the state variables rather than just the state variables themselves. The output equation serves as a placeholder for this general case.
  - Note, [6], for example, emphasizes that state variables are linearly independent such that the number of state variables needed to model a system is fixed. (However, there is no one correct choice of state variables for a system; see, for example, [7, 35].)
  - Note, systems with derivatives of the inputs can be harder to put into standard form. See Example 9.17.
  - Software may be used to represent state space models and numerically solve for the response of the systems they represent using, for example, the following commands:
    * Python: - see Appendix D.4 Representing and Solving Linear Time Invariant Systems
      · `ss`
      · `initial_response`
      · `step_response`
      · `impulse_response`
      · `forced_response`
    * MATLAB® - see Appendix E.5 Representing and Solving Linear Time Invariant Systems
      · `ss`

· `initial`
· `step`
· `impulse`
· `lsim`

Example 9.16:

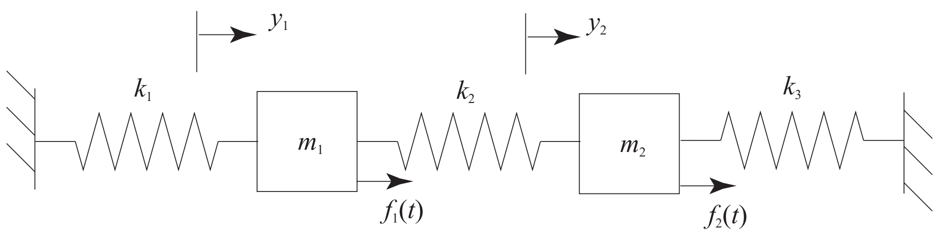


Figure 9.9:

Construct a state space model in standard form for the mass spring system with the following equations of motion,

$$m_1\ddot{y}_1 = -(k_1 + k_2)y_1 + k_2 y_2 + f_1(t) \tag{9.168}$$
$$m_2\ddot{y}_2 = k_2 y_1 - (k_2 + k_3)y_2 + f_2(t). \tag{9.169}$$

Here, $u_1 = f_1(t)$ and $u_2 = f_2(t)$ are inputs, and lets consider y_1 and y_2 as outputs

- Define state variables

$$x_1 = y_1 \tag{9.170}$$
$$x_2 = \dot{y}_1 \tag{9.171}$$
$$x_3 = y_2 \tag{9.172}$$
$$x_4 = \dot{y}_2 \tag{9.173}$$

- Write state equations

$$\dot{x}_1 = 0x_1 + 1x_2 + 0x_3 + 0x_4 + 0u_1 + 0u_2 \tag{9.174}$$
$$\dot{x}_2 = -\frac{k_1 + k_2}{m_1}x_1 + 0x_2 + \frac{k_2}{m_1}x_3 + 0x_4 + \frac{1}{m_1}u_1 + 0u_2 \tag{9.175}$$
$$\dot{x}_3 = 0x_1 + 0x_2 + 0x_3 + 1x_4 + 0u_1 + 0u_2 \tag{9.176}$$
$$\dot{x}_4 = \frac{k_2}{m_2}x_1 + 0x_2 - \frac{k_2 + k_3}{m_2}x_3 + 0x_4 + 0u_1 + \frac{1}{m_2}u_2 \tag{9.177}$$

- Write output equations

$$y_1 = 1x_1 + 0x_2 + 0x_3 + 0x_4 + 0u_1 + 0u_2 \tag{9.178}$$
$$y_2 = 0x_1 + 0x_2 + 1x_3 + 0x_4 + 0u_1 + 0u_2 \tag{9.179}$$

- Recall the equations in standard form are

$$\dot{\vec{x}} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u} \tag{9.180}$$

$$\vec{y} = \mathbf{C}\vec{x} + \mathbf{D}\vec{u} \tag{9.181}$$

  – Therefore, the matrices are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & 0 \end{bmatrix} \tag{9.182}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \tag{9.183}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{9.184}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{9.185}$$

$$\vec{u} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{9.186}$$

- What would change if we added dampers in parallel with the springs?
- What if we were interested in different outputs:
  – spring tension
  – linear momentum
  – acceleration of a mass (from the net force)

Example 9.17: derivative of the input

Construct a state space model for the following system

$$a\dot{y} + by = c\dot{u} + du \tag{9.187}$$

with input u and output y. Note, this system includes a derivative of the input; see, for example, [35, 7].

- Because u is an input (therefore it's known), we could write the following state equation:

$$\dot{y} = \frac{1}{a}\left(-by + c\dot{u} + du\right) \tag{9.188}$$

- However, $\dot{u}$ does not appear in a state variable model in standard form. So, if we want the model to be in standard we can define a new state variable x such that

$$a\dot{x} + bx = u \tag{9.189}$$

With the state equation

$$\dot{x} = \frac{1}{a}\left(-bx + u\right) \tag{9.190}$$

- Because we have a linear system, we know the solution (output, y) can be written in terms of x

$$y = c\dot{x} + dx \tag{9.191}$$

now substitute from the state equation to arrive at the output equation

$$\begin{aligned} y &= c\left(\frac{1}{a}\left(-bx + u\right)\right) + dx && (9.192)\\ &= \left(d - \frac{bc}{a}\right)x + \frac{c}{a}u && (9.193) \end{aligned}$$

- Therefore our matrices are

$$\begin{aligned} \mathbf{A} &= -\frac{b}{a} && (9.194)\\ \mathbf{B} &= \frac{1}{a} && (9.195)\\ \mathbf{C} &= d - \frac{bc}{a} && (9.196)\\ \mathbf{D} &= \frac{c}{a} && (9.197) \end{aligned}$$

9.4 System Representations and Equivalence

- Representing dynamic systems
  - There are many ways to represent dynamic systems.
    - Schematic or sketch of the physical system
    - System of differential equations
      - Scalar or matrix equations
      - State space (including standard form)
    - Input output forms
      - Transfer function
      - Single uncoupled ordinary differential equation that relates input to output
      - Block diagram
  - Choose representations and convert between representations to meet needs
    - Enter into a computer for analysis/simulation
    - Facilitate analysis

- Equivalence of dynamic system representations
  - See, for example, [35]
  - For a given linear time invariant system, the following are equivalent:
    - roots of its characteristic equation
    - eigenvalues of its state matrix (A)
    - poles of its transfer function
- Converting between representations
  - From ordinary differential equations to
    - Transfer function - see 9.1 Transfer Functions
    - State space - see 9.3 State Space
    - Block diagram -
      - obtain transfer function(s) then draw block diagram or
      - isolate highest derivative term on the left hand side, take the Laplace transform with 0 as initial conditions, move all ss to the right hand side, then draw diagram converting multiplication to blocks and addition to summers; see, for example, [35]
  - From transfer function to
    - Ordinary differential equation -
      - cross multiple to separate the input terms from the output terms
      - take the inverse Laplace transform
      - example

$$\frac{Y(s)}{U(s)} = \frac{cs+k}{ms^2+cs+k} \tag{9.198}$$

$$(ms^2+cs+k)Y(s) = (cs+k)U(s) \tag{9.199}$$

$$m\ddot{y}+c\dot{y}+ky = c\dot{u}+ku \tag{9.200}$$

    - State space
      - obtain a differential equation
      - convert differential equation to state space
    - Block diagram -
  - From state space to
    - Ordinary differential equations
    - Transfer function -
      - use methods presented in 9.1 Transfer Functions or

· apply those methods to the matrix equation. Example:

$$\dot{x} = Ax + Bu \tag{9.201}$$

$$y = Cx + Du \tag{9.202}$$

$$G(s) = C(sI - A)^{-1}B + D \tag{9.203}$$

$$G(s) = \begin{bmatrix} \frac{Y_1(s)}{U_1(s)} & \frac{Y_1(s)}{U_2(s)} & \cdots & \frac{Y_1(s)}{U_m(s)} \\ \frac{Y_2(s)}{U_1(s)} & \frac{Y_2(s)}{U_2(s)} & \cdots & \frac{Y_2(s)}{U_m(s)} \\ \vdots & \vdots & & \vdots \\ \frac{Y_n(s)}{U_1(s)} & \frac{Y_n(s)}{U_2(s)} & \cdots & \frac{Y_n(s)}{U_m(s)} \end{bmatrix} \tag{9.204}$$

- ∗ Block diagram -
  - · obtain transfer function(s) then draw diagram or
  - · isolate highest derivative term on the left hand side, take the Laplace transform with 0 as initial conditions, move all ss to the right hand side, then draw diagram converting multiplication to blocks and addition to summers; see, for example, [35]

– From block diagram to

- ∗ Ordinary differential equations -
  - · obtain transfer function (see 9.2 Transfer Functions) then write ordinary differential equations or
  - · write algebraic expression for the block diagram in the s domain and then take the inverse Laplace transform
- ∗ Transfer function - see 9.2 Transfer Functions
- ∗ State space -
  - · obtain transfer function (see 9.2 Transfer Functions)
  - · write ordinary differential equations
  - · construct state space model (see 9.3 State Space)

– Software may be used to convert between, for example, state space and transfer function representations.

- ∗ Python - see Appendix D.4 Representing and Solving Linear Time Invariant Systems
  - · `ssdata`
  - · `tfdata`
  - · `series`
  - · `parallel`
  - · `feedback`
- ∗ MATLAB® - see Appendix E.5 Representing and Solving Linear Time Invariant Systems

- `ssdata`
- `tfdata`
- `zpkdata`
- `series`
- `parallel`
- `feedback`

Example 9.18:

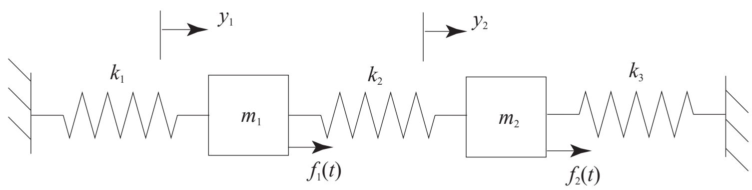


Figure 9.10:

Convert the sketch of the physical system in Figure 9.10 into several other dynamic system representations.

- System of differential equations
  - Scalar equations (equations of motion, add derivation into equations of motion chapter)

$$
\begin{aligned}
m_1\ddot{y}_1 + (k_1+k_2)y_1 - k_2y_2 &= f_1 && (9.205)\\
m_2\ddot{y}_2 - k_2y_1 + (k_2+k_3)y_2 &= f_2 && (9.206)
\end{aligned}
$$

  - Matrix equations

$$
\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k+_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{9.207}
$$

  - State space - see Example 9.16
  - State space standard form - see Example 9.16

- Input output forms
  - Transfer function (considering y_1 as output and f_1 as input) (add derivation)

$$
\frac{Y_1(s)}{F_1(s)} = \frac{m_2s^2+k_2+k_3}{m_1m_2s^4 + (m_2(k_1+k_2)+m_1(k_2+k_3))s^2 + k_1k_2 + k_1k_3 + k_2k_3} \tag{9.208}
$$

– Single ordinary differential equation that relates input to output (considering y_1 as output)
  easy to obtain from a transfer function, but here we derive an equation without using Laplace transform
  isolate y_2 in Equation 9.205

$$y_2 = \frac{1}{k_2}\left(m_1\ddot{y}_1 + (k_1+k_2)y_1 - f_1\right) \tag{9.209}$$

$$\dot{y}_2 = \frac{1}{k_2}\left(m_1\frac{d^3y_1}{dt^3} + (k_1+k_2)\dot{y}_1 - \dot{f}_1\right) \tag{9.210}$$

$$\ddot{y}_2 = \frac{1}{k_2}\left(m_1\frac{d^4y_1}{dt^4} + (k_1+k_2)\ddot{y}_1 - \ddot{f}_1\right) \tag{9.211}$$

  substitute the above into Equation 9.206

$$m_2\frac{1}{k_2}\left(m_1\frac{d^4y_1}{dt^4} + (k_1+k_2)\ddot{y}_1 - \ddot{f}_1\right) - k_2y_1 + (k_2+k_3)\frac{1}{k_2}\left(m_1\ddot{y}_1 + (k_1+k_2)y_1 - f_1\right) = f_2 \tag{9.212}$$

$$m_1m_2\frac{d^4y_1}{dt^4} + \left(m_2(k_1+k_2) + m_1(k_2+k_3)\right)\frac{d^2y_1}{dt^2} + \left(-k_2^2 + (k_1+k_2)(k_2+k_3)\right)y_1 = m_2\frac{d^2f_1}{dt^2} + (k_2+k_3)f_1 + k_2f_2 \tag{9.213}$$

$$m_1m_2\frac{d^4y_1}{dt^4} + \left(m_2(k_1+k_2) + m_1(k_2+k_3)\right)\frac{d^2y_1}{dt^2} + \left(k_1k_2 + k_1k_3 + k_2k_3\right)y_1 = m_2\frac{d^2f_1}{dt^2} + (k_2+k_3)f_1 + k_2f_2 \tag{9.214}$$

  a similar expression could be obtained for the output y_2

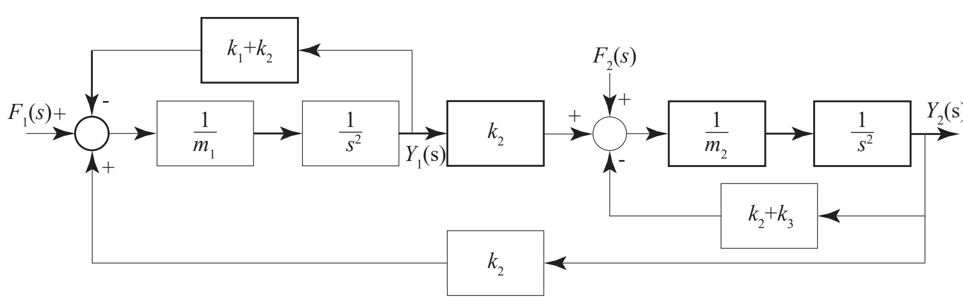


Figure 9.11:

- Block diagram - see Figure 9.11 (add process for how to construct block diagram

Part IV

Time Domain System Response and Identification

Chapter 10

Stability I

- Disclaimer:
  - **There may be differences from section to section of ME365 regarding how stability is defined.**
  - look up how stability is defined within Math 303 - it appears to parallel what I present as internal stability
  - Here we consider the stability of linear systems with constant coefficients (Linear Time Invariant systems).
  - In the case of a nonlinear system, we could evaluate stability after linearizing about an equilibrium; see Chapter 44 Nonlinear Systems and Linearization. Such analysis would give some indication of the local stability about the equilibrium, but not global stability.
- Internal stability
  - Internal stability (or Lyapunav stability) of an LTI system -
    - "... describes the behavior of the state variables rather than the output variables ... [7]"
    - Focuses on the free response or homogeneous solution
    - Here we only consider the application of Lyapunav stability to LTI systems.
  - Classifications
    - Unstable - A linear system is internally unstable if there exist finite initial conditions for which the free response of the state variables is unbounded.
    - Stable - A linear system is internally stable if for all finite initial conditions the free response of the state variables is bounded by some finite constant.

- Asymptotically stable - A linear model is internally asymptotically stable if for all finite initial conditions the free response of the state variables tends to 0 as $t \to \infty$. This is a subset of stable.
- Merely (marginally, or neutrally) stable - When a system is internally stable but not internally asymptotically stable it is termed: merely, marginally, or neutrally stable. Typically, such stability is avoided in engineering systems.

- Methods to determine the internal stability for systems described by a single scalar ordinary differential equation
  * Although we discuss the use of the 'characteristic polynomial' to determine stability, technically the 'minimal polynomial' should be used; see, for example, [37]. For the most part in ME365/375, the roots of the minimal polynomial match those of the characteristic polynomial. For example, the roots of the characteristic polynomial and the minimal polynomial match for systems described by a single scalar linear ordinary differential equation. Therefore, we focus on the familiar characteristic polynomial rather than introduce the minimal polynomial.
  * Determination of the internal stability of a system (See flowchart in Figure 10.1.)
    · Find all the roots of the characteristic equation of the system.
    · **If** any root has a positive real part **or** if there is a repeated root with zero as the real part, **then** the system is unstable.
    · **Elseif** all the roots have negative real parts **then** the system is asymptotically stable.
    · **Else** the system is stable. (Note, an asymptotically stable system is also stable.)
  * Determination of the internal stability of a system by plotting roots (poles) on the complex plane
    · Find the roots of the characteristic equation.
    · Plot points corresponding to the roots of the characteristic equation on the complex plane. Take the x coordinate to be the real part and the y coordinate to be the imaginary part.
    · The rightmost root/s (or dominant root/s) determine the stability. If the dominant root/s is/are in the open right half plane then the system is unstable. If the real part of the dominant root/s is/are zero and it's repeated, then the system is unstable. Otherwise the system is stable. If the dominant root is in the open left half plane, the system is asymptotically stable. (Note, 'open left half plane' excludes 0 and 'closed left half plane' includes 0.)
  * Tricks
    · If the coefficients of the characteristic polynomial differ in sign, the system is unstable; see, for example, [6, 8]. However, not all systems with coefficients of the same sign are stable.
    · First and second order systems are asymptotically stable if and only if the coefficients of the characteristic equation have the same sign.
- Internal stability for systems described by coupled scalar linear ordinary differential equations (e.g., state space)
  * Standard form of state variable model

$$\dot{\vec{x}} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u} \tag{10.1}$$

$$\vec{y} = \mathbf{C}\vec{x} + \mathbf{D}\vec{u} \tag{10.2}$$

  * The internal stability is determined by the eigenvalues of **A** following the same criteria as the roots of the characteristic equation. However, further analysis is required when there are repeated eigenvalues with zero as their real part. (In this case, the Jordan form or minimal polynomial may be used to determine if the multiplicity yields an unstable system. See, for example, [37].)

- ME 365/375 stability
  * Stability in ME 365/375 refers to internal stability but uses slightly different terminology.
  * Stable - We use the term "stable" in ME 365/375 to refer to systems that are internally asymptotically stable
  * Unstable or "Not Stable" - In ME365/375, there is some variation in the terms used to refer to systems that are internally stable, but not internally asymptotically stable. Although some instructors prefer the term "not stable," this author prefers the term "unstable." Others might use terms like "merely stable," "marginally stable," or "neutrally stable."
  * Unstable - we use the term "unstable" in ME 365/375 to refer to systems that have at least one pole with a positive real part and/or repeated poles on the imaginary axis.

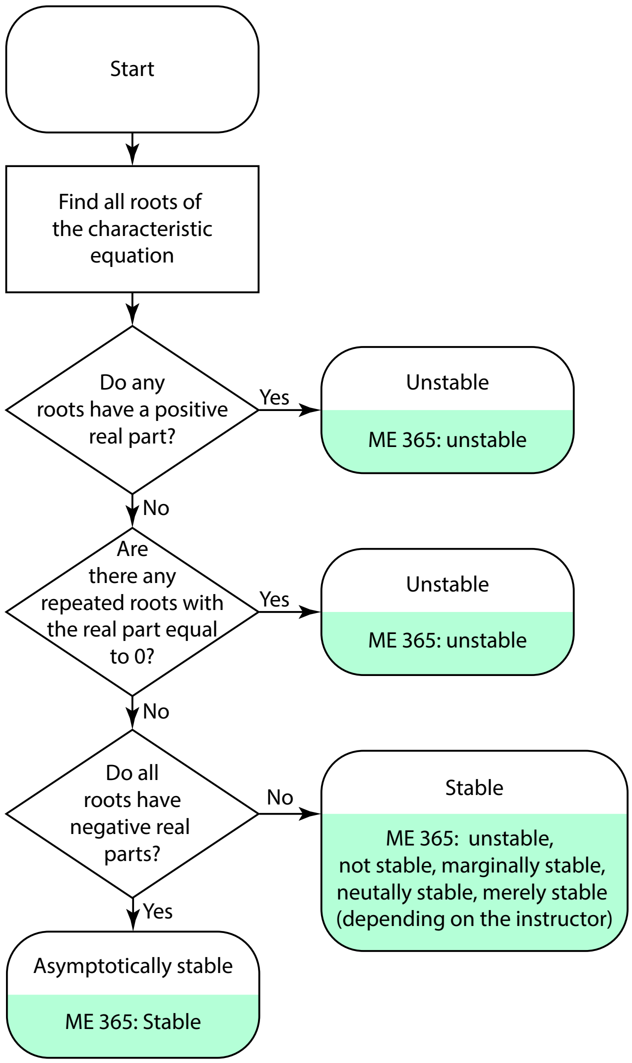


Figure 10.1: Flow chart to determine the internal stability of systems described by a single scalar linear ordinary differential equation (i.e., the roots of the characteristic polynomial match the roots of the minimal polynomial).

Example 10.1:

Characterize the stability of each of the following systems:

a. $\dot{y} = 0$:

- Characteristic equation
$$s = 0 \tag{10.3}$$
- Root of the characteristic equation (pole)
$$s = 0 \tag{10.4}$$
- Stable

b. $\tau\dot{y} + y = 0$

- Characteristic equation
$$\tau s + 1 = 0 \tag{10.5}$$
- Root of the characteristic equation (pole)
$$s = -\frac{1}{\tau} \tag{10.6}$$
- Stable if $\tau > 0$

c. $\ddot{y} = 0$:

- Characteristic equation
- Roots of the characteristic equation (poles)
$$s = \{0,\ 0\} \tag{10.7}$$
- Unstable (repeated root with a real part of 0)

d. $m\ddot{y} + c\dot{y} + ky = 0$:

- Characteristic equation
$$ms^2 + cs + k = 0 \tag{10.8}$$
- Roots of characteristic equation (poles)
$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \tag{10.9}$$
- Stability determination
  - Asymptotically stable if $m > 0$, $c > 0$, and $k > 0$ (or $m < 0$, $c < 0$, and $k < 0$)
  - Stable if $m > 0$, $c = 0$, and $k > 0$ (or $m < 0$, $c = 0$, and $k < 0$)
  - Stable if $m > 0$, and $c > 0$, and $k = 0$ (or $m < 0$, and $c < 0$, and $k = 0$)

Example 10.2: Inverted pendulum

Determine the stability of the inverted pendulum with equation of motion

$$ml^2\ddot{\theta} + K_T(\theta) - mgl\sin(\theta) = 0. \tag{10.10}$$

- Linearized equation of motion. (m, l, g, K_T, and θ_E are known constants) (Note, the linearization process is presented in Chapter 44 Nonlinear Systems and Linearization

$$ml^2\ddot{\theta} + K_T(\theta) - mgl\cos(\theta_E)\theta = mgl\sin(\theta_E) - mgl\cos(\theta_E)\theta_E \tag{10.11}$$

- Consider the homogeneous problem

$$ml^2\ddot{\theta} + K_T(\theta) - mgl\cos(\theta_E)\theta = 0 \tag{10.12}$$

- Characteristic equation

$$ml^2s^2 + (K_T - mgl^2\cos(\theta_E)) = 0 \tag{10.13}$$

- Roots of the characteristic equation

$$s = -\frac{K_T - mgl^2\cos(\theta_E)}{ml^2} \tag{10.14}$$

- Stable if $(K_T - mgl^2\cos(\theta_E)) > 0$. Note, since we linearized the nonlinear equation of motion, this test of stability is only valid near the vicinity of the equilibrium.

- External stability
  - Note, [6, 8] both address two types of stability
  - External stability
    - External stability pertains to "... the output variables, which are ... considered external [7]."
    - In the standard state space model, the output equation depends upon both the state variables and the input.
  - Bounded Input Bounded Output (BIBO) stability
    - A system is BIBO stable, "if for any bounded input, ..., the output, or response ..., is bounded for any arbitrary set of initial conditions [46]."
    - Note, some authors allow for arbitrary initial conditions in their definition of BIBO stability (e.g., [7, 46]) while others require initial conditions be set to 0 (e.g., [47]).
    - We will use the concept of BIBO stability to characterize external stability of systems.
  - Determination of BIBO stability
    - "In general, if the homogeneous solution is asymptotically stable, the forced response will be BIBO stable [46]."
    - Asymptotically stable systems are BIBO stable, but BIBO stable systems are not necessarily asymptomatically stable. (For an LTI systems with finite A, B, C, and D matrices.) this needs to be verified
    - "An LTI continuous-time system is BIBO stable if and only if the poles of its (proper) transfer function $[C(sI - A)^{-1}B + D]$ are located in the open left half complex plane [7]."
      - A system is BIBO stable if all of the poles of its transfer function are in the open left half plane. (Systems with cancelled right half plane poles are considered unstable; see, for example, [37].)
  - Proof
    - For a signal to be bounded in the time domain, it's poles must meet the following criteria:
      - The poles of a bounded signal cannot have positive real parts. (If they did, the signal would grow exponentially.)
      - The poles of a bounded signal cannot include repeats with zero as the real part. (If they did, the signal would grow with a factor of t in the corresponding terms; e.g, t or $t\cos(t)$).
    - The output (i.e., forced response) of a system can be expressed as

$$Y_{\text{forced}}(s) = G(s)U(s) \tag{10.15}$$

* Considering a partial fraction expansion, a system cannot be BIBO stable if the poles of $G(s)$ violate the criteria for a bounded signal.
* If however, $G(s)$ includes at least one pole with zero as the real part, a bounded input ($u(t)$) can be found such that the product $G(s)U(s)$ includes at least one repeated pole with zero as the real part.
* Consequently, a system is BIBO stable if all poles of its transfer function have negative real parts. (Systems with cancelled right half plane poles are considered unstable; see, for example, [37].)

Example 10.3: Stability of mass-spring system

Investigate the external stability of a mass-spring system

$$m\ddot{y} + ky = f \tag{10.16}$$

with $m > 0$ and $k > 0$.

- Transfer function considering y as output and f as input

$$\frac{Y(s)}{F(s)} = \frac{1}{ms^2 + k} \tag{10.17}$$

- External stability - distinct imaginary roots with 0 real part - therefore not BIBO stable
  - To demonstrate unbounded output given a bounded input, consider the input $f(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right)$
  - Forced response would have the following form

$$\begin{aligned}
Y_{\text{forced}}(s) &= \frac{1}{ms^2+k}\frac{As}{s^2+\frac{k}{m}} && (10.18)\\
&= \frac{\frac{A}{m}s}{(s^2+\frac{k}{m})^2} && (10.19)\\
&= \frac{A}{m}\frac{d}{ds}\left(\frac{-\frac{1}{2}}{(s^2+\frac{k}{m})^2}\right) && (10.20)\\
&= -\frac{A}{2m}\frac{d}{ds}\left(\frac{1}{(s^2+\frac{k}{m})^2}\right) && (10.21)\\
y_{\text{forced}}(t) &= \frac{A}{2m}t\sin\left(\sqrt{\frac{k}{m}}t\right) && (10.22)
\end{aligned}$$

- Here the amplitude of the output grows linearly with time. Because we can find a bounded input for which the response is unbounded, the system is not BIBO stable.

- Software may be used to assist in the determination of stability using, for example, the following commands:
  - Appendix D Python
    - `roots`
    - `eig`
    - `pole`
    - `pzmap`
  - Appendix E MATLAB®
    - `roots`
    - `eig`
    - `pole`
    - `pzmap`
- Routh-Hurowitz criterion for stability - see Chapter 29 Stability II

Example 10.4:

a. $\frac{d^4y}{dt^4} + 12\frac{d^3y}{dt^3} + 56\frac{d^2y}{dt^2} + 120\frac{dy}{dt} + 100y = 2\frac{du}{dt} + u$
$s^4 + 12s^3 + 56s^2 + 120s + 100 = 0$
$(s + 3 - j)^2(s + 3 + j)^2 = 0$

b. $\frac{d^4y}{dt^4} + 6\frac{d^3y}{dt^3} + 18\frac{d^2y}{dt^2} + 30\frac{dy}{dt} + 25y = 2\frac{du}{dt} + u$
$s^4 + 6s^3 + 18s^2 + 30s + 25 = 0$
$(s + 1 - 2j)(s + 1 + 2j)(s + 2 - j)(s + 2 + j) = 0$

c. $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - 5\frac{dy}{dt} - 6y = 2\frac{du}{dt} + u$
$s^3 + 2s^2 - 5s - 6 = 0$
$(s + 1)(s - 2)(s + 3) = 0$

d. $\frac{d^4y}{dt^4} + 2\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 2\frac{du}{dt} + u$
$s^4 + 2s^3 + 2s^2 + 10s + 25 = 0$
$(s - 1 - 2j)(s - 1 + 2j)(s + 2 - j)(s + 2 + j) = 0$

Chapter 11

Response in the Time Domain

11.1 First Order System Response

- Here we restrict our discussion to BIBO stable first order systems
- Homogeneous solution decays exponentially
  - Consider an example system

$$a_1\dot{y} + a_0 y = b_0 u,\ y(0) = y_0 \tag{11.1}$$

  - We can show that the Laplace transform of the solution will have the form

$$\begin{aligned} Y(s) &= \frac{a_1}{a_1 s + a_0} y_0 + \frac{b_0}{a_1 s + a_0} U(s) && (11.2) \\ &= \frac{C_1}{s + \frac{a_0}{a_1}} + \text{terms based on the poles of } U(s) && (11.3) \end{aligned}$$

  - Inverting the Laplace transform yields:

$$y(t) = C_1 e^{-\frac{a_0}{a_1}t} + \text{terms based on the poles of } U(s) \tag{11.4}$$

    * The first term (i.e., homogeneous solution) decays exponentially.
    * The second term takes on the form of the input (i.e., particular solution).
- The time constant ($\tau = \frac{a_1}{a_0}$) characterizes the rate decay
  - Time constant ($\tau = \frac{a_1}{a_0}$) - the reciprocal of the coefficent of time in the exponential term has units of time and is defined as the time constant.
    * Depends upon system properties (a_1 and a_0) and not initial conditions or inputs

    * The system's pole is at $p = -\frac{1}{\tau}$.

- Settling time
    * The time required for the transient response to become an insignificant contribution to the total response of a system.
    * Table 11.1 provides the value of $e^{-\frac{1}{\tau}t}$ for time evaluated at multiples of the time constant

t	$e^{-\frac{1}{\tau}t}$
0τ	1.000
1τ	0.368
2τ	0.135
3τ	0.050
3.9τ	0.020
4τ	0.018
4.6τ	0.010
5τ	0.007

Table 11.1:

    * Authors differ in where they set the border between significant and insignificant:
      1% Settling time - After 4.6 time constants, the output of the exponential function is less than 1% of its initial value. Used by [1]
      2% Settling time - After 3.9 (≈ 4) time constants, the output of the exponential function is less than 2% of its initial value. Used by [35]
      5% Settling time - After 3 time constants, the output of the exponential function is less than 5% of its initial value.

- Static sensitivity (K_0) [10, 4] (aka, static gain, DC gain [8], or zero frequency gain [5], steady-state gain)
    - Describes the steady state response for stable systems given a unit step input $f(t) = u_s(t)$

$$\lim_{t\to\infty} y(t) = K_0 u_s(t) \tag{11.5}$$

    - Defined as

$$\begin{aligned} K_0 &= G(0) && (11.6) \\ &= \frac{b_0}{a_0} && (11.7) \end{aligned}$$

Example 11.1: Free response

- Free response ($u(t) = 0$)

$$
\begin{aligned}
Y(s) &= \frac{b_0}{a_1 s + a_0}\cancelto{0}{U(s)} + \frac{a_1}{a_1 s + a_0} y_0 && (11.8)\\
&= \frac{a_1}{a_1 s + a_0} y_0 && (11.9)\\
&= \frac{1}{s + \frac{a_0}{a_1}} y_0 && (11.10)\\
y(t) &= y_0 e^{-\frac{a_0}{a_1} t} && (11.11)
\end{aligned}
$$

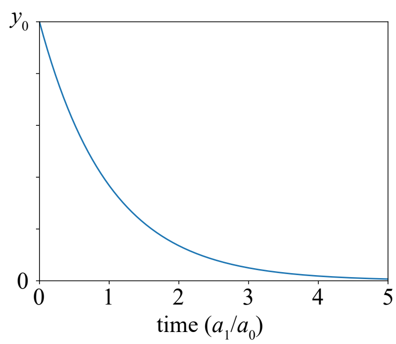


Figure 11.1:

Example 11.2: Impulse response

- Impulse $u(t) = \delta(t)$ with zero initial conditions

$$
\begin{aligned}
Y(s) &= \frac{b_0}{a_1 s + a_0} U(s) + \frac{a_1}{a_1 s + a_0}\cancelto{0}{y_0} && (11.12)\\
&= \frac{b_0}{a_1 s + a_0} 1 && (11.13)\\
&= \frac{\frac{b_0}{a_1}}{s + \frac{a_0}{a_1}} && (11.14)\\
y(t) &= \frac{b_0}{a_1} e^{-\frac{a_0}{a_1} t} && (11.15)
\end{aligned}
$$

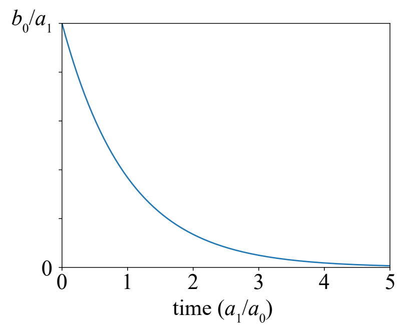


Figure 11.2:

Example 11.3: Step response

- Step response ($u(t) = u_s(t)$) with zero initial conditions
- From above, the Laplace transform of the response can be written as

$$
\begin{aligned}
Y(s) &= \frac{b_0}{a_1 s + a_0} U(s) + \frac{a_1}{a_1 s + a_0} \cancelto{0}{y_0} && (11.16)\\
&= \frac{b_0}{a_1 s + a_0} \frac{1}{s} && (11.17)\\
&= \frac{\frac{b_0}{a_1}}{s(s + \frac{a_0}{a_1})} && (11.18)\\
&= \frac{C_1}{s} + \frac{C_2}{s + \frac{a_0}{a_1}} && (11.19)
\end{aligned}
$$

- The cover up method can be use to determine the coefficients

$$
\begin{aligned}
C_1 &= \lim_{s\to 0} \frac{\frac{b_0}{a_1}}{\cancel{s}(s+\frac{a_0}{a_1})}\cancel{s} && (11.20)\\
&= \frac{\frac{b_0}{a_1}}{0+\frac{a_0}{a_1}} && (11.21)\\
&= \frac{b_0}{a_0} && (11.22)\\
C_2 &= \lim_{s\to -\frac{a_0}{a_1}} \frac{\frac{b_0}{a_1}}{s\cancel{(s+\frac{a_0}{a_1})}}\cancel{(s+\frac{a_0}{a_1})} && (11.23)\\
&= \frac{\frac{b_0}{a_1}}{-\frac{a_0}{a_1}} && (11.24)\\
&= -\frac{b_0}{a_0} && (11.25)
\end{aligned}
$$

- Finally we can obtain the response

$$
\begin{aligned}
Y(s) &= \frac{\frac{b_0}{a_0}}{s} + \frac{-\frac{b_0}{a_0}}{s+\frac{a_0}{a_1}} && (11.26)\\
y(t) &= \frac{b_0}{a_0} - \frac{b_0}{a_0}e^{-\frac{a_0}{a_1}t} && (11.27)
\end{aligned}
$$

- Interestingly, because the unit step function can be written as the integral of the impulse function, $u_s(t) = \int_{0^-}^{t} \delta(\xi)d\xi$, the step response can be obtained by integrating the impulse response.

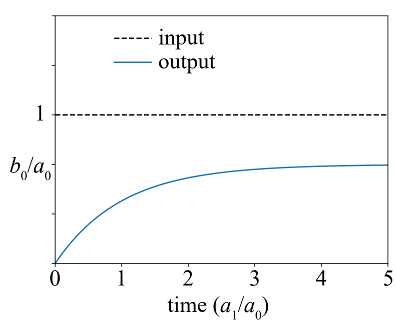


Figure 11.3:

Example 11.4: Ramp response

- Ramp response ($u(t) = tu_s(t)$) with zero initial conditions
- From above, the Laplace transform of the response can be written as

$$
\begin{aligned}
Y(s) &= \frac{b_0}{a_1 s + a_0} U(s) + \frac{a_1}{a_1 s + a_0} \cancelto{0}{y_0} && (11.28)\\
&= \frac{b_0}{a_1 s + a_0} \frac{1}{s^2} && (11.29)\\
&= \frac{\frac{b_0}{a_1}}{s^2 (s + \frac{a_0}{a_1})} && (11.30)\\
&= \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{s + \frac{a_0}{a_1}} && (11.31)
\end{aligned}
$$

- The cover up method can be use to determine the coefficients

$$\begin{aligned}
C_1 &= \lim_{s\to 0} \frac{\frac{b_0}{a_1}}{\cancel{s^2}(s+\frac{a_0}{a_1})}\cancel{s^2} && (11.32)\\
&= \frac{\frac{b_0}{a_1}}{0+\frac{a_0}{a_1}} && (11.33)\\
&= \frac{b_0}{a_0} && (11.34)\\
C_2 &= \lim_{s\to 0} \frac{d}{ds}\left(\frac{\frac{b_0}{a_1}}{\cancel{s^2}(s+\frac{a_0}{a_1})}\cancel{s^2}\right) && (11.35)\\
&= \lim_{s\to 0} -\frac{\frac{b_0}{a_1}}{(s+\frac{a_0}{a_1})^2} && (11.36)\\
&= -\frac{\frac{b_0}{a_1}}{(0+\frac{a_0}{a_1})^2} && (11.37)\\
&= -\frac{b_0 a_1}{a_0^2} && (11.38)\\
C_3 &= \lim_{s\to -\frac{a_0}{a_1}} \frac{\frac{b_0}{a_1}}{s^2\cancel{(s+\frac{a_0}{a_1})}}\cancel{(s+\frac{a_0}{a_1})} && (11.39)\\
&= \frac{\frac{b_0}{a_1}}{\left(-\frac{a_0}{a_1}\right)^2} && (11.40)\\
&= \frac{b_0 a_1}{a_0^2} && (11.41)
\end{aligned}$$

- Finally we can obtain the response

$$\begin{aligned}
Y(s) &= \frac{\frac{b_0}{a_0}}{s^2} - \frac{\frac{b_0 a_1}{a_0^2}}{s} + \frac{\frac{b_0 a_1}{a_0^2}}{s+\frac{a_0}{a_1}} && (11.42)\\
y(t) &= \frac{b_0}{a_0}t - \frac{b_0 a_1}{a_0^2} + \frac{b_0 a_1}{a_0^2}e^{-\frac{a_0}{a_1}t} && (11.43)
\end{aligned}$$

- Interestingly, because the ramp function can be written as the integral of the step function, $tu_s(t) = \int_{0^-}^{t} u_s(\xi)d\xi$, the ramp response can be obtained by integrating the step response.

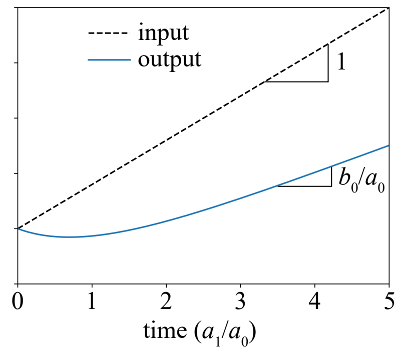


Figure 11.4:

Example 11.5: Forced response to sinusoidal input

- Forced response to sinusoidal input ($u(t) = b_0 \sin(\omega_0 t)$) and zero initial conditions
- From above, the Laplace transform of the response can be written as

$$\begin{aligned} Y(s) &= \frac{b_0}{a_1 s + a_0} U(s) + \frac{a_1}{a_1 s + a_0} \cancelto{0}{y_0} && (11.44) \\ &= \frac{b_0}{a_1 s + a_0} \frac{\omega_0}{s^2 + \omega_0^2} && (11.45) \\ &= \frac{\frac{b_0}{a_1}\omega_0}{(s^2 + \omega_0^2)(s + \frac{a_0}{a_1})} && (11.46) \\ &= C_1 \frac{\omega_0}{s^2 + \omega_0^2} + C_2 \frac{s}{s^2 + \omega_0^2} + \frac{C_3}{s + \frac{a_0}{a_1}} && (11.47) \end{aligned}$$

- Find a common denominator to determine the coefficients

$$\begin{aligned} \frac{\frac{b_0}{a_1}\omega_0}{(s^2 + \omega_0^2)(s + \frac{a_0}{a_1})} &= C_1 \frac{\omega_0}{s^2 + \omega_0^2} + C_2 \frac{s}{s^2 + \omega_0^2} + \frac{C_3}{s + \frac{a_0}{a_1}} && (11.48) \\ &= \frac{C_1 \omega_0 (s + \frac{a_0}{a_1}) + C_2 s (s + \frac{a_0}{a_1}) + C_3 (s^2 + \omega_0^2)}{(s + \frac{a_0}{a_1})(s^2 + \omega_0^2)} && (11.49) \\ &= \frac{(C_2 + C_3) s^2 + (C_1 \omega_0 + C_2 \frac{a_0}{a_1}) s + (C_1 \omega_0 \frac{a_0}{a_1} + C_3 \omega_0^2)}{(s + \frac{a_0}{a_1})(s^2 + \omega_0^2)} && (11.50) \end{aligned}$$

equating numerators on the left and right

$$s^2 \text{ terms:} \qquad 0 = C_2 + C_3 \tag{11.51}$$
$$s^1 \text{ terms:} \qquad 0 = C_1\omega_0 + C_2\frac{a_0}{a_1} \tag{11.52}$$
$$s^0 \text{ terms:} \qquad \frac{b_0}{a_1}\omega_0 = C_1\omega_0\frac{a_0}{a_1} + C_3\omega_0^2 \tag{11.53}$$

solving the system of equations yields

$$C_1 = \frac{b_0 a_0}{a_0^2 + a_1^2\omega_0^2} \tag{11.54}$$
$$C_2 = -\frac{b_0 a_1 \omega_0}{a_0^2 + a_1^2\omega_0^2} \tag{11.55}$$
$$C_3 = \frac{b_0 a_1 \omega_0}{a_0^2 + a_1^2\omega_0^2} \tag{11.56}$$

- Finally we can obtain the response

$$Y(s) = \frac{b_0 a_0}{a_0^2 + a_1^2\omega_0^2}\frac{\omega_0}{s^2+\omega_0^2} - \frac{b_0 a_1 \omega_0}{a_0^2 + a_1^2\omega_0^2}\frac{s}{s^2+\omega_0^2} + \frac{b_0 a_1 \omega_0}{a_0^2 + a_1^2\omega_0^2}\frac{1}{s+\frac{a_0}{a_1}} \tag{11.57}$$
$$= \frac{b_0}{a_0^2 + a_1^2\omega_0^2}\left(a_0\frac{\omega_0}{s^2+\omega_0^2} - a_1\omega_0\frac{s}{s^2+\omega_0^2} + a_1\omega_0\frac{1}{s+\frac{a_0}{a_1}}\right) \tag{11.58}$$
$$y(t) = \frac{b_0}{a_0^2 + a_1^2\omega_0^2}\left(a_0\sin(\omega_0 t) - a_1\omega_0\cos(\omega_0 t) + a_1\omega_0 e^{-\frac{a_0}{a_1}t}\right) \tag{11.59}$$

Example 11.6: Other examples

a. Sketch the response given an input function

b. Estimate the characteristics (τ and K_0) given a plot of the response

c. Mass damper

d. MATLAB®

11.2 Second Order System Response

- Here we focus our discussion on internally stable second order systems
- Underdamped second order system response
  - Consider an undamped system

$$a_2\ddot{y} + a_0 y = b_0 u,\ y(0) = y_0,\ \dot{y}(0) = \dot{y}_0 \tag{11.60}$$

  - The Laplace transform of the solution can be written as

$$\begin{aligned} Y(s) &= \frac{a_2 s}{a_2 s^2 + a_0} y_0 + \frac{a_2}{a_2 s^2 + a_0}\dot{y}_0 + \frac{b_0}{a_2 s^2 + a_0} U(s) & (11.61) \\ &= C_1 \frac{\sqrt{\frac{a_0}{a_2}}}{s^2 + \sqrt{\frac{a_0}{a_2}}^2} + C_2 \frac{s}{s^2 + \sqrt{\frac{a_0}{a_2}}^2} + \text{terms for poles of } U(s) & (11.62) \end{aligned}$$

  - Inverting the Laplace transform yields:

$$y(t) = C_1 \sin\left(\sqrt{\frac{a_0}{a_2}} t\right) + C_2 \cos\left(\sqrt{\frac{a_0}{a_2}} t\right) + \text{terms based on the poles of } U(s) \tag{11.63}$$

    * The first two terms (i.e., homogeneous solution) are sinusoidal.
    * The remaining terms take on the form of the input (i.e., particular solution).
      · Note, this system is not BIBO stable. Therefore, there are bounded inputs for which the response is unbounded.
  - The natural frequency characterizes the rate of oscillation of the homogeneous part of the solution
    * Natural frequency ($\omega_n = \sqrt{\frac{a_0}{a_2}}$) - the coefficent of time in the sinusoidal terms has units of time^{-1} and is defined as the natural frequency.
      · The natural requency is an angular frequency (rad/s)
      · Depends upon system properties and not initial conditions or inputs
      · The undamped systems poles are at $p = \pm\omega_n j$.
      · For a mass-spring system $a_2 = m$ and $a_0 = k$ such that $\omega_n = \sqrt{\frac{k}{m}}$
    * Period (T) of oscillation of the undamped system can be written as

$$T = \frac{2\pi}{\omega_n} \tag{11.64}$$

- BIBO stable second order systems

– Consider an damped system

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = b_0 u,\ y(0) = y_0,\ \dot{y}(0) = \dot{y}_0 \tag{11.65}$$

– We can show that the Laplace transform of the solution will have the form

$$\begin{aligned} Y(s) &= \frac{a_2 s + a_1}{a_2 s^2 + a_1 s + a_0} y_0 + \frac{a_2}{a_2 s^2 + a_1 s + a_0} \dot{y}_0 + \frac{b_0}{a_2 s^2 + a_1 s + a_0} U(s) && (11.66) \\ &= \text{terms based on the roots of } a_2 s^2 + a_1 s + a_0 \\ &\quad + \text{terms based on the poles of } U(s) && (11.67) \end{aligned}$$

– The solution depends upon the roots of $a_2 s^2 + a_1 s + a_0 = 0$:

* Find the roots of the characteristic equation and rearrange

$$\begin{aligned} 0 &= a_2 s^2 + a_1 s + a_0 && (11.68) \\ s &= \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} && (11.69) \\ &= \frac{-a_1 \pm \sqrt{4a_2 a_0}\sqrt{\frac{a_1^2}{4a_2 a_0} - 1}}{2a_2} && (11.70) \\ &= \frac{-a_1 \pm \sqrt{4a_2 a_0}\sqrt{\left(\frac{a_1}{\sqrt{4a_2 a_0}}\right)^2 - 1}}{2a_2} && (11.71) \\ &= -\frac{a_1}{2a_2} \pm \frac{\sqrt{a_0}\sqrt{\left(\frac{a_1}{\sqrt{4a_2 a_0}}\right)^2 - 1}}{\sqrt{a_2}} && (11.72) \\ &= -\frac{a_1}{2\sqrt{a_2 a_0}}\sqrt{\frac{a_0}{a_2}} \pm \sqrt{\frac{a_0}{a_2}}\sqrt{\left(\frac{a_1}{2\sqrt{a_2 a_0}}\right)^2 - 1} && (11.73) \end{aligned}$$

– The form of the homogeneous response depends upon how the damping ratio compares to 1:

* Definition of the damping ratio (ζ)
  · Damping ratio is defined as

$$\zeta = \frac{a_1}{2\sqrt{a_2 a_0}} \tag{11.74}$$

  · For a mass ($a_2 = m$) spring ($a_0 = k$) damper ($a_1 = c$) system, we can write the damping ratio as

$$\zeta = \frac{c}{2\sqrt{mk}} \tag{11.75}$$

· Substituting ζ and ω_n into the expression for the roots (poles) yields

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{11.76}$$

* if $\zeta > 1$, then two distinct real roots yielding exponential decay
* if $\zeta = 1$, then repeated real roots yielding a response dominated by exponential decay
* if $\zeta < 1$, then complex conjugate pair of roots yielding exponentially decaying sinusoid

• Overdamped ($\zeta > 1$) system response

$$\begin{aligned} Y(s) &= \frac{a_2 s + a_1}{a_2 s^2 + a_1 s + a_0} y_0 + \frac{a_2}{a_2 s^2 + a_1 s + a_0} \dot{y}_0 + G(s)U(s) && (11.77) \\ &= \frac{C_1}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} + \frac{C_1}{s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}} \\ &\quad + \text{terms based on the poles of } U(s) && (11.78) \\ y(t) &= C_1 e^{-(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t} + C_1 e^{-(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} \\ &\quad + \text{terms based on the poles of } U(s) && (11.79) \end{aligned}$$

– Overdamped systems have two distinct real roots/poles

$$p = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{11.80}$$

– Superposition of two exponential decay terms comprises the homogeneous solution

* We can identify two time constants

$$\tau_1 = \frac{1}{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} \tag{11.81}$$

$$\tau_2 = \frac{1}{\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}} \tag{11.82}$$

* For $1 < \zeta \approx 1$, the time constants are approximately equivalent and the two terms decay at about the same rate
* For $\zeta \gg 1$, the time constants differ significantly such that they decay at very different rates. The root corresponding to a larger time constant, τ_1, is considered the dominant root because its contribution to the response is significant for a longer period of time.

- Critically damped ($\zeta = 1$) system response

$$\begin{aligned}
Y(s) &= \frac{a_2 s + a_1}{a_2 s^2 + a_1 s + a_0} y_0 + \frac{a_2}{a_2 s^2 + a_1 s + a_0} \dot{y}_0 + G(s)U(s) && (11.83)\\
&= \frac{C_1}{(s+\omega_n)^2} + \frac{C_1}{s+\omega_n} \\
&\quad + \text{terms based on the poles of } U(s) && (11.84)\\
y(t) &= C_1 t e^{-\omega_n t} + C_2 e^{-\omega_n t} \\
&\quad + \text{terms based on the poles of } U(s) && (11.85)
\end{aligned}$$

  - Critically damped systems have two repeated real roots/poles

$$p = \{-\omega_n, -\omega_n\} \tag{11.86}$$

  - The homogeneous response is dominated by exponential decay
    * Despite the t in the first term, it tends to 0 at t goes to infinity.
    * Because of the t, the first term is significant for a longer period of time than the second term

- Underdamped ($\zeta < 1$) system response

$$\begin{aligned}
Y(s) &= \frac{a_2 s + a_1}{a_2 s^2 + a_1 s + a_0} y_0 + \frac{a_2}{a_2 s^2 + a_1 s + a_0} \dot{y}_0 + G(s)U(s) && (11.87)\\
&= G(s)U(s) + \frac{C_1}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} + \frac{C_1}{s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}} \\
&= \frac{C_1}{s + \zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}} + \frac{C_1}{s + \zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}} \\
&\quad + \text{terms based on the poles of } U(s) && (11.88)\\
&= \frac{C_1}{s + \zeta\omega_n - j\omega_d} + \frac{C_1}{s + \zeta\omega_n + j\omega_d} \\
&\quad + \text{terms based on the poles of } U(s) && (11.89)\\
y(t) &= C_1 e^{-(\zeta\omega_n - j\omega_d)t} + C_1 e^{-(\zeta\omega_n + j\omega_d)t} \\
&\quad + \text{terms based on the poles of } U(s) && (11.90)\\
y(t) &= e^{-\zeta\omega_n t}\left(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right) \\
&\quad + \text{terms based on the poles of } U(s) && (11.91)
\end{aligned}$$

  - Underdamped systems have a complex conjugate pair of roots/poles

$$\begin{aligned}
p &= -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} && (11.92)\\
&= -\zeta\omega_n \pm j\omega_d && (11.93)
\end{aligned}$$

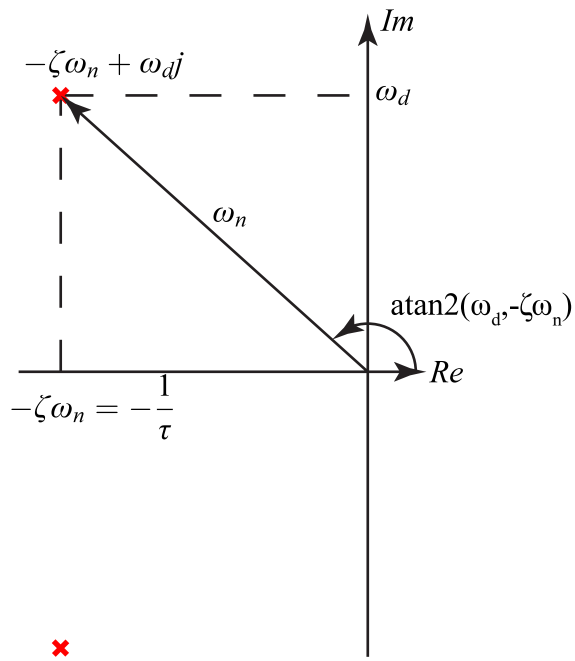


Figure 11.5: Pole locations for underdamped ($0 \leq \zeta < 1$) second order systems; see, for exmple, [35]

- The homogeneous response is an exponentially decaying sinusoid
  - Damped natural frequency characterizes the frequency of free vibrations of underdamped systems

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{11.94}$$

  - Period of free vibrations of underdamped systems

$$T = \frac{2\pi}{\omega_d} \tag{11.95}$$

  - The time constant $\tau = \frac{1}{\zeta\omega_n}$ describes the exponential rate of decay of the amplitude of the sinusoid.

Example 11.7: Free response of underdamped second order system

- free response ($U(s) = 0$)

$$
\begin{aligned}
Y(s) &= \frac{b_0}{a_1 s + a_0}\cancelto{0}{U(s)} + \frac{a_2 s + a_1}{a_2 s^2 + a_1 s + a_0} y_0 + \frac{a_2}{a_2 s^2 + a_1 s + a_0}\dot{y}_0 && (11.96)\\
&= \frac{s + \frac{a_1}{a_2}}{s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}} y_0 + \frac{1}{s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}}\dot{y}_0 && (11.97)\\
&= \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} y_0 + \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\dot{y}_0 && (11.98)\\
&= \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} y_0 + \frac{\zeta\omega_n y_0 + \dot{y}_0}{(s + \zeta\omega_n)^2 + \omega_d^2} && (11.99)\\
&= \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} y_0 + \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\frac{\zeta\omega_n y_0 + \dot{y}_0}{\omega_d} && (11.100)
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= y_0 e^{-\zeta\omega_n t}\cos(\omega_d t) + \frac{\zeta\omega_n y_0 + \dot{y}_0}{\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t) && (11.101)\\
&= e^{-\zeta\omega_n t}\left(y_0\cos(\omega_d t) + \frac{\zeta\omega_n y_0 + \dot{y}_0}{\omega_d}\sin(\omega_d t)\right) && (11.102)
\end{aligned}
$$

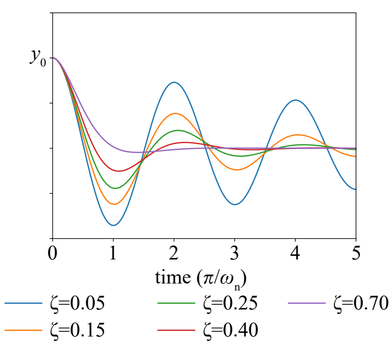


Figure 11.6: Taking $\dot{y}_0 = 0$.

11.3 Step Response of Second Order Systems

- Step response of an underdamped system
  - Equation
$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K_0\omega_n^2 f,\ y(0) = 0,\ \dot{y}(0) = 0 \tag{11.103}$$
  - Transfer function
$$G(s) = \frac{K_0\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{11.104}$$
  - Unit step response
$$\begin{aligned} Y(s) &= G(s)U_s(s) && (11.105)\\ &= \frac{K_0\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} && (11.106)\\ &= \frac{K_0\omega_n^2}{s\left((s+\zeta\omega_n)^2 + \omega_d^2\right)} && (11.107)\end{aligned}$$
  - Partial fraction expansion
    - Expansion
$$\begin{aligned} \frac{K_0\omega_n^2}{s\left((s+\zeta\omega_n)^2 + \omega_d^2\right)} &= \frac{C_1}{s} + C_2\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} + C_3\frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} && (11.108)\\ &= \frac{C_1(s^2 + 2\zeta\omega_n s + \omega_n^2) + C_2 s(s+\zeta\omega_n) + C_3 s\omega_d}{s\left((s+\zeta\omega_n)^2 + \omega_d^2\right)} && (11.109)\end{aligned}$$
    - equating numerators on the left and right
$$\begin{aligned} s^2 \text{ terms:} &\quad 0 = C_1 + C_2 && (11.110)\\ s^1 \text{ terms:} &\quad 0 = 2\zeta\omega_n C_1 + \zeta\omega_n C_2 + \omega_d C_3 && (11.111)\\ s^0 \text{ terms:} &\quad K_0\omega_n^2 = \omega_n^2 C_1 && (11.112)\end{aligned}$$
    - Solve for coefficients
$$\begin{aligned} C_1 &= K_0 && (11.113)\\ C_2 &= -K_0 && (11.114)\\ C_3 &= -\frac{\zeta\omega_n K_0}{\omega_d} && (11.115)\\ &= -\frac{\zeta K_0}{\sqrt{1-\zeta^2}} && (11.116)\end{aligned}$$
  - Inverse transform

* Laplace transform of solution

$$Y(s) = \frac{K_0}{s} - K_0\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta K_0}{\sqrt{1-\zeta^2}}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2} \quad (11.117)$$

* Solution

$$y(t) = K_0 - K_0 e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right) \quad (11.118)$$

$$= K_0 - K_0 e^{-\zeta\omega_n t}\mathbb{R}\left\{e^{j\omega_d t} - j\frac{\zeta}{\sqrt{1-\zeta^2}}e^{j\omega_d t}\right\} \quad (11.119)$$

$$= K_0 - K_0 e^{-\zeta\omega_n t}\mathbb{R}\left\{\sqrt{1^2+\frac{\zeta^2}{1-\zeta^2}}e^{j\tan^{-1}\left(\frac{-\zeta}{\sqrt{1-\zeta^2}}\right)}e^{j\omega_d t}\right\} \quad (11.120)$$

$$= K_0 - K_0 e^{-\zeta\omega_n t}\mathbb{R}\left\{\frac{1}{\sqrt{1-\zeta^2}}e^{j\left(\omega_d t - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)}\right\} \quad (11.121)$$

$$= K_0 - \frac{K_0}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_d t - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right) \quad (11.122)$$

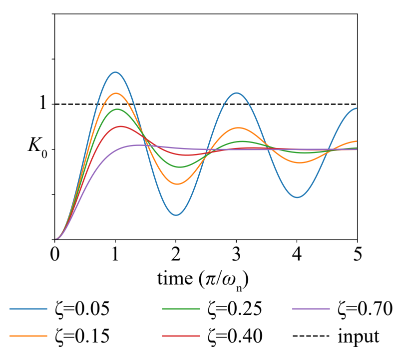


Figure 11.7:

- Characteristics
  - Static sensitivity (K_0) [10, 4] (aka, static gain, DC gain [8], or zero frequency gain [5], steady-state gain)
    * Describes the steady state response for BIBO stable systems given a unit step input $f(t) = u_s(t)$

$$\lim_{t\to\infty} y(t) = K_0 u_s(t) \quad (11.123)$$

* Defined as

$$K_0 = \frac{b_0}{a_0} = G(0) \tag{11.124}$$

- Peak time
  * Time rate of change of response, $\dot{y}(t)$

$$y(t) = K_0 - K_0\mathbb{R}\left\{\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t+j\left(\omega_d t-\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)}\right\} \quad (11.125)$$

$$\dot{y}(t) = -K_0\mathbb{R}\left\{\frac{-\zeta\omega_n+j\omega_d}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t+j\left(\omega_d t-\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)}\right\} \quad (11.126)$$

$$= -K_0\mathbb{R}\left\{\frac{\sqrt{(\zeta\omega_n)^2+\omega_d^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t+j\left(\omega_d t-\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)-\tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right)\right)}\right\} \quad (11.127)$$

$$= -K_0\mathbb{R}\left\{\frac{\sqrt{(\zeta\omega_n)^2+\omega_d^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t+j\left(\omega_d t-\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)-\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)}\right\} \quad (11.128)$$

$$= -K_0\mathbb{R}\left\{\frac{\sqrt{(\zeta\omega_n)^2+\omega_d^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t+j\left(\omega_d t-\frac{\pi}{2}\right)}\right\} \quad (11.129)$$

$$= -K_0\frac{\sqrt{(\zeta\omega_n)^2+\omega_d^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_d t-\frac{\pi}{2}\right) \quad (11.130)$$

$$= \frac{K_0\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_d t\right) \quad (11.131)$$

  * Peak occurs at $\dot{y}(t_{\text{peak}}) = 0$

$$0 = \frac{K_0\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_{\text{peak}}}\sin\left(\omega_d t_{\text{peak}}\right) \quad (11.132)$$

$$= \sin\left(\omega_d t_{\text{peak}}\right) \quad (11.133)$$

$$\omega_d t_{\text{peak}} = \pi \quad (11.134)$$

$$t_{\text{peak}} = \frac{\pi}{\omega_d} \quad (11.135)$$

  * Alternatively, because we know the response oscillates with angular frequency ω_d and the initial condition includes $\dot{y}(0) = 0$, we can reason that the peak will occur one half period after, $t_{\text{peak}} = \frac{1}{2}\frac{2\pi}{\omega_d}$.
  * Peak time decresases as the imaginary part of the complex conjugate pair of poles increase in magnitude
- Maximum (Percent) Overshoot
  * Steady state value

$$y_{ss}(t) = K_0 \quad (11.136)$$

* Maximum overshoot

$$\begin{aligned}
\text{Maximum Overshoot} &= y(t_{\text{peak}}) - y_{ss}(t) && (11.137)\\
&= K_0 - \frac{K_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_{\text{peak}}} \cos\left(\omega_d t_{\text{peak}} - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right) - K_0 && (11.138)\\
&= -\frac{K_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \cos\left(\omega_d \frac{\pi}{\omega_d} - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right) && (11.139)\\
&= -\frac{K_0}{\sqrt{1-\zeta^2}} e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}} \left(-\frac{\sqrt{1-\zeta^2}}{1}\right) && (11.140)\\
&= K_0 e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} && (11.141)
\end{aligned}$$

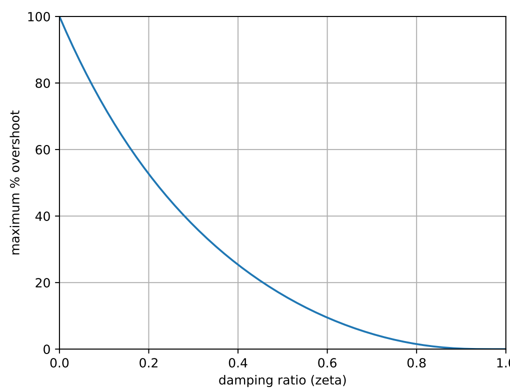


Figure 11.8: Maximum percent overshoot as a function of damping ratio.

* Maximum percent overshoot

$$\begin{aligned}
\text{Maximum Percent Overshoot} &= \frac{\text{Maximum Overshoot}}{y_{ss}(t)} 100\% && (11.142)\\
&= \frac{K_0 e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}{K_0} 100\% && (11.143)\\
&= e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} 100\% && (11.144)
\end{aligned}$$

* Maximum (percent) overshoot is determined by ζ and ζ can be obtained from the angle of the pole

$$\cos(\angle p) = -\zeta \qquad (11.145)$$

Therefore, overshoot varies with the angle of the poles

- Settling time
  * Time constant for the exponential decay of sinusoidal amplitude

$$\tau = \frac{1}{\zeta\omega_n} \tag{11.146}$$

  * 5% settling time

$$t_{s,5\%} = \frac{3}{\zeta\omega_n} \tag{11.147}$$

  * 2% settling time

$$t_{s,2\%} = \frac{3.9}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n} \tag{11.148}$$

  * 1% settling time

$$t_{s,1\%} = \frac{4.6}{\zeta\omega_n} \tag{11.149}$$

  * Settling time decreases as the real part of the rightmost pole(s) gets moves to the left in the complex plane
    · For fixed ω_n, $\zeta = 1$ pushes the dominant pole to the left the farthest and thereby yields the shortest settling time.

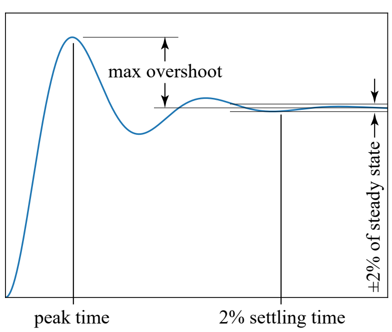


Figure 11.9:

- Rise time (not presented in the prelecture)

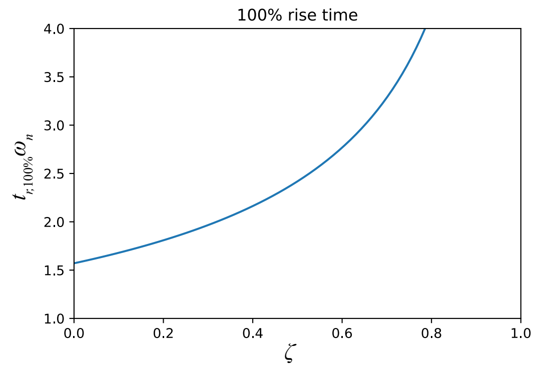


Figure 11.10:

* 100% rise time - time of first passage through steady state value

$$\cancel{K_0} = \cancel{K_0} - \frac{K_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_{r,100\%}} \cos\left(\omega_d t_{r,100\%} - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right) \tag{11.150}$$

$$0 = \cos\left(\omega_d t_{r,100\%} - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right) \tag{11.151}$$

$$\frac{\pi}{2} = \omega_d t_{r,100\%} - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{11.152}$$

$$t_{r,100\%} = \frac{\frac{\pi}{2} + \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}{\omega_d} \tag{11.153}$$

$$= \frac{\frac{\pi}{2} + \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}{\omega_n\sqrt{1-\zeta^2}} \tag{11.154}$$

  · See Figure 11.10

* 10-90% rise time - time from first passage through 10% to first passage through 90% of the steady state value
  · Solve numerically, see Figure 11.11
  · The $t_{r,10-90\%}$ for $0 \le \zeta < 1$ may be approximated by its average value over this range [8]

$$t_{r,10-90\%} \approx \frac{1.8}{\omega_n} \tag{11.155}$$

  · Increasing, ω_n, the distance of the poles from the origin of the complex plane reduces the rise time

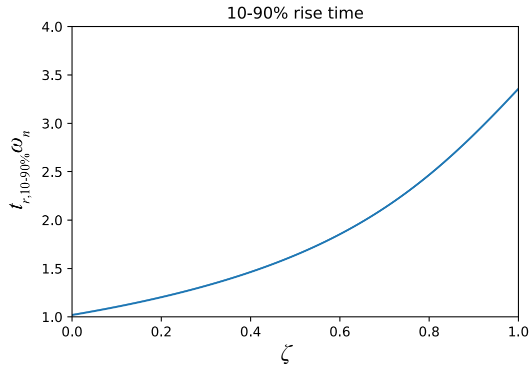


Figure 11.11:

- Step response characteristics of a linear time invariant system may be obtained using, for example, the following commands:
  * Python: - see Appendix D.4 Representing and Solving Linear Time Invariant Systems
    · `step_info`
  * MATLAB® - see Appendix E.5 Representing and Solving Linear Time Invariant Systems
    · `stepinfo`

- Characteristics and pole locations; see Figure 11.12

- Step response of overdamped second order system (this is just an aside for those that are curious)
  - Consider an overdamped second order system (characterized by two time constants)

$$\tau_1\tau_2\ddot{y} + (\tau_1 + \tau_2)\dot{y} + y = K_0 u(t) \tag{11.156}$$

  - Unit step response
    * Transfer function

$$G(s) = \frac{K_0}{\tau_1\tau_2 s^2 + (\tau_1 + \tau_2)s + s} \tag{11.157}$$

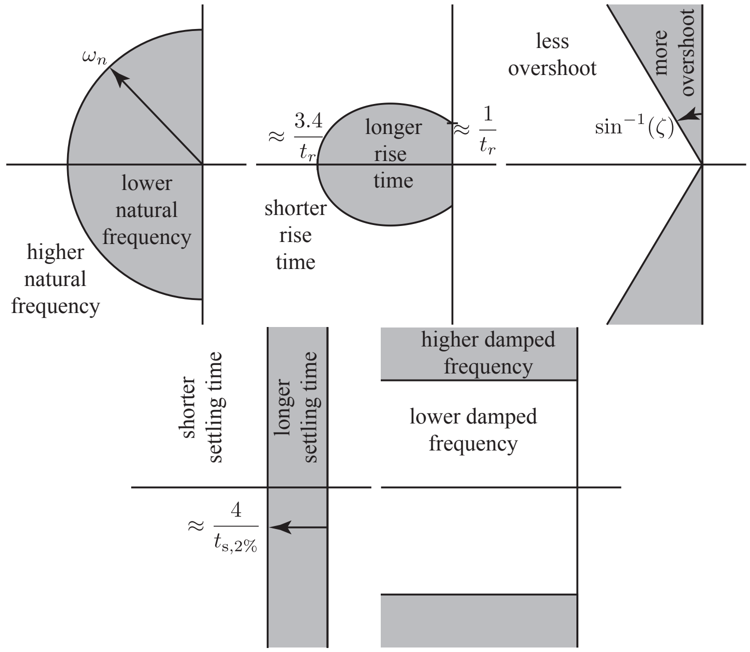


Figure 11.12: Dependence of system response characteristics on underdamped pole locations for a second order system with no zeros.

* Forced response

$$
\begin{aligned}
Y(s) &= G(s)U(s) && (11.158)\\
&= \frac{K_0}{\tau_1\tau_2 s^2 + (\tau_1+\tau_2)s + s}\frac{1}{s} && (11.159)\\
&= \frac{\frac{K_0}{\tau_1\tau_2}}{s(s+\frac{1}{\tau_1})(s+\frac{1}{\tau_2})} && (11.160)\\
&= \frac{A_1}{s} + \frac{A_2}{s+\frac{1}{\tau_1}} + \frac{A_3}{s+\frac{1}{\tau_2}} && (11.161)
\end{aligned}
$$

* Partial fraction expansion coefficients

$$A_1 = \lim_{s\to 0} \frac{\frac{K_0}{\tau_1\tau_2}}{\cancel{s}(s+\frac{1}{\tau_1})(s+\frac{1}{\tau_2})}\cancel{s} \tag{11.162}$$

$$= \lim_{s\to 0} \frac{\frac{K_0}{\tau_1\tau_2}}{\frac{1}{\tau_1}\frac{1}{\tau_2}} \tag{11.163}$$

$$= K_0 \tag{11.164}$$

$$A_2 = \lim_{s\to -\frac{1}{\tau_1}} \frac{\frac{K_0}{\tau_1\tau_2}}{s\cancel{(s+\frac{1}{\tau_1})}(s+\frac{1}{\tau_2})}\cancel{(s+\frac{1}{\tau_1})} \tag{11.165}$$

$$= \frac{\frac{K_0}{\tau_1\tau_2}}{-\frac{1}{\tau_1}(-\frac{1}{\tau_1}+\frac{1}{\tau_2})} \tag{11.166}$$

$$= -\frac{K_0\tau_1}{\tau_1-\tau_2} \tag{11.167}$$

$$A_3 = \lim_{s\to -\frac{1}{\tau_2}} \frac{\frac{K_0}{\tau_1\tau_2}}{s(s+\frac{1}{\tau_1})\cancel{(s+\frac{1}{\tau_2})}}\cancel{(s+\frac{1}{\tau_2})} \tag{11.168}$$

$$= \frac{\frac{K_0}{\tau_1\tau_2}}{-\frac{1}{\tau_2}(-\frac{1}{\tau_2}+\frac{1}{\tau_1})} \tag{11.169}$$

$$= \frac{K_0\tau_2}{\tau_1-\tau_2} \tag{11.170}$$

* Forced response

$$y(t) = K_0 - \frac{K_0}{\tau_1-\tau_2}\left(\tau_1 e^{-\frac{1}{\tau_1}t} - \tau_2 e^{-\frac{1}{\tau_2}t}\right) \tag{11.171}$$

- Dominant pole
  * Pole with the largest time constant (or rightmost pole on complex plane) dominantes the response
  * The term associated with the dominant pole has the largest amplitude
  * The term associated with the dominant pole is significant for a longer period of time

11.4 Step Response of Higher Order Systems

- See Chapter 27 Poles and Zeros

Chapter 12

System Identification in the Time Domain

- Parameter estimation for first order systems
  - First order system response to a step change
    * General equation with input $u(t)$ and output $y(t)$

$$\tau\dot{y} + y = K_0 u(t) \tag{12.1}$$

    * Step change at time $t = T_0$ from rest with $u = u_i$ and $y = y_i$ to $u = u_{ss}$ and $y = y_{ss}$

$$\tau\dot{y} + y = K_0(u_i + (u_{ss} - u_i)u_s(t - T_0)),\ y(0) = y_i \tag{12.2}$$

    * Response

$$y(t) = y_i + (y_{ss} - y_i)\left(1 - e^{-\frac{t-T_0}{\tau}}\right)u_s(t - T_0) \tag{12.3}$$

  - Static sensitivity
    * Note, we know that $y_i = K_0 u_i$ and $y_{ss} = K_0 u_{ss}$

$$\begin{aligned}\frac{y_{\text{ss}} - y_i}{u_{\text{ss}} - u_i} &= \frac{K_0 u_{\text{ss}} - K_0 u_i}{u_{\text{ss}} - u_i} &(12.4)\\ &= K_0\frac{u_{\text{ss}} - u_i}{u_{\text{ss}} - u_i} &(12.5)\\ &= K_0 &(12.6)\end{aligned}$$

  - Time constant
    * Method 1: Find the time at which the response is 63.2% of the way to the steady state

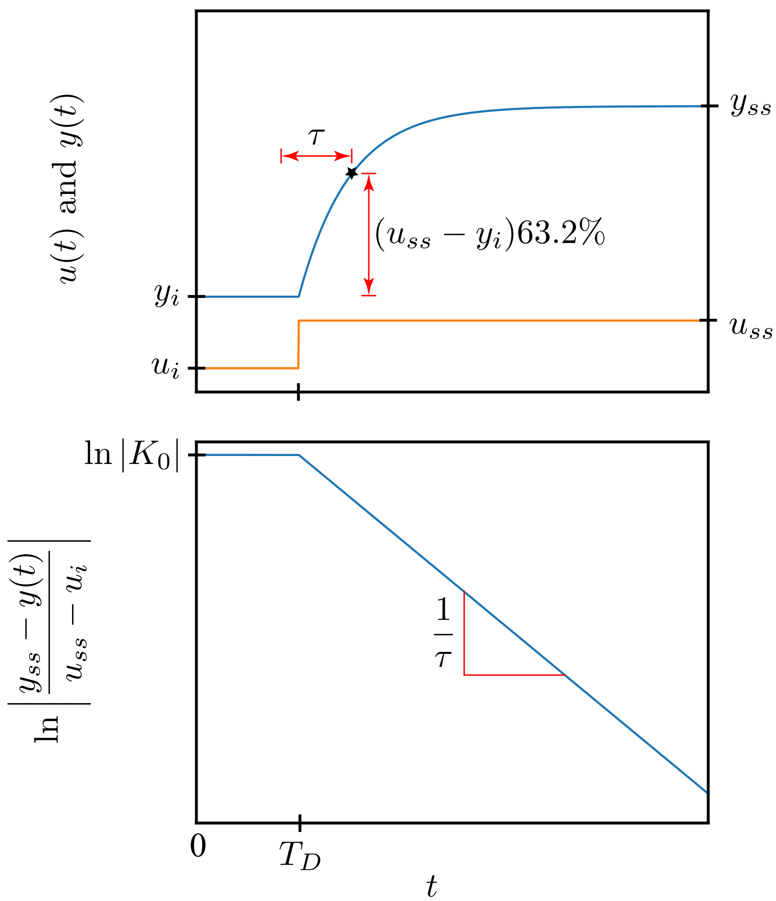


Figure 12.1:

· At $t = T_0 + \tau$

$$
\begin{aligned}
y(T_0+\tau) - y_i &= (y_{ss} - y_i)(1 - e^{-\frac{\cancel{T_0}+\tau-\cancel{T_0}}{\tau}})\cancelto{1}{u_s(T_0+\tau-T_0)} && (12.7)\\
&= (y_{ss} - y_i)(1 - e^{-1}) && (12.8)\\
y(T_0+\tau) - y_i &= (y_{ss} - y_i) - (y_{ss} - y_i)e^{-1} && (12.9)\\
\frac{y(T_0+\tau) - y_i}{(y_{ss} - y_i)} &= 1 - e^{-1} && (12.10)\\
&= 0.632 && (12.11)
\end{aligned}
$$

* Method 2: Plot $\ln|y - y_{ss}|$ and find the negative reciprocal of the slope
  · Write an expression for the natural log of the absolute value of $y - y_{ss}$

$$\ln[|y(t) - y_{ss}|] = \ln\left|y_i + (y_{ss} - y_i)\left(1 - e^{-\frac{t-T_0}{\tau}}\right)u_s(t - T_0) - y_{ss}\right| \quad (12.12)$$

$$= \begin{cases} \ln|y_i - y_{ss}| & t < T_0 \\ \ln\left|y_i + (y_{ss} - y_i)\left(1 - e^{-\frac{t-T_0}{\tau}}\right) - y_{ss}\right| & T_0 < t \end{cases} \quad (12.13)$$

$$= \begin{cases} \ln|y_i - y_{ss}| & t < T_0 \\ \ln\left|-(y_{ss} - y_i)e^{-\frac{t-T_0}{\tau}}\right| & T_0 < t \end{cases} \quad (12.14)$$

$$= \begin{cases} \ln|y_i - y_{ss}| & t < T_0 \\ \ln\left|(y_{ss} - y_i)e^{-\frac{t-T_0}{\tau}}\right| & T_0 < t \end{cases} \quad (12.15)$$

$$= \begin{cases} \ln|y_i - y_{ss}| & t < T_0 \\ \ln|(y_{ss} - y_i)| + \ln\left|e^{-\frac{t-T_0}{\tau}}\right| & T_0 < t \end{cases} \quad (12.16)$$

$$= \begin{cases} \ln|y_i - y_{ss}| & t < T_0 \\ \ln|(y_{ss} - y_i)| - \frac{t-T_0}{\tau} & T_0 < t \end{cases} \quad (12.17)$$

$$\ln|y(t) - y_{ss}| = \begin{cases} \text{constant} & t < T_0 \\ \text{constant} - \frac{1}{\tau}t & T_0 < t \end{cases} \quad (12.18)$$

- Parameter estimation for underdamped second order systems
  - Underdamped BIBO stable second order system response to a step change
    * General equation with input $u(t)$ and output $y(t)$

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K_0\omega_n^2 u(t) \quad (12.19)$$

    * Step change at time $t = T_0$ from rest with $u = u_i$, $y = y_i$, and $\dot{y} = 0$ to $u = u_{ss}$ and $y = y_{ss}$

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K_0\omega_n^2(u_i + (u_{ss} - u_i)u_s(t - T_0)),\ y(0) = y_i,\ \dot{y}(0) = 0 \quad (12.20)$$

    * Response

$$y(t) = y_i + (y_{ss} - y_i)\left(1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n(t-T_0)}\cos\left(\omega_d(t - T_0) - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)\right)u_s(t - T_0) \quad (12.21)$$

  - Static sensitivity

* Note, we know that $y_i = K_0 u_i$ and $y_{ss} = K_0 u_{ss}$

$$\begin{aligned}
\frac{y_{\text{ss}} - y_i}{u_{\text{ss}} - u_i} &= \frac{K_0 u_{\text{ss}} - K_0 u_i}{u_{\text{ss}} - u_i} && (12.22)\\
&= K_0 \frac{u_{\text{ss}} - u_i}{u_{\text{ss}} - u_i} && (12.23)\\
&= K_0 && (12.24)
\end{aligned}$$

– Damping ratio

* Method 1: Using overshoot

  · We can reason that the maximum overshoot occurs at $t = T_0 + \frac{\pi}{\omega_d}$

$$\begin{aligned}
y(T_0 + \frac{\pi}{\omega_d}) - y_{ss} &= y_i + (y_{ss} - y_i)\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n (T_0 + \frac{\pi}{\omega_d} - T_0)} \cos\left(\omega_d (T_0 + \frac{\pi}{\omega_d} - T_0) - \tan\right.\right.\\
&= -(y_{ss} - y_i) + (y_{ss} - y_i)\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \cos\left(\omega_d \frac{\pi}{\omega_d} - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-}}\right.\right.\right.\\
\frac{y(T_0 + \frac{\pi}{\omega_d}) - y_{ss}}{y_{ss} - y_i} &= -1 + \left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \cos\left(\omega_d \frac{\pi}{\omega_d} - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)\right)\\
&= -\frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \cos\left(\pi - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)\\
&= \frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \cos\left(\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)\\
&= \frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \sqrt{1-\zeta^2}\\
&= e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}
\end{aligned}$$

· Define $M_\%$ as the maximum percent overshoot as

$$
\begin{aligned}
M_\% &= \frac{y(T_0 + \frac{\pi}{\omega_d}) - y_{ss}}{y_{ss} - y_i} 100\% && (12.32)\\
&= 100\% e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} && (12.33)\\
\ln\left[\frac{M_\%}{100\%}\right] &= -\frac{\pi\zeta}{\sqrt{1-\zeta^2}} && (12.34)\\
\ln\left[\frac{M_\%}{100\%}\right]^2 &= \frac{\pi^2\zeta^2}{1-\zeta^2} && (12.35)\\
\ln\left[\frac{M_\%}{100\%}\right]^2 &= \left(\pi^2 + \ln\left[\frac{M_\%}{100\%}\right]^2\right)\zeta^2 && (12.36)\\
\zeta &= -\frac{\ln\left[\frac{M_\%}{100\%}\right]}{\sqrt{\pi^2 + \ln\left[\frac{M_\%}{100\%}\right]^2}}, \text{ for } M_\% > 0 && (12.37)
\end{aligned}
$$

· Approximation for $\zeta < 0.1$ or $M_\% > 70\%$:

$$
\zeta \approx -\frac{1}{\pi}\ln\left[\frac{M_\%}{100\%}\right], \text{ for } M_\% > 70 \text{ and } \zeta < 0.1 \qquad (12.38)
$$

∗ Method 2: Logarithmic decrement

· Peaks occur periodically after the first peak with period $\frac{2\pi}{\omega_d}$. Specifically peaks occur at $t = T_0 + \left\{\frac{\pi}{\omega_d}, \frac{3\pi}{\omega_d}, \frac{5\pi}{\omega_d}, ...\right\}$ with the n^{th} peak at $t_n = T_0 + \frac{(2n-1)\pi}{\omega_d}$

· Fractional overshoot at each peak (only changes the coefficient to π above)

$$
\begin{aligned}
\frac{y\left(T_0 + \frac{(2n-1)\pi}{\omega_d}\right) - y_{ss}}{y_{ss} - y_i} &= e^{-\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}} && (12.39)\\
\ln\left[\frac{y\left(T_0 + \frac{(2n-1)\pi}{\omega_d}\right) - y_{ss}}{y_{ss} - y_i}\right] &= -\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}} && (12.40)
\end{aligned}
$$

· Difference for n and $n = 1$

$$\begin{aligned}
\ln\left[\frac{y\left(T_0+\frac{(2\cdot 1-1)\pi}{\omega_d}\right)-y_{ss}}{y_{ss}-y_i}\right] & \\
-\ln\left[\frac{y\left(T_0+\frac{(2n-1)\pi}{\omega_d}\right)-y_{ss}}{y_{ss}-y_i}\right] &= -\frac{(2\cdot 1-1)\pi\zeta}{\sqrt{1-\zeta^2}}+\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}} \quad (12.41)\\
\ln\left[\frac{y\left(T_0+\frac{(2\cdot 1-1)\pi}{\omega_d}\right)-y_{ss}}{y\left(T_0+\frac{(2n-1)\pi}{\omega_d}\right)-y_{ss}}\right] &= \frac{2(n-1)\pi\zeta}{\sqrt{1-\zeta^2}} \quad (12.42)\\
\frac{1}{n-1}\ln\left[\frac{y_1-y_{ss}}{y_n-y_{ss}}\right] &= \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad (12.43)
\end{aligned}$$

· Note here $n-1$ is the number of periods separating the two peaks considered

· Now lets define the left hand side as the logarithmic decrement δ

$$\delta = \frac{1}{n-1}\ln\left[\frac{y_1-y_{ss}}{y_n-y_{ss}}\right] \quad (12.44)$$

· δ is obtainable from the response. And manipulating we can use δ to solve for ζ

$$\begin{aligned}
\delta &= \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad (12.45)\\
\delta^2 &= \frac{4\pi^2\zeta^2}{1-\zeta^2} \quad (12.46)\\
\delta^2 &= 4\pi^2\zeta^2+\delta^2\zeta^2 \quad (12.47)\\
\zeta^2 &= \frac{\delta^2}{4\pi^2+\delta^2} \quad (12.48)\\
\zeta &= \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \quad (12.49)
\end{aligned}$$

∗ Method 3: logarithmic decrement from negative slope of plot of $\ln[y_n - y_{ss}]$ as a function of n

– Natural frequency

∗ Obtain damped natural frequency from the period of oscillation T

$$\omega_d = \frac{2\pi}{T} \quad (12.50)$$

∗ Use ω_d and ζ to calculate ω_n

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} \quad (12.51)$$

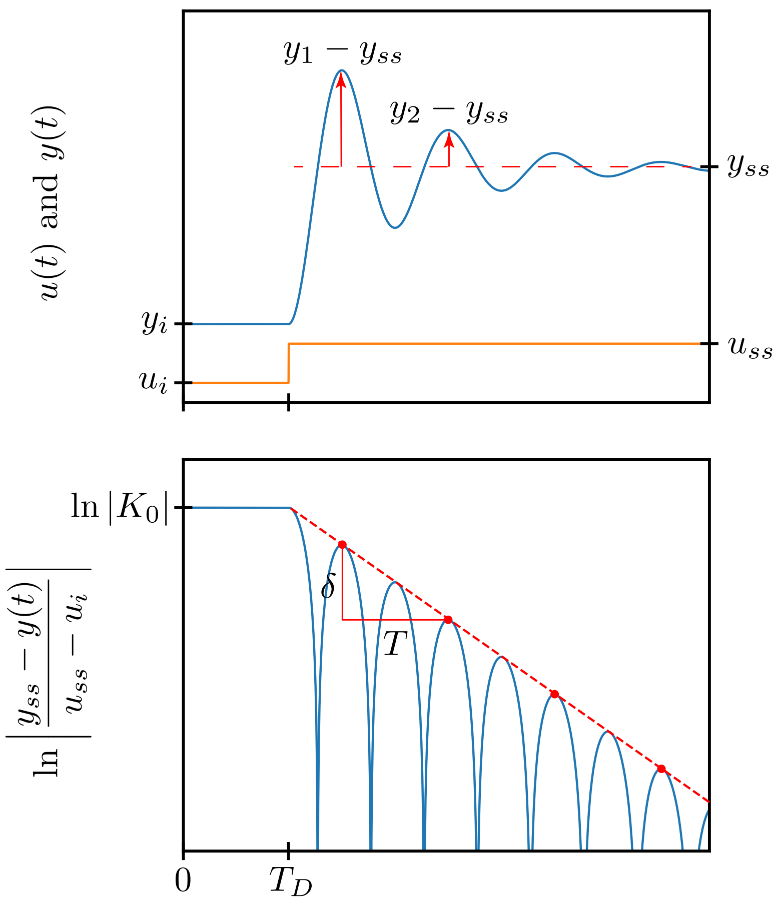


Figure 12.2:

- Parameter estimation for overdamped second order system (this is just an aside for the curious)
  - This section needs to be reviewed for possible mistakes.
  - See Web Appendix W3.7 for [1], https://www.scsolutions.com/publication/feedback-control-of-dynamic-systems-eighth-edition/
    * Systematically removes the longest time constant from the response to reveal all time constants
  - Overdamped BIBO stable second order system response to a step change (characterized by two time constants)
    * General equation with input $u(t)$ and output $y(t)$

$$\tau_1\tau_2\ddot{y} + (\tau_1 + \tau_2)\dot{y} + y = K_0 u(t) \tag{12.52}$$

  * Step change at time $t = T_0$ from rest with $u = u_i$, $y = y_i$, and $\dot{y} = 0$ to $u = u_{ss}$ and $y = y_{ss}$

$$\tau_1\tau_2\ddot{y} + (\tau_1 + \tau_2)\dot{y} + y = K_0(u_i + (u_{ss} - u_i)u_s(t - T_0)),\ y(0) = y_i,\ \dot{y}(0) = 0 \quad (12.53)$$

  * Response

$$\begin{aligned} y(t) &= y_i + (y_{ss} - y_i)\left(1 - \frac{1}{\tau_1 - \tau_2}\left(\tau_1 e^{-\frac{1}{\tau_1}(t-T_0)} - \tau_2 e^{-\frac{1}{\tau_2}(t-T_0)}\right)\right)u_s(t - T_0) && (12.54)\\ &= y_i + (y_{ss} - y_i)\left(\frac{\tau_1}{\tau_1 - \tau_2}\left(1 - e^{-\frac{1}{\tau_1}(t-T_0)}\right) - \frac{\tau_2}{\tau_1 - \tau_2}\left(1 - e^{-\frac{1}{\tau_2}(t-T_0)}\right)\right)u_s(t - T_0) && (12.55)\end{aligned}$$

- Static sensitivity
  * Note, we know that $y_i = K_0 u_i$ and $y_{ss} = K_0 u_{ss}$

$$\begin{aligned} \frac{y_{\mathrm{ss}} - y_i}{u_{\mathrm{ss}} - u_i} &= \frac{K_0 u_{\mathrm{ss}} - K_0 u_i}{u_{\mathrm{ss}} - u_i} && (12.56)\\ &= K_0 \frac{u_{\mathrm{ss}} - u_i}{u_{\mathrm{ss}} - u_i} && (12.57)\\ &= K_0 && (12.58)\end{aligned}$$

- Dominant time constant
  * Write an expression for the natural log of the absolute value of $y - y_{ss}$

$$\begin{aligned} \ln|y_{ss} - y| &= \ln\left|y_{ss} - y_i - (y_{ss} - y_i)\left(\frac{\tau_1}{\tau_1 - \tau_2}\left(1 - e^{-\frac{1}{\tau_1}(t-T_0)}\right) - \frac{\tau_2}{\tau_1 - \tau_2}\left(1 - e^{-\frac{1}{\tau_2}(t-T_0)}\right)\right)u_s\right.\\
&= \begin{cases} \ln|y_{ss} - y_i| & t \\ \ln\left|y_{ss} - y_i - (y_{ss} - y_i)\left(\frac{\tau_1}{\tau_1-\tau_2}\left(1 - e^{-\frac{1}{\tau_1}(t-T_0)}\right) - \frac{\tau_2}{\tau_1-\tau_2}\left(1 - e^{-\frac{1}{\tau_2}(t-T_0)}\right)\right)\right| & T \end{cases}\\
&= \begin{cases} \ln|K_0(u_{ss} - u_i)| \\ \ln\left|K_0(u_{ss} - u_i) - K_0(u_{ss} - u_i)\left(\frac{\tau_1}{\tau_1-\tau_2}\left(1 - e^{-\frac{1}{\tau_1}(t-T_0)}\right) - \frac{\tau_2}{\tau_1-\tau_2}\left(1 - e^{-\frac{1}{\tau_2}(t-T\right.\right.\right. \end{cases}\\
\ln\left|\frac{y_{ss} - y}{u_{ss} - u_i}\right| &= \begin{cases} \ln|K_0| & t < T_0 \\ \ln\left|K_0 - K_0\left(\frac{\tau_1}{\tau_1-\tau_2}\left(1 - e^{-\frac{1}{\tau_1}(t-T_0)}\right) - \frac{\tau_2}{\tau_1-\tau_2}\left(1 - e^{-\frac{1}{\tau_2}(t-T_0)}\right)\right)\right| & T_0 < t \end{cases}\\
&= \begin{cases} \ln|K_0| & t < T_0 \\ \ln\left|\frac{K_0}{\tau_1-\tau_2}\left(\tau_1 e^{-\frac{1}{\tau_1}(t-T_0)} - \tau_2 e^{-\frac{1}{\tau_2}(t-T_0)}\right)\right| & T_0 < t \end{cases}\\
&= \begin{cases} \ln|K_0| & t < T_0 \\ \ln\left|K_0 \frac{\tau_1 e^{-\frac{1}{\tau_1}(t-T_0)}}{\tau_1-\tau_2}\left(1 - \frac{\tau_2}{\tau_1} e^{-\left(\frac{1}{\tau_2} - \frac{1}{\tau_2}\right)(t-T_0)}\right)\right| & T_0 < t \end{cases}\\
&= \begin{cases} \ln|K_0| & t < T_0 \\ \ln\left|K_0 \frac{\tau_1}{\tau_1-\tau_2}\right| - \frac{t-T_0}{\tau_1} + \ln\left|1 - \frac{\tau_2}{\tau_1} e^{-\left(\frac{1}{\tau_2} - \frac{1}{\tau_2}\right)(t-T_0)}\right| & T_0 < t \end{cases}\end{aligned}$$

* For small $\frac{\tau_2}{\tau_1}$ (and recognizing $\ln[1-\theta] \approx \theta + O(\theta^2)$)

$$\ln\left|\frac{y_{ss}-y}{u_{ss}-u_i}\right| \approx \begin{cases} \ln|K_0| & t < T_0 \\ \ln\left|K_0\frac{\tau_1}{\tau_1-\tau_2}\right| + \frac{T_0}{\tau_1} - \frac{1}{\tau_1}t + \frac{\tau_2}{\tau_1}e^{-\left(\frac{1}{\tau_2}-\frac{1}{\tau_2}\right)(t-T_0)} & T_0 < t \end{cases} \tag{12.66}$$

  · Two terms depend upon t
  · One term is linear with slope $-\frac{1}{\tau_1}$
  · The other term decays exponentially with time constant

$$\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right)^{-1} = \frac{\tau_1\tau_2}{\tau_1-\tau_2} \tag{12.67}$$

$$= \frac{\tau_2}{1-\frac{\tau_2}{\tau_1}} \tag{12.68}$$

  · The plot is linear after a few time constants, $\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right)^{-1}$, after the step change at T_0

$$\lim_{t\to\infty} \ln\left|\frac{y_{ss}-y}{u_{ss}-u_i}\right| \approx \ln\left|K_0\frac{\tau_1}{\tau_1-\tau_2}\right| + \frac{T_0}{\tau_1} - \frac{1}{\tau_1}t \tag{12.69}$$

$$= \ln\left|K_0\frac{\tau_1}{\tau_1-\tau_2}e^{-\frac{1}{\tau_1}(t-T_0)}\right| \tag{12.70}$$

* The dominant term can be removed to reveal the non-dominant term
  · Remove dominant response (linear part from the logarithmic plot)

$$\ln\left|\frac{y_{ss}-y}{u_{ss}-u_i} - K_0\frac{\tau_1}{\tau_1-\tau_2}e^{-\frac{1}{\tau_1}(t-T_0)}\right| = \ln\left|-K_0\frac{\tau_2}{\tau_1-\tau_2}e^{-\frac{1}{\tau_2}(t-T_0)}\right| \quad T_0 < t \tag{12.71}$$

$$= \ln\left|-K_0\frac{\tau_2}{\tau_1-\tau_2}\right| - \frac{1}{\tau_2}(t-T_0) \quad T_0 < t \tag{12.72}$$

– In practice, measurement noise and uncertainty in the value y_{ss} may make it difficult to extract the dominant time constant from the step response of an overdamped system.

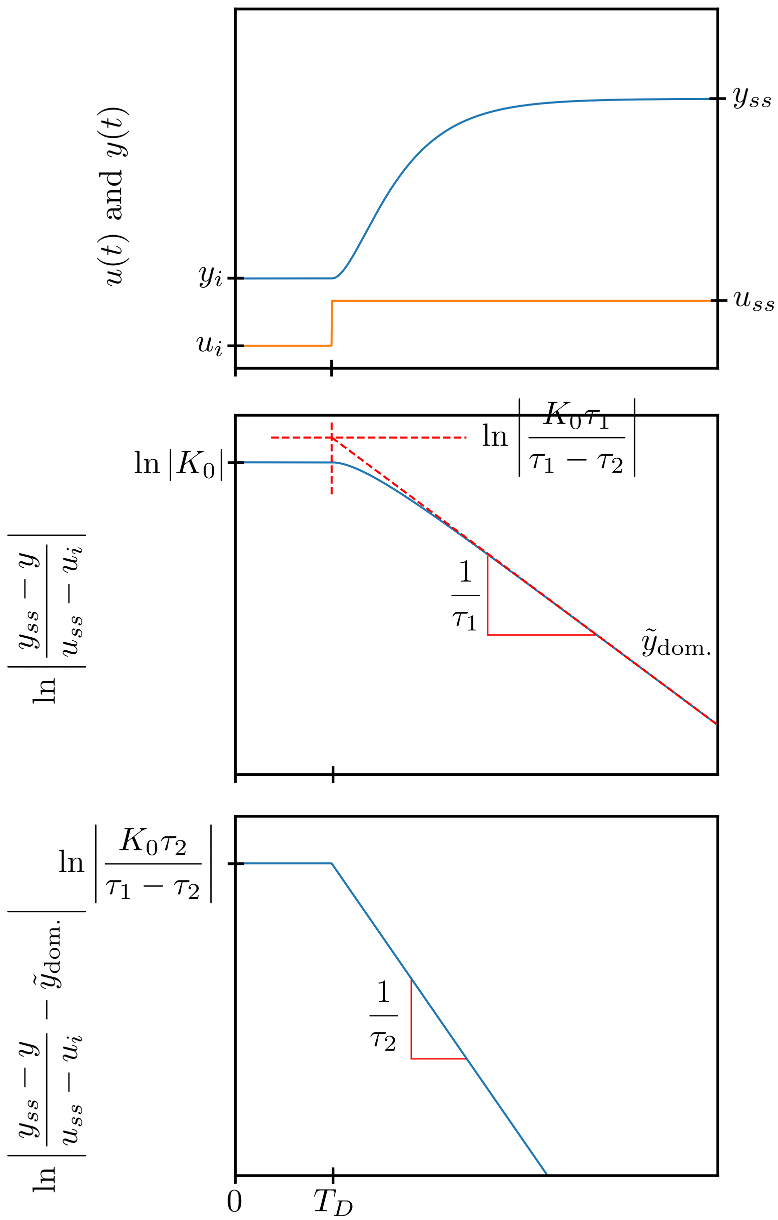


Figure 12.3: In this plot, the dominant time constant is twice that of the other time constant.

Example 12.1: first order system

Example 12.2: underdamped second order system

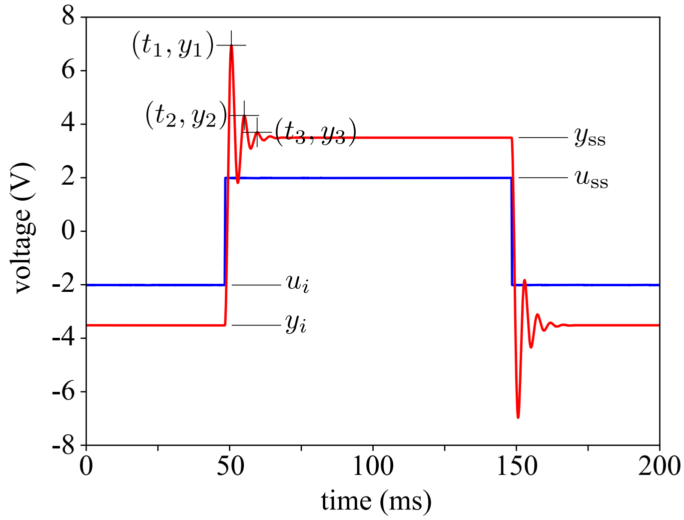


Figure 12.4:

Estimate the transfer function for the system with input and output in figure...

- Read key values off of plot

$$u_i = -2.0137 \tag{12.73}$$
$$u_{\text{ss}} = 1.9882 \tag{12.74}$$
$$y_i = -3.5169 \tag{12.75}$$
$$y_{\text{ss}} = 3.4951 \tag{12.76}$$
$$(t_1, y_1) = (0.00506, 6.9482) \tag{12.77}$$
$$(t_2, y_2) = (0.00551, 4.3262) \tag{12.78}$$
$$(t_3, y_3) = (0.00596, 3.6963) \tag{12.79}$$

- Estimate static sensitivity

$$K_0 = \frac{y_{ss} - y_i}{u_{ss} - u_i} \tag{12.80}$$
$$= \frac{3.4951 - (-3.5169)}{1.9882 - (-2.0137)} \tag{12.81}$$
$$= 1.7521 \tag{12.82}$$

- Estimate period (from two complete periods in this case)

$$T = \frac{t_3 - t_1}{2} \tag{12.83}$$
$$= \frac{0.00551 - 0.00506 \text{ s}}{2} \tag{12.84}$$
$$= 0.45 \text{ ms} \tag{12.85}$$

- Estimate damped natural frequency

$$\omega_d = \frac{2\pi}{T} \tag{12.86}$$
$$= \frac{2\pi}{0.45 \text{ ms}} \tag{12.87}$$
$$= 13960 \text{ rad/s} \tag{12.88}$$
$$\tag{12.89}$$

- Estimate damping ratio
  - Method 1: from percent overshoot
    * Maximum percent overshoot

$$M_{\%} = \frac{y_1 - y_{ss}}{y_{ss} - y_i} \tag{12.90}$$
$$= \frac{6.9482 - 3.4951}{3.4951 - (-3.5169)} \tag{12.91}$$
$$= 49.247\% \tag{12.92}$$

    * Damping ratio

$$\zeta = -\frac{\ln\left[\frac{M_{\%}}{100\%}\right]}{\sqrt{\pi^2 + \ln\left[\frac{M_{\%}}{100\%}\right]^2}} \tag{12.93}$$
$$= -\frac{\ln\left[\frac{49.247\%}{100\%}\right]}{\sqrt{\pi^2 + \ln\left[\frac{49.247\%}{100\%}\right]^2}} \tag{12.94}$$
$$= 0.2199 \tag{12.95}$$

– Method 2: from logarithmic decrement (over 2 periods)

$$\delta = \frac{1}{n-1}\ln\left[\frac{y_1 - y_{ss}}{y_n - y_{ss}}\right] \tag{12.96}$$

$$= \frac{1}{3-1}\ln\left[\frac{y_1 - y_{ss}}{y_3 - y_{ss}}\right] \tag{12.97}$$

$$= \frac{1}{2}\ln\left[\frac{6.9482 - 3.4951}{3.6963 - 3.4951}\right] \tag{12.98}$$

$$= 1.4214 \tag{12.99}$$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{12.100}$$

$$= \frac{1.4214}{\sqrt{4\pi^2 + 1.4214^2}} \tag{12.101}$$

$$= 0.2206 \tag{12.102}$$

– Method 3: from linear fit ...

$$\delta = 1.4213 \tag{12.103}$$

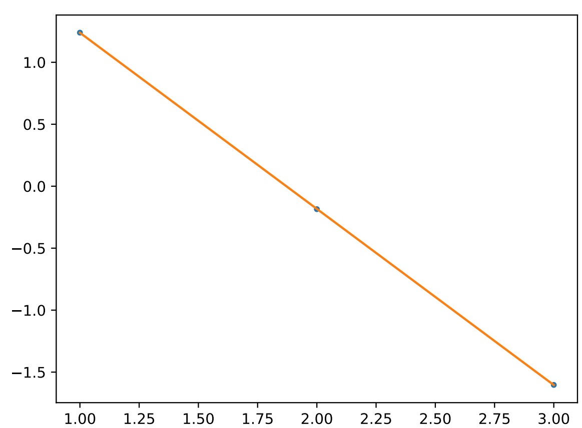


Figure 12.5:

- Estimate natural frequency

– using damping ratio from method 1

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} \tag{12.104}$$

$$= \frac{13960}{\sqrt{1-0.2199^2}} \tag{12.105}$$

$$\omega_n = 14310 \tag{12.106}$$

- Construct transfer function

$$G(s) = \frac{K_0\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{12.107}$$

(12.108)

- Compare simulation of response to measured response

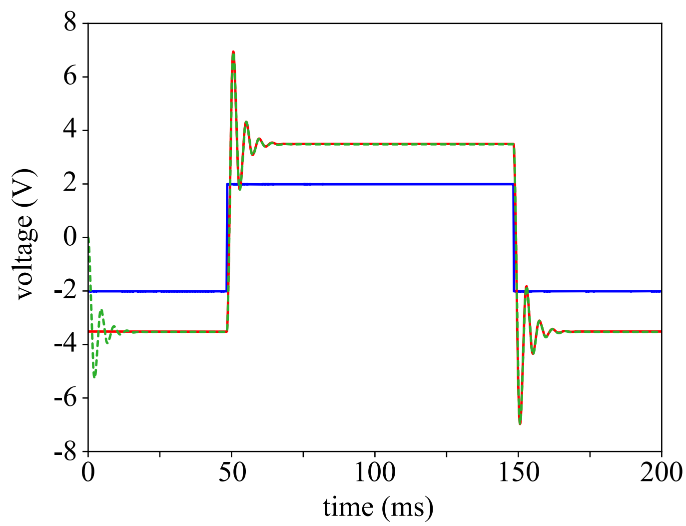


Figure 12.6: using damping ratio from percent overshoot

Example 12.3: overdamped second order system

Part V

Frequency Domain System Response and Identification

Chapter 13

Frequency Response

- Motivation
- Review complex numbers (see, for example, [35])
  - Definition and powers of the imaginary number j

$$\begin{aligned} j &= \sqrt{-1} && (13.1) \\ j^2 &= -1 && (13.2) \\ j^3 &= -j && (13.3) \\ j^4 &= 1 && (13.4) \end{aligned}$$

  - Representing a complex number
    * Rectangular form

$$N = x + yj \tag{13.5}$$

    * Graphical interpretation
    * Complex exponential form

$$\begin{aligned} N &= M(\cos(\phi) + j\sin(\phi)) && (13.6) \\ &= Me^{j\phi} && (13.7) \\ M &= |N| && (13.8) \\ &= = \sqrt{x^2+y^2} && (13.9) \\ \phi &= \angle N && (13.10) \\ \tan(\phi) &= \frac{y}{x} && (13.11) \end{aligned}$$

    * Note, the $\tan^{-1}()$ function isn't written here because it's implementation in computers and calculators generally outputs an angle (ϕ) in the range of $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$.

Therefore the user must adjust the angle to match the appropriate quadrant. The `arctan2(y,x)` function in Python or the `atan2(y,x)` function in MATLAB®, described sections D.2 and E.3 respectively, properly accounts for the quadrant of the angle.

* Complex conjugates

$$\begin{aligned} N &= x + yj && (13.12)\\ &= Me^{j\phi} && (13.13)\\ N^* &= x - yj && (13.14)\\ &= Me^{-j\phi} && (13.15) \end{aligned}$$

– Products and ratios of complex numbers

* Given complex numbers

$$N_1 = x_1 + y_1 j = M_1 e^{j\phi_1} \quad (13.16)$$

$$N_2 = x_2 + y_2 j = M_2 e^{j\phi_2} \quad (13.17)$$

* Product
  · Rectangular form - cumbersome

$$\begin{aligned} N_1 N_2 &= (x_1 + y_1 j)(x_2 + y_2 j) && (13.18)\\ &= x_1 x_2 + (x_1 y_2 + y_1 x_2)j - y_1 y_2 && (13.19)\\ &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)j && (13.20) \end{aligned}$$

  · Complex exponential form - less cumbersome

$$\begin{aligned} N_1 N_2 &= M_1 e^{j\phi_1} M_2 e^{j\phi_2} && (13.21)\\ &= M_1 M_2 e^{j(\phi_1 + \phi_2)} && (13.22)\\ |N_1 N_2| &= M_1 M_2 && (13.23)\\ \angle N_1 N_2 &= \phi_1 + \phi_2 && (13.24) \end{aligned}$$

* Quotient
  · Rectangular form - cumbersome

$$\begin{aligned} \frac{N_1}{N_2} &= \frac{x_1 + y_1 j}{x_2 + y_2 j} && (13.25)\\ &= \frac{(x_1 + y_1 j)(x_2 - y_2 j)}{(x_2 + y_2 j)(x_2 - y_2 j)} && (13.26)\\ &= \frac{x_1 x_2 + y_1 y_2 + (y_1 x_2 - x_1 y_2)j}{x_2^2 + y_2^2} && (13.27)\\ &= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + \left(\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}\right) j && (13.28) \end{aligned}$$

· Complex exponential form - less cumbersome

$$
\begin{aligned}
N &= \frac{N_1}{N_2} && (13.29)\\
&= \frac{M_1 e^{j\phi_1}}{M_2 e^{j\phi_2}} && (13.30)\\
&= \frac{M_1}{M_2} e^{j(\phi_1-\phi_2)} && (13.31)\\
\left|\frac{N_1}{N_2}\right| &= \frac{M_1}{M_2} && (13.32)\\
\angle\frac{N_1}{N_2} &= \phi_1 - \phi_2 && (13.33)
\end{aligned}
$$

– Software may be used to work with complex numbers using, for example, the following commands:

∗ Python: - see Appendix D.2 Common Mathematical Functions

· `real`

· `imag`

· `abs`

· `angle`

∗ MATLAB® - see Appendix E.3 Common Mathematical Functions

· `real`

· `imag`

· `abs`

· `angle`

- Steady state frequency response

– Consider a BIBO stable real system with input $u(t)$ and output $y(t)$ and transfer function $G(s)$

$$
\begin{aligned}
G(s) &= \frac{Y(s)}{U(s)} && (13.34)\\
u(t) &= A\left(\cos(\omega t) + j\sin(\omega t)\right) && (13.35)\\
&= Ae^{j\omega t} && (13.36)
\end{aligned}
$$

Here, A is a pure real constant.

– Forced response

$$\begin{aligned} Y(s) &= G(s)U(s) && (13.37)\\ &= G(s)\frac{A}{s-j\omega} && (13.38)\\ &= \frac{C_1}{s-j\omega} + \text{terms from poles of } G(s) && (13.39) \end{aligned}$$

* The first term is part of the steady state response
* Because $G(s)$ is BIBO stable, terms resulting from its poles must be transient
* We can solve for C_1 using the coverup method

$$\begin{aligned} C_1 &= \lim_{s\to j\omega} Y(s)(s-j\omega) && (13.40)\\ &= \lim_{s\to j\omega} G(s)\frac{A}{\cancel{s-j\omega}}\cancel{(s-j\omega)} && (13.41)\\ &= AG(j\omega) && (13.42) \end{aligned}$$

– Steady state response

$$\begin{aligned} y_{ss}(t) &= AG(j\omega)e^{j\omega t} && (13.43)\\ &= A|G(j\omega)|e^{j\angle G(j\omega)}e^{j\omega t} && (13.44)\\ &= A|G(j\omega)|e^{j(\omega t+\angle G(j\omega))} && (13.45)\\ &= A|G(j\omega)|\left(\cos\left(\omega t+\angle G(j\omega)\right) + j\sin\left(\omega t+\angle G(j\omega)\right)\right) && (13.46) \end{aligned}$$

* Now we can define the magnitude ratio $M(\omega)$ and phase angle $\phi(\omega)$ as

$$\begin{aligned} M(\omega) &= |G(j\omega)| && (13.47)\\ \phi(\omega) &= \angle G(j\omega) && (13.48) \end{aligned}$$

* Now we can rewrite the steady state response as

$$y_{ss}(t) = AM(\omega)\left(\cos\left(\omega t+\phi\right) + j\sin\left(\omega t+\phi\right)\right) \quad (13.49)$$

– Application to real systems with real inputs
  * Properties of $G(s)$
    · Linearity: In this derivation, we assume $G(s)$ is a linear system. Consequently, superposition holds.
    · Real: Usually, $G(s)$ represents a real physical system such that the coefficients in the polynomials of its numerator and denominator are real. Consequently, pure real (or imaginary) inputs must yield pure real (or imaginary) outputs.

* From these two properties we can deduce that the real (or imaginary) part of the input $u(t)$ in Equation 13.35 result in the real (or imaginary) part of the steady state response $y_{ss}(t)$ in Equation 13.49

- Summary of the result:
  * Given
    - BIBO stable and real (i.e., real coefficients) transfer function, $G(s)$, and
    - $u(t) = A\sin(\omega t)$
  * then the steady state response can be written as

$$\begin{aligned} y_{ss}(t) &= AM(\omega)\sin(\omega t + \phi(\omega)) && (13.50)\\ M(\omega) &= |G(j\omega)| && (13.51)\\ \phi(\omega) &= \angle G(j\omega) && (13.52) \end{aligned}$$

- Software may be used to help calculate steady state magnitude and phase information using, for example, the following commands:
  * Python: - see Appendix D.4 Representing and Solving Linear Time Invariant Systems
    - `freqresp`
    - `evalfr`
  * MATLAB® - see Appendix E.5 Representing and Solving Linear Time Invariant Systems
    - `freqresp`
    - `evalfr`

Example 13.1:

Find the steady state response to a system with transfer function

$$G(s) = \frac{3}{2s+3} \qquad (13.53)$$

and input

$$u(t) = 4\cos(2t) \qquad (13.54)$$

- Form of the steady state response

$$y_{ss}(t) = 4M(2)\cos(2t + \phi(2)) \qquad (13.55)$$

- Magnitude ratio

$$\begin{aligned}
M(\omega) &= |G(j\omega)| && (13.56)\\
&= \left|\frac{3}{2j\omega+3}\right| && (13.57)\\
&= \frac{|3|}{|2j\omega+3|} && (13.58)\\
&= \frac{3}{\sqrt{(2\omega)^2+3^2}} && (13.59)\\
M(2) &= \frac{3}{\sqrt{(2\cdot 2)^2+3^2}} && (13.60)\\
&= \frac{3}{5} && (13.61)
\end{aligned}$$

- Phase angle

$$\begin{aligned}
\phi(\omega) &= \angle G(j\omega) && (13.62)\\
&= \angle\frac{3}{2j\omega+3} && (13.63)\\
&= \angle 3 - \angle(2j\omega+3) && (13.64)\\
&= 0^\circ - \tan^{-1}\left(\frac{2\omega}{3}\right) && (13.65)\\
&= -\tan^{-1}\left(\frac{2\omega}{3}\right) && (13.66)\\
\phi(2) &= -\tan^{-1}\left(\frac{2\cdot 2}{3}\right) && (13.67)\\
&= -\tan^{-1}\left(\frac{4}{3}\right) && (13.68)
\end{aligned}$$

- Steady state reponse

$$\begin{aligned}
y_{ss}(t) &= 4\cdot\frac{3}{5}\cos\left(2t-\tan^{-1}\left(\frac{4}{3}\right)\right) && (13.69)\\
&= \frac{12}{5}\cos\left(2t-\tan^{-1}\left(\frac{4}{3}\right)\right) && (13.70)
\end{aligned}$$

Example 13.2:

Find the steady state response to a system with transfer function

$$G(s) = \frac{3}{2s+3} \tag{13.71}$$

and input

$$u(t) = 4\cos(2t) + 2\sin(4t) \tag{13.72}$$

- By superposition the form of the steady state response is

$$y_{ss}(t) = 4M(2)\cos(2t + \phi(2)) + 2M(4)\sin(4t + \phi(4)) \tag{13.73}$$

- Magnitude ratio

$$\begin{aligned}
M(\omega) &= |G(j\omega)| && (13.74)\\
&= \left|\frac{3}{2j\omega+3}\right| && (13.75)\\
&= \frac{|3|}{|2j\omega+3|} && (13.76)\\
&= \frac{3}{\sqrt{(2\omega)^2+3^2}} && (13.77)\\
M(2) &= \frac{3}{\sqrt{(2\cdot 2)^2+3^2}} && (13.78)\\
&= \frac{3}{5} && (13.79)\\
M(4) &= \frac{3}{\sqrt{(2\cdot 4)^2+3^2}} && (13.80)\\
&= \frac{3}{\sqrt{73}} && (13.81)
\end{aligned}$$

- Phase angle

$$\begin{aligned}
\phi(\omega) &= \angle G(j\omega) && (13.82)\\
&= \angle \frac{3}{2j\omega+3} && (13.83)\\
&= \angle 3 - \angle(2j\omega+3) && (13.84)\\
&= 0^\circ - \tan^{-1}\left(\frac{2\omega}{3}\right) && (13.85)\\
&= -\tan^{-1}\left(\frac{2\omega}{3}\right) && (13.86)\\
\phi(2) &= -\tan^{-1}\left(\frac{2\cdot 2}{3}\right) && (13.87)\\
&= -\tan^{-1}\left(\frac{4}{3}\right) && (13.88)\\
\phi(4) &= -\tan^{-1}\left(\frac{2\cdot 4}{3}\right) && (13.89)\\
&= -\tan^{-1}\left(\frac{8}{3}\right) && (13.90)
\end{aligned}$$

- Steady state reponse

$$\begin{aligned}
y_{ss}(t) &= 4\cdot\frac{3}{5}\cos\left(2t - \tan^{-1}\left(\frac{4}{3}\right)\right) + 2\cdot\frac{3}{\sqrt{73}}\sin\left(4t - \tan^{-1}\left(\frac{8}{3}\right)\right) && (13.91)\\
&= \frac{12}{5}\cos\left(2t - \tan^{-1}\left(\frac{4}{3}\right)\right) + \frac{6}{\sqrt{73}}\sin\left(4t - \tan^{-1}\left(\frac{8}{3}\right)\right) && (13.92)\\
& && (13.93)
\end{aligned}$$

Example 13.3:

Find the steady state response to a system with transfer function

$$G(s) = \frac{5}{s^2+3s+25} \tag{13.94}$$

and input

$$u(t) = 6\sin(4t) \tag{13.95}$$

- Form of the steady state response

$$y_{ss}(t) = 6M(4)\sin(4t+\phi(4)) \tag{13.96}$$

- Magnitude ratio

$$
\begin{aligned}
M(\omega) &= |G(j\omega)| && (13.97)\\
&= \left|\frac{5}{(j\omega)^2+3j\omega+25}\right| && (13.98)\\
&= \frac{|5|}{|(j\omega)^2+3j\omega+25|} && (13.99)\\
&= \frac{|5|}{|-\omega^2+3j\omega+25|} && (13.100)\\
&= \frac{|5|}{|25-\omega^2+3j\omega|} && (13.101)\\
&= \frac{5}{\sqrt{(25-\omega^2)^2+(3\omega)^2}} && (13.102)\\
M(4) &= \frac{5}{\sqrt{(25-4^2)^2+(3\cdot 4)^2}} && (13.103)\\
&= \frac{5}{15} && (13.104)\\
&= \frac{1}{3} && (13.105)
\end{aligned}
$$

- Phase angle

$$
\begin{aligned}
\phi(\omega) &= \angle G(j\omega) && (13.106)\\
&= \angle\frac{5}{(j\omega)^2+3j\omega+25} && (13.107)\\
&= \angle 5-\angle((j\omega)^2+3j\omega+25) && (13.108)\\
&= 0^\circ-\angle(-\omega^2+3j\omega+25) && (13.109)\\
&= -\tan^{-1}\left(\frac{3\omega}{25-\omega^2}\right) && (13.110)\\
\phi(4) &= -\tan^{-1}\left(\frac{3\cdot 4}{25-4^2}\right) && (13.111)\\
&= -\tan^{-1}\left(\frac{3\cdot 4}{9}\right) && (13.112)\\
&= -\tan^{-1}\left(\frac{4}{3}\right) && (13.113)
\end{aligned}
$$

- Steady state reponse

$$\begin{aligned} y_{ss}(t) &= 6 \cdot \frac{1}{3} \sin\left(4t - \tan^{-1}\left(\frac{4}{3}\right)\right) && (13.114) \\ &= 2\sin\left(4t - \tan^{-1}\left(\frac{4}{3}\right)\right) && (13.115) \end{aligned}$$

Example 13.4:

Find the steady state response to a system with transfer function

$$G(s) = \frac{5}{s^2 + 3s + 25} \tag{13.116}$$

and input

$$u(t) = 6\cos(8t) \tag{13.117}$$

- Form of the steady state response

$$y_{ss}(t) = 6M(8)\cos(8t + \phi(8)) \tag{13.118}$$

- Magnitude ratio

$$\begin{aligned} M(\omega) &= |G(j\omega)| && (13.119) \\ &= \left|\frac{5}{(j\omega)^2 + 3j\omega + 25}\right| && (13.120) \\ &= \frac{|5|}{|(j\omega)^2 + 3j\omega + 25|} && (13.121) \\ &= \frac{|5|}{|-\omega^2 + 3j\omega + 25|} && (13.122) \\ &= \frac{|5|}{|25 - \omega^2 + 3j\omega|} && (13.123) \\ &= \frac{5}{\sqrt{(25-\omega^2)^2 + (3\omega)^2}} && (13.124) \\ M(8) &= \frac{5}{\sqrt{(25-8^2)^2 + (3\cdot 8)^2}} && (13.125) \\ &= \frac{5}{\sqrt{2097}} && (13.126) \end{aligned}$$

- Phase angle

$$
\begin{aligned}
\phi(\omega) &= \angle G(j\omega) && (13.127)\\
&= \angle\frac{5}{(j\omega)^2+3j\omega+25} && (13.128)\\
&= \angle 5-\angle((j\omega)^2+3j\omega+25) && (13.129)\\
&= 0^\circ-\angle(-\omega^2+3j\omega+25) && (13.130)\\
&= -\tan^{-1}\left(\frac{3\omega}{25-\omega^2}\right) && (13.131)\\
\phi(8) &= -\tan^{-1}\left(\frac{3\cdot 8}{25-8^2}\right) && (13.132)\\
&= -\tan^{-1}\left(\frac{24}{-39}\right) && (13.133)\\
&= -\tan^{-1}\left(\frac{8}{-13}\right) && (13.134)
\end{aligned}
$$

Note, our use of the $\tan^{-1}()$ function is a little imprecise; mindlessly typing $\tan^{-1}\left(\frac{8}{-13}\right)$ into a calculator would yield -0.5517 rad. However, with a real part of -13 and an imaginary part of 8, the angle must be in the second quadrant. Consequently, we need to add π to the value for the calculator. Alternatively, the MATLAB® command `atan2(8,-13)` directly yields `2.5899` rad. Therefore,

$$\phi(8)=-2.5899 \tag{13.135}$$

- Steady state reponse

$$
\begin{aligned}
y_{ss}(t) &= 6\cdot\frac{5}{\sqrt{2097}}\cos\left(8t-2.5899\right) && (13.136)\\
&= \frac{30}{\sqrt{2097}}\cos\left(8t-2.5899\right) && (13.137)
\end{aligned}
$$

Chapter 14

Frequency Response Plots

14.1 Bode Plots

- Composite transfer functions
  - Consider a transfer funtion that consists of a product and quotient of component transfer functions
$$G(s) = \frac{G_1(s)G_2(s)}{G_3(s)G_4(s)} \tag{14.1}$$
  - The magnitude ratio and phase angle for the composite can be written as
$$\begin{aligned} M(\omega) &= |G(j\omega)| && (14.2)\\ &= \left|\frac{G_1(j\omega)G_2(j\omega)}{G_3(j\omega)G_4(j\omega)}\right| && (14.3)\\ &= \frac{|G_1(j\omega)||G_2(j\omega)|}{|G_3(j\omega)||G_4(j\omega)|} && (14.4)\\ &= \frac{M_1(\omega)M_2(\omega)}{M_3(\omega)M_4(\omega)} && (14.5)\\ \phi(\omega) &= \angle G(j\omega) && (14.6)\\ &= \angle G_1(j\omega) + \angle G_2(j\omega) - \angle G_3(j\omega) - \angle G_4(j\omega) && (14.7)\\ &= \phi_1(\omega) + \phi_2(\omega) - \phi_3(\omega) - \phi_4(\omega) && (14.8) \end{aligned}$$
  - Common transfer function components (factors)
    * These components may appear in the numerator and/or the denominator
    * Real constant
$$G(s) = K \tag{14.9}$$
    * Root at 0 with multiplicity n
$$G(s) = s^n \tag{14.10}$$

- Negative real root at $-\frac{1}{\tau}$

$$G(s) = \tau s + 1 \tag{14.11}$$

- Complex conjugate pair of roots $-\zeta\omega_n \pm j\omega_d$

$$G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2} \tag{14.12}$$

- Bode (or frequency response) plots
  - Review properties of log and introduce dB
    - Properties of log
      - Log of product

$$\log[ab] = \log[a] + \log[b] \tag{14.13}$$

      - Log of quotient

$$\log\left[\frac{a}{b}\right] = \log[a] - \log[b] \tag{14.14}$$

      - Log with exponent

$$\log\left[a^b\right] = b\log[a] \tag{14.15}$$

  - Plotting $M(\omega)$ and $\phi(\omega)$ as functions of ω on logarithmic scales is advantageous
    - Sometimes $M(\omega)$ and $\phi(\omega)$ are plotted as functions of ω with traditional linear scales
    - When we are interested in the frequency response of a system over a wide range of frequencies spanning over orders of magnitude, it can be advantageous to use a logarithmic scale along the ω axis.
    - There are advantages to using a logarithmic scale for $M(\omega)$
      - When logarithmic scales are used for both the $M(\omega)$ and ω axes, the curve is well approximated by simple piecewise linear segments
      - When $G(s)$ is a composite transfer function,

$$G(s) = \frac{G_1(s)G_2(s)}{G_3(s)G_4(s)}, \tag{14.16}$$

        $\log[M(\omega)]$ is the superposition of the component magnitude ratios,

$$\log[M(\omega)] = \log[M_1(\omega)] + \log[M_2(\omega)] - \log[M_3(\omega)] - \log[M_4(\omega)] \tag{14.17}$$

    - Linear scales are used for the phase angle, $\phi(\omega)$

· Note, when $G(s)$ is a composite transfer function,

$$G(s) = \frac{G_1(s)G_2(s)}{G_3(s)G_4(s)}, \tag{14.18}$$

$\phi(\omega)$ is the superposition of the phase angles,

$$\phi(\omega) = \phi_1(\omega) + \phi_2(\omega) - \phi_3(\omega) - \phi_4(\omega) \tag{14.19}$$

∗ Often we convert the magnitude ratio, $M(\omega)$, to decibels or dB, $m(\omega)$, as

$$\begin{aligned} m &= 10\log\left(M^2\right) && (14.20)\\ &= 20\log(M) && (14.21)\end{aligned}$$

and plot $m(\omega)$ on a linear scale

· The magnitude ratio in dB of the system is the superposition magnitude ratios

$$m(\omega) = m_1(\omega) + m_2(\omega) - m_3(\omega) - m_4(\omega) \tag{14.22}$$

· Note, M^2 is typically proportional to the power of the system

· Although we probably shouldn't take the log of M if it isn't dimensionless, we often do it anyway.

– Software may be used to create bode plots using, for example, the following commands:

∗ Python: - see Appendix D.4 Representing and Solving Linear Time Invariant Systems

· `bode`

∗ MATLAB® - see Appendix E.5 Representing and Solving Linear Time Invariant Systems

· `bode`

14.2 Bode Plots of Common Factors

- Real constant K
  - Factor

$$G(s) = K \tag{14.23}$$

  - Magnitude ratio

$$\begin{aligned} M(\omega) &= |K| && (14.24)\\ m(\omega) &= 20\log[|K|] && (14.25)\end{aligned}$$

  - Phase angle

$$\begin{aligned} \phi(\omega) &= \angle K && (14.26)\\ &= \begin{cases} 0 & K > 0 \\ -180 & K < 0 \end{cases} && (14.27)\end{aligned}$$

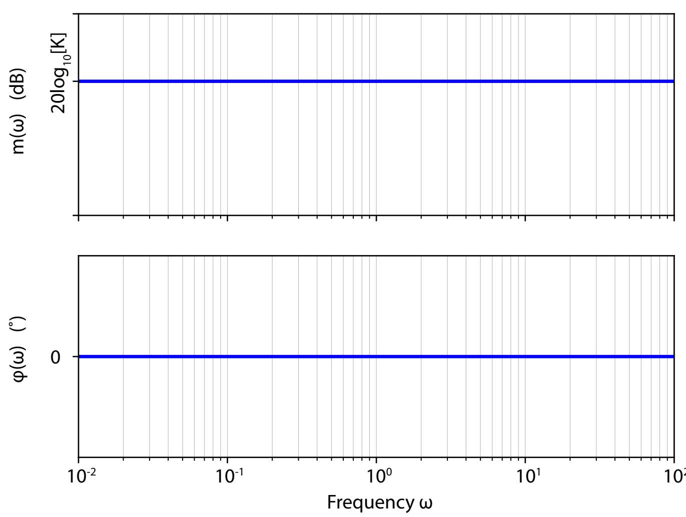


Figure 14.1: $G(s) = K > 0$

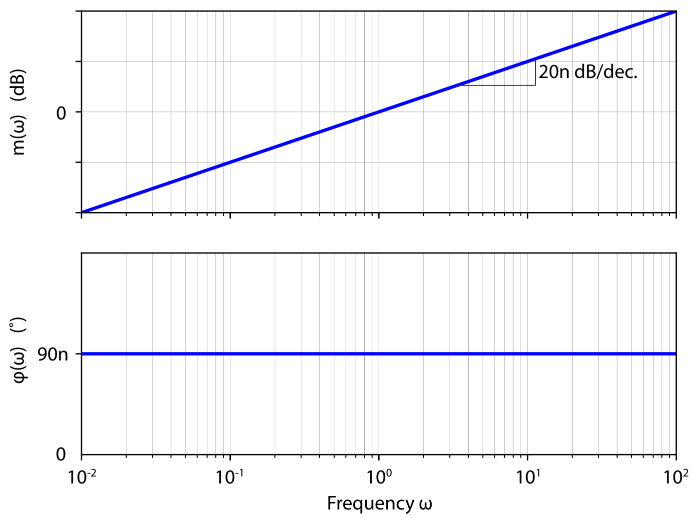


Figure 14.2: $G(s) = s^n$

- Root at 0 with multiplicity n
  - Factor

$$G(s) = s^n \tag{14.28}$$

  - Magnitude ratio

$$\begin{aligned} M(\omega) &= |(j\omega)^n| & (14.29)\\ &= |j^n|\,|\omega^n| & (14.30)\\ &= \omega^n & (14.31)\\ m(\omega) &= 20\log\left[\omega^n\right] & (14.32)\\ &= 20n\log[\omega] & (14.33) \end{aligned}$$

  - Phase angle

$$\begin{aligned} \phi(\omega) &= \angle\,(j\omega)^n & (14.34)\\ &= \angle j^n & (14.35)\\ &= 90^\circ n & (14.36) \end{aligned}$$

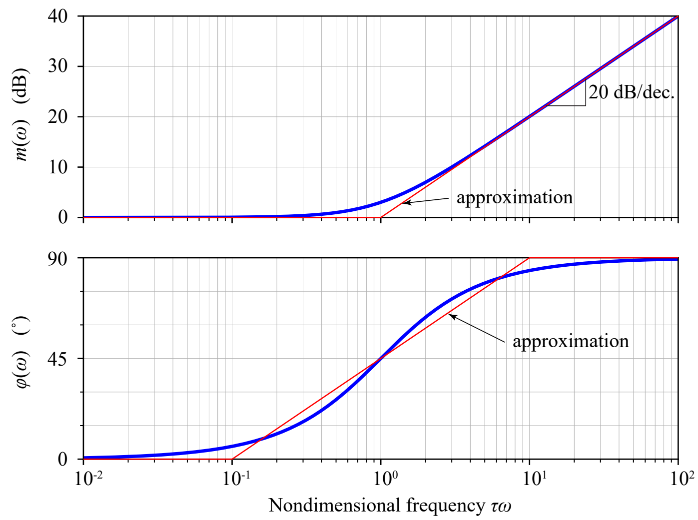


Figure 14.3: $G(s) = \tau s + 1$

- Negative real root at $-\frac{1}{\tau}$
  - Factor

$$G(s) = \tau s + 1 \tag{14.37}$$

  - Magnitude ratio

$$\begin{aligned} M(\omega) &= |\tau j\omega + 1| & (14.38)\\ &= \sqrt{(\tau\omega)^2 + 1^2} & (14.39)\\ m(\omega) &= 20\log\left[\sqrt{(\tau\omega)^2 + 1^2}\right] & (14.40)\\ &= \frac{1}{2}20\log\left[(\tau\omega)^2 + 1^2\right] & (14.41)\\ &= 10\log\left[(\tau\omega)^2 + 1^2\right] & (14.42)\end{aligned}$$

    * Low frequency asymptote ($\tau\omega \ll 1$)

$$\begin{aligned} m(\omega) &\approx 10\log\left[\cancel{(\tau\omega)^2} + 1^2\right] & (14.43)\\ &\approx 0 & (14.44)\end{aligned}$$

    * High frequency asymptote ($\tau\omega \gg 1$)

$$\begin{aligned} m(\omega) &\approx 10\log\left[(\tau\omega)^2 + \cancel{1^2}\right] & (14.45)\\ &\approx 20\log[\tau\omega] & (14.46)\end{aligned}$$

* ‘break frequency’ [6, 40] ($\omega_b = \frac{1}{\tau}$) - The break frequency separates high and low frequency regimes and forms the corner, intersection, or ‘knee’ [40] of the high and low frequency asymptotes. a.k.a., ‘Breakpoint frequency’ [35], ‘corner frequency’ [35, 40], and ‘break point’ [8]

$$\begin{aligned} m(\omega_b) &= 10 \log \left[1^2 + 1^2\right] & (14.47) \\ &= 3.01 \text{ dB} & (14.48) \end{aligned}$$

– Phase angle

$$\begin{aligned} \phi(\omega) &= \angle(\tau j\omega + 1) & (14.49) \\ &= \tan^{-1}\left(\frac{\tau\omega}{1}\right) & (14.50) \end{aligned}$$

* Low frequency asymptote ($\tau\omega \ll 1$)

$$\begin{aligned} \phi(\omega) &= \tan^{-1}\left(\frac{\cancelto{0}{\tau\omega}}{1}\right) & (14.51) \\ &= 0^\circ & (14.52) \end{aligned}$$

* High frequency asymptote ($\tau\omega \gg 1$)

$$\begin{aligned} \phi(\omega) &= \tan^{-1}\left(\frac{\cancelto{\infty}{\tau\omega}}{1}\right) & (14.53) \\ &= 90^\circ & (14.54) \end{aligned}$$

* ‘break frequency’ [6] ($\omega_b = \frac{1}{\tau}$)

$$\begin{aligned} \phi(\omega_b) &= \tan^{-1}\left(\frac{1}{1}\right) & (14.55) \\ &= 45^\circ & (14.56) \end{aligned}$$

* Simple linear approximation around break frequency - Draw a line of slope 45°/decade passing through 45° at the break frequency and extending one decade on both sides; see, for example [6].
* Better linear approximation around break frequency - “For $\omega\tau \cong 1$, the $\angle(j\omega + 1)$ curve is tangent to an asymptote going from 0° at $\omega\tau = 0.2$ to 90° at $\omega\tau = 5$ ... [8]”
* Note, the prelecture videos refer to vertices of the straight line approximation of the phase angle plot as break frequencies.

- Complex conjugate pair of roots $-\zeta\omega_n \pm j\omega_d$ (for $0 \leq \zeta < 1$)
  - Factor

$$G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2} \tag{14.57}$$

  - Magnitude ratio

$$\begin{aligned}
M(\omega) &= \left|\frac{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2}{\omega_n^2}\right| && (14.58)\\
&= \left|\frac{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2}{\omega_n^2}\right| && (14.59)\\
&= \left|1 - \frac{\omega^2}{\omega_n^2} + 2\zeta j\frac{\omega}{\omega_n}\right| && (14.60)\\
&= \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} && (14.61)\\
m(\omega) &= 20\log\left[\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right] && (14.62)\\
&= \frac{1}{2}20\log\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right] && (14.63)\\
&= 10\log\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right] && (14.64)
\end{aligned}$$

    * Low frequency asymptote ($\frac{\omega}{\omega_n} \ll 1$)

$$\begin{aligned}
m(\omega) &\approx 10\log\left[\left(1 - \cancel{\frac{\omega^2}{\omega_n^2}}\right)^2 + \cancel{\left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right] && (14.65)\\
&\approx 0 && (14.66)
\end{aligned}$$

    * High frequency asymptote ($\frac{\omega}{\omega_n} \gg 1$)

$$\begin{aligned}
m(\omega) &\approx 10\log\left[\left(\cancel{1} - \frac{\omega^2}{\omega_n^2}\right)^2 + \cancel{\left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right] && (14.67)\\
&\approx 10\log\left[\left(\frac{\omega^2}{\omega_n^2}\right)^2\right] && (14.68)\\
&\approx 40\log\left[\frac{\omega}{\omega_n}\right] && (14.69)
\end{aligned}$$

  * ‘break frequency’ [6] ($\omega_b = \omega_n$) - The break frequency separates high and low frequency regimes and forms the corner or intersection of the high and low frequency asymptotes. a.k.a., ‘Breakpoint frequency’ [35], ‘corner frequency’ [35], and ‘break point’ [8]

- Phase angle

$$
\begin{aligned}
\phi(\omega) &= \angle\left(\frac{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2}{\omega_n^2}\right) && (14.70)\\
&= \angle\left(1 - \frac{\omega^2}{\omega_n^2} + 2\zeta j\frac{\omega}{\omega_n^2}\right) && (14.71)\\
&= \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right) && (14.72)
\end{aligned}
$$

  * Low frequency asymptote ($\frac{\omega}{\omega_n} \ll 1$)

$$
\begin{aligned}
\phi(\omega) &= \tan^{-1}\left(\frac{\cancelto{0}{2\zeta\frac{\omega}{\omega_n}}}{1 - \cancelto{0}{\frac{\omega^2}{\omega_n^2}}}\right) && (14.73)\\
&= \tan^{-1}\left(\frac{0}{1}\right) && (14.74)\\
&= 0^\circ && (14.75)
\end{aligned}
$$

  * High frequency asymptote ($\frac{\omega}{\omega_n} \gg 1$)

$$
\begin{aligned}
\phi(\omega) &= \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{\cancel{1} - \frac{\omega^2}{\omega_n^2}}\right) && (14.76)\\
&= \tan^{-1}\left(\frac{2\zeta\frac{\cancel{\omega}}{\cancel{\omega_n}}}{-\frac{\omega^{\cancel{2}}}{\omega_n^{\cancel{2}}}}\right) && (14.77)\\
&= \tan^{-1}\left(\frac{2\zeta}{-\cancelto{\infty}{\frac{\omega}{\omega_n}}}\right) && (14.78)\\
&= 180^\circ && (14.79)
\end{aligned}
$$

∗ 'break frequency' [6] ($\omega_b = \omega_n$)

$$\phi(\omega_b) = \tan^{-1}\left(\frac{2\zeta\cancel{\frac{\omega}{\omega_n}}}{1-\cancelto{1}{\frac{\omega^2}{\omega_n^2}}}\right) \tag{14.80}$$

$$= 90^\circ \tag{14.81}$$

∗ Linear approximation around break frequency

· The derivative of $\phi(\omega)$ near ω_n is

$$\phi(\omega) = \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right) \tag{14.82}$$

$$\frac{d\phi}{d\omega} = \frac{1}{1+\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)^2}\left(\frac{2\zeta}{1-\frac{\omega^2}{\omega_n^2}}+\frac{2\zeta\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2}2\frac{\omega}{\omega_n}\right)\frac{1}{\omega_n} \tag{14.83}$$

$$= \frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}\left(\frac{2\zeta}{1-\frac{\omega^2}{\omega_n^2}}+\frac{2\zeta\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2}2\frac{\omega}{\omega_n}\right)\frac{1}{\omega_n} \tag{14.84}$$

$$= \frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}\left(2\zeta\left(1-\frac{\omega^2}{\omega_n^2}\right)+2\zeta\frac{\omega}{\omega_n}2\frac{\omega}{\omega_n}\right)\frac{1}{\omega_n} \tag{14.85}$$

$$\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_n} = -\frac{1}{(2\zeta)^2}(4\zeta)\frac{1}{\omega_n} \tag{14.86}$$

$$= \frac{1}{\zeta\omega_n} \tag{14.87}$$

· Viewed on a logarithmic scale, the slope is

$$\begin{aligned}
\frac{d\phi}{d\omega} &= \frac{d\phi}{d\log_{10}[\omega]}\frac{d}{d\omega}\left(\log_{10}[\omega]\right) && (14.88)\\
&= \frac{d\phi}{d\log_{10}[\omega]}\frac{d}{d\omega}\left(\frac{\ln[\omega]}{\ln[10]}\right) && (14.89)\\
&= \frac{d\phi}{d\log_{10}[\omega]}\frac{d\ln[\omega]}{d\omega}\frac{1}{\ln[10]} && (14.90)\\
&= \frac{d\phi}{d\log_{10}[\omega]}\frac{1}{\omega\ln[10]} && (14.91)\\
\frac{d\phi}{d\log_{10}[\omega]} &= \omega\ln[10]\frac{d\phi}{d\omega} && (14.92)\\
\left.\frac{d\phi}{d\log_{10}[\omega]}\right|_{\omega=\omega_n} &= \omega\ln[10]\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_n} && (14.93)\\
&= \omega_n\ln[10]\frac{1}{\zeta\omega_n} && (14.94)\\
&= \ln[10]\frac{1}{\zeta} && (14.95)
\end{aligned}$$

· A line on a logarithmic scale has the form

$$\phi_{\text{linear}}(\omega) = \alpha\log_{10}\left[\frac{\omega}{\omega_n}\right] + \beta \qquad (14.96)$$

· Fitting a line to the slope at the break frequency yields

$$\phi_{\text{linear}}(\omega) = \ln[10]\frac{1}{\zeta}\log_{10}\left[\frac{\omega}{\omega_n}\right] + \frac{\pi}{2} \qquad (14.97)$$

· The line intersects 0° at

$$\begin{aligned}
0 &= \ln[10]\frac{1}{\zeta}\log_{10}\left[\frac{\omega}{\omega_n}\right] + \frac{\pi}{2} && (14.98)\\
\frac{\omega}{\omega_n} &= 10^{-\frac{\pi\zeta}{2\ln[10]}} && (14.99)\\
&= \left(10^{-\frac{\pi}{2\ln[10]}}\right)^{\zeta} && (14.100)\\
&= (0.2079)^{\zeta} && (14.101)\\
&\approx \left(\frac{1}{5}\right)^{\zeta} && (14.102)
\end{aligned}$$

· The line intersects 180° at

$$\pi = \ln[10]\frac{1}{\zeta}\log_{10}\left[\frac{\omega}{\omega_n}\right] + \frac{\pi}{2} \tag{14.103}$$

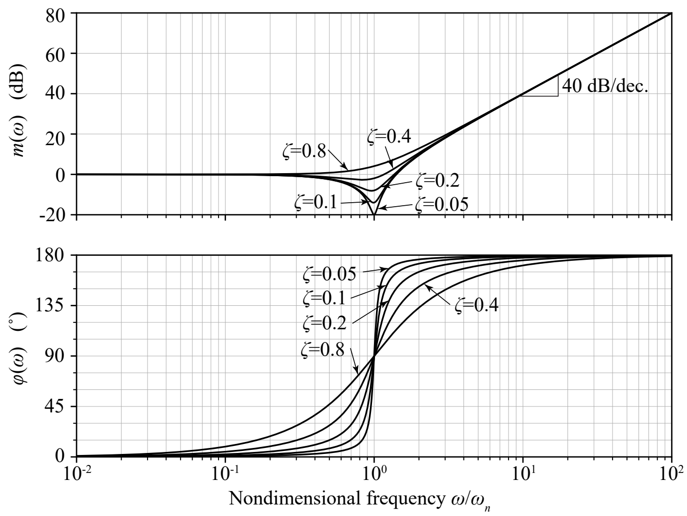
$$\frac{\omega}{\omega_n} = 10^{\frac{\pi\zeta}{2\ln[10]}} \tag{14.104}$$

$$= \left(10^{\frac{\pi}{2\ln[10]}}\right)^{\zeta} \tag{14.105}$$

$$= 4.8105^{\zeta} \tag{14.106}$$

$$\approx 5^{\zeta} \tag{14.107}$$

· Draw a line from 0° and $\omega = \omega_n 0.2^{\zeta}$ passing through 90° and $\omega = \omega_n$ then ending at 180° and $\omega = \omega_n 5^{\zeta}$; see, for example [48].



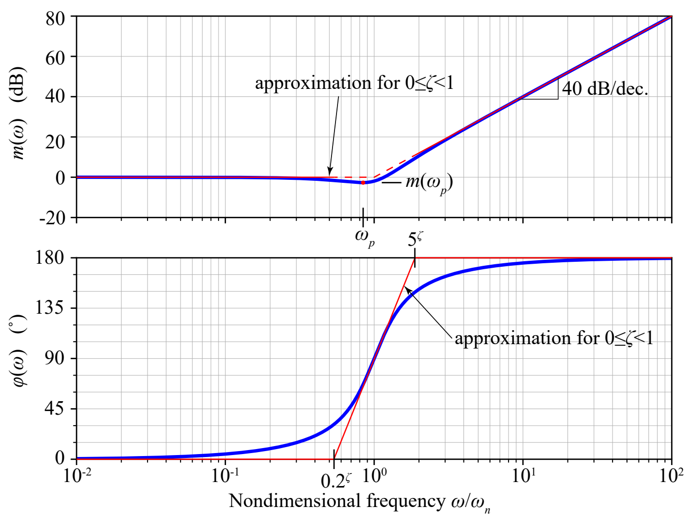


Figure 14.4: (Top) Bode plots for the transfer function $G(s) = \frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$ with several values for damping ratio. (Bottom) Straight line approximation to the bode plot for $0 \leq \zeta < 1$.

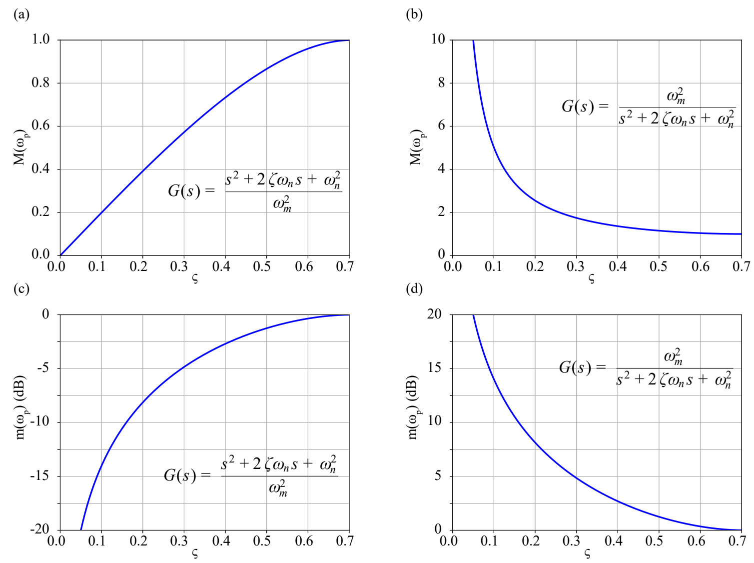


Figure 14.5: (a) Magnitude ratio at the extreme value for underdamped second order zeros. (b) Magnitude ratio at the extreme value for underdamped second order poles. (c) Magnitude ratio in dB at the extreme value for underdamped second order zeros. (d) Magnitude ratio in dB at the extreme value for underdamped second order poles.

- Extremum
  * Magnitude ratio

$$M(\omega) = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} \tag{14.108}$$

  * Extrema at $\frac{d}{d\omega}M(\omega)|_{\omega_p} = 0$

$$\begin{aligned}
0 &= \frac{\frac{1}{2}\left[2\left(1 - \left(\frac{\omega_p}{\omega_n}\right)^2\right) 2\frac{\omega_p}{\omega_n^2} + 2\left(2\zeta\frac{\omega_p}{\omega_n}\right) 2\zeta\frac{1}{\omega_n}\right]}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} && (14.109)\\
&= -2\left(1 - \left(\frac{\omega_p}{\omega_n}\right)^2\right) 2\frac{\omega_p}{\omega_n^2} + 2\left(2\zeta\frac{\omega_p}{\omega_n}\right) 2\zeta\frac{1}{\omega_n} && (14.110)\\
&= -\left(1 - \left(\frac{\omega_p}{\omega_n}\right)^2\right) + 2\zeta^2 && (14.111)\\
\omega_p &= \omega_n\sqrt{1 - 2\zeta^2} && (14.112)
\end{aligned}$$

∗ Extreme value

$$\begin{aligned}
M(\omega_p) &= \sqrt{\left(1-\sqrt{1-2\zeta^2}^2\right)^2+\left(2\zeta\sqrt{1-2\zeta^2}\right)^2} && (14.113)\\
&= \sqrt{(1-1+2\zeta^2)^2+4\zeta^2(1-2\zeta^2)} && (14.114)\\
&= \sqrt{4\zeta^4+4\zeta^2(1-2\zeta^2)} && (14.115)\\
&= \sqrt{-4\zeta^4+4\zeta^2} && (14.116)\\
&= 2\zeta\sqrt{1-\zeta^2} && (14.117)\\
m(\omega_p) &= 20\log\left[2\zeta\sqrt{1-\zeta^2}\right] && (14.118)
\end{aligned}$$

Note, this extreme value was derived from complex conjugate pair of zeros. If this underdamped factor were to appear in the denominator, the peak value would be the reciprocal and the peak value in dB would have the opposite sign.
For small ζ

$$\begin{aligned}
M(\omega_p) &\approx 2\zeta\sqrt{1-\cancelto{0}{\zeta^2}} && (14.119)\\
&\approx 2\zeta && (14.120)\\
m(\omega_p) &\approx 20\log\left[2\zeta\sqrt{1-\cancelto{0}{\zeta^2}}\right] && (14.121)\\
&\approx 20\log[2\zeta] && (14.122)
\end{aligned}$$

∗ No peak when $\zeta \geq \frac{1}{\sqrt{2}}$

∗ Resonance:
"It is important to note that resonance is defined to occur when $\omega = \omega_n$ ... This corresponds with a phase shift of $90°(\pi/2)$. Resonance does not, however, exactly correspond with the value of ω at which the peak value of the steady-stte response occurs [49]."
"... resonance is defined as the driving frequency at which the response magnitude X is the largest ... Resonance is also defined, for small damping, as the condition when the driving frequency and the system's natural frequency coincide [49]."
"The resonance angular frequency $[\omega_n]$ is defined as that at which the mechanical reactance X_m vanishes and the mechanical impedance is pure real with its minimum value $Z_m = R_m$ [41]."

14.3 Bode Plots of Less-common Factors

- Minimum vs. nonminimum-phase systems

- Minimum phase systems
  * "Bode called these systems minimum phase systems because they have the smallest phase lag of all systems with the same gain curve. For minimum phase systems the phase is uniquely given by the shape of the gain curve and vice versa ... [5]"
  * "These systems do not have time delays or poles and zeros in the right half-plan ... [5]"
  * "If all the poles and zeros of a system lie in the left-half s plane, then the system is called minimum phase [39]."
  * Does a change in the sign of the gain make a system nonminimum phase?
  * "The name minimum-phase refers to the fact that such a system has the minimum possible phase lag for the given magnitude response ... [43]."
  * "A system with no pole/zero in the ORHP is called an MP system [37]."
- Nonminimum phase systems
  * "A system with a zero in the RHP undergoes a net change in phase when evaluated for frequency inputs between zero and infinity, which, for an associated magnitude plot, is greater than if all poles and zeros were in the LHP. Such a system is called nonminimum phase [8]."
  * We won't spend a lot of time on nonminimum phase systems in this class.
  * "... a system with at least one zero/pole in the Open Right Half Plane (ORHP) is called an NMP system [37]."
  * Systems with time delays are nonminimum phase; see, for example, [37].
  * "RHP-zeros and time delays contribute additional phase lag to a system when compared to that of a minimum-phase system with the same gain ... [43]."

- Positive real root at $\frac{1}{\tau}$; see, for example, [50]
  - Factor

$$G(s) = -\tau s + 1 \tag{14.123}$$

  - Magnitude ratio

$$\begin{aligned} M(\omega) &= |-\tau j\omega + 1| && (14.124)\\ &= \sqrt{(-\tau\omega)^2 + 1^2} && (14.125)\\ m(\omega) &= 20\log\left[\sqrt{(\tau\omega)^2 + 1^2}\right] && (14.126)\\ &= \frac{1}{2}20\log\left[(\tau\omega)^2 + 1^2\right] && (14.127)\\ &= 10\log\left[(\tau\omega)^2 + 1^2\right] && (14.128) \end{aligned}$$

- * Low frequency limit ($\tau\omega \ll 1$)

$$\begin{aligned} m(\omega) &\approx 10 \log \left[\cancel{(\tau\omega)^2} + 1^2 \right] && (14.129) \\ &\approx 0 && (14.130) \end{aligned}$$

- * High frequency limit ($\tau\omega \gg 1$)

$$\begin{aligned} m(\omega) &\approx 10 \log \left[(\tau\omega)^2 + \cancel{1^2} \right] && (14.131) \\ &\approx 20 \log[\tau\omega] && (14.132) \end{aligned}$$

– Phase angle

$$\begin{aligned} \phi(\omega) &= \angle(-\tau j\omega + 1) && (14.133) \\ &= \tan^{-1}\left(\frac{-\tau\omega}{1}\right) && (14.134) \end{aligned}$$

- * Low frequency limit ($\tau\omega \ll 1$)

$$\begin{aligned} \phi(\omega) &= \tan^{-1}\left(\frac{\cancelto{0}{-\tau\omega}}{1}\right) && (14.135) \\ &= 0^\circ && (14.136) \end{aligned}$$

- * High frequency limit ($\tau\omega \gg 1$)

$$\begin{aligned} \phi(\omega) &= \tan^{-1}\left(\frac{\cancelto{\infty}{-\tau\omega}}{1}\right) && (14.137) \\ &= -90^\circ && (14.138) \end{aligned}$$

- Complex conjugate pair of roots $\zeta\omega_n \pm j\omega_d$ [50]

– Factor

$$G(s) = \frac{s^2 - 2\zeta\omega_n s + \omega_n^2}{\omega_n^2} \tag{14.139}$$

– Magnitude ratio

$$
\begin{aligned}
M(\omega) &= \left|\frac{(j\omega)^2 - 2\zeta\omega_n j\omega + \omega_n^2}{\omega_n^2}\right| && (14.140)\\
&= \left|\frac{-\omega^2 - 2\zeta\omega_n j\omega + \omega_n^2}{\omega_n^2}\right| && (14.141)\\
&= \left|1 - \frac{\omega^2}{\omega_n^2} - 2\zeta j\frac{\omega}{\omega_n}\right| && (14.142)\\
&= \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(-2\zeta\frac{\omega}{\omega_n}\right)^2} && (14.143)\\
m(\omega) &= 20\log\left[\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right] && (14.144)\\
&= \frac{1}{2}20\log\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right] && (14.145)\\
&= 10\log\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right] && (14.146)
\end{aligned}
$$

∗ Low frequency limit ($\frac{\omega}{\omega_n} \ll 1$)

$$
\begin{aligned}
m(\omega) &\approx 10\log\left[\left(1 - \cancel{\frac{\omega^2}{\omega_n^2}}\right)^2 + \cancel{\left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right] && (14.147)\\
&\approx 0 && (14.148)
\end{aligned}
$$

∗ High frequency limit ($\frac{\omega}{\omega_n} \gg 1$)

$$
\begin{aligned}
m(\omega) &\approx 10\log\left[\left(\cancel{1} - \frac{\omega^2}{\omega_n^2}\right)^2 + \cancel{\left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right] && (14.149)\\
&\approx 10\log\left[\left(\frac{\omega^2}{\omega_n^2}\right)^2\right] && (14.150)\\
&\approx 40\log\left[\frac{\omega}{\omega_n}\right] && (14.151)
\end{aligned}
$$

– Phase angle

$$\begin{aligned}
\phi(\omega) &= \angle\left(\frac{(j\omega)^2 - 2\zeta\omega_n j\omega + \omega_n^2}{\omega_n^2}\right) && (14.152)\\
&= \angle\left(1 - \frac{\omega^2}{\omega_n^2} - 2\zeta j\frac{\omega}{\omega_n^2}\right) && (14.153)\\
&= \tan^{-1}\left(\frac{-2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right) && (14.154)
\end{aligned}$$

* Low frequency limit ($\frac{\omega}{\omega_n} \ll 1$)

$$\begin{aligned}
\phi(\omega) &= \tan^{-1}\left(-\frac{2\zeta\cancelto{0}{\frac{\omega}{\omega_n}}}{1-\cancelto{0}{\frac{\omega^2}{\omega_n^2}}}\right) && (14.155)\\
&= \tan^{-1}\left(\frac{0}{1}\right) && (14.156)\\
&= 0^\circ && (14.157)
\end{aligned}$$

* High frequency limit ($\frac{\omega}{\omega_n} \gg 1$)

$$\begin{aligned}
\phi(\omega) &= \tan^{-1}\left(\frac{-2\zeta\frac{\omega}{\omega_n}}{\cancel{1}-\frac{\omega^2}{\omega_n^2}}\right) && (14.158)\\
&= \tan^{-1}\left(\frac{-2\zeta\cancel{\frac{\omega}{\omega_n}}}{-\frac{\omega^{\cancel{2}}}{\omega_n^{\cancel{2}}}}\right) && (14.159)\\
&= \tan^{-1}\left(\frac{-2\zeta}{-\cancelto{\infty}{\frac{\omega}{\omega_n}}}\right) && (14.160)\\
&= -180^\circ && (14.161)
\end{aligned}$$

- Time delay

– Factor

$$G(s) = e^{-sT_D} \tag{14.162}$$

14.4 Bode Plot Examples

Example 14.1:

- Transfer function

$$G(s) = \frac{1}{s^2 + s + 100} \tag{14.163}$$

- Decompose into common factors
  - Natural frequency

$$\begin{aligned} \omega_n &= \sqrt{\frac{100}{1}} && (14.164) \\ &= 10 && (14.165) \end{aligned}$$

  - Damping ratio (underdamped)

$$\begin{aligned} \zeta &= \frac{1}{2\sqrt{100}} && (14.166) \\ &= \frac{1}{20} && (14.167) \end{aligned}$$

  - Rewrite as common factors

$$\begin{aligned} G(s) &= \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} && (14.168) \\ &= \frac{\omega_n^2}{\omega_n^2} \cdot \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} && (14.169) \\ &= \frac{1}{\omega_n^2} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} && (14.170) \\ &= K \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} && (14.171) \\ &= \frac{1}{10^2} \cdot \frac{10^2}{s^2 + 2 \cdot \frac{1}{20} \cdot 10 s + 10^2} && (14.172) \\ &= G_1(s) G_2(s) && (14.173) \end{aligned}$$

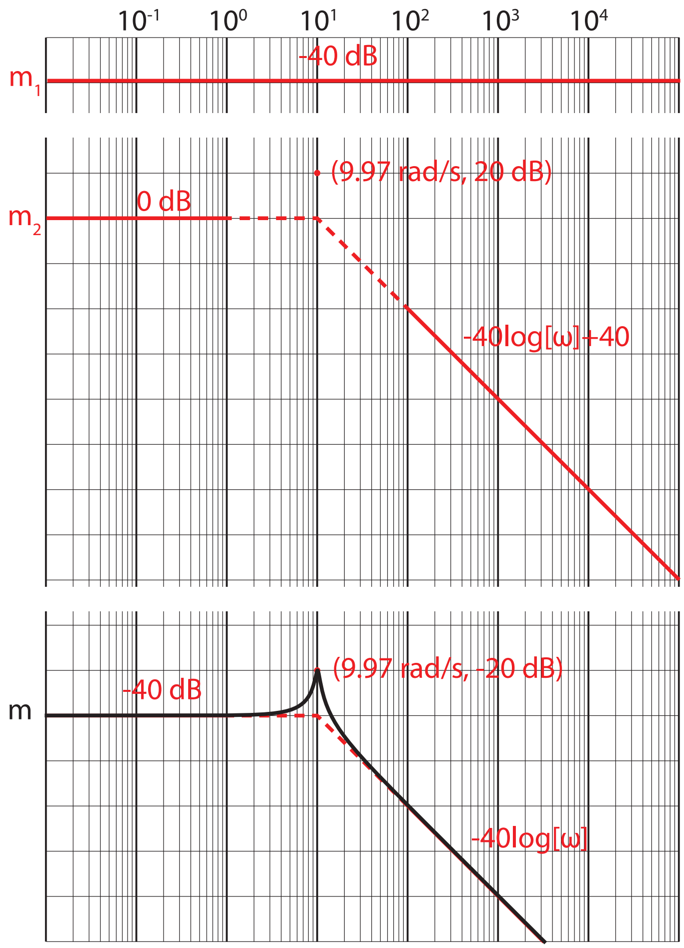


Figure 14.6: Superposition of magnitude ratio plots

- Sketch the magnitude ratio for the components and apply superposition
  - $m_1(\omega)$

$$\begin{aligned} m_1(\omega) &= 20\log[K] & (14.174)\\ &= 20\log\left[\frac{1}{100}\right] & (14.175)\\ &= -40 & (14.176) \end{aligned}$$

– $m_2(\omega)$:

$$m_2(\omega) \approx \begin{cases} 0 & \omega \ll \omega_n \\ -20\log\left[2\zeta\sqrt{1-\zeta^2}\right] & \omega = \omega_n\sqrt{1-2\zeta^2} \\ -40\log\left[\frac{\omega}{\omega_n}\right] & \omega_n \ll \omega \end{cases} \tag{14.177}$$

$$\approx \begin{cases} 0 & \omega \ll 10 \\ -20\log\left[2\cdot\frac{1}{20}\sqrt{1-\left(\frac{1}{20}\right)^2}\right] & \omega = 10\sqrt{1-2\cdot\frac{1}{20^2}} \\ -40\log\left[\frac{\omega}{10}\right] & 10 \ll \omega \end{cases} \tag{14.178}$$

$$\approx \begin{cases} 0 & \omega \ll 10 \\ 20.0 & \omega = 9.97 \\ -40\log[\omega] + 40\log[10] & 10 \ll \omega \end{cases} \tag{14.179}$$

$$\approx \begin{cases} 0 & \omega \ll 10 \\ 20.0 & \omega = 9.97 \\ -40\log[\omega] + 40 & 10 \ll \omega \end{cases} \tag{14.180}$$

– $m(\omega) = m_1(\omega) + m_2(\omega)$

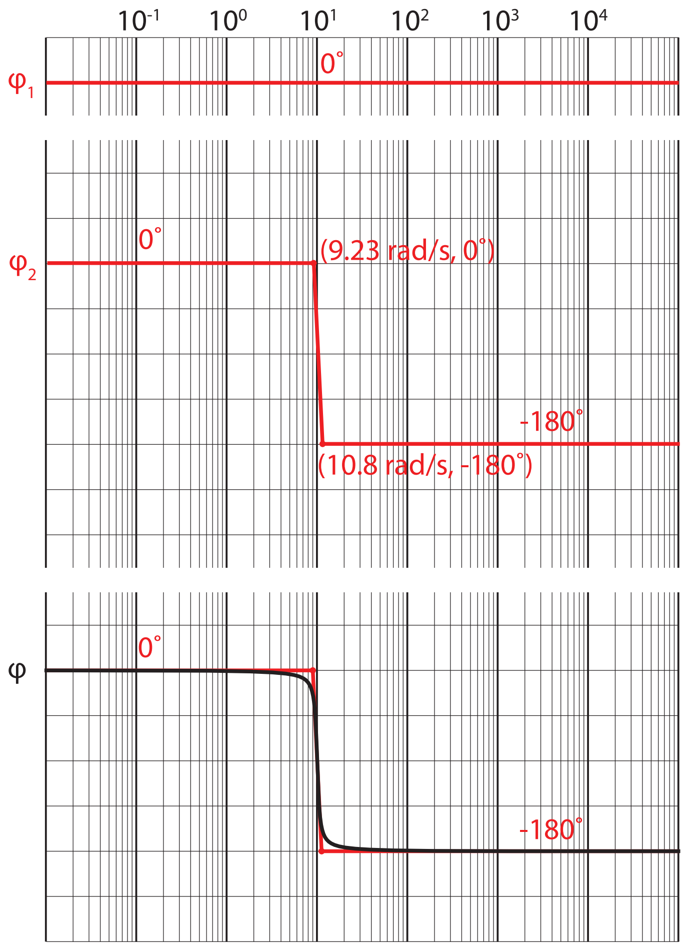


Figure 14.7: Superposition of phase angle plots

- Sketch the phase angle for the components and apply superposition
  - $\phi_1(\omega)$

$$\phi_1(\omega) \quad = \quad 0^\circ \tag{14.181}$$

- $\phi_2(\omega)$:

$$\phi_2(\omega) \approx \begin{cases} 0^\circ & \omega < \omega_n \left(\frac{1}{5}\right)^{\zeta} \\ -90^\circ & \omega = \omega_n \\ -180^\circ & \omega_n 5^{\zeta} < \omega \end{cases} \tag{14.182}$$

$$\approx \begin{cases} 0^\circ & \omega < 10 \left(\frac{1}{5}\right)^{\frac{1}{20}} \\ -90^\circ & \omega = 10 \\ -180^\circ & 10 \cdot 5^{\frac{1}{20}} < \omega \end{cases} \tag{14.183}$$

$$\approx \begin{cases} 0^\circ & \omega < 9.23 \\ -90^\circ & \omega = 10 \\ -180^\circ & 10.8 < \omega \end{cases} \tag{14.184}$$

- $\phi(\omega) = \phi_1(\omega) + \phi_2(\omega)$

Example 14.2:

- Transfer function

$$G(s) = \frac{1}{100s^2 + 101s + 1} \tag{14.185}$$

- Decompose into common factors
  - Note, the system is overdamped so there are two distinct real poles $\left\{-\frac{1}{100}, -1\right\}$.
  - Rewrite as common factors

$$G(s) = \frac{1}{(100s+1)(s+1)} \tag{14.186}$$

$$= \frac{1}{100s+1} \cdot \frac{1}{s+1} \tag{14.187}$$

$$= G_1(s)G_2(s) \tag{14.188}$$

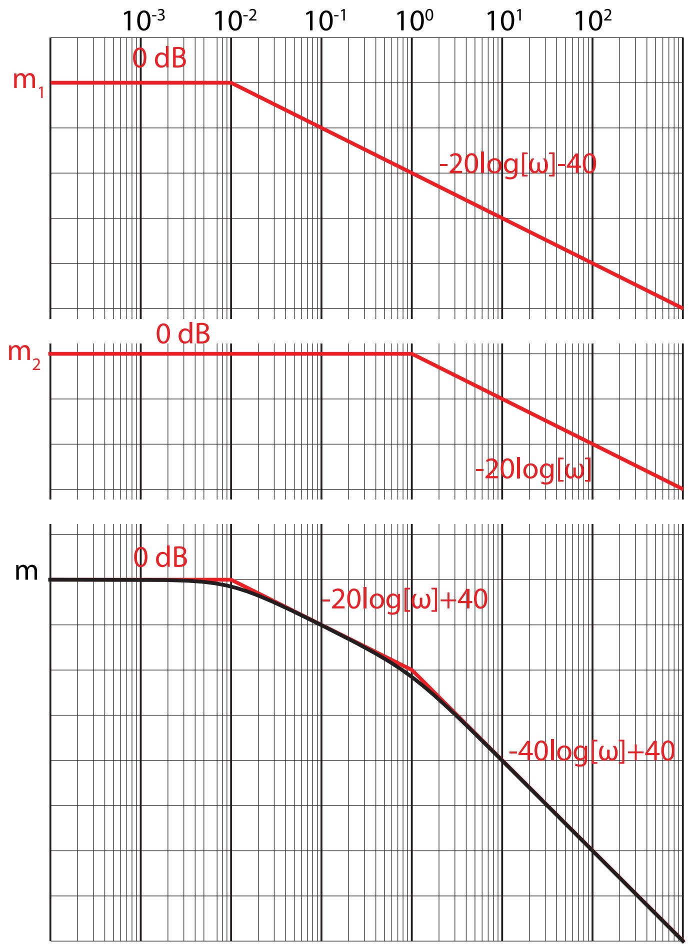


Figure 14.8: Superposition of magnitude ratio plots

- Sketch the magnitude ratio for the components and apply superposition

- $m_1(\omega)$

$$m_1(\omega) \approx \begin{cases} 0 & \omega \ll \frac{1}{\tau_1} \\ -20\log[\tau_1\omega] & \frac{1}{\tau_1} \ll \omega \end{cases} \tag{14.189}$$

$$\approx \begin{cases} 0 & \omega \ll \frac{1}{100} \\ -20\log[100\omega] & \frac{1}{100} \ll \omega \end{cases} \tag{14.190}$$

$$\approx \begin{cases} 0 & \omega \ll \frac{1}{100} \\ -20\log[\omega] - 40 & \frac{1}{100} \ll \omega \end{cases} \tag{14.191}$$

- $m_2(\omega)$:

$$m_2(\omega) \approx \begin{cases} 0 & \omega \ll \frac{1}{\tau_2} \\ -20\log[\tau_1\omega] & \frac{1}{\tau_2} \ll \omega \end{cases} \tag{14.192}$$

$$\approx \begin{cases} 0 & \omega \ll \frac{1}{1} \\ -20\log[100\omega] & \frac{1}{1} \ll \omega \end{cases} \tag{14.193}$$

$$\approx \begin{cases} 0 & \omega \ll 1 \\ -20\log[\omega] & 1 \ll \omega \end{cases} \tag{14.194}$$

- $m(\omega) = m_1(\omega) + m_2(\omega)$

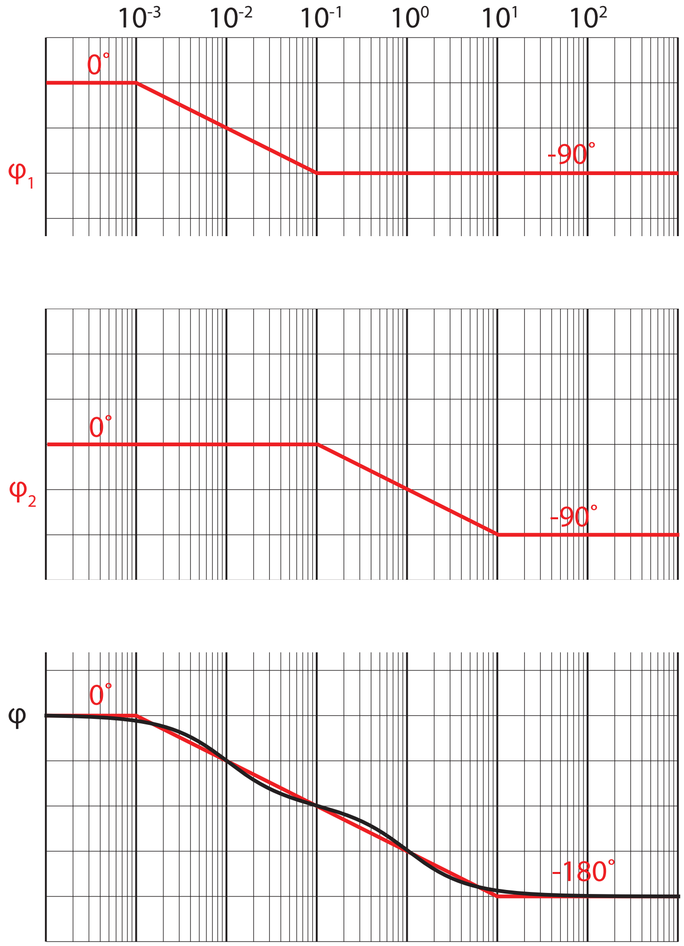


Figure 14.9: Superposition of phase angle plots

- Sketch the phase angle for the components and apply superposition

– $\phi_1(\omega)$

$$\begin{aligned}
\phi_1(\omega) &\approx \begin{cases} 0^\circ & \omega \ll \frac{1}{\tau_1} \\ -90^\circ & \frac{1}{\tau_1} \ll \omega \end{cases} && (14.195)\\
&\approx \begin{cases} 0^\circ & \omega \ll \frac{1}{100} \\ -90^\circ & \frac{1}{100} \ll \omega \end{cases} && (14.196)\\
&\approx \begin{cases} 0^\circ & \omega < \frac{1}{10}\frac{1}{100} \\ -90^\circ & 10\frac{1}{100} < \omega \end{cases} && (14.197)\\
&\approx \begin{cases} 0^\circ & \omega < \frac{1}{1000} \\ -90^\circ & \frac{1}{10} < \omega \end{cases} && (14.198)
\end{aligned}$$

– $\phi_2(\omega)$:

$$\begin{aligned}
\phi_2(\omega) &\approx \begin{cases} 0^\circ & \omega \ll \frac{1}{\tau_2} \\ -90^\circ & \frac{1}{\tau_2} \ll \omega \end{cases} && (14.199)\\
&\approx \begin{cases} 0^\circ & \omega \ll \frac{1}{1} \\ -90^\circ & \frac{1}{1} \ll \omega \end{cases} && (14.200)\\
&\approx \begin{cases} 0^\circ & \omega < \frac{1}{10} \\ -90^\circ & 10 < \omega \end{cases} && (14.201)\\
&\approx \begin{cases} 0^\circ & \omega < \frac{1}{10} \\ -90^\circ & 10 < \omega \end{cases} && (14.202)
\end{aligned}$$

– $\phi(\omega) = \phi_1(\omega) + \phi_2(\omega)$

Example 14.3: Base excitation

- Applications: accelerometer, quarter car, ...
- Governing equation

$$\begin{aligned}
m\ddot{y} + c\dot{y} + ky &= c\dot{u} + ku && (14.203)\\
\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y &= 2\zeta\omega_n\dot{u} + \omega_n^2 u && (14.204)
\end{aligned}$$

- Transfer function

$$
\begin{aligned}
G(s) &= \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} && (14.205)\\
&= \frac{\frac{2\zeta}{\omega_n}s + 1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} && (14.206)\\
&= \left(\frac{2\zeta}{\omega_n}s + 1\right)\frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} && (14.207)\\
&= \left(\frac{2\zeta}{\omega_n}s + 1\right)\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} && (14.208)\\
&= G_1(s)G_2(s) && (14.209)
\end{aligned}
$$

- Magnitude ratio

$$
\begin{aligned}
M(\omega) &= |G(j\omega)| && (14.210)\\
&= |G_1(j\omega)||G_2(j\omega)| && (14.211)
\end{aligned}
$$

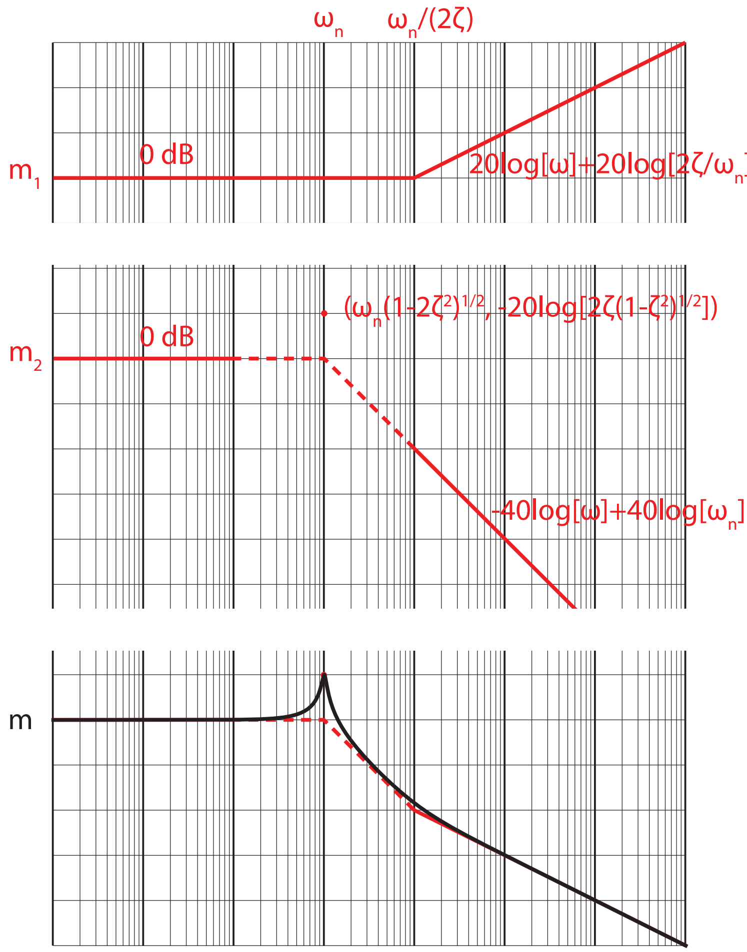


Figure 14.10: Superposition of magnitude ratio plots. Here, $\zeta = \frac{1}{20}$.

- Sketch the magnitude ratio for the components and apply superposition

$$m(\omega) \quad = \quad m_1(\omega) + m_2(\omega) \tag{14.212}$$

– $m_1(\omega)$

$$m_1(\omega) \approx \begin{cases} 0 & \omega \ll \frac{1}{\tau} \\ 20\log[\tau\omega] & \frac{1}{\tau} \ll \omega \end{cases} \tag{14.213}$$

$$\approx \begin{cases} 0 & \omega \ll \frac{1}{\tau} \\ 20\log[\omega] + 20\log[\tau] & \frac{1}{\tau} \ll \omega \end{cases} \tag{14.214}$$

$$\approx \begin{cases} 0 & \omega \ll \frac{1}{\frac{2\zeta}{\omega_n}} \\ 20\log[\omega] + 20\log\left[\frac{2\zeta}{\omega_n}\right] & \frac{1}{\frac{2\zeta}{\omega_n}} \ll \omega \end{cases} \tag{14.215}$$

$$\tag{14.216}$$

– $m_2(\omega)$:

$$m_2(\omega) \approx \begin{cases} 0 & \omega \ll \omega_n \\ -20\log\left[2\zeta\sqrt{1-\zeta^2}\right] & \omega = \omega_n\sqrt{1-2\zeta^2} \\ -40\log\left[\frac{\omega}{\omega_n}\right] & \omega_n \ll \omega \end{cases} \tag{14.217}$$

$$\approx \begin{cases} 0 & \omega \ll \omega_n \\ -20\log\left[2\zeta\sqrt{1-\zeta^2}\right] & \omega = \omega_n\sqrt{1-2\zeta^2} \\ -40\log[\omega] + 40\log[\omega_n] & \omega_n \ll \omega \end{cases} \tag{14.218}$$

– $m(\omega) = m_1(\omega) + m_2(\omega)$

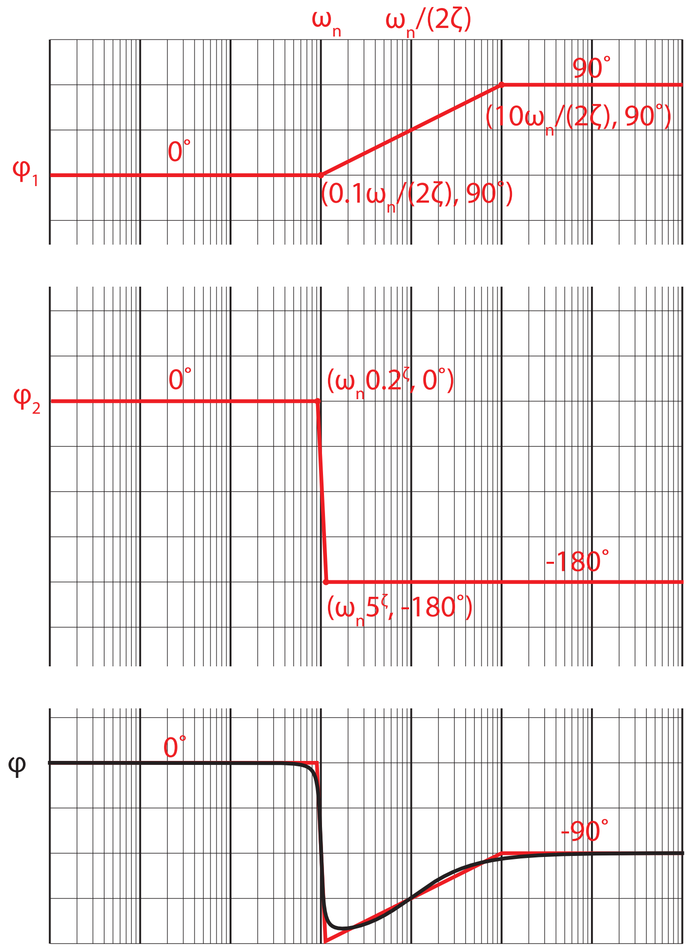


Figure 14.11: Superposition of phase angle plots. Here, $\zeta = \frac{1}{20}$.

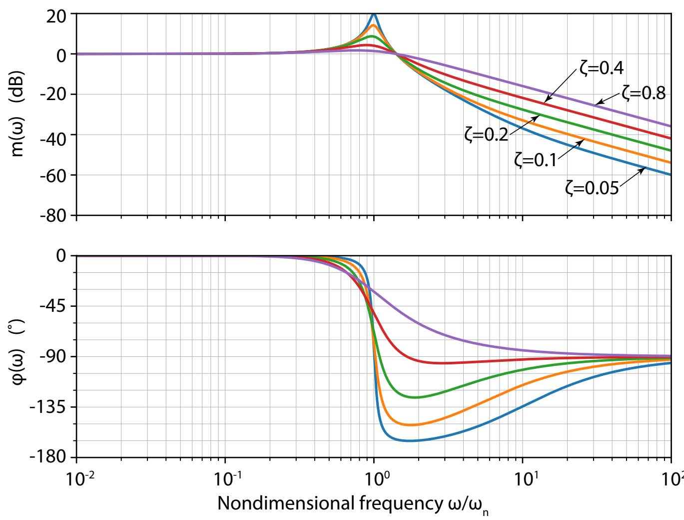


Figure 14.12:

- Sketch the phase angle for the components and apply superposition
  - $\phi_1(\omega)$

$$\phi_1(\omega) \approx \begin{cases} 0^\circ & \omega \ll \frac{1}{\tau} \\ -180^\circ & \frac{1}{\tau} \ll \omega \end{cases} \tag{14.219}$$

$$\approx \begin{cases} 0^\circ & \omega < \frac{1}{10}\frac{1}{\tau} \\ -180^\circ & 10\frac{1}{\tau} < \omega \end{cases} \tag{14.220}$$

$$\approx \begin{cases} 0^\circ & \omega < \frac{1}{10}\frac{1}{\frac{2\zeta}{\omega_n}} \\ -180^\circ & 10\frac{1}{\frac{2\zeta}{\omega_n}} < \omega \end{cases} \tag{14.221}$$

  - $\phi_2(\omega)$:

$$\phi_2(\omega) \approx \begin{cases} 0^\circ & \omega < \omega_n \left(\frac{1}{5}\right)^\zeta \\ -90^\circ & \omega = \omega_n \\ -180^\circ & \omega_n 5^\zeta < \omega \end{cases} \tag{14.222}$$

  - $\phi(\omega) = \phi_1(\omega) + \phi_2(\omega)$

- Peak derivation
  - Magnitude ratio

$$M(\omega) = \left| \frac{\frac{2\zeta}{\omega_n} j\omega + 1}{\frac{1}{\omega_n^2} - \omega^2 + \frac{2\zeta}{\omega_n} j\omega + 1} \right| \tag{14.223}$$

$$= \frac{\sqrt{\left(2\zeta\frac{\omega}{\omega_n}\right)^2 + 1}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \tag{14.224}$$

  - Square of magnitude ratio as a function of relative frequency (easier to deal with)

$$M^2(\omega) = \frac{\left(2\zeta\frac{\omega}{\omega_n}\right)^2 + 1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} \tag{14.225}$$

$$M^2(r) = \frac{(2\zeta r)^2 + 1}{(1 - r^2)^2 + (2\zeta r)^2} \tag{14.226}$$

  - Peak frequency (double check)

$$\omega = \omega_n \sqrt{\frac{1 - \sqrt{1 + 8\zeta^2}}{-4\zeta^2}} \tag{14.227}$$

  - Peak value
- Interestingly, all $m(\omega)$ curves pass through the same point at $\omega = \omega_n\sqrt{2}$.

Example 14.4: Vibration absorber

- add sketch and discuss application
- Equations of motion (input f and output $y = z_1$)

$$m_1\ddot{z}_1 + (k_1 + k_2)z_1 - k_2 z_2 = f \tag{14.228}$$

$$m_2\ddot{z}_2 - k_2 z_1 + k_2 z_2 = 0 \tag{14.229}$$

- Transfer function

$$m_1 s^2 Z_1(s) + (k_1 + k_2) Z_1(s) - k_2 Z_2(s) = F(s) \quad (14.230)$$
$$m_2 s^2 Z_2(s) - k_2 Z_1(s) + k_2 Z_2(s) = 0 \quad (14.231)$$

$$m_1 s^2 Z_1(s) + (k_1 + k_2) Z_1(s) - k_2 \frac{k_2}{m_2 s^2 + k_2} Z_1(s) = F(s) \quad (14.232)$$
$$(m_1 s^2 + (k_1 + k_2))(m_2 s^2 + k_2) Z_1(s) - k_2^2 Z_1(s) = (m_2 s^2 + k_2) F(s) \quad (14.233)$$

$$\frac{Y(s)}{F(s)} = \frac{m_2 s^2 + k_2}{(m_1 s^2 + (k_1 + k_2))(m_2 s^2 + k_2) - k_2^2} \quad (14.234)$$
$$= \frac{m_2 s^2 + k_2}{m_1 m_2 s^4 + \left[(k_1 + k_2) m_2 + k_2 m_1\right] s^2 + k_1 k_2} \quad (14.235)$$

Example 14.5: Rotating unbalance

- add sketch and discuss application

Example 14.6: Piezo-electric Accelerometer

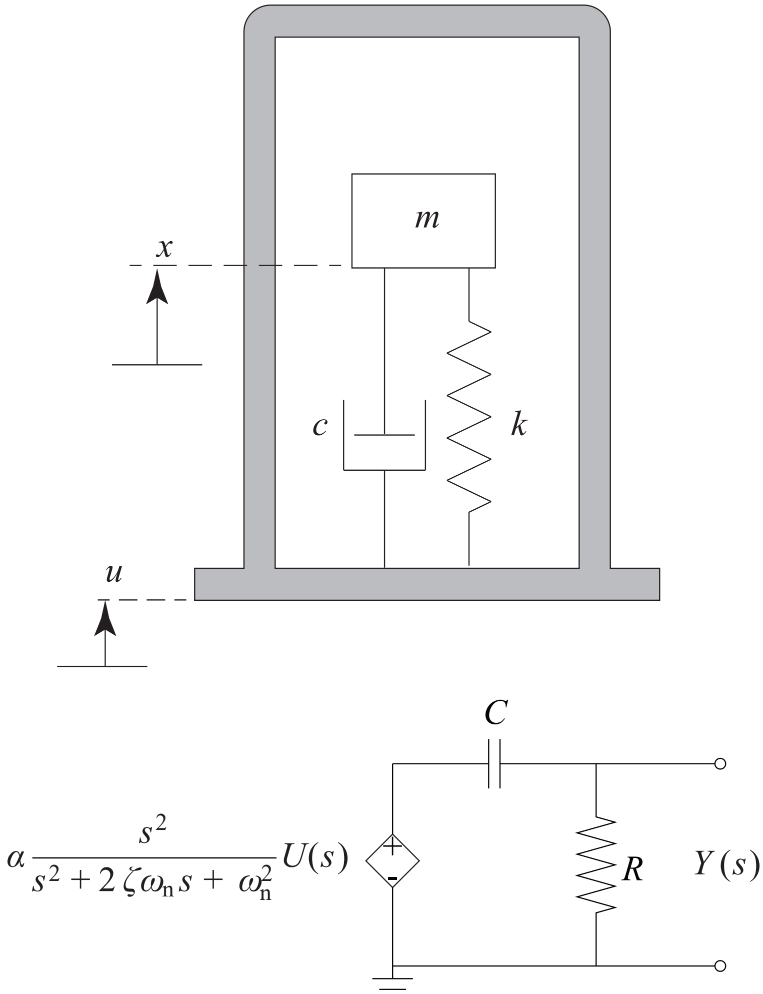


Figure 14.13:

- look at the presentation on page 182 of [9]. See also: `http://engineering.nyu.edu/mechatronics/Control_Lab/bck/VKapila/InstrumentationLabManual/lab6.pdf`
- Transfer function This model neglects loading effects (which could be negligible.

$$G(s) = \frac{Y(s)}{\alpha s^2 U(s)} = \frac{RCs}{RCs+1} \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{14.236}$$

I haven't seen a source with the zero in the numerator. Perhaps it's ignored because ζ is small?

- Parameter values: $R = 30$ MΩ, C=3.75 nF, $\zeta = 0.015$, $\omega_n = 11000 \cdot 2\pi$ rad/s

Example 14.7:

bode plot for example 9.18/9.16 ...

Example 14.8: Base excitation of cantilever beam demo

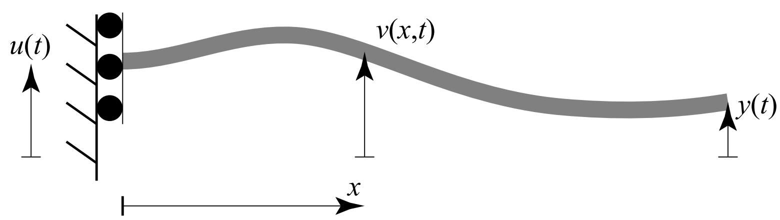


Figure 14.14:

The vibrations of an ideal cantilever beam with a moving base are best represented by a partial differential equation. However, a crude 2 degree of freedom approximation for the deflection, $v(x,t)$, of the cantilever beam may be obtained from a time dependent linear combination of rigid body motion with two trial functions $\psi_1(x) = \frac{1}{L^3}x^2(3L-x)$ and $\psi_2(x) = \frac{1}{L^4}x^3(2L-x)$,

$$v(x,t) \approx u(t) + \eta_1(t)\psi_1(x) + \eta_2(t)\psi_2(x). \tag{14.237}$$

Here, $u(t)$ is the time dependent rigid body component to the deflection (i.e., motion of the base) and $\eta_1(t)$ and $\eta_2(t)$ are time dependent weighting functions for the two trial functions. This approximation yields two governing ordinary differential equations

$$\begin{aligned} \rho L \begin{bmatrix} \frac{33}{35} & \frac{23}{56} \\ \frac{23}{56} & \frac{23}{126} \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + cL \begin{bmatrix} \frac{33}{35} & \frac{23}{56} \\ \frac{23}{56} & \frac{23}{126} \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} & \\ + \frac{EI}{L^3} \begin{bmatrix} 12 & 6 \\ 6 & \frac{24}{5} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} &= -\rho L \begin{bmatrix} \frac{3}{4} \\ \frac{3}{10} \end{bmatrix} \ddot{u} - cL \begin{bmatrix} \frac{3}{4} \\ \frac{3}{10} \end{bmatrix} \end{aligned} \tag{14.238}$$

(Need to check this equation.)

If we consider the deflection of the end of the beam to be the output, we can write

$$\begin{aligned} y &= v(L,t) & (14.239) \\ &= \psi_1(L)\eta_1 + \psi_2(L)\eta_2 + u. & (14.240) \end{aligned}$$

Construct the bode plot for the transfer function

$$G(s) = \frac{Y(s)}{U(s)} \tag{14.241}$$

Use $E = 200$ GPa, linear density $\rho =$ kg/m, thickness $h = 0.787$ mm (0.031 in), width $b = 25.146$ mm (0.99 in), $L = 0.45847$ m (18.05 in).
It might be reasonable to assume that the damping is proportional to the mass matrix.

-

Chapter 15

System Identification in the Frequency Domain

15.1 First Order System Identification

- General transfer function

$$G(s) = \frac{K_0}{\tau s + 1} \tag{15.1}$$

- Static sensitivity - obtain from magnitude ratio

$$K_0 = \begin{cases} M(0) & \phi(0) = 0^\circ \\ -M(0) & \phi(0) = -180^\circ \end{cases} \tag{15.2}$$

- Time constant
  - Relation to break frequency (ω_b).

$$\tau = \frac{1}{\omega_b} \tag{15.3}$$

  - Simple methods to approximate break frequency:
    * Break frequency is at the intersection of high and low frequency limits
    * Break frequency at $\phi(\omega_b) = -45^\circ$
    * Break frequency at $20\log[2] = 3.01$ dB below the low frequency limit, $m(\omega_b) = 20\log[|K_0|] - 20\log[2]$

15.2 Underdamped Second Order System Identification

- Equation

$$m\ddot{y} + c\dot{y} + ky = K_0 k u \tag{15.4}$$
$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K_0\omega_n^2 u \tag{15.5}$$

- General transfer function

$$G(s) = \frac{K_0\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{15.6}$$

- Static sensitivity - obtain from magnitude ratio

$$K_0 = \begin{cases} M(0) & \phi(0) = 0^\circ \\ -M(0) & \phi(0) = -180^\circ \end{cases} \tag{15.7}$$

- Natural frequency
  - Relation to break frequency (ω_b).

$$\omega_n = \omega_b \tag{15.8}$$

  - Simple methods to approximate break frequency:
    * Break frequency is at the intersection of high and low frequency limits
    * Break frequency at $\phi(\omega_b) = -90^\circ$
    * For small ζ, Break frequency is approximately the peak frequency, $\omega_b \approx \omega_p$

- Equation

$$m\ddot{y} + c\dot{y} + ky = K_0 k u \tag{15.9}$$
$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K_0\omega_n^2 u \tag{15.10}$$

- General transfer function

$$G(s) = \frac{K_0\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{15.11}$$

- Power; see, for example, [41]
  - Instantaneous power

$$P_I = \text{Force} \cdot \text{Velocity} \tag{15.12}$$
$$= K_0 k u \cdot \dot{y} \tag{15.13}$$

– Average power at steady state with $u = A\sin(\omega t)$

$$\begin{aligned}
\langle P(\omega)\rangle &= \frac{1}{T}\int_0^T K_0 k u \dot{y} dt && (15.14)\\
&= \frac{1}{T}\int_0^T K_0 k u \frac{d}{dt}(y)\, dt && (15.15)\\
&= \frac{1}{T}\int_0^T K_0 k A\sin(\omega t)\frac{d}{dt}\left(AM(\omega)\sin\left(\omega t + \phi(\omega)\right)\right) dt && (15.16)\\
&= \frac{1}{T}\int_0^T K_0 k A^2 M(\omega)\omega \sin(\omega t)\cos\left(\omega t + \phi(\omega)\right) dt && (15.17)\\
&= \frac{1}{T}\int_0^T K_0 k A^2 M(\omega)\omega \sin(\omega t)\left(\cos(\omega t)\cos(\phi(\omega)) - \sin(\omega t)\sin(\phi(\omega))\right) dt && (15.18)\\
&= \frac{1}{T}K_0 k A^2 M(\omega)\omega\left(\cancel{\int_0^T \sin(\omega t)\cos(\omega t) dt}\cos(\phi(\omega)) - \int_0^T \sin^2(\omega t) dt \sin(\phi(\omega))\right) && (15.19)\\
&= -\frac{1}{T}K_0 k A^2 M(\omega)\omega\sin(\phi(\omega))\int_0^T \sin^2(\omega t) dt && (15.20)\\
&= -\frac{1}{T}K_0 k A^2 M(\omega)\omega\sin(\phi(\omega))\frac{T}{2} && (15.21)\\
&= -\frac{1}{2}K_0 k A^2 M(\omega)\omega\sin(\phi(\omega)) && (15.22)\\
&= -\frac{1}{2}K_0 k A^2 M(\omega)\omega\left(-\frac{2\zeta\frac{\omega}{\omega_n}}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}\right) && (15.23)\\
&= \frac{1}{2}kA^2 M(\omega)\omega 2\zeta\frac{\omega}{\omega_n}\frac{K_0}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} && (15.24)\\
&= kA^2 M^2(\omega)\omega\zeta\frac{\omega}{\omega_n} && (15.25)\\
&= kA^2\zeta\frac{\omega^2}{\omega_n}M^2(\omega) && (15.26)
\end{aligned}$$

– Frequency with maximum average power, ω_r

  * Recognize that $2\zeta\frac{\omega}{\omega_n}M(\omega) = -K_0\sin(\phi(\omega))$

$$\begin{aligned}
\langle P(\omega)\rangle &= kA^2\frac{\omega_n}{4\zeta}2^2\zeta^2\frac{\omega^2}{\omega_n^2}M^2(\omega) && (15.27)\\
&= kA^2\frac{\omega_n}{4\zeta}K_0^2\sin^2(\phi(\omega)) && (15.28)
\end{aligned}$$

* Maximum average power will occur when $\phi(\omega = \omega_r) = -\frac{\pi}{2}$

$$\begin{aligned}
-\frac{\pi}{2} &= \phi(\omega_r) && (15.29)\\
&= -\tan^{-1}\left(\frac{2\zeta\frac{\omega_r}{\omega_n}}{1-\frac{\omega_r^2}{\omega_n^2}}\right) && (15.30)\\
0 &= 1-\frac{\omega_r^2}{\omega_n^2} && (15.31)\\
\omega_r &= \omega_n && (15.32)
\end{aligned}$$

- Half maximum power points, ω_1 and ω_2, would occur when

$$\begin{aligned}
\sin^2(\phi(\omega_i) &= \frac{1}{2} && (15.33)\\
\sin(\phi(\omega_i)) &= \pm\frac{1}{\sqrt{2}} && (15.34)\\
\tan(\phi(\omega_i)) &= \pm 1 && (15.35)\\
\frac{2\zeta\frac{\omega_i}{\omega_n}}{1-\frac{\omega_i^2}{\omega_n^2}} &= && (15.36)\\
\pm 2\zeta\frac{\omega_i}{\omega_n} &= 1-\frac{\omega_i^2}{\omega_n^2} && (15.37)\\
\frac{\omega_i^2}{\omega_n^2} \pm 2\zeta\frac{\omega_i}{\omega_n} - 1 &= 0 && (15.38)\\
\frac{\omega_i}{\omega_n} &= \frac{\pm 2\zeta + \sqrt{(2\zeta)^2+4}}{2} && (15.39)\\
\frac{\{\omega_2, \omega_1\}}{\omega_n} &= \pm\zeta + \sqrt{\zeta^2+1} && (15.40)
\end{aligned}$$

* recognizing $\omega_2 > \omega_1 > 0$
* Difference in half power points, $\omega_2 - \omega_1$

$$\begin{aligned}
\frac{\omega_2-\omega_1}{\omega_n} &= (\zeta + \sqrt{\zeta^2+1}) - (-\zeta + \sqrt{\zeta^2+1}) && (15.41)\\
\frac{\omega_2-\omega_1}{\omega_n} &= 2\zeta && (15.42)
\end{aligned}$$

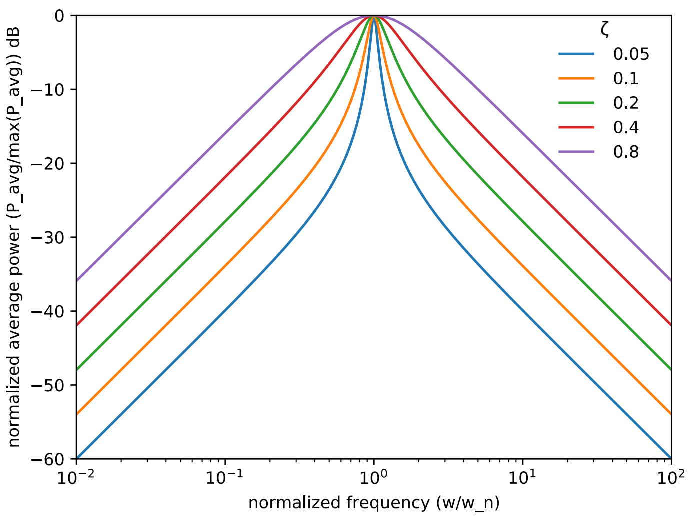

ζ
0.05
0.1
0.2
0.4
0.8
normalized average power (P_avg/max(P_avg)) dB
0
−10
−20
−30
−40
−50
−60
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
normalized frequency (w/w_n)

Figure 15.1:

- Simple methods to approximate damping ratio
  - Half power; see, for example, [51]
    * If $\langle P(\omega)\rangle$ and ω_n are available, find the half power points (ω_1 and ω_2) which occur at half the peak value of $\langle P(\omega)\rangle$

$$\zeta = \frac{\omega_2 - \omega_1}{2\omega_n}; \tag{15.43}$$

see derivation above. Unfortunately, ω_n and $\langle P(\omega)\rangle$ are not always available

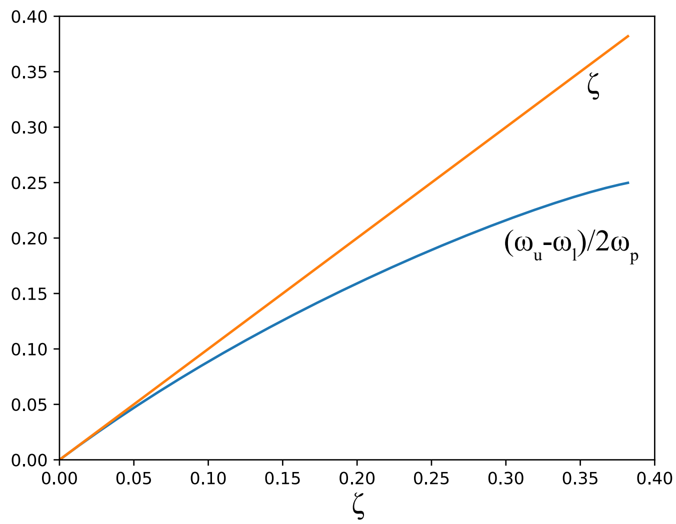


Figure 15.2: Comparison between the approximate value of ζ obtained from the half power method and exact value of zeta.

    * Approximation: Find approximate half power points (ω_l and ω_u) which occur at half the peak value of $M^2(\omega)$ or 3.01 dB below the peak in $m(\omega)$

$$\zeta \approx \frac{\omega_u - \omega_l}{2\omega_p}, \text{ for } \zeta < 0.05; \tag{15.44}$$

see, for example, [51]
    * The approximation is best for $\zeta < 0.05$ (i.e., when the peak stands out by 20 dB or more)

– Amplification at ω_n; see [52]

$$M(\omega) = \frac{K_0}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \tag{15.45}$$

$$M(\omega_n) = \frac{K_0}{\sqrt{\left(2\zeta\frac{\omega_n}{\omega_n}\right)^2}} \tag{15.46}$$

$$= \frac{K_0}{2\zeta} \tag{15.47}$$

$$\zeta = \frac{K_0}{2M(\omega_n)} \tag{15.48}$$

– Slope of phase at ω_n; see, for example, [52] The slope derivation has been copied to lecture 20 so this derivation could be dropped and replaced with a link to the earlier section.

$$\phi(\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right) \tag{15.49}$$

$$\frac{d\phi}{d\omega} = -\frac{1}{1+\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)^2}\left(\frac{2\zeta}{1-\frac{\omega^2}{\omega_n^2}} + \frac{2\zeta\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2}2\frac{\omega}{\omega_n}\right)\frac{1}{\omega_n} \tag{15.50}$$

$$= -\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\left(\frac{2\zeta}{1-\frac{\omega^2}{\omega_n^2}} + \frac{2\zeta\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2}2\frac{\omega}{\omega_n}\right)\frac{1}{\omega_n} \tag{15.51}$$

$$= -\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\left(2\zeta\left(1-\frac{\omega^2}{\omega_n^2}\right) + 2\zeta\frac{\omega}{\omega_n}2\frac{\omega}{\omega_n}\right)\frac{1}{\omega_n} \tag{15.52}$$

$$\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_n} = -\frac{1}{(2\zeta)^2}(4\zeta)\frac{1}{\omega_n} \tag{15.53}$$

$$= -\frac{1}{\zeta\omega_n} \tag{15.54}$$

$$\zeta = -\frac{1}{\omega_n}\left(\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_n}\right)^{-1} \tag{15.55}$$

* If the slope is extracted from a bode plot with frequency on a logarithmic scale, it must be converted to a linear scale (need to verify this and account for what is used

as the run when calculating slope)

$$\frac{d\phi}{d\omega} = \frac{d\phi}{d\log_{10}[\omega]}\frac{d}{d\omega}\left(\log_{10}[\omega]\right) \tag{15.56}$$
$$= \frac{d\phi}{d\log_{10}[\omega]}\frac{d}{d\omega}\left(\frac{\ln[\omega]}{\ln[10]}\right) \tag{15.57}$$
$$= \frac{d\phi}{d\log_{10}[\omega]}\frac{d\ln[\omega]}{d\omega}\frac{1}{\ln[10]} \tag{15.58}$$
$$= \frac{d\phi}{d\log_{10}[\omega]}\frac{1}{\omega\ln[10]} \tag{15.59}$$
$$\frac{\Delta\phi}{\Delta\omega} = \frac{\Delta\phi}{\Delta\left(\log_{10}[\omega]\right)}\frac{1}{\omega\ln[10]} \tag{15.60}$$

· Failure to account for the logarithmic scale can lead to errors up to about 150% depending upon what is used as the run when determining the slope.

*

– $\omega_n 0.2^{\zeta}$ to $\omega_n 5^{\zeta}$ - Note, this approach may be used for overdamped systems.

– Peak $M(\omega_p)$ - this approach requires a peak

15.3 Higher Order System Identification

- Higher order systems can be decomposed into simpler factors
- Nonminimum phase systems and sytem identification in the frequency domain
  – See, for example, 7.4.5 of [37]
- Software exists to assist in system identification
- Framework for system identification; see, for example, [6]
  – Identify all s^n factors from
    * The slope of the low frequency asymptote of $m(\omega) \approx 20n$ dB/decade

$$\lim_{\omega\to 0} m(\omega) \to 20n\log_{10}[\omega] \tag{15.61}$$

    * The phase angle at low frequencies is $n90^\circ + \angle K$. If the constant factor (K) is positive, then $\angle K = 0^\circ$.

$$\lim_{\omega\to 0} \phi(\omega) \to n90^\circ + \angle K \tag{15.62}$$

  – Estimate K

* $|K|$ may be determined from the low frequency asymptote of $m(\omega)$

$$\begin{aligned}\lim_{\omega\to 0} m(\omega) &= 20n\log_{10}[\omega] + 20\log_{10}|K| && (15.63)\\ &= m_{\text{low}}(\omega) && (15.64)\end{aligned}$$

· At $\omega = 1$

$$\begin{aligned}m_{\text{low}}(1) &= 20\log_{10}|K| && (15.65)\\ |K| &= 10^{\frac{m_{\text{low}}(1)}{20}} && (15.66)\end{aligned}$$

· Alternatively, $|K|$ may be determined from the x-intercept, ω_0, of the low frequency limit.

$$\begin{aligned}m_{\text{low}}(\omega_0) &= 0 && (15.67)\\ 20n\log_{10}[\omega_0] + 20\log_{10}|K| &= && (15.68)\\ |K| &= \omega_0^{-n} && (15.69)\end{aligned}$$

* The sign of K may be obtained by investigating the low frequency asymptote of $\phi(\omega)$

– Determine the difference between the number of poles and zeros

* From $m(\omega)$

· The slope of the low frequency asysmptote determines the s^n factors

· The slope of the high frequency asymptote can be written as $(n_z - n_p)20\log_{10}[\omega]$ and thereby reveals the difference between the number of poles and zeros.

* From $\phi(\omega)$

· The low frequency limit determines the s^n factors (assuming a minimum phase system)

· The difference between the high and low frequency phase angles determines the difference between the number of poles and zeros excluding $s = 0$ poles and zeros (assuming a minimum phase system)

– Estimate all break frequencies

* Change in slope of the asymptotic (i.e., straight line) approximation of $m(\omega)$ by an integer multiple of 20 dB/decade

* First order factors $\tau s + 1$

· Change in slope of the asymptotic approximation of $m(\omega)$ by ± 20 dB/decade

· $m(\omega_b)$ differs from the asysmptotic approximation by ± 3.01 dB for $\tau s + 1$ factors

· Halfway through a change in phase of $\pm 90°$

* Underdamped second order factors

· Change in slope of the asymptotic approximation of $m(\omega)$ by ± 40 dB/decade

· If $\zeta < \frac{1}{\sqrt{2}}$ there will be a peak in $m(\omega)$. The sharper the peak the closer it is to ω_b

· An asyptotic approximation to ϕ will yield a transition of 180° over less than 1.4 decades (or a $\frac{5^1}{\left(\frac{1}{5}\right)^1} = 25$ fold change in frequency). The break frequency will occur half way through the transition.

– Estimate damping ratios for underdamped second order factors

∗ From $M(\omega)$ or $m(\omega)$

· Magnitude at peak

· Magnitude at ω_n

· Half power

∗ From $\phi(\omega)$

· Asymptotic (straight line) approximation

· Slope of phase

– Once factors have been identify reliably, they may be removed by superposition (i.e., subtracted out of $m(\omega)$ and $\phi(\omega)$) and the above steps may be repeated to obtain remaining factors.

– Notes:

∗ Be sure to correctly distinguish the less common (or non minimum phase) factors from the more common (minimum phase) factors

∗ When break frequencies are nearby, it may not be possible to distinguish them

∗ Low signal to noise ratios in the data could make it difficult to determine asymptotic approximations over some frequency ranges (especially when $M(\omega)$ is small)

Example 15.1:

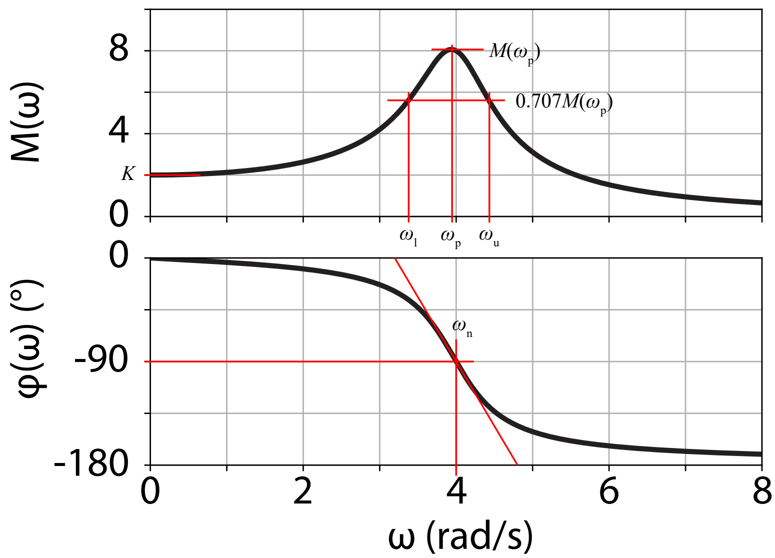


Figure 15.3:

Find a transfer function representation of the system with the above frequency response plots.

- Identify all s^n factors
  - The low frequency slope of the magnitude ratio is 0 suggesting that $n = 0$.
  - At low frequencies, the phase angle is 0°, suggesting that $n = 0$ (assuming the system is minimum phase).
- Estimate K
  - $M(0) = 2$ suggesting $|K| = 2$
  - $\phi(0) = 0$ suggesting $0 < K = 2$
- Determine the difference between the number of poles and zeros
  - The magnitude plot is on a linear scale and not in dB so it's hard to determine the high frequency slope. (We already know that there are no s^n factors.)
  - The phase angle drops by 180° suggesting that the system has two poles (assuming the system is minimum phase).
- Estimate all break frequencies

- The magnitude plot is on a linear scale and not in dB so we can't see changes in slope. However, the peak in the magnitude plot suggests an underdamped second order factor with $\omega_n \approx 4$ rad/s.
- The halfway point from 0 to -180° likely corresponds to a break frequency, $\omega_b = \omega_n = 4$ rad/s.

- Estimate damping ratio
  - Magnitude at peak $M(\omega_p) \approx 8$

$$\begin{aligned} \frac{M(\omega_p)}{K_0} &= \frac{1}{2\zeta\sqrt{1-\zeta^2}} && (15.70) \\ \frac{8}{2} &\approx && (15.71) \\ 4 &\approx && (15.72) \\ \zeta &\approx 0.126 && (15.73) \end{aligned}$$

  - Magnitude at ω_n

$$\begin{aligned} \frac{M(\omega_n)}{K_0} &= \frac{1}{2\zeta} && (15.74) \\ \frac{8}{2} &= && (15.75) \\ \zeta &= \frac{1}{8} && (15.76) \end{aligned}$$

  - Half power
    * Peak $M(\omega_p) \approx 8$
    * Half power

$$\begin{aligned} \frac{M(\omega_p)}{\sqrt{2}} &\approx \frac{8}{\sqrt{2}} && (15.77) \\ &\approx 5.66 && (15.78) \\ M(\{\omega_l, \omega_u\}) &\approx && (15.79) \\ M(\{3.3, 4.5\}) &\approx && (15.80) \\ \zeta &\approx \frac{\omega_u - \omega_l}{2\omega_p}, \text{ for } \zeta < 0.05 && (15.81) \\ &\approx \frac{4.5 - 3.3}{2 \cdot 4} && (15.82) \\ &\approx 0.15 && (15.83) \end{aligned}$$

* Slope of phase

$$\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_n} \approx \frac{-\pi-0}{4.75-3.25} \tag{15.84}$$

$$\zeta = -\frac{1}{\omega_n}\left(\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_n}\right)^{-1} \tag{15.85}$$

$$\approx -\frac{1}{4}\left(\frac{-\pi-0}{4.75-3.25}\right)^{-1} \tag{15.86}$$

$$\approx 0.119 \tag{15.87}$$

- Write transfer function

$$G(s) = K\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{15.88}$$

$$= 2\frac{4^2}{s^2+2\cdot\frac{1}{8}\cdot 4s+4^2} \tag{15.89}$$

$$= \frac{32}{s^2+s+16} \tag{15.90}$$

Example 15.2:

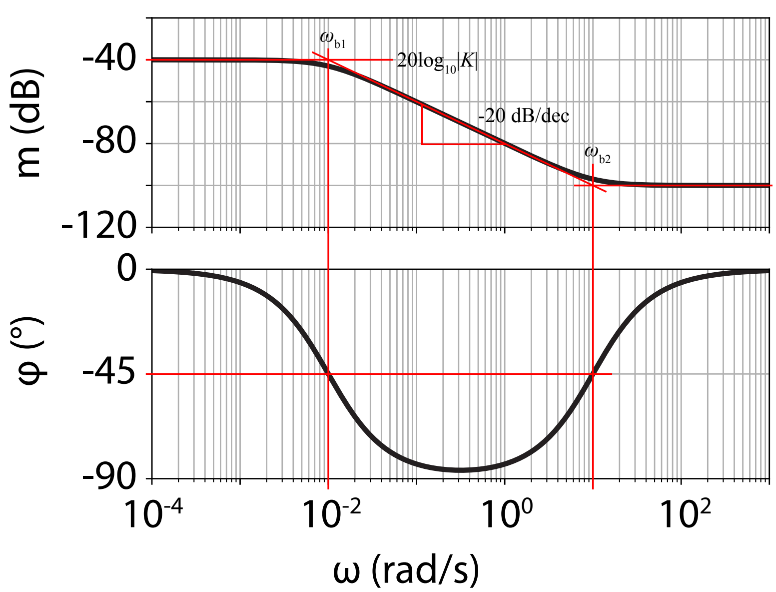


Figure 15.4:

Find a transfer function representation of the system with the above frequency response plots.

- Identify all s^n factors
  - The low frequency slope of the magnitude ratio is 0 suggesting that $n = 0$.
  - At low frequencies, the phase angle is 0°, suggesting that $n = 0$ (assuming the system is minimum phase).
- Estimate K
  - $\lim_{\omega \to 0} m(\omega) = -40$ dB suggesting $|K| = 10^{\frac{-40}{20}} = \frac{1}{100}$
  - $\phi(0) = 0$ suggesting $0 < K = \frac{1}{100}$
- Determine the difference between the number of poles and zeros
  - The high frequency slope is zero sugesting an equal number of poles and zeros.
  - The high frequency phase is 0° (which matches the low frequency limit) suggesting that the system has an equal number of poles and zeros.
- Estimate all break frequencies
  - A straight line approximation of $m(\omega)$ yields three segments and two first order break frequencies: $\omega_{b1} = 10^{-2}$ and $\omega_{b2} = 10^1$ rad/s. We know the factors are first order because the break frequencies correspond to a change in slope by 20 dB/decade.
  - The halfway points between 0 to -90° likely corresponds to two break frequencies: $\omega_{b1} = 10^{-2}$ and $\omega_{b2} = 10^1$ rad/s. We know the factors are first order because the break frequencies correspond to a change in phase by 90°.
  - First order factors
    * $\tau_1 = \frac{1}{\omega_{b1}} = \frac{1}{10^{-2}} = 100$

$$G_1(s) = \frac{1}{100s + 1} \tag{15.91}$$

    * $\tau_2 = \frac{1}{\omega_{b2}} = \frac{1}{10}$

$$G_2(s) = \frac{1}{10}s + 1 \tag{15.92}$$

- Estimate damping ratios - NA

- Write transfer function

$$G(s) = K\left(\frac{1}{\tau_1 s+1}\right)(\tau_2 s+1) \tag{15.93}$$

$$= \frac{1}{100}\left(\frac{1}{100s+1}\right)\left(\frac{1}{10}s+1\right) \tag{15.94}$$

Example 15.3:

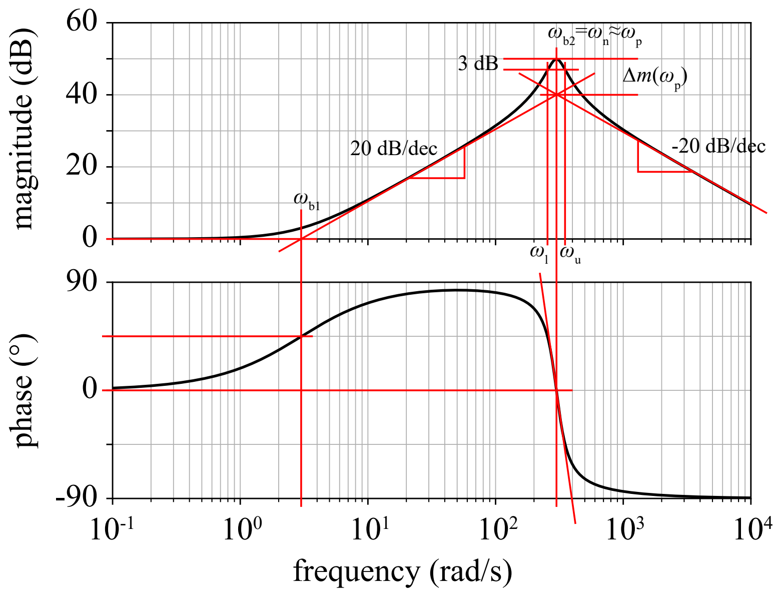


Figure 15.5:

Find a transfer function representation of the system with the above frequency response plots.

- Identify all s^n factors
  - The low frequency slope of the magnitude ratio is 0 suggesting that $n = 0$.
  - At low frequencies, the phase angle is 0°, suggesting that $n = 0$ (assuming the system is minimum phase).
- Estimate K
  - $\lim_{\omega\to 0} m(\omega) = 0$ dB suggesting $|K| = 10^{\frac{0}{20}} = 1$
  - $\phi(0) = 0$ suggesting $0 < K = 1$

- Determine the difference between the number of poles and zeros
  - The high frequency slope is -20 dB/decade sugesting one more pole than zero.
  - The high frequency phase is -90° suggesting that the system has one more pole than zero.
- Estimate all break frequencies
  - A straight line approximation of $m(\omega)$ yields three segments and two break frequencies: $\omega_{b1} = 3$ and $\omega_{b2} = 300$ rad/s. The first break frequency corresponds an increase in slope by 20 dB/decade suggesting a first order factor in the numerator. The second break frequency corresponds to a decrease in slope by 40 dB/decade suggesting an second order factor in the denominator. The peak also suggest a break frequency near 300 rad/s and an underdamped second order factor in the denominator.
  - The halfway point between 0 to 90° likely corresponds to a break frequency of $\omega_{b1} = 3$ and and the halfway point between 90 and -90° corresponds to a break frequency of $\omega_{b2} = 300$ rad/s.
  - First order factor: $\tau_1 = \frac{1}{\omega_{b1}} = \frac{1}{3}$

$$G_1(s) = \frac{1}{\frac{1}{3}s + 1} \tag{15.95}$$

  - Underdamped second order factor: $\omega_n = \omega_{b2} = 300$
- Estimate damping ratio
  - Magnitude at peak
    * Define $\Delta m(\omega_p) = 10$ dB ($\Delta M(\omega_p) = 10^{\frac{10}{20}} = 3.16$) as the height of the peak if there were no other factors in the transfer function. Therefore

$$\Delta M(\omega_p) \rightarrow \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{15.96}$$

$$3.16 = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{15.97}$$

$$\zeta \approx 0.160 \tag{15.98}$$

  - Magnitude at ω_n

* Define $\Delta m(\omega_n) \approx 10$ dB ($\Delta M(\omega_n) \approx 10^{\frac{10}{20}} = 3.16$) as the magnitude if there were no other factors in the transfer function. Therefore

$$\begin{aligned} \Delta M(\omega_n) &= \frac{1}{2\zeta} && (15.99) \\ 3.16 &\approx && (15.100) \\ \zeta &= 0.158 && (15.101) \end{aligned}$$

- Half power
  * Peak $m(\omega_p) = 50$ dB
  * Half power

$$\begin{aligned} m(\omega_p) - 3 &= 50 - 3 && (15.102) \\ &= 47 && (15.103) \\ m(\{\omega_l, \omega_u\}) &\approx && (15.104) \\ m(\{240, 330\}) &\approx && (15.105) \\ \zeta &\approx \frac{\omega_u - \omega_l}{2\omega_p}, \text{ for } \zeta < 0.05 && (15.106) \\ &\approx \frac{330 - 240}{2 \cdot 300} && (15.107) \\ &\approx 0.15 && (15.108) \end{aligned}$$

  Technically, this value violates the $\zeta < 0.05$ we suggest is best for the half power method.

- Asymptotic (straight line) approximation of phase

$$\begin{aligned} \omega_n \left(\frac{1}{5}\right)^{\zeta} &\approx 230 && (15.109) \\ 300 \left(\frac{1}{5}\right)^{\zeta} &\approx 230 && (15.110) \\ \zeta &\approx \frac{\log_{10}\left[\frac{23}{30}\right]}{-\log_{10}[5]} && (15.111) \\ &\approx 0.165 && (15.112) \\ \omega_n 5^{\zeta} &\approx 400 && (15.113) \\ 300 \cdot 5^{\zeta} &\approx 400 && (15.114) \\ \zeta &\approx \frac{\log_{10}\left[\frac{4}{3}\right]}{\log_{10}[5]} && (15.115) \\ &\approx 0.179 && (15.116) \end{aligned}$$

- Write transfer function

$$G(s) = K(\tau s+1)\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad (15.117)$$

$$\approx 1\frac{1}{\frac{1}{3}s+1}\frac{300^2}{s^2+2\cdot 0.16\cdot 300s+300^2} \quad (15.118)$$

Example 15.4:

Add an example with a nontrivial s^n factor.

15.4 Comparison of Time and Frequency Domain System Identification

- See, for example, Web Appendix W3.7 of [1] available at https://www.scsolutions.com/publication/feedb control-of-dynamic-systems-eighth-edition/
- "Often a sinusoidal input is easier to apply to a system than a step input ...[35]"
- "Another advantage of frequency response tests is that they can often be applied to a system or process without interrupting its normal operations [35]."
- A step response can be obtained experimentally from a single step input. Constructing a bode plot experimentally may necessitate a slow process of subjecting the system to a sinusoidal input one frequency at a time. (Note, superposition may facilitate subjecting the system to multiple frequencies simultaneously.)

Part VI

Physical Systems Modeling

Chapter 16

System Modeling Principles

Students will not be tested on most of the material presented in this chapter.

16.1 System Modeling Principles

- Models should match the questions under investigation
  - Engineers use models to answer specific questions like:
    - Will it fail?
    - How big should it be?
    - Is it stable?
  - Engineers use models to facilitate design
    - difficulty in coming up with a solution to a problem without models for how the world works??
    - digital twin
  - To be of use, a model must be capable of answering the relevant questions
  - Before developing a model, engineers should identify the questions that the model must answer.
  - Engineers should know the limitations of the models they use (all models have limitations) to know if the model applies to the relevant questions.
- Simple models are preferred
  - Potential disadvantages of detailed models
    - Time intensive
      - Time required to develop the model (i.e., derive equations, write code, setup software model, etc.)

- CPU time required to run the model
- ∗ Expensive
  - · Software license fees
  - · Access fees for high performance computing resources
  - · Time
- ∗ Number and accuracy of required parameters
  - · Detailed models generally require more model parameters and more accurate model parameters.
  - · The quality of the output of a model is generally no better than that of its inputs. A common saying is: "garbage in, garbage out."
  - · Obtaining accurate values for model parameters can be difficult, time consuming, and/or expensive.
- ∗ Complexity confounds relationships between inputs and outputs
- ∗ More ways in which the model can fail
- ∗ Overconfidence - Given all of the hard work put into a detailed model and all of the details represented in a model, engineers may become overconfident in the capabilities of a model and overlook model limitations.

– Potential advantages of simple models
  - ∗ Quicker, easier, and/or cheaper to develop and implement
  - ∗ Faster to execute or run the model
  - ∗ Fewer parameter values are needed.
  - ∗ Yield insightful mathematical relationships and/or strengthened intuition about a system.
  - ∗ Easier to keep model capabilities/limitations in perspective and avoid overconfidence.

– Generally, the simplest possible model capable of answering relevant questions is preferred.

– Warning: It can be tempting to use the wrong or unnecessarily complicated models.
  - ∗ CFD and FEA software output graphics with a pretty rainbow of colors.
  - ∗ Favorite and/or familiar modeling approaches aren't the best fit for all questions.
    - · "Give a boy a hammer and chisel; show him how to use them; at once he begins to hack the door posts, to take off the coners of the shutter and window frames, until you teach him a better use for them, and how to keep his activity within bounds []." write complete citation [`https://books.google.com/books?id=_leI8HnOA-8C&q=%22to+hack%22#v=snippet&q=%22hammer%22chisel&f=false`]
    - · (Using favorite and familiar modeling approaches may be better than a simpler modeling approach if the simpler approach would require a significant amount of time to learn.)

- Use multiple models
  - Applying distinct and independent models to a single question builds confidence and insight into the answers they provide (and systems they represent).
  - As the processes of system design and analysis progress, questions change and models may need to be refined or replaced.
  - System design and analysis involve many questions and often many models are needed.
- System modeling is an art
  - It requires practice/experience
  - It's not easy to learn/teach in a classroom or with a book

16.2 Methods to Develop Simple Models

- Basis
  - Theoretical model
  - Empirical
    * Empirical - "... based on, concerned with, or verifiable by observation or experience rather than theory or pure logic ..." [The Oxford Pocket Dictionary of Current English. Encyclopedia.com. 21 Oct. 2022 https://www.encyclopedia.com.]
  - Combination of theory and experiment
    * System identification
- Dimensional analysis
  - Crude but powerful models can be obtained by simply investigating the dimensions and values of system parameters
  - Buckingham Pi (Add a brief introduction to the approach.)
  - Examples:
    * The natural frequency of a pendulum can be obtained directly from system parameters.
- Approximate as a model system
  - The following thoughts on model systems are incomplete.
  - Model organisms
    * "A species that has been studied intensively over a long period and thus serves as a "model" for deriving fundamental biological principles [53]."

    * "A model organism (often shortened to model) is a non-human species that is extensively studied to understand particular biological phenomena, with the expectation that discoveries made in the model organism will provide insight into the workings of other organisms [Wikipedia. wikipedia.org. 13 June 2022 `https://en.wikipedia.org/`]."

  - Model system
    * We propose the term 'model system' to refer to simple systems that have been the focus of intensive study and exhibit behaviours representatitive of a wide range of engineering systems.
    * Model systems are among the simplest systems exhibiting specific, and widespread, physical behaviors.
    * Model systems often result when an engineer makes major simplifying assumptions to reduce a complex system into the simplest (crudest) model that retains the dominant physical phenomenon.
  - Example model systems
    * mass-spring-damper
    * cantilever beam
    * pendulum
    * vibration of a taut string
    * etc
  - Use of model systems
    * Many complex systems may be approximated by one or more model systems.
    * Studying model systems builds our ability to comprehend and predict the behaviors of complex systems.

- Reduced order models
  - Model reduction: One very simple approach to eliminating extraneous details of an existing model is to neglect poles with relatively short time constants (non dominant poles).
    * Dominant poles -
      · For a pole to be considered a dominant pole, it's contribution to the response must be significant.
      · Often, dominant poles have longer time constants such that their contribution to the response is significant for a relatively long period of time.
      · To be a dominant pole, its residue (i.e., partial fraction expansion coefficient) must be significant.

        * Poles that are sufficiently far to the left of the dominant poles (and therefore have higher break frequencies) can often be neglected (assuming their residue is sufficiently small) as their contribution to the response dies out relatively quickly.
            · Factor of 4-5 - I believe [39] suggests that a factor of 5 - let's go with a factor of 10
            · Some authors refer to this as the "dominant pole approximation" [`https://lpsa.swarthmore.edu/PZXferStepBode/DomPole.html`] find better sources?
        * Warning, [37] suggests that the rightmost poles only dominant then their residuals are relatively large. Bavafa-Toosi suggests a better approach is to use the Henkel Singular Value Decomposition to obtain a model reduction. MATLAB® implements this approach using the `balred()` command.
        * How to drop nondominant poles while maintaining static sensitivity
    – Specific reduced order approaches
        * Lumped parameter models
            · Lumped parameter models generally yield systems of ordinary differential equations and distributed parameter models generally yield systems of partial differential equations
            · Linear rather than nonlinear
            · Mathematically, these models are represented by linear time invariant ordinary differential equations.
        * Finite element, assumed mode, ...
        * Neglect higher order modes (considering a modal analysis)

- Other
- Examples (case studies)
    – Pseudo rigid body model
    – Electric solar wind sail dynamics
    – DNA dynamics (relaxation time scale,)
    – Assumed mode method of cantilever
    – revisit cantilever beam (with base excitation example)
        * lumped parameter model
        * assumed mode model
        * (pseudo rigid body model for large deflections)

Add discussion on the assembly of systems from subsystems

Example 16.1: Natural frequency of a cantilever beam

Estimate the lowest natural frequency of a cantilever beam of length L, mass per unit length ρ_L, modulus of elasticity E, and area moment of inertia I.

- Exact value for comparison (work not shown)

$$\omega_n = \sqrt{12.73\frac{EI}{\rho_L L^4}} \tag{16.1}$$

- Dimensional analysis
  - Dimensions

$$L = \text{Length} \tag{16.2}$$

$$\rho_L = \frac{\text{Mass}}{\text{Length}} \tag{16.3}$$

$$E = \frac{\text{Mass}}{\text{Time}^2 \cdot \text{Length}} \tag{16.4}$$

$$E = \text{Length}^4 \tag{16.5}$$

  - Possible combination (independent of L and I)

$$\omega_n \approx \sqrt{\frac{E}{\rho_L}} \tag{16.6}$$

  - Better combination (includes L and I)

$$\omega_n \approx \sqrt{\frac{EI}{\rho_L L^4}} \tag{16.7}$$

  - This approximation has the correct relationship but not the correct constant.

- Lumped model
  - Lumped stiffness: stiffness to deflections of the end (from mechanics of materials)

$$\delta = \frac{fL^3}{3EI} \tag{16.8}$$

$$k = \frac{f}{\delta} \tag{16.9}$$

$$= \frac{3EI}{L^3} \tag{16.10}$$

- Lumped mass:
  - The cantilever has little deflection near the wall and maximum deflection at the free end.
  - The equivalent mass of the cantilever (at the end) is some fraction of the total mass.
  - Decent estimates could be made with anything less than about half the mass of the beam. A quarter of the total mass works well.

$$m = \frac{1}{4}\rho_L L \tag{16.11}$$

- Estimate of the natural frequency

$$\begin{aligned} \omega_n &\approx \sqrt{\frac{k}{m}} && (16.12)\\ &\approx \sqrt{\frac{\frac{3EI}{L^3}}{\frac{1}{4}\rho_L L}} && (16.13)\\ &\approx \sqrt{12\frac{EI}{\rho_L L^4}} && (16.14) \end{aligned}$$

- This is an excellent approximation (in part because we used a good guess for the equivalent mass).

• Assumed mode method

- Assume the deflection of the cantilever beam may be written as the product of a function of time t and a function of spatial position x

$$y(t,x) = q(t)\phi(x) \tag{16.15}$$

- Assume a shape for the function $\phi(x)$. A good choice for $\phi(x)$ comes from the deflection given an end load

$$\begin{aligned} v(x) &= \frac{f}{EI}\left(\frac{1}{6}x^3 - \frac{1}{2}x^2\right) && (16.16)\\ &= \frac{f}{6EI}\left(x^3 - 3x^2\right) && (16.17)\\ \phi(x) &= \frac{1}{L^3}\left(x^3 - 3Lx^2\right) && (16.18) \end{aligned}$$

  Different choices could be used for $\phi(x)$. $\phi(x)$ must satisfy the geometric boundary boundary conditions, $\phi(0) = 0$ and $\phi'(x) = 0$.

- Kinetic energy

$$
\begin{aligned}
T &= \frac{1}{2}\int_0^L \rho_L \dot{y}^2(t,x)dx && (16.19)\\
&= \frac{1}{2}\int_0^L \rho_L \left(\dot{q}(t)\phi(x)\right)^2 dx && (16.20)\\
&= \frac{1}{2}\rho_L \dot{q}^2(t)\int_0^L \phi^2(x)dx && (16.21)\\
&= \frac{1}{2}\rho_L \dot{q}^2(t)\int_0^L \left(\frac{1}{L^3}\left(x^3-3Lx^2\right)\right)^2 dx && (16.22)\\
&= \frac{33}{70}\rho_L L\dot{q}^2(t) && (16.23)
\end{aligned}
$$

- Potential energy

$$
\begin{aligned}
V &= \frac{1}{2}\int_0^L EI\left(y''(t,x)\right)^2 dx && (16.24)\\
&= \frac{1}{2}\int_0^L EI\left(q(t)\phi''(x)\right)^2 dx && (16.25)\\
&= \frac{1}{2}EIq^2(t)\int_0^L \left(\phi''(x)\right)^2 dx && (16.26)\\
&= \frac{1}{2}EIq^2(t)\int_0^L \left(\frac{6}{L^3}(x-L)\right)^2 dx && (16.27)\\
&= 6\frac{EI}{L^3}q^2(t) && (16.28)
\end{aligned}
$$

- Conservation of energy

$$
\begin{aligned}
T+V &= \text{constant} && (16.29)\\
\dot{T}+\dot{V} &= && (16.30)\\
\cancel{2}\cdot\frac{33}{70}\rho_L L\cancel{\dot{q}(t)}\ddot{q}(t)+\cancel{2}\cdot 6\frac{EI}{L^3}q(t)\cancel{\dot{q}(t)} &= 0 && (16.31)\\
\frac{33}{70}\rho_L L\ddot{q}(t)+6\frac{EI}{L^3}q(t) &= && (16.32)
\end{aligned}
$$

– Estimate of natural frequency

$$\omega_n \approx \sqrt{\frac{6\frac{EI}{L^3}}{\frac{33}{70}\rho_L L}} \tag{16.33}$$

$$\approx \sqrt{\frac{140}{11}\frac{EI}{\rho_L L^4}} \tag{16.34}$$

$$\approx \sqrt{12.73\frac{EI}{\rho_L L^4}} \tag{16.35}$$

– This is an excellent approximation to the natural frequency of the system. (Note, this doesn't mean that the first mode of vibration matches the deflection for an end load.) In addition, this model could be used to represent the deflection as a function of position along the full length of the beam.

Example 16.2: Reduced order model

Consider a system with transfer function ...

$$G(s) = \frac{25}{(s+1)(s+3+4j)(s+3-4j)} \tag{16.36}$$

Chapter 17

Electrical Systems

17.1 Circuit Fundamentals

- Circuit terminology
  - Node
    - “A node consists of one or more ideal wires connected together such that an electric charge can travel between any two points on the node without traversing a circuit element... [54].”
    - “A node is simply a point of connection of two or more circuit elements [55].”
    - Voltage is uniform throughout an entire node
  - Supernode - “... closed boundary enclosing two or more nodes ... [54]”
  - Branch - “A branch is a single electrical pathway, consisting of wires and elements [54].”
  - Loop - “A loop is any closed connection of branches [18].”
  - Mesh
    - “A mesh is a closed electrical pathway that does not contain other closed physical pathways [54].”
  - Source vs. load (and sign convention)
- Modeling Circuits
  - Kirchhoff’s current law - The sum of signed currents into a node is zero. (conservation of charge)
  - Kirchhoff’s voltage law - The sum of the signed voltage differences around a circuit is zero. (conservation of energy)

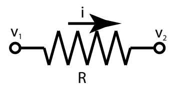
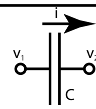
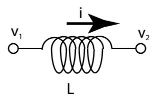
Element	Symbol	Current-voltage Relationship	Impedance $Z(s) = \frac{V(s)}{I(s)}$	Energy/Power
Resistor		$v_1 - v_2 = iR$	$Z(s) = R$	$P = Ri^2 = \frac{(v_1 - v_2)^2}{R}$
Capacitor		$C\frac{d(v_1 - v_2)}{dt} = i$	$Z(s) = \frac{1}{Cs}$	$E = \frac{1}{2}C(v_1 - v_2)^2$
Inductor		$v_1 - v_2 = L\frac{di}{dt}$	$Z(s) = Ls$	$E = \frac{1}{2}Li^2$

Figure 17.1: Electrical elements

17.2 Current-voltage Relationships

- Electrical Elements
  - Quantities
    - Voltage: volt (V)
    - Charge: coulomb (C=Nm/V)
    - Current: ampere (A=C/s)
  - Passive electrical elements
    - Resistor
      - Current-voltage relationship

$$v_1 - v_2 = iR \tag{17.1}$$

      - Transfer function (Impedance)

$$Z(s) = \frac{V(s)}{I(s)} = R \tag{17.2}$$

      - Power dissipation

$$P = Ri^2 = \frac{(v_1 - v_2)^2}{R} \tag{17.3}$$

    - Capacitor - integral vs. differential form
      - Current-voltage relationships

$$v_1 - v_2 = \frac{1}{C}\int_0^t idt + \frac{Q_0}{C} \tag{17.4}$$

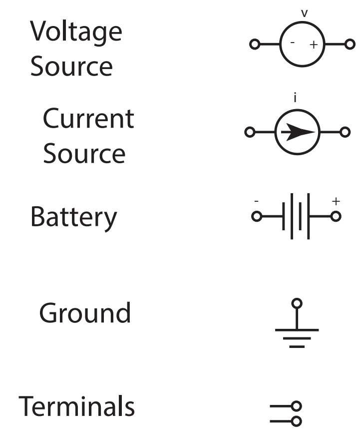


Figure 17.2:

$$i = C\frac{d(v_1 - v_2)}{dt} \tag{17.5}$$

· Transfer function (Impedance)

$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{Cs} \tag{17.6}$$

· Energy storage - stored by charge

$$E = \frac{1}{2}C(v_1 - v_2)^2 \tag{17.7}$$

∗ Inductor

· Current-voltage relationships

$$v_1 - v_2 = L\frac{di}{dt} \tag{17.8}$$

· Transfer function (Impedance)

$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{Cs} \tag{17.9}$$

· Energy storage - stored by magnetic field

$$E = \frac{1}{2}Li^2 \tag{17.10}$$

– Other common electrical symbols

* Voltage source
* Current source
* Battery
* Ground
* Terminals

- Equivalent impedance
  - Series impedance (resistance)
    - Same current through elements
    - Equivalent impedance (resistance)

$$Z(s) = Z_1(s) + Z_2(s) \tag{17.11}$$

  - Parallel impedance (resistance)
    - Same voltage across elements
    - Equivalent impedance (resistance)

$$\frac{1}{Z(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} \tag{17.12}$$

17.3 Methods to Derive Governing Equations

17.3.1 (Dr. Lillian's) Default Method

- Approach
  - Label voltage at each node on the circuit
  - Label current through each branch of circuit
  - Write expressions for Kirchhoff's current law at each independent node
  - Write current-voltage equations for each electrical element
  - Substitution
- Examples

Example 17.1:

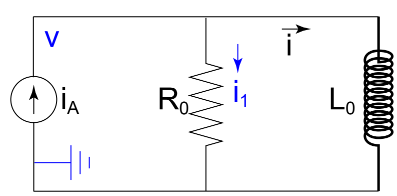


Figure 17.3: A known input current i_A is supplied to an RL circuit.

Derive the governing equation for the system in **Figure 17.3** in terms of the unknown current i.

- Label voltages: ground, v
- Label currents: i_A, i, i_1
- Kirchhoff's current law

$$i_A = i_1 + i \tag{17.13}$$

- Current-voltage relationships

Inductor

$$L_0 \frac{di}{dt} = v - 0 \tag{17.14}$$

Resistor

$$v = i_1 R_0 \tag{17.15}$$

- Substitution

Substitute Equation 17.15 into Equation 17.14

$$L_0 \frac{di}{dt} = i_1 R_0 \tag{17.16}$$

Substitute Equation 17.13 into Equation 17.16 and simplify

$$L_0 \frac{di}{dt} = (i_A - i) R_0 \tag{17.17}$$

$$L_0 \frac{di}{dt} + R_0 i = R_0 i_A \tag{17.18}$$

Example 17.2:

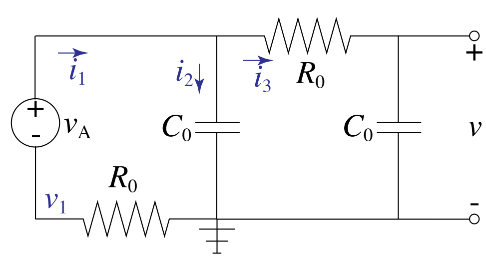


Figure 17.4:
Find governing equation for the output voltage v.

- voltage: 0, v_1, $v_1 + v_A$, v
- current: i_1, i_2, i_3
- KCL

$$i_1 - i_2 - i_3 = 0 \tag{17.19}$$

- i-v relationships

$$\begin{aligned} 0 - v_1 &= i_1 R_0 && (17.20) \\ v_1 + v_A - v &= i_3 R_0 && (17.21) \\ C_0 \frac{d(v_1 + v_A)}{dt} &= i_2 && (17.22) \\ C_0 \frac{dv}{dt} &= i_3 && (17.23) \end{aligned}$$

- substitution (need to add comments)

$$\begin{aligned} v_1 &= v - v_A + R_0 C_0 \frac{dv}{dt} && (17.24) \\ 0 &= -\frac{v - v_A + R_0 C_0 \frac{dv}{dt}}{R_0} - C_0 \frac{d(v - \cancel{v_A} + R_0 C_0 \frac{dv}{dt} + \cancel{v_A})}{dt} - C_0 \frac{dv}{dt} && (17.25) \\ &= -v + v_A - R_0^2 C_0^2 \frac{d^2 v}{dt^2} - 3 R_0 C_0 \frac{dv}{dt} && (17.26) \\ v_A &= R_0^2 C_0^2 \frac{d^2 v}{dt^2} + 3 R_0 C_0 \frac{dv}{dt} + v && (17.27) \end{aligned}$$

17.3.2 Impedance Network Methods

- (Dr. Lillian's) Default method for deriving governing equations
  - Label voltage at each node on the circuit
  - Label current through each branch of circuit
  - Write expressions for Kirchhoff's current law at each independent node
  - Write current-voltage equations for each electrical element
  - Substitution
- Node voltage method [54]
  - Label voltage at each independent node
    - Identify a reference (often ground) node - it can be advantageous for the reference node to be one with many elements connected and/or connected to a source
    - At voltage sources write one of the two voltages in terms of the other
  - Write Kirchhoff's Current Law (KCL) in terms of independent nodal voltages at each independent node.
    - No need to write KCL at dependent nodes (e.g., reference or ground node, or both nodes of a voltage source connected to ground)
    - Use a supernode when a voltage source is not connected to ground/reference
    - Use impedance relationships for each element (e.g., Ohm's law for a resistor, voltage change for voltage source, etc.) to express currents in terms of independent nodal voltages
  - Substitution
- Mesh current method [54]
  - Label each independent mesh (or loop or we might call it a supermesh, but it's unclear how supermeshes are defined in the literature) currents
    - When a current source is shared with two meshes, consider replacing one of the meshes with a loop that combines the two.
  - Write Kirchhoff's Voltage Law (KVL) in terms of independent mesh (or loop) currents.
    - No need to write KVL for dependent meshes (e.g., mesh in which current source appears exclusively)
    - Use voltage-current relationships (e.g., Ohm's law) to express voltages in terms of independent mesh currents
  - Substitution (should have as many equations as unknowns)

    * If you are trying to find a voltage, use voltage-current relationships (e.g., Ohm's law) to express voltages in terms of mesh currents

- Circuit simplification
  - Series impedance
  - Parallel impedance
  - Voltage divider
  - Current divider
  - Norton and Thévenin equivalencies
  - Superposition
    * Zeroing out a voltage source yields a short circuit
    * Zeroing out a current source yields an open circuit

- Impedance network examples

Example 17.3: Voltage divider - default method

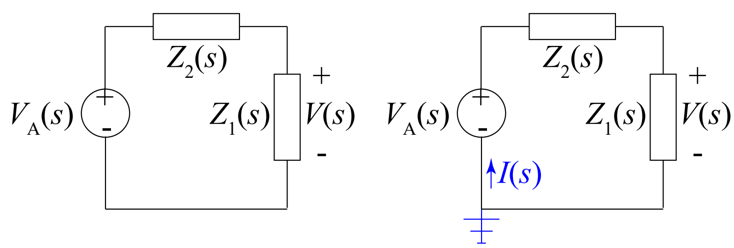


Figure 17.5:

Problem: Given $Z_1(s)$, $Z_2(s)$ and $V_A(s)$ in the circuit shown in Figure 17.5(left), find an expression for the voltage $V(s)$.

- Voltages: ground (other voltages are already labeled)
- Currents: $I(s)$
- KCL: In this case KCL is trivial at all nodes ($I(s) = I(s)$)
- $I(s)$-$V(s)$:
  - Impedance relationship for $Z_1(s)$

$$V(s) = I(s)Z_1(s) \tag{17.28}$$

- Impedance relationship for $Z_1(s)$

$$V_A(s) - V(s) = I(s)Z_2(s) \tag{17.29}$$

- Substitution
  - Isolate $I(s)$ in Eq. 17.28 and substitute into Eq. 17.29

$$\begin{aligned} V_A(s) - V(s) &= \frac{V(s)}{Z_1(s)}Z_2(s) && (17.30)\\ \left(1 + \frac{Z_2(s)}{Z_1(s)}\right)V(s) &= V_A(s) && (17.31)\\ V(s) &= \frac{Z_1(s)}{Z_1(s)+Z_2(s)}V_A(s) && (17.32) \end{aligned}$$

Example 17.4: Voltage divider - node voltage method

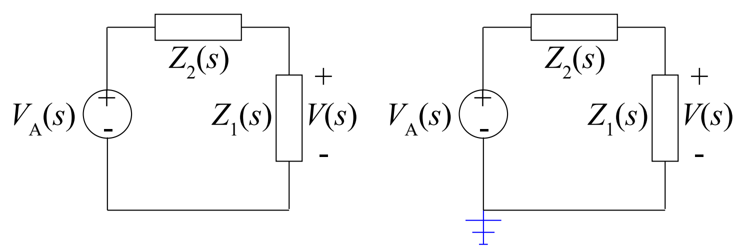


Figure 17.6:

Problem: Given $Z_1(s)$, $Z_2(s)$ and $V_A(s)$ in the circuit shown in Figure 17.6(left), find an expression for the voltage $V(s)$.

- Voltages: ground (the other independent voltage is already labeled $V(s)$)
- KCL:
  - Node with voltage $V(s)$ (this is the only independent node)

$$\frac{V_A(s) - V(s)}{Z_2(s)} = \frac{V(s)}{Z_1(s)} \tag{17.33}$$

- Substitution

– Isolate v in Eq. 17.33

$$V_A(s) - V(s) = \frac{V(s)}{Z_1(s)} Z_2(s) \tag{17.34}$$

$$\left(1 + \frac{Z_2(s)}{Z_1(s)}\right) V(s) = V_A(s) \tag{17.35}$$

$$V(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_A(s) \tag{17.36}$$

Example 17.5:

Solve the voltage divider problem again with the ground between the two resistors to demonstrate how to deal with voltage source not attached to ground and using a supernode rather than just nodes.

Example 17.6: Voltage divider - mesh current method

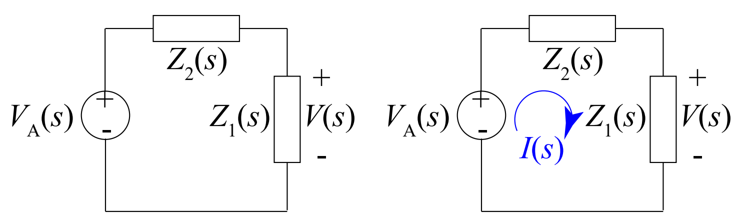


Figure 17.7:

Problem: Given $Z_1(s)$, $Z_2(s)$ and $V_A(s)$ in the circuit shown in Figure 17.5(left), find an expression for the voltage $V(s)$.

- Mesh currents: $I(s)$
- KVL in the direction of the mesh

$$V_A(s) - I(s)Z_2(s) - I(s)Z_1(s) = 0 \tag{17.37}$$

- Substitution
  – Isolate $I(s)$ in Eq. 17.37

$$V_A(s) - I(s)Z_2(s) - I(s)Z_1(s) = 0 \tag{17.38}$$

$$I(s) = \frac{V_A(s)}{Z_1(s) + Z_2(s)} \tag{17.39}$$

– Apply impedance relatiohship to find an expression for $V(s)$

$$V(s) = I(s)Z_1(s) \tag{17.40}$$
$$= \frac{Z_1(s)}{Z_1(s)+Z_2(s)}V_A(s) \tag{17.41}$$

Example 17.7: Current divider - default method

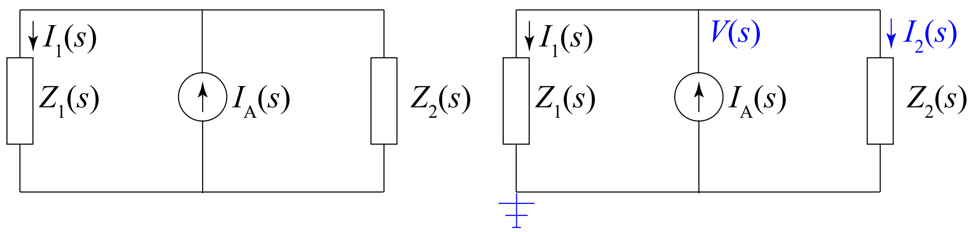


Figure 17.8:

Problem: Given $Z_1(s)$, $Z_2(s)$ and $I_A(s)$ in the circuit shown in Figure 17.8(left), find an expression for the current $I_1(s)$.

- Voltages: ground, v
- Currents: $I_2(s)$ ($I_A(s)$ and $I_1(s)$ are already labeled)
- KCL:
  - At node with voltage $V(s)$

$$I_A(s) = I_1(s) + I_2(s) \tag{17.42}$$

- i-v:
  - Impedance relationship for $Z_1(s)$

$$V(s) = I_1(s)Z_1(s) \tag{17.43}$$

  - Impedance relationship for $Z_2(s)$

$$V(s) = I_2(s)Z_2(s) \tag{17.44}$$

- Substitution

– Isolate $I_2(s)$ in Eq. 17.44 and substitute into Eq. 17.42

$$I_A(s) = I_1(s) + \frac{V(s)}{Z_2(s)} \tag{17.45}$$

– Now substitute Eq. 17.43 into Eq. 17.45

$$I_A(s) = I_1(s) + \frac{I_1(s)Z_1(s)}{Z_2(s)} \tag{17.46}$$

$$I_1(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} I_A(s) \tag{17.47}$$

Example 17.8: Current divider - node voltage method

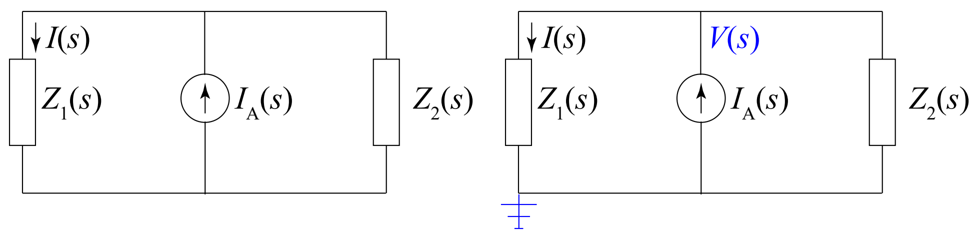


Figure 17.9:

Given $Z_1(s)$, $Z_2(s)$ and $I_A(s)$ in the circuit shown in Figure 17.9(left), find an expression for the current $I_1(s)$.

- Voltages: ground, $V(s)$
- KCL:
  – Node with voltage $V(s)$ (this is the only independent node)

$$I_A(s) = \frac{V(s)}{Z_1(s)} + \frac{V(s)}{Z_2(s)} \tag{17.48}$$

- Substitution
  – Isolate $V(s)$ in Eq. 17.48

$$V(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)} I_A(s) \tag{17.49}$$

– Since the question asks us to find $I_1(s)$, we can write the following using the impedance relationship

$$Z_1(s)I_1(s) = V(s) \tag{17.50}$$
$$= \frac{Z_1(s)Z_2(s)}{Z_1(s)+Z_2(s)} I_A(s) \tag{17.51}$$
$$I_1(s) = \frac{Z_2(s)}{Z_1(s)+Z_2(s)} I_A(s) \tag{17.52}$$

Example 17.9: Current divider - mesh current method

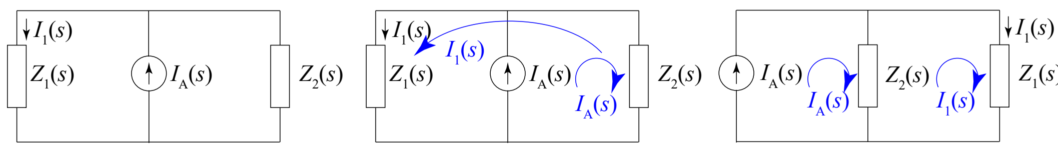


Figure 17.10:

Problem: Given $Z_1(s)$, $Z_2(s)$ and $I_A(s)$ in the circuit shown in Figure 17.8(left), find an expression for the current $I_1(s)$.

- Mesh currents: $I_A(s)$ and $I_1(s)$ (Note, it's challenging when two mesh currents pass through a single current source. For this reason, the loop current $I_1(s)$ is defined as around the perimeter of the circuit or the circuit is rearranged so that the current source is included in only one mesh.)

- KVL

  – Loop with current $I_1(s)$ in direction of loop

$$-I_1(s)Z_1(s) + (I_A(s) - I_1(s))Z_2(s) = 0 \tag{17.53}$$

- Substitution

  – Isolate $I_1(s)$ in Eq. 17.53

$$I_1(s) = \frac{Z_2(s)}{Z_1(s)+Z_2(s)} I_A(s) \tag{17.54}$$

Example 17.10: - default method

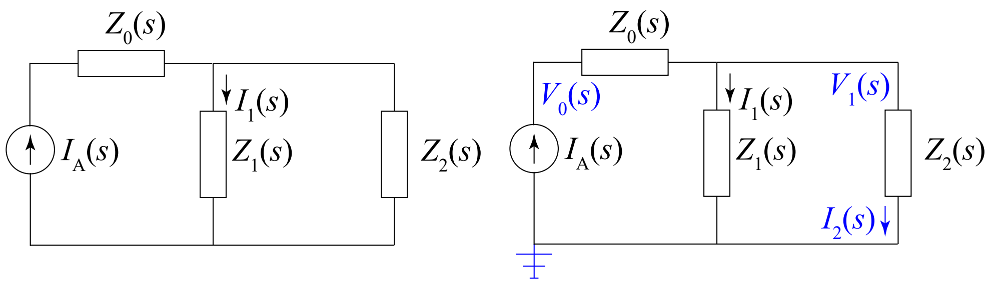


Figure 17.11:

Problem: Given $Z_0(s)$, $Z_1(s)$, $Z_2(s)$ and $I_A(s)$ in the circuit shown in Figure 17.11(left), find an expression for the current $I_1(s)$.

- Voltages: ground, $V_0(s)$, and $V_1(s)$
- Currents: $I_2(s)$ ($I_A(s)$ and $I_1(s)$ are already labeled)
- KCL:
  - the node with voltage v_0 is trivial
  - At node with voltage v_1
$$I_A(s) = I_1(s) + I_2(s) \tag{17.55}$$
- i-v:
  - Impedance relationship for $Z_0(s)$
$$V_0(s) - V_1(s) = I_A(s)Z_0(s) \tag{17.56}$$
  - Impedance relationship for $Z_1(s)$
$$V_1(s) = I_1(s)Z_1(s) \tag{17.57}$$
  - Impedance relationship for $Z_2(s)$
$$V_1(s) = I_2(s)Z_2(s) \tag{17.58}$$
- Substitution
  - Isolate $I_2(s)$ in Eq. 17.58 and substitute into Eq. 17.55
$$I_A(s) \quad = \quad I_1(s) + \frac{V_1(s)}{Z_2(s)} \tag{17.59}$$

– Now substitute Eq. 17.57 into Eq. 17.59

$$I_A(s) = I_1(s) + \frac{I_1(s)Z_1(s)}{Z_2(s)} \quad (17.60)$$

$$I_1(s) = \frac{Z_2(s)}{Z_1(s)+Z_2(s)} I_A(s) \quad (17.61)$$

Example 17.11: - node voltage method

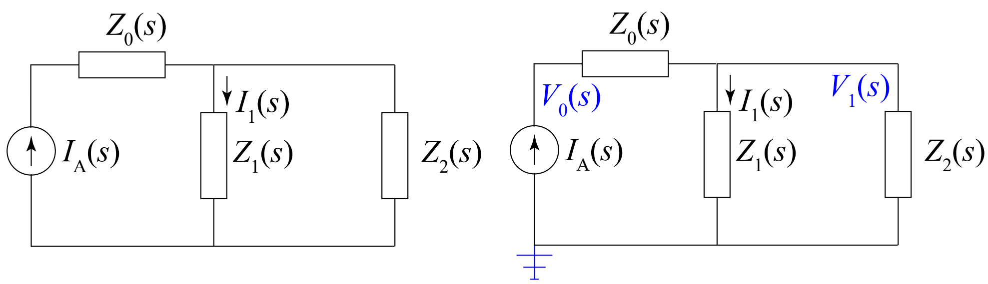


Figure 17.12:

Problem: Given $Z_0(s)$, $Z_1(s)$, $Z_2(s)$ and $I_A(s)$ in the circuit shown in Figure 17.12(left), find an expression for the current $I_1(s)$.

- Voltages: ground, $V(s)$
- KCL:
  - Node with voltage $V_0(s)$

$$I_A(s) = \frac{V_0(s) - V_1(s)}{Z_0(s)} \quad (17.62)$$

  - Node with voltage $V_1(s)$

$$\frac{V_0(s) - V_1(s)}{Z_0(s)} = \frac{V_1(s)}{Z_1(s)} + \frac{V_1(s)}{Z_2(s)} \quad (17.63)$$

- Substitution
  - Substitute $I_A(s)$ (from Eq. 17.62) in for the left hand side of Eq. 17.63

$$I_A(s) = \frac{V_1(s)}{Z_1(s)} + \frac{V_1(s)}{Z_2(s)} \quad (17.64)$$

$$V_1(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s)+Z_2(s)} I_A(s) \quad (17.65)$$

– Since the question asks us to find i_1, we can write the following using Ohm's law

$$Z_1(s)I_1(s) = V_1(s) \tag{17.66}$$
$$= \frac{Z_1(s)Z_2(s)}{Z_1(s)+Z_2(s)}I_A(s) \tag{17.67}$$
$$I_1(s) = \frac{Z_2(s)}{Z_1(s)+Z_2(s)}I_A(s) \tag{17.68}$$

Example 17.12: - mesh current method

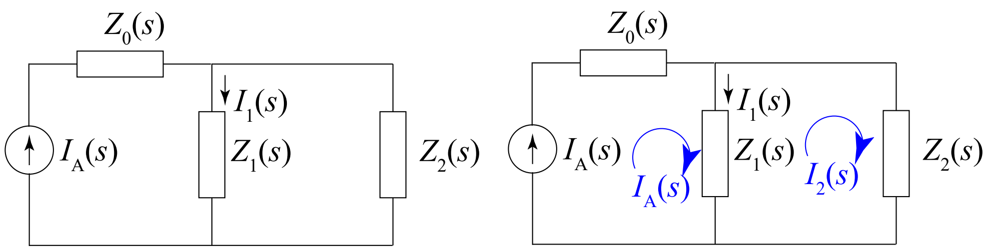


Figure 17.13:

Problem: Given $Z_0(s)$, $Z_1(s)$, $Z_2(s)$ and $I_A(s)$ in the circuit shown in Figure 17.13(left), find an expression for the current $I_1(s)$.

- Currents: $I_2(s)$ ($I_A(s)$ and $I_1(s)$ are already labeled)
- KVL:
  - Mesh with current $I_A(s)$ is trivial
  - Mesh with current $I_2(s)$ (in direction of $I_2(s)$

$$-(I_2(s) - I_A(s))Z_1(s) - I_2(s)Z_2(s) = 0 \tag{17.69}$$

- Substitution
  - Isolate $I_2(s)$ in Eq. 17.69

$$I_2(s) = \frac{Z_1(s)}{Z_1(s)+Z_2(s)}I_A(s) \tag{17.70}$$

– Now write an expression for $I_1(s)$

$$I_1(s) = I_A(s) - I_2(s) \tag{17.71}$$
$$= I_A(s) - \frac{Z_1(s)}{Z_1(s)+Z_2(s)} I_A(s) \tag{17.72}$$
$$= \frac{Z_2(s)}{Z_1(s)+Z_2(s)} I_A(s) \tag{17.73}$$

Example 17.13: - default method

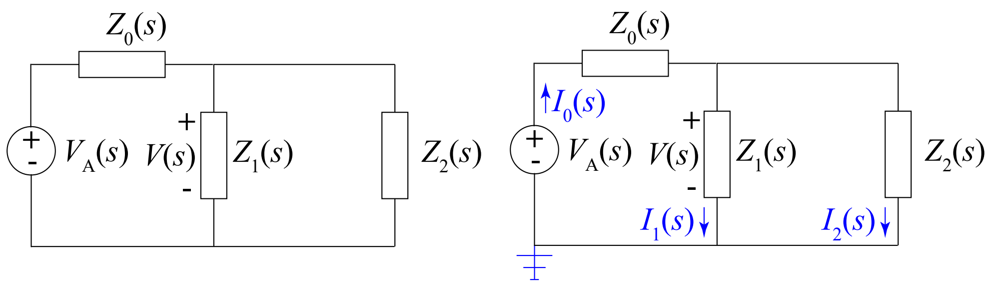


Figure 17.14:

Problem: Given $Z_0(s)$, $Z_1(s)$, $Z_2(s)$ and $V_A(s)$ in the circuit shown in Figure 17.14(left), find an expression for the voltage $V(s)$.

- Voltages: ground ($V(s)$ is already given)
- Currents: $I_0(s)$, $I_1(s)$, $I_2(s)$
- KCL:
  - the node with voltage $V_A(s)$ is trivial
  - At node with voltage $V(s)$

$$I_0(s) = I_1(s) + I_2(s) \tag{17.74}$$

- *i*-*v*:
  - Impedance relationship for $Z_0(s)$

$$V_A(s) - V(s) = I_0(s)Z_0(s) \tag{17.75}$$

  - Impedance relationship for $Z_1(s)$

$$V(s) = I_1(s)Z_1(s) \tag{17.76}$$

– Impedance relationship for $Z_2(s)$

$$V(s) = I_2(s)Z_2(s) \tag{17.77}$$

- Substitution

  – Isolate $I_0(s)$ in Eq. 17.75, $I_1(s)$ in Eq. 17.76, and $I_2(s)$ in Eq. 17.77 and substitute all into Eq. 17.74

$$\begin{aligned}
\frac{V_A(s) - V(s)}{Z_0(s)} &= \frac{V}{Z_1(s)} + \frac{V(s)}{Z_2(s)} && (17.78)\\
V(s)\left(\frac{1}{Z_0(s)} + \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}\right) &= \frac{V_A(s)}{Z_0(s)} && (17.79)\\
V(s)\left(Z_1(s)Z_2(s) + Z_0(s)Z_2(s) + Z_0(s)Z_1(s)\right) &= Z_1(s)Z_2(s)V_A(s) && (17.80)
\end{aligned}$$

$$V(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s)Z_2(s) + Z_0(s)Z_2(s) + Z_0(s)Z_1(s)}V_A(s) \tag{17.81}$$

Example 17.14: - node voltage method

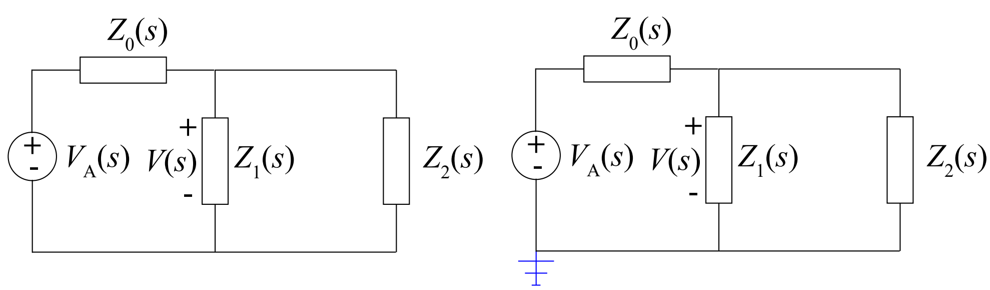


Figure 17.15:

Problem: Given $Z_0(s)$, $Z_1(s)$, $Z_2(s)$ and $V_A(s)$ in the circuit shown in Figure 17.15(left), find an expression for the voltage $V(s)$.

- Voltages: ground ($V(s)$ and $V_A(s)$ are given)
- KCL:
  – Node with voltage $V(s)$

$$\frac{V_A(s) - V(s)}{Z_0(s)} = \frac{V(s)}{Z_1(s)} + \frac{V(s)}{Z_2(s)} \tag{17.82}$$

- Substitution
  - In this case we only have one equation and one unknown. Rearranging Eq. 17.82

$$\frac{V_A(s) - V(s)}{Z_0(s)} = \frac{V(s)}{Z_1(s)} + \frac{V(s)}{Z_2(s)} \tag{17.83}$$

$$V(s)\left(\frac{1}{Z_0(s)} + \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}\right) = \frac{V_A(s)}{Z_0(s)} \tag{17.84}$$

$$V(s)\left(Z_1(s)Z_2(s) + Z_0(s)Z_2(s) + Z_0(s)Z_1(s)\right) = Z_1(s)Z_2(s)V_A(s) \tag{17.85}$$

$$V(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s)Z_2(s) + Z_0(s)Z_2(s) + Z_0(s)Z_1(s)}V_A(s) \tag{17.86}$$

Example 17.15: - mesh current method

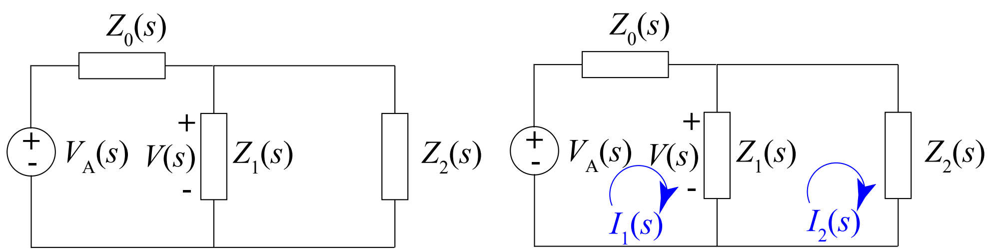


Figure 17.16:

Problem: Given $Z_0(s)$, $Z_1(s)$, $Z_2(s)$ and $V_A(s)$ in the circuit shown in Figure 17.16(left), find an expression for the voltage v.

- Currents: $I_1(s)$, $I_2(s)$
- KVL:
  - Mesh with current $I_1(s)$ in direction of current

$$V_A(s) - I_1(s)Z_0(s) - (I_1(s) - I_2(s))Z_1(s) = 0 \tag{17.87}$$

  - Mesh with current $I_2(s)$ in direction of current

$$-(I_2(s) - I_1(s))Z_1(s) - I_2(s)Z_2(s) = 0 \tag{17.88}$$

- Substitution

– Isolate $I_1(s)$ in Eq. 17.88 and substitute into Eq. 17.87

$$V_A(s) - I_2(s)\frac{Z_1(s)+Z_2(s)}{Z_1(s)}Z_0(s) - \left(I_2(s)\frac{Z_1(s)+Z_2(s)}{Z_1(s)} - I_2(s)\right)Z_1(s) = 0 \quad (17.89)$$

$$V_A(s) + I_2(s)\left(-Z_0(s) - \frac{Z_0(s)Z_2(s)}{Z_1(s)} - \cancel{Z_1(s)} - Z_2(s) + \cancel{Z_1(s)}\right) = \quad (17.90)$$

$$I_2(s) = \frac{Z_1(s)}{Z_0(s)Z_1(s) + Z_0(s)Z_2(s) + Z_1(s)Z_2(s)}V_A(s) \quad (17.91)$$

– Substitute the above into Eq. 17.88

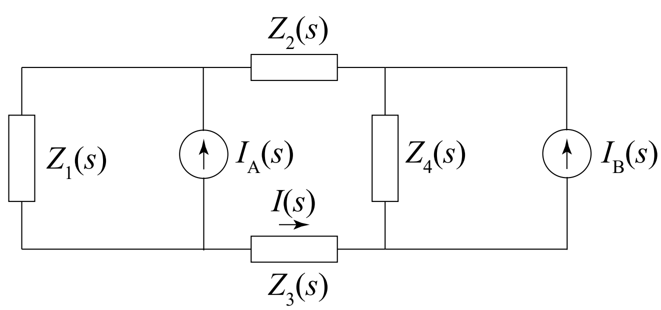
$$\begin{aligned} I_1(s) &= \frac{Z_1(s)+Z_2(s)}{Z_1(s)}I_2(s) && (17.92)\\ &= \frac{Z_1(s)+Z_2(s)}{Z_1(s)}\frac{Z_1(s)}{Z_0(s)Z_1(s) + Z_0(s)Z_2(s) + Z_1(s)Z_2(s)}V_A(s) && (17.93)\\ &= \frac{Z_1(s)+Z_2(s)}{Z_0(s)Z_1(s) + Z_0(s)Z_2(s) + Z_1(s)Z_2(s)}V_A(s) && (17.94) \end{aligned}$$

– From impedance relationship

$$\begin{aligned} V(s) &= Z_1(s)(I_1(s) - I_2(s)) && (17.95)\\ &= \frac{Z_1(s)Z_2(s)}{Z_0(s)Z_1(s) + Z_0(s)Z_2 + Z_1(s)Z_2(s)}V_A(s) && (17.96) \end{aligned}$$

Example 17.16: Additional examples

- see figure



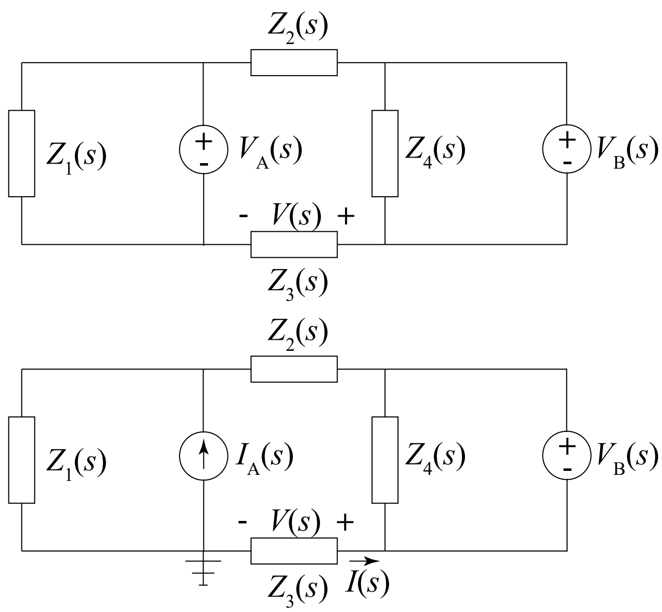


Figure 17.17:

- Even more examples from Lectures 5 and 8 of circuits notes

Chapter 18

Electromechanical Systems

18.1 Permanent Magnet DC Motors and Similar Electromechanical Systems

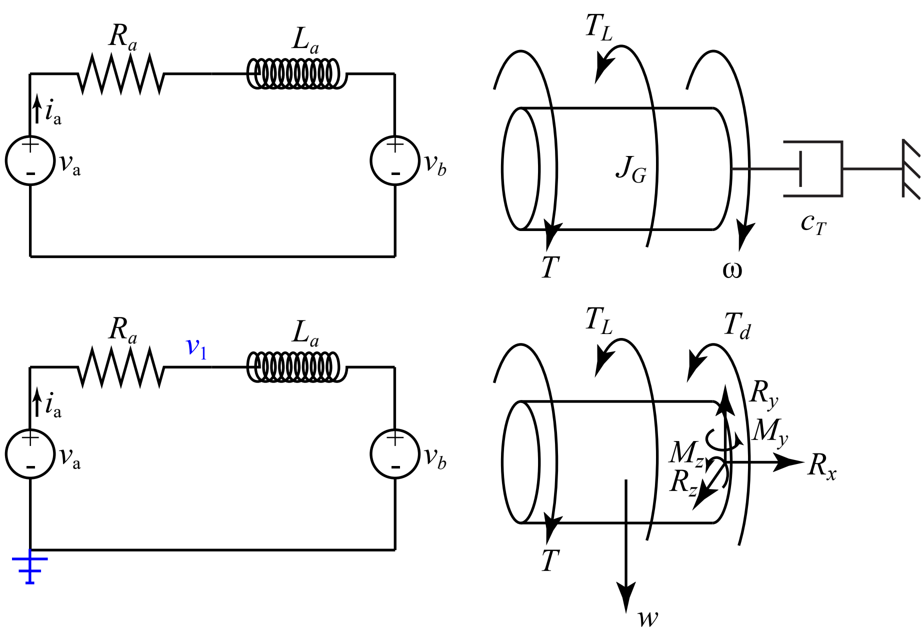


Figure 18.1: Permanent magnet DC motor. See [54]

- Permanent-Magnet DC motor model. See [54, 18, 35]
  - System parameters

  * v_a: voltage input to the motor, armature voltage
  * i_a: armature current
  * R_a: armature resistance
  * L_a: armature inductance
  * v_b: back electromotive force (emf). This is a voltage generated as the armature windings pass through the magnetic field; see, for example, [6, 35, 18].
  * T: Torque generated by the motor
  * T_L: External load torque applied to the system
  * J_G: mass moment of inertia of the motor shaft
  * c_T: viscous drag constant on the shaft

- Mechanical subsystem
  * Coordinate: ω
  * FBD
  * Force-motion relationships
    Damper

$$T_d = c_T \omega \tag{18.1}$$

    Moment equation about shaft

$$J_G \dot{\omega} = T - T_L - T_d \tag{18.2}$$

- Electrical subsystem
  * voltage: $0, v_a$, v_B
  * current: i_A
  * KCL: trivial
  * current voltage relationships
    Resistor

$$v_a - v_1 = i_a R_a \tag{18.3}$$

    Inductor

$$L_a \frac{di_a}{dt} = v_1 - v_b \tag{18.4}$$

- Coupling of mechanical and electrical subsystems
  * The torque generated by the motor, T, is proportional to armature current i_a

$$T = k_T i_a \tag{18.5}$$

  * Back emf is proportional to angular velocity of the shaft ω

$$v_b = k_b \omega \tag{18.6}$$

- Substitution (to yield governing equations)
  * Substitute Eqs. 18.3 and 18.6 into Eq. 18.4

$$L_a \frac{di_a}{dt} + R_a i_a + k_b \omega \quad = \quad v_a \tag{18.7}$$

  * Substitute Eqs. 18.1 and 18.5 into Eq. 18.2

$$J_G \frac{d\omega}{dt} + c_T \omega - k_T i_a \quad = \quad -T_{\mathrm{L}} \tag{18.8}$$

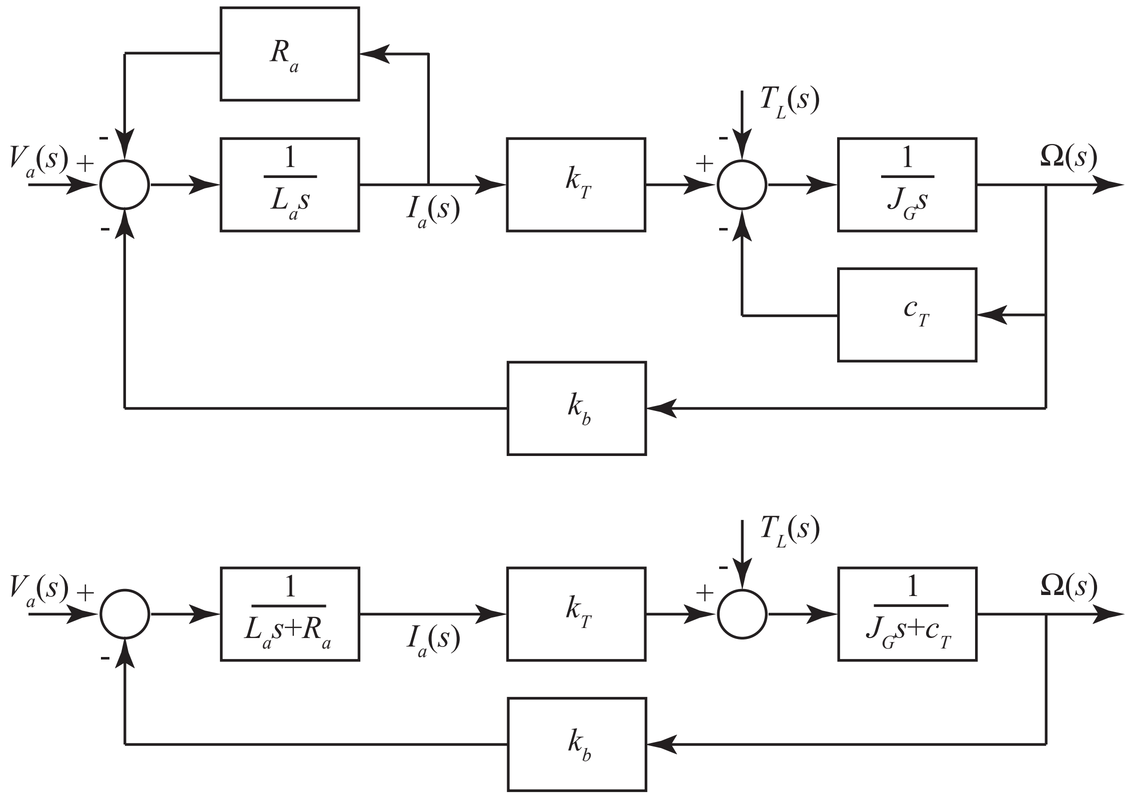


Figure 18.2:

– Block diagram

  * Laplace transform of governing equations

$$\begin{aligned} L_a s I_a(s) + R_a I_a(s) + k_b \Omega(s) &= V_a(s) && (18.9) \\ J_G s \Omega(s) + c_T \Omega(s) - k_T I_a(s) &= -T_{\mathrm{L}}(s) && (18.10) \end{aligned}$$

  * Rearrange

$$\begin{aligned} I_a(s) &= \frac{1}{L_a s}\left(-R_a I_a(s) - k_b \Omega(s) + V_a(s)\right) && (18.11) \\ \Omega(s) &= \frac{1}{J_G s}\left(-c_T \Omega(s) + k_T I_a(s) - T_{\mathrm{L}}(s)\right) && (18.12) \end{aligned}$$

  * Construct block diagram; see Figure 18.2(Above)
  * Simplify; see Figure 18.2(Below)

- Transfer functions
  * Transfer function could be obtained from the block diagram or as shown below
  * Laplace transform of governing equations

$$L_a s I_a(s) + R_a I_a(s) + k_b \Omega(s) = V_a(s) \tag{18.13}$$
$$J_G s \Omega(s) + c_T \Omega(s) - k_T I_a(s) = -T_{\mathrm{L}}(s) \tag{18.14}$$

  * Rearrange

$$(L_a s + R_a) I_a(s) + k_b \Omega(s) = V_a(s) \tag{18.15}$$
$$-k_T I_a(s) + (J_G s + c_T)\Omega(s) = -T_{\mathrm{L}}(s) \tag{18.16}$$

  * Isolate $\Omega(s)$ in Eq. 18.16 and substitute intoEq. 18.15

$$(L_a s + R_a) I_a(s) + k_b \left(\frac{k_T I_a(s) - T_{\mathrm{L}}(s)}{J_G s + c_T} \right) = V_a(s) \tag{18.17}$$
$$(L_a s + R_a)(J_G s + c_T) I_a(s) + k_b \left(k_T I_a(s) - T_{\mathrm{L}}(s) \right) = (J_G s + c_T) V_a(s) \tag{18.18}$$
$$(L_a s + R_a)(J_G s + c_T) I_a(s) + k_b k_T I_a(s) = (J_G s + c_T) V_a(s) + k_b T_{\mathrm{L}}(s) \tag{18.19}$$
$$\left(L_a J_G s^2 + (L_a c_T + J_G R_a) s + R_a c_T + k_b k_T \right) I_a(s) = (J_G s + c_T) V_a(s) + k_b T_{\mathrm{L}}(s) \tag{18.20}$$

  * Which yields two of the four transfer functions

$$\frac{I_a(s)}{V_a(s)} = \frac{J_G s + c_T}{L_a J_G s^2 + (L_a c_T + J_G R_a) s + R_a c_T + k_b k_T} \tag{18.21}$$
$$\frac{I_a(s)}{T_{\mathrm{L}}(s)} = \frac{k_b}{L_a J_G s^2 + (L_a c_T + J_G R_a) s + R_a c_T + k_b k_T} \tag{18.22}$$

  * Now isolate $I(s)$ in Eq. 18.15 and substitute intoEq. 18.16

$$-k_T \left(\frac{V_a(s) - k_b \Omega(s)}{L_a s + R_a} \right) + (J_G s + c_T)\Omega(s) = -T_{\mathrm{L}}(s) \tag{18.23}$$
$$(L_a s + R_a)(J_G s + c_T)\Omega(s) + k_T k_b \Omega(s) = k_T V_a(s) - (L_a s + R_a) T_{\mathrm{L}}(s) \tag{18.24}$$
$$\left(L_a J_G s^2 + (L_a c_T + J_G R_a) s + R_a c_T + k_b k_T \right) \Omega(s) = k_T V_a(s) - (L_a s + R_a) T_{\mathrm{L}}(s) \tag{18.25}$$

  * Which yields the remaining two of four transfer functions

$$\frac{\Omega(s)}{V_a(s)} = \frac{k_T}{L_a J_G s^2 + (L_a c_T + J_G R_a) s + R_a c_T + k_b k_T} \tag{18.26}$$
$$\frac{\Omega(s)}{T_{\mathrm{L}}(s)} = -\frac{L_a s + R_a}{L_a J_G s^2 + (L_a c_T + J_G R_a) s + R_a c_T + k_b k_T} \tag{18.27}$$

- Steady state

* Governing equations (at steady state)

$$R_a i_{a,ss} + k_b \omega_{ss} = v_{a,ss} \tag{18.28}$$
$$c_T \omega_{ss} - k_T i_{a,ss} = -T_{\mathrm{L},ss} \tag{18.29}$$

* Load torque as a function of angular velocity (substitute $i_{a,ss}$ from Eq. 18.28 into Eq. 18.29 and considering $v_{a,ss}$ to be a constant input)

$$-T_{\mathrm{L},ss} = c_T \omega_{ss} - k_T \left(\frac{v_{a,ss} - k_b \omega_{ss}}{R_a} \right) \tag{18.30}$$
$$T_{\mathrm{L},ss} = -c_T \omega_{ss} + k_T \left(\frac{v_{a,ss} - k_b \omega_{ss}}{R_a} \right) \tag{18.31}$$
$$= -\left(c_T + \frac{k_T k_b}{R_a} \right) \omega_{ss} + \frac{k_T v_{a,ss}}{R_a} \tag{18.32}$$

* Power delivered to the load

$$P_{out} = T_{\mathrm{L},ss} \omega_{ss} \tag{18.33}$$
$$= -\left(c_T + \frac{k_T k_b}{R_a} \right) \omega_{ss}^2 + \frac{k_T v_{a,ss}}{R_a} \omega_{ss} \tag{18.34}$$

* Power supplied to the system

$$P_{in} = i_{a,ss} v_{a,ss} \tag{18.35}$$
$$= \left(\frac{v_{a,ss} - k_b \omega_{ss}}{R_a} \right) v_{a,ss} \tag{18.36}$$
$$= \frac{v_{a,ss}^2}{R_a} - \frac{k_b v_{a,ss}}{R_a} \omega_{ss} \tag{18.37}$$

* Efficiency

$$E = \frac{P_{out}}{P_{in}} \tag{18.38}$$
$$= \frac{-\left(c_T + \frac{k_T k_b}{R_a} \right) \omega_{ss}^2 + \frac{k_T v_{a,ss}}{R_a} \omega_{ss}}{\frac{v_{a,ss}^2}{R_a} - \frac{k_b v_{a,ss}}{R_a} \omega_{ss}} \tag{18.39}$$
$$= \frac{-\left(c_T R_a + k_T k_b \right) \omega_{ss}^2 + k_T v_{a,ss} \omega_{ss}}{v_{a,ss}^2 - k_b v_{a,ss} \omega_{ss}} \tag{18.40}$$

Note, when $c_T = 0$ the denominator can be factored out of the numerator yielding an efficiency that grows in proportion to ω_{ss}

- DC generator and Tachometer; see, for example, [35, 18]
  - A permanent magnet DC motor may be driven mechanically to generate a voltage
  -

- Loudspeaker; see [18, 35, 8]

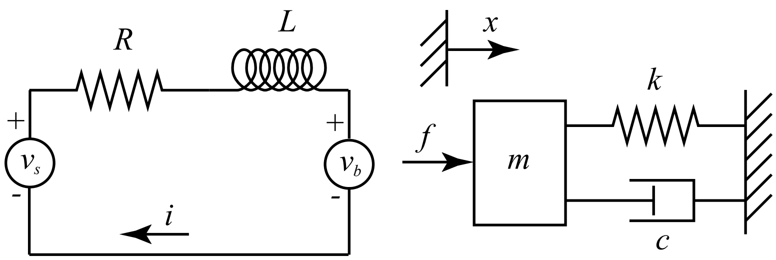


Figure 18.3: Loudspeaker

  - Coupling equations (where k_f and k_b are known constants for the system.)

$$f = k_f i \tag{18.41}$$

$$v_b = k_b \dot{x} \tag{18.42}$$

  - Possibly make an example that covers the frequency response of a two-way speaker with crossover.

18.2 Other Electromechanical Systems

- Potentiometer - covered again in ME375
- Piezoelectric transducers - covered in ME375

18.3 Reduced Order Model of DC Motor

- Reduced order modeling of DC motors
  - Neglecting L_a
    - "If we set $L_a = 0$, the second-order motor model reduces to a first-order model, which is easier to use [35]."
    - "... L_a ... is difficult to calculate or measure [35]."
    - "We conclude ... that you should be careful in using the approximation $L_a = 0$, although one sees it in common use [35]."
    - "In many cases the relative effect of the inductance is negligible compared with the mechanical motion and can be neglected ... [8]"
    - "If we assume that the armature inductance, L_a, is small compared to the armature resistance, R_a, which is usual for a dc motor ... [6]"
    - "In most DC motor applications, the dynamics of the rotor are much slower than those of the RL circuit, making the inductance L_a negligible." https://pages.hmc.edu/harris/class/e motors.pdf
  - As an alternative to neglecting armature inductance ...
    - (This section is a work in progress.)
    - We can expect one pole to dominant the response of the system when the two poles are real and sufficiently far apart
    - We might consider a factor of 5 or more to be sufficient separation; see, for example, [39]. (With a factor of 5, the contribution of the non dominant pole to the step response is 1/5 that of the dominant pole and is negligible by after one time constant of the dominant pole.) This occurs when

$$\frac{k_b k_T}{R_a c_T} < \frac{5(1 + \frac{L_a}{R_a}\frac{C_T}{J_G})^2}{36\frac{L_a}{R_a}\frac{C_T}{J_G}} - 1 \tag{18.43}$$

      Need to check this equation and decide if 5 is a sufficient separation.

- Incorporating a dynamic mechanical load into our DC motor model
  - Method 1: Simply model the mechanical and electrical subsystems and then couple them using the two equations as we did to develop our model for a DC motor
  - Method 2: Incorporate a block diagram representation of the dynamic mechanical load into the block diagram for the DC motor

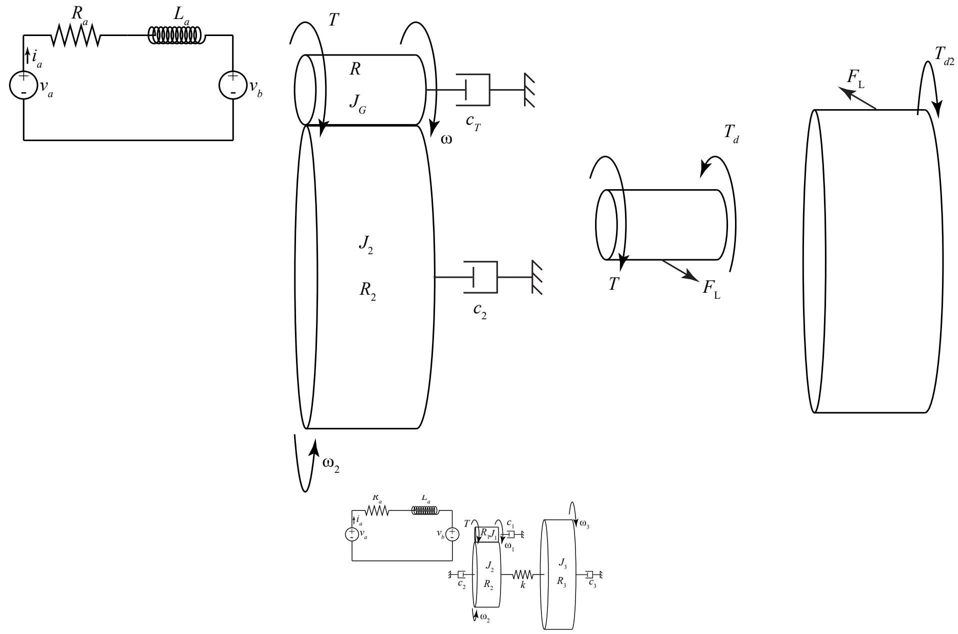


Figure 18.4: (Left) Combined system. (Right) Free-body diagrams of the shaft and load. (The reaction forces/moments at the supports were omitted because they won't enter into the governing equation.) Revise this example to parallel the servotable pre-lab.

- ∗ Example: Servotable
  - · Additional input output relationships relevant to the load

    Relationship between load torque, T_{L}, and reaction force, F_{L}, between mating gears

$$T_{\mathrm{L}} = RF_{\mathrm{L}} \tag{18.44}$$

Torque angular velocity relationship for the damper

$$T_{d2} = c_2\omega_2 \tag{18.45}$$

Moment equation for J_2 in the ω_2 direction

$$J_2\dot{\omega}_2 = R_2F_{\mathrm{L}} - T_{d2} \tag{18.46}$$

· Kinematics

$$R\omega = R_2\omega_2 \tag{18.47}$$
$$\tag{18.48}$$

· Substitution

$$J_2\dot{\omega}_2 = R_2\frac{1}{R}T_{\rm L} - c_2\omega_2 \tag{18.49}$$
$$J_2\dot{\omega}_2 + c_2\omega_2 = R_2\frac{1}{R}T_{\rm L} \tag{18.50}$$

· Transfer functions

$$\frac{T_{\rm L}(s)}{\Omega_2(s)} = \frac{R}{R_2}(J_2 s + c_2) \tag{18.51}$$
$$\frac{\Omega_2(s)}{\Omega(s)} = \frac{R}{R_2} \tag{18.52}$$

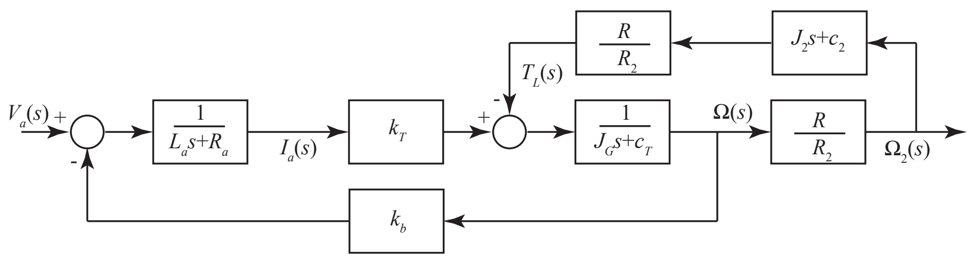


Figure 18.5:

· Draw block diagram

- Servo-table parameters
  - Gear and disk dimensions (ballpark estimates)
    - ∗ Plate
      - · 5 in diameter
      - · t_{2p} =0.25 in thick
      - · aluminum
    - ∗ Large gear
      - · 3 in diameter
      - · t_{2g} =0.125 in thick
      - · steel

* Small gear
  * R =0.5 in diameter
  * 0.125 in thick
  * steel
* Simple model of load
  * R =0.5/2 in
  * R_2 =3/2 in
  * $\frac{R}{R_2} = \frac{1}{6}$
  * J_2

$$\begin{aligned}
J_2 &= \frac{1}{2}\rho_g V_g R_2^2 + \frac{1}{2}\rho_p V_p R_p^2 && (18.53)\\
&= \frac{1}{2}\rho_g \pi t_g R_2^4 + \frac{1}{2}\rho_p t_p R_p^4 && (18.54)\\
&= \frac{1}{2}\cdot 0.3\pi 0.125\cdot 1.5^4 + \frac{1}{2}\cdot 0.1\cdot 0.25\cdot 2.5^4 \text{ lb}_\text{m}\text{in}^2 && (18.55)\\
&= 0.23\cdot 10^{-3} \text{ kgm}^2 && (18.56)
\end{aligned}$$

- DC motors
  * Pittman Ametek 8541S040, 24VDC (old version)
    * The following are my best guess regarding the parameters assuming this is the spec sheet: https://www.motionsolutions.com/store/pc/catalog/documents/pittman/8541%20DC%20Motor.p
    * Torque constant: 0.032 Nm/A
    * Voltage constant: 0.032 V/rad/s
    * Terminal resistance: 9.5 Ω
    * Inductance: 7.3 mH
    * Viscous damping factor: 1.68e-6 Nm s/rad
    * Electrical time constant: 0.77 ms
    * Mechanical time constant: 19 ms
    * Rotor inertia: 2.05e-6 kgm^2
  * Pittman Ametek DC030C-1000-A20, 24VDC
    * The following are my best guess regarding the parameters assuming this is the spec sheet: https://www.haydonkerkpittman.com/-/media/ametekhaydonkerk/downloads/data-sheets/pittman-data-sheets/dc030c-catalog_brush_dc_motor.pdf?la=en&revision=dbf2257b-49e9-4a5c-addc-1f5896e24df8
    * Torque constant: 0.0316 Nm/A
    * Voltage constant: 0.0316 V/(rad/s)
    * Terminal resistance: 9.5 Ω

- Inductance: 7.3 mH
- Viscous damping factor: Nm s/rad
- Electrical time constant: 0.77 ms
- Mechanical time constant: 19 ms
- Rotor inertia: 2.0e-6 kgm^2

Chapter 19

Hydraulic Systems

- See, for example, [2].
- Introduction to Hydraulic Systems
  - Note, this section has been updated to use volumetric flow rate rather than mass flow rate but a few typos may remain. Both [35, 8] present volumetric and mass flow rates but [35] focuses on mass flow rate while [8] seems to lean toward volumetric flow rate. However, the resistance in [8] is for mass flow rate. [40] presents volumetric flow rate.
  - Hydraulics - "Hydraulics is the study of systems in which the fluid is incompressible... [2]"
  - We will focus on hydraulic systems. And we will describe "only the gross system behavior instead of the details of the fluid motion [2]."
  - Conservation of mass
    * Equation
    $$\dot{m} = w_{in} - w_{out} \tag{19.1}$$
    * $\dot{m}$ is the time rate of change of mass of a system, w_{in} is the mass flow rate into the system, and w_{out} is the mass flow rate out of the system; see, for example, [8, 35]
    * Equivalent to conservation of volume for incompressible fluid systems
    $$\dot{m} = w_{in} - w_{out} \tag{19.2}$$
    $$\rho\dot{v} = \rho q_{in} - \rho q_{out} \tag{19.3}$$
    $$\dot{v} = q_{in} - q_{out} \tag{19.4}$$
    * $\dot{v}$ is the time rate of change of volume of a system, q_{in} is the volumetric flow rate into the system, and w_{out} is the volumetric flow rate out of the system; see, for example, [8, 35]
  - Continuity

    * “The total mass flow into a junction must equal the total flow out of the junction [2]”
    * statement of conservation of mass
    * Analogous to Kirchhoff’s current law in circuits
  - Compatibility
    * “the sum of signed pressure differences around a closed loop must be zero [2].’
    * Analogous to Kirchhoff’s voltage law in circuits
  - Hydrostatic pressure

$$p = \rho g h + p_a \tag{19.5}$$

    * Gauge pressure is the difference betweeen the absolute pressure and atmospheric pressure

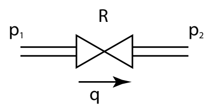
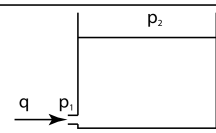
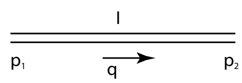
Element	Symbol	Linear Relationship
Resistance		$p_1 - p_2 = Rq$
Capacitance		$C\left(\frac{dp_1}{dt} - \frac{dp_2}{dt}\right) = q$
Inertance		$I\frac{dq}{dt} = p_1 - p_2$

Figure 19.1: Hydraulic Elements. See, for example, [2].

- Hydraulic Elements
  - Capacitance
    * Tank (p_r is just a reference point at which the expression is evaluated)

$$C = \frac{1}{\rho}\frac{dm}{dp}\bigg|_{p=p_r} \tag{19.6}$$

    * For a tank with straight vertical sides,

$$C = \frac{1}{\rho}\frac{dm}{dp} = \frac{A}{\rho g} \tag{19.7}$$

    * "When the container does not have vertical sides ..., the relations between m and h and between p and m are nonlinear [2]."
  - Resistance
    * Resistance comes from: conduit/pipe, valve, orifice
    * Laminar flow in pipe has a linear pressure-flow relation. See, for example, [2]
      · Linear relationship

$$p_1 - p_2 = Rq \tag{19.8}$$

      · Likely laminar (and linear) if Reynolds number $N_e < 2300$ [35] or less than 1000 [8]

        · Nonlinear relationship [8]

$$(p_1 - p_2)^{\frac{1}{\alpha}} = Rq \tag{19.9}$$

where $1 \leq \alpha \leq 2$ (α is 1 for linear laminar flow and 2 for turbulent flow)

- Inertance

$$I = \frac{p}{\frac{dq}{dt}} \tag{19.10}$$

  * "... inertance relates to fluid acceleration and kinetic energy, which are often negligible either because the moving fluid mass is small or because it is moving at a steady rate [2]."
  * "... inertance is larger for longer pipes and for pipes with smaller cross section [2]"

$$I = \frac{\rho L}{A} \tag{19.11}$$

- Other elements
  * flow source
  * pressure source

• Methods to derive governing equations

- Dr. Lillian's default method for hydraulic systems
  * Label pressures at each node
  * Label volumetric flow through each branch
  * Write expressions for continuity (i.e., the analog to Kirchhoff's current law)
  * Write pressure-flow relationships
    · Resistance
    · Capacitance
    · Inertance
    · Hydrostatic pressure
- Equivalent electrical circuit method
  * Sketch a circuit like diagram for the system
  * Label pressure (i.e., voltage) at each node
  * Label volumetric flow rate (i.e., current) through each branch
  * Write expressions for continuity (i.e., Kirchhoff's current law)
  * Pressure-flow (i.e., current-voltage) relationships (input output relationships)
    · Resistance
    · Capacitance
    · Inertance
    · Hydrostatic pressure
  * Substitution

- Hydraulic Examples

Example 19.1:

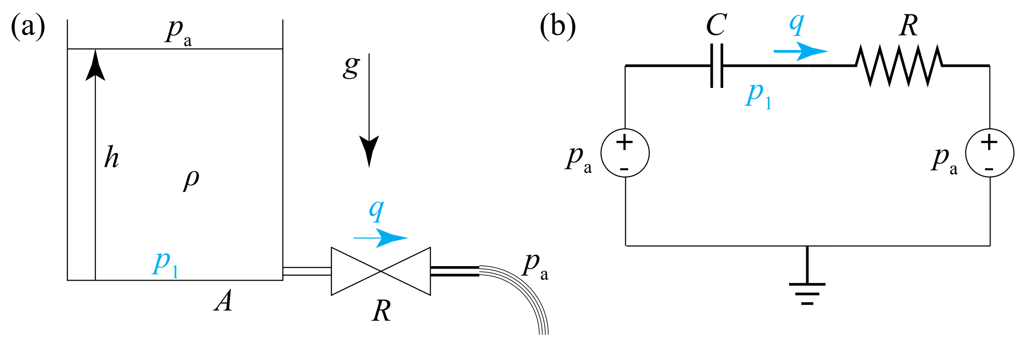


Figure 19.2: (a) A tank with cross sectional area A holds a fluid with density ρ. The top of the tank is exposed to atmospheric pressure while the fluid is allowed to exit the tank to atmospheric pressure through a pipe with hydraulic resistance R. (b) A circuit like model for the system is drawn.

Hydraulic tank system - Develop a model for the system in **Figure 19.2(a)** and derive the governing equation for the height of the tank $h(t)$.

- Dr. Lillian's default method for hydraulic systems
  - Label pressures: p_a, p_1, 0
  - Label volumetric flow rates: q
  - Continuity (KCL) - not interesting for this problem
  - Pressure-flow relationships
    * R

$$p_1 - p_a = qR \tag{19.12}$$

    * C

$$C\frac{d}{dt}(p_a - p_1) = q \tag{19.13}$$

$$C\frac{dp_1}{dt} = -q \tag{19.14}$$

    * C: Hydraulic capacitance of the tank

$$C = \frac{A}{\rho g} \tag{19.15}$$

    * Hydrostatic pressure

$$p_1 = p_\mathrm{a} + \rho g h \tag{19.16}$$

  - Substitution
    * Substitute Equation 19.12 into Equation 19.14

$$C\frac{dp_1}{dt} = -\frac{p_1 - p_\mathrm{a}}{R} \tag{19.17}$$

$$RC\frac{dp_1}{dt} = p_\mathrm{a} - p_1 \tag{19.18}$$

    * Substitute Equations 19.15 and 19.16 into the above

$$R\frac{A}{\rho g}\frac{d(p_\mathrm{a} + \rho g h)}{dt} = p_\mathrm{a} - p_\mathrm{a} - \rho g h \tag{19.19}$$

$$R\frac{A}{\cancel{\rho g}}\cancel{\rho g}\frac{dh}{dt} = -\rho g h \tag{19.20}$$

$$RA\frac{dh}{dt} = -\rho g h \tag{19.21}$$

- Equivalent electrical circuit method
  - Sketch a circuit like diagram for the system - see Figure 19.2(b).
    * C: Hydraulic capacitance of the tank

$$C = \frac{A}{\rho g} \tag{19.22}$$

    * Hydrostatic pressure

$$p_1 = p_\mathrm{a} + \rho g h \tag{19.23}$$

  - Label pressures: p_a, p_1, 0
  - Label volumetric flow rates: q
  - KCL analogy - not interesting for this problem
  - Pressure-flow relationships
    * R

$$p_1 - p_\mathrm{a} = qR \tag{19.24}$$

    * C

$$C\frac{d}{dt}(p_\mathrm{a} - p_1) = q \tag{19.25}$$

$$C\frac{dp_1}{dt} = -q \tag{19.26}$$

– Substitution

  * Substitute Equation 19.24 into Equation 19.26

$$C\frac{dp_1}{dt} = -\frac{p_1 - p_{\mathrm{a}}}{R} \tag{19.27}$$

$$RC\frac{dp_1}{dt} = p_{\mathrm{a}} - p_1 \tag{19.28}$$

  * Substitute Equations 19.22 and 19.23 into the above

$$R\frac{A}{\rho g}\frac{d(p_{\mathrm{a}} + \rho g h)}{dt} = p_{\mathrm{a}} - p_{\mathrm{a}} - \rho g h \tag{19.29}$$

$$R\frac{A}{\cancel{\rho g}}\cancel{\rho g}\frac{dh}{dt} = -\rho g h \tag{19.30}$$

$$RA\frac{dh}{dt} = -\rho g h \tag{19.31}$$

Example 19.2:

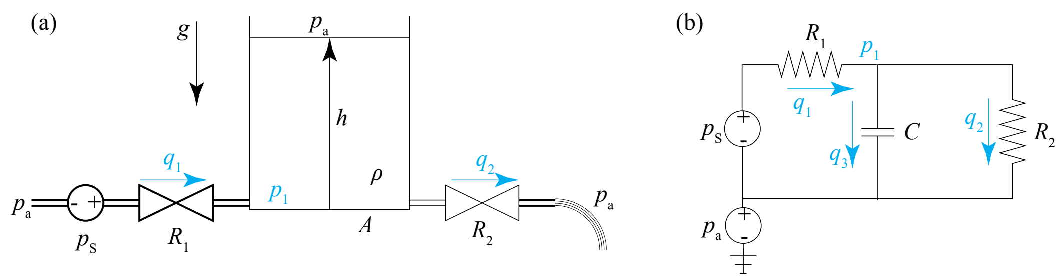


Figure 19.3: A tank with cross sectional area A holds a fluid with density ρ. The top of the tank is exposed to atmospheric pressure, p_a. A pump forces fluid into the tank through a pipe with resistance R_1 and fluid is allowed to exit the tank to atmospheric pressure through a pipe with hydraulic resistance R_2.
Derive the governing equation for height h of fluid in the tank of the system in Figure 19.3.

- Dr. Lillian's default method for hydraulic systems
  - Label pressures: p_{a}, $p_s + p_{\mathrm{a}}$, p_1
  - Label volumetric flow rates: q_1, q_2
  - Continuity (KCL) - not interesting for this problem

- Pressure-flow relationships
  * R_1
  $$p_s + p_\mathrm{a} - p_1 = q_1 R_1 \tag{19.32}$$
  * R_2
  $$p_1 - p_\mathrm{a} = q_2 R_2 \tag{19.33}$$
  * C
  $$C\frac{d}{dt}(p_1 - p_\mathrm{a}) = q_1 - q_2 \tag{19.34}$$
  $$C\frac{dp_1}{dt} = q_1 - q_2 \tag{19.35}$$
  * C: Hydraulic capacitance of the tank
  $$C = \frac{A}{\rho g} \tag{19.36}$$
  * Hydrostatic pressure
  $$p_1 = p_\mathrm{a} + \rho g h \tag{19.37}$$
- Substitution
  * Substitute Equations 19.32 and 19.33 into Equation 19.35
  $$C\frac{d}{dt}(p_1) = \frac{p_s + p_\mathrm{a} - p_1}{R_1} - \frac{p_1 - p_\mathrm{a}}{R_2} \tag{19.38}$$
  * Substitute Equations 19.36 and 19.37 into the above
  $$\begin{aligned} \frac{A}{\rho g}\frac{d}{dt}(p_a + \rho g h) &= \frac{p_s + p_\mathrm{a} - p_\mathrm{a} - \rho g h}{R_1} - \frac{p_\mathrm{a} + \rho g h - p_\mathrm{a}}{R_2} && (19.39) \\ R_1 R_2 A \dot{h} &= R_2(p_s - \rho g h) - R_1 \rho g h && (19.40) \\ R_1 R_2 A \dot{h} + \rho g (R_1 + R_2) h &= R_2 p_s && (19.41) \end{aligned}$$

- Equivalent electrical circuit method
  - Sketch a circuit like diagram for the system - see Figure 19.3(b).
    * C: Hydraulic capacitance of the tank
    $$C = \frac{A}{\rho g} \tag{19.42}$$
    * Hydrostatic pressure
    $$p_1 = p_\mathrm{a} + \rho g h \tag{19.43}$$

- Label pressures: p_a, $p_s + p_\mathrm{a}$, p_bottom
- Label volumetric flow rates: q_1, q_2, q_3
- KCL analogy

$$q_1 = q_2 + q_3 \tag{19.44}$$

- Pressure-flow relationships
  * R_1

$$p_s + p_\mathrm{a} - p_1 = q_1 R_1 \tag{19.45}$$

  * R_2

$$p_1 - p_\mathrm{a} = q_2 R_2 \tag{19.46}$$

  * C

$$C\frac{d}{dt}(p_1 - p_\mathrm{a}) = q_3 \tag{19.47}$$

$$C\frac{dp_1}{dt} = q_3 \tag{19.48}$$

- Substitution
  * Substitute Equations 19.45, 19.46, and 19.48 into Equation 19.44

$$\frac{p_s + p_\mathrm{a} - p_1}{R_1} = \frac{p_1 - p_\mathrm{a}}{R_2} + C\frac{dp_1}{dt} \tag{19.49}$$

  * Substitute Equations 19.42 and 19.43 into the above

$$\begin{aligned} \frac{p_s + \cancel{p_\mathrm{a}} - (\cancel{p_\mathrm{a}} + \rho g h)}{R_1} &= \frac{\cancel{p_\mathrm{a}} + \rho g h - \cancel{p_\mathrm{a}}}{R_2} + \frac{A}{\rho g}\frac{d(p_\mathrm{a} + \rho g h)}{dt} && (19.50) \\ \frac{p_s - \rho g h}{R_1} &= \frac{\rho g h}{R_2} + A\frac{dh}{dt} && (19.51) \\ R_1 R_2 A\dot{h} + \rho g(R_1 + R_2)h &= R_2 p_s && (19.52) \end{aligned}$$

Example 19.3:

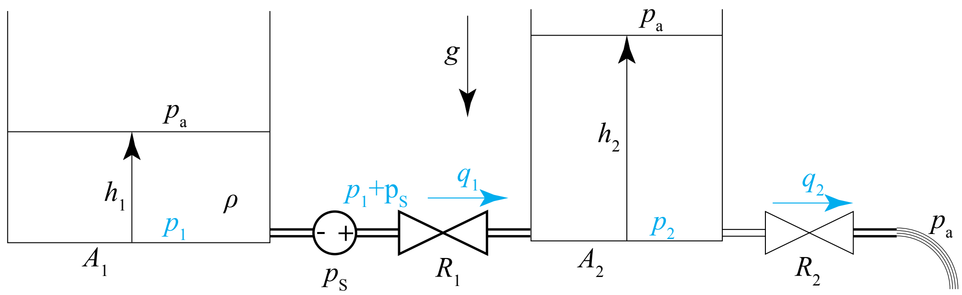


Figure 19.4: A system with two tanks with cross sectional areas A_1 and A_2 respectively, holds a fluid with density ρ. The top of each tank is exposed to atmospheric pressure, p_a. A pump forces fluid through the system which also includes two hydraulic resistors R_1 and R_2.

Derive the governing equations for heights h_1 and h_2 of the system in Figure 19.4.

- Dr. Lillian's default method for hydraulic systems
  - Label pressures: p_a, p_1, $p_1 + p_S$, p_2
  - Label volumetric flow rates: q_1, q_2
  - Continuity (KCL) - not interesting for this problem
  - Pressure-flow relationships
    - Hydrostatic pressure

$$p_1 = p_a + \rho g h_1 \tag{19.53}$$
$$p_2 = p_a + \rho g h_2 \tag{19.54}$$

    - $C_1 = \frac{A_1}{\rho g}$:

$$C_1 \frac{d}{dt}(p_1 - p_a) = -q_1 \tag{19.55}$$
$$\frac{A_1}{\rho g}\frac{d}{dt}(p_1 - p_a) = -q_1 \tag{19.56}$$

    - R_1:

$$p_1 + p_S - p_2 = q_1 R_1 \tag{19.57}$$

* $C_2 = \frac{A_2}{\rho g}$:

$$C_2 \frac{d}{dt}\left(p_2 - p_a\right) = q_1 - q_2 \tag{19.58}$$

$$\frac{A_2}{\rho g} \frac{d}{dt}\left(p_2 - p_a\right) = q_1 - q_2 \tag{19.59}$$

* R_2:

$$p_2 - p_a = q_2 R_2 \tag{19.60}$$

– Substitution

* Sub Equation 19.57 into Equation 19.56

$$\frac{A_1}{\rho g} \frac{d}{dt}\left(p_1 - p_a\right) = -\frac{p_1 + p_S - p_2}{R_1} \tag{19.61}$$

* Sub Equations 19.53 and 19.53 into the above

$$\frac{A_1}{\rho g} \frac{d}{dt}\left(p_a + \rho g h_1 - p_a\right) = -\frac{p_a + \rho g h_1 + p_S - p_a - \rho g h_2}{R_1} \tag{19.62}$$

$$A_1 \dot{h}_1 = -\frac{\rho g h_1 + p_S - \rho g h_2}{R_1} \tag{19.63}$$

$$\boxed{R_1 A_1 \dot{h}_1 = \rho g (h_2 - h_1) - p_S} \tag{19.64}$$

* Sub Equations 19.57 and 19.60 into Equation 19.59

$$\frac{A_2}{\rho g} \frac{d}{dt}\left(p_2 - p_a\right) = \frac{p_1 + p_S - p_2}{R_1} - \frac{p_2 - p_a}{R_2} \tag{19.65}$$

* Sub Equations 19.53 and 19.53 into the above

$$\frac{A_2}{\rho g} \frac{d}{dt}\left(p_a + \rho g h_2 - p_a\right) = \frac{p_a + \rho g h_1 + p_S - p_a - \rho g h_2}{R_1} - \frac{p_a + \rho g h_2 - p_a}{R_2} \tag{19.66}$$

$$A_2 \dot{h}_2 = \frac{\rho g h_1 + p_S - \rho g h_2}{R_1} - \frac{\rho g h_2}{R_2} \tag{19.67}$$

$$\boxed{R_1 R_2 A_2 \dot{h}_2 = R_2 \left(\rho g h_1 + p_S - \rho g h_2\right) - R_1 \rho g h_2} \tag{19.68}$$

Example 19.4:

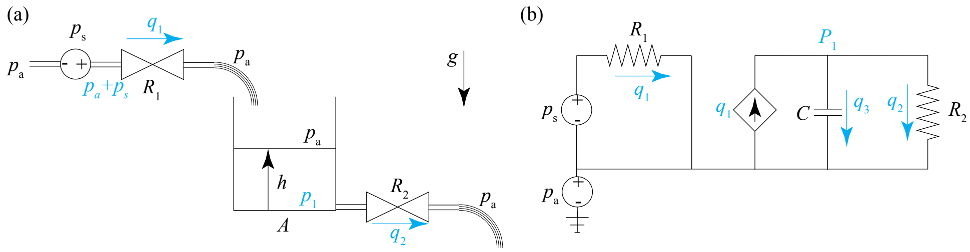


Figure 19.5:

- Dr. Lillian's default method for hydraulic systems
  - Label pressures: p_{a}, $p_s + p_{\mathrm{a}}$, p_1
  - Label volumetric flow rates: q_1, q_2
  - Continuity (KCL) - not interesting for this problem
  - Pressure-flow relationships
    * R_1

$$p_s + p_{\mathrm{a}} - p_{\mathrm{a}} = q_1 R_1 \tag{19.69}$$

$$p_s = q_1 R_1 \tag{19.70}$$

    * R_2

$$p_1 - p_{\mathrm{a}} = q_2 R_2 \tag{19.71}$$

    * $C = \frac{A}{\rho g}$

$$\frac{A}{\rho g}\frac{d}{dt}(p_1 - p_{\mathrm{a}}) = q_1 - q_2 \tag{19.72}$$

$$\frac{A}{\rho g}\frac{dp_1}{dt} = q_1 - q_2 \tag{19.73}$$

    * Hydrostatic pressure

$$p_1 = p_{\mathrm{a}} + \rho g h \tag{19.74}$$

  - Substitution
    * Substitute Equations 19.70 and 19.71 into Equation 19.73

$$\frac{A}{\rho g}\frac{d}{dt}(p_1) = \frac{p_s}{R_1} - \frac{p_1 - p_a}{R_2} \tag{19.75}$$

- Substitute Equation 19.74 into the above

$$\frac{A}{\rho g}\frac{d}{dt}(p_a + \rho g h) = \frac{p_s}{R_1} - \frac{p_a + \rho g h - p_a}{R_2} \quad (19.76)$$

$$R_1 R_2 A\dot{h} = R_2 p_s - R_1 \rho g h \quad (19.77)$$

- Equivalent electrical circuit method
  - Sketch a circuit like diagram for the system - see Figure 19.5(b).
    - C: Hydraulic capacitance of the tank

$$C = \frac{A}{\rho g} \quad (19.78)$$

    - Hydrostatic pressure

$$p_1 = p_\mathrm{a} + \rho g h \quad (19.79)$$

  - Label pressures: p_a, $p_s + p_\mathrm{a}$, p
  - Label volumetric flow rates: q_1, q_2, q_3
  - KCL analogy

$$q_1 = q_2 + q_3 \quad (19.80)$$

  - Pressure-flow relationships
    - R_1

$$p_s + p_\mathrm{a} - p_\mathrm{a} = q_1 R_1 \quad (19.81)$$

$$p_s = q_1 R_1 \quad (19.82)$$

    - R_2

$$p_1 - p_\mathrm{a} = q_2 R_2 \quad (19.83)$$

    - C

$$C\frac{d}{dt}(p_1 - p_\mathrm{a}) = q_3 \quad (19.84)$$

$$C\frac{d}{dt}(p_1) = q_3 \quad (19.85)$$

  - Substitution
    - Substitute Equations 19.82, 19.83, and 19.85 into Equation 19.80

$$\frac{p_s}{R_1} = \frac{p_1 - p_\mathrm{a}}{R_2} + C\frac{d}{dt}(p_1) \quad (19.86)$$

    - Substitute Equations 19.78 and19.79 into the above

$$\frac{p_s}{R_1} = \frac{p_\mathrm{a} + \rho g h - p_\mathrm{a}}{R_2} + \frac{A}{\rho g}\frac{d}{dt}(p_\mathrm{a} + \rho g h) \quad (19.87)$$

$$R_a R_2 A\dot{h} + R_1 \rho g h = R_2 p_s \quad (19.88)$$

Example 19.5:

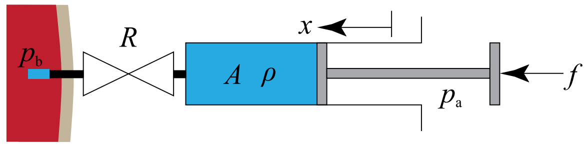


Figure 19.6: A syringe with cross sectional area A and hydraulic resistance R is used to deliver a vaccine with density ρ to the bloodstream by pressing the plunger with force f. The coordinate x measures the displacement of the plunger from it full position. Assume inertia is negligible and the cylinder is fixed to ground.

Find the governing equation for the system in Figure 19.6. I believe the answer using volumetric flow is:

$$f + (P_a - P_b)A = RA^2\dot{x} \tag{19.89}$$

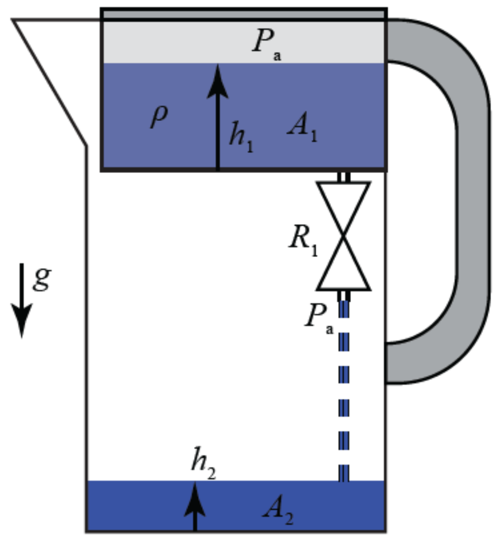


Figure 11: A water filtering pitcher consists of an upper tank that holds unfiltered water. Water passes through the filter (represented by a hydraulic resistor) before filling the lower filtered water tank. Use this figure for **Problem 11**.

11. (10 pts.) Derive the governing equations for h_1 and h_2 in **Figure 11**. Your governing equations need only to be valid for $h_1 > 0$ and $h_2 > 0$.

Chapter 20

Thermal Systems

- See, for example, [2].
- Conservation of thermal energy
  - The time rate of change in thermal energy of a system $\frac{dE}{dt}$ is equal to the net heat flow into the system (or $\sum q_{\text{in}}$)
  $$\frac{dE}{dt} = \sum q_{\text{in}} \tag{20.1}$$
  - q: Heat flow rate with units of energy/time
- Thermal Elements
  - Lumped thermal capacitance
    * In a lumped capacitance model of a body, we neglect the distribution of temperature throughout a body and assume it has a uniform, but time dependent, temperature.
    * When is this approximation appropriate?
      · When the thermal conductivity of the lump is relatively large.
      · other conditions?
      · When convective heat transfer is slow relative to conduction within the lump. Check Biot number; see, for example, [35]
    * The heat flow rate (q, with units of energy/time) into an object is related to its time rate of change in temperature
    $$q = C\frac{dT}{dt} \tag{20.2}$$
    * Capacitance
    $$C = \frac{dE}{dT} = mc_p \tag{20.3}$$

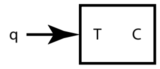
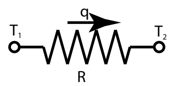
Element	Symbol	Input/output Relationship
Capacitance		$q = C\dfrac{dT}{dt}$
Resistance		$qR = T_1 - T_2$
Inductance	Does not exist	

Figure 20.1:

* Here c_p is the specific heat (often assumed constant, but is a function of temperature), and m is the mass of the object. For approximately incompressible materials, $c_p \approx c_v$.

– Thermal resistance

* Resistance equation

$$qR = T_1 - T_2 \tag{20.4}$$

* Conduction
  · Resistance

$$R = \frac{L}{kA} \tag{20.5}$$

  · Here, L is the length, A is the cross sectional area, k is the thermal conductivity (could be a function of temperature)

* Convection
  · Resistance

$$R = \frac{1}{hA} \tag{20.6}$$

  · here h is the film coefficient, or the convection coefficient

* Radiation
  · Relation is nonlinear []

$$q = \beta(T_1^4 - T_2^4) \tag{20.7}$$

  · Because of the nonlinearity, there is no defined R. However, we could linearize if there are only small changes in temperature

– Thermal inductance - **does not exist**

- Method to obtain governing equations
  - Sketch a lumped capacitance model for the system (similar to a circuit diagram)
  - Label temperatures in the system
  - Label heat flow rates
  - The net heat flow into a node is 0 (analogous to KCL)
  - Temperature-heat flow relationships (or input output equations)
  - Substitution

- Examples

Example 20.1: Veggie Burger

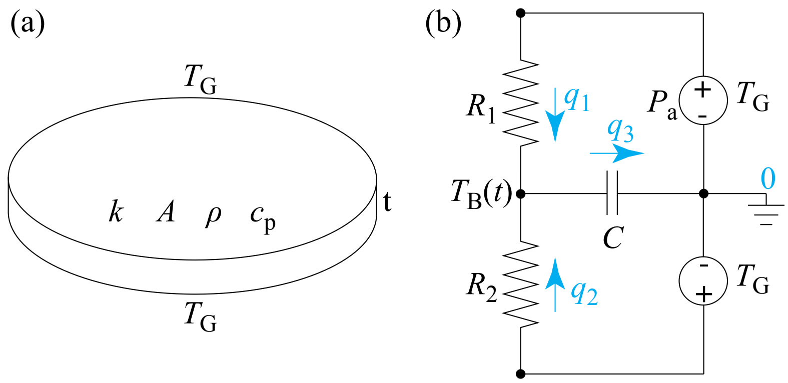


Figure 20.2: (a) An electric griddle with temperature $T_G(t)$ is in perfect thermal contact with the top and bottom surface of a frozen veggie burger of thickness t, cross sectional area A, density ρ, specific heat c_p, and thermal conductivity k. Neglect any heat transfer through the edge of the veggie burger. (b) A lumped capacitance model for the veggie burger drawn as a circuit.

Develop a lumped capacitance model of the system in **Figure 20.2(a)** and find the governing equation for an approximation to the average temperature of the interior of the veggie burger $T_B(t)$.

- Sketch a lumped capacitance model for the system - see Figure 20.2(b).
  - R_1: Thermal resistance from top surface of veggie burger to middle layer

$$R_1 = \frac{\frac{t}{2}}{kA} \tag{20.8}$$

$$= \frac{t}{2kA} \tag{20.9}$$

  - R_2: Thermal resistance from middle layer of veggie burger to bottom surface

$$R_2 = \frac{\frac{t}{2}}{kA} \tag{20.10}$$

$$= \frac{t}{2kA} \tag{20.11}$$

  - C: Thermal capacitance of veggie burger

$$C = \rho A \frac{t}{2} c_p \tag{20.12}$$

- * The temperatures of the top and bottom surfaces of the veggie burger are known (inputs) regardless of the capacitance of the burger. We can consider the average temperature of the interior of the burger to be the unknown (output) and its capacitance is less than the total capacitance of the burger. For symmetry, here we consider the capacitance of the interior half ($\frac{t}{2}$) of the burger to be associated with the temperature T_B and the top and bottom quarters ($\frac{t}{4}$) to be roughly at the same temperature as the griddle, T_G.
  * In this crude model, it wouldn't be unreasonable to use $C = \rho A t c_p$.

- Label temperatures: T_G, T_B, 0
- Label heat flow rates: q_1, q_2, q_3
- KCL analogy

$$q_1 + q_2 = q_3 \tag{20.13}$$

- Temperature-heat flow relationships
  - R_1

$$T_\mathrm{G} - T_\mathrm{B} = q_1 R_1 \tag{20.14}$$

  - R_2

$$T_\mathrm{G} - T_\mathrm{B} = q_2 R_2 \tag{20.15}$$

  - C

$$C\frac{d}{dt}(T_\mathrm{B} - 0) = q_3 \tag{20.16}$$

- Substitution
  - Substitute Equations 20.14, 20.15, and 20.16 into Equation 20.13

$$\frac{T_\mathrm{G} - T_\mathrm{B}}{R_1} + \frac{T_\mathrm{G} - T_\mathrm{B}}{R_2} = C\frac{dT_\mathrm{B}}{dt} \tag{20.17}$$

  - Substitute Equations 20.9, 20.11, and 20.12 into the above

$$\frac{T_\mathrm{G} - T_\mathrm{B}}{\frac{t}{2kA}} + \frac{T_\mathrm{G} - T_\mathrm{B}}{\frac{t}{2kA}} = \rho A\frac{t}{2}c_p\frac{dT_\mathrm{B}}{dt} \tag{20.18}$$

$$2 \cdot 2k\cancel{A}(T_\mathrm{G} - T_\mathrm{B}) = t\rho\cancel{A}\frac{t}{2}c_p\frac{dT_\mathrm{B}}{dt} \tag{20.19}$$

$$8k(T_\mathrm{G} - T_\mathrm{B}) = \rho t^2 c_p\frac{dT_\mathrm{B}}{dt} \tag{20.20}$$

- Consider a modified version of this question in which the top surface experiences convection ...

Example 20.2:

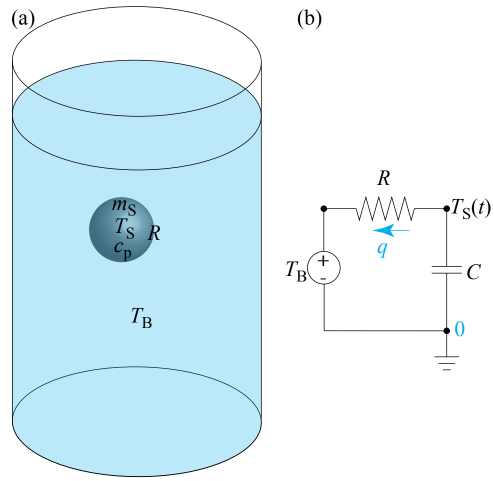


Figure 20.3: (a) A hot steel sphere with temperature T_S is dropped into a large vat of water at constant temperature T_B. The resistance to heat transfer between the sphere and the bath can be represented by resistance R. The sphere has mass m_S and specific heat c_p. Assume the thermal conductivity of the sphere is high such that the temperature is approximately uniform throughout the sphere. (b) A lumped capacitance model for the sphere drawn as a circuit. See, for example, [35]

Steel sphere in water bath - Develop a lumped capacitance model for the system in **Figure 20.3(a)** and derive the governing equation for the temperature of the sphere $T_S(t)$. See, for example, [35]

- Sketch a lumped capacitance model for the system - see Figure 20.3(b).
  - C: Thermal capacitance of sphere

$$C = m_S c_p \tag{20.21}$$

  - Note, here we assume the thermal conductivity of the sphere to be relatively large such that the temperature distribution can equalize quickly.
- Label temperatures: T_B, T_S, 0
- Label heat flow rates: q
- KCL analogy - not interesting
- Temperature-heat flow relationships

– R

$$T_{\mathrm{S}} - T_{\mathrm{B}} = qR \tag{20.22}$$

– C

$$C\frac{d}{dt}(T_{\mathrm{S}} - 0) = -q \tag{20.23}$$

- Substitution

  – Substitute Equation 20.22 into Equation 20.23

$$C\frac{dT_{\mathrm{S}}}{dt} = -\frac{T_{\mathrm{S}} - T_{\mathrm{B}}}{R} \tag{20.24}$$

$$RC\frac{dT_{\mathrm{S}}}{dt} = T_{\mathrm{B}} - T_{\mathrm{S}} \tag{20.25}$$

  – Substitute Equation 20.21 into the above

$$Rm_{\mathrm{S}}c_p\frac{dT_{\mathrm{S}}}{dt} = T_{\mathrm{B}} - T_{\mathrm{S}} \tag{20.26}$$

Example 20.3:

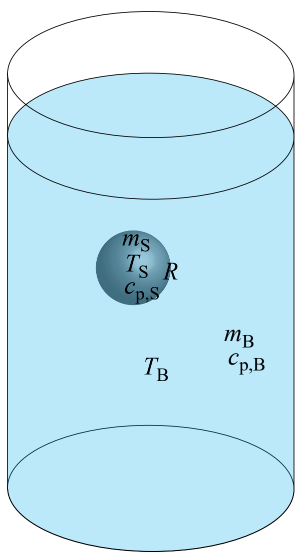


Figure 20.4: A hot steel sphere with temperature $T_{\mathrm{S}}(t)$ is dropped into a large water bath with temperature $T_B(t)$. The resistance to heat transfer between the sphere and the bath can be represented by resistance R. The sphere has mass m_{S} and

specific heat $c_{p,\mathrm{S}}$ while the bath has mass m_B and specific heat $c_{p,\mathrm{B}}$. Assume the thermal conductivity of the sphere is high such that the temperature is approximately uniform throughout the sphere. Similarly, assume the temperature of the water bath is uniform throughout. Finally, assume the system is insulated from the environment. See, for example, [35]
Derive the governing equations for the temperatures of the sphere $T_\mathrm{S}(t)$ and the bath $T_B(t)$ in Figure 20.4. See, for example, [35]

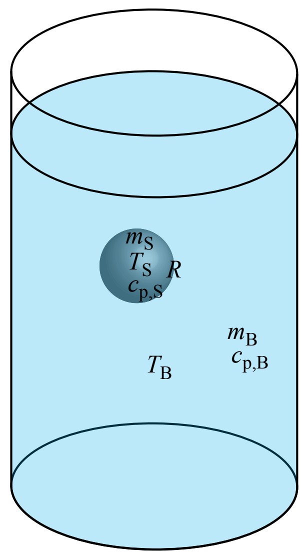
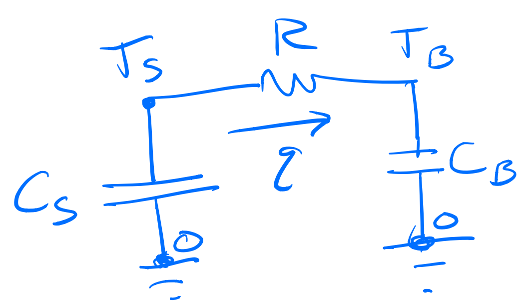


Figure 1: A hot steel sphere with temperature $T_S(t)$ is dropped into a large water bath with temperature $T_B(t)$. The resistance to heat transfer between the sphere and the bath can be represented by resistance R. The sphere has mass m_S and specific heat $c_{p,S}$ while the bath has mass m_B and specific heat $c_{p,B}$. Assume the thermal conductivity of the sphere is high such that the temperature is approximately uniform throughout the sphere. Similarly, assume the temperature of the water bath is uniform throughout. Finally, assume the system is insulated from the environment. Use this Figure for **Problem 1**.

1. Derive the governing equations for the temperatures of the sphere $T_S(t)$ and the bath $T_B(t)$ in **Figure 1**.

---

lumped capacitance model



$C_S = m_S c_{pS}$ (1)

$C_B = m_B c_{pB}$ (2)

T: T_S, T_B, 0

q: q

KCL: Not interesting

$T-q$ relationships

C_s / $\quad C_s \frac{d}{dt}(0 - T_s) = q \quad$ ③

R / $\quad T_s - T_B = Rq \quad$ ④

C_B / $\quad C_B \frac{d}{dt}(T_B - 0) = q \quad$ ⑤

substitution

① & ④ into ③

$$-m_s c_{ps} \dot{T}_s = \frac{T_s - T_B}{R}$$

$$\boxed{R m_s c_{ps} \dot{T}_s + T_s = T_B}$$

② & ④ into ⑤

$$m_B c_{pB} \dot{T}_B = \frac{T_s - T_B}{R}$$

$$\boxed{R m_B c_{pB} \dot{T}_B + T_B = T_s}$$

Example 20.4:

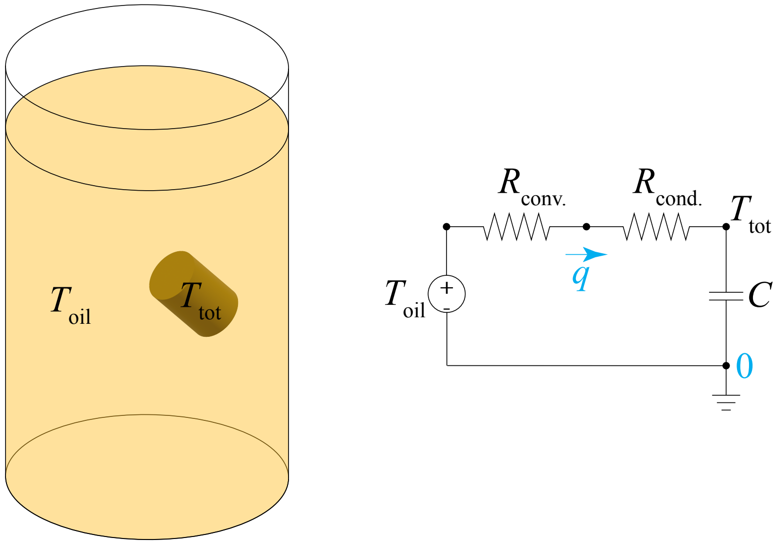


Figure 20.5:

Develop a model for the average internal temperature of the tater-tot, T_{tot}.

- Assumptions
  - Assume convection heat transfer with resistance $R_{\text{conv.}}$ describes the flow of heat into the tot.
  - Assume conduction heat transfer with resistance $R_{\text{cond.}}$ describes the flow of heat from the outer surface of the tot to the interior of the tot.

Example 20.5: Pizza oven

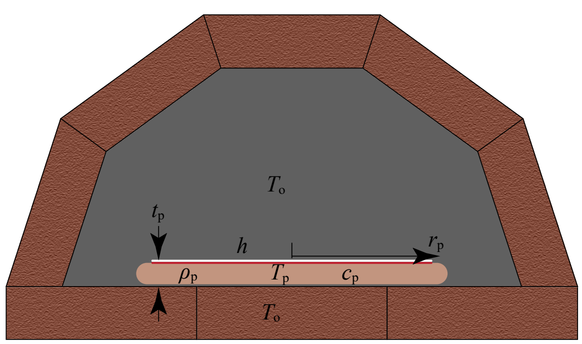
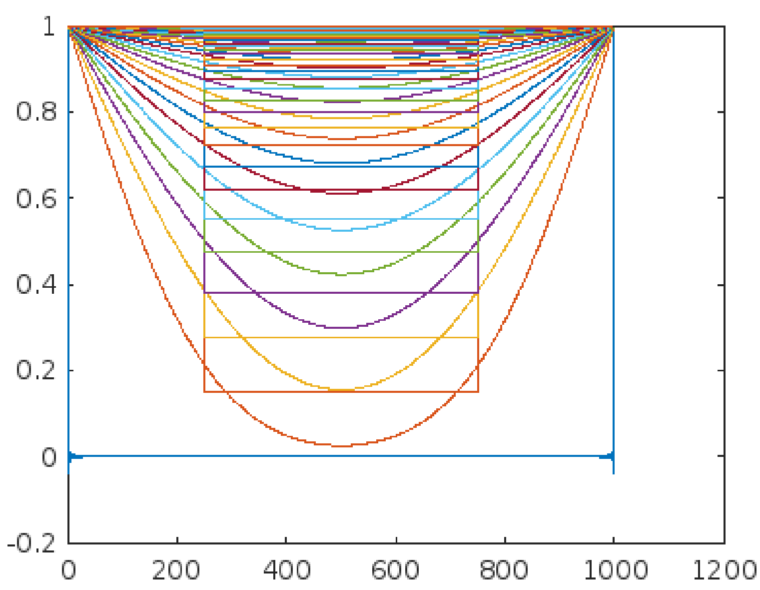
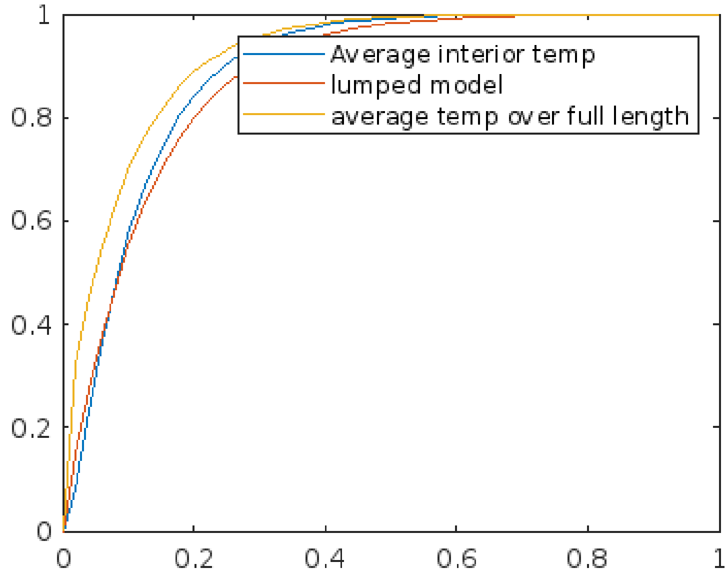


Figure 20.6: A pizza of radius r_p, thickness t_p, density ρ_p, thermal conductivity k_p, and specific heat c_p rests in the middle of a brick oven in which the bricks and air in the oven are held at a constant temperature T_o. Assume the bottom of the pizza is in perfect thermal contact with the floor of the brick oven. The convection coefficient for heat transfer between the air and the top surface of the pizza is h. Assume ρ_p, c_p, k_p, r_p, and t_p are constant. Neglect any heat transfer by radiation or through the edges of the pizza.

Example 20.6: Compare lumped model to solution of PDE


```
clear all
clc
close all
x=linspace(0,1,1001);
for n=1:2:5001
    A(n)=-4/n/pi;
    s(n)=-(n*pi)^2;

    X(n,:)=sin(n*pi*x)    ;
end
T=linspace(0,1,51)';
Eta=exp(T*s).*((T*0+1)*A);
A=Eta*X+1;
plot(A')
tau=1/8;
hold on
O=ones(1,1001);
O(1:249)=0;
O(751:end)=0;
Y=(1-exp(-T/tau))*O;
set(gca,'ColorOrderIndex',1)
Y(2:end,1:249)=1;
Y(2:end,751:end)=1;
plot(Y')
figure()

plot(T,(mean(A(:,250:750),2)), T,1-exp(-T/tau))
hold on
plot(T,(mean(A,2)))
legend('Average interior temp','lumped model','average temp over full length')
```





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Part VII

Dynamic System Measurements

Chapter 21

Fourier Series and Spectral Analysis

- Signal types; see, for example, [32]
  - Periodic vs. nonperiodic
    * Periodic
    * Nonperiodic
  - Deterministic vs. random
    * Deterministic -
    * Random - (stochastic)
- Polynomials are used often to fit data or approximate functions
  - Example: Linear fit to calibration curve
- Fourier Series are better suited to fit periodic data/functions
  - Fourier Series
    * A periodic signal $f(t)$ with period T (or an arbitrary signal considered from time $t = 0$ to time $t = T$) can be represented as an infinite sum of sine and cosine functions (or an infinite sum of complex exponential functions).
    * Fourier series: sine-cosine (or trigonometric [32]) form; see, for example, [41, 49, 35, 32, 56]

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \{a_k \cos(k\omega_T t) + b_k \sin(k\omega_T t)\} \tag{21.1}$$

· where the fundamental frequency, ω_T, can be expressed in terms of the period, T,

$$\omega_T = \frac{2\pi}{T} \tag{21.2}$$

· And coefficients can be obtained from

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_T t)dt \tag{21.3}$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_T t)dt \tag{21.4}$$

· The amplitudes a_n and b_n represent the frequency contribution to the signal at angular frequency $n\omega_T$

∗ Fourier series: cosine amplitude-phase form; see, for example, [4, 56]

$$f(t) = M_0 + \sum_{k=1}^{\infty} M_k \cos(k\omega_T t - \theta_k) \tag{21.5}$$

· where the fundamental frequency can be expressed in terms of the period

$$\omega_T = \frac{2\pi}{T} \tag{21.6}$$

· And parameters can be obtained from

$$M_0 = \frac{a_0}{2} \tag{21.7}$$

$$M_k = \sqrt{a_k^2 + b_k^2},\ k = \{1, 2, 3, ...\} \tag{21.8}$$

$$\tan(\theta_k) = \frac{b_k}{a_k} \tag{21.9}$$

· The amplitudes M_k and phase ϕ_k represent the frequency contribution to the signal at angular frequency $n\omega_T$

∗ Fourier series: sine amplitude-phase form; see, for example, [4, 56]

$$f(t) = M_0 + \sum_{k=1}^{\infty} M_k \sin(k\omega_T t + \psi_k) \tag{21.10}$$

· where the fundamental frequency can be expressed in terms of the period

$$\omega_T = \frac{2\pi}{T} \tag{21.11}$$

· And parameters can be obtained from

$$M_0 = \frac{a_0}{2} \tag{21.12}$$
$$M_k = \sqrt{a_k^2 + b_k^2},\ k = \{1, 2, 3, ...\} \tag{21.13}$$
$$\tan(\psi_k) = \frac{a_k}{b_k} \tag{21.14}$$

· The amplitudes M_k and phase ψ_k represent the frequency contribution to the signal at angular frequency $k\omega_T$

∗ Fourier series: complex exponential form; see, for example, [32, 56]

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_T t} \tag{21.15}$$

· where the fundamental frequency can be expressed in terms of the period

$$\omega_T = \frac{2\pi}{T} \tag{21.16}$$

· And parameters can be obtained from

$$c_k = \frac{1}{T}\int_0^T f(t)e^{-jk\omega_T t} \tag{21.17}$$
$$= \begin{cases} \frac{1}{2}(a_k + jb_k) & k < 0 \\ \frac{1}{2}a_0 & k = 0 \\ \frac{1}{2}(a_k - jb_k) & k > 0 \end{cases} \tag{21.18}$$

– Insights into Fourier coefficients; see, for example, [56]

∗ Odd vs. even functions

· Even function - a function of time, $f(t)$, is even if $f(t) = f(-t)$. The cosine function is even. Consequently, the fourier series of even functions will include only cosine terms, $b_k = 0$.

· Odd function - a function of time, $f(t)$, is odd if $f(t) = -f(-t)$. The sine function is odd. Consequently, the fourier series of odd functions will include only sine terms, $a_k = 0$.

∗ Limits of integration - The limits of integration may be changed to cover any complete period of the signal. Common limits include 0 to T and $-\frac{T}{2}$ to $\frac{T}{2}$.

∗ The $\frac{1}{2}a_0$ term in the Fourier series is the average, DC component, or zero frequency component of the signal

- · Equation

$$\frac{1}{2}a_k = \frac{1}{2}\frac{2}{T}\int_0^T f(t)\cos(0\omega_T t)dt \quad (21.19)$$
$$= \frac{1}{T}\int_0^T f(t)dt \quad (21.20)$$

- · $\frac{1}{2}a_0$ is the average of the periodic output

– Gibbs phenomenon:

 *

– Converting between the various forms

- Frequency content (spectrum) of a signal

 – We can plot the Fourier series coefficients (and angles) as a function of frequency

- Terms in the Fourier series are orthogonal to eachother

 – Vectors:

 * Consider two vectors

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (21.21)$$
$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (21.22)$$

 * The inner product (or dot product) of the two vectors is defined as

$$\left\langle \vec{a}, \vec{b} \right\rangle = \vec{a} \cdot \vec{b} \quad (21.23)$$
$$= a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (21.24)$$

 * $\vec{a}$ and $\vec{b}$ are orthogonal when their inner product (i.e, dot product) is zero

$$\left\langle \vec{a}, \vec{b} \right\rangle \begin{cases} = 0 & \text{if } \vec{a} \text{ and } \vec{b} \text{ are orthogonal and nonzero} \\ \neq 0 & \text{if } \vec{a} \text{ and } \vec{b} \text{ are not orthogonal and nonzero} \end{cases} \quad (21.25)$$

 * $\vec{a}$ is a unit vector when its inner product with itself is 1

$$\left\langle \vec{a}, \vec{b} \right\rangle \begin{cases} = 1 & \text{if } \vec{a} \text{ is a unit vector} \\ \neq 1 & \text{if } \vec{a} \text{ is not a unit vector} \end{cases} \quad (21.26)$$

- We can define an inner product for functions
  * Consider two non-zero functions $f_1(t)$ and $f_2(t)$
  * In the context of Fourier series, we could define the inner product of two functions as

$$\langle f_1(t), f_2(t)\rangle \quad = \quad \frac{2}{T}\int_0^T f_1(t)f_2(t)dt \tag{21.27}$$

  Note, alternative definitions of inner product could be defined.
  * $f_1(t)$ and $f_2(t)$ are orthogonal functions when their inner product is zero

$$\langle f_1(t), f_2(t)\rangle \begin{cases} = 0 & \text{if } f_1(t) \text{ and } f_2(t) \text{ are orthogonal and nonzero} \\ \neq 0 & \text{if } f_1(t) \text{ and } f_2(t) \text{ are not orthogonal and nonzero} \end{cases} \tag{21.28}$$

  * $f_1(t)$ is normalized (i.e., has unit magnitude) when its inner product with itself is 1

$$\langle f_1(t), f_1(t)\rangle \begin{cases} = 1 & \text{if } f_1(t) \text{ is normalized} \\ \neq 1 & \text{if } f_1(t) \text{ is not normalized} \end{cases} \tag{21.29}$$

- The basis functions $(\frac{1}{\sqrt{2}}, \cos\left(k\frac{2\pi}{T}t\right), \sin\left(k\frac{2\pi}{T}t\right))$ for the Fourier series are mutually orthogonal and normalized

$$\frac{2}{T}\int_0^T \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}dt \quad = \quad 1 \tag{21.30}$$

$$\frac{2}{T}\int_0^T \cos\left(k\frac{2\pi}{T}t\right)\cos\left(l\frac{2\pi}{T}t\right)dt \quad = \quad \begin{cases} 1 & l = k \\ 0 & l \neq k \end{cases}, \ k > 0,\ l > 0 \tag{21.31}$$

$$\frac{2}{T}\int_0^T \sin\left(k\frac{2\pi}{T}t\right)\sin\left(l\frac{2\pi}{T}t\right)dt \quad = \quad \begin{cases} 1 & l = k \\ 0 & l \neq k \end{cases}, \ k > 0,\ l > 0 \tag{21.32}$$

$$\frac{2}{T}\int_0^T \cos\left(k\frac{2\pi}{T}t\right)\sin\left(l\frac{2\pi}{T}t\right)dt \quad = \quad 0 \tag{21.33}$$

$$\frac{2}{T}\int_0^T \frac{1}{\sqrt{2}}\cos\left(k\frac{2\pi}{T}t\right)dt \quad = \quad 0 \tag{21.34}$$

$$\frac{2}{T}\int_0^T \frac{1}{\sqrt{2}}\sin\left(k\frac{2\pi}{T}t\right)dt \quad = \quad 0 \tag{21.35}$$

  * Proofs of the above can be obtained using trigonometric identities
  * Figure 21.1 graphically demonstrates orthogonality of trig functions.

- Present how the Fourier series of a signal changes as it passes through a system (filter).

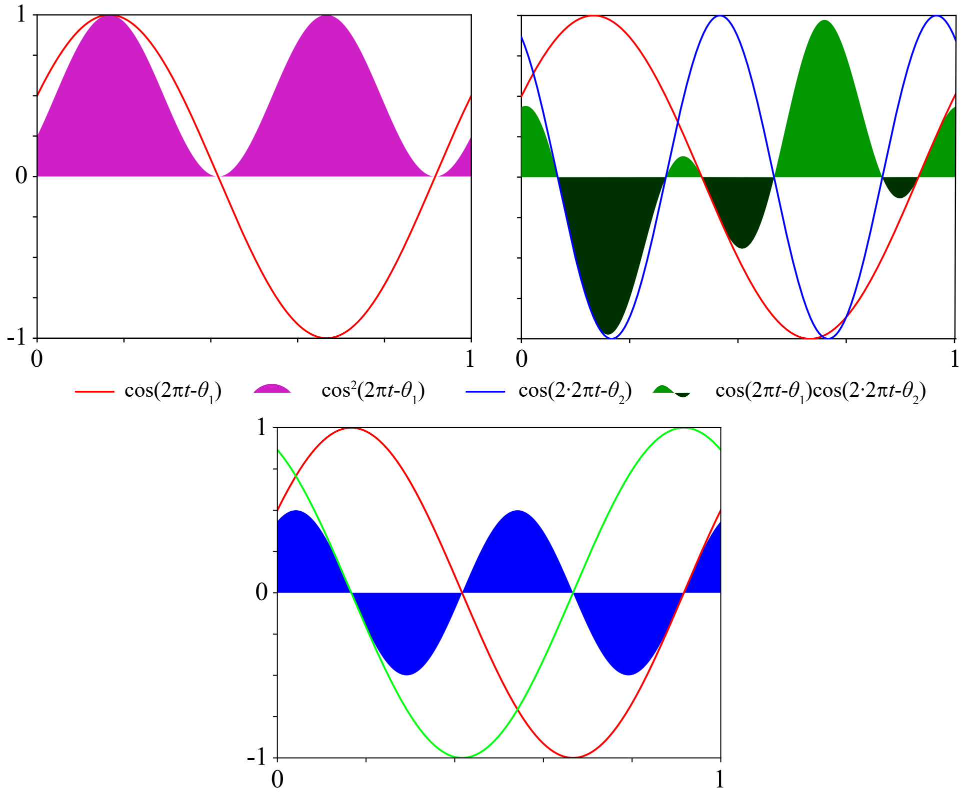


Figure 21.1: (Left) The area of a the square of a unit sinusoid is $\int_0^1 \cos^2(2\pi t - \theta_1)dt = \frac{1}{2}$. (Right) Two sinusoids with frequencies that are integer multiples of a fundamental frequency (e.g., $\cos(2\pi t - \theta_1)$ and $\cos(2 \cdot 2\pi t - \theta_2)$) are orthogonal over the period of the fundamental frequency (e.g., $\int_0^1 \cos(2\pi t - \theta_1)\cos(2 \cdot 2\pi t - \theta_2)dt = 0$). (Bottom) Update the figure to show orthogonalityof sine with cosine.

- Examples

Example 21.1: Square wave

Square wave with $T = 1$

- function

$$f(t) = \begin{cases} ... \\ 0 & -\frac{1}{2} < t < 0 \\ 1 & 0 < t < \frac{1}{2} \\ ... \end{cases} \tag{21.36}$$

- Cosine coefficients

$$\begin{aligned}
a_k &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(k\frac{2\pi}{T}t\right)dt && (21.37)\\
&= \frac{2}{1}\int_{-\frac{1}{2}}^{0}(0)\cos\left(k\frac{2\pi}{1}t\right)dt + \frac{2}{1}\int_{0}^{\frac{1}{2}}(1)\cos\left(k\frac{2\pi}{1}t\right)dt && (21.38)\\
&= \frac{1}{k2\pi}\left(2\sin\left(k2\pi t\right)\Big|_0^{\frac{1}{2}}\right) && (21.39)\\
&= \frac{\sin\left(k\pi\right) - 0}{k\pi} && (21.40)\\
&= \frac{\sin\left(k\pi\right)}{k\pi} && (21.41)\\
a_0 &= \lim_{k\to 0}\frac{\sin\left(k\pi\right)}{k\pi} && (21.42)\\
&= \lim_{k\to 0}\frac{\pi\cos\left(k\pi\right)}{\pi} && (21.43)\\
&= 1 && (21.44)\\
a_k &= 0,\ k = \{1, 2, 3, ...\} && (21.45)
\end{aligned}$$

  - Note, a_0 is nonzero because the average value is non zero.
  - The remaining a coefficients are zero because the function would be odd if a_0 were 0.

- Sine coefficients

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(k\frac{2\pi}{T}t\right)dt && (21.46)\\
&= \frac{2}{1}\int_{-\frac{1}{2}}^{0}(0)\sin\left(k\frac{2\pi}{1}t\right)dt+\frac{2}{1}\int_{0}^{\frac{1}{2}}(1)\sin\left(k\frac{2\pi}{1}t\right)dt && (21.47)\\
&= -\frac{1}{k2\pi}\left(2\cos\left(k2\pi t\right)\Big|_0^{\frac{1}{2}}\right) && (21.48)\\
&= -\frac{\cos\left(k\pi\right)-1}{k\pi} && (21.49)\\
&= -\frac{1}{k\pi}\Big((-1)^k-1\Big),\ k>0 && (21.50)\\
b_k &= \begin{cases}\frac{2}{k\pi} & k=\{1,3,5,...\}\\ 0 & k=\{2,4,6,...\}\end{cases} && (21.51)
\end{aligned}
$$

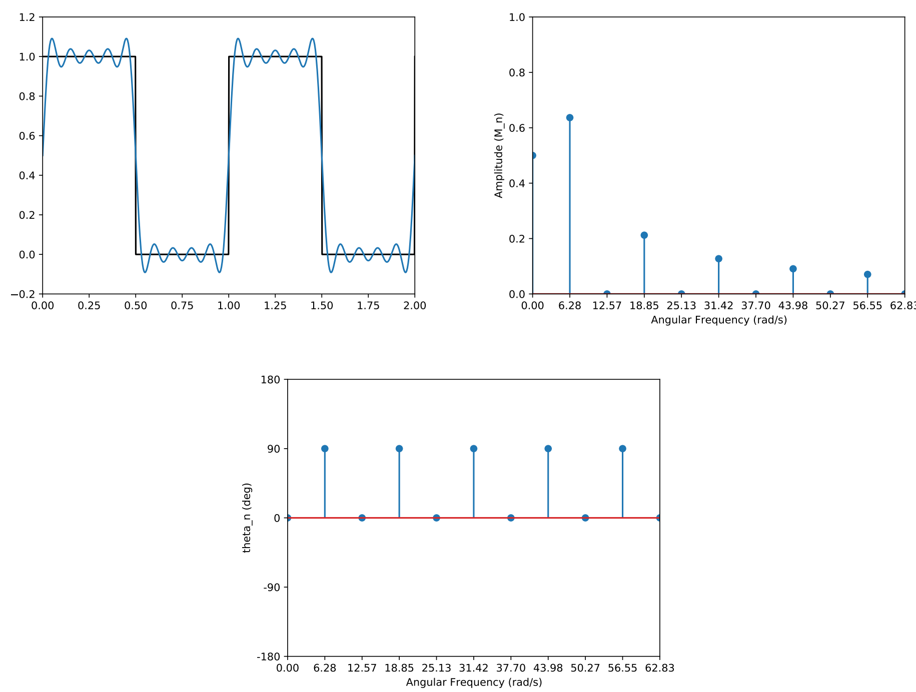


Figure 21.2: (Left) Fourier series approximation of square wave including terms up to $k = 10$. (Center) Amplitude spectrum for terms up to $k = 10$. (Right)

Phase spectrum for terms up to $k = 10$.

Example 21.2: Sawtooth wave

Sawtooth wave with $T = 1$

- function

$$f(t) = \begin{cases} ... \\ t & 0 < t < 1 \\ ... \end{cases} \tag{21.52}$$

- Cosine coefficients

$$\begin{aligned}
a_k &= \frac{2}{T}\int_0^T f(t)\cos\left(k\frac{2\pi}{T}t\right)dt && (21.53)\\
&= \frac{2}{1}\int_0^1 t\cos\left(k\frac{2\pi}{1}t\right)dt && (21.54)\\
&= 2\left(t\frac{1}{k2\pi}\sin(k2\pi t)\Big|_0^1 - \int_0^1 \frac{1}{k2\pi}\sin(k2\pi t)\,dt\right) && (21.55)\\
&= 2\left(\frac{1}{k2\pi}\sin(k2\pi) + \frac{1}{4k^2\pi^2}\cos(k2\pi t)\Big|_0^1\right) && (21.56)\\
&= \frac{2}{4k^2\pi^2}\left(k2\pi\sin(k2\pi) + \cos(k2\pi) - 1\right) && (21.57)\\
&= \frac{k2\pi\sin(k2\pi) + \cos(k2\pi) - 1}{2k^2\pi^2} && (21.58)\\
a_0 &= \lim_{k\to 0}\frac{k2\pi\sin(k2\pi) + \cos(k2\pi) - 1}{2k^2\pi^2} && (21.59)\\
&= \lim_{k\to 0}\frac{2\pi\sin(k2\pi) + k2^2\pi^2\cos(k2\pi) - 2\pi\sin(k2\pi)}{4k\pi^2} && (21.60)\\
&= \lim_{k\to 0}\frac{2^2\pi^2\cos(k2\pi) + 2^2\pi^2\cos(k2\pi) - k2^3\pi^3\sin(k2\pi) - 2^2\pi^2\cos(k2\pi)}{4\pi^2} && (21.61)\\
&= \frac{2^2\pi^2 + 2^2\pi^2 - 2^2\pi^2}{4\pi^2} && (21.62)\\
&= 1 && (21.63)\\
a_k &= \frac{k2\pi\sin(k2\pi) + \cos(k2\pi) - 1}{2k^2\pi^2} && (21.64)\\
& && (21.65)\\
&= 0,\ k = \{1, 2, 3, ...\} && (21.66)
\end{aligned}$$

- Note, a_0 is nonzero because the average value is non zero.
- The remaining a coefficients are zero because the function would be odd if a_0 were 0.

- Sine coefficients

$$b_k = \frac{2}{T}\int_0^T f(t)\sin\left(k\frac{2\pi}{T}t\right)dt \tag{21.67}$$

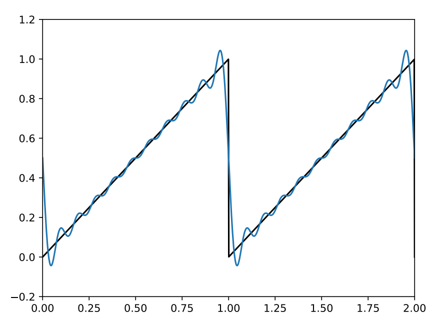
$$= \frac{2}{1}\int_0^1 t\sin\left(k\frac{2\pi}{1}t\right)dt \tag{21.68}$$

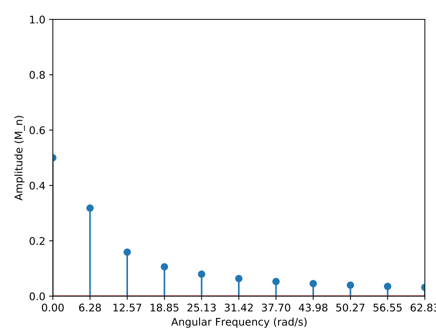
$$= 2\left(-t\frac{1}{k2\pi}\cos(k2\pi t)\Big|_0^1 + \int_0^1 \frac{1}{k2\pi}\cos\left(k2\pi t\right)dt\right) \tag{21.69}$$

$$= -\frac{1}{k\pi} + \int_0^1 \frac{1}{k\pi}\cos\left(k2\pi t\right)dt \tag{21.70}$$

$$= -\frac{1}{k\pi} + \frac{1}{2k^2\pi^2}\sin\left(k2\pi t\right)\Big|_0^1 \tag{21.71}$$

$$b_k = -\frac{1}{k\pi},\ k = \{1,2,3,...\} \tag{21.72}$$





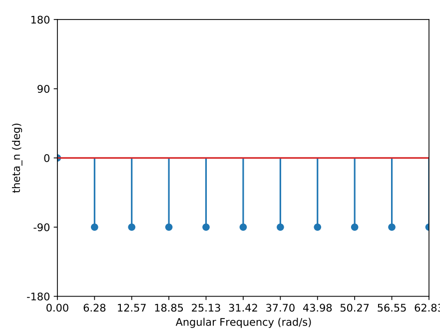


Figure 21.3: (Left) Fourier series approximation of sawtooth wave including terms up to $k = 10$. (Center) Amplitude spectrum for terms up to $k = 10$. (Right) Phase spectrum for terms up to $k = 10$.

Example 21.3: Impulse train

Impulse train with $T = 1$

- function

$$f(t) = \begin{cases} ... \\ \delta(t) & 0 < t < 1 \\ ... \end{cases} \tag{21.73}$$

- Cosine coefficients

$$\begin{aligned} a_k &= \frac{2}{T}\int_0^T f(t)\cos\left(k\frac{2\pi}{T}t\right)dt & (21.74)\\ &= \frac{2}{1}\int_0^1 \delta(t)\cos\left(k\frac{2\pi}{1}t\right)dt & (21.75)\\ &= \frac{2}{1}\cos(0) & (21.76)\\ &= 2,\ k = \{0, 1, 2, 3, ...\} & (21.77)\end{aligned}$$

  - Note, a_0 is nonzero because the average value is non zero.
  - The remaining a coefficients are zero because the function would be odd if a_0 were 0.

- Sine coefficients

$$\begin{aligned} b_k &= \frac{2}{T}\int_0^T f(t)\sin\left(k\frac{2\pi}{T}t\right)dt & (21.78)\\ &= \frac{2}{1}\int_0^1 \delta(t)\sin\left(k\frac{2\pi}{1}t\right)dt & (21.79)\\ &= \frac{2}{1}\sin(0) & (21.80)\\ b_k &= 0,\ k = \{1, 2, 3, ...\} & (21.81)\end{aligned}$$

- Comments
  - The frequency spectrum is constant (amplitude and phase)
  - An impulse includes all frequencies equally

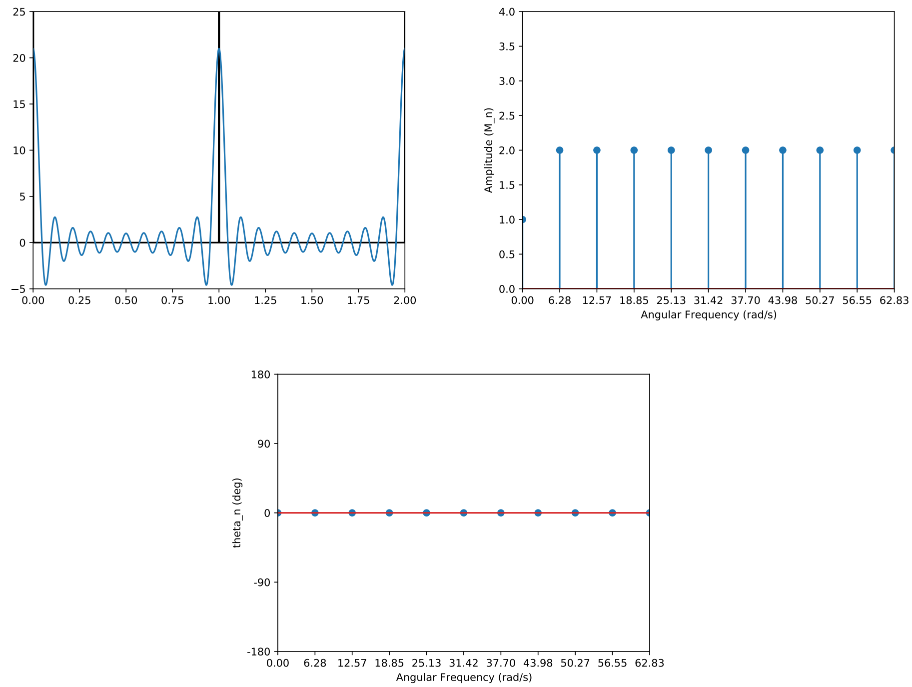


Figure 21.4: (Left) Fourier series approximation of an impulse train including terms up to $k = 10$. (Center) Amplitude spectrum for terms up to $k = 10$. (Right) Phase spectrum for terms up to $k = 10$.

Example 21.4: pulse train

Pulse train with $T = 1$

- function update this

$$f(t) = \begin{cases} \dots & \\ A_o & -T_o < t < T_o \\ \dots & \end{cases} \tag{21.82}$$

- Cosine coefficients

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(k\frac{2\pi}{T}t\right)dt && (21.83)\\
&= \frac{2}{1}\int_{-T_o}^{T_o} A_o\cos\left(k\frac{2\pi}{1}t\right)dt && (21.84)\\
&= \frac{2A_o}{k2\pi}\sin\left(k2\pi t\right)\Big|_{-T_o}^{T_o} && (21.85)\\
&= \frac{A_o}{k\pi}\left(\sin\left(k2\pi T_o\right)-\sin\left(-k2\pi T_o\right)\right) && (21.86)\\
&= \frac{2A_o\sin\left(k2\pi T_o\right)}{k\pi} && (21.87)\\
a_0 &= \lim_{k\to 0}\frac{2A_o\sin\left(k2\pi T_o\right)}{k\pi} && (21.88)\\
&= \lim_{k\to 0}\frac{2A_o 2\pi T_o\cos\left(k2\pi T_o\right)}{\pi} && (21.89)\\
&= 4A_oT_o && (21.90)\\
a_k &= \frac{2A_o\sin\left(k2\pi T_o\right)}{k\pi},\ k=\{0,1,2,3,...\} && (21.91)
\end{aligned}
$$

  - Note, a_0 is nonzero because the average value is non zero.
  - The remaining a coefficients are zero because the function would be odd if a_0 were 0.

- Sine coefficients even function, but show the math

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_0^T f(t)\sin\left(k\frac{2\pi}{T}t\right)dt && (21.92)\\
&= \frac{2}{1}\int_0^1 \delta(t)\sin\left(k\frac{2\pi}{1}t\right)dt && (21.93)\\
&= \frac{2}{1}\sin\left(0\right) && (21.94)\\
b_k &= 0,\ k=\{1,2,3,...\} && (21.95)
\end{aligned}
$$

- Special cases
  - Square wave
  - Impulse train

- Fourier series of the steady state system response can be obtained from the superposition of the Fourier series terms of the input

Example 21.5:

Determine the output of an RC filter with transfer function

$$G(s) = \frac{1}{RCs+1} \tag{21.96}$$

where $R = 1$ kΩ and $C = 1$ μF given a Fourier series expression for the 100 Hz square wave input to the system

$$u(t) = \sum_{k=1}^{\infty} b_k \sin(k\omega_T t) \tag{21.97}$$

$$= \sum_{k=\{1,3,5,\ldots\}}^{\infty} \frac{2}{k\pi} \sin(k 2\pi \cdot 100 t). \tag{21.98}$$

- Magnitude ratio

$$M(\omega) = |G(j\omega)| \tag{21.99}$$

$$= \left| \frac{1}{RCj\omega+1} \right| \tag{21.100}$$

$$= \frac{1}{\sqrt{(RC\omega)^2+1}} \tag{21.101}$$

- Phase

$$\phi(\omega) = \angle G(j\omega) \tag{21.102}$$

$$= \angle \frac{1}{RCj\omega+1} \tag{21.103}$$

$$= \angle 1 - \angle\left(RCj\omega+1\right) \tag{21.104}$$

$$= 0 - \tan^{-1}\left(\frac{RC\omega}{1}\right) \tag{21.105}$$

$$= -\tan^{-1}\left(RC\omega\right) \tag{21.106}$$

- Steady state response

$$y(t) = \sum_{k=1}^{\infty} b_k M(k\omega_T) \sin(k\omega_T t + \phi(k\omega_T)) \tag{21.107}$$

$$= \sum_{k=1}^{\infty} b_k \frac{1}{\sqrt{(RCk\omega_T)^2 + 1}} \sin(k\omega_T t - \tan^{-1}(RCk\omega_T)) \tag{21.108}$$

$$= \sum_{k=\{1,3,5,\ldots\}}^{\infty} \frac{2}{k\pi} \frac{1}{\sqrt{(RCk\omega_T)^2 + 1}} \sin(k\omega_T t - \tan^{-1}(RCk\omega_T)) \tag{21.109}$$

add plots in the time and frequency domain

Example 21.6: Beating Frequency

Find the period of the periodic signal

$$y(t) = \frac{1}{2}\cos(9 \cdot 2\pi t) - \frac{1}{2}\cos(11 \cdot 2\pi t). \tag{21.110}$$

- The signal $y(t)$ is the superposition of two periodic signals with frequencies of 9 and 11 Hz. The greatest common factor of 9 and 11 Hz is 1 Hz. Consequently, the frequency of the resultant periodic signal is 1 Hz and the period is 1 s.
- Using trig identities, we can express this signal is a product of two sinusoids

$$y(t) = \frac{1}{2}\cos(9 \cdot 2\pi t) - \frac{1}{2}\cos(11 \cdot 2\pi t) \tag{21.111}$$

$$= \sin(1 \cdot 2\pi t)\sin(10 \cdot 2\pi t), \tag{21.112}$$

to reveal the frequency, 1 Hz, of the periodic signal.

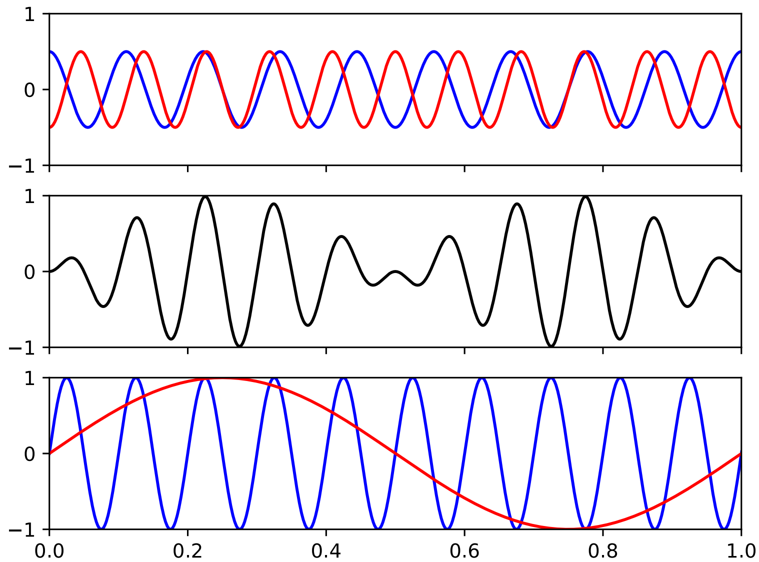


Figure 21.5: (Top) Two cosine functions: $\frac{1}{2}\cos(9 \cdot 2\pi t)$ and $-\frac{1}{2}\cos(11 \cdot 2\pi t)$. (Bottom) Two sine functions: $\sin(10 \cdot 2\pi t)$ and $\sin(1 \cdot 2\pi t)$. (Middle) This curve can be obtained by either (i) the superposition of the functions plotted on (Top) or (ii) the product of the functions plotted on (Bottom).

- The frequency of the amplitude envelope (in this case $11 - 9$ Hz= 2×1 Hz=2 Hz) is known as the beating frequency. verify this definition.

- Fourier Transform, Discrete Fourier Transform, and Fast Fourier Transform
  - Fourier series representations are used to represent continuous time signals that are usually either periodic or exist for a finite period of time.
  - Fourier transform
    - Relationship to Fourier series; see, for example, [32]
    - Mathematical definition
  - Discrete Fourier transform
    - FFT

Example 21.7: FFT of audio recording

Audio recording - see example script in Appendix. Record sound from an instrument.

Chapter 22

Data Acquisition and Aliasing

- Aperture time t_a
  - Aperture time (for ADCs without sample and hold functionality)
    - An ADC requires finite time to complete an analog to digital conversion
    - We define the aperture time as the time required by the ADC to complete an analog to digital conversion; see, of example, [10].
  - Aperture time limits signal frequency
    - (The derivation below parallels an example in [19] for ADCs that do not include sample and hold functionality.)
    - Consider a sinusoidal signal centered in the input range and sweeping across the entire input range (see Figure 22.1)

$$V_{\text{in}} = 2^{n-1}Q\sin(2\pi ft) + 2^{n-1}Q + V_{\text{in,min}} \tag{22.1}$$

    - Time rate of change of input voltage

$$\dot{V}_{\text{in}} = 2^{n-1}Q2\pi f\cos(2\pi ft) \tag{22.2}$$

    - Max time rate of change of input voltage

$$\begin{aligned}\dot{V}_{\text{in,max}} &= 2^{n-1}Q2\pi f && (22.3)\\ &= 2^{n}Q\pi f && (22.4)\end{aligned}$$

    - Max change in voltage during aperture time

$$\begin{aligned}\Delta V_{\text{in,max}} &= \dot{V}_{\text{in,max}}t_a && (22.5)\\ &= 2^{n}Q\pi ft_a && (22.6)\end{aligned}$$

    * For reliable sampling/measurements, $\Delta V_{\text{in,max}}$ must be smaller than Q; see [19].

$$Q > \Delta V_{\text{in,max}} \tag{22.7}$$
$$> 2^n Q \pi f t_a \tag{22.8}$$
$$t_a < \frac{1}{\pi f 2^n} \tag{22.9}$$
$$f < \frac{1}{\pi t_a 2^n} \tag{22.10}$$

    * Therefore the aperture time must be smaller than $\frac{1}{\pi f 2^n}$ to measure a signal with highest frequency component f
    * Alternatively, the highest frequency in the signal should be smaller than $\frac{1}{\pi t_a 2^n}$
    * Example (based on an example in [19]):
        · 12-bit ADC with aperture time of 1μs
        · The max sample rate would be something like: $\frac{1}{1\ \mu\text{s}} = 1$ MHz
        · However, using the presentation above, the maximum signal frequency is 7.7 Hz signals
        · [19] uses and example like this to demonstrate the need for a sample and hold functionality.

- ADCs with sample and hold functionality
    * Most ADC systems utilize a sample and hold amplifier to limit the effects of the aperture time; see, for example, [19]
        · A sample and hold amplifier samples the voltage of a signal over a relatively short period of time and then maintains that voltage for further processing (e.g., analog to digital conversion).
        · The voltage held on a sample and hold amplifier is approximately the average voltage over its aperture time; see, for example, [57].
    * Sample and hold functionality may be used to hold a constant voltage during analog to digital conversion and thereby avoid the frequency limit of the aperture time of the ADC.
    * For ADCs with sample and hold functionality, the term aperture time refers to the amount of time required to take an individual sample with the sample and hold amplifier rather than the length of time for analog to digital conversion; see????
    * For ADCs with sample and hold functionality, the frequency is limited by aperture jitter; see, for example [22, 58].

Example 22.1:

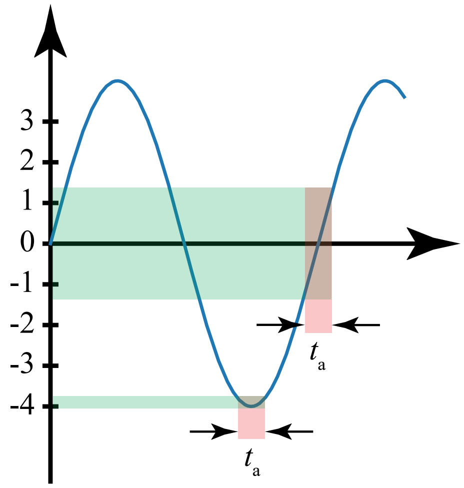


Figure 22.1:
A sine wave with 4 V amplitude and frequency $f = \frac{1}{T}$ is sampled by an analog to digital converter with ±4 V range, 3 bits, and aperture time $t_a = \frac{1}{10}T$.

- If the voltage is sampled at a time in which the slope of the signal is relatively small, the signal doesn't change much compared to the resolution of the analog to digital converter (Q) over t_a.
- However, if the slope is relatively large, the signal can change by more than Q over t_a.
- For $t_a = \frac{1}{10}T$ the max change in the signal is about $2.5Q$

$$\begin{aligned} \Delta V_{\text{in,max}} &= 2^n Q\pi f t_a && (22.11) \\ &= 2^3 Q\pi \frac{1}{T}\frac{1}{10}T && (22.12) \\ &= \frac{8\pi}{10}Q && (22.13) \\ &= 2.51Q && (22.14) \end{aligned}$$

- Aliasing
  - Demos -
    - see, for example, https://www.youtube.com/watch?v=QOwzkND_ooU

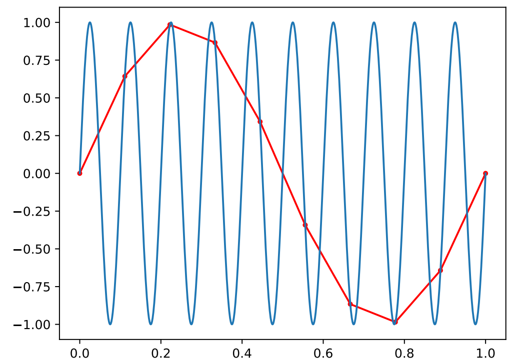


Figure 22.2:

    - AliasingDemo.m
    - Danger of using ‘Scatter with Smooth Lines’ in Microsoft Excel -

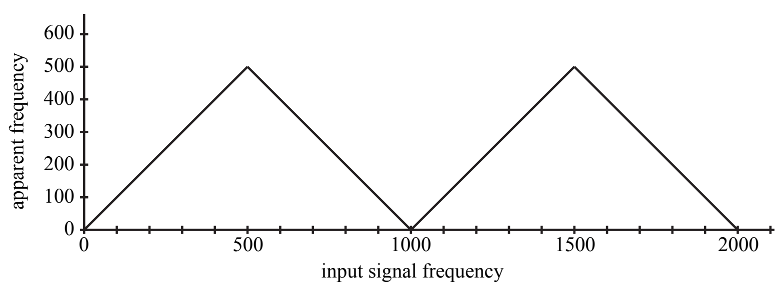


Figure 22.3: Folding diagram for a sample rate of 1000 samples/second.

  - Folding diagram
  - add an example where 400 and 600 Hz signals are sampled at 1000 S/s and show they yield opposite curves.
  - Sampling rate theorem
    - “This theorem simply states that the sampling rate must be greater than twice the highest-frequency component of the original signal in order to reconstruct the original waveform correctly [10].”

* “The sampling theorem states that to reconstruct the frequency content of a measured signal accurately, the sample rate must be more than twice the highest ferquency contained in the measured signal [4].”
* ‘Nyquist frequency’ [4] or ‘folding frequency’ [10]

$$f_N = \frac{f_s}{2} \tag{22.15}$$

where f_s is the sample rate
* In practice we probably want more like 10 samples per period or $f_s > 10 f_{\text{signal}}$

– How do we avoid aliasing?
  * Use a high sample rate
    · Practical ADC have sample rate limitations
    · Although we might have a guess about the frequency content of a signal, in general we measure a signal because it is unknown. Therefore, we don’t know the highest frequency in signals we measure before measurement.

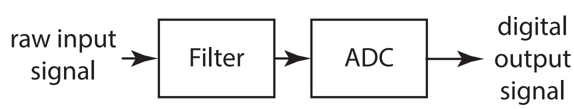


Figure 22.4: Folding diagram for a sample rate of 1000 samples/second

  * Use a low pass filter to filter out high frequency content above f_N
    · Note, the filter must be an analog filter applied before the ADC

– Filter selection; see, for example, [10]
  * Filter requirement - ideally we want the filter to reduce the amplitude of any signals with frequency exceeding f_N to an amplitude below the resolution Q

Example 22.2: First order filter

Determine the relationship between the break frequency of a first order filter, ω_b, and the Nyquist frequency, f_N, of an n-bit ADC such that all signal components with frequency larger than f_N are attenuated to less than Q by the filter.

- Assume signals to be measured are within the range of the ADC
- Consider a sinusoidal signal with frequency f_N and amplitude of half the span of the ADC

$$x(t) = Q2^{n-1} \sin(2\pi f_N t) \tag{22.16}$$

- The first order low pass filter with break frequency ω_b may be represented with transfer function

$$G(s) = \frac{1}{\frac{1}{\omega_b}s + 1} \tag{22.17}$$

- We want the amplitude of the steady state output of the filter to be no greater than the resolution of the ADC, Q

$$\begin{aligned}
\cancel{Q} &> \cancel{Q}2^{n-1}|G(j2\pi f_N)| && (22.18)\\
1 &> 2^{n-1}\frac{1}{\sqrt{\left(\frac{2\pi f_N}{\omega_b}\right)^2+1}} && (22.19)\\
\sqrt{\left(\frac{2\pi f_N}{\omega_b}\right)^2+1} &> 2^{n-1} && (22.20)\\
\left(\frac{2\pi f_N}{\omega_b}\right)^2+1 &> 2^{2(n-1)} && (22.21)\\
\left(\frac{2\pi f_N}{\omega_b}\right)^2 &> 2^{2(n-1)}-1 && (22.22)\\
\frac{2\pi f_N}{\omega_b} &> \sqrt{2^{2(n-1)}-1} && (22.23)\\
\frac{\omega_b}{2\pi f_N} &< \frac{1}{\sqrt{2^{2(n-1)}-1}} && (22.24)
\end{aligned}$$

n	$\max \frac{\omega_b}{2\pi f_N}$
8	$7.8 \cdot 10^{-3}$
12	$4.9 \cdot 10^{-4}$
16	$3.1 \cdot 10^{-5}$
32	$4.7 \cdot 10^{-10}$

- Note, with a simple first order filter, the break frequency has to be substantially smaller than the Nyquist frequency for typical values of n. Consequently, higher order filters are often used in practice.

Chapter 23

Noise and Interference

23.1 Introduction

- Notes

  Check out [9], it appears to be the basis for [59].

  - See also [60]

- Signal

  - Signal - For our purposes, signals are some physical action (e.g., voltage as a function of time) used to transmit information; see, for example, [61].
  - Signals can be corrupted by undesirable noise/interference.
  - Unfortunately, sometimes we use the term signal to refer to noise/interference (e.g., noise signal).

- Undesirable signal contributions (noise and interference)

  - Many authors use the terms ‘noise’ and ‘interference’ interchangeably. For example:
    * “Noise–meaning any undesirable signal interfering with a measurement–is an unavoidable element of all measurements [18].”
    * “It is difficult to transmit such [low] voltages over long distances because the wires tend to pick up electrical noise (interference) [10].”
  - Deterministic noise and/or interference
    * The term ‘interference’ tends to be used by some authors to refer to undesirable deterministic and/or periodic contributions to signals
      · “Interference produces undesirable deterministic trends on the measured value because of extraneous variables [4].”

- · "Coherent interference, as the name suggests, generally has its origins in periodic, manmade phenomena ... [13]."
- · "Coherent interference, unlike random noise, has narrowband PDS, often with harmonic peaks at integral multiples of the fundamental frequency [13]."
- · "Unwanted random signals are usually referred to as noise signals and unwanted deterministic signals as interference signals [9]."

∗ (Drift and offset aren't the focus right now)

– Random noise and/or interference

∗ The term 'noise' tends to be used by some authors to refer to random undesirable contributions to signals

- · "Noise is considered to arise in a circuit or measurement system from completely random phenomena [13]."
- · "The incoherent noise that we will consider is random noise from within the measurement system; coherent noise usually enters a system from without [13]." Here the term noise is used in the context of both random and deterministic/coherent processes.
- · "Noise is a random variation of the value of the measured signal as a consequence of the variation of the extraneous variables [4]."
- · "Unwanted random signals are usually referred to as noise signals ... [9]."

∗ Stationary random noise/interference

- · "A random signal is stationary if its statistical properties (usually, its average or mean square) do not change with time [49]."
- · When noise is stationary "... the physical phenomena giving rise to the noise are assumed not to change with time. When stationarity is assumed for a noise, averages over time are equivalent to ensemble averages [13]."
- · Here we will focus on stationary noise/interference
- · "Under most conditions, we assume that the random noise arising in electronic circuits is modeled by a Gaussian PDF. Many mathematical benefits follow this approximation; for example, the output $y(t)$ of an LS is Gaussian with variance σ_y^2 given the input $x(t)$ to be Gaussian with variance σ_x^2 [13]."

∗ Ergodicity

– Here we consider noise and/or interference to be superimposed on a signal [59]

$$\begin{aligned} \text{measurement} &= \text{signal} + \text{noise/interference} && (23.1)\\ y(t) &= s(t) + n(t) && (23.2) \end{aligned}$$

∗ (Contrast this with distortion, for example: "Distortion refers to a notable change in the shape of the waveform, as opposed to simply an amplitude alteration or relative phase shift [4].")

∗ Typically, we assume the signal, $s(t)$, and noise, $n(t)$, are independent functions.

23.2 Noise Quantification and Propagation

- Mean value

$$\begin{aligned}\overline{n(t)} &= \langle n(t) \rangle &(23.3)\\ &= \lim_{T\to\infty} \frac{1}{T}\int_0^T n(t)dt &(23.4)\end{aligned}$$

- Mean Squared (MS) value; see, for example, [49]

$$\begin{aligned}\overline{n^2(t)} &= \langle n^2(t) \rangle &(23.5)\\ &= \lim_{T\to\infty} \frac{1}{T}\int_0^T n^2(t)dt &(23.6)\end{aligned}$$

  - In many applications, the mean squared value is proportional to the average power and therefore it's often called average power.

- Root Mean Square (RMS) value

$$n_{\text{rms}} = \sqrt{\overline{n^2(t)}} \tag{23.7}$$

- Signal to Noise Ratio (SNR)
  - Signal to noise ratio; see, for example, [59, 12, 62] and implied by [13]

$$\text{SNR} = \frac{\overline{s^2(t)}}{\overline{n^2(t)}} \tag{23.8}$$

  - SNR in dB

$$\text{snr} = 10\log_{10}\left[\frac{\overline{s^2(t)}}{\overline{n^2(t)}}\right] \tag{23.9}$$

  - Comment on how snr is an appropriate use of dB because it comes from a dimensionless ratio of powers.

- Power spectral density or power density spectra
  - Fourier series based definition; see, for example, [59]
    * Although it's typically a key part of the power spectral density, in the following we avoid the Fourier Transform. Consequently, this 'derivation' may have some limitations.

* Average power of a random signal (assuming power is proportional to $n^2(t)$) over a time interval

$$\overline{n^2(t)} = \langle n^2(t) \rangle \tag{23.10}$$
$$= \frac{1}{T}\int_0^T n^2(t)dt \tag{23.11}$$

* Expressing $n(t)$ as a fourier series

$$\overline{n^2(t)} = \frac{1}{T}\int_0^T \left(\frac{1}{2}a_0 + \sum_{k=1}^{\infty} M_k \cos(k\omega_T t - \theta_k)\right)^2 dt \tag{23.12}$$

* Recognizing the orthogonality of terms in the Fourier series; see Figure 21.1

$$\overline{n^2(t)} = \frac{1}{T}\int_0^T \left(M_0 + \sum_{k=1}^{\infty} M_k \cos(k\omega_T t - \theta_k)\right)^2 dt \tag{23.13}$$
$$= \frac{1}{T}\int_0^T (M_0)^2\, dt + \sum_{k=1}^{\infty}\int_0^T (M_k \cos(k\omega_T t - \theta_k))^2\, dt \tag{23.14}$$
$$= M_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} M_k^2 \tag{23.15}$$

* Power spectrum (the argument of the sum)

$$= \frac{1}{2}M_k^2 \tag{23.16}$$

* In the limit as $T \to \infty$ and $\omega_T \to 0$, the above sum turns into an integral in which the integrand, $\Phi(\omega)$, is known as the power spectral density

$$\text{average power} = \int_0^{\infty} \Phi(\omega)d\omega \tag{23.17}$$

* In this definition, only positive frequencies are considered (i.e., one-sided). **This is the definition historically employed in ME365.** Other, possibly more common, definitions employ negative frequencies (see below).

– Fourier transform based definition; see, for example, [32]. **This is not the definition historically used in ME365.**

* We define the autocorrelation of the random signal as

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} n(t)n(t+\tau)dt \tag{23.18}$$

* Note, $R(0) = \overline{n^2(t)}$

$$
\begin{aligned}
R(0) &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} n(t)n(t+0)dt && (23.19)\\
&= \overline{n^2(t)} && (23.20)
\end{aligned}
$$

* The Fourier transform of $R(\tau)$ is defined as

$$\hat{R}(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau \tag{23.21}$$

(See [32] for an example of how the Fourier transform may be obtained from the Fourier series in the limit as $T \to \infty$.)

* And inverse Fourier transform of $\hat{R}(\omega)$ is defined as

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{R}(\omega)e^{j\omega\tau}d\omega \tag{23.22}$$

Authors differ in how they treat the 2π in their definitions of Fourier transform and inverse fourier transform.

* Therefore

$$
\begin{aligned}
\overline{n^2(t)} &= R(0) && (23.23)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{R}(\omega)e^{j\omega 0}d\omega && (23.24)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{R}(\omega)d\omega && (23.25)
\end{aligned}
$$

* Therefore, the power spectral density is

$$\Phi(\omega) = \frac{\hat{R}(\omega)}{2\pi} \tag{23.26}$$

* An alternative derivation is available in, for example, [23]. See also [63].
* Note, this definition for power spectral density includes both positive and negative frequencies (i.e., two sided).

- White noise
- Noise propagation
  - Input-output relationship for power spectral density; see, for example, [49]

$$
\begin{aligned}
N_{\text{out}}(s) &= G(s)N_{\text{in}}(s) && (23.27)\\
\Phi_{\text{out}}(\omega) &= |G(j\omega)|^2\Phi_{\text{in}}(\omega) && (23.28)
\end{aligned}
$$

- Noise factor/figure of systems (e.g., amplifier); see, for example
  - The signal to noise ratio changes as a signal and associated noise passes through a system
  - Mathematical definition; see, for example, [13, 59]

$$\text{NF} = \frac{\text{input SNR}}{\text{output SNR}} \tag{23.29}$$

  - or in dB (now called noise figure by [13])

$$\text{nf} = 10 \log_{10} \left[\frac{\text{input SNR}}{\text{output SNR}} \right] \tag{23.30}$$

  - "Since a real amplifier is noisy and adds noise to the signal as well as amplifying it, the output SNR ... is always less than the input SNR ... [13]."
  - [13] makes a distinction between noise factor (not in dB) from noise figure (in dB). Part of the distinction might depend on whether SNR is mean squared or root mean squared.
  - [62] seems to use a different definition for noise figure and noise factor

- Theoretical signals
- Measured signals (including sampling)

23.3 Internal Noise Sources

- Noise sources see, for example, [12]
- The literature
  - [12, 59] Divide noise sources into the categories: exterior and internal
  - [18] identifies noise sources (e.g., AC power systems, high voltage/current circuits, switching circuits) and coupling mechanisms (conductive, capacitive, inductive, and radiative)
  - [9] discusses internal, external, and coupling mechanisms
  - "Note that there are three major sources of noise in the typical measurement system shown in Figure 1.1: noise sensed along with the QUM (environmental noise), noise associated with the electronic signal conditioning system (referred to its input), and equivalent noise generated in the analog-to-digital conversion process (quantization noise) [13]."
  - For a sketch for the noise spectrum, see Figure 3-5 on page 3-8 of https://download.tek.com/document/

  - Incorporate [9, 60]

- Internal; see, for example, [12, 62, 59]
  - Thermal or Johnson's noise -
    * "This kind of noise corresponds to the electrons in resistive components [12]."
    * "Thermal noise or Johnson noise is generated by the random collision of charge carriers with a lattice under conditions of thermal equilibrium [62]"
    * Power spectral density (i.e., mean squared voltage/Hz) [12, 13]

$$\phi(f) = 4k_B TR \tag{23.31}$$

      where $k_B = 1.38 \cdot 10^{-23}$ J/K is Boltzmann's constant, T is absolute temperature, and R is resistance [12]
    * PSD is constant (white noise)
  - Shottky's or Shot noise -
    * "... is due to random instances of charge carriers crossing P-N junctions [12]."
    * "Shot noise is caused by the random emission of electrons and by the random passage of charge carriers across potential barriers [62]."
    * Power spectral density (i.e., mean squared current/Hz) [59]

$$\phi(f) = 2Ie \tag{23.32}$$

      "... where I is the dc current through the device" [62], e is electron charge
    * PSD is constant (white noise)
  - Flicker or 1/f noise -
    * "This noise has various origins. In bipolar transistors it is due to defects on the semiconductor surface in the depletion zone of the base-emitter [12]."
    * "The imperfect contact between two conducting materials causes the conductivity to fluctuate in the presence of a dc current [62]."
    * Power spectral density for current (i.e., mean squared current/Hz)[59]

$$\phi(f) = \frac{C}{f^n} \tag{23.33}$$

      where C is a constant that depends upon the material [59] and according to [62] it depends upon the DC current
  - Burst or Popcorn noise -
    * "The nature of popcorn noise is not completely known, but we can say that it is linked to the presence of contaminating heavy metal ions in the circuits [12]."

    * "Burst noise or popcorn noise is caused by a metallic impurity in a pn junction [62]."
  - Avalanche or Zener noise - "Avalanche noise is produced by creating an avalanche in the P-N junctions of Zener diodes [12]."

- Exterior [12, 62]
  - [12] identifies two sources of exterior noise: "electric disturbances transmitted by conduction" and "radiated electric disturbances"
  - [59] identifies four sources of external noise: (i) 60 Hz harmonics, (ii) radio and TV, (iii) lightining and 'spark generators', and (iv) mechanical vibrations
  - In the next section, we discuss how noise from external sources can enter into circuits

23.4 Coupling Mechanisms to External Noise/Interference Sources

- Coupling mechanisms - the physical means by which noise/interference from external sources enter circuits
- incorporate chapter 90 of [62]

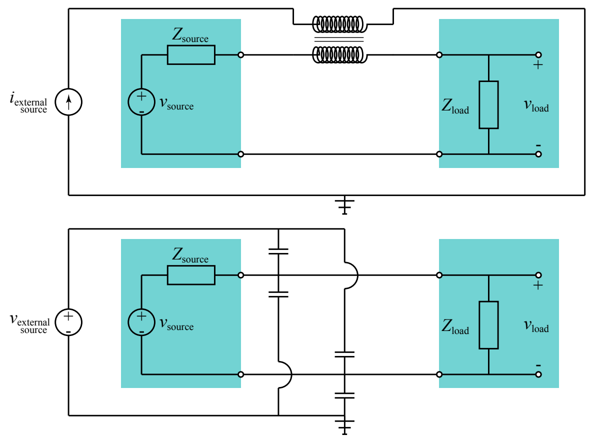


Figure 23.1: (Top) Inductive coupling with external source. (Bottom) Capacitive coupling with external source; see, for example, [64].

- Inductive coupling
  - Various names include: electromagnetic [60, 9] or inductive [18, 9, 64] or direct magnetic induction [13]
  - [60, 9, 64] imply inductive coupling yields only series mode signals.
- Capacitive coupling
  - Various names include: electrostatic [60, 9, 13, 64] or capacitive [18, 60, 9, 64]
  - [60, 9, 64] suggest capacitive coupling generally yields both series and common mode signals.
- Conductive coupling
  - Ground loops
    * Various names include: ground loops [13] or conductive [18] or shared impedance [60] or multiple earths [9, 64]

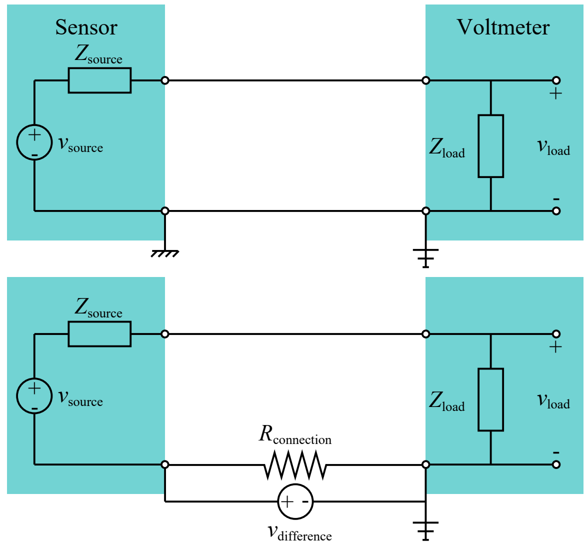


Figure 23.2: (Top) Measurement system in which the sensor and voltmeter are both grounded. As in [18], different symbols are used for these grounds as they are at different points. (Bottom) A model for a ground loop in which the potential difference between distinct grounds is represented by a voltage source and the resistance of the connection between sensor and voltmeter is shown.

    * In [9], 'multiple earths' yields both series and common mode signals. In [64], 'multiple earths' yields series mode signals.
  - Voltage transients; see, for example, [64]
    * Significant changes in power usage of heavy equipment near a measurment system can cause significant voltage spikes in power supplies for measurement system elements/stages.

- Radiative [18, 60]
  - "Normally the field strength of such transmissions is sufficiently small as to be of no importance unless an instrument is prompted to enter into oscillations as a result of a gratuitous 'tuned circuit' [60]."

23.5 Models

Models for representing noise/interference in a measurement system; see [60, 9]

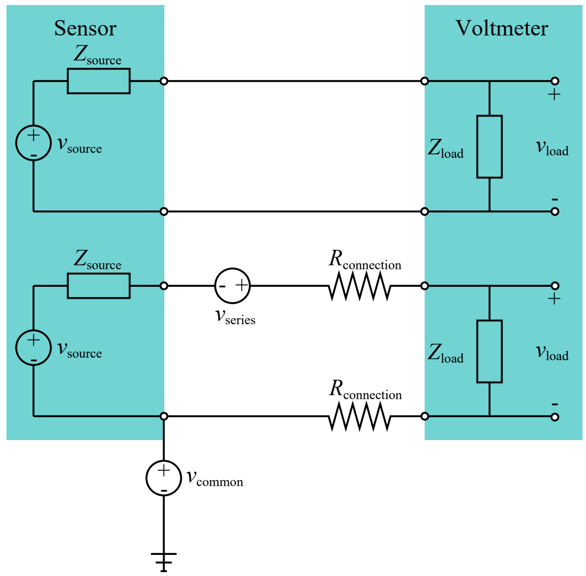


Figure 23.3: (Top) Desired circuit without noise/interference or connection resistance. (Bottom) Model of a circuit with series mode and common mode noise/interference as well as resistance in the connection between source and load; see, for example, [9, 60]. (Note, ground loops are omitted as well as details about whether the voltmeter is differential or single ended (or ground referenced [18]).)

- Components
  - Source - The source is some sensor producing voltage or current (e.g., thermocouple or accelerometer) and is represented by a Thevenin equivalent.
    * A Thevenin equivalent, with a voltage source, is shown.
    * A Norton equivalent, with current source, could be drawn instead.
    * Some authors recommend current transmission over voltage transmission, so a Norton perspective might be better. (e.g., "... all transmission signals to and from the field mounted equipment should employ current and not voltage signals, wherever a choice exists [60].")
  - Load - A voltage measurement device serves as a load with an equivalent impedance.
  - Connection resistance - Wires connecting source and load are modeled as resistors.

- (Might discuss grounded vs. floating sensors and differential vs. ground-referenced voltmeter; see, for example, [18].)

- 'Series mode' [9, 60, 64] noise/interferenece
  - To model series mode noise/interference, a voltage source representing noise/interference is placed in series with the source; see, for example, [9].
  - (Note, [13] presents an analysis of differential amplifiers and uses the term 'difference-mode.')
- Common mode noise/interferenece
  - One model for common mode noise/interference, is to add a voltage source representing noise/interference between ground and the lower end of the source; see, for example, [9]. ([60] suggests that there other models and no one model is suitable for all scenarios.)
  - "I believe the most valuable definition of a common mode signal is that of it being a signal which is present simultaneously on two input terminals of an instrument, measured with respect to a common third point [60]."
  - "The difference between the two ground point voltages is called the common-mode voltage [4]."
  - Ideal vs. practical differential measurements
    * Theoretically, measurements of the voltage difference across the load are not suceptible to common mode noise/interference because it appears in both terminals equally.
    * Practically, some of the common mode noise/interference will appear in real voltage measurements. ("... common-mode votages ... can be converted into series mode voltages ... [64].")
-

23.6 Reduction

Bentley makes the following eight recommendations for reducing noise/interference [9]. Here the recommendations reworded and some justification is offered.

1. Distance the measurement system from noise/interferences sources [9, 64]
   - Inductive and capacitive coupling is inversely proportional to the squared distance from the source [9, 64].
   - [64] recommends distances of no less than 0.3 m and suggests a better guideline is 1 m.

2. Use twisted pairs to reduce effects of inductive coupling [9, 64, 10, 18]
   - Twisted pairs is "the simplest way of reducing the effects of inductive coupling ... [9]."
   - (As an alternative to twisted pairs, magnetic shielding may be used. Note, "... shields of non magnetic materials are powerless against induced noise, and even in steel conduit there is a significant advantage to be gained in using the most twisted pair available [60].")
3. Use shielding to reduce effects of capacitive coupling [60, 9, 10]
   - "The best method of avoiding the problem of capacitive coupling to a power circuit ... is to enclose the entire measurement cicuit in an earthed metal screen or shield [9]."
   - The shield should be grounded at only one point [10], typically the signal source [4, 18, 64]
   - [60] points out that shielding reduces the effect of capacitive coupling rather than the capacitance.
4. Use stages/elements with high common mode rejection [10, 9]
   - Common mode rejection ratio
     - "The CMRR is defined as the ratio of the difference-mode gain [K_{dm}] to the common-mode gain [K_{cm}] ... [18]" often expressed in dB:

       $$\mathrm{CMRR} = 20 \log \left[\frac{K_{\mathrm{dm}}}{K_{\mathrm{cm}}} \right] \tag{23.34}$$

       see, for example, [10, 9, 18].
   - Use elements/stages with differential inputs rather than single-ended inputs
     - Single-ended - System elements/stages with single-ended (or 'ground-referenced' [18]) inputs take in the voltage at one terminal with respect to ground; see, for example, [4, 18]
     - Differential - System elements/stages with differential (or 'nonreferenced' [18]) inputs take in the voltages at two distinct terminals (e.g., the difference in voltages at the AI0+ and AI0- terminals on myRIO) and use the voltage difference between the terminals as the input; see, for example, [4, 18]
     - With some elements/stages the user can choose between single ended or differential inputs. (Often, twice as many measurements can be made when single-ended inputs are selected.)

- Amplifiers

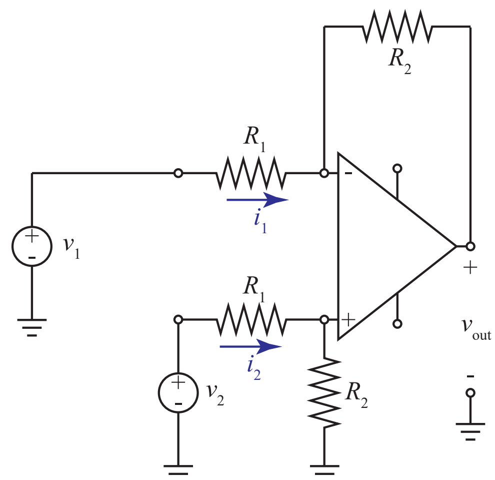


Figure 23.4: Op amp configured as a differential amplifier.

- Op amp in the differential amplifier configuration; see Figure 23.4
  * Equation

$$v_{\text{out}} = \frac{R_2}{R_1} v_{\text{diff}} \tag{23.35}$$

$$= \frac{R_2}{R_1}(v_2 - v_1) \tag{23.36}$$

  * Performance degrades when the pairs of resistors are not matched exactly; for example, [65] discusses the dependence of CMRR on resistor tolerance.
  * The input impedance of an op amp circuit in the differential amplifier configuration is generally small (i.e., $< R_1$) and coupled with its gain (i.e., $\frac{R_2}{R_1}$); see, for example, [9]. (In Figure 23.4, I believe $\frac{V_1(s)}{I_1(s)} = R_1 \frac{R_1}{R_1+R_2} < R_1$ when $v_2 = 0$ and $\frac{V_2(s)}{I_2(s)} = R_1 \frac{R_1^2+R_1R_2}{R_1^2+2R_1R_2} < R_1$ when $v_1 = 0$.)
  * (Note, the input impedance of a typical op amp by itself is on the order of 1 MΩ; see, for example, [9, 10].)
- Use instrumentation amplifiers
  * The output of an ideal instrumentation amplifier is proportional to the voltage difference accross the the input terminals and indepenent of any voltage components common to the two terminals; see, for example, [10].
  * "The advantage of the instrumentation amplifier compared with a standard operational amplifier is that its differential input impedance is much higher. In consequence, its common-mode rejection capability is much better [64]."

  * The input impedance of an instrumentation amplifier is largely independent of its gain.
  * (Note, a typical input resistance of an instrumenation amplifier is 10 GΩ; see, for example, [9].)

5. Filtering
6. Modulation
7. Averaging
8. Autocorrelation

Additional recommendations

9. Use current transmission where possible
   - Note, "... a current transmission system has far greater inherent immunity to series mode interfrence than a voltage transmission system [9]."
   - ".... the ability of this sort of circuit [curent transmission] to reject noise, or rather to limit its effect on the receiver, makes it the preferred type of installation [66]."
10. Use appropriate grounding techniques
    - See, for example, [62].

Chapter 24

Modulation

- Modulation
  - (Signals and Systems by Oppenheim and Willsky might be a good reference)
  - Motivation
    - Signals are not always suitable for transmission
      - Frequency content of the signal may be especially susceptible to noise/interference
      - Frequency content of the signal may be especially susceptible attenuation by the medium
      - It may not be practical to transmit the signal given its frequency content ("... antenna size is inversely proportional to the frequency of the signal, the lower the frequency of the signal the larger is the required antenna size [67].")
      - ...
    - Multiple signals need to be transmitted through the same medium
    - Some sensors output a modulated signal (e.g., LVDT)

- Amplitude modulation
  - Terminology and definitions; see, for example, [68]
    - Carrier signal $v(t) = V_c \sin(\omega_c t)$ (We could use $V_c \cos(\omega_c t)$ or other carrier signals instead.)
      - Carrier frequency ω_c
      - Carrier amplitude V_c
    - Modulating signal or 'baseband signal' $x(t)$
    - Modulated signal (Some authors use a slightly different equation)

$$e_m(t) = v(t)x(t) \tag{24.1}$$

- Mathematical manipulation
  * Let's represent $x(t)$ as a Fourier series (often the bandwidth of the modulating signal is significantly smaller than the carrier frequency, $N\omega_T \ll \omega_c$)

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{N} \{a_k \cos(k\omega_T t) + b_k \sin(k\omega_T t)\} \tag{24.2}$$

  * Modulated signal

$$\begin{aligned} e_m(t) &= v(t)x(t) && (24.3)\\ &= V_c \sin(\omega_c t)\left(\frac{1}{2}a_0 + \sum_{k=1}^{N} \{a_k \cos(k\omega_T t) + b_k \sin(k\omega_T t)\}\right) && (24.4)\\ &= V_c \sin(\omega_c t)\frac{1}{2}a_0 + \sum_{k=1}^{N} V_c \sin(\omega_c t)\{a_k \cos(k\omega_T t) + b_k \sin(k\omega_T t)\} && (24.5) \end{aligned}$$

  * Trig identities (derived)

$$\begin{aligned} \sin(\alpha)\sin(\beta) &= \frac{e^{j\alpha} - e^{-j\alpha}}{2j}\frac{e^{j\beta} - e^{-j\beta}}{2j} && (24.6)\\ &= \frac{e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)} - e^{j(\alpha-\beta)} - e^{-j(\alpha-\beta)}}{(2j)^2} && (24.7)\\ &= -\frac{1}{2}\frac{e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)} - e^{j(\alpha-\beta)} - e^{-j(\alpha-\beta)}}{2} && (24.8)\\ &= -\frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta) && (24.9) \end{aligned}$$

$$\begin{aligned} \sin(\alpha)\cos(\beta) &= \frac{e^{j\alpha} - e^{-j\alpha}}{2j}\frac{e^{j\beta} + e^{-j\beta}}{2} && (24.10)\\ &= \frac{e^{j(\alpha+\beta)} - e^{-j(\alpha+\beta)} + e^{j(\alpha-\beta)} - e^{-j(\alpha-\beta)}}{2^2 j} && (24.11)\\ &= \frac{1}{2}\frac{e^{j(\alpha+\beta)} - e^{-j(\alpha+\beta)} + e^{j(\alpha-\beta)} - e^{-j(\alpha-\beta)}}{2j} && (24.12)\\ &= \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta) && (24.13) \end{aligned}$$

$$
\begin{aligned}
\cos(\alpha)\cos(\beta) &= \frac{e^{j\alpha}+e^{-j\alpha}}{2}\frac{e^{j\beta}+e^{-j\beta}}{2} && (24.14)\\
&= \frac{e^{j(\alpha+\beta)}+e^{-j(\alpha+\beta)}+e^{j(\alpha-\beta)}+e^{-j(\alpha-\beta)}}{2^2} && (24.15)\\
&= \frac{1}{2}\frac{e^{j(\alpha+\beta)}+e^{-j(\alpha+\beta)}+e^{j(\alpha-\beta)}+e^{-j(\alpha-\beta)}}{2} && (24.16)\\
&= \frac{1}{2}\cos(\alpha+\beta)+\frac{1}{2}\cos(\alpha-\beta) && (24.17)
\end{aligned}
$$

* Simplifying the modulated signal

$$
\begin{aligned}
e_m(t) &= V_c\sin(\omega_c t)\frac{1}{2}a_0+\sum_{k=1}^{N}V_c\sin(\omega_c t)\left\{a_k\cos(k\omega_T t)+b_k\sin(k\omega_T t)\right\} && (24.18)\\
&= \frac{V_c a_0}{2}\sin(\omega_c t)\\
&\quad+\sum_{n=1}^{N}\frac{V_c a_k}{2}\left\{\sin((\omega_c+k\omega_T)t)+\sin((\omega_c-k\omega_T)t)\right\}\\
&\quad+\sum_{k=1}^{N}\frac{V_c b_k}{2}\left\{-\cos((\omega_c+k\omega_T)t)+\cos((\omega_c-k\omega_T)t)\right\} && (24.19)
\end{aligned}
$$

* Magnitude change
  · $M_0=\frac{A_0}{2}$, the DC magnitude, goes to V_cM_0 at ω_c
  · M_k at $k\omega_T$ goes to $\frac{1}{2}V_cM_k$ at $\omega_c-k\omega_T$ and $\omega_c+k\omega_T$
* Phase shift (for carrier $V_c\sin(\omega_c t)$)
  · The DC term goes to 90°
  · θ_k goes to $90^\circ+\theta_k$ at $\omega_c+k\omega_T$
  · θ_k goes to $90^\circ-\theta_k$ at $\omega_c-k\omega_T$
  · (Note, if the carrier were $V_c\cos(\omega_c t)$, the phases, as measured by θ_k in $M_k\cos(k\omega_T t-\theta_k)$, wouldn't change.)

– Frequency spectrum

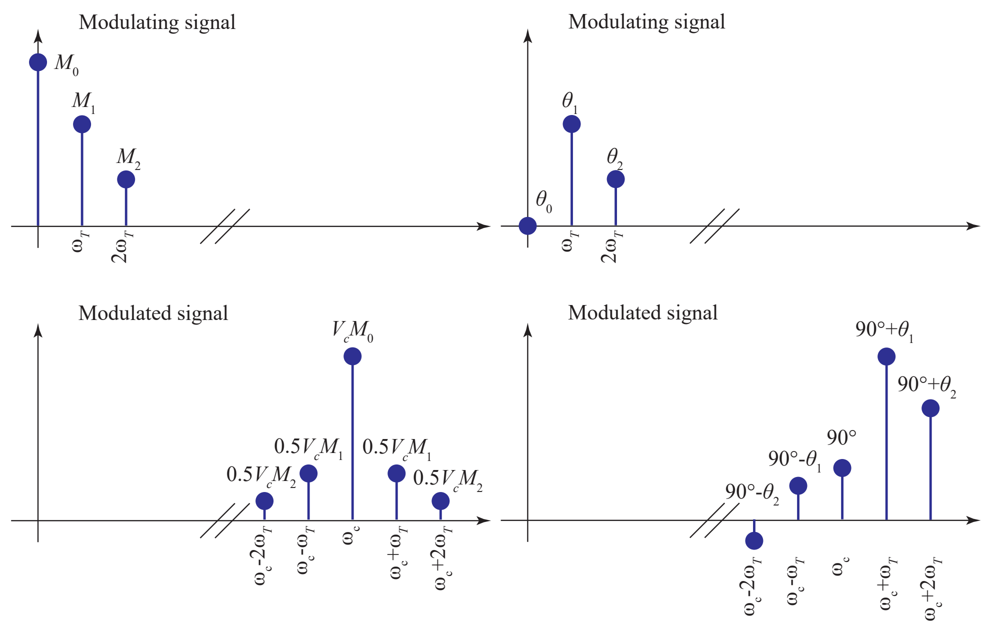


Figure 24.1: Note, the phase shifts are specific to the case in which the carrier is $V_c \sin(\omega_c t)$ and we measure phase, θ, in $M_k \cos(k\omega_T t - \theta_k)$.

- Demodulation Methods
  - multiply by carrier signal

$$
\begin{aligned}
v(t)e_m(t) &= v^2(t)x(t) && (24.20)\\
&= V_c^2\sin^2(\omega_c t)x(t) && (24.21)\\
&= \frac{V_c^2}{2}\left(1-\cos(2\omega_c t)\right)x(t) && (24.22)\\
&= \frac{V_c^2}{2}\left(1-\cos(2\omega_c t)\right)\left(\frac{a_0}{2}+\sum_{k=1}^{N}\{a_k\cos(k\omega_T t)+b_k\sin(k\omega_T t)\}\right) && (24.23)\\
&= \frac{V_c^2}{2}\left(\frac{a_0}{2}+\sum_{k=1}^{N}\{a_k\cos(k\omega_T t)+b_k\sin(k\omega_T t)\}\right)\\
&\quad -\frac{V_c^2 a_0}{4}\cos(2\omega_c t)\\
&\quad -\frac{V_c^2}{4}\sum_{k=1}^{N}\Bigg\{a_k\cos((2\omega_c+k\omega_T)t)+a_k\cos((2\omega_c-k\omega_T)t)\\
&\quad +b_k\sin((2\omega_c+k\omega_T)t)-b_k\sin((2\omega_c-k\omega_T)t)\Bigg\} && (24.24)
\end{aligned}
$$

Note a low pass filter could remove the last three lines on the right hand side of this equation to yield a scaled version of the original modulating (or baseband) signal.

  - rectification

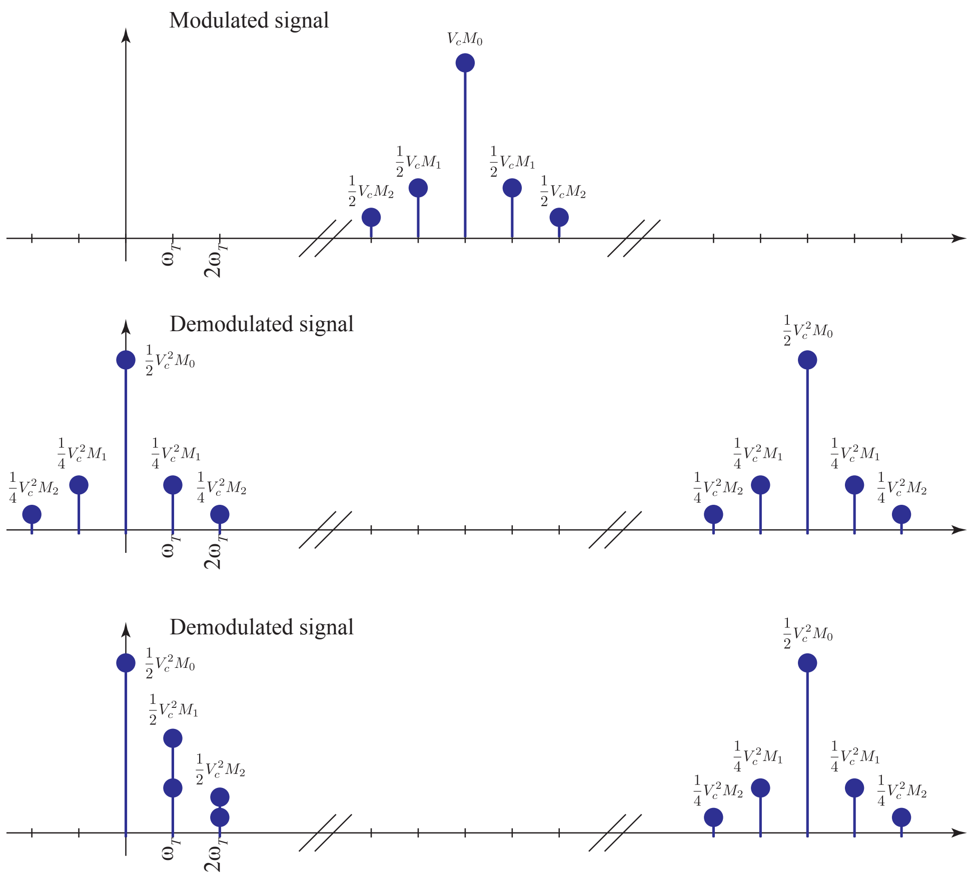


Figure 24.2: (Top) Frequency spectrum for the magnitude of an amplitude modulated signal, $e_m(t) = v(t)x(t)$. (Middle) Frequency spectrum for the magnitude of $v^2(t)x(t)$. (Bottom) Because the components with negative frequency are in phase with their positive frequency counterparts (see Figure 24.3) and $\cos(\theta) = \cos(-\theta)$, the components with negative frequency may be added to their positive frequency counterparts.

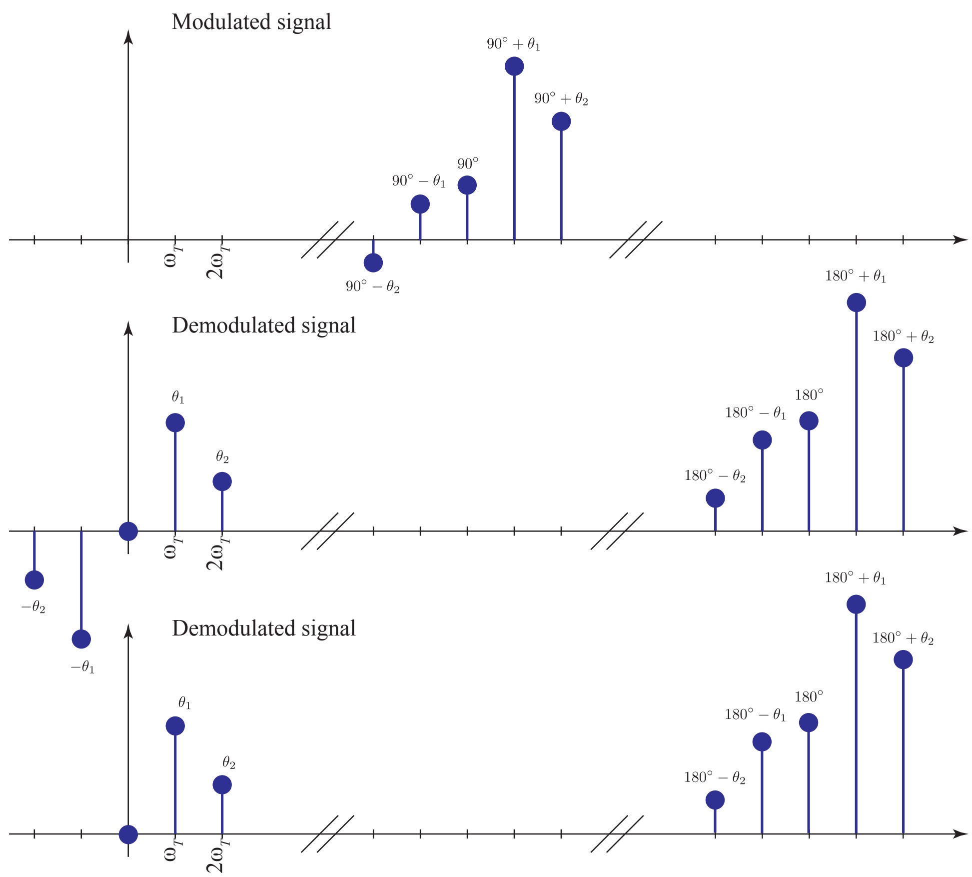


Figure 24.3: (Top) Frequency spectrum for the phase of an amplitude modulated signal, $e_m(t) = v(t)x(t)$. (Middle) Frequency spectrum for the phase of $v^2(t)x(t)$. (Bottom) Because the phases of the components with negative frequency are the opposite of their positive frequency counterparts and $\cos(\theta) = \cos(-\theta)$, the magnitudes of the components with negative phase may be added to their positive frequency counterparts (see Figure 24.2). Consequently, a low-pass filter may be used to recover a signal that is proportional to the baseband signal, $x(t)$.

Example 24.1:

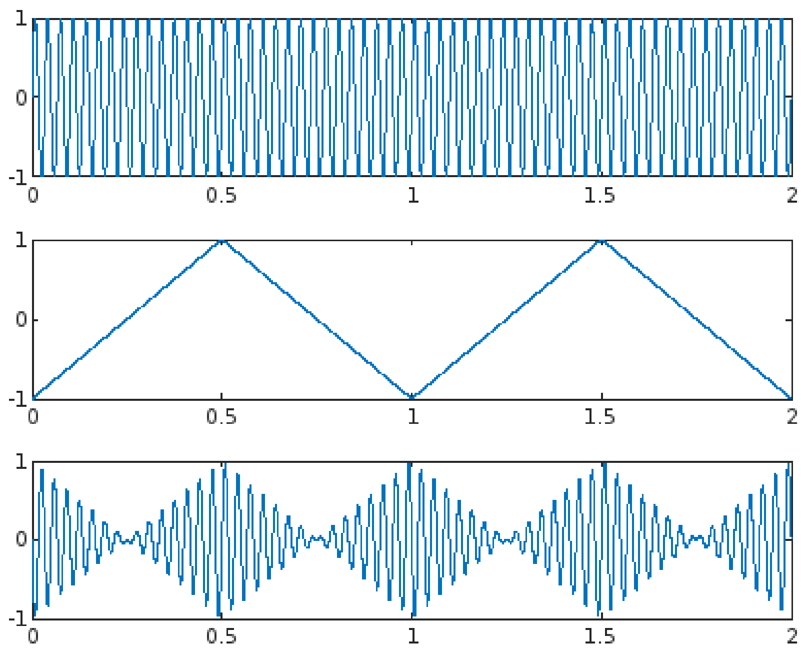
Capacitive differential pressure transducer example - see pages 132-133 and 159-160 of [18] and MATLAB® script below. update this example to parallel the example being developed in the bridge circuits section

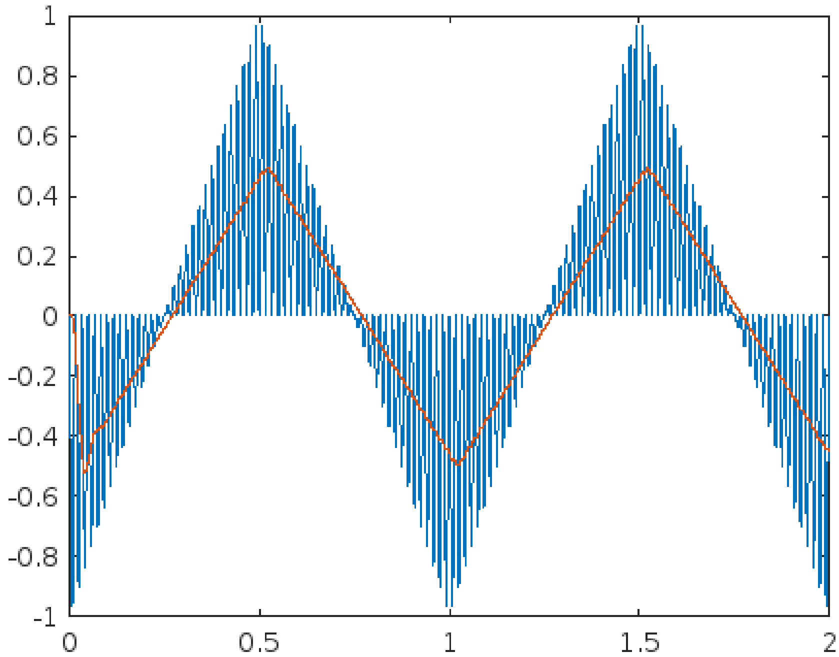

```
clear all
clc
close all

time=linspace(0,2,10000);
vin=sin(30*2*pi*time);
d=sawtooth(2*pi*time,.5);
subplot(3,1,1)
plot(time,vin)
subplot(3,1,2)
plot(time,d)
subplot(3,1,3)
plot(time,vin.*d)

figure()
plot(time,d.*vin.*vin)

hold on
%apply a 4th order butterworth filter (for example)
[z,p,k] = butter(4,2*pi*20,'s');
G=zpk(z,p,k);
[y,time]=lsim(G,d.*vin.*vin,time);
plot(time,y)
```





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Chapter 25

Loading Effects and Filtering in Measurement Systems

25.1 Equivalent Circuits and Loading Effects

- Incorporate comments on high z mode for function generator and cantilever vibration demo
- Although measurement systems typically consist of a sequence of multiple stages or elements (see Figure 2.1), they are not in series in the sense of electrical circuits (addition of impedances) or block diagrams (multiplication of transfer functions).
- Norton and Thevenin equivalents for one port networks (see [54, 18])

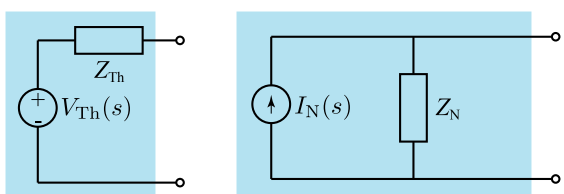


Figure 25.1: (Left) Thevenin Equivalent circuit. (Right) Norton Equivalent circuit.

  - One port network: "A network (circuit or part of a circuit) that has two, and only two, terminals at which it can attach to other networks ... is known as a one-port network [54]."
    - completely characterized by its i-v relationship [18]
    - equivalent - "Two one-port networks are electrically equivalent if their i-v characteristics are equivalent [54]."

- Here we restrict our analysis to networks consisting of sources and linear circuit elements
- Norton's Theorem - "When viewed from its terminals, any linear one-port network may be represented by an equivalent circuit consisting of an ideal current source i_N in parallel with an equivalent [impedance Z_N] ... [54]."
- Thévnin's Theorem - "When viewed from its terminals, any linear one-port network may be represented by an equivalent circuit consisting of an ideal voltage source v_T in series with an equivalent [impedance Z_T] ... [54]."
- Method to calculate Norton or Thévenin equivalent impedance of a linear one port network; see [54, 18]
  * Identify terminals of network and isolate from any connected network (separate source and load)
  * Set independent sources to 0
    · open circuit for current sources and
    · short circuit for voltage sources
  * Find the equivalent impedance between the two terminals
    · (In the case when dependent sources are present, the equivalent impedance comes from the current voltage relationship of the network when all independent sources are zeroed out [54].)
  * Note: $Z_T = Z_N$
- Method to calculate the Thévenin voltage and Norton Current
  * "The equivalent (Thévenin) source voltage is equal to the open-circuit voltage present at the load terminals (with the load removed) [18]."
  * "The Norton equivalent current is equal to the short-circuit current that would flow were the load replaced by a short circuit [18]."
  * The Thévenin voltage and Norton current are related

$$\begin{aligned} V_T(s) &= I_N(s)Z_T & (25.1) \\ &= I_N(s)Z_N & (25.2) \end{aligned}$$

  recall $Z_T = Z_N$)
- Experimental measurement of Thévenin voltage and Norton equivalents
  * "... it is often not advisable to actually short-circuit a network by inserting a series ammeter ...; permanent damage to the circuit or the ammeter may be a consequence [18]."
  * "... collect data along the device load line and extrapolate i_S from that data [54]."

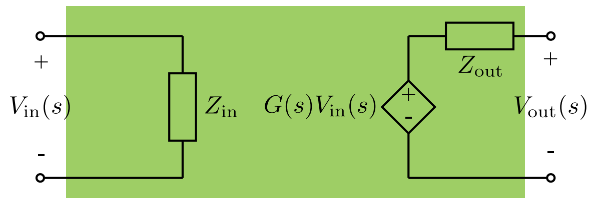


Figure 25.2: A general measurement system stage with input impedance Z_{in}, dependent voltage source with transfer function $G(s)$, and output impedance Z_{out}.

- Models for common measurement system stages/elements (represented as two port networks)
  - Two port network
    - We will consider one of the two ports to be the input and the other to be the output
  - Figure 25.2 presents a general model for a two port measurement system stage.
  - Input impedance - is the (Norton/Thévenin) equivalent impedance through the input port of the network. The output port is left as an open circuit. For the example two port networks in Figure 25.2 and assuming the left port is the input, the input impedance is Z_{in}.
  - Output impedance - is the equivalent (Norton/Thévenin) impedance through the output port of the network. (We treat the input port as an independent voltage source and zero it out yielding a short-circuit.). For the example two port networks in Figure 25.2 and assuming the right port is the output, the output impedance is Z_{out}.
  - Stage/element transfer function

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = G(s) \tag{25.3}$$

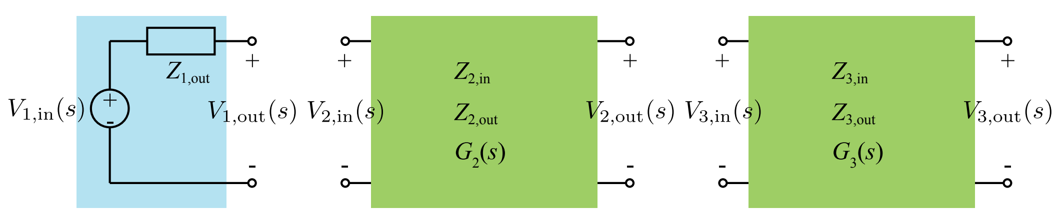


Figure 25.3:

- Loading effects

- update this section to match the presentation in Lecture worksheet 34
- "... the process of measurement always disturbs the system being measured [64]."
- Definitions
  * Loading error - "When the insertion of a sensor into a process somehow changes the physical variable being measured, a process loading error occurs [4]."
  * Interstage loading error - "If the output from one system stage is in any way affected by the subsequent stage, then the signal is affected by an interstage loading error [4]."
- Quantification
  * The unloaded source voltage is $v_{1,\text{out}} = v_{1,\text{in}}$
  * When the source and load are connected ($v_{1,\text{out}} = v_{2,\text{in}}$) a voltage divider is formed such that

$$V_{1,\text{out}}(s) = \frac{Z_{2,\text{in}}}{Z_{2,\text{in}} + Z_{1,\text{out}}} V_{1,\text{in}}(s) \tag{25.4}$$

  * Loading error; see, for example, [4]

$$\begin{aligned} L[\text{Loading Error}] &= V_{1,\text{in}}(s) - V_{1,\text{out}}(s) && (25.5)\\ &= V_{1,\text{in}}(s) - \frac{Z_{2,\text{in}}}{Z_{2,\text{in}} + Z_{1,\text{out}}} V_{1,\text{in}}(s) && (25.6)\\ &= V_{1,\text{in}}(s)\left(1 - \frac{Z_{2,\text{in}}}{Z_{2,\text{in}} + Z_{1,\text{out}}}\right) && (25.7) \end{aligned}$$

    · For a voltage source, the loading error is minimized when $Z_{2,\text{in}} \gg Z_{1,\text{out}}$
    · To analyze a current source (considering the current through $Z_{2,\text{in}}$ as the output), we would most likely use a Norton equivalent source. In this case the error would be minimized when $Z_{2,\text{in}} \ll Z_{1,\text{out}}$
  * The voltage at the output port of the load can be written as

$$\begin{aligned} V_{2,\text{out}}(s) &= G_2(s) V_{2,\text{in}}(s) && (25.8)\\ &= G_2(s) V_{1,\text{out}}(s) && (25.9)\\ &= G_2(s) \frac{Z_{2,\text{in}}}{Z_{2,\text{in}} + Z_{1,\text{out}}} V_{1,\text{in}}(s) && (25.10) \end{aligned}$$

- Accounting for multiple stages/elements
  * Start the analysis by considering all of the stages/elements separated
  * Connect the first stage/element (i.e., the source) to the second stage/element

$$V_{2,\text{out}}(s) = G_2(s) \frac{Z_{2,\text{in}}}{Z_{2,\text{in}} + Z_{1,\text{out}}} V_{1,\text{in}}(s) \tag{25.11}$$

    * Connect the second stage

$$V_{3,\text{out}}(s) = G_3(s)\frac{Z_{3,\text{in}}}{Z_{3,\text{in}}+Z_{2,\text{out}}}G_2(s)\frac{Z_{2,\text{in}}}{Z_{2,\text{in}}+Z_{1,\text{out}}}V_{1,\text{in}}(s) \quad (25.12)$$

    * Continue until all of the stages are connected

Example 25.1:

revisit cantilever beam with base excitation example (shaker amplitude changes with frequency)

- Function generator
  - Output impedance: 50 Ω
- Amplifier [see spec sheet]
  - Input impedance: > 10 kΩ
  - Output impedance: < 0.02 Ω from 5-1000 Hz
  - Gain 1-5 (I think)
- Shaker [see spec sheet]
  - Coil impedance: 0.8 Ω at 500 Hz (coils in parallel - not clear if used in parallel or series)
  - Flexure stiffness: 5.6 N/m
  - Mass of moving element: 160 g
  - Current force ratio: 0.16 A/F (parallel) and 0.08 A/F (series) - it's not clear what the units of force are?
- Adapter and cantilever

25.2 Filters

- Filters
  - Purposes
    * Filters are commonly required to address noise and aliasing [10]
    * "Provided that the power spectrum of the measurement signal occupies a different frequency range from that of the noise or interference signal, then filtering improve the signal-to-noise ratio [9]."double check the spelling in the original

    * Digital to analog to conversion
  - Passband vs. stopband [10] (and Bandwidth)
    * Passband - "The frequency band with approximately constant gain ... is known as the passband [10]."
    * Stopband - "The significantly attenuated frequency range is known as the stopband [10]."
  - Common filter types (or classifications [62])
    * Low pass - In a low pass filter, the passband is all frequencies below the cutoff frequency, ω_c, and the stopband is all frequencies above it.
    * High pass - In a high pass filter, the stopband is all frequencies below the cutoff frequency, ω_c, and the passband is all frequencies above it.
    * Band pass - Band pass filters feature two cutoff frequencies bounding the passband of the filter. The stopband includes all frequencies above and below this passband.
    * "bandstop filter" [10] - Band pass filters feature two cutoff frequencies bounding the stopband of the filter. The passband includes all frequencies above and below this stopband. ("If the band of stopped frequency is very narrow, the bandstop filter is called a notch filter [10].")
  - Cutoff frequency, corner frequency (break frequency), bandwidth
    * These terms are used in a variety of contexts such their definitions can vary depending upon context (e.g., filters vs. amplifiers) and author. (To narrow down our definitions, we could follow [9], which suggests that ideal filters have a gain of 1 and phase shift of 0 in the passband.)
    * Cutoff frequency -
      · For an ideal filter, a cutoff frequency is a boundary between the passband and the stopband; see, for example, [62, 69, 4]
      · In a practical filter approximation, the "... cutoff frequency, f_c, ... is usually defined as the frequency at which the signal power is reduced to one-half [4]."
      · Note, [69] points out that there are other ways cutoff frequency is defined in the literature depending on the filter approximation under consideration
    * Corner frequency (break frequency)
      · Although some authors use the term 'corner frequency' and 'cutoff frequency' interchangeably (see, for example, [10]), we will reserve 'corner frequency' exclusively for the intersection of asymptotic approximations to the bode plot. (In some filter approximations, the corner and cutoff frequencies may be equivalent.)
    * Bandwidth -
      · "The bandwidth of an element or system is the range of frequencies for which $|G(j\omega)|$ is greater than $1/\sqrt{2}$ [9]."

    - "... the range of sinusoidal input frequenccies over which the measuring system gives a constant ratio of output amplitde to input amplitude [10]."
- Filter order and "roll-off" [4]
  - Order - filters are dynamic systems such that the order of a filter is simply the order of the dynamic system representatoin of the system.
  - Roll-off
    - The term roll-off describes the abruptness of the transition between the passband and stopband; see, for example, [10, 4]
    - Roll-off is measured in dB/decade or dB/octave. (Recall that the order of a system is connected to the slopes on a system's bode plot. Therefore, roll-off is connected to the systems order.)
  - (Some filters have a different roll-off near the cutoff frequency and then a roll-off consistent with the order of the system far from the cutoff frequency.)
- Gain (DC gain or static sensitivity, high frequency gain) - filters can be combined with an amplifier such that the gain in the passband differs from 1.
- Ideal vs. approximation (practical)
- Common 'approximations' [62, 69] to ideal filters (other authors use the term 'class' [10] or type)
  - Butterworth - "maximally flat" [10] ("... the term maximally flat refers to its response at and just above zero frequency [69].")
  - Chebyshev - "... crisper change in slope but at the price of ripple in the passband gain ... [10]"
  - Elliptic
  - Bessel
  - others
- Passive vs. active
  - Passive - passive filters use only passive components such as resistors, capacitors, and inductors
  - Active - active filter employ active components such as opamps in addition to passive components
- Analog vs. Digital
- Common implementations
  - Low pass RC circuit -
  - High pass RC circuit -
  - Band pass (low and high pass in series)
  - Band stop (superposition of low and high filtered signals)

- Equivalent noise bandwidth
- Examples
  -
  - HW8
  - HW8 spring 20, problem 2
  - Final fall 18, problem21
  - Final fall 16, problem A9

Part VIII

Control System Analysis

Chapter 26

Common Control System Objectives

- Control system objectives
  - Common control systems objectives include (see, for example, [6, 8, 37]):
    - Stability
    - Tracking - A control system has good tracking if its output is close to the reference input; see, for example, [8]. Tracking can be quantified by
      - Transient response characteristics (e.g., overshoot, settling time, rise time, frequency)
      - Steady state response characteristics (e.g., steady state error)
    - Robustness - A control system is robust if it has relatively low sensitivity to changes in system parameters or discrepancies between model and reality; see, for example, [70, 6]
    - Disturbance and noise rejection
    - Other (financial, energetic, actuator effort, safety etc.)
  - Optimization may be used to best balance multiple objectives

Chapter 27

Poles and Zeros

- This chapter builds on the following section and chapters:
  - Section 9.1 Transfer Functions
  - Chapter 10 Stability I
  - Chapter 11 Response in the Time Domain
- Temporal response of higher order BIBO stable LTI systems
  - For BIBO stability, all poles of the transfer function must have negative real parts
  - Dominant poles
    * The dominance of a pole over other poles depends upon its relative position to the left of the imaginary axis and the relative magnitude of its residue; see, for example, [39, 3, 6].
      · Proof:
      Laplace transform of system response (assuming distinct poles)

$$Y(s) = K\frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)} \tag{27.1}$$

$$= \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + ... + \frac{C_n}{s-p_n} \tag{27.2}$$

      Time response

$$y(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} + ... + C_n e^{p_n t} \tag{27.3}$$

      · Pole location - Many textbooks suggest that poles with real parts 5-10 times to the left of a dominant pole are negligible for the purposes of design; see, for example, [38, 70, 6, 37].

  · Residue - For simplicity, often residue calculations are omitted in initial design stages but accounted for in simulations; see, for example, [6].

∗ Neglecting non-dominant poles facilitates preliminary analysis and/or conceptual design

  · When neglecting poles, care should be taken to maintain the static gain of the system; see, for example, [38].

  · Many systems can be reduced to a dominant real pole or complex conjugate pair of poles yielding familiar first order or underdamped second order systems.

∗ Simulate with all poles for detailed analysis and/or design

∗ “It should be noted that, while designing, we cannot place the insignificant poles arbitrarily far to the left in the s-plane or these may require unrealistic system parameter values when the pencil-and-paper design is implemented by physical components [38].”

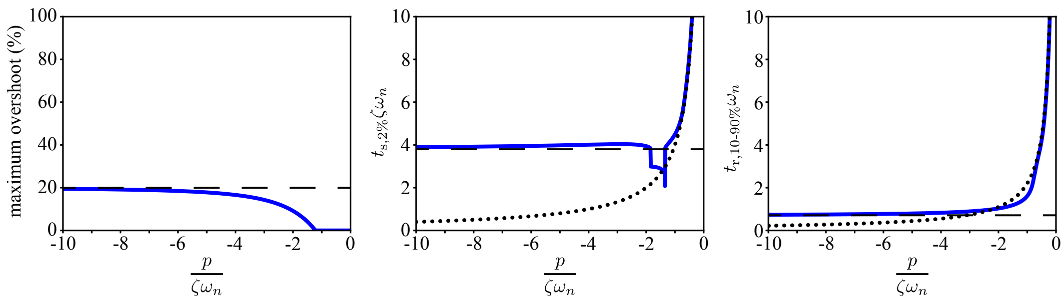


Figure 27.1: Effect of adding a pole to an underdamped second order system with $\zeta = 0.4559$. (Left) Percent overshoot as a function of relative pole position. (Middle) Nondimensional 2% settling time as a function of relative pole position. (Right) Nondimensional 10-90% rise time as a function of relative pole position. The dashed line marks the value without the added pole and the dotted curve plots the value for a first order system with the added pole. probably need to check these plots.

– Effects on the step response of incorporating a third pole to an underdamped second order system; see, for example, [8]

  * Unit step response

    · Laplace transform of step response

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{-p}{s-p}\frac{1}{s} \tag{27.4}$$

$$= C_1\frac{1}{s + \zeta\omega_n - \omega_d j} + C_1\frac{1}{s + \zeta\omega_n - \omega_d j} + C_3\frac{1}{s-p} + C_4\frac{1}{s} \tag{27.5}$$

· Partial fraction coefficients

$$
\begin{aligned}
C_1 &= \lim_{s\to-\zeta\omega_n+\omega_d j} \frac{\omega_n^2}{\cancel{(s+\zeta\omega_n-\omega_d j)}(s+\zeta\omega_n+\omega_d j)}\frac{-p}{s-p}\frac{1}{s}\cancel{(s+\zeta\omega_n-\omega_d j)} && (27.6)\\
&= \frac{\omega_n^2}{-\zeta\omega_n+\omega_d j+\zeta\omega_n+\omega_d j}\frac{-p}{-\zeta\omega_n+\omega_d j-p}\frac{1}{-\zeta\omega_n+\omega_d j} && (27.7)\\
&= \frac{\omega_n^2}{2\omega_d j}\frac{-p(-\zeta\omega_n-p-\omega_d j)}{(-\zeta\omega_n-p)^2+\omega_d^2}\frac{-\zeta\omega_n-\omega_d j}{\zeta^2\omega_n^2+\omega_d^2} && (27.8)\\
&= \frac{\omega_n^2}{2\omega_d j}\frac{-p(-\zeta\omega_n-p-\omega_d j)}{p^2+2\zeta\omega_n p+\omega_n^2}\frac{-\zeta\omega_n-\omega_d j}{\omega_n^2} && (27.9)\\
&= \frac{p}{2\omega_d j}\frac{(\zeta\omega_n+p+\omega_d j)(-\zeta\omega_n-\omega_d j)}{p^2+2\zeta\omega_n p+\omega_n^2} && (27.10)\\
&= \frac{p}{2\omega_d j}\frac{-\zeta^2\omega_n^2-\zeta\omega_n p+\omega_d^2-\omega_d(\zeta\omega_n+\zeta\omega_n+p)j}{p^2+2\zeta\omega_n p+\omega_n^2} && (27.11)\\
&= \frac{p}{2\omega_d j}\frac{-2\zeta^2\omega_n^2-\zeta\omega_n p+\omega_n^2-\omega_d(\zeta\omega_n+\zeta\omega_n+p)j}{p^2+2\zeta\omega_n p+\omega_n^2} && (27.12)\\
&= \frac{p}{2}\frac{-(\zeta\omega_n+\zeta\omega_n+p)}{p^2+2\zeta\omega_n p+\omega_n^2}+\frac{p}{2\omega_d j}\frac{-2\zeta^2\omega_n^2-\zeta\omega_n p+\omega_n^2}{p^2+2\zeta\omega_n p+\omega_n^2} && (27.13)\\
&= \frac{p}{2}\frac{-(2\zeta\omega_n+p)}{p^2+2\zeta\omega_n p+\omega_n^2}+\frac{p}{2j\sqrt{1-\zeta^2}}\frac{-2\zeta^2\omega_n-\zeta p+\omega_n}{p^2+2\zeta\omega_n p+\omega_n^2} && (27.14)\\
C_2 &= C_1^* && (27.15)
\end{aligned}
$$

$$
\begin{aligned}
C_3 &= \lim_{s\to p}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\frac{-p}{\cancel{s-p}}\frac{1}{s}\cancel{(s-p)} && (27.16)\\
&= \frac{\omega_n^2}{p^2+2\zeta\omega_n p+\omega_n^2}\frac{-p}{p} && (27.17)\\
&= -\frac{\omega_n^2}{p^2+2\zeta\omega_n p+\omega_n^2} && (27.18)\\
C_4 &= \lim_{s\to 0}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\frac{-p}{s-p}\frac{1}{\cancel{s}}\cancel{s} && (27.19)\\
&= \frac{\omega_n^2}{\omega_n^2}\frac{-p}{-p} && (27.20)\\
&= 1 && (27.21)
\end{aligned}
$$

· Step response

$$y(t) = 1 - \frac{\zeta}{\sqrt{1-\zeta^2}} \frac{p^2 + 2\zeta p\omega_n - \frac{1}{\zeta}\omega_n p}{p^2 + 2\zeta p\omega_n + \omega_n^2} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{27.22}$$

$$- \frac{p^2 + 2\zeta p\omega_n}{p^2 + 2\zeta p\omega_n + \omega_n^2} e^{-\zeta\omega_n t} \cos(\omega_d t) \tag{27.23}$$

$$- \frac{\omega_n^2}{p^2 + 2\zeta p\omega_n + \omega_n^2} e^{pt} \tag{27.24}$$

* Overshoot - the overshoot decreases as the additional pole goes from $-\infty$ towards 0 because the system goes from being dominated by a complex conjugate pair of poles to being dominated by a single real pole for which there is no overshoot.
* Settling time - the settling time generally increases as the additional pole goes from $-\infty$ towards 0 because the time constant for the additional pole increases and becomes more dominant as it approaches 0. (The dip in settling time results from changes in whether the last oscillation outside the $\pm 2\%$ band occurs above or below.)
* Rise time - the rise time increases as the additional pole goes from $-\infty$ towards 0 because the time constant for the additional pole increases and becomes more dominant as it approaches 0.
* Damped natural frequency - unchanged
* See `ThirdPole_1.py`

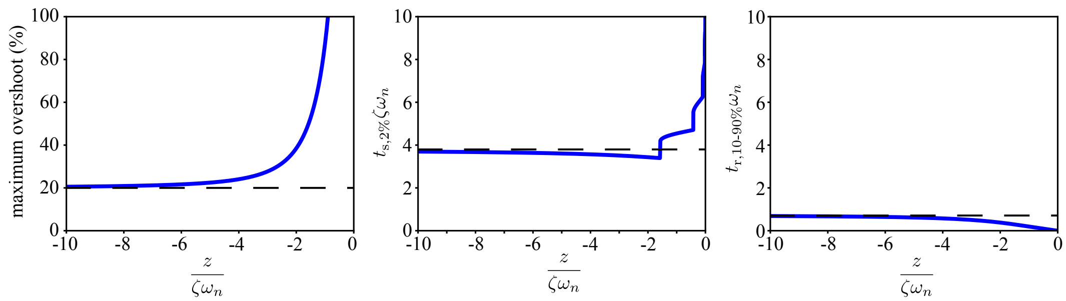


Figure 27.2: Effect of adding a zero to an underdamped second order system with $\zeta = 0.4559$. (Left) Percent overshoot as a function of relative zero position. (Middle) Nondimensional 2% settling time as a function of relative zero position. (Right) Nondimensional 10-90% rise time as a function of relative zero position. The dashed line marks the value without the added zero. probably need to check these plots.

- Effects on the step response of adding a zero to an underdamped second order system; see, for example, [8]
  - Unit step response
    - Laplace transform of step response

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{s - z}{-z} \frac{1}{s} \tag{27.25}$$

    - Step response

$$y(t) = 1 - e^{-\zeta\omega_n t} \left(\frac{1}{\sqrt{1-\zeta^2}} \left(\zeta + \frac{\omega_n}{z} \right) \sin(\omega_d t) + \cos(\omega_d t) \right) \tag{27.26}$$

  - Overshoot - the overshoot increases as the zero goes from $-\infty$ towards 0 because the system goes from looking like the step response of an underdamped system to the derivative of the step response (i.e., impulse response) of an underdamped system for which the overshoot would be infinite because the steady state value is 0.
  - Settling time -
  - Rise time - the rise time decreases as the zero goes from $-\infty$ towards 0 because:
    - the system goes from looking like the step response of an underdamped system to the derivative of the step response (i.e., impulse response) of an underdamped system and
    - the amplitude of this impulse response is proportional to the inverse of the pole position (yielding an abrupt transition toward the steady state value and short rise time).

* Damped natural frequency -
* Note, if the zero moves to the right half plane the system becomes non-minimum phase and may initially move in a direction opposite to that of the command.
* Note, the impact of a zero may be thought of as a linear combination of the response without the zero and that of the derivative of that response; see, for example, [6]
* See `AddedZero_1.py`

– Pole-zero cancellation (I believe there was a reference that inspired this presentation, but if so, I lost it.)
  * In physical systems it's not possible to add a zero to perfectly cancel a pole.
  * Pole-zero cancellation can work in the left half plane (i.e., stable pole)
    · Consider a first order system with a zero just a small distance ϵ from the pole at -1

$$G(s) = \frac{\frac{1}{1+\epsilon}s + 1}{s+1} \tag{27.27}$$

    · The step response ($R(s) = \frac{1}{s}$) can be written as

$$\begin{aligned}
Y(s) &= \frac{\frac{1}{1+\epsilon}s + 1}{s+1}\frac{1}{s} && (27.28)\\
Y(s) &= \frac{1}{1+\epsilon}\left(\frac{-\epsilon}{s+1} + \frac{1+\epsilon}{s}\right) && (27.29)\\
y(t) &= -\frac{\epsilon}{1+\epsilon}e^{-t} + 1 && (27.30)\\
\lim_{t\to\infty} y(t) &\to 1 && (27.31)
\end{aligned}$$

  * Pole-zero cancellation cannot work in the right half plane (i.e., unstable pole)
    · Consider a first order system with a zero just a small distance ϵ from the pole at 1

$$G(s) = \frac{-\frac{1}{1+\epsilon}s + 1}{-s+1} \tag{27.32}$$

    · The step response ($R(s) = \frac{1}{s}$) can be written as

$$\begin{aligned}
Y(s) &= \frac{-\frac{1}{1+\epsilon}s + 1}{-s+1}\frac{1}{s} && (27.33)\\
Y(s) &= \frac{1}{1+\epsilon}\left(\frac{-\epsilon}{-s+1} + \frac{1+\epsilon}{s}\right) && (27.34)\\
y(t) &= -\frac{\epsilon}{1+\epsilon}e^{t} + 1 && (27.35)\\
\lim_{t\to\infty} y(t) &\to -\infty && (27.36)
\end{aligned}$$

Chapter 28

Control System Configurations and Preliminary Analysis

- This chapter builds on the following section:
  - Section 9.2 Block Diagrams
- Common systems that may need to be controlled (i.e., common plant)
  - Plant - is a generic name for a system or process for which a control system will be designed
  - Often we take a transfer function representation of a plant without mentioning the physical systems that it represents. Here we present some common transfer functions and some of the physical systems they can represent.
  - First order plants
    * Transfer function
$$P(s) = \frac{1}{\tau s + 1} \tag{28.1}$$
    * Position control of DC motor (assuming fast dynamics may be ignored)
    * Servo table - velocity control of DC motor
    * Temperature control
    * Wind tunnel (motor voltage to air speed)
  - Second order plants
    * One pole at zero
      · Transfer function
$$P(s) = \frac{1}{s(s+1)} \tag{28.2}$$
      · Cruise control

  · Drone speed and attitude control
  · Satelite attitude control
  · Servo table - position control
- ∗ No first order term
  · Transfer functions

$$P^+(s) = \frac{1}{s^2+1} \tag{28.3}$$

$$P^-(s) = \frac{1}{s^2-1} \tag{28.4}$$

  · Pendulum, $P^+(s)$, or inverted pendulum, $P^-(s)$
  · Onewheel?
  · Balance beam (cart position to balance beam angle for small motions about equilibrium)
  · Simple model of bicycle steering angle, α, to tilt angle, θ,

$$\ddot{\theta} - \frac{g}{h}\theta = -\frac{bv}{ha}\dot{\alpha} - \frac{v^2}{ha}\alpha; \tag{28.5}$$

  see [71].
- ∗ General
  · DC motor with armature inductance
  · Mass spring damper

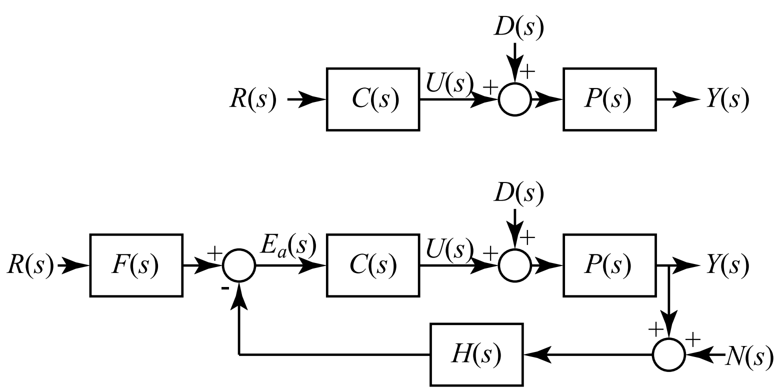


Figure 28.1: (Above) Open loop control configuration. (Below) Closed loop control configuration.

- Common control system configurations and terminology (see, for example, [8, 6, 5, 37])

- Configuration
  - Open loop control configuration. See block diagram in Figure 28.1(Above).
  - Closed loop control configuration. See block diagram in Figure 28.1(Below).
- Signals
  - $R(s)$: Reference or input (i.e., desired output)
  - $Y(s)$: Output or controlled variable
  - $U(s)$: Control signal
  - $E_a(s)$: Actuating signal
  - $E(s) = R(s) - Y(s)$: Error. Note, $E(s) = E_a(s)$ when $N(s) = 0$ and $F(s) = H(s) = 1$. Also, the error doesn't make sense if the reference, $R(s)$, and output, $Y(s)$, have different units.
  - $D(s)$: Disturbance (or Plant Disturbance)
  - $N(s)$: Sensor noise
- Blocks
  - $P(s)$: Plant or process
  - $C(s)$: Controller
  - $H(s)$: Output transducer or sensor
  - $F(s)$: Input transducer [6], filter/prefilter [8, 70, 37], forward compensation [38], or feedforward [5]
- Degrees of freedom of control system - number of control blocks in the system; see, for example, [38]

- Preliminary analysis of open loop control configuration
  - Forced response of output, control signal, and error. ((s) notation dropped from transfer functions for compactness.)

$$\begin{aligned} Y(s) &= CPR(s) + PD(s) & (28.6) \\ U(s) &= CPR(s) & (28.7) \\ E(s) &= (1 - CP)R(s) - PD(s) & (28.8) \end{aligned}$$

  - How well does open-loop control address typical control objectives?
    - Stability
      - Imperfections in physical systems and system models prevent exact cancellation of unstable poles in $P(s)$ with right half plane zeros in $C(s)$.
      - A system with an unstable plant will remain unstable with respect to disturbances regardless of $C(s)$
    - Tracking

  - Mathematically (but not physically), perfect tracking could be achieved by choosing $C(s)$ such that $P(s)C(s) = 1$
- Robustness
  - Imperfections in physical systems and system models prevent exact cancellation, $P(s)C(s) = 1$.
  - High sensitivity to changes in system parameters
- Disturbance and noise rejection
  - An open loop controller, $C(s)$, has no ability to address unmodeled disturbances (and noise). The full effect of any disturbance or noise is fed directly through to the output.

- Preliminary analysis of closed loop control configuration
  - Definitions (In these definitions we focus on the loop in Figure 28.1(Below) and ignore $F(s)$ (i.e., set $F(s) = 1$).)
    - Open loop transfer function (aka, loop gain or loop transfer function) - the product of all blocks in the loop (i.e., $H(s)P(s)C(s)$); see, for example, [39, 3, 6]
    - Forward transfer function (aka, feedforward transfer function or direct transfer function) - product of all blocks on the forward path from input to output without regard to the feedback path (i.e., $P(s)C(s)$); see, for example, [39, 3]
  - Forced response of output, actuating signal, control signal, and error; see, for example, [5, 37, 8]. ((s) notation dropped from transfer functions for compactness.)

$$Y(s) = \frac{CPF}{1+CPH}R(s) + \frac{P}{1+CPH}D(s) - \frac{CPH}{1+CPH}N(s) \tag{28.9}$$

$$E_a(s) = \frac{F}{1+CPH}R(s) - \frac{HP}{1+CPH}D(s) - \frac{H}{1+CPH}N(s) \tag{28.10}$$

$$U(s) = \frac{CF}{1+CPH}R(s) - \frac{CHP}{1+CPH}D(s) - \frac{CH}{1+CPH}N(s) \tag{28.11}$$

$$E(s) = \frac{1+CP(H-F)}{1+CPH}R(s) - \frac{P}{1+CPH}D(s) + \frac{CPH}{1+CPH}N(s) \tag{28.12}$$

    - Note, the 12 transfer functions above share the same denominator $1 + CPH$ and depending upon the numerators, they can share many of the same poles.

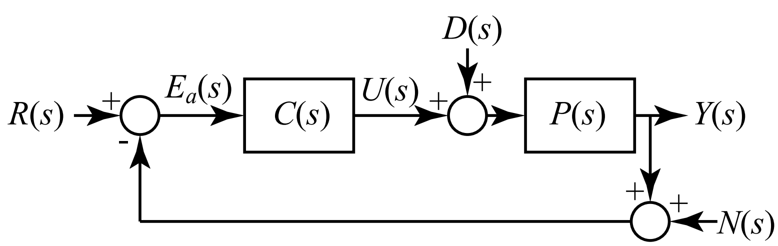


Figure 28.2:

  - For simplicity, we focus our attention on a basic closed loop system with $F(s) = H(s) = 1$; see Figure 28.2.
    - Note, this is also known as series or cascade compensation as the controller is in series (or cascaded) with the plant.
  - How well does closed-loop control address typical control objectives? (for $F(s) = H(s) = 1$)
    - Stability

- Closed loop stability depends upon the closed loop poles (which we can think of being the roots of $1 + C(s)P(s) = 0$)
- A controller, $C(s)$, has the potential to influence the closed loop poles and stabilize an otherwise unstable plant

* Tracking
  - The transfer function for the error with respect to reference is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + C(s)P(s)} \tag{28.13}$$

  - The transient response characteristics (e.g., overshoot, rise time, settling time, frequency) are largely determined by the locations of the poles and zeros of $\frac{E(s)}{R(s)}$ which the controller, $C(s)$, has the potential to influence
  - The output will track the reference (i.e., the error is small) when $|C(j\omega)P(j\omega)| \gg 1$ for the range of frequencies dominant in $r(t)$; see, for example, [37]. For example, if the static sensitivity of $C(s)P(s)$ is large, then a step input for the reference would yield a small steady state error.

* Robustness
  - In the context of controls, the sensitivity of a function F with respect to a parameter P is defined as the "... ratio of the fractional change in the function to the fractional change in the parameter as the fractional change in the parameter approaches zero [6]."

$$S_{F:P} = \frac{P}{F}\frac{\delta F}{\delta P} \tag{28.14}$$

  - Sensitivity of $\frac{Y(s)}{R(s)}$ with respect to changes in $P(s)$

$$\begin{aligned}
S_{\frac{Y(s)}{R(s)}:P(s)} &= \frac{P(s)}{\frac{Y(s)}{R(s)}}\frac{\delta\frac{Y(s)}{R(s)}}{\delta P(s)} && (28.15)\\
&= \frac{P(s)}{\frac{C(s)P(s)}{1+C(s)P(s)}}\left(\frac{C(s)}{1+C(s)P(s)} - \frac{C^2(s)P(s)}{(1+C(s)P(s))^2}\right) && (28.16)\\
&= \frac{1+C(s)P(s)}{C(s)}\frac{C(s)(1+C(s)P(s)) - C^2(s)P(s)}{(1+C(s)P(s))^2} && (28.17)\\
&= \frac{1+C(s)P(s)}{C(s)}\frac{C(s)}{(1+C(s)P(s))^2} && (28.18)\\
&= \frac{1}{1+C(s)P(s)} && (28.19)
\end{aligned}$$

  - The output, $y(t)$, will have low sensitivity to changes in the plant, $P(s)$, when $|C(j\omega)P(j\omega)| \gg 1$ for the range of frequencies dominant in $r(t)$. Note, this is the same requirement for good tracking. See, for example, [37].

* Disturbance rejection
  · The transfer function for the error with respect to the disturbance is

$$\frac{E(s)}{D(s)} = -\frac{P(s)}{1 + C(s)P(s)} \tag{28.20}$$

  · Disturbances will be rejected when $|1 + C(j\omega)P(j\omega)| \gg |P(j\omega)|$ for the range of frequencies dominant in $d(t)$; see, for example, [37, 70]
* Noise rejection
  · The transfer function for the error with respect to the noise is

$$\frac{E(s)}{N(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)} \tag{28.21}$$

  · Noise will be rejected when $|C(j\omega)P(j\omega)| \ll 1$ for the range of frequencies dominant in $n(t)$; see, for example, [37]
* Note, there are several potential benefits of closed loop over open loop control (i.e, stability, robustness, and disturbance and noise rejection. However, there is no guarantee that all potential benefits are realizable when implementing a closed loop controller
  · For example, good tracking and noise suppression can only be achieved when the frequency content of $r(t)$ and $n(t)$ don't overlap; see, for example, [37]. (Often, $r(t)$ and $n(t)$ occupy low and high frequencies respectively.)

- Some additional control configurations
  - Control systems take on many different configurations. Below are a few more configurations.
  - Feedback compensation (in minor loop; also called parallel compensation); see, for example, [38, 3, 6, 39]
  - Feedforward compensation
    * See, for example, [5, 43]
    * Incorporates an open-loop (feedforward) contribution to the input of the plant (for improved tracking, [5]) while maintaining the potential to have the benefits of closed-loop control
    * If $P(s)C_2(s) = 1$ (which isn't physically possible, but it might be possible to get close enough), the open-loop part could take care of tracking while the closed loop part could take care of stability, robustness and disturbance rejection.
  - Feedforward compensation for the disturbance
    * If the disturbance can be measured and/or anticipated, it may be possible to alleviate the effects of disturbances; see, for example, [37, 5, 43]

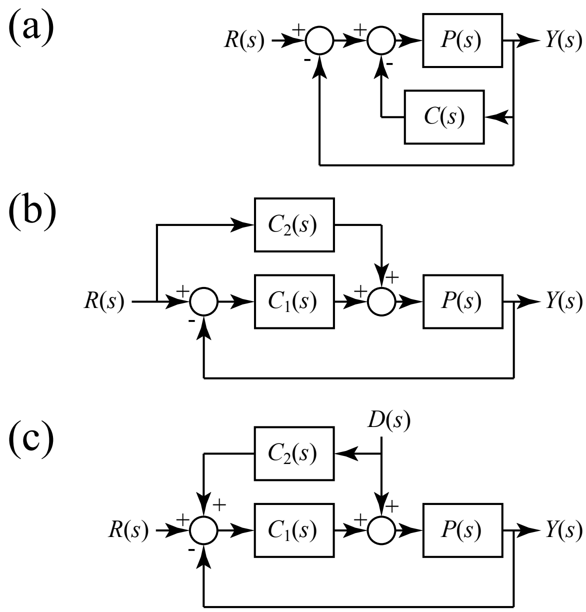


Figure 28.3: (a) Feedback compensation (see, for example, [38, 3, 6, 39]). (b) Feedforward compensation (see, for example, [38]). (c) Feedforward compensation for the disturbance (see, for example, [37, 5]).

- In Chapter 43 we introduce another configuration, the Smith Predictor; see Figure 43.1. Note, the control blocks in the Smith predictor could be combined into a single control block and thereby reduced to a configuration similar to Figure 28.2.

Chapter 29

Stability II

- This chapter builds on the following chapter:
  - Chapter 10 Stability I
- Comment on unstable pole-zero cancellations in transfer functions
  - Physically, perfect pole-zero cancellations are practically impossible or just temporary.
  - Such unstable cancellations likely will not occur in the free response of the system - unstable
  - "... if a system is stable in the transfer function formulation (which means that all its poles are stable) but has had an unstable pole-zero cancellation, it is in fact unstable [37]."
  - "... stability of a system cannot be assessed from its transfer function unless we know that there has been no unstable pole-zero cancellation in it [37]."
- Computer algorithms to determine stability (or roots of characteristic equation)
  - Usage
    * Convenient if there are no unknown (symbolic) parameters in the characteristic equation; see, for example, [8]
  - Functions that calculate roots and poles are described in Python and MATLAB®

- Analytical methods to determine stability
  - Usage
    - Used before the widespread availability of computer methods
    - Can be useful when there are unknown (symbolic) parameters in the characteristic equation; see, for example, [8]

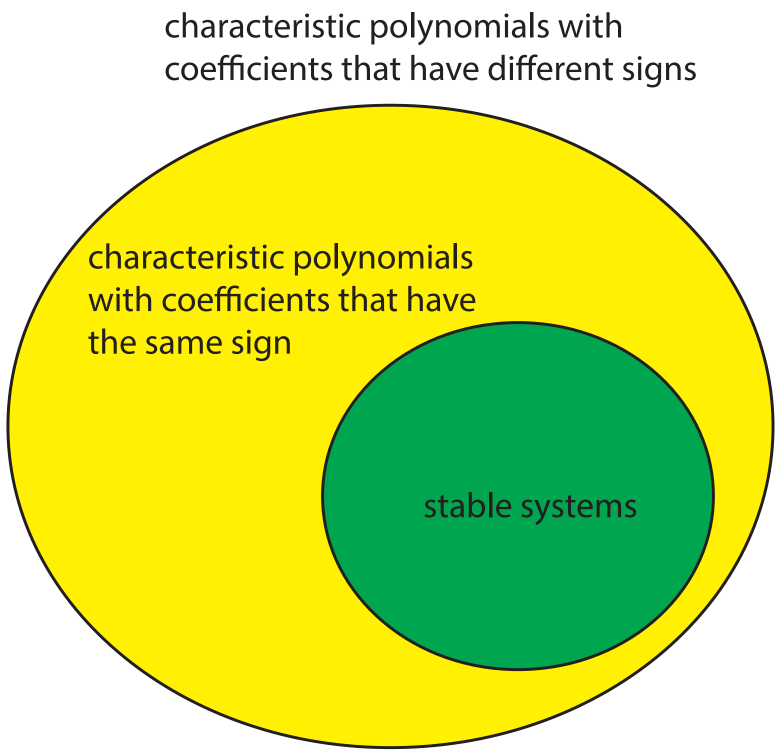


Figure 29.1:

  - A necessary but **insufficient** condition for (BIBO) stability
    - All coefficients of the characteristic polynomial must exist (i.e., be nonzero) and have the same sign; see, for example, [70, 8, 39, 37]
    - Notes
      - This condition does **not** guarantee stability (unless the system is first or second order)
      - A proof can be obtained by starting with a factored version of the characteristic polynomial; see, for example, [70]
      - (A sufficient condition for **instability** is for any of the coefficients to not exist or have different signs.)

- Routh-Hurwitz criterion is a common analytical method to determine stability

- Routh-Hurwitz criterion (a necessary and sufficient condition)
  - "... the Routh-Hurwitz criterion declares that the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column [of the Routh array] [6]."

- Definition of Routh array (defined using examples)
  - Third order system example: $a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

$$
\begin{array}{lcc}
s^3\text{:} & a_3 & a_1 \\
s^2\text{:} & a_2 & a_0 \\
s^1\text{:} & -\dfrac{\begin{vmatrix} a_3 & a_1 \\ a_2 & a_0 \end{vmatrix}}{a_2} = b_1 & 0 \\
s^0\text{:} & -\dfrac{\begin{vmatrix} a_2 & a_0 \\ b_1 & 0 \end{vmatrix}}{b_1} = c_1 & 0
\end{array}
$$

or

$$
\begin{array}{lcc}
s^3\text{:} & a_3 & a_1 \\
s^2\text{:} & a_2 & a_0 \\
s^1\text{:} & b_1 & 0 \\
s^0\text{:} & c_1 & 0
\end{array}
$$

– Fourth order system example: $a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

$$
\begin{array}{llll}
s^4\text{:} & a_4 & a_2 & a_0 \\
s^3\text{:} & a_3 & a_1 & 0 \\
s^2\text{:} & -\dfrac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 & -\dfrac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 & \\
s^1\text{:} & -\dfrac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 & 0 & \\
s^0\text{:} & -\dfrac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 & &
\end{array}
$$

or

$$
\begin{array}{llll}
s^4\text{:} & a_4 & a_2 & a_0 \\
s^3\text{:} & a_3 & a_1 & 0 \\
s^2\text{:} & b_1 & b_2 & \\
s^1\text{:} & c_1 & c_2 & \\
s^0\text{:} & d_1 & &
\end{array}
$$

– Fifth order system example: $a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

$$\begin{array}{llll}
s^5: & a_5 & a_3 & a_1 \\
s^4: & a_4 & a_2 & a_0 \\
s^3: & -\dfrac{\begin{vmatrix} a_5 & a_3 \\ a_4 & a_2 \end{vmatrix}}{a_4} = b_1 & -\dfrac{\begin{vmatrix} a_5 & a_1 \\ a_4 & a_0 \end{vmatrix}}{a_4} = b_2 & 0 \\
s^2: & -\dfrac{\begin{vmatrix} a_4 & a_2 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 & -\dfrac{\begin{vmatrix} a_4 & a_0 \\ b_1 & 0 \end{vmatrix}}{b_1} = c_2 & \\
s^1: & -\dfrac{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{c_1} = d_1 & 0 & \\
s^0: & -\dfrac{\begin{vmatrix} c_1 & c_2 \\ d_1 & 0 \end{vmatrix}}{d_1} = e_1 & &
\end{array}$$

or

$$\begin{array}{llll}
s^5: & a_5 & a_3 & a_1 \\
s^4: & a_4 & a_2 & a_0 \\
s^3: & b_1 & b_2 & \\
s^2: & c_1 & c_2 & \\
s^1: & d_1 & 0 & \\
s^0: & e_1 & &
\end{array}$$

– Tricks

  * We can multiply a row of the Routh array by an arbitrary positive constant to simplify calculations of subsequent rows without changing the stability criteria; see, for example, [8, 6, 39]

• Special cases:

  – 0 arises in the first element of a row but at least one other entry in the row is nonzero (or there are no more entires in the row (see, for example, [39])

    * External stability: Not BIBO stable
    * Internal stability: Either unstable or, at best, marginally, neutrally, or merely stable.
    * To complete the Routh array and characterize pole locations

- · Substitute a small positive number, ϵ, for the zero in the first element of that row and proceed
- · The number of RHP poles may be determined by skipping the row with the 0 when counting the number of sign changes in the first column of the Routh array. If there are no sign changes, there is a pair of pure imaginary poles.

– An entire row is zero -

- ∗ External stability: Not BIBO stable
- ∗ Internal stability: Either unstable or, at best, marginally, neutrally, or merely stable.
- ∗ Cause: at least one pair of poles is symmetric about the origin (i.e., equal and opposite real poles, equal and opposite imaginary poles, or 2 complex conjugate pairs); see, for example, [38]
  - · At best, the instability is due to a complex conjugate pair of poles on the imaginary axis and not poles with positive real part
- ∗ To complete the Routh array and characterize stability (see, for example, [6]):
  - · Replace the row of zeros with the coefficients for the derivative of the polynomial representation of the previous row.
  - · Consider the Routh array in two parts: (i) the first row through the row two rows above the row of zeros and (ii) the row above the row of zeros to the end.
  - · Sign changes in the upper part are treated as usual.
  - · The order associated with the row above the row of zeros indicates the number of poles associated with the lower part. Poles associated with this lower part come in pairs that are opposite through about the origin; see, for example, [39]. If there are no sign changes in the lower part, all poles are pure imaginary. Any sign changes in the lower part will come in multiples of two. Every two sign changes corresponds to four poles. If a leading zero or a second row of zeros appears in the lower part, it seems that there would be repeated poles on the imaginary axis?

Example 29.1:

Characterize the internal stability of the following dynamic system

$$\frac{d^4x}{dt^4} + 12\frac{d^3x}{dt^3} + 56\frac{d^2x}{dt^2} + 120\frac{dx}{dt} + 100x = 0 \tag{29.1}$$

- Characteristic equation

$$s^4 + 12s^3 + 56s^2 + 120s + 100 = 0 \tag{29.2}$$

- Routh's stability criterion

- Routh array

$$\begin{array}{llll} s^4: & 1 & 56 & 100 \\ s^3: & 12 & 120 & 0 \\ s^2: & -\frac{1\cdot 120-56\cdot 12}{12}=46 & -\frac{1\cdot 0-100\cdot 12}{12}=100 & 0 \\ s^1: & -\frac{12\cdot 100-120\cdot 46}{46}=\frac{2160}{23} & 0 & \\ s^0: & -\frac{46\cdot 0-100\cdot\frac{2160}{23}}{\frac{2160}{23}}=100 & & \end{array}$$

- Asymptotically stable (no sign changes in the first column of the Routh array)

- Stability from python

```
In: np.roots([1,12,56,120,100])
Out: array([-3+1j, -3-1j,-3+1j, -3-1j])
```

  - Asymptotically stable (each root has a negative real part)

Example 29.2: Special case: leading zero in row of Routh array (pair of pure imaginary poles)

Characterize the internal stability of the following dynamic system

$$\frac{d^2x}{dt^2}+0\frac{dx}{dt}+4x=0 \tag{29.3}$$

- Characteristic equation

$$s^2+0s+4=0 \tag{29.4}$$

- Routh's stability criterion
  - Routh array

$$\begin{array}{lll} s^2: & 1 & 4 \\ s^1: & 0 & 0 \end{array}$$

    * Note, there is a leading zero in this row. The second element in the row must be zero because it would be associated with a s^{-1} term. Therefore, we don't need to consider this as the case with an entire row of zeros. We replace the leading zero in this row with a small positive number ϵ.

$$\begin{array}{lll} s^1: & \epsilon & 0 \\ s^0: & -\frac{1\cdot 0-4\cdot\epsilon}{\epsilon}=4 & 0 \end{array}$$

- Two pure imaginary poles (skipping the s^1 row, there are no sign changes in the first column of the Routh array).

- We can factor the characteristic equation as

$$(s-2j)(s+2j)=0 \tag{29.5}$$

Example 29.3: Special case: leading zero in row of Routh array

Characterize the internal stability of the following dynamic system

$$\frac{d^4x}{dt^4}+2\frac{d^3x}{dt^3}+2\frac{d^2x}{dt^2}+4\frac{dx}{dt}+x=0 \tag{29.6}$$

- Characteristic equation

$$s^4+2s^3+2s^2+4s+1=0 \tag{29.7}$$

- Routh's stability criterion
  - Routh array

$$\begin{array}{llll} s^4\colon & 1 & 2 & 1 \\ s^3\colon & 2 & 4 & 0 \\ s^2\colon & -\frac{1\cdot 4-2\cdot 2}{2}=0 & -\frac{1\cdot 0-1\cdot 2}{2}=1 & 0 \end{array}$$

    * Replace the leading zero in this row with a small positive number ϵ

$$\begin{array}{lll} s^2\colon & \epsilon & 1 \quad 0 \\ s^1\colon & -\frac{2\cdot 1-4\cdot\epsilon}{\epsilon}=-\frac{2}{\epsilon}+4 & 0 \\ s^0\colon & -\frac{\epsilon\cdot 0-1\cdot\left(-\frac{2}{\epsilon}+4\right)}{-\frac{2}{\epsilon}+4}=1 & \end{array}$$

  - Unstable with two unstable poles (skipping the s^1 row, in the limit as $\epsilon\to 0^+$ there are two sign changes in the first column of the Routh array)

- Stability from python

```
In: np.roots([1,2,2,4,1])
Out: array([-1.9070+0.j, 0.0933+1.3660j, 0.0933-1.3660j, -0.2797+0.j])
```

  - Unstable with two unstable poles (two poles have positive real parts)

Example 29.4: Special case: row of zeros in Routh array (two RHP poles)

Characterize the internal stability of the following dynamic system

$$\frac{d^5x}{dt^5} + \frac{d^4x}{dt^4} + 2\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 3x = 0 \tag{29.8}$$

- Characteristic equation

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 3 = 0 \tag{29.9}$$

- Routh's stability criterion
  - Routh array

$$\begin{array}{llll} s^5: & 1 & 2 & 3 \\ s^4: & 1 & 2 & 3 \\ s^3: & -\frac{1\cdot 2-2\cdot 1}{1} = 0 & -\frac{1\cdot 3-3\cdot 1}{1} = 0 & 0 \end{array}$$

    * Replace this row of zeros with the coefficients for the derivative of the polynomial representation of the previous row (i.e., $\frac{d}{ds}\left(s^4 + 2s^2 + 3\right) = 4s^3 + 4s + 0$).

$$\begin{array}{llll} s^3: & 4 & 4 & 0 \\ s^2: & -\frac{1\cdot 4-2\cdot 4}{4} = 1 & -\frac{1\cdot 0-3\cdot 4}{4} = 3 & 0 \\ s^1: & -\frac{4\cdot 3-4\cdot 1}{1} = -8 & 0 & \\ s^0: & -\frac{1\cdot 0-3\cdot(-8)}{-8} = 3 & & \end{array}$$

  - Unstable with two unstable poles
    * There are no sign changes in the first column of the upper part (s^5 row only)
    * There are two sign changes in the first column of the lower part (s^4 row and below). s^4 is associated with 4 poles. Because of the sign change, two of the poles are in the right half plane and the other two are in the left half plane.

- Stability from python

```
In: np.roots([1,1,2,2,3,3])
Out: array([0.6050+1.1688j, 0.6050-1.1688j, -0.6050+1.1688j, -0.6050-1.1688j, -1.+0.j])
```

  - Unstable with two unstable poles (two poles have positive real parts)

Example 29.5: Special case: row of zeros in Routh array (two pair of pure imaginary poles)

Characterize the internal stability of the following dynamic system

$$\frac{d^5x}{dt^5} + \frac{d^4x}{dt^4} + 3\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0 \tag{29.10}$$

- Characteristic equation

$$s^5 + s^4 + 3s^3 + 3s^2 + 2s + 2 = 0 \tag{29.11}$$

- Routh's stability criterion
  - Routh array

$$\begin{array}{llll} s^5: & 1 & 3 & 2 \\ s^4: & 1 & 3 & 2 \\ s^3: & -\frac{1\cdot3-3\cdot1}{1} = 0 & -\frac{1\cdot2-2\cdot1}{1} = 0 & 0 \end{array}$$

    * Replace this row of zeros with the coefficients for the derivative of the polynomial representation of the previous row (i.e., $\frac{d}{ds}\left(s^4 + 3s^2 + 2\right) = 4s^3+6s+0$).

$$\begin{array}{llll} s^3: & 4 & 6 & 0 \\ s^2: & -\frac{1\cdot6-3\cdot4}{4} = \frac{3}{2} & -\frac{1\cdot0-2\cdot4}{4} = 2 & 0 \\ s^1: & -\frac{4\cdot2-6\cdot\frac{3}{2}}{\frac{3}{2}} = \frac{2}{3} & 0 & \\ s^0: & -\frac{1\cdot0-2\cdot\frac{2}{3}}{\frac{2}{3}} = 2 & & \end{array}$$

  - Unstable with four pure imaginary poles
    * There are no sign changes in the first column of the upper part (s^5 row only)
    * There are no sign changes in the first column of the lower part (s^4 row and below). s^4 is associated with 4 poles. Because there are no sign changes, there are four pure imaginary poles.

- Stability from python

```
In: np.roots([1,1,3,3,2,2])
Out: array([-1.0000+0.0000j, 0.0000+1.4142j, 0.0000-1.4142j, 0.0000+1.0000j, 0.0000
```

  - Two pairs of poles on the imaginary axis

Example 29.6: Special case: row of zeros in Routh array (repeated poles on the imaginary axis)

Characterize the internal stability of the following dynamic system

$$\frac{d^4x}{dt^4} + 0\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + 0\frac{dx}{dt} + x = 0 \tag{29.12}$$

- Characteristic equation

$$s^4 + 0s^3 + 2s^2 + 0s + 1 = 0 \tag{29.13}$$

- Routh's stability criterion
  - Routh array

$$\begin{array}{lccc} s^4\!: & 1 & 2 & 1 \\ s^3\!: & 0 & 0 & 0 \end{array}$$

    - Replace this row of zeros with the coefficients for the derivative of the polynomial representation of the previous row (i.e., $\frac{d}{ds}\left(s^4 + 2s^2 + 1\right) = 4s^3 + 4s + 0$).

$$\begin{array}{lccc} s^3\!: & 4 & 4 & 0 \\ s^2\!: & -\frac{1\cdot4-2\cdot4}{4} = 1 & -\frac{1\cdot0-1\cdot4}{4} = 1 & 0 \\ s^1\!: & -\frac{4\cdot1-4\cdot1}{1} = \overset{\epsilon}{\cancel{0}} & 0 & \\ s^0\!: & -\frac{1\cdot0-1\cdot\epsilon}{\epsilon} = 1 & & \end{array}$$

  - Unstable with repeated (unstable poles) on the imaginary axis
    - There is no upper part (s^5 and above)
    - There are no sign changes in the first column of the lower part but there is a leading zero. Because there are no sign changes, there are four pure imaginary poles. The leading zero seems to suggest that there are repeated poles on the imaginary axis (but it's not clear if the leading zero is guaranteed every time there are repeated imaginary poles).

Example 29.7:

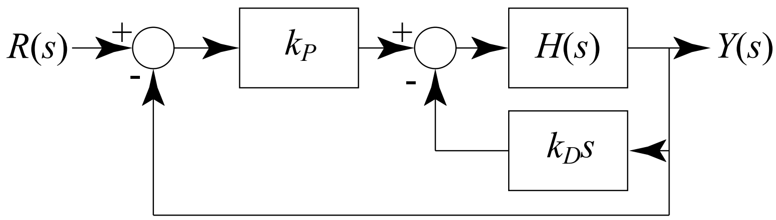


Figure 29.2:

A feedback loop is applied to an unstable system with transfer function

$$H(s) = \frac{1}{s^2 - 1}$$

yielding the block diagram in **Figure 29.2**. Use the Routh's stability criterion to find the range of values for k_P and k_D such that the new system with transfer function $\frac{Y(s)}{R(s)}$ is BIBO stable.

- Block diagram

$$\begin{aligned}
Y(s) &= H(s)\left(k_P(R(s) - Y(s)) - k_D s Y(s)\right) && (29.14)\\
\left(H(s)\left(k_D s + k_P\right) + 1\right) Y(s) &= H(s) k_P R(s) && (29.15)\\
\frac{Y(s)}{R(s)} &= \frac{k_P H(s)}{(k_D s + k_P) H(s) + 1} && (29.16)\\
&= \frac{k_P \frac{1}{s^2-1}}{(k_D s + k_P)\frac{1}{s^2-1} + 1} && (29.17)\\
&= \frac{k_P}{k_D s + k_P + s^2 - 1} && (29.18)\\
&= \frac{k_P}{s^2 + k_D s + k_P - 1} && (29.19)
\end{aligned}$$

- Characteristic equation

$$s^2 + k_D s + k_P - 1 = 0 \tag{29.20}$$

- Routh's stability criterion

$$\begin{array}{lcc}
s^2: & 1 & k_P - 1 \\
s^1: & k_D & 0 \\
s^0: & -\frac{1\cdot 0 - (k_P - 1)k_D}{k_D} = k_P - 1 &
\end{array}$$

  – stability requirements: $k_D > 0$ and $k_P > 1$

Chapter 30

Steady State Error

- Steady-state error -
  - Steady state error - "Steady-state error is the difference between the input and the output for a prescribed test input as $t \to \infty$ [6]."
  - Common test inputs

Impulse	$\delta(t)$	1	
Step	$u_s(t)$	$\frac{1}{s}$	(position input)
Ramp	$tu_s(t)$	$\frac{1}{s^2}$	(velocity input)
Parabolic	$\frac{1}{2}t^2u_s(t)$	$\frac{1}{s^3}$	(acceleration input)
n^{th} order polynomial	$\frac{1}{n}t^n u_s(t)$	$\frac{1}{s^{n+1}}$	

  - Assume stability - Here we restrict our steady state error analysis to BIBO stable systems, because stability is a desired characteristic in nearly all control systems; see, for example, [6]. Also, remember that the final value theorem is not valid when, for example, $sE(s)$ is unstable.
  - Steady state error contributions
    - Sources
      - Reference (command, input, set point)
      - Disturbance
      - Noise
    - Laplace transform of the error

$$E(s) = \frac{1+CP(H-F)}{1+CPH}R(s) - \frac{P}{1+CPH}D(s) + \frac{CPH}{1+CPH}N(s) \quad (30.1)$$

      - The error really only makes sense when the units of input and output are identical. When the units are different, we may use the actuating signal instead.

  - In many controls applications, we want the steady state error to be 0 (or at worst a small constant) given a test input representative of the inputs the system would recieve in operation.

- Steady state error contributions from polynomial test inputs
  - Assume stability - assume all poles of the closed loop transfer function are in the open left half plane (except up to one pole at the origin) so that the final value theorem may be applied
  - Consider polynomial test inputs for reference, disturbance, and noise
  - Steady state error contributions from reference, disturbance, noise
    * contribution from reference

$$\lim_{t\to\infty} e_{\text{reference}}(t) \rightarrow \lim_{s\to 0} s\frac{1-CP(H-F)}{1+CPH}\frac{1}{s^{n+1}} \tag{30.2}$$

$$\rightarrow \lim_{s\to 0} \frac{1-CP(H-F)}{1+CPH}\frac{1}{s^{n}} \tag{30.3}$$

    * contribution from disturbance

$$\lim_{t\to\infty} e_{\text{disturbance}}(t) \rightarrow \lim_{s\to 0} -s\frac{P}{1+CPH}\frac{1}{s^{n+1}} \tag{30.4}$$

$$\rightarrow \lim_{s\to 0} -\frac{P}{1+CPH}\frac{1}{s^{n}} \tag{30.5}$$

    * contribution from noise

$$\lim_{t\to\infty} e_{\text{noise}}(t) \rightarrow \lim_{s\to 0} s\frac{CPH}{1+CPH}\frac{1}{s^{n+1}} \tag{30.6}$$

$$\rightarrow \lim_{s\to 0} \frac{CPH}{1+CPH}\frac{1}{s^{n}} \tag{30.7}$$

  - Three cases arise
    * $\lim_{t\to\infty} e_{\text{contribution}}(t) \rightarrow 0$
      · Occurs when there are more than n zeros at the origin in the associated closed loop transfer function
    * $\lim_{t\to\infty} e_{\text{contribution}}(t) \rightarrow \text{constant}$
      · Occurs when there are exactly n zeros at the origin in the associated closed loop transfer function
    * $|\lim_{t\to\infty} e_{\text{contribution}}(t)| \rightarrow \infty$
      · Occurs when there are fewer than n zeros at the origin in the associated closed loop transfer function

- Special case: unity feedback $H(s) = 1$, $F(s) = 1$, and ignoring disturbance $D(s) = 0$ and noise $N(s) = 0$
  - Assume stability - all poles of the closed loop transfer function are in the open left half plane (except up to one pole at the origin) so that the final value theorem may be applied

$$\begin{aligned} \lim_{t\to\infty} e_{\text{reference}}(t) &\to \lim_{s\to 0} s\frac{E(s)}{R(s)}R(s) && (30.8)\\ &\to \lim_{s\to 0} s\frac{1}{1+CP}R(s) && (30.9)\end{aligned}$$

  - Consider a polynomial test input for the reference of order n

$$\begin{aligned} \lim_{t\to\infty} e_{\text{reference}}(t) &\to \lim_{s\to 0} \frac{1}{1+CP}\frac{1}{s^{n+1}}s && (30.10)\\ &\to \lim_{s\to 0} \frac{1}{1+CP}\frac{1}{s^{n}} && (30.11)\end{aligned}$$

  - Represent the product CP as a ratio of polynomials $\frac{B(s)}{s^k A(s)} = C(s)P(s)$ in which the denominator has k roots at the origin (i.e., k integrators)

$$\begin{aligned} \lim_{t\to\infty} e_{\text{reference}}(t) &\to \lim_{s\to 0} \frac{1}{1+\frac{B}{s^k A}}\frac{1}{s^{n}} && (30.12)\\ &\to \lim_{s\to 0} \frac{s^k}{s^k+\frac{B}{A}}\frac{1}{s^{n}} && (30.13)\\ &\to \begin{cases} 0 & n<k \\ \frac{1}{s^k+\frac{B(0)}{A(0)}} & n=k \\ \infty & k<n \text{ (final value theorem is invalid)} \end{cases} && (30.14)\\ &\to \begin{cases} 0 & n<k \\ \frac{1}{s^n+(\lim_{s\to 0} s^n C(0)P(0))} & n=k \\ \infty & k<n \text{ (final value theorem is invalid)} \end{cases} && (30.15)\end{aligned}$$

  - In this case the poles of the open loop transfer function, CP, correspond to zeros of the closed loop transfer functions.
    * Therefore, any open loop transfer function poles at the origin (i.e., integrations) will yield zeros at the origin of the closed loop transfer functions.
    * These zeros at the origin can cancel with inverse powers of s in the test input.
  - System type
    * Definition of system type for unity feedback systems

        · Number of poles at the origin in the forward transfer function, PC; see, for example, [6, 38, 70, 37].
        · **As with many textbooks, we restrict our definition of type to unity feedback systems**
        · Also, we restrict our definition to consider only the contribution to the error from the reference
    * More general definitions
        · "... we classify systems as to "type" according to the degree of the polynomial that they can reasonably track [8]."
- Static error constants for input at the reference
    * Position constant
        · Definition

$$K_p = \lim_{s \to 0} P(s)C(s) \tag{30.16}$$

        · Steady state error for a step (position) intput

$$e_{ss} = \frac{1}{1 + K_p} \tag{30.17}$$

    * Velocity constant
        · Definition

$$K_v = \lim_{s \to 0} sP(s)C(s) \tag{30.18}$$

        · Steady state error for a ramp (velocity) intput

$$e_{ss} = \frac{1}{K_v} \tag{30.19}$$

    * Acceleration constant
        · Definition

$$K_a = \lim_{s \to 0} s^2 P(s)C(s) \tag{30.20}$$

        · Steady state error for a parabolic (acceleration) intput

$$e_{ss} = \frac{1}{K_a} \tag{30.21}$$

    * Possible values and interpretation for these constants
        · infinite - the steady state error for the associated input is 0
        · finite nonzero constant - the steady state error for the associated input is a constant
        · 0 - the steady state error for the associated input is infinite

Table 30.1: Steady state error due to a polynomial reference input $R(s)$ for various system types. See, for example, [6, 8].

Type	Step $(\frac{1}{s})$	Ramp $(\frac{1}{s^2})$	Parabola $(\frac{1}{s^3})$	$(\frac{1}{s^4})$	$\dots$	$(\frac{1}{s^{n+1}})$
0	$\frac{1}{1+K_p}$	∞	∞	∞	∞	∞
1	0	$\frac{1}{K_v}$	∞	∞	∞	∞
2	0	0	$\frac{1}{K_a}$	∞	∞	∞
3	0	0	0	$\frac{1}{K_3}$	∞	∞
$\vdots$						
n	0	0	0	0	$\dots$	$\frac{1}{K_n}$

  – Table 30.1 gives the dependence of steady state error on system type and reference input for unity feedback system ($H(s) = 1$) with $F(s) = 1$.

- Special case: unity feedback $H(s) = 1$, $F(s) = 1$, and ignoring reference $R(s) = 0$ and noise $N(s) = 0$

Part IX

Control System Design

Chapter 31

Direct Pole Placement

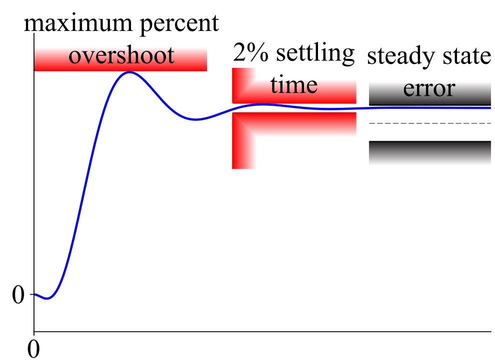


Figure 31.1: Many specifications for the design of a control system may be thought of as forming an envelope in which the system's unit step response must remain; see, for example, [8]. Limits on maximum percent overshoot and 2% settling time (red) are measured relative to the steady state response of the system whereas the steady state error (black) is measured with respect to the reference (dashed line).

- Control system specifications
  - Common control system objectives include stability and tracking (i.e., transient and steady state response characteristics).
  - Time domain specifications - For the design of a control system, we might set requirements for its response to a test input (e.g., step input). Although there are many test inputs and characteristics that could be considered as part of the design of a control system, in this coursepack we focus on the following step response characteristics: 2% settling time, maximum percent overshoot, and steady state error; see Figure 31.1.

Note, the stability of a control system is not always readily determined from its step response.

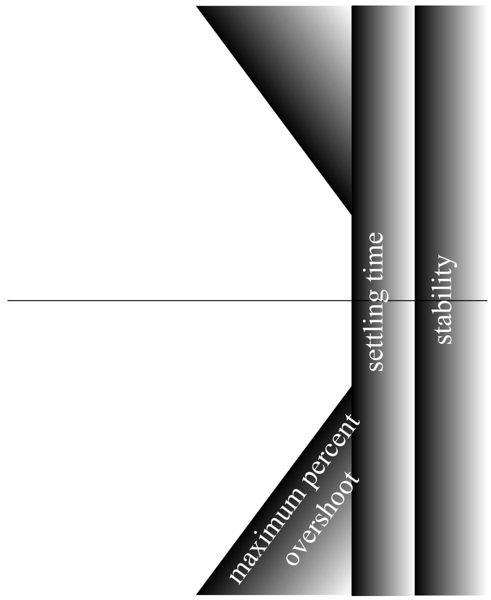


Figure 31.2: Control system specifications may be used to construct a design envelope of acceptable pole locations within the complex plane; see, for example, [8]. This sketch depicts limitations on the the stability, settling time, and maximum percent overshoot of a system.

– Pole position specifications - Assuming a system's step response is dominated by a first order pole or a complex conjugate pair of poles (and all zeros are insignificant), the design envelope in the time domain (e.g., Figure 31.1) may be mapped into an envelope in the complex plane; see Figure 31.2. Steady state error, which depends upon the gain of the transfer function, may not be determined from a pole zero-map. However, stability may be determined by inspecting a system's pole zero map.

- We can design controllers that dictate closed loop pole locations (i.e., pole placement or pole assignment); see, for example, [8, 72]
  - Assumptions
    - Plant
      - Transfer function

$$
\begin{aligned}
P(s) &= \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_1 s + b_0}{1s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0} && (31.1)\\
&= \frac{B(s)}{A(s)} && (31.2)
\end{aligned}
$$

      - strictly proper ($n > m$)
      - coprime (no common factors in numerator and denominator)
      - monic denominator - leading coeficient in the denominator is 1 ($a_n = 1$)
    - Controller is represented by a biproper transfer function
      - transfer function

$$
\begin{aligned}
C(s) &= K\frac{(s-z_1)(s-z_2)...(s-z_k)}{(s-p_1)(s-p_2)...(s-p_k)} && (31.3)\\
&= \frac{\beta_k s^k + \beta_{k-1}s^{k-1} + ... + \beta_1 s + \beta_0}{1s^k + \alpha_{k-1}s^{k-1} + ... + \alpha_1 s + \alpha_0} && (31.4)\\
&= \frac{\beta(s)}{\alpha(s)} && (31.5)
\end{aligned}
$$

      - biproper: we consider biproper transfer functions because of all realizable (i.e., proper) transfer functions they offer the most design freedom (i.e., free parameters) per given order
      - Total number design parameters (i.e., constants) ($\{\alpha_0,\ \alpha_1,\ ...\alpha_{k-1},\ \beta_0,\ \beta_1,\ ...,\ \beta_k\}$ or $\{K,\ z_1,\ z_2,\ ...z_k,\ p_1,\ p_2,\ ...,\ p_k\}$): $2k+1$
      - We take the denominator to be monic ($\alpha_k = 1$). If we consider $\alpha_k \neq 1$, the controller will be overparameterized. As an example of an overparameterized controller, consider the zeroth order controller

$$C(s) = \frac{\beta_0}{\alpha_0}. \tag{31.6}$$

        In this case, two parameters is overkill because $C(s)$ depends upon one quantity (i.e., the ratio of β_0 to α_0).
    - Unitary feedback, $H(s) = 1$, and no filter, $F(s) = 1$

- Closed loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{CP}{1+CP} \tag{31.7}$$

$$= \frac{\frac{\beta(s)}{\alpha(s)}\frac{B(s)}{A(s)}}{1+\frac{\beta(s)}{\alpha(s)}\frac{B(s)}{A(s)}} \tag{31.8}$$

$$= \frac{\beta(s)B(s)}{\alpha(s)A(s)+\beta(s)B(s)} \tag{31.9}$$

- Characteristic equation
  * Characteristic equation

$$\alpha(s)A(s)+\beta(s)B(s)=0 \tag{31.10}$$

  * Order: $n+k$
    · due to the assumptions (i.e., $n>m$ and monic $\alpha(s)$ and $A(s)$), the characterisitc equation is monic (i.e., the leading coefficient is 1)
    · $n+k$ coefficients that can be adjusted (while remaining monic)
  * Note, poles in the plant (or controller) canceled by zeros in the controller (or plant) remain in the closed loop characteristic equation
    · In practice they will differ, even if by a small amount
    · Canceled poles may also remain part of the free response.
  * Open loop zeros remain zeros in the transfer function $\frac{Y(s)}{R(s)}$
  * (Note, if $m=n$ and the characteristic equation isn't monic, divide by the leading term and compare to a monic desired characteristic equation. In other words, scale the characteristic equation such that it becomes monic.)
- Controller design options
  * Controller parameters may be chosen to yield (place) desired poles for the closed loop characteristic equation
  * The number of poles that can be placed is limited by the order of the controller
    · A biproper controller of order k has $2k+1$ parameters
    · A biproper controller of order k can prescribe $2k+1$ poles
    · There is no guarantee about the location of unprescribed poles (i.e., prescribing only some of the poles could yield an unstable system).
  * All closed loop poles can be prescribed if the order of the controller is sufficiently high
    · A biproper controller of order k has $2k+1$ parameters
    · A biproper controller of order k yields a closed loop characteristic equation of order $n+k$

· minimum order of controller to prescribe all poles

$$\begin{aligned} 2k+1 &\geq n+k && (31.11)\\ k &\geq n-1 && (31.12)\end{aligned}$$

* To place all closed loop poles, equate the closed loop characteristic polynomial $(\alpha(s)A(s)+\beta(s)B(s))$ to the desired characteristic polynomial $(A_{\text{desired}}(s))$

$$\alpha(s)A(s)+\beta(s)B(s) = A_{\text{desired}}(s). \tag{31.13}$$

The above is known as a polynomial Diophantine equation in which the polynomials $\alpha(s)$ and $\beta(s)$ are unknown. Potentially helpful references for the term polynomial Diophantine equation include [73, 8] and `https://en.wikipedia.org/wiki/Polynomial_Diophantine_equation`

- Direct pole placement controller design method
  - Determine the desired dominant pole location(s).
  - Select a form for the controller, $C(s)$.
  - Solve for all unknown parameters in $C(s)$.
    * Write the closed loop characteristic equation with a monic polynomial
    * Write the desired characteristic equation with a monic polynomial. If necessary, the desired characteristic equation may be augmented with non-dominant poles.
    * Expressions for a static error constant may be used to write additional relationships for parameters in $C(s)$.
    * We can use zeros in $C(s)$ to cancel poles of $P(s)$ (which are also closed loop poles). Alternatively, we can use poles of $C(s)$ to cancel zeros of $P(s)$. (Pole-zero cancellation should not be used on unstable poles.)
  - Evaluate performance.
  - Iterate on the design.
- Additional design considerations
  - Comments on common control objectives
    * Stability - placing all the poles directly determines stability
    * Tracking
      · Transient - both poles and zeros determine transient response characteristics. Use simulations to be sure the system behaves properly
      · Steady state error - poles, zeros, and gain determine steady state error. Steady state error can be adjusted by (i) including pure integrations in the controller or (ii) using a constant $F(s)$ to adjust the static sensitivity of the closed loop system.

    * Robustness
    * Disturbance and noise rejection - "... the selection of the desired closed-loop poles ... is a compromise between the rapidity of the response of the error vector and the sensitivity to disturbances and measurement noises [39]."
  - Simulate to check performance
    * Zeros impact the transient response of a system by determining how residuals will be distributed across the poles. Simulations may be used to check that the closed loop transient response characteristics are acceptable.
    * If transfer functions are constructed in MATLAB®, use the feedback function (i.e., `sys=feedback(G,H)`) rather than `sys=G/(1+G*H)`. The command `sys=G/(1+G*H)` might not cancel unnecessary common factors in numerator and denominator.
  - Control effort
    * "To move the poles a long way requires large gains [8]."
  - "... NMP zeros in general play the role of the supreme antagonist in the stabilization task as well as performance achievement [37]."
  - Pole zero cancellations between plant and controller don't always work for all relevant transfer functions. Pole-zero cancellations should not be used with unstable poles.

Example 31.1: Place a single pole

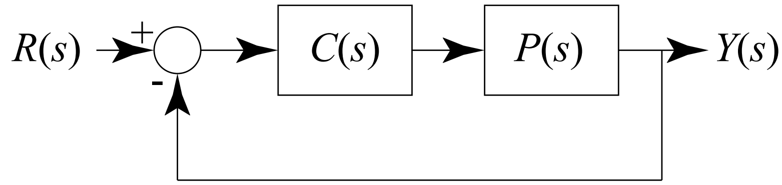


Figure 31.3:

Given the transfer function for the plant

$$P(s) = \frac{1}{s+1} \tag{31.14}$$

in the closed loop configuration of Figure 31.3, design a controller, $C(s)$, such that the unit step response of the resultant system has a 2% settling time of 2 s.
See Example 32.5 for a root locus approach to this problem.

- Determine the desired dominant pole location(s).

- 2% settling time

$$
\begin{aligned}
t_{\mathrm{s},2\%} &= 4\tau && (31.15)\\
2 &= 4\tau && (31.16)\\
\tau &= \frac{1}{2} && (31.17)
\end{aligned}
$$

- Desired dominant pole

$$
\begin{aligned}
p_{\mathrm{des.,dom.}} &= -\frac{1}{\tau} && (31.18)\\
&= -\frac{1}{\frac{1}{2}} && (31.19)\\
&= -2 && (31.20)
\end{aligned}
$$

- Select a form for the controller, $C(s)$.
  - The plant is first order, $n = 1$;
  - Required order of the controller to place all poles

$$
\begin{aligned}
k &\geq n - 1 && (31.21)\\
k &\geq 0 && (31.22)
\end{aligned}
$$

  - Form of controller

$$C(s) = K \tag{31.23}$$

  - Note, with a first order plant and zeroeth order controller, we expect the resultant closed loop characteristic equation to be first order. If everything goes as planned, the one parameter in the controller, will allow us to arbitrarily place the one pole of the closed loop system.

- Solve for all unknown parameters in $C(s)$.
  - Write the closed loop characteristic equation with a monic polynomial

$$
\begin{aligned}
1 + C(s)P(s) &= 0 && (31.24)\\
1 + K\frac{1}{s+1} &= && (31.25)\\
s + 1 + K &= && (31.26)
\end{aligned}
$$

  - Write the desired characteristic equation with a monic polynomial.

$$
\begin{aligned}
s - p_{\mathrm{des.,dom.}} &= 0 && (31.27)\\
s + 2 &= && (31.28)
\end{aligned}
$$

- Solve for unknown parameters in the controller by equating coefficients in the two expressions.

$$K = 1 \tag{31.29}$$

- Therefore

$$C(s) = 1 \tag{31.30}$$

- Evaluate performance.
  - Closed loop transfer function

$$\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{C(s)P(s)}{1 + C(s)P(s)} && (31.31)\\
&= \frac{1\frac{1}{s+1}}{1 + 1\frac{1}{s+1}} && (31.32)\\
&= \frac{1}{s+2} && (31.33)
\end{aligned}$$

  - The pole is correctly located to yield the desired 2% settling time
  - The static sensitivity is

$$\begin{aligned}
K_0 &= \lim_{s\to 0} \frac{Y(s)}{R(s)} && (31.34)\\
&= \frac{1}{2} && (31.35)
\end{aligned}$$

  - Consequently, the steady state error will be $\frac{1}{2}$ given a unit step input. Depending upon the application, this may be a problem.

- Iterate on the design.
  - Example 31.3 iterates on this controller to reduce steady state error.

Example 31.2: Place poles (and eliminate steady state error)

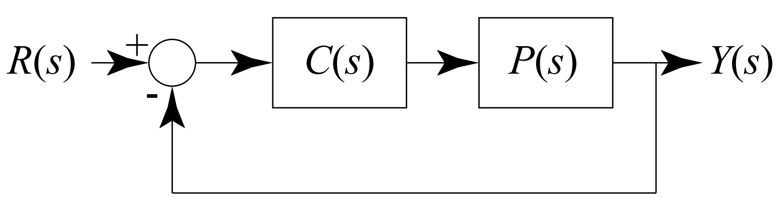


Figure 31.4:

Given the transfer function for the plant

$$P(s) = \frac{1}{s+1} \tag{31.36}$$

in the closed loop configuration of Figure 31.4, design a controller, $C(s)$ such that the unit step response of the resultant system has maximum overshoot of 21%, 2% settling time of 2 s, and no steady state error.
See Example 32.6 for a root locus approach to this problem.

- Determine the desired dominant pole location(s).
  - 2% settling time

$$\begin{aligned} t_{\mathrm{s},2\%} &= \frac{4}{\zeta\omega_n} && (31.37) \\ 2 &= && (31.38) \\ \zeta\omega_n &= 2 && (31.39) \end{aligned}$$

  - Maximum percent overshoot

$$\begin{aligned} \zeta &= \frac{-\ln\left[\frac{M_{\%}}{100\%}\right]}{\sqrt{\pi^2 + \ln\left[\frac{M_{\%}}{100\%}\right]^2}} && (31.40) \\ &= \frac{-\ln\left[\frac{21\%}{100\%}\right]}{\sqrt{\pi^2 + \ln\left[\frac{21\%}{100\%}\right]^2}} && (31.41) \\ &= 0.4449 && (31.42) \end{aligned}$$

  - Natural frequency and damped natural frequency

$$\begin{aligned} \zeta\omega_n &= 2 && (31.43) \\ \omega_n &= \frac{2}{0.4449} && (31.44) \\ &= 4.4954 && (31.45) \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} && (31.46) \\ &= 4.4954\sqrt{1-0.4449^2} && (31.47) \\ &\approx 4 && (31.48) \end{aligned}$$

- Desired dominant pole

$$
\begin{aligned}
p_{\text{des.,dom.}} &= -\zeta\omega_n \pm \omega_d j && (31.49)\\
&= -2 \pm 4j && (31.50)
\end{aligned}
$$

- Select a form for the controller, $C(s)$.
  - The plant is first order, $n = 1$;
  - Required order of the controller to place all poles

$$
\begin{aligned}
k &\geq n-1 && (31.51)\\
k &\geq 0 && (31.52)
\end{aligned}
$$

  - To place two poles (and achieve 0 steady state error), we need at least a first order controller (and at least one integrator).
  - Form of controller

$$
C(s) = \frac{b_1 s + b_0}{s} \tag{31.53}
$$

  - Note, with a first order plant and a first order controller, we expect the resultant closed loop characteristic equation to be second order. If everything goes as planned, the two parameters in the controller, will allow us to arbitrarily place the two poles of the closed loop system.
- Solve for all unknown parameters in $C(s)$.
  - Write the closed loop characteristic equation with a monic polynomial

$$
\begin{aligned}
1 + C(s)P(s) &= 0 && (31.54)\\
1 + \frac{b_1 s + b_0}{s}\frac{1}{s+1} &= && (31.55)\\
s(s+1) + b_1 s + b_0 &= && (31.56)\\
s^2 + (1+b_1)s + b_0 &= && (31.57)
\end{aligned}
$$

  - Write the desired characteristic equation with a monic polynomial.

$$
\begin{aligned}
(s - p_{\text{des.,dom.}})(s - p^*_{\text{des.,dom.}}) &= 0 && (31.58)\\
(s + 2 - 4j)(s + 2 + 4j) &= && (31.59)\\
s^2 + 4s + 20 &= && (31.60)
\end{aligned}
$$

- Solve for unknown parameters in the controller by equating coefficients in the two expressions.

$$\begin{aligned} b_0 &= 20 && (31.61)\\ 1+b_1 &= 4 && (31.62)\\ b_1 &= 3 && (31.63) \end{aligned}$$

- Therefore

$$C(s) = \frac{3s+20}{s} \tag{31.64}$$

- Evaluate performance.
  - Closed loop transfer function

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{C(s)P(s)}{1+C(s)P(s)} && (31.65)\\ &= \frac{\frac{3s+20}{s}\frac{1}{s+1}}{1+\frac{3s+20}{s}\frac{1}{s+1}} && (31.66)\\ &= \frac{3s+20}{s(s+1)+3s+20} && (31.67)\\ &= \frac{3s+20}{s^2+3s+20} && (31.68)\\ &= \frac{3\left(s+\frac{20}{3}\right)}{(s+2-4j)(s+2+4j)} && (31.69) \end{aligned}$$

  - The poles are correctly located at $-2 \pm 4j$
  - However, there is a zero at $-\frac{20}{3} = -6.67$
  - 2% settling time: 1.7 s
  - Maximum percent overshoot: 27.3%
- Iterate on the design.

Example 31.3: Place poles and set steady state error

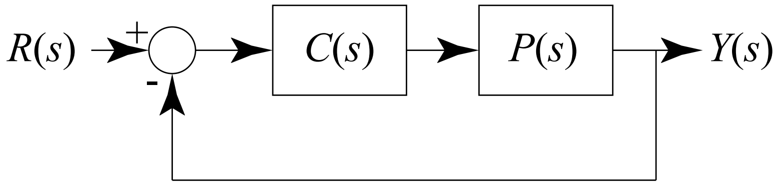


Figure 31.5:

Given the transfer function for the plant

$$P(s) = \frac{1}{s+1} \tag{31.70}$$

in the closed loop configuration of Figure 31.5, design a controller, $C(s)$, such that the unit step response of the resultant system has a 2% settling time of 2 s and steady state error of 0.01.

This is a continuation of Example 31.1

- Determine the desired dominant pole location(s).
  - 2% settling time

$$t_{\text{s},2\%} = 4\tau \tag{31.71}$$
$$2 = 4\tau \tag{31.72}$$
$$\tau = \frac{1}{2} \tag{31.73}$$

  - Desired dominant pole

$$p_{\text{des.,dom.}} = -\frac{1}{\tau} \tag{31.74}$$
$$= -\frac{1}{\frac{1}{2}} \tag{31.75}$$
$$= -2 \tag{31.76}$$

- Select a form for the controller, $C(s)$.
  - The plant is first order, $n = 1$;
  - We need at least two parameters in $C(s)$ to achieve a steady state error of 0.01 and place a pole.

- To have two independent parameters, we need at least a first order controller (which have three parameters). A first order controller will yield two poles. So we can use two parameters to place the poles and one parameter to set the steady state error.
- Required order of the controller to place all poles

$$\begin{aligned} k &\geq n-1 && (31.77) \\ k &\geq 0 && (31.78) \end{aligned}$$

- Form of controller

$$C(s) = \frac{b_1 s + b_0}{s + a_0} \tag{31.79}$$

- Note, with a first order plant and a first order controller, we expect the resultant closed loop characteristic equation to be second order. If everything goes as planned, the three parameters in the controller, will allow us to arbitrarily place the two poles and specify the steady state error of the closed loop system.

• Solve for all unknown parameters in $C(s)$.

- Write the closed loop characteristic equation with a monic polynomial

$$\begin{aligned} 1 + C(s)P(s) &= 0 && (31.80) \\ 1 + \frac{b_1 s + b_0}{s + a_0}\frac{1}{s+1} &= && (31.81) \\ (s + a_0)(s + 1) + b_1 s + b_0 &= && (31.82) \\ s^2 + (1 + a_0 + b_1)s + a_0 + b_0 &= && (31.83) \end{aligned}$$

- Write the desired characteristic equation with a monic polynomial.
  * Because the closed loop characteristic equation is second order, we will need to augment the single desired pole with a second pole.
  * In an attempt to force the second pole to be negligible, we want it to be a large multiple of the desired dominant pole. (If the factor is too large however, the control signal may exceed the capabilities of the actuator.) As a compromise, in this case we choose a factor of 5.

$$\begin{aligned} (s - p_{\text{des.,dom.}})(s - 5p_{\text{des.,dom.}}) &= 0 && (31.84) \\ (s + 2)(s + 5 \cdot 2) &= && (31.85) \\ s^2 + 12s + 20 &= && (31.86) \end{aligned}$$

– Write relationships for unknown parameters in the controller by equating coefficients in the two expressions for the characteristic equation.

$$\begin{aligned} a_0 + b_0 &= 20 && (31.87) \\ 1 + a_0 + b_1 &= 12 && (31.88) \end{aligned}$$

– Steady state error for type 0 system

$$\begin{aligned} e_{\mathrm{ss}} &= \frac{1}{1+K_p} && (31.89) \\ 0.01 &= && (31.90) \\ K_p &= 99 && (31.91) \\ \lim_{s\to 0} C(s)P(s) &= && (31.92) \\ \frac{b_0}{a_0}1 &= && (31.93) \\ -99a_0 + b_0 &= 0 && (31.94) \end{aligned}$$

– Solve for the parameters

$$\begin{aligned} a_0 &= 0.2 && (31.95) \\ b_1 &= 10.8 && (31.96) \\ b_0 &= 19.8 && (31.97) \end{aligned}$$

– Therefore

$$C(s) = \frac{10.8s + 19.8}{s + 0.2} \tag{31.98}$$

- Evaluate performance.

  – Closed loop transfer function

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{C(s)P(s)}{1 + C(s)P(s)} && (31.99) \\ &= \frac{\frac{10.8s+19.8}{s+0.2}\frac{1}{s+1}}{1 + \frac{10.8s+19.8}{s+0.2}\frac{1}{s+1}} && (31.100) \\ &= \frac{10.8s + 19.8}{(s+0.2)(s+1) + 10.8s + 19.8} && (31.101) \\ &= \frac{10.8s + 19.8}{s^2 + 12s + 20} && (31.102) \\ &= \frac{10.8\left(s + \frac{19.8}{10.8}\right)}{(s+2)(s+10)} && (31.103) \end{aligned}$$

- The poles are correctly located with one at -2 and one at -10
- However, there is a zero at $-\frac{19.8}{10.8} = -1.83$ suggesting that the desired dominant pole at -2 will be at least partially cancelled and the remaining pole at -10 may instead dominate.
- 2% settling time: 0.86 s
- Maximum percent overshoot: 3.4%

- Iterate on the design.
  - We could iterate on this design to reduce/eliminate the effects of the zero.
  - Example 31.4 iterates on this controller to eliminate the zero.

Example 31.4: Place pole, cancel open loop pole of the plant, and set steady state error

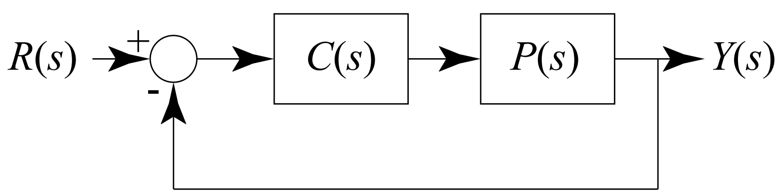


Figure 31.6:

Given the transfer function for the plant

$$P(s) = \frac{1}{s+1} \tag{31.104}$$

in the closed loop configuration of Figure 31.6, design a controller, $C(s)$, such that the unit step response of the resultant system has a 2% settling time of 2 s and steady state error of 0.01.

This is a continuation of Example 31.3

- Determine the desired dominant pole location(s).
  - 2% settling time

$$t_{s,2\%} = 4\tau \tag{31.105}$$

$$2 = 4\tau \tag{31.106}$$

$$\tau = \frac{1}{2} \tag{31.107}$$

- Desired dominant pole

$$p_{\text{des.,dom.}} = -\frac{1}{\tau} \tag{31.108}$$
$$= -\frac{1}{\frac{1}{2}} \tag{31.109}$$
$$= -2 \tag{31.110}$$

- Select a form for the controller, $C(s)$.
  - The plant is first order, $n = 1$;
  - We need at least two parameters in $C(s)$ to cancel the pole of the plant and place a pole.
  - To have two independent parameters, we need at least a first order controller (which have three parameters). A first order controller will yield two poles. So we can use two parameters to place the poles and one parameter to cancel the pole of the plant.
  - Required order of the controller to place all poles

$$k \geq n - 1 \tag{31.111}$$
$$k \geq 0 \tag{31.112}$$

  - Form of controller

$$C(s) = \frac{b_0(s+1)}{s+a_0} \tag{31.113}$$

  - Note, with a first order plant and a first order controller, we expect the resultant closed loop characteristic equation to be second order. The zero in the numerator is used to cancel the open loop pole of the plant. If everything goes as planned, the two remaining parameters in the controller allow us to arbitrarily place the one pole and dictate the steady state error.
- Solve for all unknown parameters in $C(s)$.
  - Write the closed loop characteristic equation with a monic polynomial

$$1 + C(s)P(s) = 0 \tag{31.114}$$
$$1 + \frac{b_0(s+1)}{s+a_0}\frac{1}{s+1} = \tag{31.115}$$
$$(s+1)(s+a_0+b_0) = \tag{31.116}$$

  - Write the desired characteristic equation with a monic polynomial.

* Because the closed loop characteristic equation is second order, we will need to augment the single desired pole with a second pole. In this case the second pole will be the cancelled pole
* In an attempt to force the second pole to be negligible, we want it to be a large multiple of the desired dominant pole. (If the factor is too large however, the control signal may exceed the capabilities of the actuator.) As a compromise, in this case we choose a factor of 5.

$$\begin{aligned} (s - p_{\text{des.,dom.}})(s+1) &= 0 && (31.117) \\ (s+2)(s+1) &= && (31.118) \end{aligned}$$

- Write relationships for unknown parameters in the controller by equating coefficients in the two expressions for the characteristic equation.

$$a_0 + b_0 = 2 \quad (31.119)$$

- Steady state error for type 0 system

$$\begin{aligned} e_{\text{ss}} &= \frac{1}{1+K_p} && (31.120) \\ 0.01 &= && (31.121) \\ K_p &= 99 && (31.122) \\ \lim_{s\to 0} C(s)P(s) &= && (31.123) \\ \frac{b_0}{a_0}1 &= && (31.124) \\ -99a_0 + b_0 &= 0 && (31.125) \end{aligned}$$

- Solve for the parameters

$$\begin{aligned} a_0 &= 0.02 && (31.126) \\ b_0 &= 1.98 && (31.127) \end{aligned}$$

- Therefore

$$C(s) = \frac{1.98(s+1)}{s+0.02} \quad (31.128)$$

- Evaluate performance.

- Closed loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1+C(s)P(s)} \tag{31.129}$$

$$= \frac{\frac{1.98(s+1)}{s+0.02}\frac{1}{s+1}}{1+\frac{1.98(s+1)}{s+0.02}\frac{1}{s+1}} \tag{31.130}$$

$$= \frac{1.98}{s+2}\frac{s+1}{s+1} \tag{31.131}$$

- The poles are correctly located with one at -2 and a cancelled pole at -1
- Pole zero cancellation
  * A pole zero cancelation occurs in the closed loop transfer function, $\frac{Y(s)}{R(s)}$. In effect, there is no zero in this transfer function
  * However, the pole zero cancellation does not occur for all transfer functions. For example

$$\frac{Y(s)}{D(s)} = \tag{31.132}$$

has a pole at -1 without a cancelling zero
- Steady state error requirements are met.

• Iterate on the design.

–

Example 31.5:

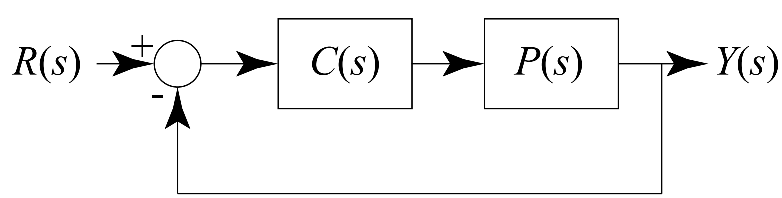


Figure 31.7:

Given the transfer function for the plant

$$P(s) = \frac{1}{s(s+1)} \tag{31.133}$$

in the closed loop configuration of Figure 31.7, design a controller, $C(s)$, such that

the unit step response of the resultant system has a maximum percent overshoot of 21% and a 2% settling time of 2 s.
See Examples 32.7 and 32.9 for a root locus approach to this problem and Example 33.2 for a frequency domain approach to this problem.

- Determine the desired dominant pole location(s).
  - 2% settling time

$$
\begin{aligned}
t_{\mathrm{s},2\%} &= \frac{4}{\zeta\omega_n} && (31.134)\\
2 &= && (31.135)\\
\zeta\omega_n &= 2 && (31.136)
\end{aligned}
$$

  - Maximum percent overshoot

$$
\begin{aligned}
\zeta &= \frac{-\ln\left[\frac{M_\%}{100\%}\right]}{\sqrt{\pi^2+\ln\left[\frac{M_\%}{100\%}\right]^2}} && (31.137)\\
&= \frac{-\ln\left[\frac{21\%}{100\%}\right]}{\sqrt{\pi^2+\ln\left[\frac{21\%}{100\%}\right]^2}} && (31.138)\\
&= 0.4449 && (31.139)
\end{aligned}
$$

  - Natural frequency and damped natural frequency

$$
\begin{aligned}
\zeta\omega_n &= 2 && (31.140)\\
\omega_n &= \frac{2}{0.4449} && (31.141)\\
&= 4.4954 && (31.142)\\
\omega_d &= \omega_n\sqrt{1-\zeta^2} && (31.143)\\
&= 4.4954\sqrt{1-0.4449^2} && (31.144)\\
&\approx 4 && (31.145)
\end{aligned}
$$

  - Desired dominant pole

$$
\begin{aligned}
p_{\mathrm{des.,dom.}} &= -\zeta\omega_n \pm \omega_d j && (31.146)\\
&= -2 \pm 4j && (31.147)
\end{aligned}
$$

- Select a form for the controller, $C(s)$.
  - The plant is second order, $n = 2$
  - Required order of the controller to place all poles

$$k \geq n - 1 \tag{31.148}$$
$$k \geq 1 \tag{31.149}$$

  - Form of controller

$$C(s) = \frac{b_1 s + b_0}{s + a_0} \tag{31.150}$$

  - Note, with a second order plant and a first order controller, we expect the resultant closed loop characteristic equation to be third order. If everything goes as planned, the three parameters in the controller allow us to arbitrarily place the three resultant poles.

- Solve for all unknown parameters in $C(s)$.
  - Write the closed loop characteristic equation with a monic polynomial

$$1 + C(s)P(s) = 0 \tag{31.151}$$
$$1 + \frac{b_1 s + b_0}{s + a_0} \frac{1}{s(s+1)} = \tag{31.152}$$
$$s(s+1)(s+a_0) + b_1 s + b_0 = \tag{31.153}$$
$$s^3 + (1 + a_0)s^2 + (a_0 + b_1)s + b_0 = \tag{31.154}$$

  - Write the desired characteristic equation with a monic polynomial.
    * Because the closed loop characteristic equation is third order, we will need to augment the complex pair of desired poles with a third pole.
    * In an attempt to force the third pole to be negligible, we want it to be a large multiple of the real part of the desired dominant pole. (If the factor is too large however, the control signal may exceed the capabilities of the actuator.) As a compromise, in this case we choose a factor of 5.

$$(s - p_{\text{des.,dom.}})(s - p^*_{\text{des.,dom.}})(s + 10) = 0 \tag{31.155}$$
$$(s + 2 - 4j)(s + 2 + 4j)(s + 10) = 0 \tag{31.156}$$
$$(s^2 + 4s + 20)(s + 10) = \tag{31.157}$$
$$s^3 + 14s^2 + 60s + 200 = \tag{31.158}$$

- Write relationships for unknown parameters in the controller by equating coefficients in the two expressions for the characteristic equation.

$$\begin{aligned} b_0 &= 200 && (31.159) \\ a_0 + b_1 &= 60 && (31.160) \\ 1 + a_0 &= 14 && (31.161) \end{aligned}$$

- Solve for the parameters

$$\begin{aligned} a_0 &= 13 && (31.162) \\ b_0 &= 200 && (31.163) \\ b_1 &= 47 && (31.164) \end{aligned}$$

- Therefore

$$C(s) = \frac{47s + 200}{s + 13} \tag{31.165}$$

- Evaluate performance.
  - 2% settling time: 1.8 s
  - Maximum percent overshoot: 33.8%
- Iterate on the design; see Example 31.6.

Example 31.6: Use of controller zero to cancel closed loop pole

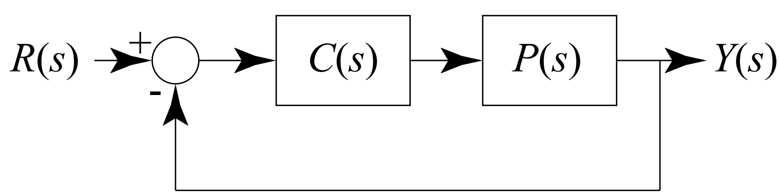


Figure 31.8:

Given the transfer function for the plant

$$P(s) = \frac{1}{s(s+1)} \tag{31.166}$$

in the closed loop configuration of Figure 31.8, iterate on the design of Example 31.5 to reduce the maximum percent overshoot to 21% and maintain a 2% settling time of 2 s.

- Select a form for the controller, $C(s)$.

- The plant is second order, $n = 2$
- Required order of the controller to place all poles

$$k \geq n - 1 \tag{31.167}$$
$$k \geq 1 \tag{31.168}$$

- Form of controller

$$C(s) = \frac{b_1 s + b_0}{s + a_0} \tag{31.169}$$

- As a consequence of the closed loop zero in Example 31.5, the maximum percent overshoot is larger than the desired 21%. Incidentally, we can show that the closed loop zero is in the same place as the zero of $C(s)P(s)$. To avoid the impact of zeros on the closed loop system, let's cancel the zero in the controller with a pole of the plant.
- Form of the controller

$$C(s) = \frac{b_0(s + 1)}{s + a_0} \tag{31.170}$$

• Determine the desired dominant pole location(s).

- 2% settling time

$$t_{\mathrm{s},2\%} = \frac{4}{\zeta\omega_n} \tag{31.171}$$
$$2 = \tag{31.172}$$
$$\zeta\omega_n = 2 \tag{31.173}$$

- Maximum percent overshoot

$$\zeta = \frac{-\ln\left[\frac{M_{\%}}{100\%}\right]}{\sqrt{\pi^2 + \ln\left[\frac{M_{\%}}{100\%}\right]^2}} \tag{31.174}$$
$$= \frac{-\ln\left[\frac{21\%}{100\%}\right]}{\sqrt{\pi^2 + \ln\left[\frac{21\%}{100\%}\right]^2}} \tag{31.175}$$
$$= 0.4449 \tag{31.176}$$

– Natural frequency and damped natural frequency

$$\zeta\omega_n = 2 \tag{31.177}$$

$$\omega_n = \frac{2}{0.4449} \tag{31.178}$$

$$= 4.4954 \tag{31.179}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{31.180}$$

$$= 4.4954\sqrt{1-0.4449^2} \tag{31.181}$$

$$\approx 4 \tag{31.182}$$

– Desired dominant pole

$$p_{\text{des.,dom.}} = -\zeta\omega_n \pm \omega_d j \tag{31.183}$$

$$= -2 \pm 4j \tag{31.184}$$

- Solve for all unknown parameters in $C(s)$.

– Write the closed loop characteristic equation with a monic polynomial

$$1 + C(s)P(s) = 0 \tag{31.185}$$

$$1 + \frac{b_0(s+1)}{s+a_0}\frac{1}{s(s+1)} = \tag{31.186}$$

$$(s+1)\left(s(s+a_0)+b_0\right) = \tag{31.187}$$

$$(s+1)\left(s^2+a_0 s+b_0\right) = \tag{31.188}$$

– Write the desired characteristic equation with a monic polynomial.

* Because the closed loop characteristic equation is third order, we will need to augment the complex pair of desired poles with a third pole.
* In this case, the third pole is used to cancel the zero of the controller.

$$(s+1)(s-p_{\text{des.,dom.}})(s-p^*_{\text{des.,dom.}}) = 0 \tag{31.189}$$

$$(s+1)(s+2-4j)(s+2+4j) = 0 \tag{31.190}$$

$$(s+1)(s^2+4s+20) = \tag{31.191}$$

– Write relationships for unknown parameters in the controller by equating coefficients in the two expressions for the characteristic equation.

$$a_0 = 4 \tag{31.192}$$

$$b_0 = 20 \tag{31.193}$$

- Therefore

$$C(s) = \frac{20(s+1)}{s+4} \tag{31.194}$$

- Evaluate performance.
  - 2% settling time: 1.9 s
  - Maximum percent overshoot: 20.8%
  - Comments:
    * The pole zero cancellation improved the performance in this case.
    * Unfortunately, other transfer functions we might construct for this system may not benefit from the same pole zero cancellation.
    * Note, the pole at $s = -1$ remains a root of the characteristic equation. Therefore, the poles (or zeros) of $C(s)$ should not be used to cancel right half plane zeros (or poles) of $P(s)$ because the closed loop system would be unstable.

Example 31.7: Non-monic characteristic equation

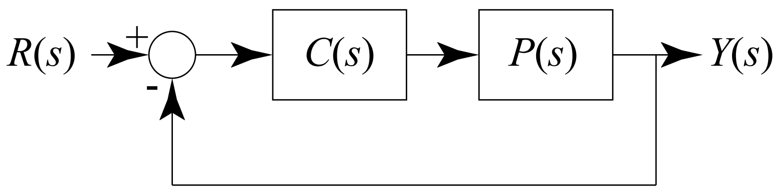


Figure 31.9:

Given the transfer function for the plant

$$P(s) = \frac{2s+5}{s+1} \tag{31.195}$$

in the closed loop configuration of Figure 31.9, design a controller, $C(s)$, such that the unit step response of the resultant system has a 2% settling time of 2 s.

- Determine the desired dominant pole location(s).
  - 2% settling time

$$t_{s,2\%} = 4\tau \tag{31.196}$$

$$2 = 4\tau \tag{31.197}$$

$$\tau = \frac{1}{2} \tag{31.198}$$

– Desired dominant pole

$$
\begin{aligned}
p_{\text{des.,dom.}} &= -\frac{1}{\tau} && (31.199)\\
&= -\frac{1}{\frac{1}{2}} && (31.200)\\
&= -2 && (31.201)
\end{aligned}
$$

- Select a form for the controller, $C(s)$.
  – The plant is first order, $n = 1$;
  – Required order of the controller to place all poles

$$
\begin{aligned}
k &\geq n-1 && (31.202)\\
k &\geq 0 && (31.203)
\end{aligned}
$$

  – Form of controller

$$C(s) = K \qquad (31.204)$$

  – Note, with a first order plant and zeroeth order controller, we expect the resultant closed loop characteristic equation to be first order. If everything goes as planned, the one parameter in the controller, will allow us to arbitrarily place the one pole of the closed loop system.

- Solve for all unknown parameters in $C(s)$.
  – Write the desired characteristic equation with a monic polynomial.

$$
\begin{aligned}
s - p_{\text{des.,dom.}} &= 0 && (31.205)\\
s + 2 &= && (31.206)
\end{aligned}
$$

  – Write the closed loop characteristic equation

$$
\begin{aligned}
1 + C(s)P(s) &= 0 && (31.207)\\
1 + K\frac{2s+5}{s+1} &= && (31.208)\\
s + 1 + K(2s+5) &= && (31.209)\\
(1+2K)s + 1 + 5K &= && (31.210)
\end{aligned}
$$

  – Solve for unknown parameters in the controller by equating coefficients in the two expressions.

    * At this point, it appears as if we have two equations and one unknown.

$$1 = 1 + 2K \tag{31.211}$$
$$2 = 1 + 5K \tag{31.212}$$

This is incorrect.

    * We must normalize the characteristic equations to make them monic (i.e., leading coefficient of 1). In this case we need to normalize the actual characteristic polynomial

$$s + \frac{1+5K}{1+2K} = 0 \tag{31.213}$$

    * Now we can solve for the unknown parameters

$$\frac{1+5K}{1+2K} = 2 \tag{31.214}$$
$$1 + 5K = 2 + 4K \tag{31.215}$$
$$K = 1 \tag{31.216}$$

  - Therefore

$$C(s) = 1 \tag{31.217}$$

- Evaluate performance.
  - Closed loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1+C(s)P(s)} \tag{31.218}$$
$$= \frac{1\frac{2s+5}{s+1}}{1+1\frac{2s+5}{s+1}} \tag{31.219}$$
$$= \frac{2s+5}{s+1+2s+5} \tag{31.220}$$
$$= \frac{2s+5}{3s+6} \tag{31.221}$$
$$= \frac{2s+5}{3(s+2)} \tag{31.222}$$
$$= \frac{\frac{2}{3}s+\frac{5}{3}}{s+2} \tag{31.223}$$

  - 2% settling time: 1.2 s

- – maximum percent overshoot: 0%
  - – steady state error: 0.17

- Comments
  - – The biproper plant resulted in a non monic characteristic polynomial. We had to make the characteristic polynomial monic in order to solve for the coefficients.

Example 31.8: Non-coprime plant

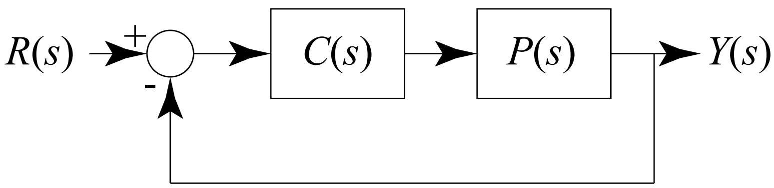


Figure 31.10:

Given the transfer function for the plant

$$P(s) = \frac{s+1}{s^2-1} \tag{31.224}$$

Example 31.9: Requirements for both step and ramp response

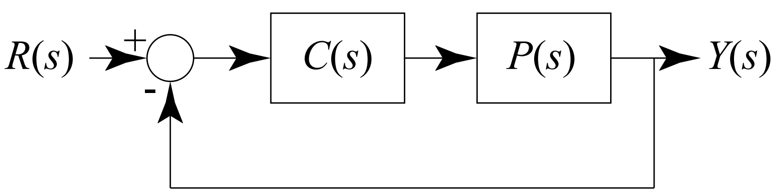


Figure 31.11:

Given the transfer function for the plant

$$P(s) = \frac{1}{s(s+1)} \tag{31.225}$$

in the closed loop configuration of Figure 31.11, design a controller, $C(s)$, such that:

a. the 2% settling time for a unit *step* input is 1 s, should this be adjusted to 2 s to be in line with the other example problems

b. the maximum percent overshoot for a unit *step* input is 21%, and

c. the steady state error from the unit *ramp* response is 0.01.

- Determine the desired dominant pole location(s).
  - 2% settling time

$$
\begin{aligned}
t_{\mathrm{s},2\%} &= \frac{4}{\zeta\omega_n} && (31.226)\\
1 &= && (31.227)\\
\zeta\omega_n &= 4 && (31.228)
\end{aligned}
$$

  - Maximum percent overshoot

$$
\begin{aligned}
\zeta &= \frac{-\ln\left[\frac{M_{\%}}{100\%}\right]}{\sqrt{\pi^2+\ln\left[\frac{M_{\%}}{100\%}\right]^2}} && (31.229)\\
&= \frac{-\ln\left[\frac{21\%}{100\%}\right]}{\sqrt{\pi^2+\ln\left[\frac{21\%}{100\%}\right]^2}} && (31.230)\\
&= 0.4449 && (31.231)
\end{aligned}
$$

  - Natural frequency and damped natural frequency

$$
\begin{aligned}
\zeta\omega_n &= 4 && (31.232)\\
\omega_n &= \frac{4}{0.4449} && (31.233)\\
&= 8.9908 && (31.234)\\
\omega_d &= \omega_n\sqrt{1-\zeta^2} && (31.235)\\
&= 8.9908\sqrt{1-0.4449^2} && (31.236)\\
&\approx 8 && (31.237)
\end{aligned}
$$

  - Desired dominant pole

$$
\begin{aligned}
p_{\mathrm{des.,dom.}} &= -\zeta\omega_n \pm \omega_d j && (31.238)\\
&= -4 \pm 8j && (31.239)
\end{aligned}
$$

- Select a form for the controller, $C(s)$.
  - The plant is second order, $n = 2$;

– Required order of the controller to place all poles

$$
\begin{aligned}
k &\geq 2-1 && (31.240)\\
k &\geq 1 && (31.241)
\end{aligned}
$$

– Form of controller

$$C(s) = \frac{b_2 s + b_1 s + b_0}{s^2 + a_1 s + a_0} \quad (31.242)$$

– Note, with a second order plant and a second order controller, we expect the resultant closed loop characteristic equation to be fourth order. If everything goes as planned, the five parameters in the controller, will allow us to arbitrarily place all four poles and specify the steady state error of the closed loop system.

- Solve for all unknown parameters in $C(s)$.

– Write the closed loop characteristic equation with a monic polynomial

$$
\begin{aligned}
1 + C(s)P(s) &= 0 && (31.243)\\
1 + \frac{b_2 s + b_1 s + b_0}{s^2 + a_1 s + a_0}\frac{1}{s(s+1)} &= && (31.244)\\
(s^2 + a_1 s + a_0)s(s+1) + b_2 s^2 + b_1 s + b_0 &= && (31.245)\\
s^4 + (1+a_1)s^3 + (a_1 + a_0 + b_2)s^2 + (a_0 + b_1)s + b_0 &= && (31.246)
\end{aligned}
$$

– Write the desired characteristic equation with a monic polynomial.

* Because the closed loop characteristic equation is fourth order, we will need to augment the complex pair of poles with two additional poles sufficiently far to the left to be insignificant.
* Characteristic equation.

$$
\begin{aligned}
(s+4-8j)(s+4+8j)(s+5\cdot 4)^2 &= 0 && (31.247)\\
(s^2 + 8s + 80)(s^2 + 40s_4 00) &= && (31.248)\\
s^4 + 48s^3 + 800s^2 + 6400s + 32000 &= && (31.249)
\end{aligned}
$$

– Write relationships for unknown parameters in the controller by equating coefficients in the two expressions for the characteristic equation.

$$
\begin{aligned}
a_1 + 1 &= 48 && (31.250)\\
a_1 + a_0 + b_2 &= 800 && (31.251)\\
a_0 + b_1 &= 6400 && (31.252)\\
b_0 &= 32000 && (31.253)
\end{aligned}
$$

- Steady state error for type 1 system

$$
\begin{aligned}
e_{\text{ss}} &= \frac{1}{K_v} && (31.254)\\
0.01 &= && (31.255)\\
K_v &= 100 && (31.256)\\
\lim_{s\to 0} sC(s)P(s) &= && (31.257)\\
\frac{b_0}{a_0}1 &= && (31.258)\\
-100a_0 + b_0 &= 0 && (31.259)
\end{aligned}
$$

- Solve for the parameters

$$
\begin{aligned}
a_0 &= 320 && (31.260)\\
a_1 &= 47 && (31.261)\\
b_0 &= 32000 && (31.262)\\
b_1 &= 6080 && (31.263)\\
b_2 &= 433 && (31.264)
\end{aligned}
$$

- Therefore

$$C(s) = \frac{433s^2 + 6080s + 32000}{s^2 + 47s + 320} \tag{31.265}$$

- Evaluate performance.
  - Closed loop transfer function
  - The poles are correctly located
  -
- Iterate on the design.
  - We could iterate on this design to reduce/eliminate the effects of the zero.

Example 31.10: Direct pole placement for an alternative control configuration

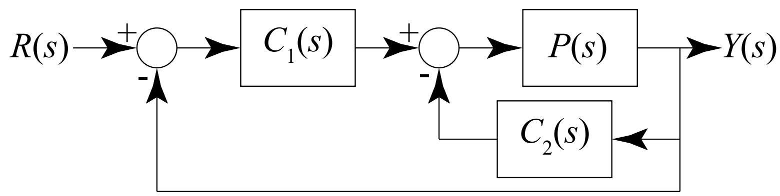


Figure 31.12:

Given the transfer function for the plant

$$P(s) = \frac{1}{s^2 - 1} \tag{31.266}$$

in the closed loop configuration of Figure 31.12, design controllers $C_1(s)$ and $C_2(s)$ such that the unit step response of the resultant system has a maximum percent overshoot of 21%, a 2% settling time of 2 s, and no steady state error.

This problem needs to be completed with details

- Determine the desired dominant pole location(s).

$$p_{\text{des.}} = -2 \pm 4j \tag{31.267}$$

- Select the form of the controllers
  - $C_1(s)$
    * Use $C_1(s)$ set the system type to 1.
    * Also, keep the zero of $C_1(s)$ sufficiently far to the left that it has little impact on the system response. For this problem a zero at -10 should work because it is a factor of 5 times to the left of the desired pole location.
    * Form

$$C_1(s) = K\frac{s+10}{s} \tag{31.268}$$

  - $C_2(s)$
    * Use the parameters in $C_2(s)$ and K from $C_1(s)$ to place the closed loop system poles
    * The plant is of order 2, so lets use a first order controller

$$C_2(s) = \frac{b_1 s + b_0}{s + a_0} \tag{31.269}$$

- Solve for all unknown parameters in $C_1(s)$ and $C_2(s)$
  - Write the closed loop characteristic equation

$$1 + (C_1(s) + C_2(s))P(s) = 0 \tag{31.270}$$

  - Write the desired characteristic equation with a monic polynomial.
- ...
- Solution?

$$C_1(s) = 8.\bar{3}\frac{s+10}{s} \tag{31.271}$$

$$C_2(s) = \frac{192.\bar{6}s + 540.\bar{6}}{s+24} \tag{31.272}$$

Chapter 32

Root Locus Design

32.1 Root Locus

- Introduction to root locus
  - Root locus - The root locus is the set of all points (in the complex plane) available as closed loop system poles considering all possible values of a parameter K (usually for $K > 0$).
  - The root locus can be used as a control system design tool to visualize or anticipate how closed loop system poles change as a parameter is varied
    * Recall closed loop system poles determine stability and play a major role in the transient response.
  - Closed loop characteristic equation
    * We consider systems for which the closed loop characteristic equation can be wrriten in terms of the parameter K and a transfer function $L(s)$

$$1 + KL(s) = 0 \tag{32.1}$$

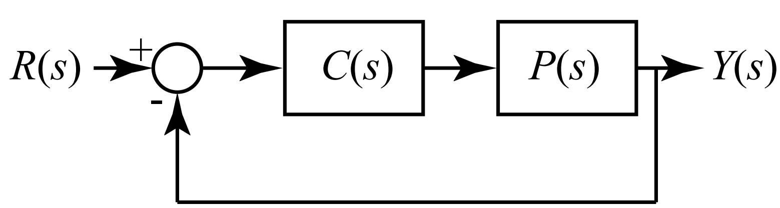


Figure 32.1:

* Note, the parameter K can appear in many places and still yield a characteristic equation of the above form.
  · Example (see Figure 32.1): $C(s) = K\bar{C}(s)$

$$1 + C(s)P(s)H(s) = 0 \tag{32.2}$$
$$1 + K\bar{C}(s)P(s)H(s) = 0 \tag{32.3}$$
$$1 + KL(s) = 0 \tag{32.4}$$

  · Example: $P(s) = K\bar{P}(s)$

$$1 + C(s)P(s)H(s) = 0 \tag{32.5}$$
$$1 + C(s)K\bar{P}(s)H(s) = 0 \tag{32.6}$$
$$1 + KL(s) = 0 \tag{32.7}$$

  · Example: $H(s) = K\bar{H}(s)$

$$1 + C(s)P(s)H(s) = 0 \tag{32.8}$$
$$1 + C(s)P(s)K\bar{H}(s) = 0 \tag{32.9}$$
$$1 + KL(s) = 0 \tag{32.10}$$

  · Example: $C(s) = \frac{1}{Ks+1}\bar{C}(s)$

$$1 + C(s)P(s)H(s) = 0 \tag{32.11}$$
$$1 + \frac{1}{Ks+1}\bar{C}(s)P(s)H(s) = 0 \tag{32.12}$$
$$Ks + 1 + \bar{C}(s)P(s)H(s) = 0 \tag{32.13}$$
$$1 + K\frac{s}{1 + \bar{C}(s)P(s)H(s)} = 0 \tag{32.14}$$
$$1 + KL(s) = 0 \tag{32.15}$$

  · Example: $C(s) = (Ks + 1)\bar{C}(s)$

- Root locus: mathematical foundation
  - Modify this section to emphasize (i) characteristic equation, (ii) magnitude, and (iii) angle
  - Consider the product of root locus transfer function $L(s)$ and parameter K

$$KL(s) = K\frac{(s - z_i)(s - z_2)...(s - z_m)}{(s - p_1)(s - p_2)...(s - p_n)} \tag{32.16}$$
$$= K\frac{\prod_{i-1}^{m}(s - z_i)}{\prod_{i-1}^{n}(s - p_i)} \tag{32.17}$$
$$= K\frac{B(s)}{A(s)} \tag{32.18}$$

- Assumptions
  * Assume any gain in $L(s)$ is absorbed in K
  * Assume $L(s)$ is proper (i.e., $n \geq m$)
  * Assume $L(s)$ is co-prime?
  * Assume K is real and $K \geq 0$
  * Assume the coefficients to $A(s)$ and $B(s)$ are real (i.e., any complex poles (or zeros) come in complex conjugate pairs).
- Characteristic equation (polynomial perspective)

$$1 + KL(s) = 0 \tag{32.19}$$
$$1 + K\frac{B(s)}{A(s)} = 0 \tag{32.20}$$
$$A(s) + KB(s) = \tag{32.21}$$

  * There will be n poles because the order of the characteristic polynomial is n and $n \geq m$.
  * When $K = 0$, the poles are the roots of $A(s)$ (i.e., poles of $L(s)$)
  * As $K \to \infty$, m of the poles approach the roots of $B(s)$ (i.e., zeros of $L(s)$). (The remaining $n - m$ poles asymptotically approach a line on the complex plane.)
- Characteristic equation (complex number perspective)

$$1 + KL(s) = 0 \tag{32.22}$$
$$KL(s) = -1 \tag{32.23}$$
$$|KL(s)|e^{j\angle KL(s)} = e^{\pi j} \tag{32.24}$$

  * Magnitude - In the complex plane, a point on the root locus, s, corresponds to one value of K because $L(s)$ yields one number. (However, one value of K may correspond to multiple distinct points on the root locus)

$$|KL(s)| = 1 \tag{32.25}$$

  * Angle
    · For a complex number s to be on the root locus, $\angle L(s)$ must be an odd multiple of π (or 180°)

$$\angle KL(s) = \pi\{\pm 1, \pm 3, ...\} \tag{32.26}$$
$$\angle L(s) = \tag{32.27}$$

· For a complex number, s, to be on the root locus, the signed sum of angles to s from all other poles and zeros of $L(s)$ must be an odd multiple of π

$$\angle L(s) = \pi\{\pm 1, \pm 3, ...\} \tag{32.28}$$

$$\angle K \frac{\prod_{i-1}^{m}(s - z_i)}{\prod_{i-1}^{n}(s - p_i)} = \tag{32.29}$$

$$\sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = \tag{32.30}$$

- Root locus: mathematical implications
  - Many sources provide a list of mathematical implications (often termed rules) for sketching the root locus. The list below, for example, parallels that in [6].
  - Implications
    1. Root locus is symmetric about the real axis
       * Proof: for a system with real coefficients in $A(s)$ and $B(s)$, all closed loop poles must either be real or come in complex conjugate pairs
    2. The root locus will have n branches
       * Proof: The closed loop system will have n poles when the denominator of $L(s)$ is of order n (assuming $n \geq m$).
    3. Each branch starts at a pole of $L(s)$ and m of the branches end at a zero of $L(s)$ (assuming $n \geq m$).
       * Proof: The characteristic equation in polynomial form reduces to $A(s) = 0$ for $K = 0$ and approaches $B(s) = 0$ as $K \to \infty$
    4. Points on the real axis are part of the root locus when they are to the left of an odd number of poles and zeros of $L(s)$ (i.e., the sum of the number of poles and the number of zeros is odd)
       * Recall that for a point, s_{RL}, to be on the root locus it must satisfy

$$L(s_{\mathrm{RL}}) = \pm\pi\{1, 3, ...\} \tag{32.31}$$

$$\sum_{i=1}^{m} \angle(s_{\mathrm{RL}} - z_i) - \sum_{i=1}^{n} \angle(s_{\mathrm{RL}} - p_i) = \tag{32.32}$$

       * if s_{RL} is on the real axis, zeros and poles of $L(s)$ to the left of s_{RL} have a net zero contribution to the sum of angles because
         · the poles or zeros are on the real axis and contribute an angle of 0 and/or
         · complex conjugate pairs of poles or zeros make a net zero contribution (assuming all complex poles/zeros come with their complex conjugate)
       * if s is on the real axis, zeros and poles of $L(s)$ that are to the right of s have a net nonzero contribution to the sum of angles if and only if s is to the left of an odd number of poles and zeros because
         · The angles from a complex conjugate pair (even) will net 0 and
         · real axis poles and zeros contribute 180° each for which even multiples are equivalent to 0° and odd multiples are equivalent to 180°
    5. $n - m$ branches asymptotically approach lines directed outward from a point on the real axis
       * Results:

· Angle of asymptote

$$\theta = \frac{\pm\pi\{1, 3, ...\}}{n - m} \tag{32.33}$$

· Real axis intersection of asymptote

$$\sigma_0 = \frac{-\sum_{i=1}^{m} z_i + \sum_{i=1}^{n} p_i}{n - m} \tag{32.34}$$

∗ Proof:

· Consider the closed loop characteristic equation in polynomial form

$$A(s) + KB(s) = 0 \tag{32.35}$$

· As $K \to \infty$, the magnitude, $|s|$, of the $n - m$ poles that don't approach the zeros of $L(s)$ must approach ∞ for $A(s)$ to remain relevant in the characteristic equation.

· As $|s| \to \infty$ on any of the $n - m$ asymptotic branches, the vectors to s from any of the open loop poles and zeros are approximately equivalent

· Angle of asymptote θ

$$\sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = \pm\pi\{1, 3, ...\} \tag{32.36}$$

$$\sum_{i=1}^{m} \theta - \sum_{i=1}^{n} \theta = \tag{32.37}$$

$$m\theta - n\theta = \tag{32.38}$$

$$(m - n)\theta = \tag{32.39}$$

$$(n - m)\theta = \tag{32.40}$$

$$\theta = \frac{\pm\pi\{1, 3, ...\}}{n - m} \tag{32.41}$$

· Consider a point $s(d)$ on the asymptote with real axis intersection σ_0, angle θ, and distance d from real axis intersection;

$$s(d) = \sigma_0 + de^{j\theta}. \tag{32.42}$$

· Substitute this expression in for s considering d sufficiently large that the distance to the nearest point on the root locus may be neglected

$$\sum_{i=1}^{m} \angle(\sigma_0 + de^{j\theta} - z_i) - \sum_{i=1}^{n} \angle(\sigma_0 + de^{j\theta} - p_i) = \pm\pi\{1, 3, ...\} \tag{32.43}$$

$$\sum_{i=1}^{m} \tan^{-1}\left(\frac{d\sin(\theta) - \mathbb{I}\{z_i\}}{\sigma_0 + d\cos(\theta) - \mathbb{R}\{z_i\}}\right) - \sum_{i=1}^{n} \tan^{-1}\left(\frac{d\sin(\theta) - \mathbb{I}\{p_i\}}{\sigma_0 + d\cos(\theta) - \mathbb{R}\{p_i\}}\right) = \pm\pi\{1, 3, ...\} \tag{32.44}$$

· The terms with d dominate as $d \to \infty$, yielding

$$\sum_{i=1}^{m} \tan^{-1}\left(\frac{\sin(\theta)}{\cos(\theta)}\right) - \sum_{i=1}^{n} \tan^{-1}\left(\frac{\sin(\theta)}{\cos(\theta)}\right) = \pm\pi\{1, 3, ...\} \tag{32.45}$$

$$\sum_{i=1}^{m} \theta - \sum_{i=1}^{n} \theta = \tag{32.46}$$

$$(m - n)\theta = \tag{32.47}$$

$$(n - m)\theta = \tag{32.48}$$

$$\theta = \frac{\pm\pi\{1, 3, ...\}}{n - m} \tag{32.49}$$

· Note, multiple nontrivial angles may result from the above.

· Now we account for the small impact of the poles (or zeros) not being colocated ($\sigma_0 - p_i$) by recognizing (Here we assume that d is sufficiently large that the differences between a point on the asymptote and nearest point on the root locus are small compared to the differences in real part of the position of the poles and zeros.)

$$\tan^{-1}\left(\frac{d\sin(\theta) - \mathbb{I}\{p\}}{d\cos(\theta) + \sigma_0 - \mathbb{R}\{p\}}\right) \approx \theta + \frac{1}{1 + \frac{\sin^2(\theta)}{\cos^2(\theta)}} \frac{(-\sin(\theta))}{\cos^2(\theta)}(\sigma_0 - \mathbb{R}\{p\}) + \frac{1}{1 + \frac{\sin^2(\theta)}{\cos^2(\theta)}}(-\mathbb{I}\{p\}) \tag{32.50}$$

recognizing that poles (and zeros) come in complex conjugate pairs

$$\tan^{-1}\left(\frac{d\sin(\theta) - \mathbb{I}\{p\}}{d\cos(\theta) + \sigma_0 - \mathbb{R}\{p\}}\right) \approx \theta + \frac{-\sin(\theta)}{\cos^2(\theta) + \sin^2(\theta)}(\sigma_0 - \mathbb{R}\{p\}) \tag{32.51}$$

$$\approx \theta - \sin(\theta)(\sigma_0 - \mathbb{R}\{p\}) \tag{32.52}$$

therefore

$$\sum_{i=1}^{m} -\sin(\theta)(\sigma_0 - \mathbb{R}\{z_i\}) + \sum_{i=1}^{n} \sin(\theta)(\sigma_0 - \mathbb{R}\{p_i\}) = 0 \tag{32.53}$$

$$\sum_{i=1}^{m} -(\sigma_0 - \mathbb{R}\{z_i\}) + \sum_{i=1}^{n}(\sigma_0 - \mathbb{R}\{p_i\}) = \tag{32.54}$$

$$\sum_{i=1}^{m} -(\sigma_0 - \mathbb{R}\{z_i\}) + \sum_{i=1}^{n}(\sigma_0 - \mathbb{R}\{p_i\}) = \tag{32.55}$$

$$\sigma_0 = \frac{-\sum_{i=1}^{m}\mathbb{R}\{z_i\} + \sum_{i=1}^{n}\mathbb{R}\{p_i\}}{n-m} \tag{32.56}$$

again recognizing that complex poles come in complex conjugate pairs

$$\sigma_0 = \frac{-\sum_{i=1}^{m} z_i + \sum_{i=1}^{n} p_i}{n-m} \tag{32.57}$$

· While the above proof yields the correct answer, there may be some errors in reasoning that need to be resolved.

6. Angles of departure and arrival
   * Results:
     · Angle of departure from a pole, p_k, with multiplicity q

$$\theta_{\text{depart},k} = \frac{\sum_{i=1}^{m}\angle(p_k - z_i) - \sum_{i=1,p_i\neq p_k}^{n}\angle(p_k - p_i) \pm \pi\{1,3,....\}}{q} \tag{32.58}$$

     · Angle of arrival at a zero, z_k, with multiplicity q (Note, this angle points in the direction of decreasing K.)

$$\theta_{\text{arrive},k} = \frac{-\sum_{i=1,z_i\neq z_k}^{m}\angle(z_k - z_i) + \sum_{i=1}^{n}\angle(z_k - p_i) \pm \pi\{1,3,....\}}{q} \tag{32.59}$$

   * Proofs:

· A point on the root locus near an open loop pole p_k with multiplicity q

$$\sum_{i=1}^{m} \angle(p_k + \epsilon - z_i) - \sum_{i=1}^{n} \angle(p_k + \epsilon - p_i) = \pm\pi\{1, 3, ...\} \tag{32.60}$$

$$\sum_{i=1}^{m} \angle(p_k + \epsilon - z_i) - \sum_{i=1, p_i \neq p_k}^{n} \angle(p_k + \epsilon - p_i) - q\angle(p_k + \epsilon - p_i) = \tag{32.61}$$

for small ϵ, ϵ may be ignored at all poles and zeros except for $p = p_k$ for which the angle will be $\theta_{\text{depart},k}$ (assuming $L(s)$ is coprime)

$$\sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{i=1, p_i \neq p_k}^{n} \angle(p_k - p_i) - q\theta_{\text{depart},k} = \pm\pi\{1, 3, ...\} \tag{32.62}$$

$$\theta_{\text{depart},k} = \frac{\sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{i=1, p_i \neq p_k}^{n} \angle(p_k - p_i) \pm \pi\{1, 3, ...\}}{q} \tag{32.63}$$

· Similar reasoning yields the angle of arrival

7. Root locus intersection with the imaginary axis
   * Substitution method: the location of the imaginary axis crossing, $j\omega$ and value of K may be obtained by substituting $s = j\omega$ into the characteristic equation [39]
   * Routh array method: The value of K may be determined by finding the values of K that yield a row of zeros in the Routh array. The frequencies may be obtained from the row above the row of zeros; see, for example, [39, 6]
   * Angle method [6]
   * Note, these crossings are significant because they the imaginary axis serves as the boundary between stable and unstable systems.
8. Break-in and break-away points
   * Results
     · break-in and break-away points may be obtained by solving either of the following for real values of s

$$\frac{d}{ds} L(s) = 0 \tag{32.64}$$

$$\frac{d}{ds} \frac{1}{L(s)} = 0 \tag{32.65}$$

     · To be break-in or break-away points, they must also be on the the root locus; see, for example, [38, 39]

* Proof: (The reasoning below follows that provided in [6] but also appears in other sources; see, for example, [70].)
    · As K increases before break-away, two poles move toward each other. Consequently, along that region of the real axis $K = -\frac{1}{L(s)}$ is maximized at the break-away point. Therefore, along that region of the real axis $L(s)$ is minimized at the break-away point to satisfy the characteristic equation.
    · As K increases after break-in, two poles move away from each other. Consequently, along that region of the real axis $K = -\frac{1}{L(s)}$ is minimized at the break-in point. Therefore, along that region of the real axis $L(s)$ is maximized at the break-in point to satisfy the characteristic equation.
    · Note the root locus can cross (or intersect) the real axis. (It's unclear if this is considered both a break-in and a break-out point or neither.) When this occurs, the above criteria may be used to find the intersection. Proof: Because of symmetry, at an intersection there must be a root of the characteristic equation ($1 + KL(s) = 0$) with multiplicity greater than 1 (for the associated constant value of K). Therefore, the derivative of the characteristic equation ($K\frac{d}{ds}L(s) = \frac{d}{ds}L(s) = 0$) will also have a root at the intersection.

Example 32.1:

Sketch the root locus given the root locus transfer function

$$L(s) = \frac{s+2}{s+1} \tag{32.66}$$

- symmetric about the real axis
- n (i.e., number of poles) branches
    - There is one branch because $L(s)$ is first order
- real axis to the left of an odd number of poles and zeros
    - The root locus exists along the real axis between the pole at -1 and the zero at -2
- branches start (with $K = 0^+$) at a pole and as $K \to \infty$ approach
    - one of the m zeros or
        * The branch starts at the pole (-1) and ends at the zero (-2)
    - an asymptote
        * NA

- angle of departure and arrival
  - departure (trivial in this case)

$$\theta_{\text{dep.},k} = \frac{\sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{i=1,p_i\neq p_k}^{n} \angle(p_k - p_i) \pm \pi\{1,3,....\}}{q} \tag{32.67}$$

$$\theta_{\text{dep.},1} = \frac{\sum_{i=1}^{1} \angle(p_1 - z_i) - \sum_{i=1,p_i\neq p_1}^{1} \angle(p_1 - p_i) \pm \pi\{1,3,....\}}{1} \tag{32.68}$$

$$= \angle(-1-(-2)) \pm \pi\{1,3,....\} \tag{32.69}$$

$$= \angle(1) \pm \pi\{1,3,....\} \tag{32.70}$$

$$= 0 \pm \pi\{1,3,....\} \tag{32.71}$$

$$= \pi \tag{32.72}$$

  - arrival (trivial in this case)

$$\theta_{\text{arr.},k} = \frac{-\sum_{i=1,z_i\neq z_k}^{m} \angle(z_k - z_i) + \sum_{i=1}^{n} \angle(z_k - p_i) \pm \pi\{1,3,....\}}{q} \tag{32.73}$$

$$\theta_{\text{arr.},1} = \frac{-\sum_{i=1,z_i\neq z_1}^{1} \angle(z_1 - z_i) + \sum_{i=1}^{1} \angle(z_1 - p_i) \pm \pi\{1,3,....\}}{1} \tag{32.74}$$

$$= \angle(-2-(-1)) \pm \pi\{1,3,....\} \tag{32.75}$$

$$= \angle(-1) \pm \pi\{1,3,....\} \tag{32.76}$$

$$= \pi \pm \pi\{1,3,....\} \tag{32.77}$$

$$= 0 \tag{32.78}$$

- imaginary axis crossings may be found by substituting $j\omega$ for s
  - NA
- break-in and break-away points are on RL and satisfy
  - NA

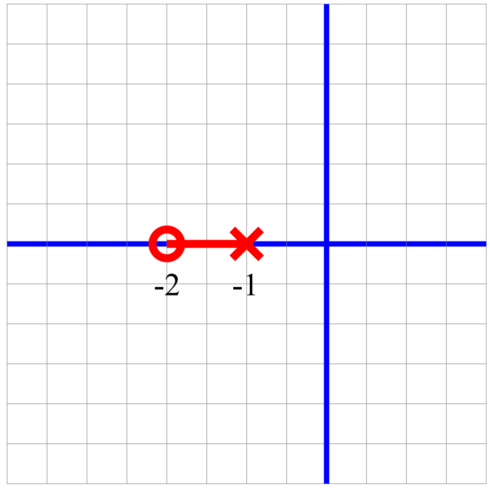


Figure 32.2:

Example 32.2:

Sketch the root locus given the root locus transfer function

$$L(s) = \frac{1}{(s+1)(s-1)} \tag{32.79}$$

- symmetric about the real axis
- n (i.e., number of poles) branches
  - There are two branches because $L(s)$ is second order
- real axis to the left of an odd number of poles and zeros
  - The root locus exists along the real axis between the pole at -1 and the pole at 1
- branches start (with $K = 0^+$) at a pole and as $K \to \infty$ approach
  - one of the m zeros or
    - NA

- an asymptote with

$$\theta = \frac{\pm\pi\{1,3,....\}}{n-m} \tag{32.80}$$
$$= \frac{\pm\pi\{1,3,....\}}{2-0} \tag{32.81}$$
$$= \frac{\pm\pi\{1,3,....\}}{2} \tag{32.82}$$
$$= \pm\frac{\pi}{2} \tag{32.83}$$

$$\sigma_0 = \frac{-\sum_{i=1}^{m} z_i + \sum_{i=1}^{n} p_i}{n-m} \tag{32.84}$$
$$= \frac{-1+1}{2-0} \tag{32.85}$$
$$= 0 \tag{32.86}$$

- angle of departure and arrival
  - departure (trivial in this case)
    - ∗ Equation

$$\theta_{\text{dep.},k} = \frac{\sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{i=1,p_i\neq p_k}^{n} \angle(p_k - p_i) \pm \pi\{1,3,....\}}{q} \tag{32.87}$$

    - ∗ pole at -1

$$\theta_{\text{dep.},1} = \frac{\sum_{i=1}^{0} \angle(p_1 - z_i) - \sum_{i=1,p_i\neq p_1}^{2} \angle(p_1 - p_i) \pm \pi\{1,3,....\}}{1} \tag{32.88}$$
$$= -\angle(-1-1) \pm \pi\{1,3,....\} \tag{32.89}$$
$$= -\angle(-2) \pm \pi\{1,3,....\} \tag{32.90}$$
$$= -\pi \pm \pi\{1,3,....\} \tag{32.91}$$
$$= 2\pi \tag{32.92}$$
$$= 0 \tag{32.93}$$

    - ∗ pole at 1

$$\theta_{\text{dep.},2} = \frac{\sum_{i=1}^{0} \angle(p_2 - z_i) - \sum_{i=1,p_i\neq p_2}^{2} \angle(p_2 - p_i) \pm \pi\{1,3,....\}}{1} \tag{32.94}$$
$$= -\angle(1-(-1)) \pm \pi\{1,3,....\} \tag{32.95}$$
$$= \angle(2) \pm \pi\{1,3,....\} \tag{32.96}$$
$$= 0 \pm \pi\{1,3,....\} \tag{32.97}$$
$$= \pi \tag{32.98}$$

- arrival - NA

- imaginary axis crossings may be found by substituting $j\omega$ for s
  - (trivial in this case)

$$1 + KL(j\omega) = 0 \tag{32.99}$$

$$1 + K\frac{1}{(j\omega + 1)(j\omega - 1)} = \tag{32.100}$$

$$1 + K\frac{1}{-\omega^2 - 1} = \tag{32.101}$$

$$-\omega^2 - 1 + K = \tag{32.102}$$

$$\omega = \pm\sqrt{K - 1},\ K > 1 \tag{32.103}$$

$$\geq \qquad 0 \tag{32.104}$$

- break-in and break-away points are on RL and satisfy
  - break-away point at (trivial in this case)

$$\frac{d}{ds}\frac{1}{L(s)} = 0 \tag{32.105}$$

$$\frac{d}{ds}\left((s + 1)(s - 1)\right) = \tag{32.106}$$

$$(s - 1) + (s + 1) = 0 \tag{32.107}$$

$$2s = 0 \tag{32.108}$$

$$s = 0 \tag{32.109}$$

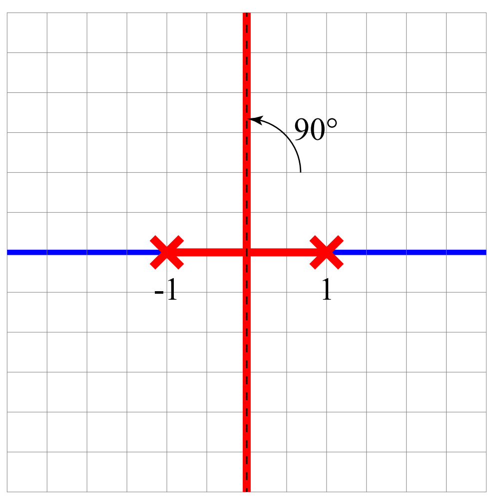


Figure 32.3:

Example 32.3:

Sketch the root locus given the root locus transfer function

$$L(s) = \frac{1}{s\left((s+3)^2 + 3^2\right)} \tag{32.110}$$

- symmetric about the real axis
- n (i.e., number of poles) branches
  - There are three branches because $L(s)$ is third order
- real axis to the left of an odd number of poles and zeros
  - The root locus exists along the real axis to the left of the pole at the origin
- branches start (with $K = 0^+$) at a pole and as $K \to \infty$ approach
  - one of the m zeros or
    - NA
  - an asymptote with

$$\begin{aligned} \theta &= \frac{\pm\pi\{1,3,....\}}{n-m} && (32.111)\\ &= \frac{\pm\pi\{1,3,....\}}{3-0} && (32.112)\\ &= \frac{\pm\pi\{1,3,....\}}{3} && (32.113)\\ &= \left\{\pm\frac{\pi}{3}, \pi\right\} && (32.114) \end{aligned}$$

$$\begin{aligned} \sigma_0 &= \frac{-\sum_{i=1}^{m} z_i + \sum_{i=1}^{n} p_i}{n-m} && (32.115)\\ &= \frac{0 - 3 + 3j - 3 - 3j}{3-0} && (32.116)\\ &= -2 && (32.117) \end{aligned}$$

- angle of departure and arrival
  - departure
    - Equation

$$\theta_{\text{dep.},k} = \frac{\sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{i=1, p_i \neq p_k}^{n} \angle(p_k - p_i) \pm \pi\{1,3,....\}}{q} \tag{32.118}$$

* pole at $-3+3j$

$$\theta_{\text{dep.},1} = \frac{\sum_{i=1}^{0} \angle(p_1 - z_i) - \sum_{i=1, p_i \neq p_1}^{3} \angle(p_1 - p_i) \pm \pi\{1,3,....\}}{1} \tag{32.119}$$

$$= -\angle(-3+3j-0) - \angle(-3+3j-(-3-3j)) \pm \pi\{1,3,....\} \tag{32.120}$$

$$= -\angle(-3+3j) - \angle(6j) \pm \pi\{1,3,....\} \tag{32.121}$$

$$= -\tan^{-1}\left(\frac{3}{-3}\right) - \tan^{-1}\left(\frac{6}{0}\right) \pm \pi\{1,3,....\} \tag{32.122}$$

$$= -\frac{3}{4}\pi - \frac{1}{2}\pi \pm \pi\{1,3,....\} \tag{32.123}$$

$$= -\frac{1}{4}\pi \tag{32.124}$$

$$= \frac{7}{4}\pi \tag{32.125}$$

* pole at 0 (trivial)

$$\theta_{\text{dep.},2} = \frac{\sum_{i=1}^{0} \angle(p_2 - z_i) - \sum_{i=1, p_i \neq p_2}^{3} \angle(p_2 - p_i) \pm \pi\{1,3,....\}}{1} \tag{32.126}$$

$$= -\angle(0-(-3+3j)) - \angle(0-(-3-3j)) \pm \pi\{1,3,....\} \tag{32.127}$$

$$= -\angle(3-3j) - \angle(3+3j) \pm \pi\{1,3,....\} \tag{32.128}$$

$$= -\tan^{-1}\left(\frac{-3}{3}\right) - \tan^{-1}\left(\frac{3}{3}\right) \pm \pi\{1,3,....\} \tag{32.129}$$

$$= \frac{1}{4}\pi - \frac{1}{4}\pi \pm \pi\{1,3,....\} \tag{32.130}$$

$$= \pi \tag{32.131}$$

* pole at $-3-3j$ (symmetric)

$$\theta_{\text{dep.},3} = -\theta_{\text{dep.},1} \tag{32.132}$$

$$= -\frac{7}{4}\pi \tag{32.133}$$

$$= \frac{1}{4}\pi \tag{32.134}$$

– arrival - NA

- imaginary axis crossings may be found by substituting $j\omega$ for s

$$1 + KL(j\omega) = 0 \tag{32.135}$$
$$1 + K\frac{1}{j\omega\left((j\omega+3)^2+3^2\right)} = \tag{32.136}$$
$$1 + K\frac{1}{j\omega\left(-\omega^2+6j\omega+18\right)} = \tag{32.137}$$
$$1 + K\frac{1}{-j\omega^3-6\omega^2+18j\omega} = \tag{32.138}$$
$$-j\omega^3-6\omega^2+18j\omega+K = \tag{32.139}$$
$$\left(K-6\omega^2\right)+j\left(18\omega-\omega^3\right) = \tag{32.140}$$

- Intersections (from imaginary part)

$$18\omega-\omega^3 = 0 \tag{32.141}$$
$$\omega\left(18-\omega^2\right) = 0 \tag{32.142}$$
$$\omega = \left\{0, 3\sqrt{2}, -3\sqrt{2}\right\} \tag{32.143}$$

- value of K at intersections (from real part)

$$K-6\omega^2 = 0 \tag{32.144}$$
$$K = 6\omega^2 \tag{32.145}$$
$$= 6\left\{0, 3\sqrt{2}, -3\sqrt{2}\right\}^2 \tag{32.146}$$
$$= \{0, 108, 108\} \tag{32.147}$$

- break-in and break-away points are on RL and satisfy
  - NA

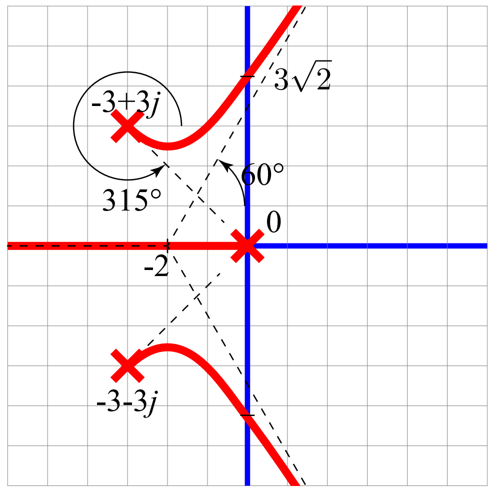


Figure 32.4:

Example 32.4:

Sketch the root locus given the root locus transfer function

$$L(s) = \frac{(s+1)(s+4)}{(s-1)(s-4)} \tag{32.148}$$

- symmetric about the real axis
- n (i.e., number of poles) branches
  - There are two branches because $L(s)$ is second order
- real axis to the left of an odd number of poles and zeros
  - The root locus exists along the real axis between -4 and -1 and between 1 and 4
- branches start (with $K = 0^+$) at a pole and as $K \to \infty$ approach
  - one of the $m = 2$ zeros or
  - an asymptote with ... NA
- angle of departure and arrival
  - departure ... trivial
  - arrival - trivial

- imaginary axis crossings may be found by substituting $j\omega$ for s

$$\begin{aligned}
1 + KL(j\omega) &= 0 && (32.149)\\
1 + K\frac{(j\omega+1)(j\omega+4)}{(j\omega-1)(j\omega-4)} &= && (32.150)\\
(j\omega-1)(j\omega-4) + K(j\omega+1)(j\omega+4) &= && (32.151)\\
-\omega^2 - 5j\omega + 4 + K(-\omega^2 + 5j\omega + 4) &= && (32.152)\\
\left(-\omega^2 + 4\right)(1+K) + j5\omega\,(-1+K) &= && (32.153)
\end{aligned}$$

  - Intersections (from real part)

$$\begin{aligned}
\left(-\omega^2 + 4\right)(1+K) &= 0 && (32.154)\\
\left(-\omega^2 + 4\right) &= && (32.155)\\
\omega &= \pm 2 && (32.156)
\end{aligned}$$

  - Intersections (from imaginary part)

$$\begin{aligned}
5\omega\,(-1+K) &= 0 && (32.157)\\
\omega &= 0 && (32.158)
\end{aligned}$$

  Note, $\omega = 0$ is not on the root locus for $K > 0$. But this equation may be used to solve for the value of K at the intersection, $K = 1$.

- break-in and break-away points are on RL and satisfy

$$\frac{d}{ds}L(s) = 0 \tag{32.159}$$

$$\frac{(s+1)+(s+4)}{(s-1)(s-4)} - \frac{(s+1)(s+4)}{(s-1)^2(s-4)^2}(s+1+s+4) = \tag{32.160}$$

$$-s^2 + 4 = \tag{32.161}$$

$$s = \pm 2 \tag{32.162}$$

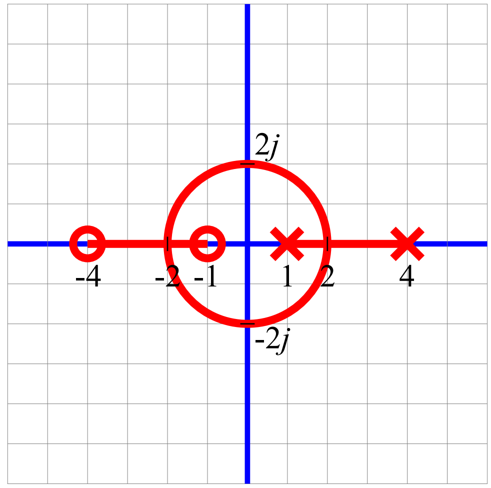


Figure 32.5:

32.2 Compensation

- This section could be moved somewhere else, because it's not specific to Root locus design
- Controller vs. compensator
  - Controller -
    - "An automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value [39]." (See also, Figure 3-7 of [39].)
  - Compensator -
    - "A compensator is an additional component ... that is inserted into a control system to compensate for a deficient performance [70]."
    - "A device inserted into the system for the purpose of satisfying the specifications is called a compensator. The compensator compensates for deficit performance of the original system [39]."
    - Figure 9.2 of [6] shows a compensator as a block in series with the original controller all within a feedback loop.
  - Distinction
    - The above definitions suggest that the purpose of a compensator is to improve an existing control system; see, for example, [70, 39, 6].
    - The 'existing control system' could consist only of a feedback path and summer (i.e., $C(s) = 1$).
    - In the context of root locus design, we prefer the term compensator because we address (i.e., compensate for) only one part of the response at a time. For example, a compensator might:
      - Move only the dominant poles
      - improve only the steady state error
      - etc.
    - We will use these terms almost interchangeably.
- Common points in which compensators are incorporated into the overall feedback structure: (See, for example, [70, 6].)
  - Cascade (or series) compensation - the compensator is placed in the forward path of the feedback system
    - We will focus on this method
    - Usage: Often used to improve steady state and/or transient response characteristics

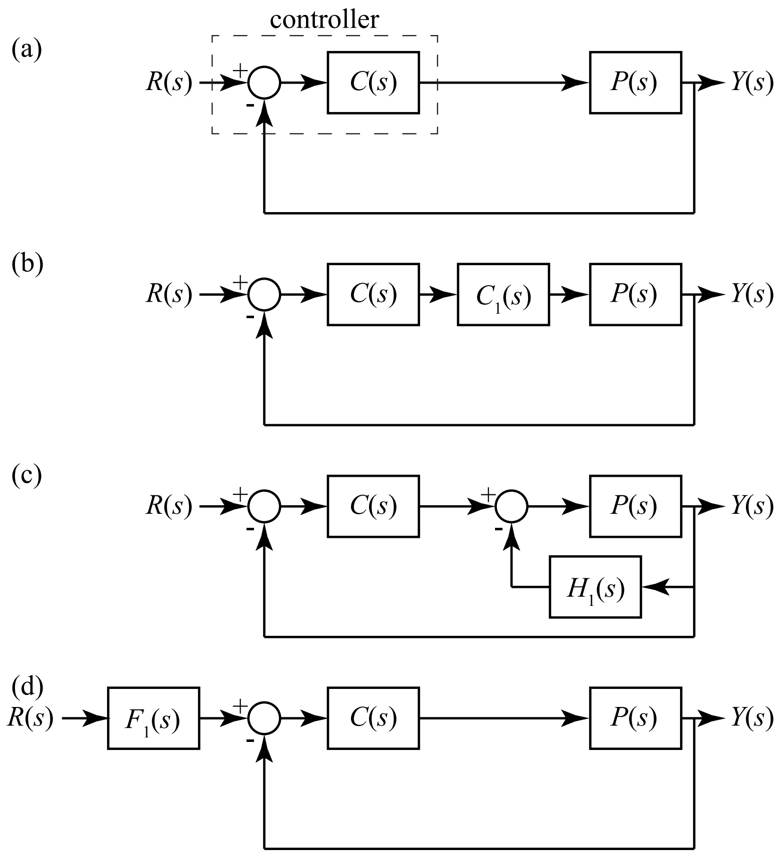


Figure 32.6: (a) An existing control system (see, for example, [39]) and various compensation schemes: (b) cascade or series compensator ($C_1(s)$), (c) feedback compensator ($H_1(s)$), and (d) input compensator ($F_1(s)$).

- Feedback compensation -
  * A common configuration with feedback compensation includes a feedback compensator placed along the inner (minor) of two feedback paths; see, for example, [39, 6, 38]
  * Usage:
- Input compensation; see, for example, [70]
  * Usage: potentially useful at alleviating (or cancelling) zeros in the closed loop system; see, for example, [38].
- Output compensation; see, for example, [70]
- Note, often a “fixed-configuration design” approach is taken in which the configuration

is fixed and the controller/compensator is designed [38].

32.3 Pole Repositioning

- Justification
  - Closed loop poles directly impact stability and transient response characteristics (which are common control system design objectives)
    * Chapter 26 Common Control System Objectives
    * Chapter 27 Poles and Zeros
- Root locus pole repositioning problem statement
  - Given an existing feedback system, find a compensator $C(s)$ such that a closed loop pole (or complex conjugate pair of poles) is (are) repositioned to a more desirable location, $p_{\text{des.}}$.
    * The approach here is to place only one pole (and hope all other poles land in, or remain near, satisfactory positions).
    * The pole may be pure real or a complex pole (the complex conjugate will automatically be placed). For complex conjugate pairs, typically we focus on the one with positive imaginary component (i.e., $\mathbb{I}\{p_{\text{des.}}\} \geq 0$)
  - Here the term compensator may be appropriate because we will dictate the position of a single pole (or a complex conjugate pair of poles) rather than all closed loop poles.
- Theory/method for repositioning a pole using root locus
  - Identify the desired closed loop pole location
    * Determine the desired pole location $p_{\text{des.}}$ (one real pole or the upper of a complex conjugate pair of poles) based on transient performance specifications.
  - Calculate the angular deficiency (angle criterion)
    * Determine the angular contribution (i.e., angular deficiency) from the the necessary for $p_{\text{des.}}$ to be on the root locus

$$\angle KL(p_{\text{des.}}) = \pm\pi\{1, 2, ...\} \tag{32.163}$$
$$\angle C(p_{\text{des.}})P(p_{\text{des.}}) = \tag{32.164}$$
$$\angle C(p_{\text{des.}}) = -\angle P(p_{\text{des.}}) \pm \pi\{1, 2, ...\} \tag{32.165}$$
$$\phi_{\text{def.}} = -\angle P(p_{\text{des.}}) \pm \pi\{1, 2, ...\} \tag{32.166}$$

    * Typically, we want to improve performance by, for example, reducing settling time and decreasing overshoot (i.e., moving the dominant pole towards the left and towards the real axis. Often this corresponds to a positive angular deficiency, $\phi_{\text{def.}} > 0$; see Figure 32.7.

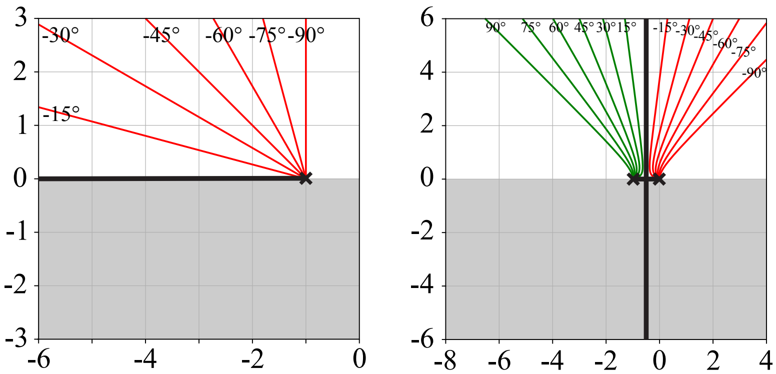


Figure 32.7: Root locus (solid black curves) and curves of constant angular deficiency (considering $-90^\circ \leq \phi_{\text{def.}} \leq 90^\circ$) for two example systems: (Left) first order system and (Right) second order system with pole at the origin. The bottom half plane is ignored in both cases (gray).

– Select the form necessary to alleviate the angular deficiency. Common forms include:
  * Proportional compensator (applicable if $\phi_{\text{def.}} = 0^\circ$)

$$C(s) = K \tag{32.167}$$

  * Lead or lag compensator (applicable if $-90^\circ < \phi_{\text{def.}} < 90^\circ$)

$$C(s) = K\frac{s-z}{s-p} \tag{32.168}$$

    · Typically, the angle requirement will only dictate a relationship between z and p.
    · Consequently, another design specification may be used to select one pole-zero pair.
    · This is considered a lead compensator when $p < z$ and a lag compensator when $z < p$.
    · A lead compensator may be used to relieve a positive angular deficiency, $0 < \phi_{\text{def.}} < 90^\circ$ (and a lag compensator may be used to relieve a negative angular deficiency).
  * PD compensator

$$C(s) = K(s-z) = K_P + K_D s \tag{32.169}$$

    · A PD compensator may be used to relieve a positive angular deficiency, $0 < \phi_{\text{def.}} < 90^\circ$.
    · Note, this is an improper transfer function and there may be issues realizing a physical system with this transfer function

– Find the appropriate value for K to place the closed loop pole (magnitude criterion)

$$|KL(p_{\text{des.}})| = 1 \tag{32.170}$$
$$|C(p_{\text{des.}})P(p_{\text{des.}})| = \tag{32.171}$$

– Evaluate system performance and iterate if necessary

  * This approach is designed to place one pole. All other pole and zero locations are not specified.
  * Other unplaced poles may dominate the response.
  * Therefore it is essential to verify system performance and stability.

Example 32.5: Reposition a real pole.

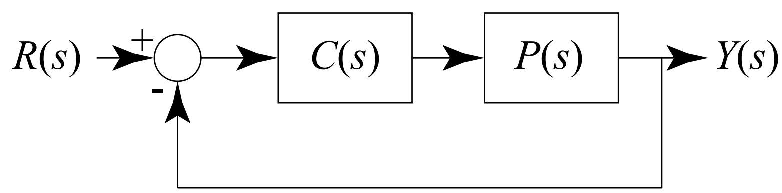


Figure 32.8:

Given the transfer function for the plant

$$P(s) = \frac{1}{s+1}$$

in the closed loop configuration of Figure 32.8, use root locus techniques to design a compensator, $C(s)$ such that the unit step response of the resultant system has a 2% settling time of 2 s.

See Example 31.1 for a direct pole placement approach to this problem.

- Identify the desired closed loop pole location

$$p_{\text{des.}} = -2 \tag{32.172}$$

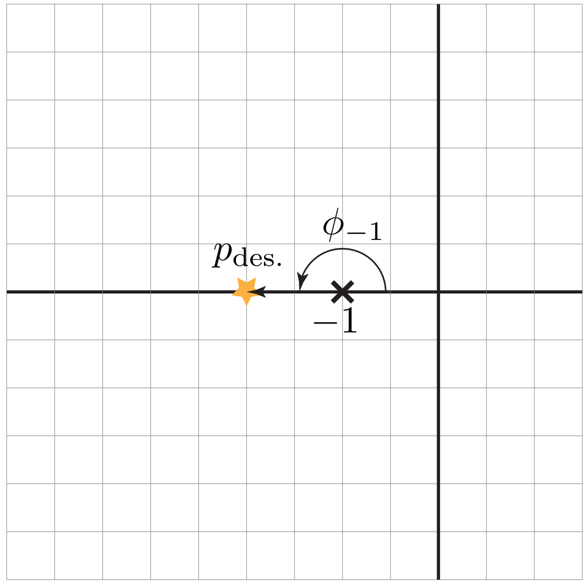


Figure 32.9:

- Calculate the angular deficiency (angle criterion)

$$1 + KL(s) = 0 \tag{32.173}$$
$$KL(s) = -1 \tag{32.174}$$
$$\angle KL(p_{\text{des.}}) = \angle -1 \tag{32.175}$$
$$= \pm\pi \tag{32.176}$$
$$\angle C(p_{\text{des.}})P(p_{\text{des.}}) = \tag{32.177}$$
$$\angle C(p_{\text{des.}}) + \angle P(p_{\text{des.}}) = \tag{32.178}$$
$$\angle C(p_{\text{des.}}) = -\angle P(p_{\text{des.}}) \pm \pi \tag{32.179}$$
$$\phi_{\text{def.}} = \tag{32.180}$$
$$= \angle \left(\frac{1}{p_{\text{des.}} + 1}\right) \pm \pi \tag{32.181}$$
$$= -\left(\angle 1 - \angle(p_{\text{des.}} + 1)\right) \pm \pi \tag{32.182}$$
$$= \angle(p_{\text{des.}} + 1) \pm \pi \tag{32.183}$$
$$= \phi_{-1} \pm \pi \tag{32.184}$$

quantitatively

$$\phi_{\text{def.}} = \angle(-2 + 1) \pm \pi \tag{32.185}$$
$$= 0^\circ \tag{32.186}$$

- Select the compensator form necessary to alleviate the angular deficiency.
  - A proportional compensator may be used to place a pole when the angular deficiency is $\phi_{\text{def.}}0°$.

$$C(s) = K \tag{32.187}$$

- Find the appropriate value for K to place the closed loop pole (magnitude criterion)

$$1 + KL(s) = 0 \tag{32.188}$$
$$KL(s) = -1 \tag{32.189}$$
$$-1 = \frac{1}{KL(s)} \tag{32.190}$$
$$1 = \left|\frac{1}{C(p_{\text{des.}})P(p_{\text{des.}})}\right| \tag{32.191}$$
$$= \left|\frac{1}{K\frac{1}{p_{\text{des.}}+1}}\right| \tag{32.192}$$
$$K = |p_{\text{des.}} + 1| \tag{32.193}$$

quantitatively

$$K = |-2 + 1| \tag{32.194}$$
$$= 1 \tag{32.195}$$

- Evaluate system performance and iterate if necessary
  - The system is stable.
  - 2% settling time 1.96
  - steady state error for step input 0.5

Example 32.6: Reposition a pole off the real axis (and eliminate steady state error)

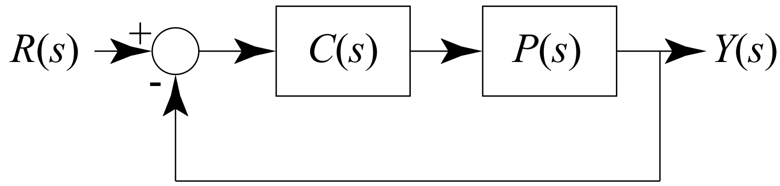


Figure 32.10:

Given the transfer function for the plant

$$P(s) = \frac{1}{s+1}$$

in the closed loop configuration of Figure 32.10, use root locus techniques to design a compensator, $C(s)$ such that the unit step response of the resultant system has maximum overshoot of 21% and 2% settling time of 2 s.

See Example 31.2 for a direct pole placement approach to this problem.

- Identify the desired closed loop pole location

$$p_{\text{des.}} = -2 \pm 4j \tag{32.196}$$

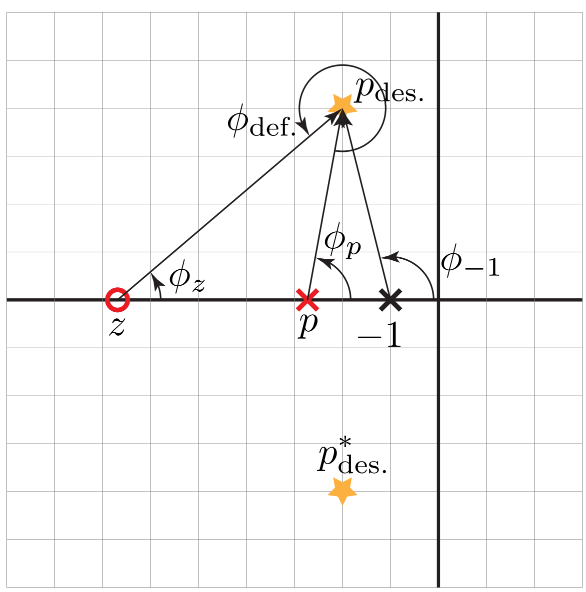


Figure 32.11:

- Calculate the angular deficiency (angle criterion)

$$\begin{aligned} 1 + KL(s) &= 0 && (32.197)\\ KL(s) &= -1 && (32.198)\\ \angle KL(p_{\text{des.}}) &= \angle -1 && (32.199)\\ &= \pm\pi && (32.200)\\ \angle C(p_{\text{des.}})P(p_{\text{des.}}) &= && (32.201)\\ \angle C(p_{\text{des.}}) + \angle P(p_{\text{des.}}) &= && (32.202)\\ \angle C(p_{\text{des.}}) &= -\angle P(p_{\text{des.}}) \pm \pi && (32.203)\\ \phi_{\text{def.}} &= && (32.204)\\ &= \angle\left(\frac{1}{p_{\text{des.}}+1}\right) \pm \pi && (32.205)\\ &= -\left(\angle 1 - \angle(p_{\text{des.}}+1)\right) \pm \pi && (32.206)\\ &= \angle(p_{\text{des.}}+1) \pm \pi && (32.207)\\ &= \phi_{-1} \pm \pi && (32.208) \end{aligned}$$

quantitatively

$$\begin{aligned} \phi_{\text{def.}} &= \angle(-2+4j+1) \pm \pi && (32.209)\\ &= -75.96^\circ && (32.210) \end{aligned}$$

- Select the compensator form necessary to alleviate the angular deficiency.
  - A lag compensator may be used to alleviate a negative angular deficiency with $-90^\circ < \phi_{\text{def.}} < 0^\circ$. (Note, in ME375 it is unusual to get a negative angular deficiency when we are trying to place poles.)
  - There are an infinite number of lag compensators that could place a pole at the desired location. (This may be explained by the fact that we are trying to satisfy two requirements, the real and imaginary parts of the desired pole location, with three parameters.)
  - If we select the compensator's pole location, p, the compensator's zero location,

z, will be uniquely determined from the angular deficiency.

$$C(s) = K\frac{s-z}{s} \tag{32.211}$$

$$\phi_{\text{def.}} = \angle C(p_{\text{des.}}) \tag{32.212}$$

$$= \angle\left(K\frac{p_{\text{des.}}-z}{p_{\text{des.}}-p}\right) \tag{32.213}$$

$$= \angle(p_{\text{des.}}-z) - \angle(p_{\text{des.}}-p) \tag{32.214}$$

$$\phi_{-1} \pm \pi = \phi_z - \phi_p \tag{32.215}$$

$$\phi_z = \phi_p + \phi_{-1} - \pi \tag{32.216}$$

– If we place the pole at the origin, we can eliminate steady state error. Now solve for the zero quantitatively

$$\phi_{\text{def.}} = \angle(-2+4j-z) - \angle(-2+4j) \tag{32.217}$$

$$\angle(-2+4j-z) = \phi_{\text{def.}} + \angle(-2+4j) \tag{32.218}$$

$$\tan^{-1}\left(\frac{4}{-2-z}\right) = \tag{32.219}$$

$$\frac{4}{-2-z} = \tan\left(\phi_{\text{def.}} + \angle(-2+4j)\right) \tag{32.220}$$

$$z = -2 - \frac{4}{\tan\left(\phi_{\text{def.}} + \angle(-2+4j)\right)} \tag{32.221}$$

$$= -2 - \frac{4}{\tan\left(-75.96^\circ + \angle(-2+4j)\right)} \tag{32.222}$$

$$= -6.\bar{6} \tag{32.223}$$

- Find the appropriate value for K to place the closed loop pole (magnitude criterion)

$$1 + KL(s) = 0 \tag{32.224}$$

$$KL(s) = -1 \tag{32.225}$$

$$-1 = \frac{1}{KL(s)} \tag{32.226}$$

$$1 = \left|\frac{1}{C(p_{\text{des.}})P(p_{\text{des.}})}\right| \tag{32.227}$$

$$= \left|\frac{1}{K\frac{p_{\text{des.}}-z}{p_{\text{des.}}-p}\frac{1}{p_{\text{des.}}+1}}\right| \tag{32.228}$$

$$K = \left|\frac{(p_{\text{des.}}-p)(p_{\text{des.}}+1)}{(p_{\text{des.}}-z)}\right| \tag{32.229}$$

quantitatively

$$K = \left| \frac{(-2+4j+0)(-2+4j+1)}{-2+4j+6.\bar{6}} \right| \tag{32.230}$$

$$= 3 \tag{32.231}$$

- Evaluate system performance and iterate if necessary
  - the system is stable
  - 2% settling time 1.73
  - 27.3% overshoot
  - steady state error for step input 0

Example 32.7: Lead compensation pole repositioning I

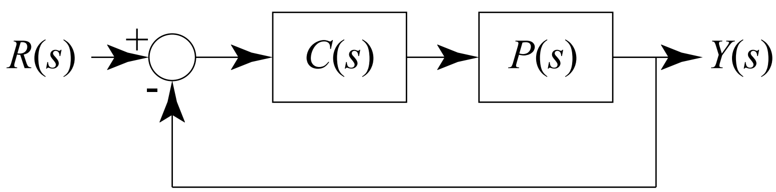


Figure 32.12:

Given the transfer function for the plant

$$P(s) = \frac{1}{s(s+1)}$$

in the closed loop configuration of Figure 32.12, use root locus techniques to design a compensator, $C(s)$ such that the unit step response of the resultant system has has maximum overshoot of 21% and 2% settling time of 2 s.
See Example 32.9 for a PD compensator design for this problem, Example 31.5 for a direct pole placement approach to this problem, and Example 33.2 for a frequency domain approach to this problem.

- Identify the desired closed loop pole location

$$p_{\text{des.}} = -2 \pm 4j \tag{32.232}$$

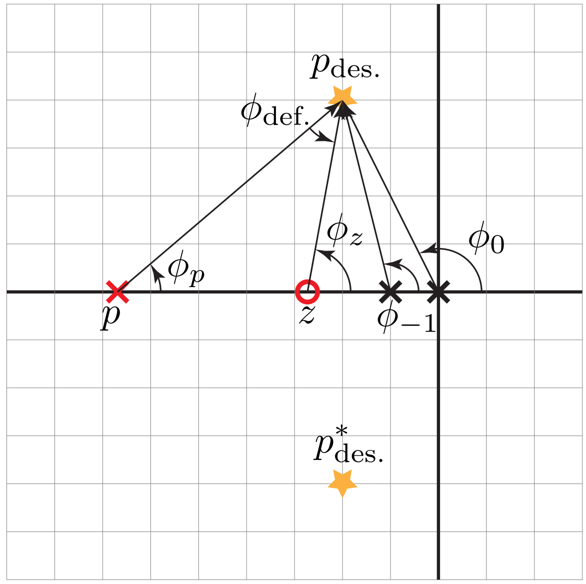


Figure 32.13:

- Calculate the angular deficiency (angle criterion)

$$\begin{aligned}
1 + KL(s) &= 0 && (32.233)\\
KL(s) &= -1 && (32.234)\\
\angle KL(p_{\text{des.}}) &= \angle -1 && (32.235)\\
&= \pm\pi && (32.236)\\
\angle C(p_{\text{des.}})P(p_{\text{des.}}) &= && (32.237)\\
\angle C(p_{\text{des.}}) + \angle P(p_{\text{des.}}) &= && (32.238)\\
\angle C(p_{\text{des.}}) &= -\angle P(p_{\text{des.}}) \pm \pi && (32.239)\\
\phi_{\text{def.}} &= && (32.240)\\
&= \angle\left(\frac{1}{p_{\text{des.}}(p_{\text{des.}}+1)}\right) \pm \pi && (32.241)\\
&= -\left(\angle 1 - \angle(p_{\text{des.}}) - \angle(p_{\text{des.}}+1)\right) \pm \pi && (32.242)\\
&= \angle(p_{\text{des.}}) + \angle(p_{\text{des.}}+1) \pm \pi && (32.243)\\
&= \phi_0 + \phi_{-1} \pm \pi && (32.244)
\end{aligned}$$

quantitatively

$$\begin{aligned}
\phi_{\text{def.}} &= \angle(-2+4j) + \angle(-2+4j+1) \pm \pi && (32.245)\\
&= 40.60^\circ && (32.246)
\end{aligned}$$

- Select the compensator form necessary to alleviate the angular deficiency.
  - A lead compensator may be used to alleviate a positive angular deficiency with $0° < \phi_{\text{def.}} < 90°$.
  - There are an infinite number of lead compensators that could place a pole at the desired location. (This may be explained by the fact that we are trying to satisfy two requirements, the real and imaginary parts of the desired pole location, with three parameters.)
  - If we select the compensator's zero location, z, the compensator's pole location, p, will be uniquely determined from the angular deficiency.

$$C(s) = K\frac{s-z}{s-p} \tag{32.247}$$

$$\phi_{\text{def.}} = \angle C(p_{\text{des.}}) \tag{32.248}$$

$$= \angle\left(K\frac{p_{\text{des.}}-z}{p_{\text{des.}}-p}\right) \tag{32.249}$$

$$= \angle(p_{\text{des.}}-z) - \angle(p_{\text{des.}}-p) \tag{32.250}$$

$$\phi_0 + \phi_{-1} \pm \pi = \phi_z - \phi_p \tag{32.251}$$

$$\phi_p = \phi_z - \phi_0 - \phi_{-1} + \pi \tag{32.252}$$

  - If we place the zero at the real part of the desired pole (ME375 rule of thumb)

$$\phi_{\text{def.}} = \angle(-2+4j+2) - \angle(-2+4j-p) \tag{32.253}$$

$$\phi_{\text{def.}} - \angle(-2+4j+2) = -\angle(-2+4j-p) \tag{32.254}$$

$$= -\tan^{-1}\left(\frac{4}{-2-p}\right) \tag{32.255}$$

$$\tan\left(-\phi_{\text{def.}} - \angle(-2+4j+2)\right) = \frac{4}{-2-p} \tag{32.256}$$

$$p = -2 - \frac{4}{\tan\left(-\phi_{\text{def.}} - \angle(-2+4j+2)\right)} \tag{32.257}$$

$$= -5.4286 \tag{32.258}$$

- Find the appropriate value for K to place the closed loop pole (magnitude criterion)

$$1 + KL(s) = 0 \tag{32.259}$$
$$KL(s) = -1 \tag{32.260}$$
$$-1 = \frac{1}{KL(s)} \tag{32.261}$$
$$1 = \left|\frac{1}{C(p_{\text{des.}})P(p_{\text{des.}})}\right| \tag{32.262}$$
$$= \left|\frac{1}{K\frac{p_{\text{des.}}-z}{p_{\text{des.}}-p}\frac{1}{p_{\text{des.}}(p_{\text{des.}}+1)}}\right| \tag{32.263}$$
$$K = \left|\frac{(p_{\text{des.}}-p)p_{\text{des.}}(p_{\text{des.}}+1)}{(p_{\text{des.}}-z)}\right| \tag{32.264}$$

quantitatively

$$K = \left|\frac{(-2+4j+5.4286)(-2+4j)(-2+4j+1)}{-2+4j+2}\right| \tag{32.265}$$
$$= 24.2858 \tag{32.266}$$

- Evaluate system performance and iterate if necessary
  - the system is stable
  - 2% settling time 1.86
  - 30.8% overshoot
  - steady state error for step input 0

Example 32.8: Lead compensation pole repositioning II

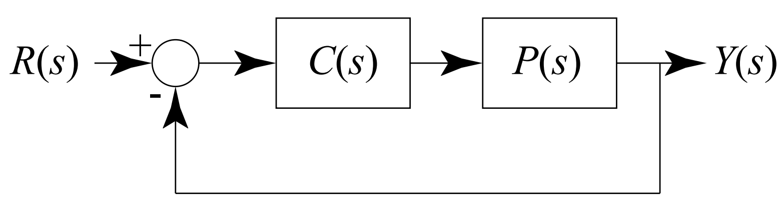


Figure 32.14:

Given the transfer function for the plant

$$P(s) = \frac{1}{s^2+1}$$

in the closed loop configuration of Figure 32.14, use root locus techniques to design a compensator, $C(s)$ such that the unit step response of the resultant system has has maximum overshoot of 21% and 2% settling time of 2 s.
See Example 32.10 for a PD compensator design for this problem.

- Identify the desired closed loop pole location

$$p_{\text{des.}} = -2 \pm 4j \tag{32.267}$$

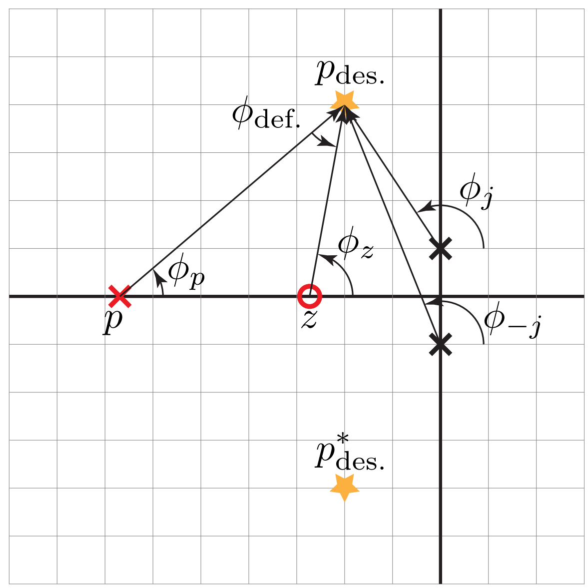


Figure 32.15:

- Calculate the angular deficiency (angle criterion)

$$\begin{aligned}
1 + KL(s) &= 0 && (32.268)\\
KL(s) &= -1 && (32.269)\\
\angle KL(p_{\text{des.}}) &= \angle -1 && (32.270)\\
&= \pm\pi && (32.271)\\
\angle C(p_{\text{des.}})P(p_{\text{des.}}) &= && (32.272)\\
\angle C(p_{\text{des.}}) + \angle P(p_{\text{des.}}) &= && (32.273)\\
\angle C(p_{\text{des.}}) &= -\angle P(p_{\text{des.}}) \pm \pi && (32.274)\\
\phi_{\text{def.}} &= -\angle \left(\frac{1}{p_{\text{des.}}^2 + 1}\right) \pm \pi && (32.275)\\
&= -\angle 1 + \angle(p_{\text{des.}}^2 + 1) \pm \pi && (32.276)\\
&= \angle(p_{\text{des.}} - j) + \angle(p_{\text{des.}} + j) \pm \pi && (32.277)\\
&= \phi_j + \phi_{-j} \pm \pi && (32.278)
\end{aligned}$$

quantitatively

$$\begin{aligned}
\phi_{\text{def.}} &= \angle(-2 + 4j - j) + \angle(-2 + 4j + j) \pm \pi && (32.279)\\
&= 55.4915^\circ && (32.280)
\end{aligned}$$

- Select the compensator form necessary to alleviate the angular deficiency.
  - A lead compensator may be used to alleviate a positive angular deficiency with $0^\circ < \phi_{\text{def.}} < 90^\circ$.
  - There are an infinite number of lead compensators that could place a pole at the desired location. (This may be explained by the fact that we are trying to satisfy two requirements, the real and imaginary parts of the desired pole location, with three parameters.)
  - If we select the compensator's zero location, z, the compensator's pole location, p, will be uniquely determined from the angular deficiency.

$$\begin{aligned}
C(s) &= K\frac{s - z}{s - p} && (32.281)\\
\phi_{\text{def.}} &= \angle C(p_{\text{des.}}) && (32.282)\\
&= \angle \left(K\frac{p_{\text{des.}} - z}{p_{\text{des.}} - p}\right) && (32.283)\\
&= \angle(p_{\text{des.}} - z) - \angle(p_{\text{des.}} - p) && (32.284)\\
\phi_j + \phi_{-j} \pm \pi &= \phi_z - \phi_p && (32.285)\\
\phi_p &= \phi_z - \phi_j - \phi_{-j} \pm \pi && (32.286)
\end{aligned}$$

- If we place the zero at the real part of the desired pole (ME375 rule of thumb)

$$\begin{aligned}
\phi_{\text{def.}} &= \angle(-2+4j+2) - \angle(-2+4j-p) && (32.287)\\
\angle(-2+4j-p) &= \angle(-2+4j+2) - \phi_{\text{def.}} && (32.288)\\
\tan^{-1}\left(\frac{4}{-2-p}\right) &= && (32.289)\\
\frac{4}{-2-p} &= \tan\left(\angle(-2+4j+2) - \phi_{\text{def.}}\right) && (32.290)\\
p &= -2 - \frac{4}{\tan\left(\angle(-2+4j+2) - \phi_{\text{def.}}\right)} && (32.291)\\
&= -2 - \frac{4}{\tan\left(90^\circ - 55.4915^\circ\right)} && (32.292)\\
&= -7.8182 && (32.293)
\end{aligned}$$

- Find the appropriate value for K to place the closed loop pole (magnitude criterion)

$$\begin{aligned}
1 + KL(s) &= 0 && (32.294)\\
KL(s) &= -1 && (32.295)\\
-1 &= \frac{1}{KL(s)} && (32.296)\\
1 &= \left|\frac{1}{C(p_{\text{des.}})P(p_{\text{des.}})}\right| && (32.297)\\
&= \left|\frac{1}{K\frac{p_{\text{des.}}-z}{p_{\text{des.}}-p}\frac{1}{(p_{\text{des.}}-j)(p_{\text{des.}}+j)}}\right| && (32.298)\\
K &= \left|\frac{(p_{\text{des.}}-p)(p_{\text{des.}}-j)(p_{\text{des.}}+j)}{(p_{\text{des.}}-z)}\right| && (32.299)
\end{aligned}$$

quantitatively

$$\begin{aligned}
K &= \left|\frac{(-2+4j+7.8182)(-2+4j-j)(-2+4j+j)}{-2+4j+2}\right| && (32.300)\\
&= 34.2728 && (32.301)
\end{aligned}$$

- Evaluate system performance and iterate if necessary
  - 2% settling time 1.86

- 52.2% overshoot. This overshoot is substantially larger than our design specifications dictate. It might be worth adjusting the zero or pole position to try to improve performance.
- steady state error for step input 0.1024
- We might consider redesigning:
  * following the rule of thumb presented in [8] in which the zero is placed at the closed loop natural frequency and/or
  * adjusting the desired closed loop pole locations.

See Example 32.12 in which a lag compensator is incorporated into this design to reduce the steady state error.

Example 32.9: PD compensation pole repositioning I

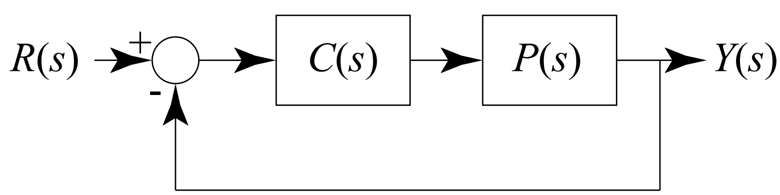


Figure 32.16:

Given the transfer function for the plant

$$P(s) = \frac{1}{s(s+1)}$$

in the closed loop configuration of Figure 32.16, use root locus techniques to design a compensator, $C(s)$ such that the unit step response of the resultant system has has maximum overshoot of 21% and 2% settling time of 2 s.
See Example 32.7 for a lead compensator design for this problem, Example 31.5 for a direct pole placement approach to this problem, and Example 33.2 for a frequency domain approach to this problem.

- Identify the desired closed loop pole location

$$p_{\text{des.}} = -2 \pm 4j \tag{32.302}$$

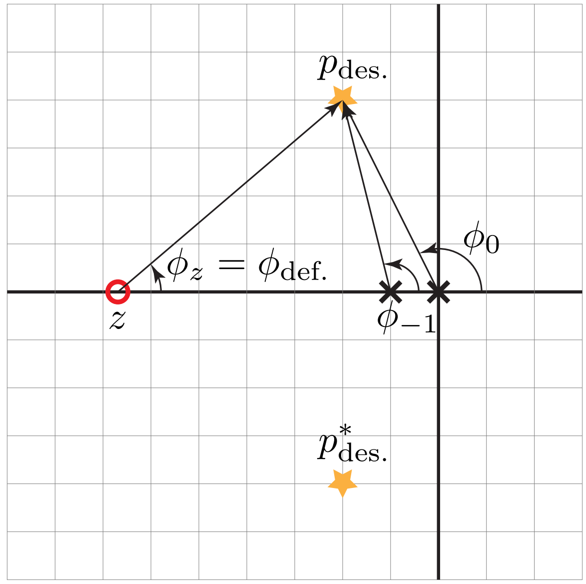


Figure 32.17:

- Calculate the angular deficiency (angle criterion)

$$
\begin{aligned}
1 + KL(s) &= 0 && (32.303)\\
KL(s) &= -1 && (32.304)\\
\angle KL(p_{\text{des.}}) &= \angle -1 && (32.305)\\
&= \pm\pi && (32.306)\\
\angle C(p_{\text{des.}})P(p_{\text{des.}}) &= && (32.307)\\
\angle C(p_{\text{des.}}) + \angle P(p_{\text{des.}}) &= && (32.308)\\
\angle C(p_{\text{des.}}) &= -\angle P(p_{\text{des.}}) \pm \pi && (32.309)\\
\phi_{\text{def.}} &= && (32.310)\\
&= \angle\left(\frac{1}{p_{\text{des.}}(p_{\text{des.}}+1)}\right) \pm \pi && (32.311)\\
&= -\left(\angle 1 - \angle(p_{\text{des.}}) - \angle(p_{\text{des.}}+1)\right) \pm \pi && (32.312)\\
&= \angle(p_{\text{des.}}) + \angle(p_{\text{des.}}+1) \pm \pi && (32.313)\\
&= \phi_0 + \phi_{-1} \pm \pi && (32.314)
\end{aligned}
$$

quantitatively

$$
\begin{aligned}
\phi_{\text{def.}} &= \angle(-2+4j) + \angle(-2+4j+1) \pm \pi && (32.315)\\
&= 40.60^\circ && (32.316)
\end{aligned}
$$

- Select the compensator form necessary to alleviate the angular deficiency.
  - A PD compensator may be used to alleviate a positive angular deficiency with $0° < \phi_{\text{def.}} < 90°$.
  - Solve for the compensator's zero location, z, as dictated by the angular deficiency.

$$\begin{aligned} C(s) &= K(s-z) && (32.317)\\ \phi_{\text{def.}} &= \angle C(p_{\text{des.}}) && (32.318)\\ &= \angle\left(K(p_{\text{des.}}-z)\right) && (32.319)\\ &= \angle(p_{\text{des.}}-z) && (32.320)\\ \phi_0+\phi_{-1}\pm\pi &= \phi_z && (32.321)\\ \phi_z &= \phi_0+\phi_{-1}\pm\pi && (32.322) \end{aligned}$$

  - Solving

$$\begin{aligned} \phi_{\text{def.}} &= \angle(-2+4j-z) && (32.323)\\ \angle(-2+4j+z) &= \phi_{\text{def.}} && (32.324)\\ \frac{4}{-2+z} &= \tan\left(\phi_{\text{def.}}\right) && (32.325)\\ z &= -2-\frac{4}{\tan\left(\phi_{\text{def.}}\right)} && (32.326)\\ &= -6.\bar{6} && (32.327) \end{aligned}$$

- Find the appropriate value for K to place the closed loop pole (magnitude criterion)

$$\begin{aligned} 1+KL(s) &= 0 && (32.328)\\ KL(s) &= -1 && (32.329)\\ -1 &= \frac{1}{KL(s)} && (32.330)\\ 1 &= \left|\frac{1}{C(p_{\text{des.}})P(p_{\text{des.}})}\right| && (32.331)\\ &= \left|\frac{1}{K(p_{\text{des.}}-z)\frac{1}{p_{\text{des.}}(p_{\text{des.}}+1)}}\right| && (32.332)\\ K &= \left|\frac{p_{\text{des.}}(p_{\text{des.}}+1)}{(p_{\text{des.}}-z)}\right| && (32.333) \end{aligned}$$

quantitatively

$$K = \left| \frac{(-2+4j)(-2+4j+1)}{-2+4j+6.\bar{6}} \right| \tag{32.334}$$

$$= 3 \tag{32.335}$$

- Evaluate system performance and iterate if necessary
  - 2% settling time 1.74
  - 27.3% overshoot
  - steady state error for step input 0

Example 32.10: PD compensation pole repositioning II

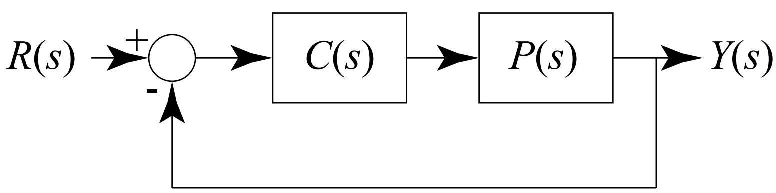


Figure 32.18:

Given the transfer function for the plant

$$P(s) = \frac{1}{s^2+1}$$

in the closed loop configuration of Figure 32.18, use root locus techniques to design a compensator, $C(s)$ such that the unit step response of the resultant system has has maximum overshoot of 21% and 2% settling time of 2 s.
See Example 32.8 for a lead compensator design for this problem.

- Identify the desired closed loop pole location

$$p_{\text{des.}} = -2 \pm 4j \tag{32.336}$$

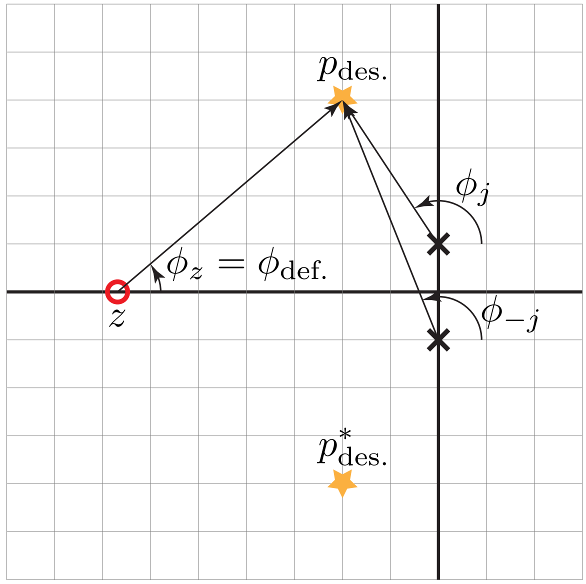


Figure 32.19:

- Calculate the angular deficiency (angle criterion)

$$\begin{aligned}
1 + KL(s) &= 0 && (32.337)\\
KL(s) &= -1 && (32.338)\\
\angle KL(p_{\text{des.}}) &= \angle -1 && (32.339)\\
&= \pm\pi && (32.340)\\
\angle C(p_{\text{des.}})P(p_{\text{des.}}) &= && (32.341)\\
\angle C(p_{\text{des.}}) + \angle P(p_{\text{des.}}) &= && (32.342)\\
\angle C(p_{\text{des.}}) &= -\angle P(p_{\text{des.}}) \pm \pi && (32.343)\\
\phi_{\text{def.}} &= && (32.344)\\
&= \angle(p^2_{\text{des.}} + 1) \pm \pi && (32.345)\\
&= \angle((-2+4j)^2 + 1) \pm \pi && (32.346)\\
&= 55.4915^\circ && (32.347)
\end{aligned}$$

- Select the compensator form necessary to alleviate the angular deficiency.
  - A PD compensator may be used to alleviate a positive angular deficiency with $0^\circ < \phi_{\text{def.}} < 90^\circ$.

- Solve for the compensator's zero location, z, as dictated by the angular deficiency.

$$C(s) = K\frac{s+2}{s-p} \tag{32.348}$$
$$\phi_{\text{def.}} = \angle C(p_{\text{des.}}) \tag{32.349}$$
$$= \angle(K(p_{\text{des.}} - z)) \tag{32.350}$$
$$= \angle(p_{\text{des.}} - z) \tag{32.351}$$
$$\tan(\phi_{\text{def.}}) = \frac{4}{-2-z} \tag{32.352}$$
$$z = -2 - \frac{4}{\tan(\phi_{\text{def.}})} \tag{32.353}$$
$$= -4.75 \tag{32.354}$$

- Find the appropriate value for K to place the closed loop pole (magnitude criterion)

$$1 + KL(s) = 0 \tag{32.355}$$
$$KL(s) = -1 \tag{32.356}$$
$$-1 = \frac{1}{KL(s)} \tag{32.357}$$
$$1 = \left|\frac{1}{C(p_{\text{des.}})P(p_{\text{des.}})}\right| \tag{32.358}$$
$$= \left|\frac{1}{K(p_{\text{des.}} - z)\frac{1}{p_{\text{des.}}(p_{\text{des.}}+1)}}\right| \tag{32.359}$$
$$K = \left|\frac{1}{(-2+4j-z)\frac{1}{((-2+4j)^2+1)}}\right| \tag{32.360}$$
$$= 4 \tag{32.361}$$

- Evaluate system performance and iterate if necessary
  - 2% settling time 1.71
  - 34.4% overshoot
  - steady state error for step input 0.05

Example 32.11: Pole repositioning may inadvertently yield an unstable system

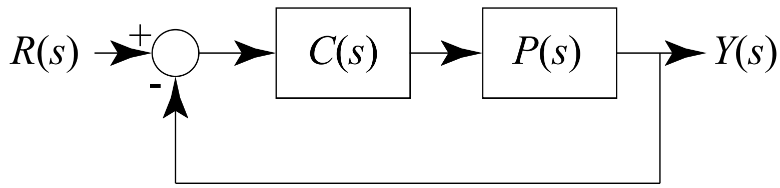


Figure 32.20:

Given the transfer function for the plant

$$P(s) = \frac{s-1}{s(s+1)^2}$$

in the closed loop configuration of Figure 32.20, use root locus techniques to design a compensator, $C(s)$ such that the unit step response of the resultant system has has maximum overshoot of 21% and 2% settling time of 2 s.

The purpose of this example is to yield an unstable system in order to emphasize the need to evaluate system performance and iterate if necessary.

- Identify the desired closed loop pole location

$$p_{\text{des.}} = -2 \pm 4j \tag{32.362}$$

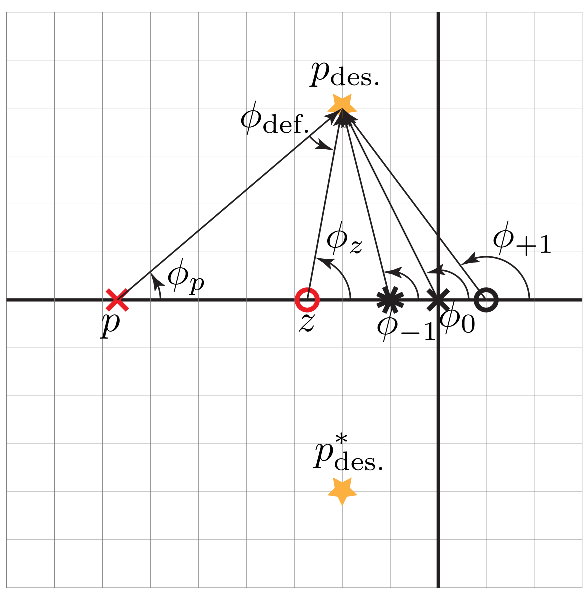


Figure 32.21:

- Calculate the angular deficiency (angle criterion)

$$
\begin{aligned}
1 + KL(s) &= 0 && (32.363)\\
KL(s) &= -1 && (32.364)\\
\angle KL(p_{\text{des.}}) &= \angle -1 && (32.365)\\
&= \pm\pi && (32.366)\\
\angle C(p_{\text{des.}})P(p_{\text{des.}}) &= && (32.367)\\
\angle C(p_{\text{des.}}) + \angle P(p_{\text{des.}}) &= && (32.368)\\
\angle C(p_{\text{des.}}) &= -\angle P(p_{\text{des.}}) \pm \pi && (32.369)\\
\phi_{\text{def.}} &= && (32.370)\\
&= \angle\left(\frac{p_{\text{des.}} - 1}{p_{\text{des.}}(p_{\text{des.}} + 1)^2}\right) \pm \pi && (32.371)\\
&= -\left(\angle(p_{\text{des.}} - 1) - \angle(p_{\text{des.}}) - 2\angle(p_{\text{des.}} + 1)\right) \pm \pi && (32.372)\\
&= \angle(p_{\text{des.}}) + \angle(p_{\text{des.}} + 1) \pm \pi && (32.373)\\
&= -\phi_{+1} + \phi_0 + 2\phi_{-1} \pm \pi && (32.374)
\end{aligned}
$$

quantitatively

$$
\begin{aligned}
\phi_{\text{def.}} &= -\angle(-2 + 4j - 1) + \angle(-2 + 4j) + 2\angle(-2 + 4j + 1) \pm \pi && (32.375)\\
&= 17.77^\circ && (32.376)
\end{aligned}
$$

- Select the form necessary to alleviate the angular deficiency.
  - A lead compensator may be used to alleviate a positive angular deficiency with $0^\circ < \phi_{\text{def.}} < 90^\circ$.
  - There are an infinite number of lead compensators that could place a pole at the desired location. (This may be explained by the fact that we are trying to satisfy two requirements, the real and imaginary parts of the desired pole location, with three parameters.)
  - If we select the 's zero location, z, the 's pole location, p, will be uniquely

determined from the angular deficiency.

$$C(s) = K\frac{s-z}{s} \tag{32.377}$$
$$\phi_{\text{def.}} = \angle C(p_{\text{des.}}) \tag{32.378}$$
$$= \angle\left(K\frac{p_{\text{des.}}-z}{p_{\text{des.}}-p}\right) \tag{32.379}$$
$$= \angle(p_{\text{des.}}-z) - \angle(p_{\text{des.}}-p) \tag{32.380}$$
$$-\phi_{+1}+\phi_0+2\phi_{-1}\pm\pi = \phi_z - \phi_p \tag{32.381}$$
$$\phi_p = \phi_z + \phi_{+1} - \phi_0 - 2\phi_{-1} \pm \pi \tag{32.382}$$

– If we place the zero at the real part of the desired pole (ME375 rule of thumb)

$$\phi_{\text{def.}} = \angle(-2+4j+2) - \angle(-2+4j-p) \tag{32.383}$$
$$\phi_{\text{def.}} - \angle(-2+4j+2) = -\angle(-2+4j-p) \tag{32.384}$$
$$= -\tan^{-1}\left(\frac{4}{-2-p}\right) \tag{32.385}$$
$$\tan\left(-\phi_{\text{def.}} - \angle(-2+4j+2)\right) = \frac{4}{-2-p} \tag{32.386}$$
$$p = -2 - \frac{4}{\tan\left(-\phi_{\text{def.}} - \angle(-2+4j+2)\right)} \tag{32.387}$$
$$= -3.2818 \tag{32.388}$$

- Find the appropriate value for K to place the closed loop pole (magnitude criterion)

$$1 + KL(s) = 0 \tag{32.389}$$
$$KL(s) = -1 \tag{32.390}$$
$$-1 = \frac{1}{KL(s)} \tag{32.391}$$
$$1 = \left|\frac{1}{C(p_{\text{des.}})P(p_{\text{des.}})}\right| \tag{32.392}$$
$$= \left|\frac{1}{K\frac{p_{\text{des.}}-z}{p_{\text{des.}}-p}\frac{(p_{\text{des.}}-1)}{p_{\text{des.}}(p_{\text{des.}}+1)^2}}\right| \tag{32.393}$$
$$K = \left|\frac{(p_{\text{des.}}-p)p_{\text{des.}}(p_{\text{des.}}+1)}{(p_{\text{des.}}-z)(p_{\text{des.}}-1)}\right| \tag{32.394}$$

quantitatively

$$K = \left| \frac{(-2+4j+3.2818)(-2+4j)(-2+4j+1)^2}{(-2+4j+2)(-2+4j-1)} \right| \tag{32.395}$$
$$= 15.9669 \tag{32.396}$$

- Evaluate system performance and iterate if necessary
  - the system is unstable with poles at: $\{-2+4j,\ -2-4j,\ 0.7759,\ -2.0577\}$

32.4 Improving Steady State Error

- Steady state error problem statement
  - Given an existing feedback system with satisfactory transient performance (i.e., dominant poles are well-placed), find a compensator $C(s)$ that provides sufficient improvement to steady state error without degrading the transient performance
  - To improve steady state error, here we will employ a compensator of the form

$$C(s) = \frac{s-z}{s-p} \tag{32.397}$$

- Theory for improving steady state error without degrading transient response
  - Assume the transfer function for the existing controller and plant combination is represented by $P(s)$
  - Static error constant
    * Denote $\bar{P}(s)$ as the transfer function $P(s)$ without any pure integrator factors
    * Static error constant

$$K_n = \lim_{s\to 0} C(s)\bar{P}(s) \tag{32.398}$$
$$= \lim_{s\to 0} \frac{s-z}{s-p}\bar{P}(s) \tag{32.399}$$
$$= \frac{0-z}{0-p}\bar{P}(0) \tag{32.400}$$
$$= \frac{z}{p}\bar{P}(0) \tag{32.401}$$

* To substantially decrease the relevant steady state error, we want

$$|K_n| \gg |\bar{P}(0)| \tag{32.402}$$

$$\left|\frac{K_n}{\bar{P}(0)}\right| \gg 1 \tag{32.403}$$

$$\left|\frac{z}{p}\right| \gg \tag{32.404}$$

$$|z| \gg |p| \tag{32.405}$$

* Typically, we want

$$z \ll p \leq 0; \tag{32.406}$$

i.e., a lag compensator (or a PI compensator if $p = 0$).

– Maintaining transient performance

* To avoid degrading the transient performance of the system, we want to keep the closed loop poles very near their current positions
* Therefore, the contribution from the compensator to the angle and magnitude at the dominant closed loop pole positions must be negligible.
* This can be accomplished by choosing z and p sufficiently near the origin that their contributions to angle and magnitude nearly cancel but $|\frac{z}{p}| \gg 1$.
* The following needs to be verified: If the steady state error needs significant improvement, the pole from the lag compensator (necessary to reduce steady state error) will likely have a relatively large residual and time constant. Therefore, incorporating a lag compensator is likely to degrade the transient performance significantly. Instead, it might be best to place the poles and reduce steady state error with a single controller.

Example 32.12: Reduce steady state error

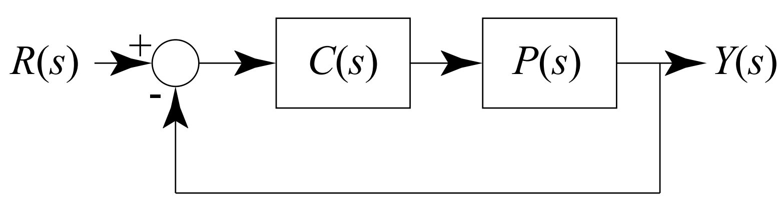


Figure 32.22:

Given the transfer function for an existing plant controller combination

$$P(s) = 34.2728\frac{s+2}{s+7.8182}\frac{1}{s^2+1}$$

in the closed loop configuration of Figure 32.22, use root locus techniques to design a lag compensator, $C(s)$, such that the steady state error of the resultant system is reduced by a factor of approximately $\frac{1}{20}$ the transient response characteristics remain largely unchanged. This is a continuation of Example 32.8.

- Identify the current dominant pole location

$$p_{\text{des.}} = -2 \pm 4j \tag{32.407}$$

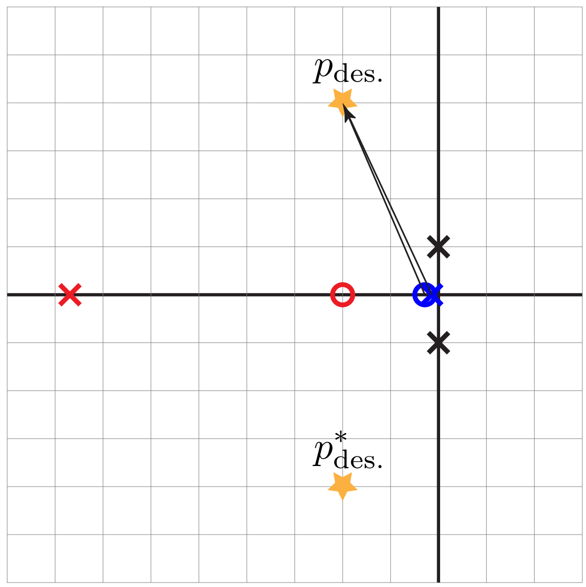


Figure 32.23:

- Following a rule of thumb, set the lag compensator zero to

$$0 > z \geq \frac{1}{10}\mathbb{R}\{p_{\text{des.}}\} \tag{32.408}$$
$$z = -0.2 \tag{32.409}$$

  – Following this rule of thumb, the magnitudes and angles of the vectors from the pole and zero be approximately equal and thereby nearly cancel out.

- Determine the pole location based on the desired reduction in steady state error (here we desire a reduction of about $\frac{1}{20}$.

$$p = \frac{1}{20}z \tag{32.410}$$
$$= -.01 \tag{32.411}$$

- Technically, this is a type 0 system so we should use

- Construct lag compensator

$$C(s) = \frac{s + 0.2}{s + 0.01} \tag{32.412}$$

- Simulate to verify satisfactory response and iterate if necessary
  - 2% settling time 9.28
    - A closed loop pole close to the origin serves to reduce steady state error, however it corresponds to rather slow dynamics.
    - These slow dynamics may require an adjustment to the lag compensator design if not both the lag and lead compensator designs.
  - 42.2% overshoot
  - steady state error for step input 0.0057
    - The steady state error is reduced by a factor of about $\frac{1}{20}$
  - pole locations: $\{-1.8341 + 4.0021j, -1.8341 - 4.0021j, -3.9811, -0.1787\}$
  - zero locations: $\{-2., -0.2\}$

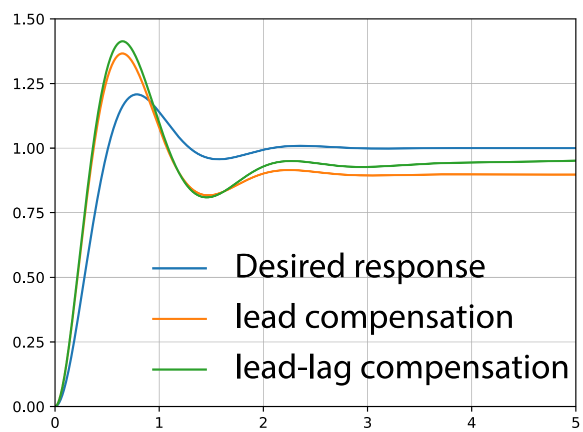


Figure 32.24:

32.5 Compensators: Design Processes and Guidelines

- this section needs to be streamlined and better synchronized with (or possibly incorporated into) the previous sections
- Proportional compensation: fill this out.
- Lead compensation: improving transient response and stability with a cascade compensator
  - Lead compensator transfer function

$$C_{\text{lead}}(s) = K\frac{s-z}{s-p} \tag{32.413}$$

    * Note, this is a biproper transfer function similar to what we used for direct pole placement
    * To be considered a lead compensator, the pole p must be to the left of the zero z, $p < z$. (typically $p < z \le 0$)
  - (Iterative) design approach; see, for example, [39, 70]
    * Identify desired dominant pole location(s), $p_{\text{des.}}$
    * Calculate the angular deficiency [39] (i.e., the angular contribution necessary from the lead compensator).

$$\angle KL(p_{\text{des.}}) = \pm\pi\{1,3,...\} \tag{32.414}$$

$$\angle C_{\text{lead}}(p_{\text{des.}}) + \angle\left(C(p_{\text{des.}})P(p_{\text{des.}})\right) = \tag{32.415}$$

$$\angle C_{\text{lead}}(p_{\text{des.}}) = -\angle\left(C(p_{\text{des.}})P(p_{\text{des.}})\right) \pm \pi\{1,3,...\} \tag{32.416}$$

    * (If the angular deficiency is nonzeror then the desired pole location is not on the root locus. Therefore,) select zero (z) and pole (p) locations for the lead compensator such that the angular deficiency is alleviated
    * Find the appropriate value for K to place the closed loop poles

$$K = \frac{1}{|L(p_{\text{des.}})|} \tag{32.417}$$

    * Simulate to verify satisfactory response and iterate if necessary
  - Design considerations and guidelines
    * Selecting the zero position
      · Franklin et. al suggests that the zero is often placed near the desired closed loop natural frequency [8].

- The common ME375 rule of thumb is to select a zero with the same real part as the desired closed loop poles; see lecture slides from other sections.

$$z = \mathbb{R}\{p_{\text{des.}}\} \tag{32.418}$$

- While using the zero to cancel a pole of the plant is tempting, the cancellation may not be advisable in all relevant system transfer functions.

* Noise
  - As $p \to -\infty$ the gain at high frequencies increases
  - This is problematic if high frequency noise is present
  - Franklin et. al suggests that the pole should generally be 5-20 times that of the zero to balance competing objectives of improving transient response and satisfactory noise rejection [8].
* May be implemented with passive elements for finite poles (i.e., ideal PD compensation requires active elements); see, for example, [6].
* Limitations for single stage lead compensator [38]
  - bandwidth considerations [38]
  - maximum phase lead is 90° [38]

– Multiple lead compensators may be cascaded
  * See, for example, discussion on multistage phase lead in [38]
  * It is possible to use more than one lead compensator in series
  * Merging a pair of lead compensators offers the freedom for poles and zeros to be complex conjugate pairs.
  * Notch compensator (or notch filter)
    - Transfer function; see, for example, [38, 8]

$$C_{\text{notch}}(s) = K\frac{s^2 + 2\zeta_z\omega_0 s + \omega_0^2}{s^2 + 2\zeta_p\omega_0 s + \omega_0^2} \tag{32.419}$$

    - With $\zeta_z < \zeta_p \leq 1$, the zeros (and possibly the poles) come in complex conjugate pairs
    - Notch compensators attenuate input signals with frequencies near ω_0
    - “If a plant ... has high-frequency vibration modes, then a desired closed-loop response may be difficult to obtain [6].”
    - “One way of eliminating the high-frequency oscillations is to cascade a *notch filter* with the plant... [6].”

• Ideal proportional + derivative (PD) compensation: improving transient response and stability with a cascade compensator

– Ideal PD compensator transfer function can be obtained by substituting $K = -K_D p$ and taking the limit as $p \to -\infty$

$$C_{\mathrm{PD}}(s) = \lim_{p\to-\infty} C_{\mathrm{lead}}(s) \tag{32.420}$$
$$= \lim_{p\to-\infty} -K_D p \frac{s-z}{s-p} \tag{32.421}$$
$$= K_D(s-z) \tag{32.422}$$
$$= K_P + K_D s \tag{32.423}$$

where we substitute $K_P = -K_D z$

– (Iterative) design approach (following the design for lead compensators; see, for example [39, 70])

  * Identify desired dominant pole location, $p_{\mathrm{des.}}$
  * Calculate the angular deficiency

$$\angle L(p_{\mathrm{des.}}) = \pi\{\pm 1, \pm 3, ...\} \tag{32.424}$$
$$\angle C_{\mathrm{PD}}(p_{\mathrm{des.}}) + \angle\left(C(p_{\mathrm{des.}})P(p_{\mathrm{des.}})\right) = \tag{32.425}$$
$$\angle C_{\mathrm{PD}}(p_{\mathrm{des.}}) = -\angle\left(C(p_{\mathrm{des.}})P(p_{\mathrm{des.}})\right) + \pi\{\pm 1, \pm 3, ...\} \tag{32.426}$$

  * (If the desired pole location is not on the root locus) select zero (z) location for the PD compensator such that the angular deficiency is alleviated
  * Find the appropriate value for K to place the closed loop poles
  * Simulate to verify satisfactory response and iterate if necessary

– Design considerations

  * Note, ideal PD compensators correspond to improper transfer functions and are not realizable.
  * Unbounded high frequency gain: One reason PD is not realizable is that the high frequency gain is unbounded. Physical systems have limits on the signal magnitude they can generate and/or tolerate.
  * Requires active elements which are suceptible to saturation; see, for example, [6]
  * Derivative kick:
  * Sometimes derivative action is used in a feedback compensator to avoid derivative kick with abrubt changes to the reference input
  * A practical PD compensator has the same form as that of a lead compensator where the denominator serves as a low pass filter.

• Lag compensation: improving steady state response with a cascade compensator

- Lag compensator transfer function

$$C_{\text{lag}}(s) = K\frac{s-z}{s-p} \tag{32.427}$$

  * Note, this is a biproper transfer function similar to what we used for direct pole placement
  * To be considered a lag compensator, the zero z must be to the left of the pole p, $z < p \leq 0$.

- (Iterative) design approach; see, for example, [39, 70]
  * (We assume the root locus for the system already passes through/near the desired poles.)
  * Determine the necessary amplification of the relevant static error constant to get satisfactory steady state response. (Note, the static error constant is inversely related to the steady state error.)
  * Set the ratio $\frac{z}{p}$ to the necessary amplification.
  * Select z near the origin and find p using the ratio. (If z and p are sufficiently close to the origin, the net angular contribution from the pole and zero will be negligible such that the desired poles will remain sufficiently close to the root locus.)
  * Find the necessary gain to maintain closed loop poles near the desired pole locations (adjusting the gain probably not necessary)
  * Simulate to verify satisfactory response and iterate if necessary

- Design considerations and guidelines
  * The common ME375 rule of thumb is to select a zero with the real part less than 1/10 in magnitude of the real part of the desired closed loop poles; see lecture slides from other sections.
  * With respect to the reference, the pole and zero nearly cancel. However, this cancellation doesn't appear with respect to disturbance, so "... it is important to place the lag pole-zero combination at as high a frequency as possible without causing major shifts in the dominant root locations [8]."
  * Note, the time constant associated with the lag compensator will be long because the pole is relatively close to the origin. Consequently, steady state error reduction can take significant time.
  * May be implemented with passive elements for nonzero poles (i.e., ideal PI compensation requires active elements); see, for example, [6].

• Ideal proportional + integral (PI) compensation: improving steady state response with a cascade compensator

- Ideal PI compensator transfer function can be obtained by substituting $p \to 0$

$$C_{\mathrm{PI}}(s) = \lim_{p\to 0} K\frac{s-z}{s-p} \tag{32.428}$$

$$= K\frac{s-z}{s} \tag{32.429}$$

$$= K_P + K_I\frac{1}{s} \tag{32.430}$$

  where we substitute $K_P = K$ and $K_I = -Kz$

- (Iterative) design approach (following the design for lag compensators; see, for example [39, 70])
  * (We assume the root locus for the system does not already passes through/near the desired poles.)
  * Select z sufficiently close close to the origin such that the net angular contribution from the zero and the pole at the origin will be negligible and the desired poles will remain sufficiently close to the root locus.
  * Find the necessary gain to maintain closed loop poles near the desired pole locations
  * Simulate to verify satisfactory response and iterate if necessary
- Design considerations
  * Requires active elements; see, for example, [6]
  * "Integral windup" see for example [5]

• Lag-lead compensation: improving transient response, stability, and steady state response by cascading compensators

- Lag and lead compensators may be cascaded together to somewhat 'independently' [6] improve both transient and steady state response characteristics
- Lag-lead transfer function

$$C_{\text{lag-lead}}(s) = K\left(\frac{s-z_{\text{lag}}}{s-p_{\text{lag}}}\right)\left(\frac{s-z_{\text{lead}}}{s-p_{\text{lead}}}\right); \tag{32.431}$$

  see, for example, [6].
  * Typically $p_{\text{lead}} < z_{\text{lead}} < 0$
  * Typically $z_{\text{lag}} < p_{\text{lag}} \leq 0$
  * Typically $p_{\text{lead}} < z_{\text{lead}} < z_{\text{lag}} < p_{\text{lag}} \leq 0$
- (Iterative) design approach
  * Design a lead (or lag) compensator to meet transient response characteristics (or meet steady state response characteristics). (Note, in some circumstances, it may be advantageous to design the lag compensator first.)

    * Design a lag (or lead) compensator, placed in series with the lead compensator, to meet steady state response characteristics (or meet transient response characteristics)
    * Simulate to verify satisfactory response and iterate if necessary.
  - Design considerations and guidelines
    * Note, lag and lead compensators may be combined in series to 'independently' improve both steady state and transient response characteristics [6].
      · Technically, lag and lead compensators impact both transient and steady state response characteristics.
      · However, when its pole and zero are both relatively close to the origin, a lag compensator does little to alter the root locus of the system. Consequently, the transient response of the system would be nearly unchanged.

- Ideal proportional + integral + differential (PID) compensation: improving transient response, stability, and steady state response with a cascade compensation
  - The limit as $p_1 \to 0$ and $p_2 \to -\infty$ yields the transfer function

$$
\begin{aligned}
C_{\text{PID}}(s) &= \lim_{p_1 \to 0,\ p_2 \to -\infty} C_{\text{lag-lead}}(s) && (32.432)\\
&= \lim_{p_1 \to 0,\ p_2 \to -\infty} K \left(\frac{s - z_1}{s - p_1}\right)\left(\frac{s - z_2}{s - p_2}\right) && (32.433)\\
&= \lim_{p_1 \to 0,\ p_2 \to -\infty} -K_D p_2 \left(\frac{s - z_1}{s - p_1}\right)\left(\frac{s - z_2}{s - p_2}\right) && (32.434)\\
&= K_D \left(\frac{s - z_1}{s}\right)(s - z_2) && (32.435)\\
&= K_D \frac{s^2 - (z_1 + z_2)s + z_1 z_2}{s} && (32.436)\\
&= K_P + K_I \frac{1}{s} + K_D s && (32.437)
\end{aligned}
$$

  - (Iterative) design approaches (see ME 375 slides)
    * Cascade PD and PI compensators
      · Design a PD compensator
      · Design a PI compensator to be cascaded with the PD compensator
      · Simulate to verify satisfactory response and iterate if necessary
    * Repeated zero ($z_1 = z_2$)
      · Identify the desired dominant pole location
      · Calculate the angular deficiency (properly account for the pole at the origin in the PID compensator)

        - Select repeated zero location for the PID compensator such that the angular deficiency is alleviated
        - Find the appropriate value for K_D to place the closed loop poles
        - Simulate to verify satisfactory response and iterate if necessary
    - Complex conjugate zeros ($z_1 = z_2^*$)
        - Identify the desired dominant pole location
        - Calculate the angular deficiency (properly account for the pole at the origin in the PID compensator)
        - Select complex zero locations for the PID compensator such that the angular deficiency is alleviated. (Often, an infinite number of zero locations, z_1, could be used to alleviate the angular deficiency, one equation, by using two unknowns, the real and imaginary parts of the zero.)
        - Find the appropriate value for K_D to place the closed loop poles
        - Simulate to verify satisfactory response and iterate if necessary
    - Distinct real zeros ($z_1 \neq z_2$)
- Design considerations and guidelines
    - Note, the ideal PID compensator is improper. Practically, PID may be implemented approximately with a lag-lead compensator (with potentially complex zeros).
    - K_P: Proportional control acts on the present error and serves to speed up the response
    - K_I: Integral control acts on the history of the error (i.e., the integral of the error) and serves to reduce steady state error (and eliminate error for step inputs)
    - K_D: Derivative control acts on the anticipated future error (i.e., the derivative of the error) and serves to reduce oscilations.

Chapter 33

Frequency Domain Design

- This chapter builds on the following chapters:
  - Chapter 13 Frequency Response
  - Chapter 14 Frequency Response Plots

33.1 Stability margin as a measure of robustness

- Robustness is low sensitivity to changes in: system parameters or discrepancies between model and reality
- Robustness is a common objective in the design of control systems.
- Margin of error - "... a permissible or tolerable degree of deviation from a correct or exact value or target; a small allowance made for the possibility of miscalculation, change in circumstances ..." ["margin, n." OED Online, Oxford University Press, December 2021, www.oed.com/view/Entry/114041. Accessed 22 February 2022.].
  - The stability margin of a system is a measure of how much deviation it can withstand before it becomes unstable. Therefore, stability margin is a measure of robustness.

33.2 Root locus and introduction to stability margins

- Root locus may be used to investigate control system robustness
  - With root locus we explore how closed loop pole locations (and stability) depend upon a system parameter, K.
  - A system may be considered robust, for example, if relatively large changes in the parameter, K, are required to adversely change its performance (e.g., destabilize the system).

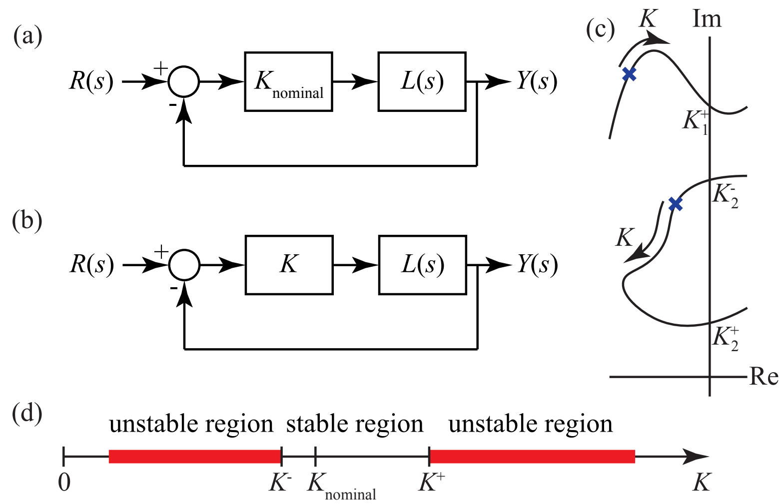


Figure 33.1: (a) Nominal closed loop system. (b) Closed loop system of (a) in which the gain, K, has been changed from it's nominal value. For the nominal value of the gain ($K_{\text{nominal}} = 1$), the system in (a) is recovered. (c) In this sketch of a segment of the root locus for the system in (b), the nominal locations for two closed loop poles and the directions of increasing open loop gain are indicated. (d) Depiction of the stable range of K about its nominal value.

- Gain margin -
  - Consider a control system that was designed using a root locus approach such that the nominal gain, K_{nominal}, yields the desired performance.
  - Gain margin for increasing K, GM^+, is obtained by increasing the gain from it's nominal value, K_{nominal}, and finding the gain at which the system first becomes unstable, $K^+ = \min\{K_1^+, K_2^+, ...\}$ (i.e., at least one closed loop pole moves into the right half plane).
$$\text{GM}^+ = \frac{K^+}{K_{\text{nominal}}} \tag{33.1}$$
    See, for example, [8] for the connection between root locus and gain margin and [37] for a discussion of gain margins for increasing versus decreasing K.
  - Gain margin for decreasing K, GM^-, is obtained by decreasing the gain from it's nominal value, K_{nominal}, and finding the gain at which the system first becomes unstable, $K^- = \max\{K_1^-, K_2^-, ...\}$ (i.e., at least one closed loop pole moves into the right half plane).
$$\text{GM}^- = \frac{K^-}{K_{\text{nominal}}} \tag{33.2}$$

- When we want to report only one value we can report gain margin as

$$\mathrm{GM} = \begin{cases} \mathrm{GM}^- & \mathrm{GM}^+ > \frac{1}{\mathrm{GM}^-} \\ \mathrm{GM}^+ & \mathrm{GM}^+ < \frac{1}{\mathrm{GM}^-} \end{cases} \tag{33.3}$$

- Phase crossover frequencies
  * To obtain the gain margins from a root locus, we determine the gains at the imaginary axis crossings (i.e., points along the imaginary axis that satisfy the angle criterion, $\angle L(j\omega) = \pm\pi\{1, 3, ...\}$).
  * The frequencies associated with the imaginary axis crossings are known as the phase crossover frequencies, ω_{pc}.

- Phase margin
  - Phase margin is directly related to our root locus design approach to placing poles.
  - To obtain the phase margin in the context of root locus we:
    * find points along the positive imaginary axis, $j\omega_{\mathrm{gc}}$, that satisfy the magnitude criterion $|L(j\omega_{\mathrm{gc}})| = 1$
    * find the angular deficit, $\phi_{\mathrm{deficit}}(j\omega_{\mathrm{gc}})$, at these gain crossover frequencies, ω_{gc} (where $-\pi < \phi_{\mathrm{deficit}}(j\omega_{\mathrm{gc}}) < \pi$)
    * Phase margin is the negative of the angular deficit with smallest magnitude

33.3 Bode plots and stability margin

- Bode plots
  - Bode plots are obtained by substituting $j\omega$ for s in a transfer function and plotting the magnitude, $|L(j\omega)|$, and angle, $\angle L(j\omega)$, of the resulting complex number.
  - The magnitude and angle criteria must be satisfied for a point, $j\omega$, to be a closed loop pole

$$|L(j\omega)| = 1 \tag{33.4}$$
$$\angle L(j\omega) = \pm\pi \tag{33.5}$$

  - Bode plots may be used to measure how close a system is to having a pole on the imaginary axis (i.e., stable/unstable border).
  - Stability margins measure how close the closed loop poles are to the border of instability (imaginary axis)
    * Gain margin is a measure of how close to instability a system is by investigating points, $j\omega$, that satisfy the angle criterion

    * Phase margin is a measure of how close to instability a system is by investigating points, $j\omega$, that satisfy the magnitude criterion
    * Vector margin (not a focus of ME375) is a measure of how close to instability a system is by considering both magnitude and angle criteria simultaneously

- Gain margin
  - Gain margin definition
    * "The amount by which the gain can be increased or decreased before the system becomes unstable if stable, or stable if unstable [37]."
    * We restrict our discussion of gain margin to closed loop systems that are nominally stable. Hence, gain margin is a measure of the robustness of the stability of the closed loop system.
  - Method to obtain gain margins from open loop bode plots
    * Find all phase crossover frequencies ω_{pc}
      · The phase crossover frequencies are the frequencies at which the phase angle plot crosses any of the angles $\pm\pi\{1, 3, ...\}$ (i.e., satisfies the angle criterion)
      · Note, these are imaginary axis crossings of the root locus for $L(s)$.
    * Determine the magnitude in dB at the phase crossover frequencies, $m(\omega_{\text{pc}})$ from the magnitude plot
    * Determine which of the phase crossover frequencies are destabilizing, $\omega_{\text{pc,destabilizing}}$
      · Given that we assume that the system is stable in its nominal configuration, this step may not really be necessary.
      · The root locus plot, the Routh-Hurwitz criterion, Nyquist plot or other tools may be used to determine which of these crossings are destabilizing.
      · Note, some crossings may simply indicate movement of poles into or out of the right half plane when there is at least one pole already in the right half plane.
    * Gain margin for increasing gain, GM^+_{dB}; see, for example, [37]:
      · Identify all destabilizing phase crossover frequencies for which $m(\omega_{\text{pc,destabilizing}}) <$ 0 dB and label them $\{m_1^+, m_2^+, ...\}$

$$\text{GM}^+_{\text{dB}} = -\max\{m_1^+, m_2^+, ...\} \tag{33.6}$$

      · If there are no destabilizing gains 0 dB exists, $\text{GM}^+ = \infty$
    * Gain margin for decreasing gain, GM^-; see, for example, [37]:
      · Identify all destabilizing phase crossover frequencies for which $m(\omega_{\text{pc,destabilizing}}) >$ 0 dB and label them $\{m_1^-, m_2^-, ...\}$

$$\text{GM}^-_{\text{dB}} = -\min\{m_1^-, m_2^-, ...\} \tag{33.7}$$

- If there are no destabilizing gains 0 dB exists, $\mathrm{GM}^+ = \infty$
- (Note, here we only consider $K > 0$)

* Gain margin (GM)
  - If we want just one number to represent the gain margin, we might state that the gain margin in dB, is the lesser in magnitude of GM^+ and GM^- both in dB

$$\mathrm{GM}_{\mathrm{dB}} = \begin{cases} \mathrm{GM}^+_{\mathrm{dB}}, & \mathrm{GM}^+_{\mathrm{dB}} < |\mathrm{GM}^-_{\mathrm{dB}}| \\ \mathrm{GM}^-_{\mathrm{dB}}, & \mathrm{GM}^+_{\mathrm{dB}} > |\mathrm{GM}^-_{\mathrm{dB}}| \end{cases}; \tag{33.8}$$

  see, for example, [37].

- Comments on the literature regarding gain margin
  * Based on the discussion in [37], there seems to be a lot of inaccuracies in the literature (and even some in MATLAB®) regarding stability margins. Identify some common inaccuracies.
  * A more robust approach to stability margin is obtained by means of the Nyquist plot. In fact, most controls textbooks present the Nyquist plot rather than exclusively using Bode plots.

- Phase margin
  - Phase margin definition
    * "The phase margin is the amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability [39]."
    * We will restrict our discussion of phase margin to closed loop systems that are nominally stable. Hence, phase margin is a measure of the robustness of the stability of the closed loop system.
  - Method to obtain phase margins from open loop bode plots
    * Find all gain crossover frequencies, ω_{gc}
      - These are the frequencies at which the magnitude plot crosses 0 dB
    * Determine which of these crossings are destabilizing, $\omega_{\mathrm{gc,destabilizing}}$
      - The root locus plot, the Routh-Hurwitz criterion, Nyquist plot, or other tools may be used to determine which of these crossings are destabilizing.
      - Because we are restricting our discussion to nominally stable systems, I don't believe we need to be overly concerned about which are destabilizing.
    * Determine the angular deficiency at these destabilizing crossover frequencies from the phase part of the bode plot

$$\begin{aligned} \phi_{\mathrm{deficiency}} &= \pm\pi\{1,3,...\} - \angle L(j\omega_{\mathrm{gc,destabilizing}}) \qquad (33.9) \\ &= \pm\pi\{1,3,...\} - \phi(\omega_{\mathrm{gc,destabilizing}}) \qquad (33.10) \end{aligned}$$

  · Use the odd multiple of π that yields the angular deficiency with smallest absolute value; see, for example, [37].

* Phase margin (PM)
  · If there is only one gain crossover frequency, the phase margin is the negative of the angular deficiency
  · If there are multiple gain crossover frequencies corresponding to what would be positive PMs, the one with smallest ratio $\frac{\mathrm{PM}_i}{\omega_{\mathrm{gc},i}}$ is the PM; see [37]. (To calculate the ratio, the phase margin should be in radians.)
  · If there are multiple gain crossover frequencies corresponding to what would be positive and negative PMs, a couple of options are presented in [37].
  · Note, $-\pi \leq \mathrm{PM} \leq \pi$; see, for example, [37]
  · Example 6.38 of [37] shows a minimum phase system in which the phase margin is negative.

Example 33.1: Gain and Phase Margins

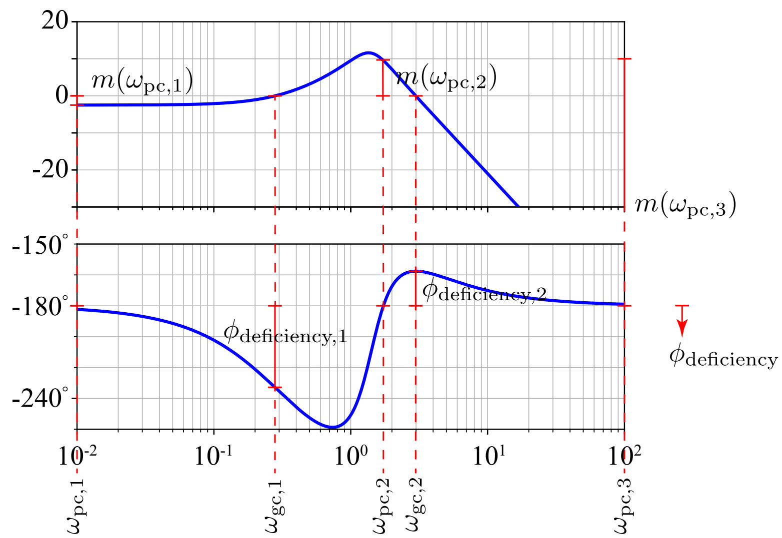


Figure 33.2:

Determine the GM and PM for the system with open loop transfer function

$$L(s) = \frac{9s - 3}{s^3 + s^2 + 4} \tag{33.11}$$

and Bode plot in Figure 33.2.

- Gain margin
  - Find all phase crossover frequencies

  * Frequencies at which the phase plot crosses $\pm\pi$

$$\omega_{\text{pc}} = \{0, 1.7321, \infty\} \tag{33.12}$$

  * (Mathematically, these may be obtained by solving for the imaginary axis crossings of the root locus.)

- Determine the magnitude at the phase crossover frequencies

$$m(\omega_{\text{pc}}) = \{-2.4988, 9.0502, -\infty\} \text{ dB} \tag{33.13}$$

- destabilizing
- Gain margin for increasing gain

$$\text{GM}^{+}_{\text{dB}} = 2.4988 \text{ dB} \tag{33.14}$$

- Gain margin for decreasing gain

$$\text{GM}^{-}_{\text{dB}} = -9.0502 \text{ dB} \tag{33.15}$$

- Gain margin

$$\text{GM}_{\text{dB}} = 2.4988 \text{ dB} \tag{33.16}$$

- Phase margin
  - Find all gain crossover frequencies
    * Frequencies at which the magnitude plot crosses 0 dB

$$\omega_{\text{gc}} = \{0.2806, 2.9842\} \tag{33.17}$$

  - destabilizing
  - Angular deficiency from the phase part of the bode plot

$$\begin{aligned}
\phi_{\text{deficiency}} &= \pm\pi - \phi(\omega_{\text{gc}}) && (33.18)\\
&= \pm\pi - \{2.4475, -2.8478\} && (33.19)\\
&= \{0.6941, -0.2938\} && (33.20)\\
&= \{39.769^\circ, -16.833^\circ\} && (33.21)
\end{aligned}$$

  - Phase margin

$$\text{PM} = 16.833^\circ \tag{33.22}$$

33.4 Nyquist Plots and Stability

This section needs to be overhauled. Double check that proper references are made.

- Better tools to investigate stability margins
  - Closed loop stability cannot be determined from the open loop bode plot
  - Proximity to the unstable boundary could be closer than gain and phase margins suggest
    * With gain and phase margins, we explore points along the imaginary axis that already satisfied either the magnitude ($|L(j\omega| = 1$) or angle condition ($\angle L(j\omega) = \pm\pi\{1, 3, ...\}$) of the characteristic equation
    * There may exist points along $s = j\omega$ which are close to satisfying both of these conditions but do not satisfy either.
  - Nyquist plots are better suited to investigate stability margins
    * Nyquist plots - we can think of Nyquist plots as combining magnitude and phase information from the Bode plot into a single polar plot.
    * Closed loop system stability can be determined easily from Nyquist plots (but not easily from Bode plots); see, for example, [37].
    * The proximity to instability can be visualized with a Nyquist plot
    * See the appendix if [8] for proof of the argument principle
  - Vector margin; see, for example, [8].
- Nyquist Plots
  - Similar to the bode plot
  - Sketching
    * Typically, and in all physical systems, the $\lim_{\omega\to\infty} M(\omega) \to 0$
    * Plot $G(j\omega)$ on the complex plane for $0 \leq \omega < \infty$
  - MATLAB® command
    * Right click in plot area>>Characteristics>>All Stability Margins or
      `h=nyquistplot()`
      `h.showCharacteristic('AllStabilityMargins')`
  - Python command - add to appendix
    * `control.nyquist_plot(sys,omega)`
    * `omega` is a vector of frequencies at which to evaluate the Nyquist plot. Example `omega=np.logspace(-3,3,10000)`
  - Stability references...

    * N: number of clockwise encirclements of -1
    * P: number of unstable open loop poles
    * $Z = N + P$: number of unstable closed loop poles

- Stability margin from Nyquist plot
  - Gain margin
  - Phase margin
  - Vector margin

33.5 Closed Loop Performance From Open Loop Frequency Response

- Closed loop response characteristics may be estimated from open loop bode plots
- Steady state error from low frequency limit
  - Recall the common factors in transfer functions
    - Gain: K
    - Integrator/differentiator: s^k
    - First order: $\tau s + 1$
    - Underdamped second order: $\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$
  - Express the open loop (unity feedback) transfer function in terms of common factors

$$L(s) = G_1(s)G_2(s)G_3(s)... \tag{33.23}$$
$$= K\frac{1}{s^k}G_3(s)... \tag{33.24}$$

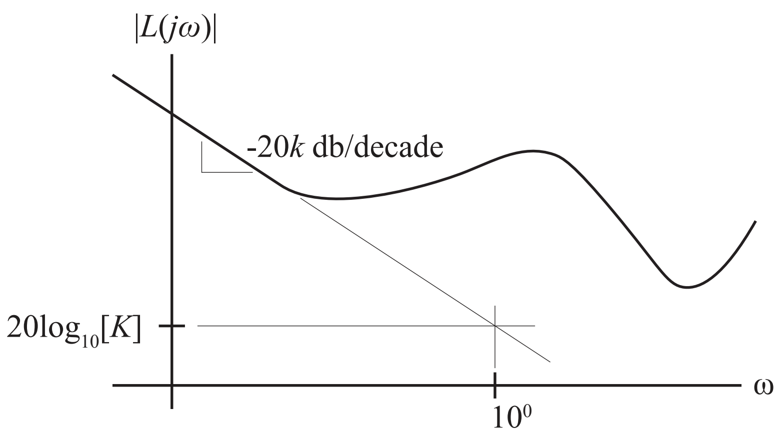


Figure 33.3:

  - System type is determined by the number of pure integrators k
    - The slope of the magnitude ratio at low frequencies for gain, first order, and second order factors is 0
    - A type k system will have a slope of $-20k$ dB/decade at low frequencies
  - Static error constant

  * Static error constant may be obtained from the $\omega = 10^0$ intersection of the low frequency limit of the magnitude ratio plot
  * Type 0: position error constant
$$K_p = K \tag{33.25}$$
  * Type 1: velocity error constant
$$K_v = K \tag{33.26}$$

- Closed loop systems dominated by first order poles
  - Consider a system with the following open loop transfer function
$$L(s) = \frac{1}{\tau s} \tag{33.27}$$
  - The resultant closed loop transfer function is
$$\frac{Y(s)}{R(s)} = \frac{L(s)}{1 + L(s)} \tag{33.28}$$
$$= \frac{1}{\tau s + 1} \tag{33.29}$$
  - Gain crossover frequency of open loop system
$$|L(j\omega_{\text{gc}})| = 1 \tag{33.30}$$
$$\left|\frac{1}{\tau j\omega_{\text{gc}}}\right| = \tag{33.31}$$
$$\omega_{\text{gc}} \approx \frac{1}{\tau} \tag{33.32}$$

- Closed loop systems dominated by underdamped second order poles
  - Consider a system with the following open loop transfer function
$$L(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}; \tag{33.33}$$
    see, for example, [8, 6].
  - The resultant closed loop transfer function is an underdamped second order system
$$\frac{Y(s)}{R(s)} = \frac{L(s)}{1 + L(s)} \tag{33.34}$$
$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{33.35}$$

- Closed loop natural frequency from open loop gain crossover frequency, ω_{gc} (i.e.,)
  * Gain crossover frequency corresponds to an open loop magnitude ratio of 1

$$
\begin{aligned}
|L(j\omega_{\text{gc}})| &= 1 && (33.36)\\
\left|\frac{\omega_n^2}{j\omega_{\text{gc}}(j\omega_{\text{gc}}+2\zeta\omega_n)}\right| &= && (33.37)\\
\omega_n^2 &= |j\omega_{\text{gc}}(j\omega_{\text{gc}}+2\zeta\omega_n)| && (33.38)\\
&= |-\omega_{\text{gc}}^2+j2\zeta\omega_n\omega_{\text{gc}}| && (33.39)\\
&= \sqrt{\omega_{\text{gc}}^4+4\zeta^2\omega_n^2\omega_{\text{gc}}^2} && (33.40)\\
\omega_n^4 &= \omega_{\text{gc}}^4+4\zeta^2\omega_n^2\omega_{\text{gc}}^2 && (33.41)\\
\omega_{\text{gc}}^4+4\zeta^2\omega_n^2\omega_{\text{gc}}^2-\omega_n^4 &= 0 && (33.42)\\
\omega_{\text{gc}}^2 &= \frac{-4\zeta^2\omega_n^2+\sqrt{(4\zeta^2\omega_n^2)^2+4\omega_n^4}}{2} && (33.43)\\
\omega_{\text{gc}}^2 &= -2\zeta^2\omega_n^2+\omega_n^2\sqrt{4\zeta^4+1} && (33.44)\\
\omega_{\text{gc}} &= \omega_n\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}} && (33.45)
\end{aligned}
$$

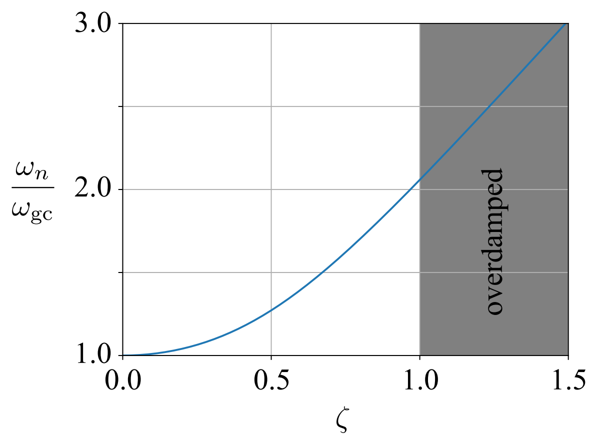


Figure 33.4: Closed loop natural frequency relative to gain crossover frequency as a function of damping ratio.

  * Note, the closed loop natural frequency may be estimated from the open loop gain crossover frequency

$$\omega_{\text{gc}} \lesssim \omega_n \lesssim 2\omega_{\text{gc}}, \text{ for } 0 \le \zeta \le 1 \tag{33.46}$$

- Estimates of damping ratio from phase margin

* Phase margin

$$
\begin{aligned}
\text{PM} &= \angle\left(L(j\omega_c)\right) - (-180^\circ) && (33.47)\\
&= 180^\circ + \angle\left(\frac{\omega_n^2}{j\omega_c(j\omega_c + 2\zeta\omega_n)}\right) && (33.48)\\
&= 180^\circ + \angle\omega_n^2 - \angle j\omega_c - \angle(j\omega_c + 2\zeta\omega_n) && (33.49)\\
&= 180^\circ + 0 - 90^\circ - \tan^{-1}\left(\frac{\omega_c}{2\zeta\omega_n}\right) && (33.50)\\
&= 90^\circ - \tan^{-1}\left(\frac{\omega_c}{2\zeta\omega_n}\right) && (33.51)\\
&= 90^\circ - \tan^{-1}\left(\frac{\omega_n\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}{2\zeta\omega_n}\right) && (33.52)\\
&= 90^\circ - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}{2\zeta}\right) && (33.53)\\
&= \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}\right) && (33.54)
\end{aligned}
$$

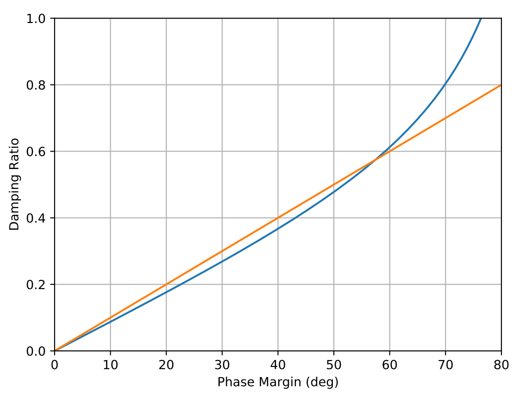


Figure 33.5: See, for example, [8].

* Phase margin approximation (for $0^\circ < \text{PM} < 70^\circ$). See Figure 33.5; see, for example, [8]

$$\zeta \approx \frac{\text{PM}}{100^\circ} \tag{33.55}$$

– Estimates of bandwidth from gain crossover frequency

* Closed loop bandwidth, ω_{BW} (i.e., closed loop magnitude of $1/\sqrt{2}$)

$$
\begin{aligned}
\left|\frac{C(j\omega_{\text{BW}})P(j\omega_{\text{BW}})}{1+C(j\omega_{\text{BW}})P(j\omega_{\text{BW}})}\right| &= \frac{1}{\sqrt{2}} && (33.56)\\
\left|\frac{\omega_n^2}{(j\omega_{\text{BW}})^2+2\zeta\omega_n j\omega_{\text{BW}}+\omega_n^2}\right| &= && (33.57)\\
\frac{\omega_n^2}{\sqrt{(\omega_n^2-\omega_{\text{BW}}^2)^2+(2\zeta\omega_n\omega_{\text{BW}})^2}} &= && (33.58)\\
\frac{1}{\sqrt{(1-\frac{\omega_{\text{BW}}^2}{\omega_n^2})^2+(2\zeta\frac{\omega_{\text{BW}}}{\omega_n})^2}} &= \frac{1}{\sqrt{2}} && (33.59)\\
\frac{1}{(1-\frac{\omega_{\text{BW}}^2}{\omega_n^2})^2+(2\zeta\frac{\omega_{\text{BW}}}{\omega_n})^2} &= \frac{1}{2} && (33.60)\\
\left(1-\frac{\omega_{\text{BW}}^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega_{\text{BW}}}{\omega_n}\right)^2 &= 2 && (33.61)\\
1-2\frac{\omega_{\text{BW}}^2}{\omega_n^2}+\frac{\omega_{\text{BW}}^4}{\omega_n^4}+4\zeta^2\frac{\omega_{\text{BW}}^2}{\omega_n^2} &= 2 && (33.62)\\
\frac{\omega_{\text{BW}}^4}{\omega_n^4}+(4\zeta^2-2)\frac{\omega_{\text{BW}}^2}{\omega_n^2}-1 &= 0 && (33.63)
\end{aligned}
$$

* applying quadratic equation

$$
\begin{aligned}
\frac{\omega_{\text{BW}}^2}{\omega_n^2} &= \frac{-(4\zeta^2-2)+\sqrt{(4\zeta^2-2)^2+4}}{2} && (33.64)\\
\frac{\omega_{\text{BW}}^2}{\omega_n^2} &= \frac{-(4\zeta^2-2)+\sqrt{16\zeta^4-16\zeta^2+8}}{2} && (33.65)\\
\frac{\omega_{\text{BW}}^2}{\omega_n^2} &= -(2\zeta^2-1)+\sqrt{4\zeta^4-4\zeta^2+2} && (33.66)\\
\omega_{\text{BW}} &= \omega_n\sqrt{-(2\zeta^2-1)+\sqrt{4\zeta^4-4\zeta^2+2}}; && (33.67)
\end{aligned}
$$

* this result also appears in [6], for example. See Figure 33.6

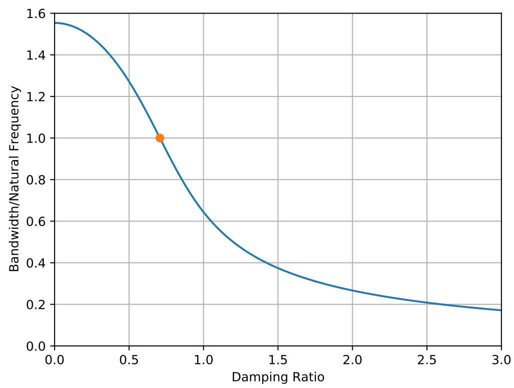


Figure 33.6: Dependence of closed loop bandwidth/natural frequency on closed loop damping ratio.

* using the relationship for the crossover frequency from Equation 33.45

$$\omega_{\text{BW}} = \frac{\omega_{\text{gc}}}{\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}}\sqrt{-(2\zeta^2-1)+\sqrt{4\zeta^4-4\zeta^2+2}} \tag{33.68}$$

$$= \omega_{\text{gc}}\frac{\sqrt{-(2\zeta^2-1)+\sqrt{4\zeta^4-4\zeta^2+2}}}{\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}} \tag{33.69}$$

see Figure 33.7

* The closed loop bandwidth is approximately equal to the open loop crossover frequency

$$\omega_{\text{gc}} \lesssim \omega_{\text{BW}} \lesssim 2\omega_{\text{gc}}, \text{ for } 0 \le \zeta \le 1 \tag{33.70}$$

* Must check the bandwidth for your particular problem

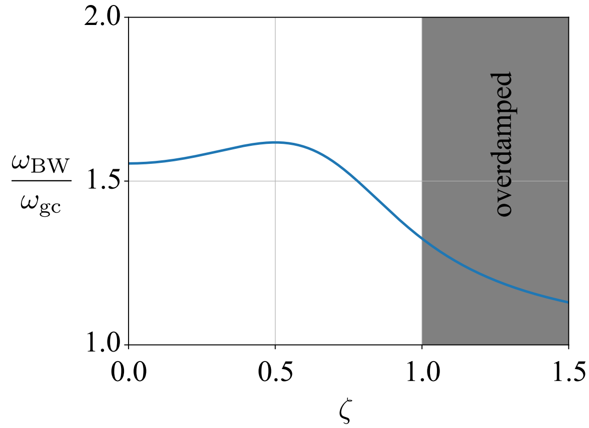


Figure 33.7: Closed loop bandwidth relative to gain crossover frequency as a function of damping ratio.

- General systems
  - See [8, 6] and http://control.asu.edu/Classes/MAE318/318Lecture21.pdf
  - Franklin [8] suggests that near the gain crossover frequency, the closed loop magnitude is closely related to phase margin.

$$\begin{aligned} L(j(\omega_{\text{gc}}+\Delta\omega)) &= |L(j(\omega_{\text{gc}}+\Delta\omega))|e^{j\angle L(j(\omega_{\text{gc}}+\Delta\omega))} && (33.71)\\ &\approx Ke^{j\angle L(j(\omega_{\text{gc}}+\Delta\omega))} && (33.72)\end{aligned}$$

  - Closed loop transfer function and frequency response function near ω_{gc}

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{L(s)}{1+L(s)} && (33.73)\\ \frac{Y(j(\omega_{\text{gc}}+\Delta\omega))}{R(j(\omega_{\text{gc}}+\Delta\omega))} &= \frac{L((\omega_{\text{gc}}+\Delta\omega))}{1+L((\omega_{\text{gc}}+\Delta\omega))} && (33.74)\\ &\approx \frac{e^{j\angle L(j(\omega_{\text{gc}}+\Delta\omega))}}{1+e^{j\angle L(j(\omega_{\text{gc}}+\Delta\omega))}} && (33.75)\end{aligned}$$

  - Bandwidth

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{L(s)}{1+L(s)} && (33.76)\\ \left|\frac{Y(j(\omega_{\text{gc}}+\Delta\omega))}{R(j(\omega_{\text{gc}}+\Delta\omega))}\right| &\approx \frac{Ke^{j\angle L(j(\omega_{\text{gc}}+\Delta\omega))}}{1+Ke^{j\angle L(j(\omega_{\text{gc}}+\Delta\omega))}} && (33.77)\\ &\approx \frac{Ke^{j\angle L(j(\phi_{\text{gc}}+\Delta\phi))}}{1+Ke^{j\angle L(j(\phi_{\text{gc}}+\Delta\phi))}} && (33.78)\\ \frac{1}{\sqrt{2}} &\approx \left|\frac{K\cos(\phi_{\text{gc}}+\Delta\phi)+jK\sin(\phi_{\text{gc}}+\Delta\phi)}{1+K\cos(\phi_{\text{gc}}+\Delta\phi)+jK\sin(\phi_{\text{gc}}+\Delta\phi)}\right| && (33.79)\\ \frac{1}{\sqrt{2}} &\approx \frac{K}{|1+K\cos(\phi_{\text{gc}}+\Delta\phi)+jK\sin(\phi_{\text{gc}}+\Delta\phi)|} && (33.80)\\ \frac{1}{2} &\approx \frac{K^2}{(1+K\cos(\phi_{\text{gc}}+\Delta\omega))^2+K^2\sin^2(\phi_{\text{gc}}+\Delta\phi)} && (33.81)\\ &\approx \frac{K^2}{1+2K\cos(\phi_{\text{gc}}+\Delta\phi)+K^2\cos^2(\phi_{\text{gc}}+\Delta\phi)+K^2\sin^2(\omega_{\text{gc}}+\Delta\omega)} && (33.82)\\ \frac{1}{2} &\approx \frac{K^2}{1+2K\cos(\phi_{\text{gc}}+\Delta\phi)+K^2} && (33.83)\\ 0 &\approx K^2-2K\cos(\phi_{\text{gc}}+\Delta\phi)-1 && (33.84)\\ K &\approx \frac{2\cos(\phi_{\text{gc}}+\Delta\phi)\pm\sqrt{2^2\cos^2(\phi_{\text{gc}}+\Delta\phi)+4}}{2} && (33.85)\\ &\approx \cos(\phi_{\text{gc}}+\Delta\phi)\pm\sqrt{\cos^2(\phi_{\text{gc}}+\Delta\phi)+1} && (33.86)\end{aligned}$$

– See Figure33.8. To use, find a point along the open loop bode plot that matches a phase magnitude relationship on this curve. The frequency at which this occurs is an estimate of the closed loop bandwidth.

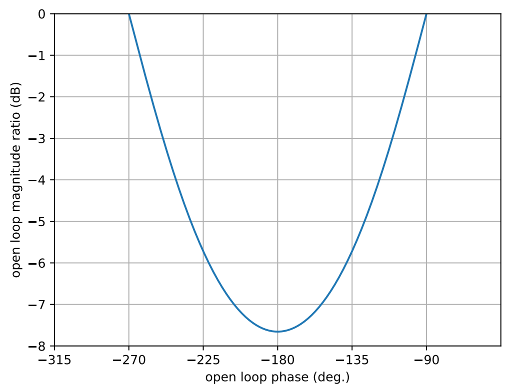


Figure 33.8: See for example [6].

33.6 Increasing Phase Margin and Bandwidth

- Justification
  - Generally, increasing phase margin is desirable because it improves robustness citation?.
    - Also, increasing phase margin tends to increase damping ratio and reduce overshoot
  - Increasing bandwidth tends to increase the speed of the response
    - Increasing bandwidth tends to reduce rise time
    - Gain crossover frequency is a measure of bandwidth
    - Comment on balance of high bandwidth and poor performance in presence of noise.
- Problem statement
  - Given an existing feedback system, find a compensator $C(s)$ that increases phase margin and gain crossover frequency
- Proportional compensation:
- Lead compensator
  - Form

$$C_{\text{lead}}(s) = K\frac{T_{\text{lead}}s+1}{\alpha_{\text{lead}}T_{\text{lead}}s+1} \tag{33.87}$$

    This notation is an adaptation of that in [8].

  - Theory
    - Bode plot
      - low frequency ($\omega \to 0$):

$$m(0) = 20\log[K] \tag{33.88}$$

$$\phi(0) = 0 \tag{33.89}$$

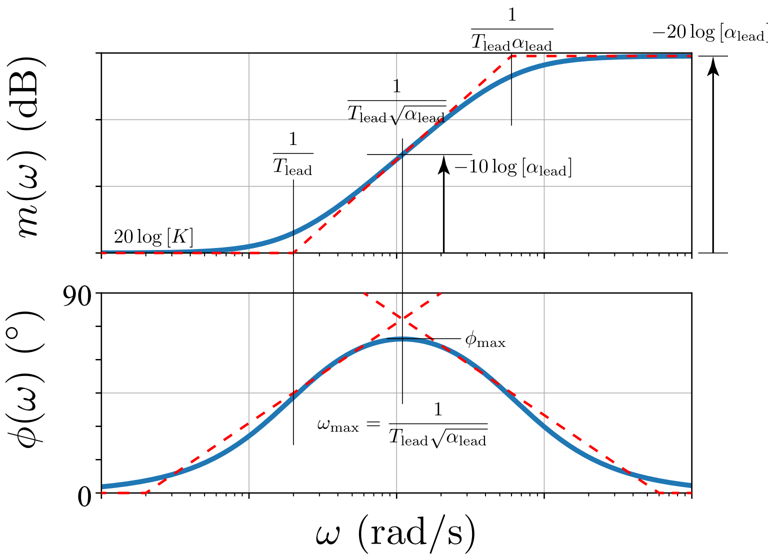


Figure 33.9: (solid blue) Bode plot for a lead compensator. (dashed red) Straight line approximation.

· mid frequency ($\omega = \omega_{\max} = \frac{1}{T_{\text{lead}}\sqrt{\alpha_{\text{lead}}}}$):

$$m(\omega_{\max}) = 20\log\left[K\frac{1}{\sqrt{\alpha_{\text{lead}}}}\right] \tag{33.90}$$

$$\begin{aligned}
\phi(\omega_{\max}) &= \angle K\frac{T_{\text{lead}}s+1}{\alpha_{\text{lead}}T_{\text{lead}}s+1} &(33.91)\\
&= \angle K\frac{\frac{1}{\omega_{\max}\sqrt{\alpha_{\text{lead}}}}s+1}{\alpha_{\text{lead}}\frac{1}{\omega_{\max}\sqrt{\alpha_{\text{lead}}}}s+1} &(33.92)\\
&= \angle K\frac{1}{\alpha_{\text{lead}}}\frac{j\omega_{\max}+\omega_{\max}\sqrt{\alpha_{\text{lead}}}}{j\omega_{\max}+\frac{\omega_{\max}}{\sqrt{\alpha_{\text{lead}}}}} &(33.93)\\
&= \angle K\frac{1}{\alpha_{\text{lead}}}+\angle\frac{j+\sqrt{\alpha_{\text{lead}}}}{j+\frac{1}{\sqrt{\alpha_{\text{lead}}}}} &(33.94)\\
&= 0+\angle\frac{\left(j+\sqrt{\alpha_{\text{lead}}}\right)\left(-j+\frac{1}{\sqrt{\alpha_{\text{lead}}}}\right)}{\left(j+\frac{1}{\sqrt{\alpha_{\text{lead}}}}\right)\left(-j+\frac{1}{\sqrt{\alpha_{\text{lead}}}}\right)} &(33.95)\\
&= \angle\frac{1+\left(\frac{1}{\sqrt{\alpha_{\text{lead}}}}-\sqrt{\alpha_{\text{lead}}}\right)j+1}{1+\frac{1}{\alpha_{\text{lead}}}} &(33.96)\\
&= \angle\left(2+\left(\frac{1}{\sqrt{\alpha_{\text{lead}}}}-\sqrt{\alpha_{\text{lead}}}\right)j\right)-\angle\left(1+\frac{1}{\alpha_{\text{lead}}}\right) &(33.97)
\end{aligned}$$

writing in terms of arcsine rather than arctangent

$$
\begin{aligned}
\phi(\omega_{\max}) &= \sin^{-1}\left(\frac{\frac{1}{\sqrt{\alpha_{\text{lead}}}} - \sqrt{\alpha_{\text{lead}}}}{\sqrt{2^2 + \left(\frac{1}{\sqrt{\alpha_{\text{lead}}}} - \sqrt{\alpha_{\text{lead}}}\right)^2}}\right) - 0 &(33.98)\\
&= \sin^{-1}\left(\frac{\frac{1}{\sqrt{\alpha_{\text{lead}}}} - \sqrt{\alpha_{\text{lead}}}}{\sqrt{4 + \frac{1}{\alpha_{\text{lead}}} - 2 + \alpha_{\text{lead}}}}\right) &(33.99)\\
&= \sin^{-1}\left(\frac{\frac{1}{\sqrt{\alpha_{\text{lead}}}} - \sqrt{\alpha_{\text{lead}}}}{\sqrt{\frac{1}{\alpha_{\text{lead}}} + 2 + \alpha_{\text{lead}}}}\right) &(33.100)\\
&= \sin^{-1}\left(\frac{\frac{1}{\sqrt{\alpha_{\text{lead}}}} - \sqrt{\alpha_{\text{lead}}}}{\sqrt{\left(\frac{1}{\sqrt{\alpha_{\text{lead}}}} + \sqrt{\alpha_{\text{lead}}}\right)^2}}\right) &(33.101)\\
&= \sin^{-1}\left(\frac{\frac{1}{\sqrt{\alpha_{\text{lead}}}} - \sqrt{\alpha_{\text{lead}}}}{\frac{1}{\sqrt{\alpha_{\text{lead}}}} + \sqrt{\alpha_{\text{lead}}}}\right) &(33.102)\\
&= \sin^{-1}\left(\frac{1 - \alpha_{\text{lead}}}{1 + \alpha_{\text{lead}}}\right) &(33.103)\\
&= \phi_{\max} &(33.104)
\end{aligned}
$$

· high frequency ($\omega \to \infty$):

$$
\begin{aligned}
m(\infty) &= 20\log\left[K\frac{1}{\alpha_{\text{lead}}}\right] &(33.105)\\
&= 20\log[K] - 20\log[\alpha_{\text{lead}}] &(33.106)\\
\phi(\infty) &= 0 &(33.107)
\end{aligned}
$$

∗ Phase can be added to increase phase margin near $\omega_{\max}$ without altering the phase at low and high frequencies

∗ Generally, increasing the gain K tends to increase *both* the gain crossover frequency (i.e., bandwidth) and the static error constant.

  · If the plant is strictly proper, the gain is expected to decrease as $\omega \to \infty$ (i.e., a downward slope at high frequencies). Therefore increasing K tends to increase the point at which the gain passes through 0 dB (i.e., gain crossover frequency).

  · Therefore, with a lead compensator we cannot generally use K to increase the bandwidth without increasing the static error constant (or vice versa).

– Design approach to increase phase margin and bandwidth

  * Determine how much phase is needed, $\phi_{\max}$, at the desired gain crossover frequency, $\omega_{\max} = \omega_{gc}$ to meet PM specifications
    · Graphical approach

$$\begin{aligned} \text{PM} &= (\phi_{\text{plant}}(\omega_{\max}) - (-180°)) + \phi_{\max} && (33.108)\\ \phi_{\max} &= \text{PM} - (\phi_{\text{plant}}(\omega_{\max}) - (-180°)) && (33.109) \end{aligned}$$

    · Algebraic approach

$$\begin{aligned} \text{PM} &= \angle P(j\omega_{\max}) - (-180°) + \phi_{\max} && (33.110)\\ \phi_{\max} &= \text{PM} - \angle P(j\omega_{\max}) - 180° && (33.111) \end{aligned}$$

  * Solve for α_{lead} and T_{lead}

$$\alpha_{\text{lead}} = \frac{1 - \sin(\phi_{\max})}{1 + \sin(\phi_{\max})} \qquad (33.112)$$

$$T_{\text{lead}} = \frac{1}{\omega_{\max}\sqrt{\alpha_{\text{lead}}}} \qquad (33.113)$$

  * Set K to yield the desired gain crossover frequency ω_{gc} of $C_{\text{lead}}(s)P(s)$
    · Graphical approach

$$\begin{aligned} 0 &= m_{\text{plant}}(\omega_{\max}) + 20\log[K] - 10\log[\alpha_{\text{lead}}] && (33.114)\\ 20\log[K] &= -m_{\text{plant}}(\omega_{\max}) + 10\log[\alpha_{\text{lead}}] && (33.115)\\ K &= 10^{\frac{-m_{\text{plant}}(\omega_{\max}) + 10\log[\alpha_{\text{lead}}]}{20}} && (33.116) \end{aligned}$$

    · Algebraic approach

$$\begin{aligned} 1 &= |P(j\omega_{\max})C(j\omega_{\max})| && (33.117)\\ &= \left|P(j\omega_{\max})K\frac{1}{\sqrt{\alpha_{\text{lead}}}}\right| && (33.118)\\ K &= \frac{\sqrt{\alpha_{\text{lead}}}}{|P(j\omega_{\max})|} && (33.119) \end{aligned}$$

  * Assemble the lead compensator

$$C_{\text{lead}}(s) = K\frac{T_{\text{lead}}s + 1}{\alpha_{\text{lead}}T_{\text{lead}}s + 1} \qquad (33.120)$$

  * Evaluate performance and iterate if necessary

– Design approach to prescribe PM and steady state error; (Note, variations of this seem to be a common approach described in the literature [8, 6, 39, 70, 38].)

* Set K using steady state error specifications
* Find the gain crossover frequency for $KP(s)$ (we will represent it with $\omega_{\max}$)
  · Graphical approach

$$\begin{aligned} 0 \text{ dB} &= m_{\text{plant}}(\omega_{\max}) + 20\log[K] && (33.121)\\ m_{\text{plant}}(\omega_{\max}) &= -20\log[K] && (33.122)\\ K &= 10^{\frac{-m_{\text{plant}}(\omega_{\max})}{20}} && (33.123)\end{aligned}$$

  · Algebraic approach

$$\begin{aligned} 1 &= |KP(j\omega_{\max})| && (33.124)\\ K &= \frac{1}{|P(j\omega_{\max})|} && (33.125)\end{aligned}$$

* Determine how much phase is needed from the lead compensator, $\phi_{\max}$, at $\omega_{\max}$ to meet PM specifications. (Add a 5-12° extra [39] to account for a shift in gain crossover frequency due to using the the lead compensator form that maintains a gain of K at low frequencies; see, for example, [39, 8, 6].)
  · Graphical approach

$$\begin{aligned} \text{PM} &= (\phi_{\text{plant}}(\omega_{\max}) - (-180^\circ)) + \phi_{\max} && (33.126)\\ \phi_{\max} &= \text{PM} - (\phi_{\text{plant}}(\omega_{\max}) - (-180^\circ)) && (33.127)\end{aligned}$$

  · Algebraic approach

$$\begin{aligned} \text{PM} &= \angle P(j\omega_{\max}) - (-180^\circ) + \phi_{\max} && (33.128)\\ \phi_{\max} &= \text{PM} - \angle P(j\omega_{\max}) - 180^\circ && (33.129)\end{aligned}$$

* Solve for α_{lead} and T_{lead}

$$\alpha_{\text{lead}} = \frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})} \qquad (33.130)$$

$$T_{\text{lead}} = \frac{1}{\omega_{\max}\sqrt{\alpha_{\text{lead}}}} \qquad (33.131)$$

* Assemble the lead compensator

$$C_{\text{lead}}(s) = K\frac{T_{\text{lead}}s+1}{\alpha_{\text{lead}}T_{\text{lead}}s+1} \qquad (33.132)$$

* Evaluate performance and iterate if necessary
  · Often this approach is modified such that $\omega_{\max}$ is located at the resultant gain crossover frequency such that the maximum phase lead is added at crossover.

- Design considerations and guidelines
  - add notes
  - Some textbooks find the frequency at which to place $\omega_{\max}$ so that all the added phase is at the gain crossover frequency; see, for example, [38, 70, 6, 39].
  - What is the advantage?
  - Note, the conventional ME375 approach increases the gain at $\omega_{\max}$ which tends to increase the gain crossover frequency of the final design.
  - Lead compensation is like a high pass filter.
    - "... lead compensation may generate large signals ... [39]"
    - lead compensation tends to amplify high frequencies more than low frequencies. Hence, the resultant system may be more suceptible to noise.
  - Number of design parameters
    - Lead compensators have three design parameters (K, α_{lead}, and $\omega_{\max}$)
    - There are three common performance metrics: (i) damping (ζ, phase margin, % overshoot), (ii) steady state error, and (iii) response speed (bandwidth, natural frequency, gain crossover frequency). See, for example, [8].
    - Two design parameters are required to meet damping specifications and only one design parameter is required to meet steady state error or response speed specifications. (Note, the wording here isn't intended to imply that all performance specifications can be met simply because there are enough design parameters to meet the specifications. Lead compensators do have fundamental limitations.)
    - With three design parameters in a lead compensator, we can meet two of the three performance characteristics. This is why there are two design approaches outlined above.
  - See also discussion under root locus design
- Multiple lead compensators may be cascaded
  - add notes and see discussion under root locus design
  - See figure 6.54 of [8] and the associated discussion suggesting that a single lead compensator should contribute at most 60° due to the associated significant high frequency gain.

- Ideal proportional + derivative (PD) compensation:
  - Ideal PD compensation is not realizable.
    - The transfer function is improper
    - It has infinite high frequency gain.
    - A step change in reference will yield an infinite control signal which is impossible for physical systems.
  - Practical PD compensation is achieved with a lead compensator.

Example 33.2: Prescribed phase margin and gain crossover frequency

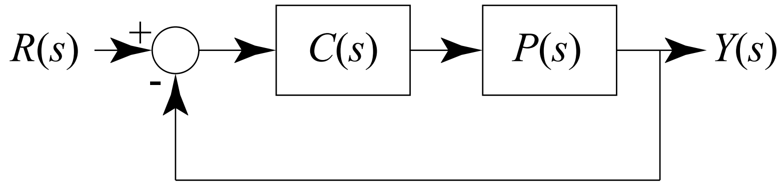


Figure 33.10:

Given the transfer function for the plant

$$P(s) = \frac{1}{s(s+1)}$$

in the closed loop configuration of Figure 33.10, use frequency domain design techniques to design a compensator, $C(s)$ to yield a unit step response with 21% maximum overshoot and 2 s 2% settling time.

Historically, this doesn't seem to be a common question type for ME 375. See Example 33.3 for a more conventional ME 375 question.

See Example 31.5 for a direct pole placement approach to this problem and Examples 32.7 and 32.9 for a root locus approach to this problem.

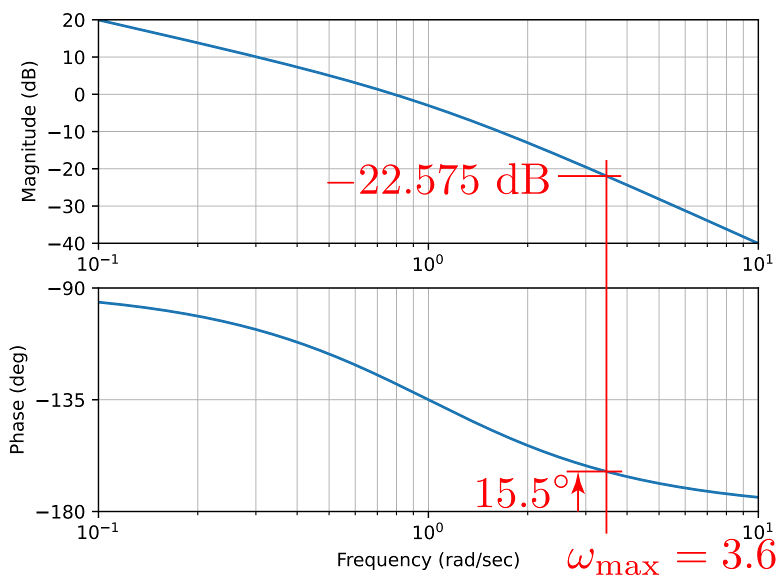


Figure 33.11:

- Determine desired phase margin and gain crossover frequency

- 21% overshoot maps to damping ratio

$$\zeta = -\frac{\ln\left[\frac{M_{\%}}{100\%}\right]}{\sqrt{\pi^2 + \ln\left[\frac{M_{\%}}{100\%}\right]^2}} \tag{33.133}$$

$$= -\frac{\ln[0.21]}{\sqrt{\pi^2 + \ln[0.21]^2}} \tag{33.134}$$

$$= 0.445 \tag{33.135}$$

- Map damping ratio to PM

$$\zeta \approx \frac{\text{PM}}{100^\circ} \tag{33.136}$$

$$\text{PM} \approx 100^\circ\zeta \tag{33.137}$$

$$\approx 100^\circ \cdot 0.445 \tag{33.138}$$

$$\approx 45^\circ \tag{33.139}$$

- Map maximum overshoot and settling time into natural frequency

$$t_{s,2\%} = \frac{4}{\zeta\omega_n} \tag{33.140}$$

$$\omega_n = \frac{4}{\zeta t_{s,2\%}} \tag{33.141}$$

$$= \frac{4}{0.445 \cdot 2} \tag{33.142}$$

$$= 4.5 \tag{33.143}$$

- Map natural frequency to gain crossover frequency
  * $\omega_{\text{gc}} \lesssim \omega_n \lesssim 2\omega_{\text{gc}}$
  * A crude approximation: $\omega_{\text{gc}} = \omega_n$
  * From Figure 33.4, for $\zeta \approx 0.5$, $\frac{\omega_n}{\omega_{\text{gc}}} \approx 1.25$

$$\frac{\omega_n}{\omega_{\text{gc}}} \approx 1.25 \tag{33.144}$$

$$\omega_{\text{gc}} \approx \frac{\omega_n}{1.25} \tag{33.145}$$

$$\approx 3.6 \tag{33.146}$$

- Find $\phi_{\max}$

- Graphical approach
  * Contribution to PM from plant at $\omega_{\max}$: 15.5°
  * Necessary contribution from lead compensator

$$\begin{aligned} \phi_{\max} &= 45^\circ - 15.5^\circ && (33.147)\\ &= 29.5^\circ && (33.148) \end{aligned}$$

- Algebraic approach
  * Desired gain crossover frequency is $\omega_{\max} = 3.6$
  * Current phase at $\omega_{\max}$

$$\begin{aligned} \phi(\omega_{\max}) &= \angle P(j\omega_{\max}) && (33.149)\\ &= \angle \frac{1}{j\omega_{\max}(j\omega_{\max}+1)} && (33.150)\\ &= \angle 1 - \angle j\omega_{\max} - \angle(j\omega_{\max}+1) && (33.151)\\ &= 0^\circ - 90^\circ - \tan^{-1}\left(\frac{\omega_{\max}}{1}\right) && (33.152)\\ &= -90^\circ - \tan^{-1}\left(\frac{3.6}{1}\right) && (33.153)\\ &= -164.5^\circ && (33.154) \end{aligned}$$

  * Phase required from compensator

$$\begin{aligned} \text{PM} &= \phi_{\max} + (\phi_{\text{plant}}(\omega_{\max}) - (-180^\circ)) && (33.155)\\ &= \phi_{\max} + (\phi_{\text{plant}}(\omega_{\max}) + 180^\circ) && (33.156)\\ \phi_{\max} &= \text{PM} - (\phi_{\text{plant}}(\omega_{\max}) + 180^\circ) && (33.157)\\ &= \text{PM} - (-164.5^\circ + 180^\circ) && (33.158)\\ &= 45^\circ - 15.5^\circ && (33.159)\\ &= 29.5^\circ && (33.160) \end{aligned}$$

- Solve for α_{lead} and T_{lead}

$$\begin{aligned}
\alpha_{\text{lead}} &= \frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})} && (33.161)\\
&= \frac{1-\sin(29.5°)}{1+\sin(29.5°)} && (33.162)\\
&= 0.3401 && (33.163)\\
T_{\text{lead}} &= \frac{1}{\omega_{\max}\sqrt{\alpha_{\text{lead}}}} && (33.164)\\
&= \frac{1}{3.6\sqrt{0.3401}} && (33.165)\\
&= 0.4763 \text{ s} && (33.166)
\end{aligned}$$

- Solve for K to yield desired gain crossover frequency
  - Graphical approach

$$\begin{aligned}
0 &= m_{\text{plant}}(\omega_{\max}) + m_{\text{lead}}(\omega_{\max}) && (33.167)\\
&= m_{\text{plant}}(\omega_{\max}) + 20\log[K] - 10\log[\alpha_{\text{lead}}] && (33.168)\\
K &= 10^{\frac{10\log[\alpha_{\text{lead}}]-m_{\text{plant}}(\omega_{\max})}{20}} && (33.169)\\
&= 10^{\frac{-4.684+22.575}{20}} && (33.170)\\
&= 7.844 && (33.171)
\end{aligned}$$

  - Algebraic approach

$$\begin{aligned}
1 &= |C_{\text{lead}}(j\omega_{\max})P(j\omega_{\max})| && (33.172)\\
&= \left|K\frac{1}{\sqrt{\alpha_{\text{lead}}}}j\omega_{\max} + \frac{\omega_{\max}}{\sqrt{\alpha_{\text{lead}}}}\frac{1}{j\omega_{\max}(j\omega_{\max}+1)}\right| && (33.173)\\
&= \frac{K}{\sqrt{\alpha_{\text{lead}}}}\frac{1}{\omega_{\max}\sqrt{\omega_{\max}^2+1}} && (33.174)\\
K &= \sqrt{\alpha_{\text{lead}}}\omega_{\max}\sqrt{\omega_{\max}^2+1} && (33.175)\\
&= \sqrt{0.3401}\cdot 3.6\sqrt{3.6^2+1} && (33.176)\\
&= 7.844 && (33.177)
\end{aligned}$$

- Construct compensator

$$\begin{aligned}
C_{\text{lead}}(s) &= 7.844\frac{0.4763s+1}{0.3401\cdot 0.4763s+1} && (33.178)\\
&= 7.844\frac{0.4763s+1}{0.1620s+1} && (33.179)
\end{aligned}$$

- Evaluate performance
  - 2% settling time 1.97 s
  - Maximum percent overshoot: 26.7%
  - Steady state error

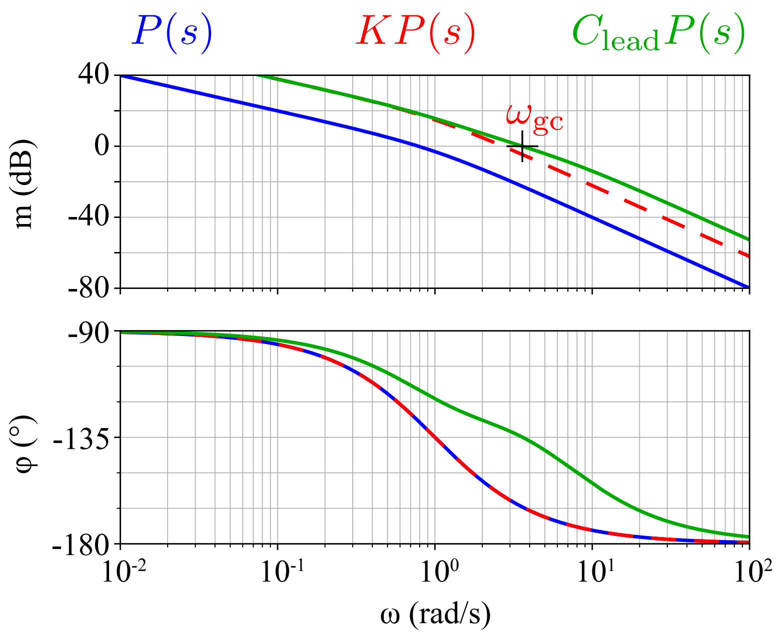


Figure 33.12:

Example 33.3: Prescribed phase margin and steady state error

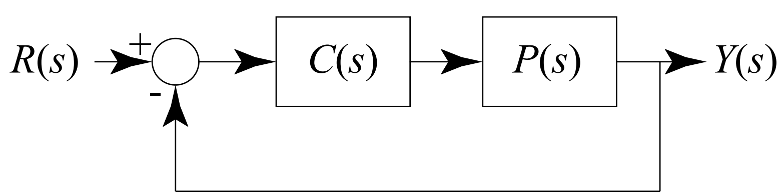


Figure 33.13:

Given the transfer function for the plant

$$P(s) = \frac{1}{s(s+1)}$$

in the closed loop configuration of Figure 33.13, use frequency domain design techniques to design a compensator, $C(s)$ to yield a phase margin of 45° and a steady state error of 0.05 given a unit ramp input.

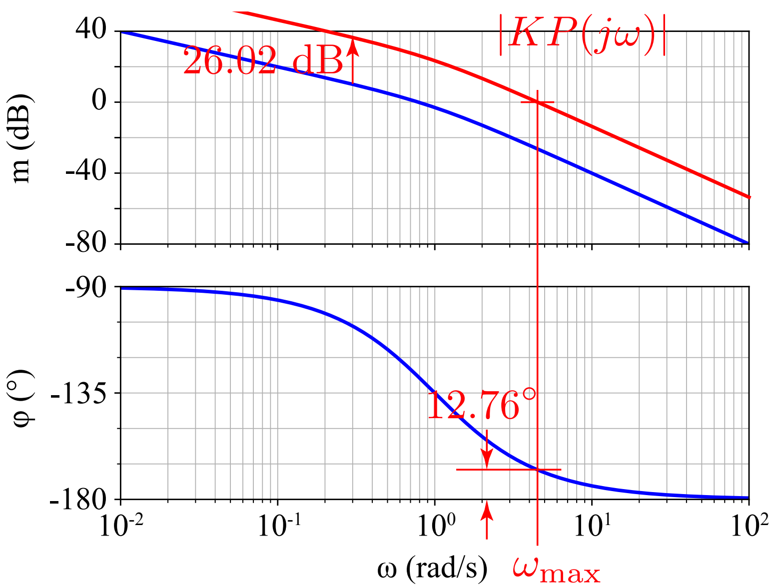


Figure 33.14:

- Find K
  - velocity error constant

$$K_v = \lim_{s\to 0} sP(s)K \tag{33.180}$$

$$= \lim_{s\to 0} s\frac{1}{s(s+1)}K \tag{33.181}$$

$$= K \tag{33.182}$$

  - steady state error for ramp

$$e_{\text{ss}} = \frac{1}{K_v} \tag{33.183}$$

$$= \frac{1}{K} \tag{33.184}$$

$$0.05 = \frac{1}{K} \tag{33.185}$$

$$K = 20 \tag{33.186}$$

$$= 26.02 \text{ dB} \tag{33.187}$$

- Find $\phi_{\max}$
  - Graphical method:
    * Gain crossover frequency for $KP(s)$ is $\omega_{\max} = 4.4166$
    * Contribution to PM from the plant at $\omega_{\max}$ is $12.76°$
    * $\phi_{\max}$

$$\begin{aligned} \text{PM} &= \angle KP(j\omega_{\max}) - (-180°) + \phi_{\max} && (33.188)\\ \phi_{\max} &= \text{PM} - \angle KP(j\omega_{\max}) + 180° && (33.189)\\ &= 45° - 12.76° && (33.190)\\ &= 32.24° && (33.191) \end{aligned}$$

Add $\sim 7°$ of extra

$$\begin{aligned} \phi_{\max} &= 32.24° + 7° && (33.192)\\ &\approx 39° && (33.193) \end{aligned}$$

Add $7°$ of extra
  - Algebraic method
    * Gain crossover frequency for $KP(s)$

$$\begin{aligned} 1 &= |KP(j\omega_{\max})| && (33.194)\\ &= \left|20\frac{1}{j\omega_{\max}(j\omega_{\max}+1)}\right| && (33.195)\\ &= \frac{20}{\omega_{\max}\sqrt{\omega_{\max}^2+1}} && (33.196)\\ \omega_{\max}^4 + \omega_{\max}^2 - 20^2 &= 0 && (33.197)\\ \omega_{\max} &= 4.4166 && (33.198) \end{aligned}$$

* Phase angle

$$
\begin{aligned}
\text{PM} &= \angle KP(j\omega_{\text{max}}) - (-180^\circ) + \phi_{\text{max}} && (33.199)\\
\phi_{\text{max}} &= \text{PM} - \angle KP(j\omega_{\text{max}}) - 180^\circ && (33.200)\\
&= 45^\circ - \angle 100\frac{1}{j\omega_{\text{max}}(j\omega_{\text{max}}+1)} - 180^\circ && (33.201)\\
&= 45^\circ - \angle 100 + \angle j\omega_{\text{max}} + \angle(j\omega_{\text{max}}+1) - 180^\circ && (33.202)\\
&= 45^\circ - 0^\circ + 90^\circ + \tan^{-1}\left(\frac{\omega_{\text{max}}}{1}\right) - 180^\circ && (33.203)\\
&= -45^\circ + \tan^{-1}\left(\frac{\omega_{\text{max}}}{1}\right) && (33.204)\\
&= -45^\circ + \tan^{-1}\left(\frac{4.4166}{1}\right) && (33.205)\\
&= 32.24^\circ && (33.206)
\end{aligned}
$$

Add $\sim 7^\circ$ of extra

$$
\begin{aligned}
\phi_{\text{max}} &= 32.24^\circ + 7^\circ && (33.207)\\
&\approx 39^\circ && (33.208)
\end{aligned}
$$

- Solve for α_{lead} and T_{lead}

$$
\begin{aligned}
\alpha_{\text{lead}} &= \frac{1-\sin(\phi_{\text{max}})}{1+\sin(\phi_{\text{max}})} && (33.209)\\
&= \frac{1-\sin(39^\circ)}{1+\sin(39^\circ)} && (33.210)\\
&= 0.2275 && (33.211)\\
T_{\text{lead}} &= \frac{1}{\omega_{\text{max}}\sqrt{\alpha_{\text{lead}}}} && (33.212)\\
&= \frac{1}{4.4166\sqrt{0.2275}} && (33.213)\\
&= 0.4747\text{ s} && (33.214)
\end{aligned}
$$

- Construct compensator

$$
\begin{aligned}
C_{\text{lead}}(s) &= 20\frac{0.4747s+1}{0.2275\cdot 0.4747s+1} && (33.215)\\
&= 20\frac{0.4747s+1}{0.1080s+1} && (33.216)
\end{aligned}
$$

- Evaluate performance

- 2% settling time 0.88 s
- Maximum percent overshoot: 29.6%
- $\omega_{\text{gc}} = 7.57$ rad/s
- PM= 42.7°:
- Note, the resultant gain crossover frequency is considerably larger than $\omega_{\max}$ as a result of the lead compensator increasing the gain beyond K at high frequencies. The phase margin is smaller than our target of 45°. We could iterate so that $\phi_{\max}$ is added at the resultant gain crossover frequency.

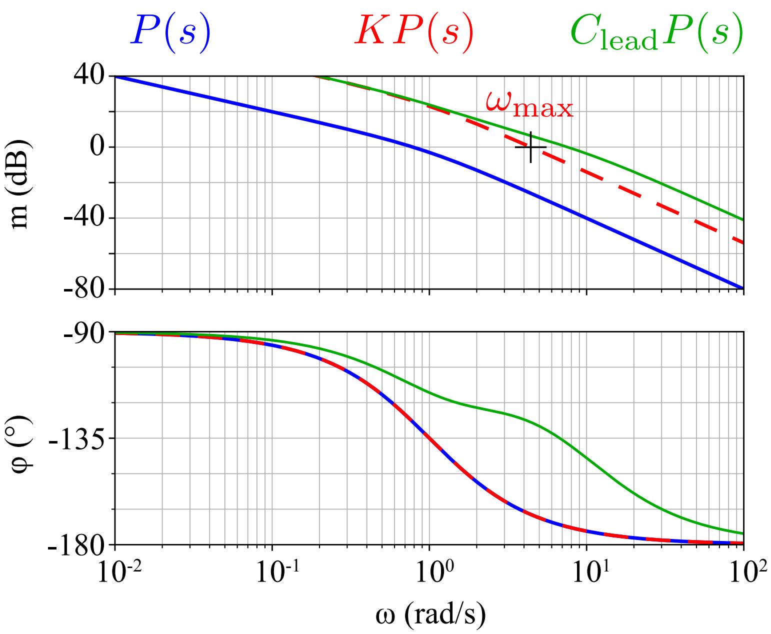


Figure 33.15:

33.7 Improving Steady State Error

- Lag compensation: improving steady state response with a cascade compensator
  - Notation; see, for example, [8]
    - Ratio of zero to pole: $\alpha_{\text{lag}} = \frac{z}{p}$ (for lag compensators $\alpha_{\text{lag}} > 1$)
  - Lag compensator
    - Magnitude ratio of K at low frequencies

$$C_{\text{lag}}(s) = K\alpha_{\text{lag}} \frac{T_{\text{lag}}s + 1}{\alpha_{\text{lag}} T_{\text{lag}} s + 1} \tag{33.217}$$

    - Magnitude ratio
      - Low frequency: $K\alpha_{\text{lag}}$
      - High frequency: K
    - Phase angle
      - Low frequency: $0°$
      - High frequency: $0°$
  - (Iterative) design approach; see, for example, [39, 6, 38, 70]
    - Find the frequency that would have the appropriate PM and assign it to be the *new* gain crossover frequency, ω_{gc}. (Include a cushion of about 5-12° extra [6, 39] to account for a small phase lag from the lag compensator.)
    - Solve for K to satisfy gain crossover frequency

$$1 = |KP(j\omega_{\text{gc}})| \tag{33.218}$$

    - Solve for α_{lag} to satisfy steady state error
    - Set the time constant associated with the lag zero to about

$$T_{\text{lag}} = \frac{10}{\omega_{\text{gc}}}; \tag{33.219}$$

      see, for example, [38, 6, 70]. Note, [8, 39] suggest $\frac{\omega_{\text{gc}}}{10} \leq \omega_z \leq \frac{\omega_{\text{gc}}}{2}$.
    - Construct the lag compensator of the form

$$C_{\text{lag}}(s) = K\alpha_{\text{lag}} \frac{T_{\text{lag}}s + 1}{\alpha_{\text{lag}} T_{\text{lag}} s + 1} \tag{33.220}$$

    - Evaluate performance and iterate if necessary
  - Design considerations and guidelines

* Recognize that our lag compensator in the root locus design approach omits the gain K
* add notes and see discussion under root locus design
* The pole near the origin introduces a slow contribution to the transient. Forunately, the zero partially cancels this pole such that the amplitude of this slow contribution is small.
  - "Lag compensation will introduce a pole-zero combination near the origin that will generate a long tail with small amplitude in the transient response [39]."

- Ideal proportional + integral (PI) compensation: improving steady state response with a cascade compensator
  - PI compensator transfer function

$$C_{\mathrm{PI}}(s) \quad = \quad K \frac{1}{T_{\mathrm{PI}}} \frac{T_{\mathrm{PI}} s + 1}{s} \tag{33.221}$$

  - (Iterative) design approach; see, for example, [38]
    * Find the frequency that would have the appropriate PM and assign it to be the *new* gain crossover frequency, ω_{gc}. (As with a lag compensator, include a cushion of about 5-12° extra [6, 39] to account for a small phase lag from the PI compensator.)
    * Select K to make ω_{gc} the gain crossover frequency
    * Set the frequency associated with the PI zero to about

$$T_{\mathrm{PI}} = \frac{10}{\omega_{\mathrm{gc}}}; \tag{33.222}$$

      see, for example, [38]. need to check this reference
    * Construct the PI compensator

      .
    * Evaluate performance and iterate if necessary
  - Design considerations and guidelines
    * add notes and see discussion under root locus design

Example 33.4: Lag compensation with prescribed PM and steady state error

Example 33.5: PI compensation with prescribed PM

33.8 Compensators: additional topics and guidelines

- Lag-lead compensation: improving transient response, stability, and steady state response by cascading compensators
  - (Iterative) design approach. (Note, this approach is a standard ME375 approach and is a variation of the approach presented in [6]. check that this reference is still valid)
    - Identify the *desired* gain crossover frequency, ω_{gc}
    - Determine how much phase is needed in $P(s)$ from the lead compensator, ϕ_{max}, at $\omega_{\text{max}} = \omega_{\text{gc}}$ to meet PM specifications. (Include a cushion of about 5-12° extra [6, 39] to account for a small phase lag from the lag compensator.)
    - Solve for α_{lead} and T_{lead}

$$\alpha_{\text{lead}} = \frac{1 - \sin(\phi_{\text{max}})}{1 + \sin(\phi_{\text{max}})} \tag{33.223}$$

$$T_{\text{lead}} = \frac{1}{\omega_{\text{max}}\sqrt{\alpha_{\text{lead}}}} \tag{33.224}$$

    - Construct the lead compensator

$$C_{\text{lead}}(s) \quad = \quad \frac{T_{\text{lead}}s + 1}{\alpha_{\text{lead}}T_{\text{lead}}s + 1} \tag{33.225}$$

    - Find K to satisfy desired gain crossover frequency

$$1 = |KC_{\text{lead}}(j\omega_{\text{gc}})P(j\omega_{\text{gc}})| \tag{33.226}$$

    - Solve for α_{lag} to satisfy steady state error
    - Set the time constant with the lag zero to about

$$T_{\text{lag}} = \frac{10}{\omega_{\text{gc}}}; \tag{33.227}$$

      see, for example, [38, 6, 70]. Note, [8, 39] suggest $\frac{\omega_{\text{gc}}}{10} \leq \omega_z \leq \frac{\omega_{\text{gc}}}{2}$. check references
    - Construct lag compensator

$$C_{\text{lag}}(s) = \alpha_{\text{lag}}\frac{T_{\text{lag}}s + 1}{\alpha_{\text{lag}}T_{\text{lag}}s + 1} \tag{33.228}$$

    - Construct the lag-lead compensator

$$C_{\text{lag-lead}}(s) \quad = \quad KC_{\text{lead}}(s)C_{\text{lag}}(s) \tag{33.229}$$

    - Evaluate performance and iterate if necessary
  - Design considerations and guidelines
    - add notes and see discussion under root locus design
- Ideal proportional + integral + differential (PID) compensation: improving transient response, stability, and steady state response with a cascade compensation

Example 33.6:

Chapter 34

Brief Comparison of Control System Design Methods

- This chapter is a work in progress.
- Direct pole placement
  - Insights and potential advantages:
    * All poles may be placed at specified locations
    * Loosely speaking, the number of design requirements (e.g., pole location, steady state error, ...) dictates the number of parameters necessary in the controller to meet the requirements
  - Potential disadvantages:
    * Requires a relatively accurate system model
    * Algebraically intensive
    * Our approach doesn't directly control zero locations
    * Our approach doesn't directly consider steady state error
    * Provides no insight into how system performance will change with changes in controller parameters
- Root locus
  - Insights and potential advantages:
    * Relatively simple graphical (or geometrical) approach to reposition closed loop poles
    * Ability to consider a variety of controller zero positions
    * Ability to graphically to investigate stability
    * Ability to visualize and predict effects of parameter changes on system performance
  - Potential disadvantages:

    * Requires a relatively accurate system model
    * Our approach is not suitable for placing all closed loop poles
    * Steady state error cannot be obtained easily from closed loop pole locations or a root locus
    * Our approach may tend to utilize more controller parameters than the number of system requirements being met

- Frequency domain
  - Insights and potential advantages:
    * Relatively simple graphical approach
    * Ability to graphically investigate stability (and stability margins)
    * Applicable to experimentally characterized systems (including measurement uncertainty)
    * Closed loop performance specifications including steady state error are easy to estimate from a bode plot
    * It's estimated that frequency domain controller design methods are the most often used in industry [8].
    * Some ability to visualize and predict effects of parameter changes on system performance
  - Potential disadvantages:
    * Our approach is not suitable for placing all closed loop poles
    * Our approach may tend to utilize more controller parameters than the number of system requirements being met

Part X

Electronic Systems

Chapter 35

Basic Electronics

- Electronics -
  - "Electronics deals with the flow of electrons through vacuum, gas or semiconductors [74]."
  - "An electronic device consists of integrated circuits which have several diodes, transistors, resistors, capacitors, etc. mounted on a single chip [74]."
- Atomic structure and nomenclature; see, for example, [75]
  - Shells:
    * "a grouping of orbitals according to principal quantum number [75]"
    * Shells are denoted by number, n, (principal quantum number): 1, 2, 3, 4, ...
  - Subshell
    * "a gouping of orbitals according to angular momentum quantum number [75]"
    * within the n^{th} shell there are n orbitals
    * Subshells are denoted by (angular momentum quantum number or) the designations for the orbital shapes: s, p, d, f, g, ...; see, for example, [75]
    * within the n^{th} orbital; see, for example, [75]
      · the s subshell has 1 orbital
      · the p subshell has 3 orbitals (for subshell d to have electrons, typically $n > 1$)
      · the d subshell has 5 orbitals (for subshell d to have electrons, typically $n > 2$)
      · ...
  - Orbital:
    * "a solution to the Schrödinger wave equation; describes a region of space where an electron is likely to be found [75]"
    * Each orbital can hold up to two electrons; see, for example, [75]

- Valence electrons -
  * "The number of electrons present in the outermost shell are called valence electrons [74]."
  * Octet rule - "Main-group elements tend to undergo reactions that leave them with eight valence electrons. That is, main-group elements react so that they attain a noble-gas electron configuration with filled s and p sublevels in their valence electron shell [75]."
    · Note, "There are many exceptions to the octet rule ... [75]."
  * Holes/vacancies

• Semiconductors

- Conductors, insulators, and semiconductors
  * Conductors
  * Insulators
  * Semiconductors
- Example semiconductor materials; see, for example, [74]
  * Germanium
  * Silicon
  * Carbon

• Silicon

- Abundant - "... approximately 28 percent of the earth's crust [76]"
- Electrons; see, for example, [74]
  * 14 electrons
    · Electron configuration: $1s^2 2s^2 2p^2 3s^2 3p^2$
  * 14 protons (neutral)
  * 4 valence electrons ($3s^2 3p^2$)
  * 4 holes or vacancies in the outer shell ($3p$ could hold 4 more electrons)
- Crystaline silicon
  * Valence electrons are shared (covalent bond) with four other silicon atoms to fill vacancies (or holes) in outer orbitals; see, for example, [74]
  * At low temperatures, electrons cannot conduct; see, for example, [74]
  * At high temperatures, heat energy allows electrons to leave a hole conduct and fall into another hole; see, for example, [74]
- Doping

* Doping - "Doping consists of adding impurities to the crystalline structure of the semiconductor [18]."
* "N-Type semiconductor is formed by doping a pure silicon or germanium crystal with a material having five valence electrons [74]."
  · A pentavalent impurity leaves a free electron capable of conducting; see, for example, [74].
  · Electrons serve as the carrier; see, for example, [18, 74].
  · Electrically neutral - same number of protons and electrons; see, for example, [74].
* "P-Type material is formed when silicon or germanium crystal is doped with ... a small percentage of trivalent impurity material like boron, gallium or indium [74]."
  · A trivalent impurity leaves a hole; see, for example, [74].
  · Holes serve as the carrier; see, for example, [18, 74].
  · Electrically neutral - same number of protons and electrons; see, for example, [74].
* Doping can be done by heating silicon surrounded with, for example, boron or phosphorous gas. These atoms diffuse into the silicon. See, for example, [74].
* Conductivity
  · Pure silicon doesn't conduct; see, for example, [74]
  · Types P and N silicon conduct; see, for example, [74].
  · "Free electrons move far more easily around the lattice than holes [18]."

- Semiconductor diodes

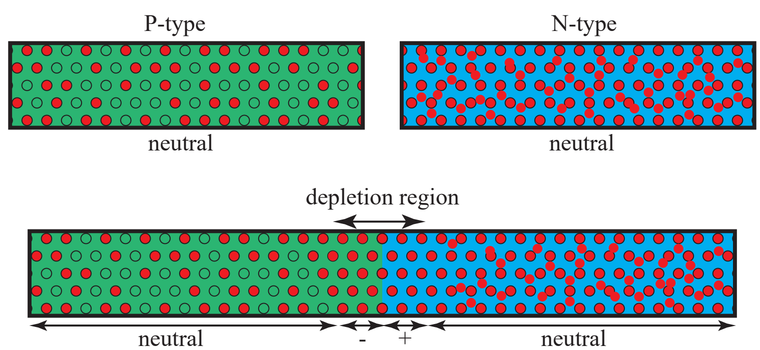


Figure 35.1: This is just a cartoon and it's ability to describe the physics is likely very limited.

  – PN junction
    * PN junctions are formed when P-type and N-type semiconductor material is joined

    * “If a junction between P-type and N-type semiconductor material is created within a single crystal, in such a way that the crystalline structure is preserved across the junction, the result is a junction diode [77].”
    * At the joint, free electrons from the N-type side diffuse into the P-type side. As a result of this diffusion, a barrier voltage develops preventing further diffusion. In silicon, the voltage barrier is about 0.7 V. See, for example, [74]
    * Depletion region; see, for example, [18]
        · Region near the P-type/N-type interface
        · Within the depletion region, holes in P-type are filled by free electrons in N-type
        · Within the depletion region, P-type becomes negatively ionized (due to filling the holes with electrons) and the N-type becomes ionized by the loss of free electrons; see, for example, [18].
- Anode and cathode; see, for example, [74]
    * Anode: P-type side of junction. Anode is connected to the positive terminal of source when forward biased.
    * Cathode: N-type side of junction. Cathode is connected to the negative terminal of source when forward biased.
- Current voltage relationship

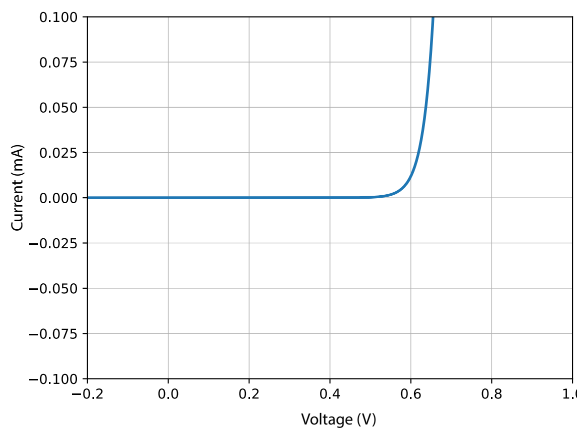


Figure 35.2: Current voltage relationship following the diode equation presented in [18].

    * Sketch of current voltage relationship
    * Forward biased
        · Voltage applied to the diode in which p-type side is at higher voltage; see, for example, [74, 77]
        · Low resistance to current flow; see, for example, [74]

· thickness of depletion layer is reduced; see, for example, [74]

· There seems to be slight inconsistencies among the definitions of forward biased. Some definitions suggest that forward biased is when the voltage is greater than or equal to 0.6 or 0.7 V and others when voltage is greater than 0 V.

∗ Reverse biased -

· Voltage applied to the diode in which p-type side is at lower voltage; see, for example, [74, 77]

· High resistance to current flow; see, for example, [74]

· thickness of depletion layer is increased; see, for example, [74]

∗ Reverse breakdown; see, for example, [18]

· "Under ... a large reverse bias ... the diode conducts current again, this time in the reverse direction [18]."

– Diode limitations; see, for example, [18]

∗ Maximum forward current (average and surge)

∗ Maximum forward voltage

∗ Maximum reverse voltage

∗ "A very high forward current or a very high reverse voltage can destroy a diode [74]."

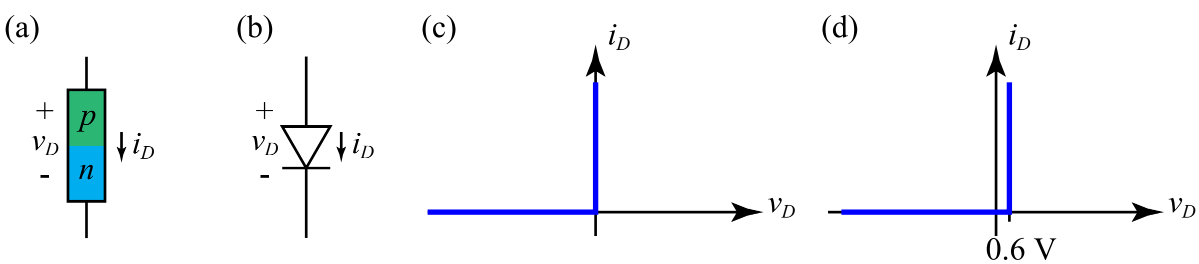


Figure 35.3: (a) Cartoon of a semiconductor diode. (b) Circuit symbol for a diode. (c) Ideal diode model; see, for example, [18]. (d) Offset diode model; see, for example, [18]. Add cartoon of what a diode looks like and label anode and cathode.

– Large signal diode models; see, for example, [18]

∗ Ideal diode model; see, for example, [18]

· When $v_D > 0$ (forward biased), the diode is approximated as a short circuit

· When $v_D < 0$ (reversed biased), the diode is approximated as an open circuit

· Warning, this is a simplified model of the $i - v$ relationship of a diode applicable for large signals.

- · In [18], ideal diodes are represented with a filled in diode symbol while other diodes are represented with an open symbol.
- ∗ Offset diode model; see, for example, [18]
  - · This is the primary model used in ME375
  - · When $v_D \geq 0.6$ V (forward biased), the voltage drop is diode is approximated as a constant v_D =0.6 V voltage drop
  - · When $v_D < 0.6$ V (reversed biased), the diode is approximated as an open circuit
  - · Warning, this is a simplified model of the $i - v$ relationship of a diode applicable for large signals.

- Diode applications

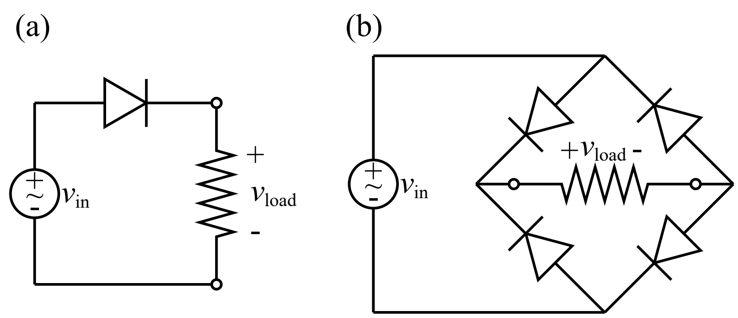


Figure 35.4: (a) Half-wave rectifier; see, for example, [18]. (b) Bridge rectifier; see, for example, [18].

- – Rectification
  - ∗ Rectification - "... the ability to convert an AC signal with zero average (DC) value to a signal with a nonzero DC value [18]."
  - ∗ Half wave rectifier
  - ∗ Bridge rectifier
    - · DC power supplied typically include a bridge rectifier; see, for example, [18]
- – find references to: Snubber diode, freewheel diode, or suppressor diode
- – LED
- – Diode clipper (limiter); see, for example, [18]
- – Other applications

Example 35.1: Half-wave rectifier

Example 35.2: Half-wave rectifier with capacitor

- Consider two states
  - Forward biased
  - Reversed biased - free response

- Transistors
  - Bipolar junction transistors (BJT)
    * Two types
      · NPN
      · PNP (not a focus of ME375)
    * NPN structure
      · 3 layers
      · Collector - N-type, lightly doped
      · Base - P-type, thin, lightly doped; see, for example, [74]
      · Emitter - N-type, more heavily doped;

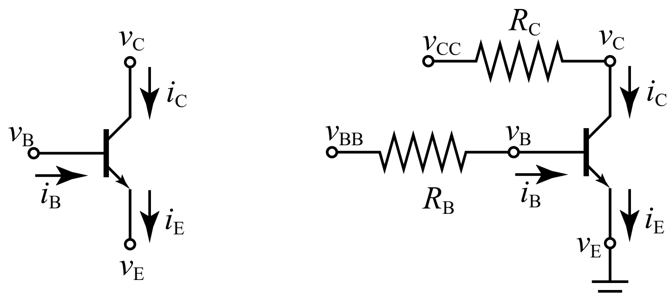


Figure 35.5: (Left) Circuit symbol for an NPN BJT. (Right) Simple circuit in which a NPN BJT is used as a switch.

    * Large signal model: npn BJT; see, for example, [18]
      · The factor β is typically in the range of 20 to 200; see, for example, [18]
      · Cutoff region - in the cutoff region, both junctions are reverse biased; see, for example, [18]

$$v_B - v_E < 0.6 \text{ V} \tag{35.1}$$
$$i_B = 0 \tag{35.2}$$
$$i_C = 0 \tag{35.3}$$
$$v_C - v_E \geq 0 \tag{35.4}$$

· Active state (not a focus of ME375)

$$v_B - v_E \approx 0.6 \text{ V} \tag{35.5}$$
$$i_B > 0 \tag{35.6}$$
$$i_C = \beta i_B \tag{35.7}$$
$$v_C - v_E > 0.6 \text{ V} \tag{35.8}$$

· Saturation region - in the saturation region, both junctions are forward biased; see, for example, [18]

$$v_B - v_E \approx 0.6 \text{ V} \tag{35.9}$$
$$i_B > 0 \tag{35.10}$$
$$i_C < \beta i_B \tag{35.11}$$
$$v_C - v_E \approx 0.2 \text{ V} \tag{35.12}$$

∗ Uses
  · Switch - toggle between cutoff and saturation
  · Current amplifier (not a focus in ME375)

∗ Switch design with NPN BJT
  · See Figure 35.5
  · Design to toggle between cutoff and saturation
  · Assume v_{BB}, V_{CC}, and R_C are fixed
  · Neglecting the voltage drop $v_C - v_E$, the max current i_C at saturation would be

$$i_{C,\max} = \frac{V_{CC}}{R_C} \tag{35.13}$$

  · Select R_B such that the base current will yield saturation

$$i_B = \frac{i_{C,\max}}{\beta} \tag{35.14}$$
$$\approx \frac{i_{C,\max}}{10} \tag{35.15}$$

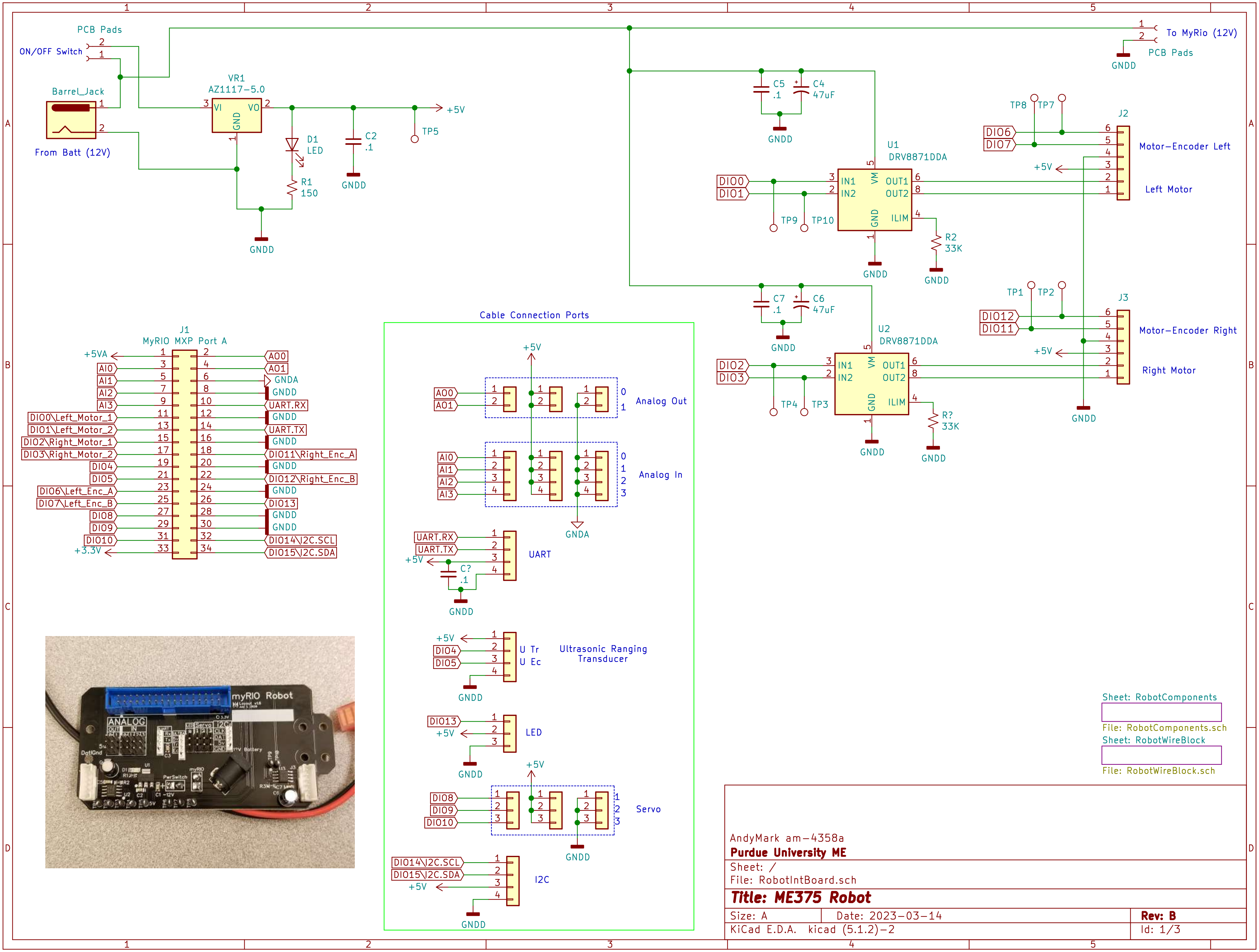
To increase the likelihood that the system is in saturation, we substitute in a conservative value for β, $\beta = 10$. (Recall that β is typically greater than 20; see, for example, [18].)

∗ Darlington pair; see, for example, [77]
  · Two transistors in series; see, for example, [77]
  · Yields high input impedance at the base and low base current; see, for example, [77]

- Field effect transistors (FET); not a focus of ME375
  * n-channel
  * p-channel

Example 35.3: Design R in transistor circuit

- Robot Kit



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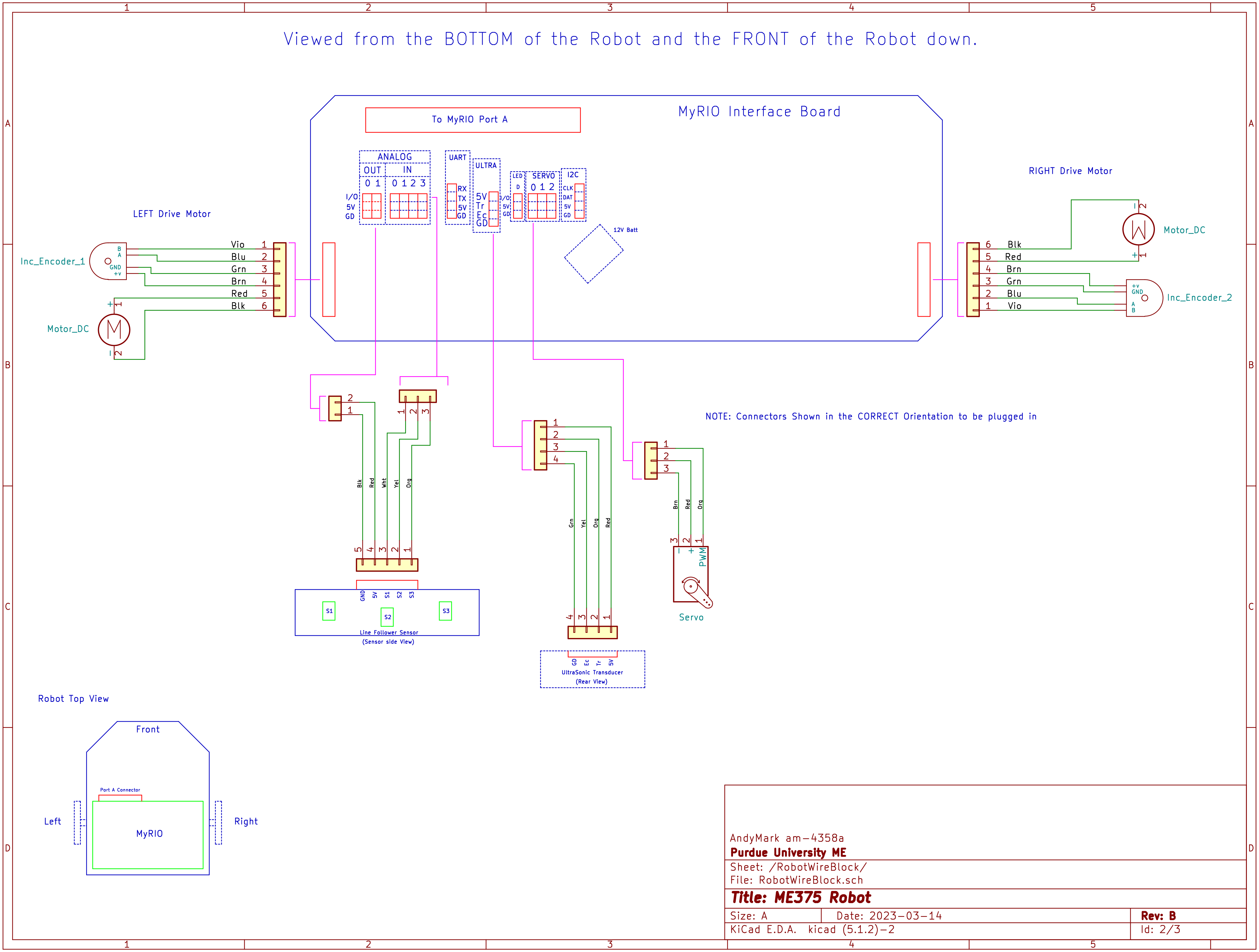

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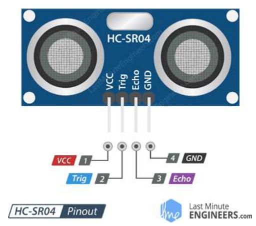

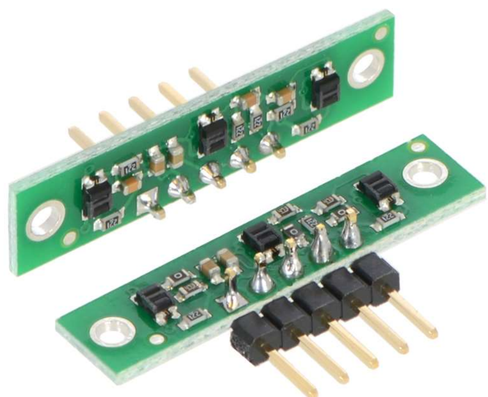

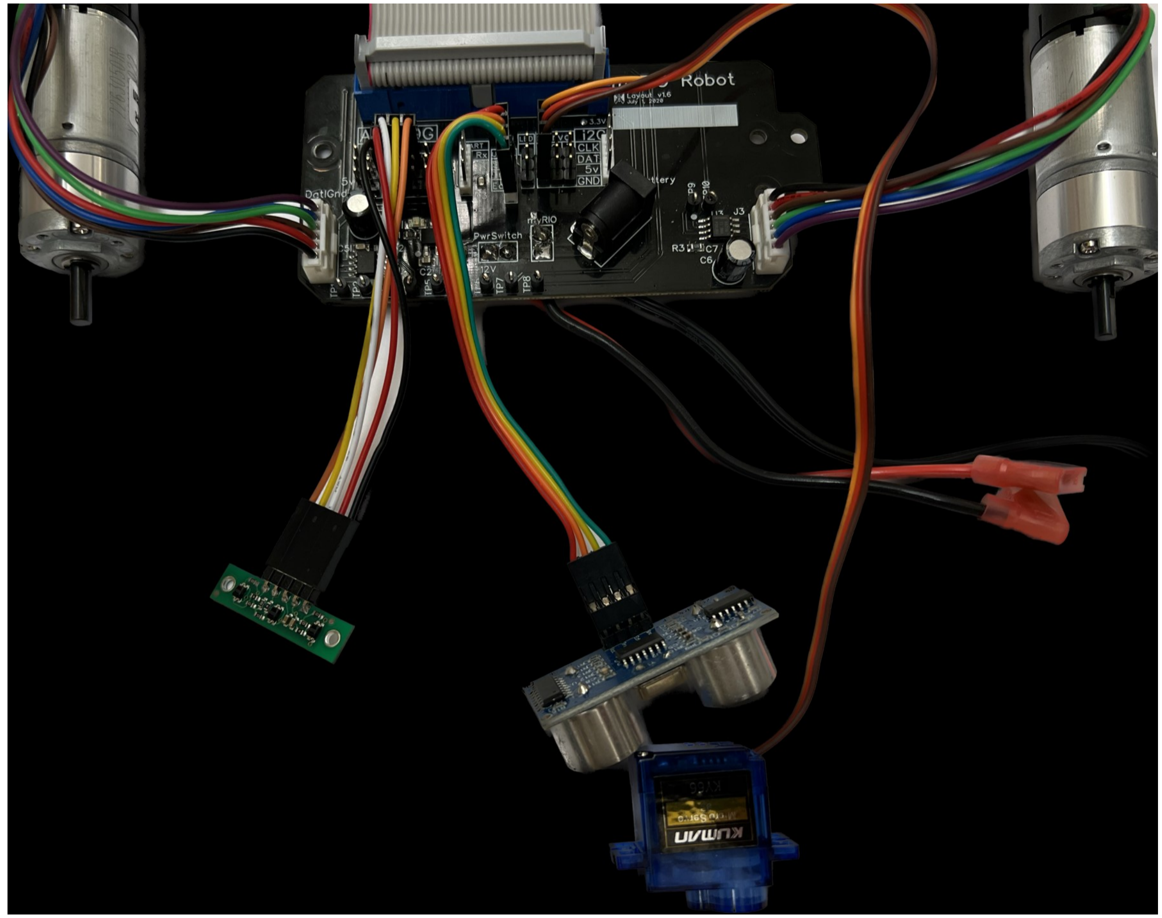

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File: RobotComponents.sch		
Title: ME375 Robot		
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Chapter 36

Analog Electronics

- This chapter builds on the following chapters:
  - Chapter 17 Electrical Systems
  - Chapter 18 Electromechanical Systems
  - Chapter 3 Analog to Digital Conversion
    - Example integrated circuits
      - MCP3002: ADC
      - MCP4911: DAC
  - Chapter 22 Data Acquisition and Aliasing
- Transistor applications
  - Pulse width modulation
    - Rectangular pulse
    - Rectangular pulse train
    - Duty cycle
    - Pulse width modulation
  - Application of a PWM signal to the base of a transistor to effectively vary the average voltage across a load (e.g., DC motor)
    - If the PWM frequency is high relative to the bandwidth of the load, the load will effectively filter the high frequency component of the PWM signal leaving only the DC component of the signal.
  - H-bridge
    - System of four transistors and diodes
    - Used with PWM to vary both the average voltage across a load (e.g., DC motor)

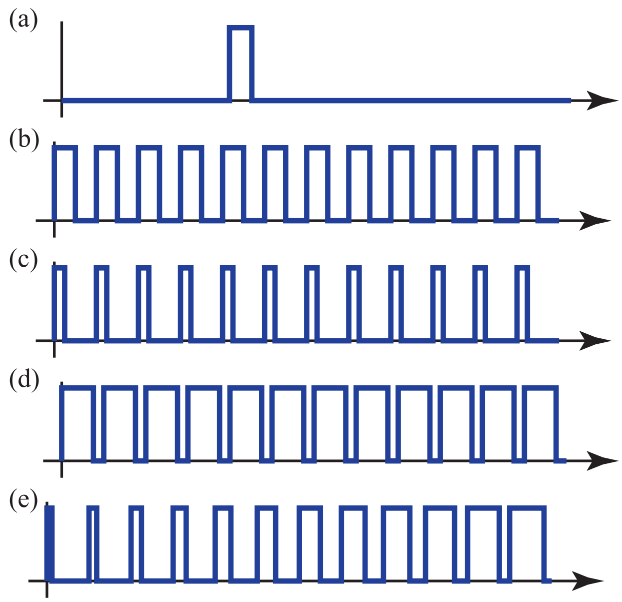


Figure 36.1: (a) Rectangular pulse. (b) Rectangular pulse train with 50% duty cycle (i.e., square wave). (c) Rectangular pulse train with about 20% duty cycle. (d) Rectangular pulse train with about 80% duty cycle. (e) Rectangular pulse train with pulse width modulation.

    * The system allows the polarity of that voltage across the load to be changed (to, for example, change the direction of rotation of a DC motor)
    * The current ME375 robot kit includes two H-bridges on the circuit board.

Example 36.1: Adjust average voltage to a load

- Operational amplifiers
  - Ideal operational amplifier
    * Gain is very large: $A \to \infty$

$$v_{\text{out}} = Av_{\text{in}} \tag{36.1}$$

    * Input impedance is very large: $R_{\text{in}} \to \infty$
    * Output impedance is very small: $R_{\text{out}} \to 0$
  - Practical operational amplifier

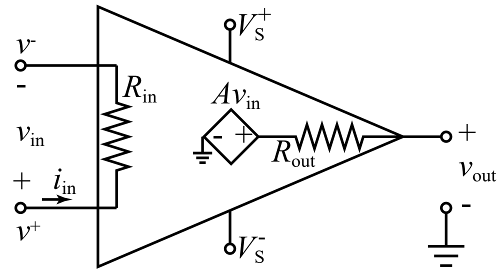


Figure 36.2: Ideal OpAmp. See [18]

  * Gain is very large: $A \approx 10^5 - 10^7$; see, for example, [18]
  * Input impedance is very large: $R_{\text{in}} > 10^7\ \Omega$ [4]
  * Output impedance is very small: $R_{\text{out}} < 100\ \Omega$; see, for example, [4, 40]

- Active element - Operational amplifiers are active electrical elements (they require power to operate).

  * Saturation - Output range limited by supply voltage V_s^- and V_s^+

- Major assumptions:

  * $i_{\text{in}} = 0$
  * $v^+ = v^-$ (when there is a pathway connecting v^- and v_{out})

- Examples

  * 741

- Example circuits

  * Follower
  * inverting amplifier
  * noninverting amplifier
  * Generalize inverting amplifier for impedance?

Example 36.2: Op amp circuit

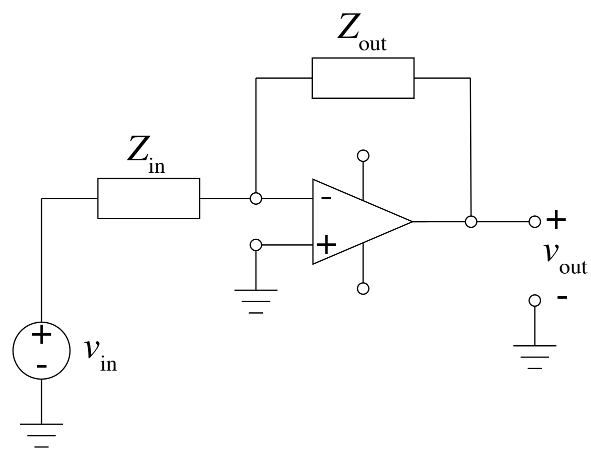


Figure 36.3:

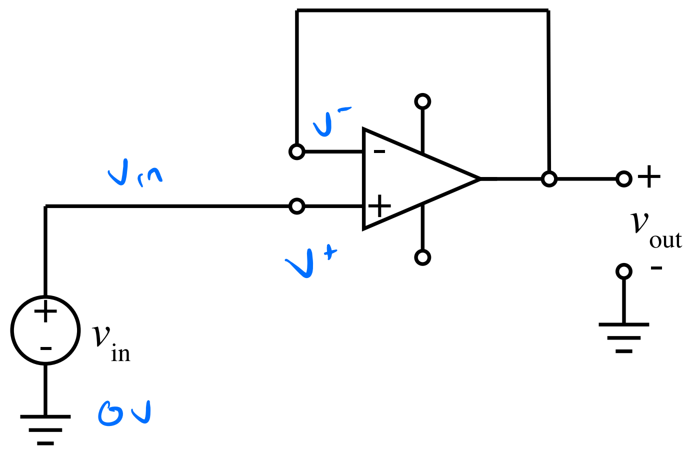


Figure 3:

5. Find the input output relationship for the op-amp circuit in **Figure 3**.

$v^+ = v^-$

$v_{in} = v_{out}$

$\boxed{v_{out} = v_{in}}$

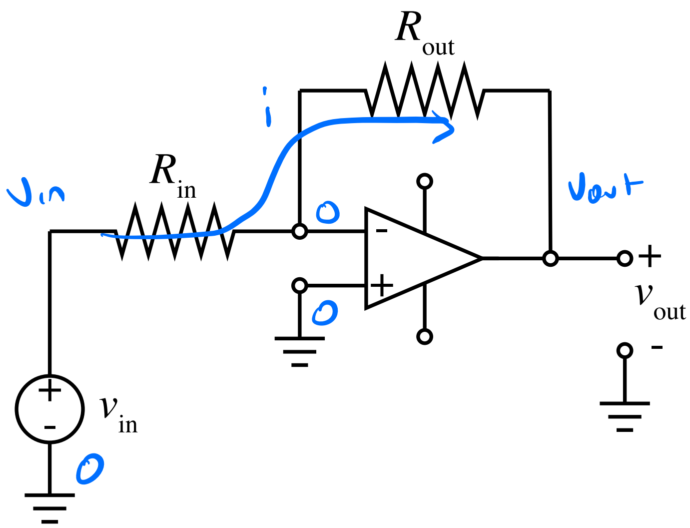


Figure 4:

6. Find the input output relationship for the op-amp circuit in **Figure 4**.

$$v^- = v^+$$
$$= 0$$

$$v_{in} - 0 = i R_{in}$$
$$0 - v_{out} = i R_{out}$$

$$\frac{v_{in} - 0}{R_{in}} = \frac{0 - v_{out}}{R_{out}}$$

$$\boxed{v_{out} = -\frac{R_{out}}{R_{in}} v_{in}}$$

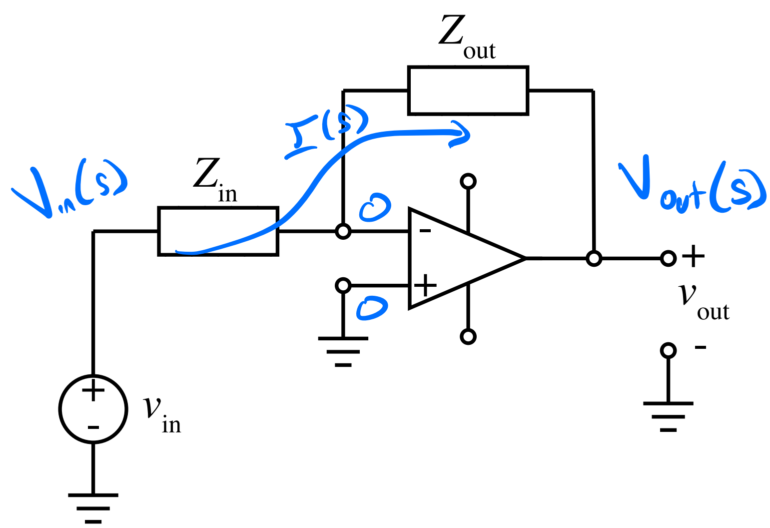


Figure 5:

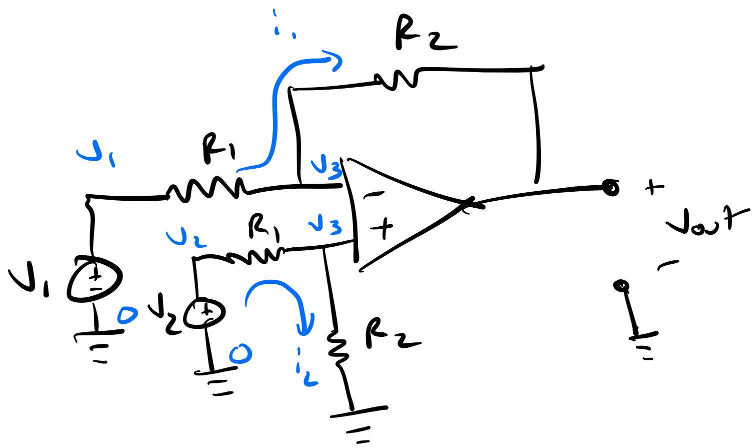
7. Find the input output relationship for the op-amp circuit in **Figure 5**.

$$V_{in}(s) - 0 = I(s)\,Z_{in}$$

$$0 - V_{out}(s) = I(s)\,Z_{out}$$

$$\frac{V_{in}(s)}{Z_{in}} = -\frac{V_{out}}{Z_{out}}$$

$$\boxed{V_{out}(s) = -\frac{Z_{out}}{Z_{in}}V_{in}(s)}$$



$$V_1 - V_3 = i_1 R_1 \quad (1)$$

$$V_3 - V_{out} = i_1 R_2 \quad (2)$$

$$V_2 - V_3 = i_2 R_1 \quad (3)$$

$$V_3 - 0 = i_2 R_2 \quad (4)$$

(1) and (2)

$$\frac{V_1 - V_3}{R_1} = \frac{V_3 - V_{out}}{R_2}$$

$$R_1 V_{out} = -R_2 V_1 + (R_1 + R_2) V_3 \quad (5)$$

(3) and (4)

$$\frac{V_2 - V_3}{R_1} = \frac{V_3}{R_2}$$

$$V_3(R_1+R_2) = R_2 V_2 \quad (6)$$

(6) into (5)

$$R_1 V_{out} = -R_2 V_1 + R_2 V_2$$

$$\boxed{V_{out} = \frac{R_2}{R_1}(V_2 - V_1)}$$

Chapter 37

Digital Electronics

- This chapter builds on the following chapter:
  - Chapter 3 Analog to Digital Conversion
- Combinational logic vs. sequential logic
  - "... combinational logic circuits provide outputs that are based on a combination of present inputs only [18]."
    * Combinational logic is the focus of this chapter.
  - "... sequential logic circuits depend on present and past input values [18]."

- Diodes and transistors can be used to construct logic gates

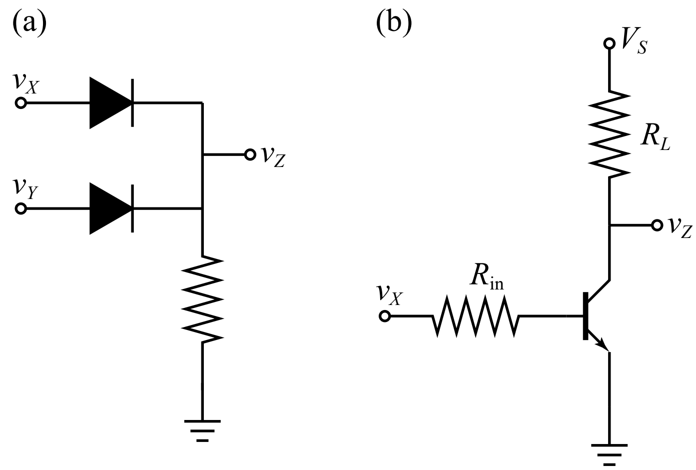


Figure 37.1: Example logic gates using diodes and transistors. (a) Or gate constructed from ideal diodes. (Following the notation of [18], symbols for ideal diodes include filled triangles.) (b) Not gate constructed from a transistor. See, for example, [18, 74]

  - Ideal diodes as an OR gate; see Figure 37.1(a)

v_X (V)	v_Y (V)	v_Z (V)
0	0	0
0	5	5
5	0	5
5	5	5

  - Transistor switch as a NOT gate ($V_S = 5$ V); see Figure 37.1(b)

v_X (V)	v_Z (V)
0	5
5	0

- Boolean algebra and logic gates

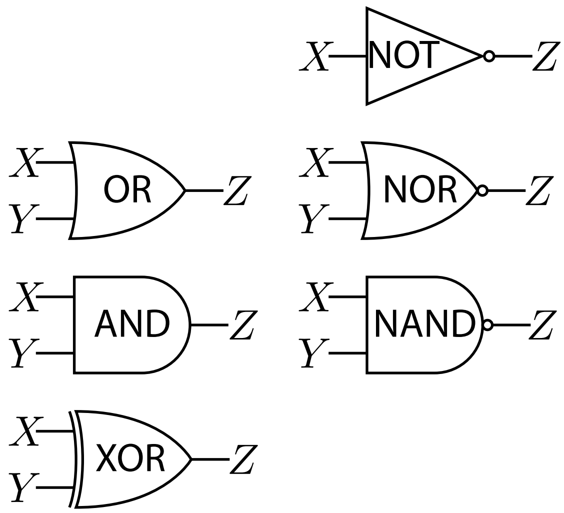


Figure 37.2: Logic gates. See, for example, [54]. Note, it appears XOR might not be a focus of ME375.

  - Positive logic; see, for example, [18, 74]

			Notes
Boolean	True	False	
Binary	1	0	
Binary	High	Low	
TTL	2.4-5 V	0-0.4 V	see, for example, [18]
CMOS			
Switch	On	Off	see, for example, [74]
Switch	Closed	Open	see, for example, [74]

    * negative logic can be used, but we will focus on positive logic.
  - Logic circuit diagrams
  - Logic gate -
  - Logic function -

- OR gate - logical addition; see, for example, [18, 74]
    * mathematical expression

$$Z = X + Y \tag{37.1}$$

    * symbol
    * truth table

X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

    * Differs from algebraic and binary addition ($1 + 1 = 1$).
- AND gate - logical multiplication; see, for example, [18]
    * mathematical expression

$$Z = X \cdot Y \tag{37.2}$$

    * symbol
    * truth table

X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

    * Follows algebraic and binary multiplication.
- NOT gate; see, for example, [18, 74]
    * mathematical notation

$$Z = \bar{X} \tag{37.3}$$

    * symbol
    * truth table

X	Z
0	1
1	0

- NOR gate; see, for example, [18, 74]
  - mathematical notation

$$Z = (\overline{X + Y}) = \bar{X} \cdot \bar{Y} \tag{37.4}$$

  - symbol
  - truth table

X	Y	Z
0	0	1
0	1	0
1	0	0
1	1	0

- NAND gate; see, for example, [18, 74]
  - mathematical notation

$$Z = (\overline{X \cdot Y}) = \bar{X} + \bar{Y} \tag{37.5}$$

  - symbol
  - truth table

X	Y	Z
0	0	1
0	1	1
1	0	1
1	1	0

– XOR gate; see, for example, [18]
  * mathematical notation

$$Z = X \oplus Y \tag{37.6}$$

  * symbol
  * truth table

X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	0

  * Note, XOR doesn't seem to be a focus of ME375.

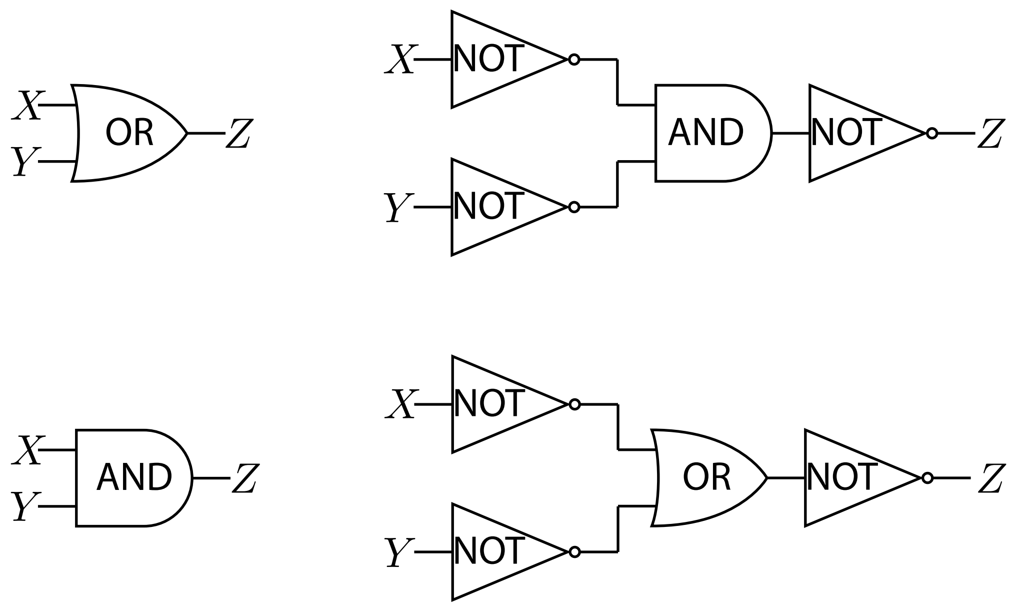


Figure 37.3: Logic gates equivalents. (Top) Implementation of an OR gate using NOT and AND gates. (Bottom) Implementation of an AND gate using Not and OR gates.

- De Morgan's Laws
  - De Morgan's Theorems; see, for example, [18, 74].

$$\overline{X+Y} = \bar{X}\cdot\bar{Y} \tag{37.7}$$
$$\overline{X\cdot Y} = \bar{X}+\bar{Y} \tag{37.8}$$

  - "Any logic function can be implemented using only OR and NOT gates, or using only AND and NOT gates [18]."
  - "... any function can be realized by just one of the two basic operations, plus the complement operation [18]."
  - Universal gates [74]: NAND and NOR
  - Applications
    * OR - an OR gate may be implemented with NOT and AND gates

$$X+Y=\overline{\bar{X}\cdot\bar{Y}} \tag{37.9}$$

    * AND - an AND gate may be implemented with NOT and OR gates

$$X\cdot Y=\overline{\bar{X}+\bar{Y}} \tag{37.10}$$

- Properties of Boolean algebra; see, for example, [76, 18]
  - operator precedence - logical multiply takes precedence over logical addition; see, for example, [76]
  - idempotent; see, for example, [76]

$$X \cdot X = X \tag{37.11}$$
$$X + X = X \tag{37.12}$$

  - complementary; see, for example, [76]

$$X \cdot \bar{X} = 0 \tag{37.13}$$
$$X + \bar{X} = 1 \tag{37.14}$$

  - involution

$$\bar{\bar{X}} = X \tag{37.15}$$

  - commutative

$$X \cdot Y = Y \cdot X \tag{37.16}$$
$$X + Y = Y + X \tag{37.17}$$

  - associative

$$X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z \tag{37.18}$$
$$X + (Y + Z) = (X + Y) + Z \tag{37.19}$$

  - distributive; see, for example, [76]

$$X \cdot (Y + Z) = X \cdot Y + X \cdot Z \tag{37.20}$$
$$(X + Y) \cdot (X + Z) = X + (Y \cdot Z) \tag{37.21}$$

  - simplification; see, for example, [76]
    * absorption; see, for example, [18]

$$X + X \cdot Y = X \tag{37.22}$$

    * Other simplification

$$X \cdot (X + Y) = X \tag{37.23}$$
$$X + \bar{X} \cdot Y = X + Y \tag{37.24}$$
$$X \cdot (\bar{X} + Y) = X \cdot Y \tag{37.25}$$

- the following property was found in [18]

$$X \cdot Y + Y \cdot Z + \bar{X} \cdot Z = X \cdot Y + \bar{X} \cdot Z \tag{37.26}$$

  - Proof

$$\begin{aligned}
X \cdot Y + Y \cdot Z + \bar{X} \cdot Z &= X \cdot Y + \bar{X} \cdot Z && (37.27)\\
X \cdot Y + Y \cdot Z \cdot (X + \bar{X}) + \bar{X} \cdot Z &= && (37.28)\\
X \cdot Y + Y \cdot Z \cdot X + Y \cdot Z \cdot \bar{X} + \bar{X} \cdot Z &= && (37.29)\\
X \cdot Y \cdot (1 + Z) + \bar{X} \cdot Z \cdot (Y + 1) &= && (37.30)\\
X \cdot Y + \bar{X} \cdot Z &= && (37.31)
\end{aligned}$$

- Methods to construct logic functions from truth tables
  - Inspection
  - Sum of products method; see, for example, [76, 18]
    - For each True output, write a product of the corresponding inputs (negating False inputs)
    - Sum the above products
    - Simplify
  - Product of sums method; see, for example, [76, 18]
    - For each False output, write a sum of the corresponding inputs (negating True inputs)
    - Multiply the above sums
    - Simplify

Chapter 38

Finite State Machines

- Programming paradigms
  - Programming paradigms -
    - Programming paradigm - "A programming paradigm is a style, or "way," of programming [78]."
    - Many programming languages often support a range of programming paradigms; see, for example, [78]
      - "Some languages make it easy to write in some paradigms but not others [78]."
      - "Many existing languages can be classified into families based on their model of computation [79]."
      - "It should be emphasized that the distinctions among language classes are not clear-cut [79]."
  - Imperative vs. declarative programming paradigms
    - "The top-level division distinguishes between the declarative languages ... and the imperative languages ... [79]."
    - Imperative programming paradigm
      - "... the focus is on how the computer should do it [79]."
      - "Programming with an explicit sequence of commands that update the state [78]."
    - Declarative programming paradigm
      - "... the focus is on what the computer is to do [79]."
      - "Programming by specifying the result you want, not how to get it [78]."
  - Event driven
    - "Programming with emitters and listeners for asynchronous actions [78]."
    - See discussion on event driven programming at `https://web.stanford.edu/class/cs123/materials.html`

  - Connection to robot control

- Finite state machines -
  - Finite State Machines
    - “The actual implementation of a function such as the controller is called a state machine [76].”
    - “In fact, when the number of states is constrained and finite, this is more usually called a finite state machine (FSM) [76].”
    - “An FSM consists of two blocks of combinational logic, next state logic and output logic, and a register that stores the state. On each clock edge, the FSM advances to the next state, which was computed based on the current state and inputs [80].”
    - “Finite state machines are so named because of the sequential logic that implements them can be in only a fixed number of possible states [81].”
    - “Their outputs and states are identical ... [81].”
    - “... the outputs and next state of a finite state machine are combinational logic functions of their inputs and present state [81].”
  - State (transition) diagrams; see, for example, [76, 80]
    - State diagram -
    - Circles - states
    - Arcs (arrows) - state transitions
  - State (transition) tables -
    - “... similar to a truth table (inputs on the left and corresponding outputs on the right), but it also includes the current state as an input and the next state as an output [76].”
    - “... indicates, for each state and input, what the next state ... should be [80].”
  - Example state transition diagrams
    - Microwave - a few different state transition diagrams for a microwave
      - `http://mason.gmu.edu/~hgomaa/swe760/SWE760-4-StateMachineModeling-RT.pdf`
      - `https://link.springer.com/content/pdf/10.1007%2F978-1-84996-522-4_10.pdf`
      - `https://www.cs.utexas.edu/users/browne/uvvf2008/IntroductiontoModelsandAbstractio pdf`
    - Garage door opener
    - Gas furnace and thermostat
  - See discussion on finite state machines at `https://web.stanford.edu/class/cs123/materials.html`

Part XI

Measurements: Sensors

Chapter 39

Sensors

- Categorization of sensors
  - "There is no standard and universally accepted classification of sensors. Depending on one's viewpoint, sensors may be grouped according to their physical characteristics ... or by the physical variable or quantity measured by the sensor. ... Other classifications are also possible [18]."
  - "There are two approaches to categorizing sensors. One way ... is to group together all those different types of sensors used for a given application ..." and the other "... is to group sensors by the mechanism by which they work ... [13]."
  - Note, here we focus on senors that yield electrical outputs.
  - Sensors may be categorized by whether or not they require external power to function.
    - It appears, the terms 'active sensor' and 'passive sensor' are used inconsistently in the literature to refer to these two categories of sensors (i.e., those that depend upon external power and those that do not).
      - Differences in definitions might originate from the distinction between passive and active circuit elements. (Passive electrical elements including resistors, capacitors, and inductors dissipate or store electrical energy while active elements like op amps and power supplies generate electrical energy; see, for example, [18].) Specifically, sensors that primarily employ changes in impedance of a *passive* circuit element to convert the measurand into an electrical signal require use of an *active* circuit element (i.e., external power source).
    - Typical definition for ME 375
      - "All sensors may be of two kinds: passive and active. A passive sensor does not need any additional energy source [11]."
      - "Instruments are divided into active or passive ones according to whether the instrument output is entirely produced by the quantity being measured [passive]

or whether the quantity being measured simply modulates the magnitude of some external power source [active] [64]."

    * Opposite definition (not used in ME375)
        · "... the delivered electrical signals of passive sensors are impedance variations because these sensors require an electrical energy source in order to read s. ... All other sensors are active [12]."
        · "Passive devices ... require an external power supply in order to give a voltage or current output signal; active devices ... need no external power supply [9]."

- Categorization of sensors by mechanism; see, for example, [13, 9]
  - Resistive
    * Potentiometer
        · Linear
        · Rotational
        · Note, measuring the voltage output of the potentiometer creates a current divider and thereby introduces nonlinearity into the system. Find a reference.
    * Temperature measurement
        · Resistance temperature detectors (RTD) - Sensors that employ the dependence on temperature of the resistance of a metal (or allow) to measure temperature.
        · Thermistor - Sensors that employ the dependence on temperature of the resistance of a semiconductor to measure temperature.
    * Photoresistor - Sensors that employ the dependence on light intensity of the resistance of photoconductive material. The resistance of photoconductive material depends upon the intensity of the light on the material; see, for example, [13, 11].
    * Strain gage
  - Capacitive
    * Pressure
    * Distance; see for example, edge sensors for segmented telescopes `https://exoplanets.nasa.gov/internal_resources/433/`
    * Capacitive (or condenser) microphone; see, for example, [11].
    * Differential pressure transducer [18, 13]
    * Capacitive touch sensors (e.g., touch screen phones); see, for example, [11].
    * Capacitive accelerometer; see, for example, [13]
  - Inductive
    * Linear variable differential transformer (LVDT)
    * Variable inductance displacement sensors; see, for example, [9]
    * Vehicle detection by inductive loop; see, for example, [11].

    * edge detection on segmented mirror; see, for example, `https://www.bluelineengineering.com/theory-of-operation` and `https://www.spiedigitallibrary.org/conference-proceedings-of-4003/1/Development-of-the-segment-alignment-maintenance-system-SAMS-for-the/10.1117/12.391509.full`
  - Hall effect sensor
    * senses magnetic field
    * used in the encoders of the robot motors
    * How two sensors are used to get direction
  - Piezoelectric effect
    * Accelerometer
    *
  - Thermoelectric effect
    * Thermocouple
    * Thermopile

- Sensors on the ME375 robot
  - 3-axis accelerometer built into myRIO
    * The accelerometer in the myRIO is likely similar to that described by the following specification sheet: `https://www.nxp.com/docs/en/data-sheet/MMA8452Q.pdf`
  - Reflectance sensor array
    * `https://www.pololu.com/product/2456`
  - Phototransistor on old ME 375 robot (Spring 2022)
    * The phototransistors in the line follower is likely similar to that described by the following specification sheet: `https://www.vishay.com/docs/83760/tcrt5000.pdf`
  - Hall effect sensor
  - Ultrasonic range finder
    * Specification sheet: `https://cdn.sparkfun.com/datasheets/Sensors/Proximity/HCSR04.pdf`
  - button
  - touch sensor

- Categorization of sensors by measured quantity

- Strain gauge
  - Need a picture of a strain gauge and need to discuss the applications, because this may be the first time students learn about strain gauges

- Strain and Poisson's ratio
  * Engineering or nominal strain
  * Poisson's ratio
    · Ratio of lateral strain to axial strain with axial loading

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon} \tag{39.1}$$

- Gauge factor derivation
  * Consider the strain gauge as a single bar with axial loading
  * Resistance
    · Resistance relationship

$$R = \rho \frac{l}{bh} \tag{39.2}$$

    · ρ: Resistivity
    · L: Length
    · b: width of cross sectional area
    · h: height of cross sectional area
  * Sensitivity of resistance to resistivity, length, and cross sectional area

$$\begin{aligned}
\Delta R &= \frac{\partial R}{\partial \rho}\Delta\rho + \frac{\partial R}{\partial L}\Delta L + \frac{\partial R}{\partial b}\Delta b + \frac{\partial R}{\partial h}\Delta h && (39.3)\\
&= \frac{L}{bh}\Delta\rho + \rho\frac{1}{bh}\Delta L - \rho\frac{L}{b^2h}\Delta b - \rho\frac{L}{bh^2}\Delta h && (39.4)\\
&= R\left(\frac{\Delta\rho}{\rho} + \frac{\Delta L}{L} - \frac{\Delta b}{b} - \frac{\Delta h}{h}\right) && (39.5)\\
&= R\left(\frac{\Delta\rho}{\rho} + \epsilon - \epsilon_{\text{lat}} - \epsilon_{\text{lat}}\right) && (39.6)
\end{aligned}$$

Note, if we stretch the bar we expect $\epsilon > 0$ and $\epsilon_{\text{lat}} < 0$ and $\epsilon_{\text{lat}} < 0$

$$\begin{aligned}
&= R\left(\frac{\Delta\rho}{\rho} + \epsilon + \nu\epsilon + \nu\epsilon\right) && (39.7)\\
\frac{\Delta R}{R} &= \frac{\Delta\rho}{\rho} + (1+2\nu)\epsilon && (39.8)
\end{aligned}$$

    · The first term in this last expression is due to the piezoreistive effect; see, for example, [9, 4]
  * Gauge factor; see, for example, [9, 10]

$$\begin{aligned}
\frac{1}{\epsilon}\frac{\Delta R}{R} &= \frac{1}{\epsilon}\frac{\Delta\rho}{\rho} + 1 + 2\nu && (39.9)\\
G_f &= && (39.10)
\end{aligned}$$

Chapter 40

Bridge Circuits

- This chapter builds on the following chapters:
  - Chapter 17 Electrical Systems
  - Chapter 21 Fourier Series and Spectral Analysis
  - Chapter 24 Modulation
- Bridge Circuits

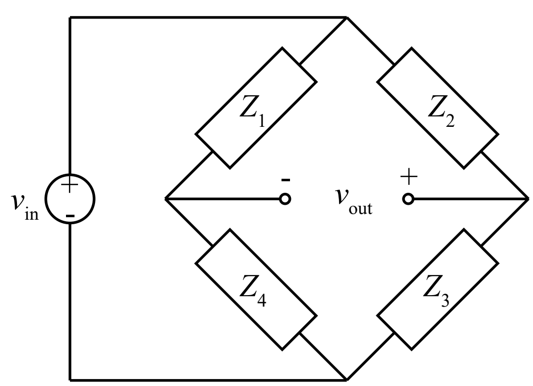


Figure 40.1: Bridge circuit constructed from four impedance elements. Note, this numbering scheme is consistent with, for example, [10, 9].

– Transfer function

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{Z_3}{Z_2+Z_3} - \frac{Z_4}{Z_1+Z_4} \tag{40.1}$$

$$= \frac{Z_3(Z_1+Z_4) - Z_4(Z_2+Z_3)}{(Z_1+Z_4)(Z_2+Z_3)} \tag{40.2}$$

$$= \frac{Z_1Z_3 - Z_2Z_4}{(Z_1+Z_4)(Z_2+Z_3)} \tag{40.3}$$

– Special cases: Wheatstone bridge (all resistive elements)

* Relationship

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_3}{R_2+R_3} - \frac{R_4}{R_1+R_4} \tag{40.4}$$

$$= \frac{R_1R_3 - R_2R_4}{(R_1+R_4)(R_2+R_3)} \tag{40.5}$$

* For bridges constructed from pure resistive elements, the transfer function is independent of s and we can write

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_1R_3 - R_2R_4}{(R_1+R_4)(R_2+R_3)} \tag{40.6}$$

$$\frac{v_{\text{out}}(t)}{v_{\text{in}}(t)} = \tag{40.7}$$

* Quarter bridge (3 fixed resistors, $R_2 = R_3 = R_4 = R_0$, and 1 variable resistor, $R_1 = R_0 + \Delta R$); see, for example, [10]

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_0}{R_0+R_0} - \frac{R_0}{R_0+\Delta R+R_0} \tag{40.8}$$

$$= \frac{1}{2} - \frac{1}{2+\frac{\Delta R}{R_0}} \tag{40.9}$$

· Linearization

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{1}{2} - \frac{1}{2+\frac{\Delta R}{R_0}} \tag{40.10}$$

$$\approx \left.\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)}\right|_{\frac{\Delta R}{R_0}=0} + \left.\frac{\partial}{\partial\frac{\Delta R}{R_0}}\left(\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)}\right)\right|_{\frac{\Delta R}{R_0}=0} \frac{\Delta R}{R_0} \tag{40.11}$$

$$\approx \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{2^2}\right)\frac{\Delta R}{R_0} \tag{40.12}$$

$$\approx \frac{1}{4}\frac{\Delta R}{R_0} \tag{40.13}$$

* Half bridge - opposite branches (2 fixed resistors, $R_2 = R_4 = R_0$, and 2 variable but equivalent resistors, $R_1 = R_3 = R_0 + \Delta R$)

$$\begin{aligned}
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} &= \frac{R_0 + \Delta R}{R_0 + \Delta R + R_0} - \frac{R_0}{R_0 + \Delta R + R_0} && (40.14)\\
&= \frac{1 + \frac{\Delta R}{R_0}}{2 + \frac{\Delta R}{R_0}} - \frac{1}{2 + \frac{\Delta R}{R_0}} && (40.15)\\
&= \frac{\frac{\Delta R}{R_0}}{2 + \frac{\Delta R}{R_0}} && (40.16)
\end{aligned}$$

· Linearization

$$\begin{aligned}
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} &= \frac{\frac{\Delta R}{R_0}}{2 + \frac{\Delta R}{R_0}} && (40.17)\\
&\approx \left.\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)}\right|_{\frac{\Delta R}{R_0}=0} + \left.\frac{\partial}{\partial \frac{\Delta R}{R_0}}\left(\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)}\right)\right|_{\frac{\Delta R}{R_0}=0} \frac{\Delta R}{R_0} && (40.18)\\
&\approx \left(\frac{0}{2}\right) + \left(\frac{1}{2} - \frac{0}{2^2}\right)\frac{\Delta R}{R_0} && (40.19)\\
&\approx \frac{1}{2}\frac{\Delta R}{R_0} && (40.20)
\end{aligned}$$

* Half bridge - same branch opposite direction (2 fixed resistors, $R_2 = R_3 = R_0$, and 2 variable resistors with opposite changes, $R_1 = R_0 + \Delta R$ and $R_4 = R_0 - \Delta R$)

$$\begin{aligned}
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} &= \frac{R_0}{R_0 + R_0} - \frac{R_0 - \Delta R}{R_0 + \Delta R + R_0 - \Delta R} && (40.21)\\
&= \frac{1}{2} - \frac{1}{2} + \frac{\Delta R}{2R_0} && (40.22)\\
&= \frac{\Delta R}{2R_0} && (40.23)
\end{aligned}$$

· Linearization (already linear)

* Full bridge (4 variable resistors, $R_1 = R_3 = R_0 + \Delta R$ and $R_2 = R_4 = R_0 - \Delta R$)

$$\begin{aligned}
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} &= \frac{R_0 + \Delta R}{R_0 + \Delta R + R_0 - \Delta R} - \frac{R_0 - \Delta R}{R_0 + \Delta R + R_0 - \Delta R} && (40.24)\\
&= \frac{\Delta R}{R} && (40.25)
\end{aligned}$$

· Linearization (already linear)

    * Gauge factor allows the conversion from Recall gauge factor for strain gauges

  - Bridge circuits with capacitive or inductive elements

    * Assumptions:

      · The voltage source is sinusoidal

$$v_{\text{in}} = V_{\text{S}} \sin(\omega_{\text{S}} t) \tag{40.26}$$

      · **Impedance changes of bridge elements are much slower than the frequency of the source.** That is if we express the changes of impedance of an element as the Fourier series

$$\Delta Z(t) \quad = \quad \frac{1}{2}a_0 + \sum_{n=1}^{N} \{a_n \cos(n\omega_Z t) + b_n \sin(n\omega_Z t)\} \tag{40.27}$$

then we are assuming that the highest frequency component in the change of the impedance is much smaller than the frequency of the supply, $N\omega_Z \ll \omega_{\text{S}}$. This allows us to use phaser or transfer function methods to analyze the circuit even though the impedance is changing with time.

      · Impedance changes are small relative to the nominal impedance values.

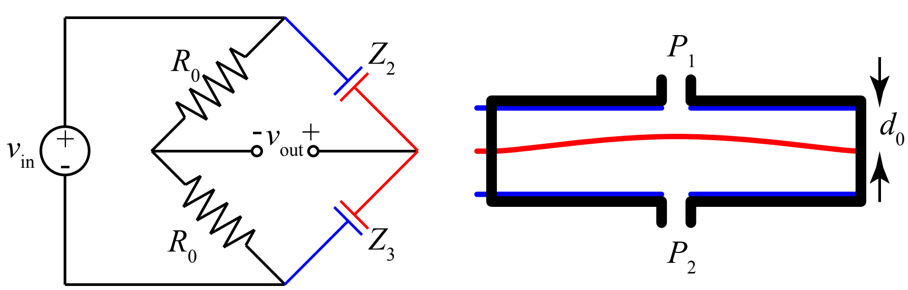


Figure 40.2:

    * Example: Capacitive differential pressure transducer; see, for example, [18, 13]

      · Capacitance dependent upon distance d, permittivity ϵ, and cross sectional area A; see, for example, [18, 13]

$$C = \frac{\epsilon A}{d} \tag{40.28}$$

· Impedance (assuming the distance changes from the nominal value $d = d_0 + \Delta d$)

$$\begin{aligned}
Z_1 &= R_0 && (40.29)\\
Z_4 &= R_0 && (40.30)\\
Z_2 &= \frac{1}{\frac{\epsilon A}{d_0 - \Delta d} s} && (40.31)\\
&= \frac{d_0 - \Delta d}{\epsilon A s} && (40.32)\\
Z_3 &= \frac{1}{\frac{\epsilon A}{d_0 + \Delta d} s} && (40.33)\\
&= \frac{d_0 + \Delta d}{\epsilon A s} && (40.34)
\end{aligned}$$

· Transfer function

$$\begin{aligned}
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} &= \frac{Z_1 Z_3 - Z_2 Z_4}{(Z_1 + Z_4)(Z_2 + Z_3)} && (40.35)\\
&= \frac{R_0 \frac{d_0 + \Delta d}{\epsilon A s} - \frac{d_0 - \Delta d}{\epsilon A s} R_0}{(R_0 + R_0)\left(\frac{d_0 - \Delta d}{\epsilon A s} + \frac{d_0 + \Delta d}{\epsilon A s}\right)} && (40.36)\\
&= \frac{2\Delta d}{2 \cdot 2 d_0} && (40.37)\\
&= \frac{\Delta d}{2 d_0} && (40.38)
\end{aligned}$$

Note, our assumption that the frequency of the input is much faster than changes in impedance (i.e., changes of Δd) yields a transfer function that is a pure gain. Consequently, the converting back to the time domain is trivial.

· Result

$$\begin{aligned}
v_{\text{out}}(t) &= \frac{1}{2 d_0} v_{\text{in}}(t) \Delta d(t) && (40.39)\\
&= K_C v_{\text{in}}(t) \Delta d(t) && (40.40)
\end{aligned}$$

In ME375 we define $K_C = \frac{1}{2 d_0}$.
Note, the output is an amplitude modulated signal in which voltage source, $v_{\text{in}}(t)$, is treated as the carrier signal and $\Delta d(t)$ is the modulating signal.
One approach to demodulating this signal in order to recover $\Delta d(t)$ is to (i) multiply by the carrier signal, (ii) filter out high frequencies, and (iii) divide by a factor dependent upon the transfer function and carrier amplitude squared; see ME375 slides and lecture worksheet.

∗ Example: Variable/differential reluctance sensor; see, for example, [9, 13, 11]

- Inductance dependent upon distance d, L_0, and α; see, for example, [9]

$$L = \frac{L_0}{1+\alpha d} \tag{40.41}$$

- Impedance (assuming the distance changes from the nominal value $d = d_0 + \Delta d$)

$$\begin{aligned}
Z_1 &= \frac{L_0}{1+\alpha(d_0 - \Delta d)} s && (40.42)\\
Z_4 &= \frac{L_0}{1+\alpha(d_0 + \Delta d)} s && (40.43)\\
Z_2 &= R_0 && (40.44)\\
Z_3 &= R_0 && (40.45)
\end{aligned}$$

- Transfer function

$$\begin{aligned}
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} &= \frac{Z_1 Z_3 - Z_2 Z_4}{(Z_1 + Z_4)(Z_2 + Z_3)} && (40.46)\\
&= \frac{\frac{L_0}{1+\alpha(d_0-\Delta d)} s R_0 - R_0 \frac{L_0}{1+\alpha(d_0+\Delta d)} s}{\left(\frac{L_0}{1+\alpha(d_0-\Delta d)} s + \frac{L_0}{1+\alpha(d_0+\Delta d)} s\right)(R_0 + R_0)} && (40.47)\\
&= \frac{\frac{1}{1+\alpha(d_0-\Delta d)} - \frac{1}{1+\alpha(d_0+\Delta d)}}{\left(\frac{1}{1+\alpha(d_0-\Delta d)} + \frac{1}{1+\alpha(d_0+\Delta d)}\right) 2} && (40.48)\\
&= \frac{(1+\alpha(d_0+\Delta d)) - (1+\alpha(d_0-\Delta d))}{((1+\alpha(d_0+\Delta d)) + (1+\alpha(d_0-\Delta d)))\, 2} && (40.49)\\
&= \frac{\alpha \Delta d}{2 + 2\alpha d_0} && (40.50)
\end{aligned}$$

  Note, our assumption that the frequency of the input is much faster than changes in impedance (i.e., changes of Δd) yields a transfer function that is a pure gain. Consequently, the converting back to the time domain is trivial.

- Result

$$\begin{aligned}
v_{\text{out}}(t) &= \frac{\alpha}{2+2\alpha d_0} v_{\text{in}}(t) \Delta d(t) && (40.51)\\
&= K_L v_{\text{in}}(t) \Delta d(t) && (40.52)
\end{aligned}$$

  In ME375 we define $K_L = \frac{\alpha}{2+2\alpha d_0}$
  Note, the output is an amplitude modulated signal in which voltage source, $v_{\text{in}}(t)$, is treated as the carrier signal and $\Delta d(t)$ is the modulating signal.
  One approach to demodulating this signal in order to recover $\Delta d(t)$ is to (i) multiply by the carrier signal, (ii) filter out high frequencies, and (iii) divide by a factor dependent upon the transfer function and carrier amplitude squared; see ME375 slides.

Part XII

Control System Sensitivity

Chapter 41

Sensitivity

- This chapter builds on the following chapter:
  - Chapter 26 Common Control System Objectives
- Sensitivity
  - Definition
    * "Sensitivity is the ratio of the fractional change in the function to the fractional change in the parameter as the fractional change of the parameter approaches zero [6]."
    * Note, this definition differs from definitions for 'sensitivity' in the context of static calibration or 'static sensitivity' of a transfer function
    * [3] present alternate names for this sensitivity: relative sensitivity, normalized sensitivity, and Bode sensitivity.
  - Mathematical definition
    * S_P^F - sensitivity of the function F to the function (or parameter) P near its nominal value P_{nom}

$$S_P^F = \left[\lim_{\Delta P \to 0} \frac{\frac{\Delta F}{F}}{\frac{\Delta P}{P}}\right]_{P_{\text{nom}}} \tag{41.1}$$

$$= \left[\lim_{\Delta P \to 0} \frac{\Delta F}{\Delta P}\frac{P}{F}\right]_{P_{\text{nom}}} \tag{41.2}$$

$$= \left.\frac{P}{F}\frac{\partial F}{\partial P}\right|_{P_{\text{nom}}} \tag{41.3}$$

Although some authors use variational/functional derivative notation, $\frac{\delta F}{\delta P}$ (see, for example, [8, 6]), here we use partial derivative notation (see, for example, [38, 37]).

– Properties
  * Analog to chain rule - sensitivity of a function F with respect to P and assuming F is a function of C which is a function of P

$$
\begin{aligned}
S_P^F &= \left.\frac{\partial F}{\delta P}\frac{P}{F}\right|_{\text{nominal}} && (41.4)\\
&= \left.\frac{\partial F}{\partial C}\frac{\partial C}{\partial P}\frac{P}{F}\right|_{\text{nominal}} && (41.5)\\
&= \left.\left(\frac{\partial F}{\partial C}\frac{C}{F}\right)\frac{F}{C}\left(\frac{\partial C}{\partial P}\frac{P}{C}\right)\frac{C}{P}\frac{P}{F}\right|_{\text{nominal}} && (41.6)\\
&= \left.S_C^F\frac{\cancel{F}}{\cancel{C}}S_P^C\frac{\cancel{C}}{\cancel{P}}\frac{\cancel{P}}{\cancel{F}}\right|_{\text{nominal}} && (41.7)\\
&= S_C^F S_P^C && (41.8)
\end{aligned}
$$

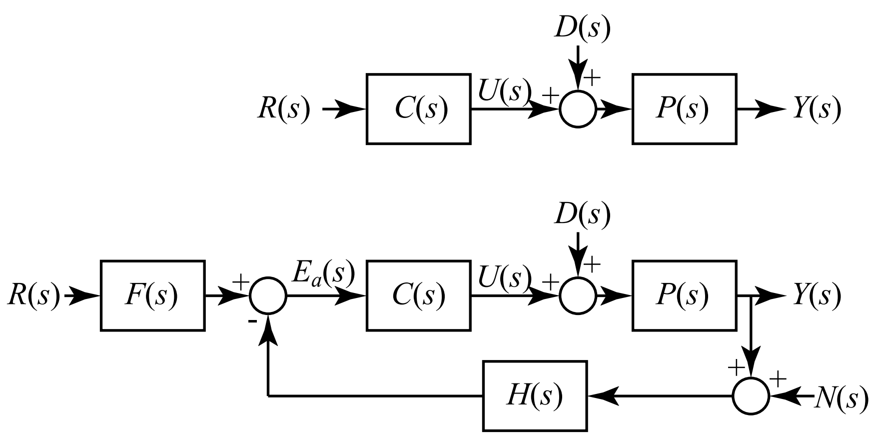


Figure 41.1:

– Sensitivity functions for open and closed loop control
  * Open loop control
    · Open loop transfer function

$$T_{\text{ol}}(s) = C(s)P(s) \tag{41.9}$$

    · Sensitivity to the controller

$$
\begin{aligned}
S_{C(s)}^{T_{\text{ol}}(s)} &= \left.\frac{C(s)}{T_{\text{ol}}(s)}\frac{\partial T_{\text{ol}}(s)}{\partial C(s)}\right|_{\text{nominal}} && (41.10)\\
&= \left.\frac{C(s)}{C(s)P(s)}P(s)\right|_{\text{nominal}} && (41.11)\\
&= 1 && (41.12)
\end{aligned}
$$

With a relative sensitivity of 1, the open loop transfer function feels the full effect of any change in the controller.

· Sensitivity to the plant

$$\begin{aligned}
S_{P(s)}^{T_{\rm ol}(s)} &= \left.\frac{P(s)}{T_{\rm ol}(s)}\frac{\partial T_{\rm ol}(s)}{\partial P(s)}\right|_{\rm nominal} && (41.13)\\
&= \left.\frac{P(s)}{C(s)P(s)}C(s)\right|_{\rm nominal} && (41.14)\\
&= 1 && (41.15)
\end{aligned}$$

With a relative sensitivity of 1, the open loop transfer function feels the full effect of any change in the plant.

∗ Closed loop control

· Closed loop transfer function

$$T_{\rm cl}(s) = \frac{C(s)P(s)}{1+C(s)P(s)H(s)} \tag{41.16}$$

· Sensitivity to the controller

$$\begin{aligned}
S_{C(s)}^{T_{\rm cl}(s)} &= \left.\frac{C(s)}{T_{\rm cl}(s)}\frac{\partial T_{\rm cl}(s)}{\partial C(s)}\right|_{\rm nominal} && (41.17)\\
&= \left.\frac{C(s)}{T_{\rm cl}(s)}\left(\frac{P(s)}{1+C(s)P(s)H(s)} - \frac{C(s)P(s)}{(1+C(s)P(s)H(s))^2}P(s)H(s)\right)\right|_{\rm nominal} && (41.18)\\
&= \left.\frac{C(s)}{T_{\rm cl}(s)}\left(\frac{T_{\rm cl}(s)}{C(s)} - \frac{T_{\rm cl}^2(s)H(s)}{C(s)}\right)\right|_{\rm nominal} && (41.19)\\
&= 1 - T_{\rm cl}(s)H(s)|_{\rm nominal} && (41.20)\\
&= 1 - \left.\frac{C(s)P(s)H(s)}{1+C(s)P(s)H(s)}\right|_{\rm nominal} && (41.21)\\
&= \left.\frac{1}{1+C(s)P(s)H(s)}\right|_{\rm nominal} && (41.22)
\end{aligned}$$

Assuming $H(s) = 1$ and steady state error is small (i.e., $C(s)P(s)$ is large at low frequencies), this sensitivity will be small such that the closed loop system will be relatively insensitive to controller changes at low frequencies.

· Sensitivity to the plant

$$
\begin{aligned}
S_{P(s)}^{T_{\mathrm{cl}}(s)} &= \left.\frac{C(s)}{T_{\mathrm{cl}}(s)}\frac{\partial T_{\mathrm{cl}}(s)}{\partial P(s)}\right|_{\text{nominal}} && (41.23)\\
&= \left.\frac{P(s)}{T_{\mathrm{cl}}(s)}\left(\frac{C(s)}{1+C(s)P(s)H(s)} - \frac{C(s)P(s)}{(1+C(s)P(s)H(s))^2}C(s)H(s)\right)\right|_{\text{nominal}} && (41.24)\\
&= \left.\frac{P(s)}{T_{\mathrm{cl}}(s)}\left(\frac{T_{\mathrm{cl}}(s)}{P(s)} - \frac{T_{\mathrm{cl}}^2(s)H(s)}{P(s)}\right)\right|_{\text{nominal}} && (41.25)\\
&= \left.1 - T_{\mathrm{cl}}(s)H(s)\right|_{\text{nominal}} && (41.26)\\
&= \left.1 - \frac{C(s)P(s)H(s)}{1+C(s)P(s)H(s)}\right|_{\text{nominal}} && (41.27)\\
&= \left.\frac{1}{1+C(s)P(s)H(s)}\right|_{\text{nominal}} && (41.28)
\end{aligned}
$$

Assuming $H(s) = 1$ and steady state error is small (i.e., $C(s)P(s)$ is large at low frequencies), this sensitivity will be small such that the closed loop system will be relatively insensitive to plant changes at low frequencies.

· Sensitivity to the sensor

$$
\begin{aligned}
S_{H(s)}^{T_{\mathrm{cl}}(s)} &= \left.\frac{C(s)}{T_{\mathrm{cl}}(s)}\frac{\partial T_{\mathrm{cl}}(s)}{\partial H(s)}\right|_{\text{nominal}} && (41.29)\\
&= \left.\frac{H(s)}{T_{\mathrm{cl}}(s)}\left(-\frac{C(s)P(s)}{(1+C(s)P(s)H(s))^2}C(s)P(s)\right)\right|_{\text{nominal}} && (41.30)\\
&= \left.\frac{H(s)}{T_{\mathrm{cl}}(s)}\left(-T_{\mathrm{cl}}(s)\right)\right|_{\text{nominal}} && (41.31)\\
&= \left.-\frac{C(s)P(s)H(s)}{1+C(s)P(s)H(s)}\right|_{\text{nominal}} && (41.32)
\end{aligned}
$$

Interestingly,

$$
\left|S_{H(s)}^{T_{\mathrm{cl}}(s)}\right| + \left|S_{P(s)}^{T_{\mathrm{cl}}(s)}\right| = 1. \qquad (41.33)
$$

Therefore, for frequencies for which the closed loop transfer function is relatively insensitive to changes in $C(s)$ or $P(s)$ (i.e., $C(s)P(s)$ is large), the closed loop transfer function will be very sensitive to changes in $H(s)$. Hence, Franklin et. al state "... it is particularly important that the transfer function of the sensor be not only low in noise but also very stable in gain [8]."

– Usage

∗ If we have the sensitivity S_P^F, then we can estimate the change in the function, ΔF, from its nominal value, F, due to a change in the parameter, ΔP, from its nominal value, P.

$$S_P^F = \frac{P}{F}\frac{\partial F}{\partial P} \tag{41.34}$$

$$S_P^F \approx \frac{\Delta F}{F}\frac{P}{\Delta P} \tag{41.35}$$

$$\frac{\Delta F}{F} \approx S_P^F \frac{\Delta P}{P} \tag{41.36}$$

$$\Delta F \approx S_P^F \frac{\Delta P}{P} F \tag{41.37}$$

$$\Delta F \approx S_P^F \frac{F}{P} \Delta P \tag{41.38}$$

- Possible applications
  * Sensitivity of a transfer function to a parameter
  * Sensitivity of the control system transfer function to a subsystem transfer function
    · Forward path
    · Feedback path
  * Sensitivity of pole location to a parameter - Note, this is similar to root locus.
  * Sensitivity of static sensitivity to a parameter

Part XIII

Systems with Delay

Chapter 42

Delay

- Transfer function for delay
  - Laplace transform of a delayed unit step $u_s(t-T_0)$; repeated from ME 365 Lecture 6: Laplace Transform
    - Delayed unit step

$$u_s(t-T_0) = \begin{cases} 0 & t < T_0 \\ \text{(undefined but finite)} & t = T_0 \\ 1 & T_0 < t \end{cases} \tag{42.1}$$

    - Laplace transform

$$\begin{aligned}
L[u_s(t-T_0)] &= \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-st}u_s(t)dt\right] && (42.2)\\
&= \lim_{T\to\infty}\left[\int_{0^-}^{T} e^{-st}\begin{cases} 0 & t < T_0 \\ \text{(undefined but finite)} & t = T_0 \\ 1 & T_0 < t \end{cases} dt\right] && (42.3)\\
&= \cancelto{0}{\int_{0^-}^{T_0^-} e^{-st}0dt} + \cancelto{0}{\int_{T_0}^{T_0} e^{-st}\text{(undefined but finite)}dt} + \lim_{T\to\infty}\int_{T_0^+}^{T} e^{-st}1dt && (42.4)\\
&= \lim_{T\to\infty}\int_{T_0^+}^{T} e^{-st}1dt && (42.5)\\
&= -\frac{1}{s}e^{-st}\bigg|_{T_0^+}^{\infty} && (42.6)\\
&= \cancelto{0}{-\frac{1}{s}e^{-s\infty}} + \frac{1}{s}e^{-sT_0} && (42.7)\\
&= \frac{1}{s}e^{-sT_0} && (42.8)
\end{aligned}$$

- Note, here we require $\mathbb{R}\{s\} > 0$ for the limit to be obtained.

– Delay in an arbitrary function and transfer function for a delay

* Delay in function $f(t)$

$$x(t) = \begin{cases} 0 & t < T_0 \\ f(t-T_0) & t > T_0. \end{cases} \tag{42.9}$$

$$= u_s(t-T_0)f(t-T_0) \tag{42.10}$$

* Laplace transform (see, for example, [8]); repeated from ME 365 Lecture 6: Laplace Transform

$$L[x(t)] = \lim_{T\to\infty}\int_{0^-}^{T} e^{-st}u_s(t-T_0)f(t-T_0)dt \tag{42.11}$$

$$= \lim_{T\to\infty}\int_{-T_0}^{T} e^{-s(\tau+T_0)}u_s(\tau)f(\tau)d\tau \tag{42.12}$$

$$= e^{-sT_0}\lim_{T\to\infty}\int_{0^-}^{T} e^{-s\tau}f(\tau)d\tau \tag{42.13}$$

$$= e^{-sT_0}F(s) \tag{42.14}$$

Here we defined $\tau = t - T_0$. Note, $u_s(\tau)f(\tau)$ is 0 for $\tau < T_0$.

– Transfer function for a delay

$$G_{\mathrm{d}}(s) = \frac{L[u_s(t-T_0)f(t-T_0)]}{L[f(t)]} \tag{42.15}$$

$$= e^{-sT_0} \tag{42.16}$$

- Delay and feedback control loops

– Introduction to delay

* "Such a delay in measuring, delay in controller action, or delay in actuator operation, and the like, is called transport lag or dead time [39]."
* All systems experience delay considering signals travel with finite speed (limited by the speed of light); see, for example, [37]
* Microcontrollers have clocks with finite cycle times

– Delays in the forward path

* Examples:
  - A microcontroller requires time to compute control actions given sensor readings.
  - An actuator is upstream from the output and sensor (e.g., a furnace is separated by a length of ductwork from the thermostat in a room; see, for example, [39]).

- * Delays in the forward path (i.e., C or P) show up in both the numerator and denominator of the closed loop transfer function.

$$\frac{Y(s)}{R(s)} = \frac{CP}{1 + CPH} \tag{42.17}$$

    - · Delays in the forward path change the dynamics of the system from their undelayed counterparts.
    - · Abrupt changes to the input have a delayed impact on the output (i.e., changes are detectable only after T_0).

– Delays in the feedback path
  - * Examples:
    - · A sensor measurement occurs downstream from the output (e.g., the thickness of metal being rolled is measured at a distance after the roller position; see, for example, [70, 38, 37]).
  - * Delays in the feedback path (i.e., H) only show up in the denominator of the closed loop transfer function.

$$\frac{Y(s)}{R(s)} = \frac{CP}{1 + CPH} \tag{42.18}$$

    - · Delays in the feedback path change the dynamics of the system from their undelayed counterparts.
    - · Unlike delays in the forward path, abrupt changes to the input can impact the output immediately when the delay is in the feedback path.

- Root locus perspective on delay
  - Computational methods can be used to construct the root locus for delays; see, for example, [8].
  - The Taylor series for a time delay is a polynomial of infinite order.
    * Taylor series
$$e^{-sT_0} = 1 - sT_0 + \frac{1}{2!}\left(sT_0\right)^2 - \frac{1}{3!}\left(sT_0\right)^3 + ... \tag{42.19}$$
    * The delay transfer function has no finite poles or zeros. However, it has infinite poles and zeros:
      · $\lim_{\mathrm{Re}\{s\}\to-\infty} e^{-sT_0} \to \infty$
      · $\lim_{\mathrm{Re}\{s\}\to\infty} e^{-sT_0} \to 0$
    * When a delay is incorporated into an open loop transfer function, the number of root locus branches is infinite; see, for example, [82].
  - Padé approximant
    * Definition - "It consists of matching the series expansion of the transcendental function [e^{-sT_0}] with the series expansion of a rational function ... [8]."
    * Common Padé approximants
      · First order lag - strictly proper (crude)
$$e^{-sT_0} \approx \frac{1}{1+T_0 s}; \tag{42.20}$$
      see, for example, [8, 39]
      · First order - biproper (good)
$$e^{-sT_0} \approx \frac{1-\frac{T_0}{2}s}{1+\frac{T_0}{2}s}; \tag{42.21}$$
      see, for example, [8, 39]
      · Second order - biproper (better)
$$e^{-sT_0} \approx \frac{1-\frac{T_0}{2}s+\frac{T_0^2}{12}s^2}{1+\frac{T_0}{2}s+\frac{T_0^2}{12}s^2}; \tag{42.22}$$
      see, for example, [8]
      · Higher order - biproper (best)
    * Poles and zeros
      · Biproper Padé approximants add right half plane zeros and symmetric left half plane poles

- The number of poles (and root locus branches) increases with the order of the approximation and the approximation improves with increasing order
- With right half plane zeros, the closed loop systems will be non-minimum phase.

– Delays tend to reduce the stability margin.

  * Gain margin -
    - Delays decrease the open loop phase and thereby reduce the phase crossover frequencies from the undelayed phase crossover frequencies.
    - Assuming the open loop transfer function has monotonically decreasing gain with increasing frequency and the undelayed gain margin is greater than 1 (or 0 dB), delay must reduce the gain margin.
    - Also, delays correspond to an infinite number of zeros in the right half plane. With an infinite number of branches ending in the right half plane, there exists an upper bound on the root locus parameter K to maintain stability.
  * Phase margin -
    - Delays tend to reduce the phase at the gain crossover frequencies.
    - Assuming the phase margin of the undelayed system is positive, a reduction in phase by more than the phase margin leads to instability.

Example 42.1:

Construct the root locus for a system with plant

$$P(s) = \frac{1}{s+1} \tag{42.23}$$

and controller

$$C(s) = e^{-s\frac{1}{3}}. \tag{42.24}$$

- Root locus for system without delay
- Root locus for system using Pade approximant - overlay on the exact root locus
- Exact root locus
  - Poles and zeros
    * one finite pole at -1
    * infinite poles at $\mathbb{R}\{s\} \to -\infty$
    * infinite zeros at $\mathbb{R}\{s\} \to \infty$
  - Segment of real axis to the left of -1

– Imaginary axis crossings must satisfy the angle criteria

$$
\begin{aligned}
1 + KL(j\omega) &= 0 && (42.25)\\
\angle KL(j\omega) &= \angle -1 && (42.26)\\
\angle L(j\omega) &= \pm\pi\{1,3,5,...\} && (42.27)\\
\angle e^{-j\omega\frac{1}{3}}\frac{1}{j\omega+1} &= \pm\pi\{1,3,5,...\} && (42.28)\\
-\omega\frac{1}{3} - \tan^{-1}\left(\frac{\omega}{1}\right) &= && (42.29)
\end{aligned}
$$

* For large $\omega \to \infty$

$$
\begin{aligned}
-\omega\frac{1}{3} - \frac{\pi}{2} &= -\pi\{1,3,5,...\} && (42.30)\\
\omega &= \frac{1}{\frac{1}{3}}\left(\pi\{1,3,5,...\} - \frac{\pi}{2}\right) && (42.31)\\
&= \frac{\pi}{2\cdot\frac{1}{3}}\left(2\{1,3,5,...\} - 1\right) && (42.32)\\
&= \frac{\pi}{2\cdot\frac{1}{3}}\{1,5,9,...\},\ \text{for large } \omega && (42.33)
\end{aligned}
$$

· Symmetry

$$
= \pm\frac{\pi}{2\cdot\frac{1}{3}}\{1,5,9,...\} \qquad (42.34)
$$

· Gain, K, at imaginary axis crossing

$$
\begin{aligned}
K &= \frac{1}{|L(j\omega)|} && (42.35)\\
&= \frac{1}{\left|e^{-j\omega\frac{1}{3}}\frac{1}{j\omega+1}\right|} && (42.36)\\
&= \frac{1}{\cancel{\left|e^{-j\omega\frac{1}{3}}\right|}\left|\frac{1}{j\omega+1}\right|} && (42.37)\\
&= \sqrt{\omega^2+1} && (42.38)\\
&= \sqrt{\left(\frac{\pi}{2\cdot\frac{1}{3}}\{1,5,9,...\}\right)^2+1} && (42.39)\\
&\approx \frac{\pi}{2\cdot\frac{1}{3}}\{1,5,9,...\},\ \text{for large } \omega && (42.40)
\end{aligned}
$$

increases with increasing ω

* For smaller ω use solver
  · specific values
    `fsolve(@(x) tan(x/3)+x,1*3*pi/2)` $\rightarrow 5.2745$
    `fsolve(@(x) tan(x/3)+x,5*3*pi/2)` $\rightarrow 23.6885$
  · Gain, K, at imaginary axis crossing

$$\begin{aligned}
K &= \frac{1}{|L(j\omega)|} && (42.41)\\
&= \frac{1}{\left|e^{-j\omega\frac{1}{3}}\frac{1}{j\omega+1}\right|} && (42.42)\\
&= \frac{1}{\cancel{\left|e^{-j\omega\frac{1}{3}}\right|}\left|\frac{1}{j\omega+1}\right|} && (42.43)\\
&= \sqrt{\omega^2+1} && (42.44)\\
&= \{\sqrt{5.2745^2+1}, \sqrt{23.6885+1}, ...\} && (42.45)\\
&= \{5.3685, 23.7096, ...\} && (42.46)
\end{aligned}$$

    increases with increasing ω
* Because the infinite open loop poles are at $\mathbb{R}\{s\} = -\infty$ and the finite open loop pole is at -1, the system must be stable for small K suggesting that the system is stable for $0 < K < 5.3685$.

– Break away and break in
  * Given a pole at -1 and a pole at -∞, there must be a break-away point

$$\begin{aligned}
\frac{d}{ds}(L(s)) &= 0 && (42.47)\\
\frac{d}{ds}\left(e^{-s\frac{1}{3}}\frac{1}{s+1}\right) &= && (42.48)\\
-\cancel{e^{-s\frac{1}{3}}}\frac{1}{3}\frac{1}{\cancel{s+1}} - e^{-s\frac{1}{3}}\frac{1}{(s+1)^{\cancel{2}}} &= && (42.49)\\
-(s+1)-3 &= && (42.50)\\
s &= -4 && (42.51)
\end{aligned}$$

– Root locus behavior at $\mathbb{R}\{s\} \rightarrow -\infty$ and $\mathbb{R}\{s\} \rightarrow \infty$
  * Consider

$$s = \lim_{\alpha\rightarrow\infty} \alpha + j\beta \qquad (42.52)$$

- angle criteria

$$\begin{aligned}
\angle L(s) &= \pm\pi\{1,3,5,...\} && (42.53)\\
\angle e^{-s\frac{1}{3}}\frac{1}{s+1} &= && (42.54)\\
\lim_{\alpha\to\infty}\angle e^{-(\alpha+j\beta)\frac{1}{3}}\frac{1}{\alpha+j\beta+1} &= && (42.55)\\
\lim_{\alpha\to\infty}\angle e^{-\alpha\frac{1}{3}}e^{-j\beta\frac{1}{3}}\frac{1}{\alpha+j\beta+1} &= && (42.56)\\
\lim_{\alpha\to\infty}\angle e^{-\alpha\frac{1}{3}}+\angle e^{-j\beta\frac{1}{3}}-\angle(\alpha+j\beta+1) &= && (42.57)\\
0-\beta\frac{1}{3}-\cancelto{0}{\lim_{\alpha\to\infty}\tan^{-1}\left(\frac{\beta}{\alpha+1}\right)} &= && (42.58)\\
\beta &= \pm\frac{\pi}{\frac{1}{3}}\{1,3,5,...\} && (42.59)
\end{aligned}$$

* Consider

$$s=\lim_{\alpha\to-\infty}\alpha+j\beta \tag{42.60}$$

- angle criteria

$$\begin{aligned}
\angle L(s) &= \pm\pi\{1,3,5,...\} && (42.61)\\
\angle e^{-s\frac{1}{3}}\frac{1}{s+1} &= && (42.62)\\
\lim_{\alpha\to-\infty}\angle e^{-(\alpha+j\beta)\frac{1}{3}}\frac{1}{\alpha+j\beta+1} &= && (42.63)\\
\lim_{\alpha\to-\infty}\angle e^{-\alpha\frac{1}{3}}e^{-j\beta\frac{1}{3}}\frac{1}{\alpha+j\beta+1} &= && (42.64)\\
\lim_{\alpha\to-\infty}\angle e^{-\alpha\frac{1}{3}}+\angle e^{-j\beta\frac{1}{3}}-\angle(\alpha+j\beta+1) &= && (42.65)\\
0-\beta\frac{1}{3}-\cancelto{\pi}{\lim_{\alpha\to-\infty}\tan^{-1}\left(\frac{\beta}{\alpha+1}\right)} &= && (42.66)\\
\beta &= \frac{\pi\pm\pi\{1,3,5,...\}}{\frac{1}{3}} && (42.67)\\
&= \pm\frac{\pi}{\frac{1}{3}}\{0,2,4,...\} && (42.68)
\end{aligned}$$

– See root locus in Figure 42.1

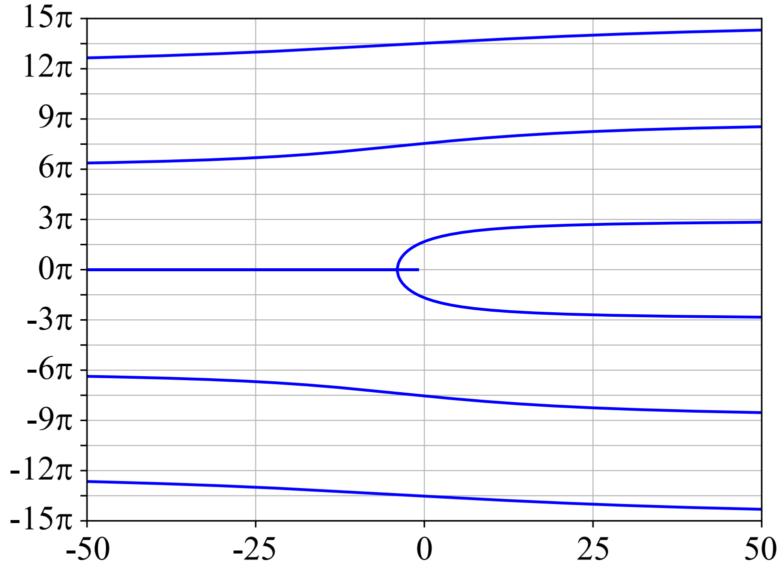


Figure 42.1: See, for example, [39, 37]

- Frequency domain perspective on delay

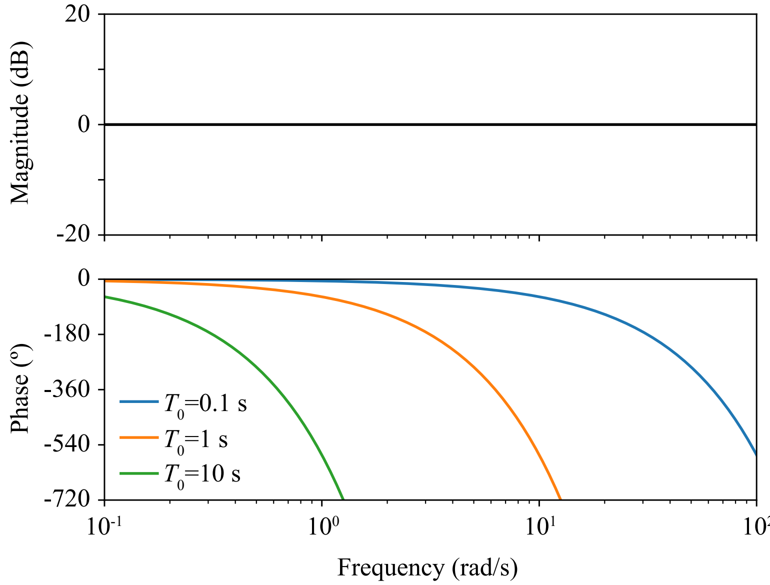


Figure 42.2:

  – Exact representation in the frequency domain
    ∗ Transfer function

$$G_{\mathrm{d}}(s) = e^{-sT_0} \tag{42.69}$$

    ∗ Magnitude

$$\begin{aligned} M(\omega) &= |G_{\mathrm{d}}(j\omega)| && (42.70)\\ &= \left|e^{-j\omega T_0}\right| && (42.71)\\ &= 1 && (42.72)\end{aligned}$$

    ∗ Phase

$$\begin{aligned} \phi(\omega) &= \angle G_{\mathrm{d}}(j\omega) && (42.73)\\ &= \angle e^{-j\omega T_0} && (42.74)\\ &= -\omega T_0 && (42.75)\end{aligned}$$

  – Notes
    ∗ It's easier to represent time delays in the frequency domain.

- * Time delays add an infinite number of phase crossover frequencies.
  * Time delays tend to reduce the phase margin suggesting that time delays tend to push systems toward instability.

- MATLAB® representations of delay
  - `[num,den]=pade(T_0,N)`
    * `num`: coefficients of the numerator for the Padé approximant
    * `den`: coefficients of the denominator for the Padé approximant
    * `T_0`: delay time
    * `N`: order of the approximant
  - LTI objects
    * `sys=tf(num,den,'InputDelay',T_0)`

- Design guidelines for systems with delay (Possibly move to previous lecture.)
  - Delay margin
    * Delay margin - the amount of delay that can be incorporated into a system before the sytem becomes unstable.
    * Relationship

$$\mathrm{DM} = \frac{\mathrm{PM}}{\omega_{\mathrm{gc}}}; \tag{42.76}$$

      see, for example, [37]. Note, the units must be consistent; for example, express PM in radians and ω_{gc} in radians/time. Also, does this formula always work given that stable systems can have negative PM?
    * Here, PM and ω_{gc} are the phase margin and gain crossover frequency, respectively, for the system without delay.
    * The delay here is in the forward path.
    * Proof
      · Delays reduce the phase angle following

$$\phi_{\mathrm{delay}} = -\omega T_0 \tag{42.77}$$

      · If we consider PM_0 as the phase margin in the absence of delay and PM the phase margin in the presence of a delay, then

$$\mathrm{PM} = \mathrm{PM}_0 - \omega_{\mathrm{gc}} T_0 \tag{42.78}$$

      where ω_{gc} is the gain crossover frequency of both the delayed and undelayed open loop transfer function

· Typically we want the phase margin to be greater than 0, therefore

$$0 < \mathrm{PM}_0 - \omega_{\mathrm{gc}} T_0 \tag{42.79}$$
$$T_0 < \frac{\mathrm{PM}_0}{\omega_{\mathrm{gc}}} \tag{42.80}$$

· If we define the delay margin, DM, as the largest delay the system can withstand before becoming unstable, then

$$\mathrm{DM} = \frac{\mathrm{PM}_0}{\omega_{\mathrm{gc}}}; \tag{42.81}$$

see, for example, [83, 84]

– Maximum gain crossover frequency (i.e., bandwidth) for systems with delay

∗ Consider the contribution to the PM of a system from the delay,

$$\mathrm{PM} = \mathrm{PM}_0 - \omega_{\mathrm{gc}} T_0 \tag{42.82}$$

Where ω_{gc} is the gain crossover frequency, PM_0 is the phase margin of the system excluding the delay, and PM is the phase margin including the delay.

∗ Phase margin budget

· Assume, for the sake of argument, that the plant (without delay) may be approximated by $P(s) \approx K\frac{1}{s}$. Therefore, the plant contributes 90° to the phase margin.

· Assume a second-order lead compensator, without delay, is used to increase the phase margin an additional 180°. (Note, a second order lead compensator increases high frequency gain making the system more susceptible to noise. Although this may not be the limit, there is likely a practical limit as to how much phase can be added.)

· In many applications we want a phase margin in the ballpark of 90°. Therefore, the delay can reduce the phase margin by approximately 180° at the gain crossover frequency.

$$\mathrm{PM} - \mathrm{PM}_0 = -\omega_{\mathrm{gc}} T_0 \tag{42.83}$$
$$\frac{\pi}{2} - \frac{3\pi}{2} = \tag{42.84}$$
$$-\pi = -\omega_{\mathrm{gc}} T_0 \tag{42.85}$$
$$\omega_{\mathrm{gc}} = \frac{\pi}{T_0} \tag{42.86}$$

∗ Support, in the literature, for this result

- "... it would be virtually impossible to stabilize a system ... with a crossover frequency greater than $\omega = \frac{5}{T_d}$, and it would be difficult for frequencies greater than $\omega \cong \frac{3}{T_d}$ [8]."
- "The fundamental limitation that delay causes is setting an upper bound on the achievable bandwidth as $BW < 1/T$ [37]."
- "The closed loop bandwidth is limited to be less than $1/\theta$, approximately [43]."

Chapter 43

Smith Predictor

- Smith Predictor; see, for example, [8]
  - look into the connection with internal model control (IMC); see, for example, [37]
  - Consider a plant with a pure delay

$$P(s) = P_0(s)e^{-sT_0} \tag{43.1}$$

    * We require $P_0(s)$ to be stable; see, for example, [85].
  - Using system identification methods, we approximate the plant as

$$P(s) \approx \tilde{P}_0(s)e^{-s\tilde{T}_0} \tag{43.2}$$

  - Assume we design a controller $C(s)$ to control our model of the plant, $\tilde{P}_0(s)$, neglecting delay.

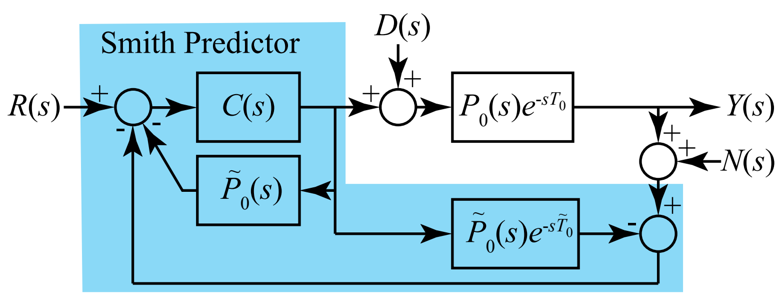


Figure 43.1: Here we define the Smith Predictor with the model of the plant, $\tilde{P}_0(s)e^{-s\tilde{T}_0}$, purposefully placed next to the actual plant, $P_0(s)e^{-sT_0}$.

  - We define the Smith Predictor as shown in Figure 43.1 with the model of the plant, $\tilde{P}_0(s)e^{-s\tilde{T}_0}$, purposefully placed next to the actual plant, $P_0(s)e^{-sT_0}$.

* This block diagram emphasizes the cancellation that occurs when the plant and model blocks perfectly match.
* The Smith Predictor cannot stabilize an unstable plant. Perfect cancellation effectively removes the plant from the feedback loop yielding open loop control (see Figure 43.2(b)), which cannot stabilize an unstable plant. (c) We avoid simplifying the Smith Predictor into a single block within the feedback loop because it hide potential unstable internal dynamics.

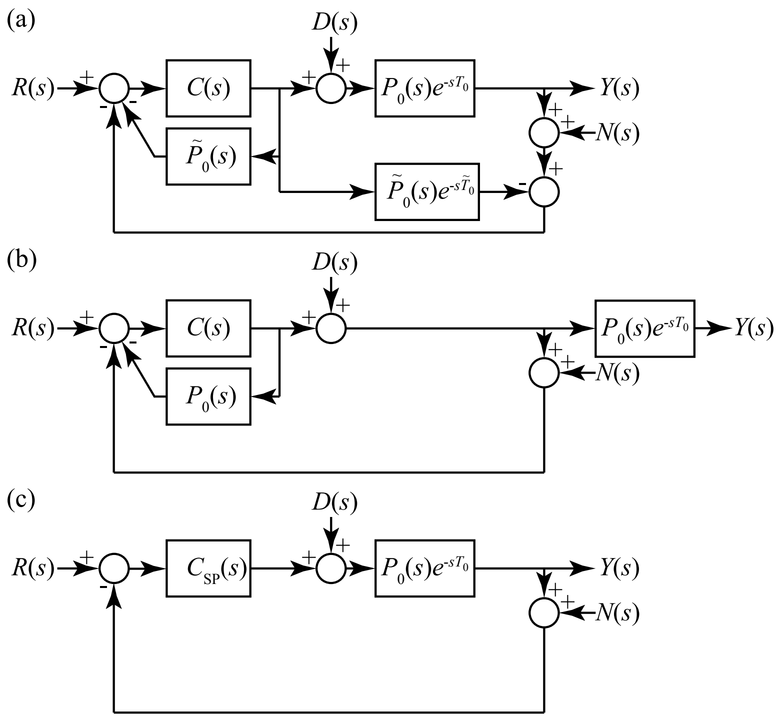


Figure 43.2: (a) Preferred representation of the Smith Predictor. (b) When model, $\tilde{P}_0(s)e^{-s\tilde{T}_0}$, and plant, $P_0(s)e^{-sT_0}$, perfectly match, cancellation yields open loop control. (c) Simplifying the Smith Predictor into a single block hides potential instabilities.

* Therefore, we avoid representing the Smith Predictor as a single block or transfer function (see Figure 43.2(c)) because it hides potential unstable internal dynamics. (Similarly, to avoid hiding unstable internal dynamics, we don't remove unstable pole-zero cancellations from transfer functions.)
* However, for some analysis, we may get away using the transfer function

$$C_{\text{SP}}(s) \quad = \quad \frac{C(s)}{1 + C(s)\tilde{P}_0(s)\left(1 - e^{-s\tilde{T}_0}\right)}; \tag{43.3}$$

see, for example, [8, 84, 39].

- Closed loop transfer function ($R(s)$ to $Y(s)$)
  * Closed loop transfer function (assuming $P_0(s)$ and $\tilde{P}_0(s)$ are stable)

$$\frac{Y(s)}{R(s)} = \frac{C_{\text{SP}}(s)P(s)}{1+C_{\text{SP}}(s)P(s)} \tag{43.4}$$

$$= \frac{\frac{C(s)}{1+C(s)\tilde{P}_0(s)\left(1-e^{-s\tilde{T}_0}\right)}P_0(s)e^{-sT_0}}{1+\frac{C(s)}{1+C(s)\tilde{P}_0(s)\left(1-e^{-s\tilde{T}_0}\right)}P_0(s)e^{-sT_0}} \tag{43.5}$$

$$= \frac{C(s)P_0(s)e^{-sT_0}}{1+C(s)\tilde{P}_0(s)\left(1-e^{-s\tilde{T}_0}\right)+C(s)P_0(s)e^{-sT_0}} \tag{43.6}$$

  * If the model matches the plant (i.e., if $\tilde{P}(s)=P(s)$ and $\tilde{T}_0=T_0$), the closed loop transfer function reduces to

$$\left.\frac{Y(s)}{R(s)}\right|_{\tilde{P}(s)=P(s),\tilde{T}_0=T_0} = \frac{C(s)P_0(s)}{1+C(s)P_0(s)}e^{-sT_0} \tag{43.7}$$

    · In effect, the delay has been removed from the feedback loop; see, for example, [8].

- Disturbance to output transfer function
  * Disturbance to output transfer function(assuming $P_0(s)$ and $\tilde{P}_0(s)$ are stable)

$$\frac{Y(s)}{D(s)} = \frac{P(s)}{1+C_{\text{SP}}(s)P(s)} \tag{43.8}$$

$$= \frac{P_0(s)e^{-sT_0}}{1+\frac{C(s)}{1+C(s)\tilde{P}_0(s)\left(1-e^{-s\tilde{T}_0}\right)}P_0(s)e^{-sT_0}} \tag{43.9}$$

$$= \frac{\left(1+C(s)\tilde{P}_0(s)\left(1-e^{-s\tilde{T}_0}\right)\right)P_0(s)e^{-sT_0}}{1+C(s)\tilde{P}_0(s)\left(1-e^{-s\tilde{T}_0}\right)+C(s)P_0(s)e^{-sT_0}} \tag{43.10}$$

  * If the model matches the plant (i.e., if $\tilde{P}(s)=P(s)$ and $\tilde{T}_0=T_0$), the transfer function reduces to

$$\frac{Y(s)}{D(s)} = \frac{P_0(s)e^{-sT_0}+C(s)P_0^2(s)\left(1-e^{-sT_0}\right)e^{-sT_0}}{1+C(s)P_0(s)} \tag{43.11}$$

    · Given the term that includes $P_0^2(s)$ in the numerator, we conclude that the poles of the plant remain part of the disturbance transfer function. (When we factor these two sets of poles out of the numerator, one set remains a factor of the denominator.)

- “... the poles of $P(s)$ cannot be eliminated from the disturbance rejection transfer function ... except for a pole at $s = 0$ [83].”

  Even if a pole at 0 is ‘cancelled’ from the disturbance rejection transfer function, it seems initial conditions may excite the unstable mode and the system remains unstable.
- Note, a pole at $s = 0$ may be eliminated in the final value theorem of the step disturbance:

$$\begin{aligned} y_{\text{ss}} &= \lim_{s\to 0} \cancel{s}\frac{Y(s)}{D(s)}\frac{1}{\cancel{s}} && (43.12)\\ &= \lim_{s\to 0} \frac{P_0(s)e^{-sT_0} + C(s)P_0^2(s)\left(1 - e^{-sT_0}\right)e^{-sT_0}}{1 + C(s)P_0(s)} && (43.13)\\ &= \lim_{s\to 0} P_0(s)e^{-sT_0}\frac{1 + C(s)P_0(s)\left(1 - e^{-sT_0}\right)}{1 + C(s)P_0(s)} && (43.14)\\ & && (43.15) \end{aligned}$$

The term $\lim_{s\to 0}\left(1 - e^{-sT_0}\right)$ can cancel one pole of the plant at $s = 0$

– Sensitivity - (The following section needs to be verified)

  ∗ Sensitivity of the closed loop transfer function to changes in $\tilde{T}_0$ (assuming $P_0(s)$ and $\tilde{P}_0(s)$ are stable)

$$\begin{aligned} S_{\tilde{T}_0}^{\frac{Y(s)}{R(s)}} &= \left.\frac{\partial \frac{Y(s)}{R(s)}}{\partial \tilde{T}_0}\frac{\tilde{T}_0}{\frac{Y(s)}{R(s)}}\right|_{\tilde{P}(s)=P(s),\tilde{T}_0=T_0} && (43.16)\\ &= -\left.\frac{C(s)P_0(s)e^{-sT_0}C(s)\tilde{P}_0(s)e^{-s\tilde{T}_0}s}{\left(1 + C(s)\tilde{P}_0(s)\left(1 - e^{-s\tilde{T}_0}\right) + C(s)P_0(s)e^{-sT_0}\right)^2}\frac{\tilde{T}_0}{\frac{Y(s)}{R(s)}}\right|_{\tilde{P}(s)=P(s),\tilde{T}_0=T_0} && (43.17)\\ &= -\frac{C(s)P_0(s)}{1 + C(s)P_0(s)}e^{-sT_0}sT_0 && (43.18) \end{aligned}$$

- Assuming the bandwidth of the system without delay is ω_{BW} and evaluating the magnitude of the sensitivity for $\omega < \omega_{\text{BW}}$,

$$\begin{aligned} \left|S_{\tilde{T}_0}^{\frac{Y(s)}{R(s)}}(j\omega)\right| &\approx \left|-\frac{C(j\omega)P_0(j\omega)}{1 + C(j\omega)P_0(j\omega)}e^{-j\omega T_0}j\omega T_0\right|,\ \omega < \omega_{\text{BW}} && (43.19)\\ &\approx \cancelto{\approx 1}{\left|\frac{C(j\omega)P_0(j\omega)}{1 + C(j\omega)P_0(j\omega)}\right|}\cancelto{\approx 1}{\left|e^{-j\omega T_0}\right|}\left|j\omega T_0\right|,\ \omega < \omega_{\text{BW}} && (43.20)\\ &\approx \omega T_0,\ \omega < \omega_{\text{BW}} && (43.21) \end{aligned}$$

        - This sensitivity will approximately reach or exceed 1 if $\omega_{\mathrm{BW}} T_0 \geq 1$. Consequently, the closed loop bandwidth must be limited to avoid excessive sensitivity.
    - Sensitivity of the closed loop transfer function to changes in $\tilde{P}_0(s)$ (assuming $P_0(s)$ and $\tilde{P}_0(s)$ are stable)

$$
\begin{aligned}
S_{\tilde{P}_0(s)}^{\frac{Y(s)}{R(s)}} &= \left. \frac{\partial \frac{Y(s)}{R(s)}}{\partial \tilde{P}_0(s)} \frac{\tilde{P}_0(s)}{\frac{Y(s)}{R(s)}} \right|_{\tilde{P}(s)=P(s), \tilde{T}_0=T_0} && (43.22)\\
&= \left. -\frac{C(s)P_0(s)e^{-sT_0}C(s)C(s)(1-e^{-s\tilde{T}_0})}{\left(1+C(s)\tilde{P}_0(s)\left(1-e^{-s\tilde{T}_0}\right)+C(s)P_0(s)e^{-sT_0}\right)^2} \frac{\tilde{P}_0(s)}{\frac{Y(s)}{R(s)}} \right|_{\tilde{P}(s)=P(s), \tilde{T}_0=T_0} && (43.23)\\
&= \frac{C(s)P_0(s)}{1+C(s)P_0(s)}(e^{-sT_0}-1) && (43.24)
\end{aligned}
$$

        - Assuming the bandwidth of the system without delay is ω_{BW} and evaluating the magnitude of the sensitivity for $\omega < \omega_{\mathrm{BW}}$,

$$
\begin{aligned}
\left| S_{\tilde{P}_0(s)}^{\frac{Y(s)}{R(s)}}(j\omega) \right| &\approx \left| -\frac{C(j\omega)P_0(j\omega)}{1+C(j\omega)P_0(j\omega)}(e^{-j\omega T_0}-1) \right|, \ \omega < \omega_{\mathrm{BW}} && (43.25)\\
&\approx \left| \cancelto{\approx 1}{\frac{C(j\omega)P_0(j\omega)}{1+C(j\omega)P_0(j\omega)}} \right| \left| e^{-j\omega T_0}-1 \right|, \ \omega < \omega_{\mathrm{BW}} && (43.26)\\
&\approx \left| e^{-j\omega T_0}-1 \right|, \ \omega < \omega_{\mathrm{BW}} && (43.27)
\end{aligned}
$$

        - This sensitivity will exceed 1 if $\omega_{\mathrm{BW}} T_0 \geq 1$. Consequently, the closed loop bandwidth needs to be limited to avoid excessive sensitivity.
- Limitations
    - Plant must be stable and non integrating
        - Smith predictor "... does not work for integrating processes [84]."
        - "... SP controller cannot be used, at least in its original form, with integrative processes [83]."
        - "... the poles of $P(s)$ cannot be eliminated from the disturbance rejection transfer function ... except for a pole at $s = 0$ [83]."
    - Sensitivity and robustness
        - "... the closed-loop poles of $\frac{Y(s)}{R(s)}$ cannot be arbitrarily defined but the robustness condition must be used to impose an upper limit on the closed-loop performance [83]."
        - "... when designing a Smith predictor using only the classical robustness margins (GM and PM) you easily end up with an aggressive controller with very small DM [84]."

* Due to its limitations and the performance possible with PID controllers, some authors argue we should "forget the Smith predictor"; see, for example, [84].

Part XIV

Additional Topics

Chapter 44

Nonlinear Systems and Linearization

- Introduction
  - In general, linear differential equations are easier to solve than nonlinear differential equations.
  - Often we are interested in small motions about an equilibrium point (θ_E). In this case we can approximate a nonlinear ordinary differential equation with a linear ordinary differential equation by linearizing about the equilibrium.

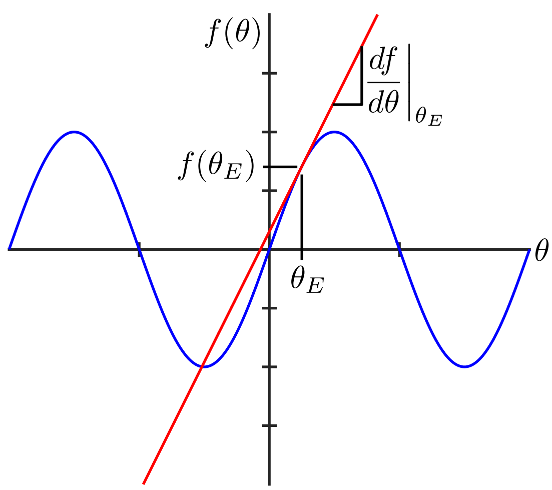


Figure 44.1:

  - Linearizing a nonlinear function about a point θ_E consists of finding a line tangent to the function at the point. Such a line approximates the function in the neighborhood of the point. See Figure 44.1.

- Function of one variable
  - The Taylor series expansion for a function $f(\theta)$ about some value θ_E can be written as

$$\begin{aligned} f(\theta) &= f(\theta_E + x) && (44.1) \\ f(\theta_E + \epsilon_\theta) &= f(\theta_E) + \left(\frac{df}{d\theta}\right)\bigg|_{\theta=\theta_E} x + \frac{1}{2}\left(\frac{d^2 f}{d\theta^2}\right)\bigg|_{\theta=\theta_E} x^2 \\ &\quad + \sum_{k=3}^{\infty} \frac{1}{k!}\left(\frac{d^k f}{d\theta^k}\right)\bigg|_{\theta=\theta_E} x^k && (44.2) \end{aligned}$$

  - Here we use x to represent small departures from the constant value θ_E
  - Linearization approximates $f(\theta)$ using the first two terms of a Taylor series expansion. This is justified when θ is close θ_E (i.e., x is small such that x^k is very small for $k > 1$. Consequently the linearization of a function $f(\theta)$ near θ_E is

$$f(\theta) \approx f(\theta_E) + \left(\frac{df}{d\theta}\right)\bigg|_{\theta=\theta_E} x \qquad (44.3)$$

  - Examples. Linearize the following functions $f(\theta)$ about $\theta = \theta_E$:
    * $f(\theta) = \sin(\theta) \approx \sin(\theta_E) + \cos(\theta_E)x$
    * $f(\theta) = \cos(\theta) + 3\theta + 4 \approx \cos(\theta_E) + \sin(\theta_E)x + 3(\theta_E + x) + 4$
    * $f(\theta) = 3\theta^2 \approx 3\theta_E^2 + 6\theta_E x$

- Function of two variables
  - The Taylor series expansion for a function of two variables $f(\theta, \phi)$ about some point $(\theta_E,\ \phi_E)$ can be written as

$$\begin{aligned} f(\theta,\ \phi) = f(\theta_E,\ \phi_E) + \left(\frac{\partial f}{\partial \theta}\right)\bigg|_{\theta=\theta_E,\ \phi=\phi_E} x_\theta + \left(\frac{\partial f}{\partial \phi}\right)\bigg|_{\theta=\theta_E,\ \phi=\phi_E} x_\phi \\ + O\left(x_\theta^2,\ x_\phi^2,\ x_\theta x_\phi\right) \qquad (44.4) \end{aligned}$$

  - Linearization approximates $f(\theta,\ \phi)$ using the first three terms of a Taylor series expansion. This is justified when θ is close θ_E and ϕ is close ϕ_E. Consequently the linearization of a function $f(\theta,\ \phi)$ near $(\theta_E,\ \phi_E)$ is

$$\begin{aligned} f(\theta,\ \phi) \quad &\approx \quad f(\theta_E,\ \phi_E) + \left(\frac{\partial f}{\partial \theta}\right)\Bigg|_{\theta=\theta_E,\ \phi=\phi_E} x_\theta \\ &\quad + \left(\frac{\partial f}{\partial \phi}\right)\Bigg|_{\theta=\theta_E,\ \phi=\phi_E} x_\phi \end{aligned} \tag{44.5}$$

- Vector functions
  - Linearization of a vector function $\vec{F}(\vec{\theta})$ of the vector $\vec{\theta}$ is based upon its Taylor series expansion. The first two terms of a Taylor series expansion for a vector function $\vec{F}(\vec{\theta})$ about some value $\vec{\theta}_E$ can be written as

$$\vec{F}(\vec{\theta}) \approx \vec{F}(\vec{\theta}_E) + \frac{\partial \vec{F}}{\partial \vec{\theta}}\Bigg|_{\vec{\theta}=\vec{\theta}_E} \vec{x} \tag{44.6}$$

$$\vec{F}(\vec{\theta}) \approx \vec{F}(\vec{\theta}_E) + \left[\begin{array}{ccc} \frac{\partial F_1}{\partial \theta_1} & \frac{\partial F_1}{\partial \theta_2} & \ldots \\ \frac{\partial F_2}{\partial \theta_1} & \frac{\partial F_2}{\partial \theta_2} & \ldots \\ \ldots & \ldots & \end{array}\right]\Bigg|_{\vec{\theta}=\vec{\theta}_E} \vec{x} \tag{44.7}$$

  - Here, the matrix is called the Jacobian (J)

$$J = \left[\begin{array}{ccc} \frac{\partial F_1}{\partial \theta_1} & \frac{\partial F_1}{\partial \theta_2} & \ldots \\ \frac{\partial F_2}{\partial \theta_1} & \frac{\partial F_2}{\partial \theta_2} & \ldots \\ \ldots & \ldots & \end{array}\right]\Bigg|_{\vec{\theta}=\vec{\theta}_E} \tag{44.8}$$

  - Examples. Linearize the following vector functions $\vec{F}(\vec{\theta})$ about $\vec{\theta}_E$ given a two-dimensional vector valued variable $\vec{\theta}$

$$\vec{\theta} \quad = \quad \left[\begin{array}{c} \theta_1 \\ \theta_2 \end{array}\right] \tag{44.9}$$

with

$$\vec{\theta}_e \quad = \quad \left[\begin{array}{c} \theta_{1E} \\ \theta_{2E} \end{array}\right] \tag{44.10}$$

and two dimensional vector function $\vec{F}(\vec{\theta})$

$$\vec{F}(\vec{\theta}) \quad = \quad \left[\begin{array}{c} F_1(\vec{\theta}) \\ F_2(\vec{\theta}) \end{array}\right] \tag{44.11}$$

$$\quad = \quad \left[\begin{array}{c} F_1(\theta_1, \theta_2) \\ F_2(\theta_1, \theta_2) \end{array}\right] \tag{44.12}$$

∗ example 1

$$\vec{F}(\vec{\theta}) = \begin{bmatrix} \theta_1^2 + \theta_2 + \pi \\ 4\theta_1 + \theta_2^3 \end{bmatrix} \tag{44.13}$$

$$\vec{F}(\vec{\theta}_E) = \begin{bmatrix} \theta_{1E}^2 + \theta_{2E} + \pi \\ 4\theta_{1E} + \theta_{2E}^3 \end{bmatrix} \tag{44.14}$$

$$J = \left.\begin{bmatrix} \frac{\partial F_1}{\partial \theta_1} & \frac{\partial F_1}{\partial \theta_2} \\ \frac{\partial F_2}{\partial \theta_1} & \frac{\partial F_2}{\partial \theta_2} \end{bmatrix}\right|_{\vec{\theta}=\vec{\theta}_E} \tag{44.15}$$

$$= \left.\begin{bmatrix} 2\theta_1 & 1 \\ 4 & 3\theta_2^2 \end{bmatrix}\right|_{\vec{\theta}=\vec{\theta}_E} \tag{44.16}$$

$$= \begin{bmatrix} 2\theta_{1E} & 1 \\ 4 & 3\theta_{2E}^2 \end{bmatrix} \tag{44.17}$$

$$\vec{F}(\vec{\theta}) \approx \vec{F}(\vec{\theta}_E) + J\vec{\epsilon}_\theta \tag{44.18}$$

$$\approx \begin{bmatrix} \theta_{1E}^2 + \theta_{2E} + \pi \\ 4\theta_{1E} + \theta_{2E}^3 \end{bmatrix} + \begin{bmatrix} 2\theta_{1E} & 1 \\ 4 & 3\theta_{2E}^2 \end{bmatrix} \vec{x} \tag{44.19}$$

- ODE linearization about equilibrium point
  - Solve for the value of the dependent variables at static equilibrium by setting all time derivatives to 0.
  - Substitute the equilibrium value plus a small amount ($\theta_E + x$) of the dependent variable and linearize.
  - Note, the time derivative of the dependent variable will look like $\dot{\theta} = \dot{x}$ because the equilibrium value is constant.

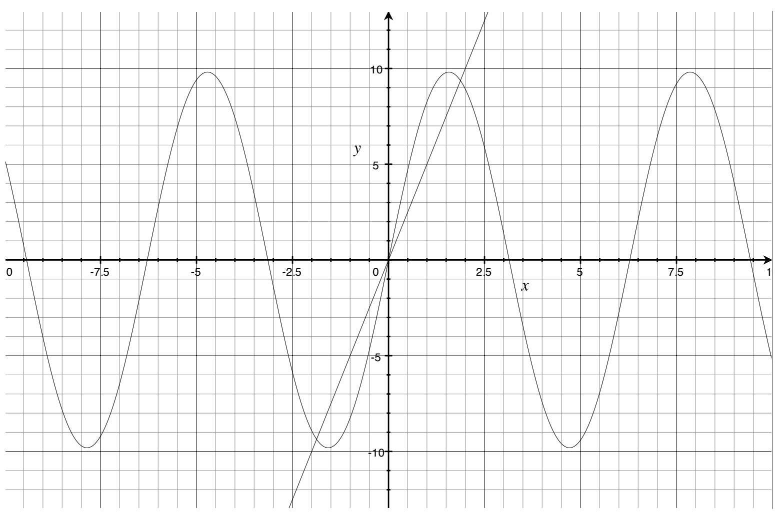
- State variable linearization about equilibrium point
  - Solve for the equilibrium value of the state variables by setting all time derivatives to 0.
  - Linearize the RHS of the state equations about the equilibrium value of interest.
  - Substitute the linearized RHS into the state variable model and write the linearized state variable model.

Example 44.1: Linearization of inverted pendulum

- need to add a forcing term to the system.
- Equation of motion
$$ml^2\ddot{\theta} + K_T\theta - mgl\sin(\theta) = 0 \tag{44.20}$$
- Find equilibrium value of the dependent variable θ (i.e., find θ_E):
  - let $l = 1$, $m = 1$, $g = 9.81$, $K_T = 5$
$$K_T\theta_E - mgl\sin(\theta_E) = 0 \tag{44.21}$$
  - Solving yields: $\theta_E = \{-1.8731,\ 0,\ 1.8731\}$
- Consider x to be a small angular displacement from the equilibrium value, θ_E, and linearize.

$$\begin{aligned}
ml^2\ddot{\theta} + K_T\theta - mgl\sin(\theta) &= 0 \quad (44.22)\\
ml^2\ddot{\theta}\Big|_{\ddot{\theta}=\cancelto{0}{\ddot{\theta}_E}} + \left.\frac{\partial ml^2\ddot{\theta}}{\partial\ddot{\theta}}\right|_{\ddot{\theta}=\ddot{\theta}_E}\ddot{x} + K_T\theta|_{\theta=\theta_E} + \left.\frac{\partial K_T\theta}{\partial\theta}\right|_{\theta=\theta_E}x & \\
- mgl\sin(\theta)|_{\theta=\theta_E} - \left.\frac{\partial mgl\sin(\theta)}{\partial\theta}\right|_{\theta=\theta_E}x &\approx 0 \quad (44.23)\\
ml^2\ddot{x} + \cancel{K_T\theta_E} + K_Tx - \cancel{mgl\sin(\theta_E)} - mgl\cos(\theta_E)x &\approx \quad (44.24)\\
ml^2\ddot{x} + K_Tx - mgl\cos(\theta_E)x &\approx 0 \quad (44.25)
\end{aligned}$$

- Note we were able to cancel out the equilibrium equation



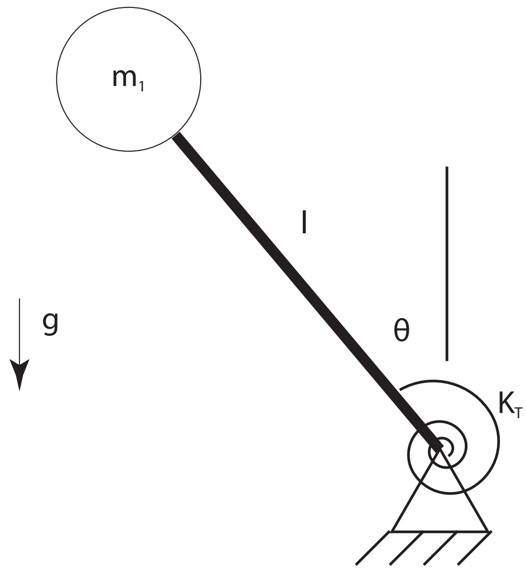


Figure 44.2:

Example 44.2: Linearization of the state variable model for an inverted pendulum

- State variable model of the inverted pendulum ($z_1 = \theta$ and $z_2 = \dot{\theta}$)

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{K_T}{ml^2} z_1 + \frac{g}{l}\sin(z_1) \end{bmatrix} \tag{44.26}$$

- Solve for the equilibrium value of the state variables (use the following equations to solve for z_{1E} and z_{2E})

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z_{2E} \\ -\frac{K_T}{ml^2} z_{1E} + \frac{g}{l}\sin(z_{1E}) \end{bmatrix} \tag{44.27}$$

- Linearize the RHS of the state equations about the equilibrium value of interest

(z_{1E} and z_{2E})

$$
\begin{aligned}
\begin{bmatrix} z_2 \\ -\frac{K_T}{ml^2} z_1 + \frac{g}{l}\sin(z_1) \end{bmatrix} &= \cancelto{0 \text{ by equilibrium}}{\begin{bmatrix} z_{2E} \\ -\frac{K_T}{ml^2} z_{1E} + \frac{g}{l}\sin(z_{1E}) \end{bmatrix}} \\
&\quad + \begin{bmatrix} 0 & 1 \\ -\frac{K_T}{ml^2} + \frac{g}{l}\cos(z_{1E}) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} && (44.28) \\
&= \begin{bmatrix} 0 & 1 \\ -\frac{K_T}{ml^2} + \frac{g}{l}\cos(z_{1E}) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} && (44.29)
\end{aligned}
$$

- Now substitute back into the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_T}{ml^2} + \frac{g}{l}\cos(z_{1E}) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{44.30}$$

- Compare ode45 solution to nonlinear inverted pendulum with linearization

Example 44.3: Linearization of balance beam

Example 44.4: Linearization of nonlinear mass-spring

- Find equation of motion for Nonlinear spring $F = k_1 x + k_3 x^3$
  - Coordinate x from unstretched length of spring
  - Draw a free body diagram
  - Write $\sum F = ma$

$$-k_1 x - k_3 x^3 + mg = m\ddot{x} \tag{44.31}$$

  - Kinematics here are trivial
  - Rewrite equation of motion

$$m\ddot{x} + k_1 x + k_3 x^3 = mg \tag{44.32}$$

- Solve for the equilibrium value of the dependent variables (x). Let's assume $m = 1$, $k_1 = -10$, $k_3 = 1$, $g = 9.81$
  - Let $m = 1$, $k_1 = -10$, $k_3 = 1$, $g = 9.81$

$$k_1 x_E + k_3 x_E^3 = mg \tag{44.33}$$

- $x_E = \{-2.4481, -1.1223, 3.5704\}$

- Substitute the equilibrium value plus a small amount ($x_E + \epsilon_x$) of the dependent variable and linearize.

$$-k_1 x - k_3 x^3 + mg = m\ddot{x} \tag{44.34}$$

$$-\cancel{k_1 x_E} - k_1 \epsilon_x - \cancel{k_3 x_E^3} - \left.\frac{\partial k_3 x^3}{\partial x}\right|_{x_E} \epsilon_x + \cancel{mg} = \cancel{m\ddot{x}_E} + \left.\frac{\partial m\ddot{x}}{\partial \ddot{x}}\right|_{x_E} \ddot{\epsilon}_x \tag{44.35}$$

$$-k_1 \epsilon_x + 3k_3 x_E^2 \epsilon_x \approx m\ddot{\epsilon}_x \tag{44.36}$$

Chapter 45

Actuator Limitations

- “actuator and sensor limitations” - See, for example, [37]
  - “Maximum actuator movement” - saturation
    - “In a situation in which maximal actuator movement will be a restriction it is usually the case that the closed-loop bandwidth is much larger than the plant bandwidth [37].”
    - “On the other hand it should be noted that sometimes actuator limitation turns out to have a good effect on reducing the large overshoots in the output at the expense of only a slightly slower response [37].”
  - “Minimum actuator movement”
  -
- Actuator saturation
  - “In a situation in which maximal actuator movement will be a restriction it is usually the case that the closed-loop bandwidth is much larger than the plant bandwidth [37].”
  - “On the other hand it should be noted that sometimes actuator limitation turns out to have a good effect on reducing the large overshoots in the output at the expense of only a slightly slower response [37].”
  - Saturation opens the loop
    - “Whenever actuator saturation happens, the control signal to the process stops changing and the feedback path is effectively opened [8].”
    - At the actuator limit, “... the feedback loop is broken and the system runs in open loop because the actuator remains at its limit independently of the process output as long as the actuator remains saturated [5].”
  - Integral windup; see, for example, [8, 5]

    * The term integral windup refers to a situation in which the integral of the error grows relatively large in magnitude. This occurs when the sign of the error signal remains unchanged for a period of time due to actuator saturation (i.e., the actuator is unable to quickly reduce the error because of saturation). Windup can lead to significant overshoot when the controller continues to call for significant integral action after the error changes sign. See, for example, [5, 8].
  - A crude model for actuator saturation is that it reduces the effective loop gain is this always true?
    * check out [8]
    * set up a python script to adjust actuator limit and see change in overshoot and settling time...
  - see the example on the python control library page `https://python-control.readthedocs.io/en/latest/cruise.html`

- Different transfer functions including $\frac{U(s)}{R(s)}$
  - How does $U(s)$ compare to the desired signal (considering open loop control and the desired pole locations)

- Simulink for simulation

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Part XV

Appendices

Appendix A

Dr. Lillian's Philosophy on Teaching and Learning

- Goal: maximize the efficiency of student learning
  - Learning requires work (it's unavoidable)
    - No one would expect to finish a marathon by watching a few YouTube videos. One must train regularly to build up strength and endurance.
  - Class time should be used to learn from each other, practice applying new knowledge, and build lifelong relationships
    - The potential for learning in the classroom is unparalleled because of the people it brings together into one room.
      - Each student (and instructor) brings knowledge, experience, and perspective from which everyone can learn.
      - The internet with all of its text, pictures, and videos and/or the library with its shelves of books cannot match the potential for learning available in the classroom.
      - People (students and instructor) are the most valuable tool we have in the learning process.
    - Practicing in the classroom offers opportunities to ask questions, receive immediate feedback, and learn from eachother
    - Learning is more fun and effective among friends. Positive relationships, built in the classroom, can last far beyond graduation.
  - One-way learning activities should be completed outside the classroom
    - One-way learning activities include copying notes or following along as a problem is solved. (The first exposure to new material is often one-way.)
    - Reading textbooks (or course notes) and/or watching video lectures outside of class should replace traditional one-way classroom activities.

    * Technology such as the copy machine, course website, and email can be used to replace some traditional one-way activities.
  - Two-way learning opportunities exist outside of the classroom
    * 20 Minute Rule - If you find yourself stuck on a specific problem for more than about 20 minutes, get help.
    * Office hours -
    * Study groups

- Instructor's responsibility
  - Maintain a positive learning environment of respect and inclusion
  - Be prepared for and direct classroom activities
  - Develop and curate quality resources to facilitate learning both inside and outside the classroom
  - Be available for office hours
  - Assess learning fairly and promptly

- Students' responsibility
  - Build positive relationships with classmates
  - Prepare for and fully participate in classroom activities (e.g., ask questions for which the answer might be obvious to others and give answers even if they might be wrong).
    * "... we find that students in the active classroom learn more, but they feel like they learn less [86]."
    * "Students' judgments of their own learning are often misled by intuitive yet false ideas about how people learn [87]."
    * How can you tell if you are learning?
      · What does it feel like when you are learning?
      · What does it feel like when you build muscle mass?
  - Seek out, identify, and use the best resources (classmates, office hours, course notes, textbooks, video lectures, podcasts, etc.)
  - Practice
  - Seek out, identify, own, and overcome deficiencies in understanding
    * Seek out opportunities to test the limits of your understanding in small groups of students and/or with the instructor.
    * Study and learn from graded feedback on assignments, quizzes, and exams.
    * Critically evaluate your own understanding.

- · When viewing solutions, for example, avoid falsely concluding that problems are easy.
- · Consider the following extreme example: "In 1995, McArthur Wheeler walked into two Pittsburgh banks and robbed them in broad daylight, with no visible attempt at disguise. He was arrested later that night, less than an hour after videotapes of him taken from surveillance cameras were broadcast on the 11 o'clock news. When police later showed him the surveillance tapes, Mr. Wheeler stared in incredulity. "But I wore the juice," he mumbled. Apparently, Mr. Wheeler was under the impression that rubbing one's face with lemon juice rendered it invisible to videotape cameras (Fuocco, 1996) [88]."
- · Example: The photograph of Figure A.1 suggests that a student likely felt confident locking up his/her bicycle. How did he/she do?

– Feynman technique?

– Workload expectation

- ∗ Purdue University: "Another widely repeated standard states that each in-class hour of college work should require two hours of preparation or other outside work [Semester Credit Hours Guidelines. Purdue.edu/registrar. 6 Jan. 2023. https://www.purdue.edu/registrar/forms/Semester_Credit_Hours_Guidelines.html]."
- ∗ Purdue mechanical engineering expectation: approximately 3 hours outside class/week/credit [Plan of Study 2022. engineering.purdue.edu/ME. 6 Jan. 2023. https://engineering.purdue.edu/ME/Under

Figure A.1: But my bike was locked up!

Appendix B

Dimensions and Units

- Unit systems
  - Fundamental units

Dimension	SI System
Length	meter (m)
Time	second (s)
Mass	kilogram (kg)
Current	Ampere (A)
Temperature	Kelvin (K)

  - Derived units

Dimension	SI System
Charge	coulomb (C, 1 C=1 A·s)
Energy	joule (J, 1 J=1 N·m)
Electrical potential	volt (V, 1 V=1 $\frac{\text{J}}{\text{C}}$)
Power	watt (W, 1 W=1 $\frac{\text{J}}{\text{s}}$)

- Mathematical operations
  - Quantities to be added or subtracted must have the same dimensions.
  - The dimensions of quantities that are multiplied (divided) are themselves multiplied (divided)
  - Arguments to sin(), cos(), tan(), exp(), ln() are dimensionless. Note, radians are considered to be dimensionless.

  - Exponents are dimensionless
  - Quantities that are equivalent have equivalent dimensions. An equation must have the same dimensions on both sides.

- Advice
  - For every equation that you write, verify that the dimensions on both sides agree.
  - Determine the dimensions associated with all the quantities in your problem. Dimensions for common quantities in mechanical systems

Quantitiy	Dimension
Mass (m)	Mass
Spring constant (k)	$\frac{\text{Force}}{\text{Length}}$
Damping coefficient (c)	$\frac{\text{Force}\cdot\text{Time}}{\text{Length}}$
Mass moment of inertia (I_G)	$\text{Mass} \cdot \text{Length}^2$
Torsional spring constant (k_T)	$\text{Force} \cdot \text{Length}$
Torsional damping coefficient (c_T)	$\text{Force} \cdot \text{Length} \cdot \text{Time}$

Appendix C

Method of Undetermined Coefficients (or Trial Solution Method)

- Introduction
  - See, for example, [35]
  - Some Preliminaries
    - The exponential function (e^t) is the only non-trivial function whose derivative is itself ($\dot{y} = y$)
    - The time derivative of e^{st} is se^{st} for constant s
    - Euler equation:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{C.1}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{C.2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{C.3}$$

  - The Method undetermined coefficients is for linear time invariant systems of ordinary differential equations

- Solution strategy
  - Determine the homogeneous solution
    - Write the homogeneous equation
    - Assume the solution is of the form $y_h = De^{st}$

    * Substitute into the homogeneous equation and cancel terms to obtain the "characteristic equation" (Note, with experience you can write the characteristic equation by inspection.)
    * Solve for the roots $\{s_1,\ s_2,\ s_3,\ ...\}$ of the "characteristic" equation
    * Write the homogeneous solution as $y_h = De^{st}$ for one value of s, or $y_h = D_1e^{s_1t} + D_2e^{s_2t} + D_3e^{s_3t} + ...$ for multiple values of s. (Caution, if there are multiple values of s that are equivalent. If for example there are 4 values for s say $s_1 = s_2 = s_3 \neq s_4$, then $y_h = D_1e^{s_1t} + D_2te^{s_1t} + D_3t^2e^{s_1t} + D_4e^{s_4t}$.) Note, at this point the coefficients D_1, D_2, ... remain unknown.
- Determine the particular solution y_p
    * Guess the form of the particular solution, y_p, using experience and/or Table C.1. (Special care is needed if the particular solution duplicates the homogeneous solution. Basically, the treatment is like the repeated root case, you multiply by the next highest factor of t.)

Table C.1:

RHS	y_p
n^{th} degree polynomial	$A_0 + A_1t + A_2t^2 + ... + A_nt^n$
$\sin(at)$	$A_1\cos(at) + A_2\sin(at)$
$\cos(at)$	$A_1\cos(at) + A_2\sin(at)$
e^{at}	Ae^{at}

    * Find all necessary derivatives of the guess
    * Substitute the guess into the ODE and solve for unknown coefficients. If specific values for the coefficients can be found, then the guess is the particular solution.
    * Write the particular solution
- Determine the total solution
    * Write the form of the total solution as the sum of the homogeneous y_h and particular y_p solutions.
    * Solve for the unknown coefficients of y_h by applying initial conditions
    * Write the total solution
- Check total solution by substitution into ODE

- Method of Undetermined Coefficients: Examples
  - Example: First order with polynomial RHS

$$\dot{y} + y = t^2,\ y(0) = 1 \tag{C.4}$$

    * Determine the homogeneous solution
      - Write the homogeneous equation

$$\dot{y}_h + y_h = 0 \tag{C.5}$$

      - Assume the solution is of the form $y_h = De^{st}$
      - Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}} s + \cancel{De^{st}} = 0 \tag{C.6}$$

$$s + 1 = 0 \tag{C.7}$$

      - Solve for the roots of the characteristic equation

$$s = -1 \tag{C.8}$$

      - Write the homogeneous solution

$$y_h = De^{-t} \tag{C.9}$$

    * Determine the particular solution.
      - Guess the form of the particular solution

$$y_p = C_0 + C_1 t + C_2 t^2 \tag{C.10}$$

      - Find all necessary derivatives

$$\dot{y}_p = C_1 + 2C_2 t \tag{C.11}$$

      - Substitute the guess into the ODE and solve for unknown coefficients

$$(C_1 + 2C_2 t) + (C_0 + C_1 t + C_2 t^2) = t^2 \tag{C.12}$$

        Quadratic terms (t^2)

$$C_2 = 1 \tag{C.13}$$

        Linear terms (t^1)

$$2C_2 + C_1 = 0 \tag{C.14}$$

        Constant terms (t^0)

$$C_1 + C_0 = 0 \tag{C.15}$$

        $C_0 = 2$, $C_1 = -2$, $C_2 = 1$,

- Write the particular solution

$$y_p = 2 - 2t + t^2 \tag{C.16}$$

* Total solution
  - Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$\begin{aligned} y &= y_h + y_p && \text{(C.17)} \\ &= De^{-t} + 2 - 2t + t^2 && \text{(C.18)} \end{aligned}$$

  - Solve for the unknown coefficients by applying initial conditions

$$y(0) = 1 = D + 2 \tag{C.19}$$

$$D = -1 \tag{C.20}$$

  - Write the total solution

$$y = -e^{-t} + 2 - 2t + t^2 \tag{C.21}$$

* Check the solution by substitution into the ODE

– Example: First order with trig RHS

$$\dot{y} + y = \cos(t),\ y(0) = 1 \tag{C.22}$$

* Determine the homogeneous solution
  · Write the homogeneous equation

$$\dot{y}_h + y_h = 0 \tag{C.23}$$

  · Assume the solution is of the form $y_h = De^{st}$
  · Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}}s + \cancel{De^{st}} = 0 \tag{C.24}$$
$$s + 1 = 0 \tag{C.25}$$

  · Solve for the roots of the characteristic equation

$$s = -1 \tag{C.26}$$

  · Write the homogeneous solution

$$y_h = De^{-t} \tag{C.27}$$

* Determine the particular solution.
  · Guess the form of the particular solution

$$y_p = C_1 \cos(t) + C_2 \sin(t) \tag{C.28}$$

  · Find all necessary derivatives

$$\dot{y}_p = -C_1 \sin(t) + C_2 \cos(t) \tag{C.29}$$

  · Substitute the guess into the ODE and solve for the unknown coefficients

$$(-C_1 \sin(t) + C_2 \cos(t)) + (C_1 \cos(t) + C_2 \sin(t)) = \cos(t) \tag{C.30}$$

cos terms ($\cos(t)$)

$$C_2 + C_1 = 1 \tag{C.31}$$

sin terms ($\sin(t)$)

$$-C_1 + C_2 = 0 \tag{C.32}$$

$C_1 = \frac{1}{2}$, $C_2 = \frac{1}{2}$,

· Write the particular solution

$$y_p = \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) \tag{C.33}$$

∗ Total solution

· Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$\begin{aligned} y &= y_h + y_p && \text{(C.34)} \\ &= De^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) && \text{(C.35)} \end{aligned}$$

· Solve for the unknown coefficients by applying initial conditions

$$y(0) = D + \frac{1}{2} = 1 \tag{C.36}$$

$$D = \frac{1}{2} \tag{C.37}$$

· Write the total solution

$$y = \frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) \tag{C.38}$$

∗ Check the solution by substitution into the ODE

– Example: First order with exp RHS

$$\dot{y} + y = e^{-2t},\ y(0) = 1 \tag{C.39}$$

* Determine the homogeneous solution
  · Write the homogeneous equation

$$\dot{y}_h + y_h = 0 \tag{C.40}$$

  · Assume the solution is of the form $y_h = De^{st}$
  · Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}}s + \cancel{De^{st}} = 0 \tag{C.41}$$
$$s + 1 = 0 \tag{C.42}$$

  · Solve for the roots of the characteristic equation

$$s = -1 \tag{C.43}$$

  · Write the homogeneous solution

$$y_h = De^{-t} \tag{C.44}$$

* Determine the particular solution.
  · Guess the form of the particular solution

$$y_p = Ce^{-2t} \tag{C.45}$$

  · Find all necessary derivatives

$$\dot{y}_p = -2Ce^{-2t} \tag{C.46}$$

  · Substitute the guess into the ODE and solve for the unknown coefficients

$$(-2Ce^{-2t}) + (Ce^{-2t}) = e^{-2t} \tag{C.47}$$
$$-C = 1 \tag{C.48}$$
$$C = -1 \tag{C.49}$$

  · Write the particular solution

$$y_p = -e^{-2t} \tag{C.50}$$

* Total solution

· Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$
\begin{aligned}
y &= y_h + y_p && \text{(C.51)} \\
&= De^{-t} - e^{-2t} && \text{(C.52)}
\end{aligned}
$$

· Solve for the unknown coefficients by applying initial conditions

$$y(0) = D - 1 = 1 \tag{C.53}$$

$$D = 2 \tag{C.54}$$

· Write the total solution

$$y = 2e^{-t} - e^{-2t} \tag{C.55}$$

∗ Check the solution by substitution into the ODE

– Example: First order with exp RHS (duplicates homogeneous solution)

$$\dot{y} + y = e^{-t},\ y(0) = 1 \tag{C.56}$$

∗ Determine the homogeneous solution

· Write the homogeneous equation

$$\dot{y}_h + y_h = 0 \tag{C.57}$$

· Assume the solution is of the form $y_h = De^{st}$

· Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}}s + \cancel{De^{st}} = 0 \tag{C.58}$$
$$s + 1 = 0 \tag{C.59}$$

· Solve for the roots of the characteristic equation

$$s = -1 \tag{C.60}$$

· Write the homogeneous solution

$$y_h = De^{-t} \tag{C.61}$$

∗ Determine the particular solution.

· Guess the form of the particular solution

$$y_p = Cte^{-t} \tag{C.62}$$

· Find all necessary derivatives

$$\dot{y}_p = Ce^{-t} - Cte^{-t} \tag{C.63}$$

· Substitute the guess into the ODE and solve for the unknown coefficients

$$(Ce^{-t} - \cancel{Cte^{-t}}) + (\cancel{Cte^{-t}}) = e^{-t} \tag{C.64}$$
$$C = 1 \tag{C.65}$$

· Write the particular solution

$$y_p = e^{-t} \tag{C.66}$$

∗ Total solution

· Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$y = y_h + y_p \tag{C.67}$$

$$= De^{-t} + te^{-t} \tag{C.68}$$

· Solve for the unknown coefficients by applying initial conditions

$$y(0) = D = 1 \tag{C.69}$$

$$D = 1 \tag{C.70}$$

· Write the total solution

$$y = e^{-t} + te^{-t} \tag{C.71}$$

∗ Check the solution by substitution into the ODE

- Example: Second order ODE (homogeneous) with distinct roots

$$\ddot{y} + 3\dot{y} + 2y = 0,\ y(0) = 1,\ \dot{y}(0) = 1 \tag{C.72}$$

  * Determine the homogeneous solution
    · Write the homogeneous equation

$$\ddot{y}_h + 3\dot{y}_h + 2y_h = 0 \tag{C.73}$$

    · Assume the solution is of the form $y_h = De^{st}$
    · Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}}s^2 + 3\cancel{De^{st}}s + 2\cancel{De^{st}} = 0 \tag{C.74}$$
$$s^2 + 3s + 2 = 0 \tag{C.75}$$

    · Solve for the roots of the characteristic equation (using quadratic formula)

$$s = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2}}{2} \tag{C.76}$$
$$= -\frac{3}{2} \pm \frac{1}{2} \tag{C.77}$$
$$= \{-1,\ -2\} \tag{C.78}$$

    · Write the homogeneous solution

$$y_h = D_1 e^{-1t} + D_2 e^{-2t} \tag{C.79}$$

  * Determine the particular solution.
    · Guess the form of the particular solution

$$y_p = 0 \tag{C.80}$$

    · Find all necessary derivatives

$$\dot{y}_p = 0 \tag{C.81}$$
$$\ddot{y}_p = 0 \tag{C.82}$$

    · Substitute the guess into the ODE and solve for the unknown coefficients

$$(0) + 3(0) + 2(0) = 0 \tag{C.83}$$

    · Write the particular solution

$$y_p = 0 \tag{C.84}$$

* Total solution
  · Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$\begin{aligned} y &= y_h + y_p \quad &\text{(C.85)} \\ &= D_1 e^{-1t} + D_2 e^{-2t} \quad &\text{(C.86)} \end{aligned}$$

  · Solve for the unknown coefficients by applying initial conditions

$$\begin{aligned} y(0) &= D_1 + D_2 = 1 \quad &\text{(C.87)} \\ \dot{y}(0) &= -D_1 - 2D_2 = 1 \quad &\text{(C.88)} \end{aligned}$$

  $D_1 = 3$ and $D_2 = -2$
  · Write the total solution

$$y = 3e^{-t} - 2e^{-2t} \tag{C.89}$$

* Check the solution by substitution into the ODE

– Example: Second order ODE (homogeneous) with repeated roots

$$\ddot{y} + 2\dot{y} + y = 0,\ y(0) = 1,\ \dot{y}(0) = 1 \tag{C.90}$$

* Determine the homogeneous solution
  · Write the homogeneous equation

$$\ddot{y}_h + 2\dot{y}_h + y_h = 0 \tag{C.91}$$

  · Assume the solution is of the form $y_h = De^{st}$
  · Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}}s^2 + 2\cancel{De^{st}}s + \cancel{De^{st}} = 0 \tag{C.92}$$
$$s^2 + 2s + 1 = 0 \tag{C.93}$$

  · Solve for the roots of the characteristic equation (using quadratic formula)

$$s = \frac{-2 \pm \sqrt{2^2 - 4}}{2} \tag{C.94}$$
$$= -1 \pm 0 \tag{C.95}$$
$$= \{-1,\ -1\} \tag{C.96}$$

  · Write the homogeneous solution

$$y_h = D_1 e^{-t} + D_2 t e^{-t} \tag{C.97}$$

* Determine the particular solution.
  · Guess the form of the particular solution

$$y_p = 0 \tag{C.98}$$

  · Find all necessary derivatives

$$\dot{y}_p = 0 \tag{C.99}$$
$$\ddot{y}_p = 0 \tag{C.100}$$

  · Substitute the guess into the ODE and solve for the unknown coefficients

$$(0) + 2(0) + (0) = 0 \tag{C.101}$$

  · Write the particular solution

$$y_p = 0 \tag{C.102}$$

* Total solution
  - Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$
\begin{aligned}
y &= y_h + y_p && \text{(C.103)}\\
&= D_1 e^{-t} + D_2 t e^{-t} && \text{(C.104)}
\end{aligned}
$$

  - Solve for the unknown coefficients by applying initial conditions

$$
\begin{aligned}
y(0) &= D_1 = 1 && \text{(C.105)}\\
\dot{y}(0) &= -D_1 + D_2 = 1 && \text{(C.106)}
\end{aligned}
$$

  $D_1 = 1$ and $D_2 = 2$
  - Write the total solution

$$
y = e^{-t} + 2te^{-t} \tag{C.107}
$$

* Check the solution by substitution into the ODE

– Example: Imaginary roots homogeneous

$$\ddot{y} + 4y = 0,\ y(0) = 1,\ \dot{y}(0) = 1 \tag{C.108}$$

* Determine the homogeneous solution
  · Write the homogeneous equation

$$\ddot{y}_h + 4y_h = 0 \tag{C.109}$$

  · Assume the solution is of the form $y_h = De^{st}$
  · Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}}s^2 + 4\cancel{De^{st}} = 0 \tag{C.110}$$
$$s^2 + 4 = 0 \tag{C.111}$$

  · Solve for the roots of the characteristic equation (using quadratic formula)

$$s = \pm 2j \tag{C.112}$$

  · Write the homogeneous solution

$$y_h = D_1 e^{2jt} + D_2 e^{-2jt} \tag{C.113}$$

Substituting in the Euler identity $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$, we have

$$y_h = D_1(\cos(2t) + j\sin(2t)) + D_2(\cos(2t) - j\sin(2t) \tag{C.114}$$
$$y_h = (D_1 + D_2)\cos(2t) + j(D_1 - D_2)\sin(2t) \tag{C.115}$$

Because D_1 and D_2 are arbitrary complex constants, we can write this in terms of new constants as $B_1 = D_1 + D_2$ and $B_2 = j(D_1 - D_2)$.

$$y_h = B_1\cos(2t) + B_2\sin(2t) \tag{C.116}$$

* Determine the particular solution.
  · Guess the form of the particular solution

$$y_p = 0 \tag{C.117}$$

  · Find all necessary derivatives

$$\dot{y}_p = 0 \tag{C.118}$$
$$\ddot{y}_p = 0 \tag{C.119}$$

· Substitute the guess into the ODE and solve for the unknown coefficients

$$(0) + 4(0) = 0 \tag{C.120}$$

· Write the particular solution

$$y_p = 0 \tag{C.121}$$

∗ Total solution

· Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$\begin{aligned} y &= y_h + y_p & \text{(C.122)} \\ &= B_1 \cos(2t) + B_2 \sin(2t) & \text{(C.123)} \end{aligned}$$

· Solve for the unknown coefficients by applying initial conditions

$$\begin{aligned} y(0) &= B_1 = 1 & \text{(C.124)} \\ \dot{y}(0) &= 2B_2 = 1 & \text{(C.125)} \end{aligned}$$

$B_1 = 1$ and $B_2 = \frac{1}{2}$

· Write the total solution

$$y = \cos(2t) + \frac{1}{2}\sin(2t) \tag{C.126}$$

∗ Check the solution by substitution into the ODE

– Example: Imaginary roots driven at natural frequency

$$\ddot{y} + 4y = \sin(2t),\ y(0) = 0,\ \dot{y}(0) = 0 \tag{C.127}$$

* Determine the homogeneous solution
  · Write the homogeneous equation

$$\ddot{y}_h + 4y_h = 0 \tag{C.128}$$

  · Assume the solution is of the form $y_h = De^{st}$
  · Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}}s^2 + 4\cancel{De^{st}} = 0 \tag{C.129}$$

$$s^2 + 4 = 0 \tag{C.130}$$

  · Solve for the roots of the characteristic equation (using quadratic formula)

$$s = \pm 2j \tag{C.131}$$

  · Write the homogeneous solution

$$y_h = D_1e^{2jt} + D_2e^{-2jt} \tag{C.132}$$

Substituting in the Euler identity $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$, we have

$$y_h = D_1(\cos(2t) + j\sin(2t)) + D_2(\cos(2t) - j\sin(2t) \tag{C.133}$$

$$y_h = (D_1 + D_2)\cos(2t) + j(D_1 - D_2)\sin(2t) \tag{C.134}$$

Because D_1 and D_2 are arbitrary complex constants, we can write this in terms of new constants as $B_1 = D_1 + D_2$ and $B_2 = j(D_1 - D_2)$.

$$y_h = B_1\cos(2t) + B_2\sin(2t) \tag{C.135}$$

* Determine the particular solution.
  · Guess the form of the particular solution

$$y_p = t(C_1\cos(2t) + C_2\sin(2t)) \tag{C.136}$$

  · Find all necessary derivatives

$$\dot{y}_p = (C_1\cos(2t) + C_2\sin(2t)) + 2t(-C_1\sin(2t) + C_2\cos(2t)) \tag{C.137}$$

$$\begin{aligned}\ddot{y}_p &= 2(-C_1\sin(2t) + C_2\cos(2t)) + 2(-C_1\sin(2t) + C_2\cos(2t)) \\ &\quad +4t(-C_1\cos(2t) - C_2\sin(2t))\end{aligned} \tag{C.138}$$

· Substitute the guess into the ODE and solve for the unknown coefficients

$$\begin{aligned} 2(-C_1\sin(2t)+C_2\cos(2t))+2(-C_1\sin(2t)+C_2\cos(2t)) & \\ +4t(-C_1\cos(2t)-C_2\sin(2t))+4t(C_1\cos(2t)+C_2\sin(2t)) &= \sin(2t) \quad (C.139) \\ -4C_1\sin(2t)+4C_2\cos(2t) &= sin(2t) \quad (C.140) \end{aligned}$$

sin terms ($\sin(2t)$)

$$-4C_1 = 1 \quad (C.141)$$

cos terms ($\cos(2t)$)

$$4C_2 = 0 \quad (C.142)$$

$C_1 = -\frac{1}{4}$, $C_2 = 0$

· Write the particular solution

$$y_p = -\frac{1}{4}t\cos(2t) \quad (C.143)$$

∗ Total solution

· Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$\begin{aligned} y &= y_h + y_p \quad (C.144) \\ &= B_1\cos(2t)+B_2\sin(2t)-\frac{1}{4}t\cos(2t) \quad (C.145) \end{aligned}$$

· Solve for the unknown coefficients by applying initial conditions

$$\begin{aligned} y(0) &= B_1 = 0 \quad (C.146) \\ \dot{y}(0) &= 2B_2 - \frac{1}{4} = 0 \quad (C.147) \end{aligned}$$

$B_1 = 0$ and $B_2 = \frac{1}{8}$

· Write the total solution

$$y = \frac{1}{8}\sin(2t) - \frac{1}{4}t\cos(2t) \quad (C.148)$$

∗ Check the solution by substitution into the ODE

– Example: Complex roots homogeneous

$$\ddot{y} + 2\dot{y} + 2y = 0,\ y(0) = 1,\ \dot{y}(0) = 1 \tag{C.149}$$

* Determine the homogeneous solution
  · Write the homogeneous equation

$$\ddot{y}_h + 2\dot{y}_h + 2y_h = 0 \tag{C.150}$$

  · Assume the solution is of the form $y_h = De^{st}$
  · Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}}s^2 + 2\cancel{De^{st}}s + 2\cancel{De^{st}} = 0 \tag{C.151}$$

$$s^2 + 2s + 2 = 0 \tag{C.152}$$

  · Solve for the roots of the characteristic equation (using quadratic formula)

$$s = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} \tag{C.153}$$

$$= -1 \pm j \tag{C.154}$$

  · Write the homogeneous solution

$$y_h = \left(D_1 e^{(-1+j)t} + D_2 e^{(-1-j)t}\right) \tag{C.155}$$

$$= e^{-t}\left(D_1 e^{jt} + D_2 e^{-jt}\right) \tag{C.156}$$

Substituting in the Euler identity $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$, we have

$$y_h = e^{-t}\left(D_1(\cos(t) + j\sin(t)) + D_2(\cos(t) - j\sin(t)\right) \tag{C.157}$$

$$y_h = e^{-t}\left((D_1 + D_2)\cos(2t) + j(D_1 - D_2)\sin(2t)\right) \tag{C.158}$$

Because D_1 and D_2 are arbitrary complex constants, we can write this in terms of new constants as $B_1 = D_1 + D_2$ and $B_2 = j(D_1 - D_2)$.

$$y_h = e^{-t}\left(B_1\cos(t) + B_2\sin(t)\right) \tag{C.159}$$

* Determine the particular solution.
  · Guess the form of the particular solution

$$y_p = 0 \tag{C.160}$$

· Find all necessary derivatives

$$\dot{y}_p = 0 \tag{C.161}$$
$$\ddot{y}_p = 0 \tag{C.162}$$

· Substitute the guess into the ODE and solve for the unknown coefficients

$$(0) + 2(0) + 2(0) = 0 \tag{C.163}$$

· Write the particular solution

$$y_p = 0 \tag{C.164}$$

∗ Total solution

· Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$y = y_h + y_p \tag{C.165}$$
$$= e^{-t}\left(B_1 \cos(t) + B_2 \sin(t)\right) \tag{C.166}$$

· Solve for the unknown coefficients by applying initial conditions

$$y(0) = B_1 = 1 \tag{C.167}$$
$$\dot{y}(0) = -B_1 + B_2 = 1 \tag{C.168}$$

$B_1 = 1$ and $B_2 = 2$

· Write the total solution

$$y = e^{-t}\left(\cos(t) + 2\sin(t)\right) \tag{C.169}$$

∗ Check the solution by substitution into the ODE

- Example: Complex roots constant RHS

$$\ddot{y} + 2\dot{y} + 2y = 1, \; y(0) = 0, \; \dot{y}(0) = 0 \tag{C.170}$$

  * Determine the homogeneous solution
    - Write the homogeneous equation

$$\ddot{y}_h + 2\dot{y}_h + 2y_h = 0 \tag{C.171}$$

    - Assume the solution is of the form $y_h = De^{st}$
    - Substitute into the homogeneous equation and cancel terms to obtain the characteristic equation

$$\cancel{De^{st}}s^2 + 2\cancel{De^{st}}s + 2\cancel{De^{st}} = 0 \tag{C.172}$$

$$s^2 + 2s + 2 = 0 \tag{C.173}$$

    - Solve for the roots of the characteristic equation (using quadratic formula)

$$s = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} \tag{C.174}$$

$$= -1 \pm j \tag{C.175}$$

    - Write the homogeneous solution

$$y_h = \left(D_1 e^{(-1+j)t} + D_2 e^{(-1-j)t}\right) \tag{C.176}$$

$$= e^{-t}\left(D_1 e^{jt} + D_2 e^{-jt}\right) \tag{C.177}$$

    Substituting in the Euler identity $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$, we have

$$y_h = e^{-t}\left(D_1(\cos(t) + j\sin(t)) + D_2(\cos(t) - j\sin(t)\right) \tag{C.178}$$

$$y_h = e^{-t}\left((D_1 + D_2)\cos(t) + j(D_1 - D_2)\sin(t)\right) \tag{C.179}$$

    Because D_1 and D_2 are arbitrary complex constants, we can write this in terms of new constants as $B_1 = D_1 + D_2$ and $B_2 = j(D_1 - D_2)$.

$$y_h = e^{-t}\left(B_1\cos(t) + B_2\sin(t)\right) \tag{C.180}$$

  * Determine the particular solution.
    - Guess the form of the particular solution

$$y_p = C \tag{C.181}$$

· Find all necessary derivatives

$$\dot{y}_p = 0 \tag{C.182}$$
$$\ddot{y}_p = 0 \tag{C.183}$$

· Substitute the guess into the ODE and solve for the unknown coefficients

$$(0) + 2(0) + 2(C) = 1 \tag{C.184}$$
$$C = \frac{1}{2} \tag{C.185}$$

· Write the particular solution

$$y_p = \frac{1}{2} \tag{C.186}$$

∗ Total solution

· Write the form of the total solution as the sum of the homogeneous and particular solutions.

$$y = y_h + y_p \tag{C.187}$$
$$= e^{-t}\left(B_1 \cos(t) + B_2 \sin(t)\right) + \frac{1}{2} \tag{C.188}$$

· Solve for the unknown coefficients by applying initial conditions

$$y(0) = B_1 + \frac{1}{2} = 0 \tag{C.189}$$
$$\dot{y}(0) = -B_1 + B_2 = 0 \tag{C.190}$$

$B_1 = -\frac{1}{2}$ and $B_2 = -\frac{1}{2}$

· Write the total solution

$$y = -e^{-t}\left(\frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)\right) + \frac{1}{2} \tag{C.191}$$

∗ Check the solution by substitution into the ODE

Appendix D

Python

D.1 Python Software Options

- Anaconda distribution with Spyder IDE (preferred method)
  - Download and install the appropriate version of Anaconda for your operating system.
  - Open the Anaconda prompt (Note, if you have trouble with this link, replace the %23 in the link with #.)
  - Type the following into the command line prompt to install the Python Control Systems Library:
    `conda install -c conda-forge control`
  - Close the Anaconda prompt
  - Open the Anaconda Navigator (Note, if you have trouble with this link, replace the %23 in the link with #.)
  - Open Spyder (Note, if you have trouble with this link, replace the %23 in the link with #.)
  - Problems: Some people experience a problem installing the Python Control Systems Library. It was resolved by create a new environment in the Anaconda Navigator. The following numpy, scipy, and matplotlib should be installed in the environment. Then use the command:
    `conda install -n EnvironmentName -c conda-forge control` to install the Python Control Systems Library.
- Google's Colab (alternate method)
  - The Control Systems Library may be installed using one of the following
    * run the command:
      `!pip install -q control`

- or the commands:
  `!pip install slycot`
  `!pip install -q control`

D.2 Common Mathematical Functions

- Vectors
  - `linspace`: creates a vector of linearly spaced values
    - Syntax:
      ```
      import numpy as np
      v=np.linspace(vi,vf,N)
      ```
    - Intputs and outputs:
      `v` : a vector of `N` values evenly spaced starting with `vi` and ending with `vf`
      `vi` : starting value
      `vf` : ending value
      `N` : number of values
  - `logspace`: creates a vector of logarithmically spaced values
    - Syntax:
      ```
      import numpy as np
      v=np.logspace(ei,ef,N)
      ```
    - Intputs and outputs:
      `v` : a vector of `N` values evenly spaced on a logarithmic scale starting with `10^ei` and ending with `10^ef`
      `ei` : starting value
      `ef` : ending value
      `N` : number of values

- `roots`: find the roots of a polynomial equation
  - To find the roots of an equation

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 = 0 \tag{D.1}$$

  - Syntax:
    ```
    import numpy as np
    r=np.roots(a)
    ```
  - Inputs and outputs:
    `r` : one dimensional array of roots
    `a` : one dimensional array of coefficients of the polynomial in descending order (`[a_2, a_1, a_0]`)

- `eig`: find the eigenvalues of a matrix
  - To find the eigenvalues of a square matrix `A`
  - Syntax:
    ```
    import numpy as np
    e=np.linalg.eig(A)[0]
    ```
  - Inputs and outputs:
    `e` : one dimensional array of eigenvalues
    `A` : square matrix

- `convolve`: to find the product of two polynomials
  - To multiply polynomials

$$(a_3 s^3 + a_2 s^2 + a_1 s^1 + a_1 s + a_0)(b_2 s^2 + b_1 s + b_0) \tag{D.2}$$

  - syntax:
    ```
    [c_5,c_4,c_3,c_2,c_1,c_0]=np.convolve([a_3,a_2,a_1,a_0],[b_2,b_1,b_0])
    ```

- `atan2`: inverse tangent in the correct quadrant
  - To find the angle, in the correct quadrant, given the opposite, y, and adjacent, x, sides
  - Syntax:
    ```
    import numpy as np
    theta=np.arctan2(y,x)
    ```
  - Inputs and outputs:
    `theta` : angle
    `y` : opposite
    `x` : adjacent

- Complex numbers
  - `real`: real part of complex number
    * To find the real part of a complex number
    * Syntax:
      ```
      import numpy as np
      x=np.real(N)
      ```
    * Inputs and outputs:
      `x` : real part of a complex number
      `N` : complex number
  - `imag`: imaginary part of complex number

        * To find the imaginary part of a complex number
        * Syntax:
          ```
          import numpy as np
          y=np.imag(N)
          ```
        * Inputs and outputs:
          `y` : imaginary part of a complex number
          `N` : complex number

    - `abs`: magnitude of complex number
        * To find the magnitude of a complex number
        * Syntax:
          ```
          import numpy as np
          M=np.abs(N)
          ```
        * Inputs and outputs:
          `M` : magnitude of a complex number
          `N` : complex number

    - `angle`: angle of complex number
        * To find the angle of a complex number
        * Syntax:
          ```
          import numpy as np
          phi=np.angle(N)
          ```
        * Inputs and outputs:
          `phi` : angle of a complex number
          `N` : complex number

- `polyfit`: polynomial fit
    - Syntax:
      ```
      p=np.polyfit(x,y,n)
      ```
    - Inputs and outputs:
      `p` : array of polynomial coefficients
      `x` : x data
      `y` : y data
      `n` : order of polynomial

D.3 Numeric Solution of Initial Value Problems Via State Space

- Note, the presentation here was originally based on that given for MATLAB® in [35]

- Use for both linear and non-linear systems of ordinary differential equations (ODE's)
- `solve_ivp`: solve an initial value problem
  - Syntax
    `sol=solve_ivp(yourfunction,tspan,y0,t_eval=T);`
  - `yourfunction`: is a function that you define with two inputs
    * `t` time (scalar)
    * `y` a one dimensional array of n state variables evaluated at `t`

    and one output
    * `dydt` a one dimensional array of the time derivatives of the n state variables evaluated at `t`
  - `tspan`: is a one dimensional array with two entries, the initial and final time
  - `T`: is a one dimensional array with elements being times at which the solution is to be evaluated. The first and last entires should match `tspan`. `T` has q entires. Note that the increment in time might not be constant. (optional)
  - `y0`: is a one dimensional array with initial conditions for the n state variables.
  - `sol.y`: is a two dimensional array with each row representing a state variable and each column corresponding to an instant in time. Thus it has n rows and q columns, where n is the number of state variables.
  - `sol.t`: is a one dimensional array with elements being times at which the solution is evaluated
- `def`: define a function
  - Syntax
    ```
    def yourfunction(t,y):
    return dydx
    ```
  - `dydx`: a one dimensional array with n entries corresponding to the current values of the derivatives of state variables
  - `t`: a scalar value for the current time
  - `y`: a one dimensional array, with n entries corresponding to the current values of state variables

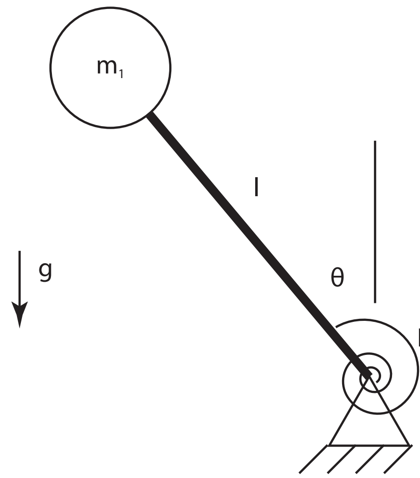

Figure D.1:

- Example: Inverted pendulum. See Figure D.1.
  - Equation of motion
$$ml^2\ddot{\theta} + K_T\theta - mgl\sin(\theta) = 0 \tag{D.3}$$
    Note, this is a nonlinear ordinary differential equation
  - Let's define our state variables: $x_1 = \theta$ and $x_2 = \dot{\theta}$
  - Now we can rewrite the equation of motion as a state variable model
$$\dot{x}_1 = x_2 \tag{D.4}$$
$$\dot{x}_2 = \frac{1}{ml^2}(-K_T x_1 + mgl\sin(x_1)) \tag{D.5}$$
$$\tag{D.6}$$
    Note this state variable model is not linear
  - Python script:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
def pendulum(t, y): #function representing state equations
 m=1
 l=1
 g=9.81
 k_T=5
```


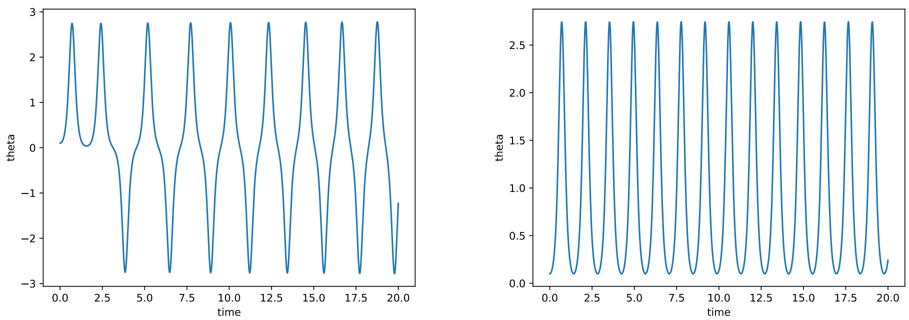

Figure D.2:

```
 dydt=[y[1],g/(m*l**2)*(-k_T*y[0]+m*g*l*np.sin(y[0]))]
 return dydt
tspan=[0, 20]#time range over which to solve
y0=[0.1,0]#initial conditions
T=np.linspace(tspan[0],tspan[1],1000)#time points at which to solve
sol = solve_ivp(pendulum, tspan, y0, t_eval=T)
plt.plot(sol.t,sol.y[0,:])
plt.xlabel('time')
plt.ylabel('theta')
plt.show()
```

– **Warning, the computed response in Figure D.2a is nonphysical.** The mechanical energy in the initial state when the pendulum starts from rest with $\theta(0) = 0.1$ and $\dot{\theta}(0) = 0$ should be conserved. However, the plot shows that the nature of the oscillations of the pendulum change sometime after about 3 seconds. In other words, the mechanical energy in the simulation was not conserved but increased from its initial value.

– A somewhat better solution is presented in Figure D.2(b) and is obtained by imposing a stricter tolerance requirement on the solver:
`sol = solve_ivp(pendulum, tspan, y0, t_eval=T, rtol=1e-6)`

D.4 Representing and Solving Linear Time Invariant Systems

- Note, the presentation here was originally based on that given for MATLAB® in [35]

- `residue`: partial fraction expansion

- For rational function like

$$X(s)=\frac{N(s)}{D(s)}=\frac{b_ms^m+b_{m-1}s^{m-1}+...+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+...+a_1s+a_0} \tag{D.7}$$

- Syntax:

```
from scipy.signal import residue
r,p,K=residue(num,den)
```

- Inputs and outputs:
  * `r`: residues - one dimensional array of expansion coefficients
  * `p`: poles - one dimensional array of poles (roots of the denominator)
  * `K`: direct polynomial term - one dimensional array of the coefficients of the direct polynomial(i.e., quotient of polynomial long division). Is nonzero if the order of the numerator is greater than or equal to the denominator (We won't consider this often.)
  * `num`: one dimensional array of coefficients of the numerator in order of highest to lowest order terms
  * `den`: one dimensional array of coefficients of the denominator in order of highest to lowest order terms
- Inpterpretation:
  * The expanded expression would then be (for all distinct roots):

$$X(s)=\frac{r_1}{s-p_1}+\frac{r_2}{s-p_2}+...+\frac{r_n}{s-p_n} \tag{D.8}$$

  * The expanded expression would then be (and for k repeated roots):

$$X(s)=\frac{r_1}{s-p_1}+\frac{r_2}{(s-p_1)^2}+...+\frac{r_k}{(s-p_1)^k} \\ +\frac{r_{k+1}}{s-p_{k+1}}+...+\frac{r_n}{s-p_n} \tag{D.9}$$

- Example - repeated poles
  * Given
$$X(s) = \frac{5s^2 + 20s + 23}{s^3 + 5s^2 + 7s + 3} \tag{D.10}$$
  * Enter into python:
```
from scipy.signal import residue
r,p,k=residue([5, 20, 23],[1, 5, 7, 3])
```
  * Python returns (to numerical precision):
```
r=[3, 4, 2]
p=[-1, -1, -3]
K=[]
```
  * This means
$$X(s) = \frac{3}{s+1} + \frac{4}{(s+1)^2} + \frac{2}{s+3} \tag{D.11}$$
- Example - complex poles
  * Given
$$X(s) = \frac{-2}{s^2 + 4s + 5} \tag{D.12}$$
  * Enter into python:
```
from scipy.signal import residue
r,p,k=residue([-2],[1, 4, 5])
```
  * Python returns (to numerical precision):
```
r=[1.j, -1.j]
p=[-2.+1.j,-2.-1.j]
K=[]
```
  * This means
$$X(s) = \frac{j}{s+2-j} + \frac{-j}{s+2+j} \tag{D.13}$$

- Representing LTI systems
  - `ss`: construct a state space object
    * Syntax
      `sys=control.ss(A,B,C,D)`
    * `sys`: a variable representing an LTI system
    * `A`: n by n system matrix (n is the number of state variables)
    * `B`: n by m input matrix (m is the number of inputs)
    * `C`: p by n output matrix (p is the number of outputs)
    * `D`: p by m 'Feedforward matrix' [6], 'direct transmission term' [8], or 'feedthrough matrix' [7]

- `tf`: construct a transfer function object
  - Syntax
    `sys=control.tf(num,den)`
  - `sys`: a variable representing an LTI system
  - `num`: coefficients of the polynomial in the numerator of the transfer function
  - `den`: coefficients of the polynomial in the denominator of the transfer function
- `pade`: determine the coefficients for a the Padé approximation
  - Syntax
    `num,den=control.pade(Td,N)`
  - `num`: coefficients of the polynomial in the numerator of the approximant
  - `den`: coefficients of the polynomial in the denominator of the approximant
  - `Td`: time delay
  - `N`: order of the approximation

- Combining systems as block diagrams
  - `series`: combine transfer function objects in series
    - Syntax: `sys3=control.series(sys1,sys2)`
    - equivalent to: `sys3=sys2*sys1`
  - `parallel`: combine transfer function objects in parallel
    - Syntax: `sys3=control.parallel(sys1,sys2)`
    - equivalent to: `sys3=sys1+sys2`
  - `feedback`: combine transfer function objects in a feedback configuration
    - `sys3=control.feedback(sys1,sys2)` – default is for negative feedback. For positive feedback `sys3=control.feedback(sys1,sys2,+1)`
  - for gain, simply `sys2=K*sys1;`

- Accessing system data
  - `ssdata`:
    - Syntax: `A,B,C,D=control.ssdata(sys)`
  - `tfdata`:
    - Syntax: `num,den=control.tfdata(sys)`
  - `pole`:
    - Syntax: `p=control.pole(sys)`
  - `zero`:

    - Syntax: `z=control.zero(sys)`
  - `pzmap`:
    - Syntax: `control.pzmap(sys)`
  - `dcgain`: get the static sensitivity of a system
    - Syntax: `K0=control.dcgain(sys)`
  - `root_locus`: generate a root locus plot
    - Syndax: `control.root_locus(sys)`
- System response in the time domain
  - `initial_response`: calculate the free response of a linear system
    - Syntax
      `T, Y=control.initial_response(sys,None,xo)`
  - `step_response`: calculate the step response of a linear system
    - Syntax
      `T, Y=control.step_response(sys)`
  - `impulse_response`: calculate the impulse response of a linear system
    - Syntax
      `T, Y=control.impulse_response(sys)`
  - `forced_response`: calculate the forced response of a linear system
    - Syntax
      `T, Y, X=control.forced_response(sys,T,U)`
  - Array dimensions
    - `T`: one dimensional array with q entries representing sequential points in time at which the solution is evaluated.
    - `Y`: two dimensional array with p rows and q columns that contains the outputs evaluated at each point in time.
    - `X`: two dimensional array n rows by q columns that contains the state variables evaluated at each point in time
    - `U`: two dimensional array with m rows and q columns that contains the inputs evaluated at each point in time.
    - `xo`: one dimensional array with n entries containing the initial conditions.
  - `step_info`: calculate step response characteristics of a linear system
    - Syntax
      `control.step_info(sys)`

    * Output structure with fields:
        · Overshoot: Maximum % overshoot
        · Peak: Maximum absolute value of the output
        · PeakTime: time at which the output reaches its peak
        · RiseTime: 10-90% rise time
        · SettlingMin: Minimum value of the response after it has risen to 100%
        · SettlingMax: Maximum value of the response after it has risen to 100%
        · SettlingTime: 2% settling time
        · SteadyStateValue: Steady state value
        · Undershoot: Maximum % undershoot (amplitude traveled in direction opposite that of the steady state response)

- System response in the frequency domain
  - `freqresp`: evaluate the frequency response function of a linear system
    * Syntax:
```
import control
M,Phi,omega=control.freqresp(sys,omega)
```
    * Inputs and outputs:
      `M` : vector of magnitude ratios, $|G(j\omega)|$
      `Phi` : vector of phase angles, $\angle G(j\omega)$
      `omega` : vector of frequencies at which to evaluate the transfer function $G(j\omega)$
      `sys` : variable representing an LTI system
  - `evalfr`: evaluate the frequency response function of a linear system at a point in the complex plane
    * double check this
    * Syntax:
```
import control
H=control.evalfr(sys,X)
```
    * Inputs and outputs:
      `H` : complex number representing, $G(X)$
      `sys` : variable representing an LTI system
      `X` : point in the complex plane at which to evaluate the transfer function
  - `bode`: generate the bode plot of a linear system
    * Syntax:
```
import control
M,Phi,w=control.bode(sys,dB=True,omega=w)
```

    * Inputs and outputs:
      `M` : vector of magnitudes
      `Phi` : vector of phase angles
      `w` : vector of angular frequencies (i.e., rad/s) at which the response is evaluated. [optional as an input]
      `sys` : variable representing an LTI system
      `dB=True` : optional argument to plot in dB

  - `margin`: stability margin information
    * Syntax:
      `GM,PM,wpc,wgc=control.margin(sys)`
    * Inputs and outputs:
      `GM` : gain margin
      `PM` : phase margin
      `wpc` : phase crossover frequency
      `wgc` : gain crossover frequency
      `sys` : variable representing an LTI system

  - `nyquist_plot`: plots the Nyquist plot for a system
    * This function plots the Nyquist plot for a system
    * Syntax:
      `control.nyquist_plot(sys)`
    * Inputs and outputs:
      `sys` : variable representing an LTI system

- Example Python code. Try several values for α, inputs $f(t)$, and initial conditions. Plots are provided in Figure D.3.

```
import numpy as np
import matplotlib.pyplot as plt
import control
#%%define system
alpha=2
A=np.array([[0,1],[-1,-alpha]])
B=np.array([[0],[1]])
C=np.array([[1,0]])
D=np.array([[0]])
sys=control.ss(A,B,C,D)

#%%free response
x0=[1,0]
T,Y=control.initial_response(sys,None,x0)
```



```
plt.plot(T,Y)


#\%\%step response
T,Y=control.step_response(sys)
plt.figure()
plt.plot(T,Y)


#%%impulse response
T,Y=control.impulse_response(sys)
plt.figure()
plt.plot(T,Y)


#%%forced response
T=np.linspace(0,30,1000)
U=np.sin(np.pi/4*T)
T,Y,X=control.forced_response(sys,T,U)
plt.figure()
plt.plot(T,Y)


plt.show()
```

- Example - Plot the unit step response given a system
  - transfer function
$$G(s) = \frac{5}{s^2 + 2s + 5} \tag{D.14}$$
  - Syntax
```
import control
import matplotlib.pyplot as plt
sys=control.tf(5,[1,2,5])
T,Y=control.step_response(sys)
plt.plot(T,Y)
plt.show()
```
- Example - construct a state space model and obtain the numerator and denominator coefficients for the corresponding transfer function

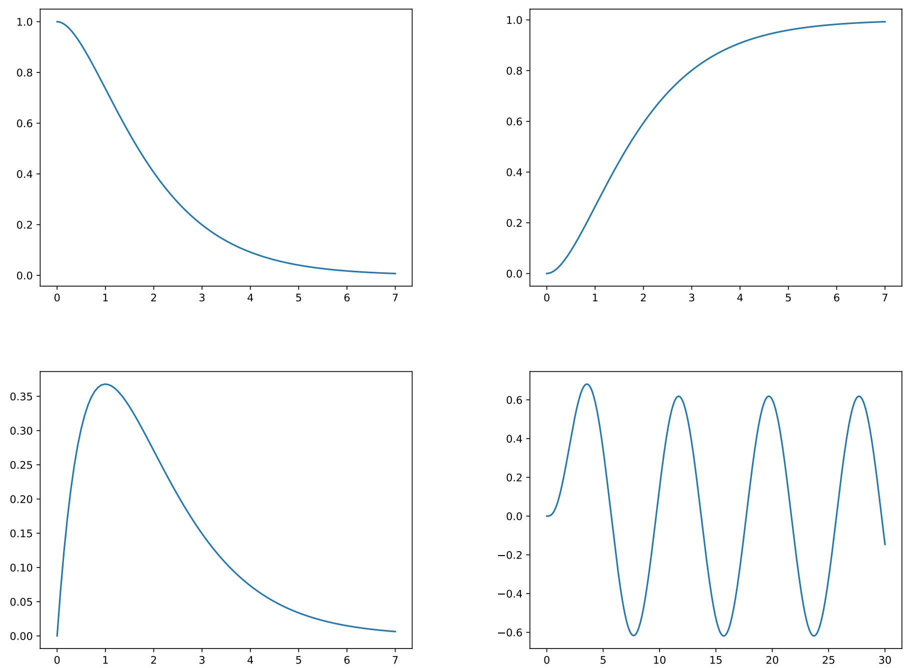

Figure D.3: Free, step, impulse, and forced response examples.

– state space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{D.15}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u(t) \tag{D.16}$$

– Syntax

```
import control
A=[[-2,1],[1,-2]]
B=[[1],[0]]
C=[[1,0]]
D=[[0]]
sys=control.ss(A,B,C,D)
num,den=control.tfdata(sys)
print(num)
print(den)
```

D.5 Probability, Statistics, and Uncertainty

- Summary statistics for an array of values `a`
  - `np.mean(a)`: calculate the mean of the elements of an array
  - `np.var(a,ddof=1)`: calculate the sample variance of the elements of an array
  - `np.var(a)`: calculate the population variance of the elements of an array
  - `np.std(a,ddof=1)`: calculate the sample standard deviation of the elements of an array
  - `np.std(a)`: calculate the population standard deviation of the elements of an array
- Pseudo random number generation
  - Construct (and optionally seed) a new random number generator: `rng=np.random.default_rng()` (the generator may be seeded by providd an optional integer argument)
  - Generate an array of random numbers with standard normal distribution: `rng.standard_normal(siz`
  - Generate an array of random numbers with uniform distribution: `rng.uniform(size=)`
  - Generate an array of random integers with uniform distribution: `rng.integers(Low, high=High,siz`
- Import data from csv file
  - `np.loadtxt()`:
- Visualization
  - `plt.plot(x,y)`:
  - `plt.hist(x)`: generate a histogram for an array array
  - Cumulative distribution function `cdfplot(x)`
    * (Function written by Dr. T. Lillian)

```
import numpy as np
import matplotlib.pyplot as plt
def cdfplot(x):
print(np.size(x,axis=0))
if x.ndim==2: #check if the input is a 2-dimensional array with possibly more than
    r,c=np.shape(x)
    X=np.zeros((r*2,c))
    for i in range(0,c):
        X[0::2,i]=np.sort(x[:,i])
        X[1::2,i]=np.sort(x[:,i])
```

```
    elif x.ndim==1: #check if the input is a 1-dimensional array with
        r=np.size(x)
        X=np.zeros((r*2,1))
        X[0::2,0]=np.sort(x[:])
        X[1::2,0]=np.sort(x[:])
    else:
        print('cdfplot will not work properly with a multidimensional array')
    Y=np.zeros((2*r,1))
    Y[0::2,0]=np.linspace(0,1.-1./r,r)
    Y[1::2,0]=np.linspace(1./r,1.,r)
    plt.plot(X,Y)
    return (X,Y)
```

  - `plt.boxplot(x)`:
  - `plt.errorbar(x,y,err)`:

D.6 Spectral Analysis

- FFT
- PSD estimates
  - `scipy.signal.periodogram()`:
  - `scipy.signal.welch()`:
- Visualization
  - `plt.stem(x,y)`:

Appendix E

MATLAB®

E.1 MATLAB® Software Options

- The following text is copied from [`https://www.mathworks.com/content/dam/mathworks/mathworks-dot-com/support/books/members/pdfs/mathworks-book-program-brand-guidelines.pdf`]
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E.2 MATLAB® Software Options

- Install on your own machine (preferred method)
  - Install the following tool boxes: control systems, ...
- MATLAB® Online™ (alternate method)
- Octave and Octave Online

E.3 Common Mathematical Functions

- Vectors
  - `linspace`: creates a vector of linearly spaced values
    - Syntax:
      `v=linspace(vi,vf,N)`
    - Intputs and outputs:
      `v` : a vector of `N` values evenly spaced starting with `vi` and ending with `vf`
      `vi` : starting value
      `vf` : ending value
      `N` : number of values
  - `logspace`: creates a vector of logarithmically spaced values
    - Syntax:
      `v=logspace(ei,ef,N)`
    - Intputs and outputs:
      `v` : a vector of `N` values evenly spaced on a logarithmic scale starting with `10^ei` and ending with `10^ef`
      `ei` : starting value
      `ef` : ending value
      `N` : number of values

- `roots`: find the roots of a polynomial equation
  - To find the roots of an equation

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 = 0 \tag{E.1}$$

  - Syntax:
    `r=roots(a)`
  - Inputs and outputs:
    `r` : one dimensional array of roots
    `a` : one dimensional array of coefficients of the polynomial in descending order (`[a_2, a_1, a_0]`)

- `eig`: find the eigenvalues of a matrix
  - To find the eigenvalues of a square matrix `A`
  - Syntax:
    `e=eig(A)`

- Inputs and outputs:
  `e` : one dimensional array of eigenvalues
  `A` : square matrix

- `conv`: to find the product of two polynomials
  - To multiply polynomials

$$(a_3 s^3 + a_2 s^2 + a_1 s^1 + a_1 s + a_0)(b_2 s^2 + b_1 s + b_0) \tag{E.2}$$

  - syntax:
    `[c_5,c_4,c_3,c_2,c_1,c_0]=conv([a_3,a_2,a_1,a_0],[b_2,b_1,b_0])`

- `atan2`: inverse tangent in the correct quadrant
  - To find the angle, in the correct quadrant, given the opposite, y, and adjacent, x, sides
  - Syntax:
    `theta=atan2(y,x)`
  - Inputs and outputs:
    `theta` : angle
    `y` : opposite
    `x` : adjacent

- Complex numbers
  - `real`: real part of complex number
    * To find the real part of a complex number
    * Syntax:
      `x=real(N)`
    * Inputs and outputs:
      `x` : real part of a complex number
      `N` : complex number
  - `imag`: imaginary part of complex number
    * To find the imaginary part of a complex number
    * Syntax:
      `y=imag(N)`
    * Inputs and outputs:
      `y` : imaginary part of a complex number
      `N` : complex number
  - `abs`: magnitude of complex number

    * To find the magnitude of a complex number
    * Syntax:
      `M=abs(N)`
    * Inputs and outputs:
      `M` : magnitude of a complex number
      `N` : complex number

  - `angle`: angle of complex number

    * To find the angle of a complex number
    * Syntax:
      `phi=angle(N)`
    * Inputs and outputs:
      `phi` : angle of a complex number
      `N` : complex number

- `polyfit`: polynomial fit

  - Syntax:
    `p=polyfit(x,y,n)`
  - Inputs and outputs:
    `p` : array of polynomial coefficients
    `x` : x data
    `y` : y data
    `n` : order of polynomial

E.4 Numeric Solution of Initial Value Problems Via State Space

- Note, the presentation here was originally based on that given for MATLAB® in [35]

- Use for both linear and non-linear systems of ordinary differential equations (ODE's)

- `od45`: solve an initial value problem

  - Syntax
    `[T,Y]=ode45(@yourfunction,tspan,y0);`
  - `yourfunction`: is a function that you define with two inputs

    * `t` time (scalar)
    * `y` a one dimensional array of n state variables evaluated at `t`

and one output

    * `dydt` a one dimensional array of the time derivatives of the n state variables evaluated at `t`

  - `tspan`: is a one dimensional array with two entries, the initial and final time
  - `y0`: is a one dimensional array with initial conditions for the n state variables.
  - `T`: is a one dimensional array with elements being times at which the solution is evaluated. `T` has q entires. Note that the increment in time might not be constant.
  - `Y`: is a two dimensional array with each column representing a state variable and each row corresponding to an instant in time. Thus it has q rows and n columns, where n is the number of state variables.

- `function`: define a function

  - Syntax

```
function dydx=yourfunction(t,y)
dydx=
```

  - `dydx`: a one dimensional array with n entries corresponding to the current values of the derivatives of state variables
  - `t`: a scalar value for the current time
  - `y`: a one dimensional array, with n entries corresponding to the current values of state variables

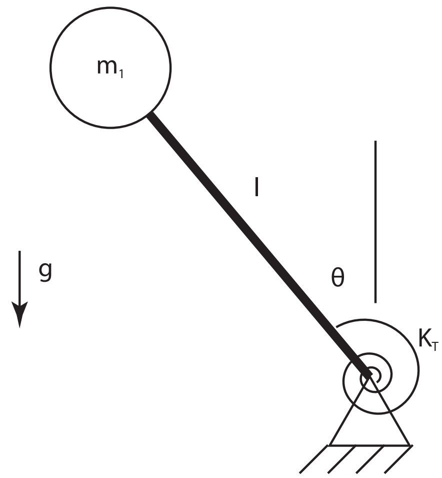

Figure E.1:

- Example: Inverted pendulum. See Figure E.1.
  - Equation of motion
$$ml^2\ddot{\theta} + K_T\theta - mgl\sin(\theta) = 0 \tag{E.3}$$
    Note, this is a nonlinear ordinary differential equation
  - Let's define our state variables: $x_1 = \theta$ and $x_2 = \dot{\theta}$
  - Now we can rewrite the equation of motion as a state variable model
$$\dot{x}_1 = x_2 \tag{E.4}$$
$$\dot{x}_2 = \frac{1}{ml^2}(-K_T x_1 + mgl\sin(x_1)) \tag{E.5}$$
$$\tag{E.6}$$
    Note this state variable model is not linear
  - MATLAB® function file 'pendulum.m':

```
function dydt=pendulum(t, y) %function representing state equations
 m=1;
 l=1;
 g=9.81;
 k_T=5;
 dydt=[y(2,1);g/(m*l^2)*(-k_T*y(1,1)+m*g*l*sin(y(1,1)))];
```


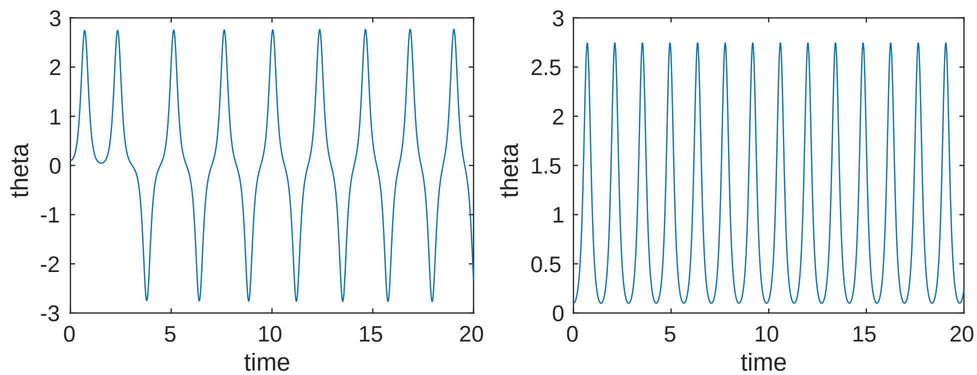

Figure E.2:

- MATLAB® script:

```
clear all
clc
close all
tspan=[0, 20];%time range over which to solve
y0=[0.1,0];%initial conditions
[T,Y]= ode45(@pendulum, tspan, y0);
plot(T,Y(:,1))
xlabel('time')
ylabel('theta')
```

- **Warning, the computed response in Figure E.2a is nonphysical.** The mechanical energy in the initial state when the pendulum starts from rest with $\theta(0) = 0.1$ and $\dot{\theta}(0) = 0$ should be conserved. However, the plot shows that the nature of the oscillations of the pendulum change sometime after about 3 seconds. In other words, the mechanical energy in the simulation was not conserved but increased from its initial value.

- A somewhat better solution is presented in Figure E.2(b) and is obtained by imposing a stricter tolerance requirement on the solver:

```
options=odeset('RelTol',1e-6);
[T,Y]= ode45(@pendulum, tspan, y0,options);
```

E.5 Representing and Solving Linear Time Invariant Systems

- Note, the presentation here was originally based on that given for MATLAB® in [35]

- `residue`: partial fraction expansion
  - For rational function like

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0} \tag{E.7}$$

  - Syntax:
    `[r,p,K]=residue(num,den)`
  - Inputs and outputs:
    * `r`: residues - one dimensional array of expansion coefficients
    * `p`: poles - one dimensional array of poles (roots of the denominator)
    * `K`: direct polynomial term - one dimensional array of the coefficients of the direct polynomial(i.e., quotient of polynomial long division). Is nonzero if the order of the numerator is greater than or equal to the denominator (We won't consider this often.)
    * `num`: one dimensional array of coefficients of the numerator in order of highest to lowest order terms
    * `den`: one dimensional array of coefficients of the denominator in order of highest to lowest order terms
  - Interpretation:
    * The expanded expression would then be (for all distinct roots):

$$X(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + ... + \frac{r_n}{s-p_n} \tag{E.8}$$

    * The expanded expression would then be (and for k repeated roots):

$$X(s) = \frac{r_1}{s-p_1} + \frac{r_2}{(s-p_1)^2} + ... + \frac{r_k}{(s-p_1)^k} + \frac{r_{k+1}}{s-p_{k+1}} + ... + \frac{r_n}{s-p_n} \tag{E.9}$$

- Example - repeated poles
  - Given
$$X(s) = \frac{5s^2 + 20s + 23}{s^3 + 5s^2 + 7s + 3} \tag{E.10}$$
  - Enter into MATLAB®:
`[r,p,k]=residue([5, 20, 23],[1, 5, 7, 3])`
  - MATLAB® returns (to numerical precision):
```
r=
2.0000
3.0000
4.0000

p=
-3.0000
-1.0000
-1.0000

K=
[]
```
  - This means
$$X(s) = \frac{2}{s+3} + \frac{3}{s+1} + \frac{4}{(s+1)^2} \tag{E.11}$$
- Example - complex poles
  - Given
$$X(s) = \frac{-2}{s^2 + 4s + 5} \tag{E.12}$$
  - Enter into MATLAB®:
`[r,p,k]=residue([-2],[1, 4, 5])`
  - MATLAB® returns (to numerical precision):
```
r=
0.0000+1.0000i
0.0000-1.0000i
p=
-2.0000+1.0000i
-2.0000-1.0000i
K=
[]
```
  - This means
$$X(s) = \frac{j}{s+2-j} + \frac{-j}{s+2+j} \tag{E.13}$$

- Representing LTI systems
  - `ss`: construct a state space object
    - Syntax
      `sys=ss(A,B,C,D)`
    - `sys`: a variable representing an LTI system
    - `A`: n by n system matrix (n is the number of state variables)
    - `B`: n by m input matrix (m is the number of inputs)
    - `C`: p by n output matrix (p is the number of outputs)
    - `D`: p by m 'Feedforward matrix' [6], 'direct transmission term' [8], or 'feedthrough matrix' [7]
  - `tf`: construct a transfer function object
    - Syntax
      `sys=tf(num,den)`
    - `sys`: a variable representing an LTI system
    - `num`: coefficients of the polynomial in the numerator of the transfer function
    - `den`: coefficients of the polynomial in the denominator of the transfer function
    - An optional argument may be used for systems with delay
      - `sys=tf(num,den,'InputDelay',Td)`
  - `zpk`: construct a transfer function object using the zero pole gain form
    - Syntax
      `sys=zpk(z,p,k)`
    - `sys`: a variable representing an LTI system
    - `z`: array of zeros of the transfer function
    - `p`: array of poles of the transfer function
    - `k`: gain
  - `pade`: determine the coefficients for a the Padé approximation
    - Syntax
      `[num,den]=pade(Td,N)`
    - `num`: coefficients of the polynomial in the numerator of the approximant
    - `den`: coefficients of the polynomial in the denominator of the approximant
    - `Td`: time delay
    - `N`: order of the approximation
- Combining systems as block diagrams
  - `series`: combine transfer function objects in series

    - Syntax: `sys3=series(sys1,sys2)`
    - equivalent to: `sys3=sys2*sys1`
  - `parallel`: combine transfer function objects in parallel
    - Syntax: `sys3=parallel(sys1,sys2)`
    - equivalent to: `sys3=sys1+sys2`
  - `feedback`: combine transfer function objects in a feedback configuration
    - `sys3=feedback(sys1,sys2)` – default is for negative feedback. For positive feedback `sys3=feedback(sys1,sys2,+1)`
  - for gain, simply `sys2=K*sys1;`
- Accessing system data
  - `ssdata`:
    - Syndax: `[A,B,C,D]=ssdata(sys)`
  - `tfdata`:
    - Syndax: `[num,den]=tfdata(sys)`
  - `zpkdata`:
    - Syndax: `[z,p,k]=zpkdata(sys)`
  - `pole`:
    - Syndax: `p=pole(sys)`
  - `zero`:
    - Syndax: `z=zero(sys)`
  - `pzmap`:
    - Syndax: `pzmap(sys)`
  - `dcgain`:
    - Syndax: `K0=dcgain(sys)`
  - `rlocus`:
    - Syndax: `rlocus(sys)`
- System response in the time domain
  - `initial`: calculate the free response of a linear system
    - Syntax
      `[Y,T]=initial(sys)`
  - `step`: calculate the step response of a linear system

    * Syntax
      `[Y,T]=step(sys)`

  - `impulse`: calculate the impulse response of a linear system

    * Syntax
      `[Y,T]=impulse(sys)`

  - `lsim`: calculate the forced response of a linear system

    * Syntax
      `[Y,T, X]=lsim(sys,U,T,x0)`

  - Array dimensions

    * `T`: one dimensional array with q entries representing sequential points in time at which the solution is evaluated.
    * `Y`: two dimensional array with q rows and p columns that contains the outputs evaluated at each point in time.
    * `X`: two dimensional array q rows by n columns that contains the state variables evaluated at each point in time
    * `U`: two dimensional array with q rows and m columns that contains the inputs evaluated at each point in time.
    * `xo`: one dimensional array with n entries containing the initial conditions.

  - `stepinfo`: calculate step response characteristics of a linear system

    * Syntax
      `control.step_info(sys)`
    * Output dictionary with entries:
      · RiseTime: 10-90% rise time
      · SettlingTime: 2% settling time
      · SettlingMin: Minimum value of the response after it has risen to 100%
      · SettlingMax: Maximum value of the response after it has risen to 100%
      · Overshoot: Maximum % overshoot
      · Undershoot: Maximum % undershoot (opposite direction of the steady state response)
      · Peak: Peak (absolute value) value of the output (in direction of the steady state response)
      · PeakTime: time at which the output reaches its peak value

- System response in the frequency domain

  - `freqresp`: evaluate the frequency response function of a linear system

  - Syntax:
    `H=freqresp(sys,omega)`
  - Inputs and outputs:
    `H` : complex number representing, $G(j\omega)$
    `sys` : variable representing an LTI system
    `omega` : vector of frequencies at which to evaluate the transfer function $G(j\omega)$

- `evalfr`: evaluate the frequency response function of a linear system
  - double check this
  - Syntax:
    `H=evalfr(sys,X)`
  - Inputs and outputs:
    `H` : complex number representing, $G(X)$
    `sys` : variable representing an LTI system
    `X` : point in the complex plane at which to evaluate the transfer function
- `bode`: generate the bode plot of a linear system
  - Syntax:
    `[M,Phi,w]=bode(sys,w)`
  - Inputs and outputs:
    `M` : vector of magnitudes
    `Phi` : vector of phase angles
    `w` : vector of angular frequencies (i.e., rad/s) at which the response is evaluated. [optional as an input]
    `sys` : variable representing an LTI system
- `margin`: stability margin information
  - This function outputs a bode plot showing stability margins and optionally outputs stability margin information
  - Syntax:
    `[GM,PM,wpc,wgc]=margin(sys)`
  - Inputs and outputs:
    `GM` : gain margin
    `PM` : phase margin
    `wpc` : phase crossover frequency
    `wgc` : gain crossover frequency
    `sys` : variable representing an LTI system
- add the allmargin() command
- `nyquist`: plots the Nyquist plot for a system
  - Syntax:
    `nyquist(sys)`

- Inputs and outputs:
  `sys` : variable representing an LTI system

- Example MATLAB® code. Try several values for α, inputs $f(t)$, and initial conditions. Plots are provided in Figure E.3.

```
clear all
clc
close all
%%define system
alpha=2;
A=[0,1;-1,-alpha];
B=[0;1];
C=[1,0];
D=[0];
sys=ss(A,B,C,D);

%%free response
x0=[1;0];
[Y,T,X]=initial(sys,x0);
plot(T,Y)

%%step response
[Y,T,X]=step(sys);
figure()
plot(T,Y)

%%impulse response
[Y,T,X]=impulse(sys);
figure()
plot(T,Y)

%%forced response
T=linspace(0,30,1000);
U=sin(pi/4*T);
[Y,T,X]=lsim(sys,U,T);
figure()
plot(T,Y)
```


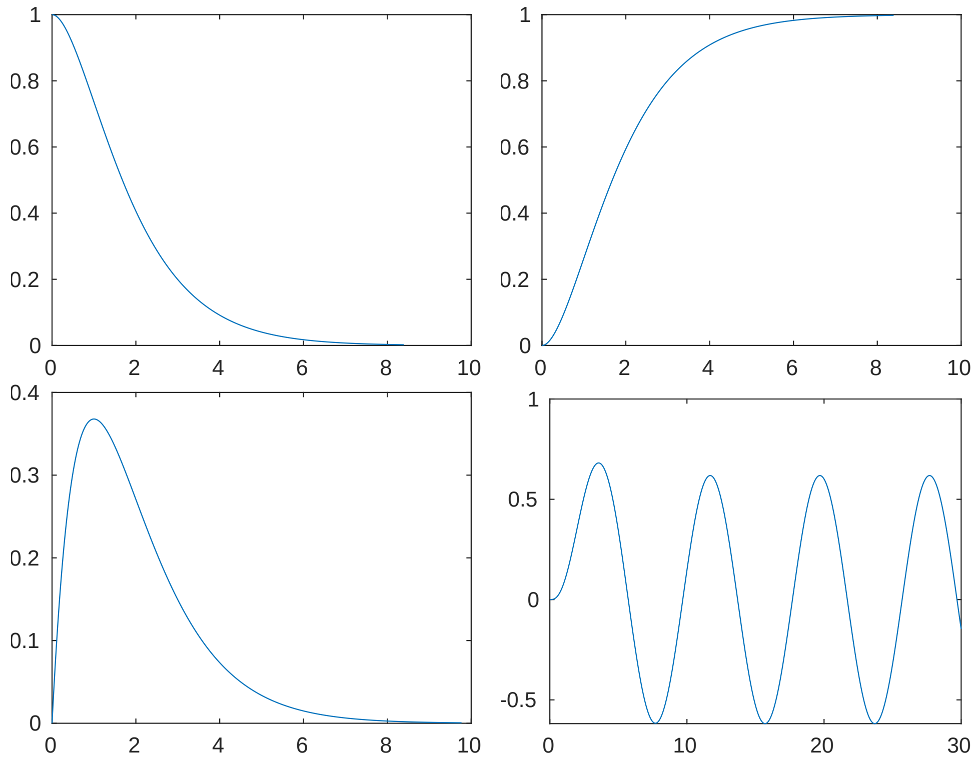

Figure E.3: Free, step, impulse, and forced response examples.

- Example - Plot the unit step response given a system
  - transfer function
  $$G(s) = \frac{5}{s^2 + 2s + 5} \tag{E.14}$$
  - Syntax
    ```
    sys=tf(5,[1,2,5]);
    [Y,T]=step(sys);
    plot(T,Y)
    ```

- Example - construct a state space model and obtain the numerator and denominator coefficients for the corresponding transfer function
  - state space model
  $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{E.15}$$
  $$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u(t) \tag{E.16}$$
  - Syntax
    ```
    A=[-2,1;1,-2];
    B=[1;0];
    C=[1,0];
    D=[0];
    sys=ss(A,B,C,D);
    [num,den]=tfdata(sys)
    ```

E.6 Probability, Statistics, and Uncertainty

- Summary statistics for an array of values `a`
  - `mean(a)`: calculate the mean of the elements of an array
  - Sample variance: `var(a)`
  - `var(a)`: calculate the sample variance of the elements of an array
  - `var(a,1)`: calculate the population variance of the elements of an array
  - `std(a)`: calculate the sample standard deviation of the elements of an array
  - `std(a,1)`: calculate the population standard deviation of the elements of an array

- Pseudo random number generation
  - Seed the random number generator: `rng(i)`
  - Generate an array of random numbers with standard normal distribution: `randn(n)`
  - Generate an array of random numbers with uniform distribution: `rand(n)`
  - Generate an array of random integers with uniform distribution: `randi(imax,n)`
- Import data from csv file
  - `readmatrix()`:
- Visualization
  - `plot(x,y)`:
  - `histogram(x)`: generate a histogram for an array array
  - `cdfplot(x)`: generate a cumulative distribution function for an array
  - `boxplot(x)`:
  - `boxchart(x)`:
  - `errorbar(x,y,err)`:

E.7 Spectral Analysis

- Audio files
  - `audioread`
  - `sound`
- FFT
  - Example MATLAB® script; see examples at [`https://www.mathworks.com/help/matlab/ref/fft.html`] :

```
[A,Fs]=audioread("AudioFile.m4a");%read in audio file
N=length(A);%number of samples read
t=(0:N-1)/Fs;%create time vector
plot(t,A);%plot audio signal as a function of time
figure()

Y=fft(A);%compute FFT of audio signal
%process FFT to obtain single sided magnitude spectrum
P2=abs(Y/N);
if mod(N,2)==0%if an even number of samples
```

```
     P1 = P2(1:N/2+1);
     P1(2:end-1) = 2*P1(2:end-1);
     f = Fs*(0:(N/2))/N;
else %if an odd number of samples
     P1 = P2(1:(N+1)/2);
     P1(2:end) = 2*P1(2:end);
     f = Fs*(0:(N-1)/2)/N;
end
semilogx(f,P1)%plot magnitude of frequency spectrum on a log scale
```

- PSD estimates
  - `periodogram()`:
  - `pwelch()`:
- Visualization
  - `stem(x,y)`:

Appendix F

Written Work Guidelines

- Pictures/scans must be of excellent quality
- Handwriting must be neat and clear
- Single column work: The sequence of steps used to solve a problem should progress in order down a column rather than scattered over the page.
- Mistakes:
  - Erase without a trace, or
  - neatly cross out mistakes.
- Make comments:
  - Describe where an equation came from or how it was applied (e.g., KCL, Ohm's law, etc.).
  - Number significant equations and use equation numbers to guide the reader through the algebra (e.g., substitute equation 2 into equation 3).
  - Interpret the result of a calculation.
- Cite your sources (e.g., name of classmates with whom you consulted), url for website at which you checked your answer, etc.
- When necessary, sketch diagrams and define variables that you introduce
- Indicate problem number from book or assignment
- Circle or box answers