This is Start

Vine

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7 微分方程

7.1 基本概念

$\begin{cases} \frac{dy}{dx}=2x \\ x=1,y=2 \end{cases} \Rightarrow y=2x+1$

函數，函數導數，自變量關□的方程**微分方程**

最高階導數的階數**微分方程的階**

$F\left(x,y^{'},\ldots,y^{(n)}\right)=0$ **一般形式**

函數**微分方程的解**

函數含常數，常數個數同階數**微分方程的通解**

確定常數的通解**微分方程的通解**

7.2 可分離變量的微分方程

$\frac{dy}{dx}=2x \Rightarrow dy=2xdx \Rightarrow$

$\frac{dy}{dx}=2xy^2 \Rightarrow \frac{dy}{y^2}=2xdx \Rightarrow$

$g\left(y\right)dy=f\left(x\right)dx \Rightarrow y=\varphi\left(x\right)$

$G\left(y\right)=F\left(x\right)+C \Rightarrow y=\Phi\left(x\right)$

7.3 齊次方程

$\frac{dy}{dx}=\varphi\left(\frac{y}{x}\right)$

$\frac{dy}{dx}=\frac{ax+by+c}{a_1x+b_1y+c_1}$

7.4 一階□性微分方程

$\frac{dy}{dx}+P\left(x\right)y=Q\left(x\right)$ 關於函數，函數導數是一次方程**一階□性微分方程**

$Q\left(x\right)=0$ 時**齊次方程**

$y=Ce^{-\int P(x)dx}$ **齊次通解**, 設 $C=\mu\left(x\right)$

$\mu\left(x\right)=\int Q\left(x\right)e^{\int P(x)\,dx}\,dx+C_2$

$y=\left(\int Q\left(x\right)e^{\int P(x)\,dx}\,dx+C_2\right)e^{-\int P(x)dx}$ **非齊次通解**

$\frac{dy}{dx}+P\left(x\right)y=Q\left(x\right)y^n$ **伯努利方程**

$\frac{dy}{dx}y^{-n}+P\left(x\right)yy^{-n}=Q\left(x\right)\xrightarrow{z=y^{1-n},z^{'}=\left(1-n\right)y^{-n}\frac{dy}{dx}}\frac{z^{'}}{(1-n)}+P\left(x\right)z=Q\left(x\right)$ **一階□性微分方程**

7.5 可降階的高階微分方程

$y^{(n)}=f\left(x\right) \quad \Rightarrow \quad y^{(n-1)}=\int f\left(x\right)\,dx$

$y^{''}=f\left(x,y^{'}\right) \quad \xrightarrow{y^{'}=p,y^{''}=p^{'}} \quad p^{'}=f\left(x,p\right)$

$y^{''}=f\left(y,y^{'}\right) \quad \xrightarrow{y^{'}=p,y^{''}=p\frac{dp}{dy}} \quad p\frac{dp}{dy}=f\left(y,p\right) \quad \xrightarrow{p=y^{'}=\varphi\left(y,c_1\right)}$

7.6 高階性微分方程

二阶微分方程

$$\frac{d^2y}{dx^2}+P\left(x\right)\frac{dy}{dx}+Q\left(x\right)y=f\left(x\right)$$

解的结构

$y=C_1y_1\left(x\right)+C_2y_2\left(x\right)$ 是解

$y=C_1y_1(x)+C_2y_2(x)$ 无关特解是通解
$y=C_1y_1(x)+C_2y_2(x)+\cdots+C_ny_n(x)$ 无关特解是通解
$y=Y(x)+y^*(x)$ 齐次通，非齐特，非齐通

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=f_1(x)+f_2(x)$$

$y=y_1^*(x)+y_2^*(x)$ 特解，特解，特解
常数变易法
$Y(x)=C_1y_1(x)+C_2y_2(x)$
$Y(x)=v_1(x)y_1(x)+v_2(x)y_2(x)$
$\begin{cases} y_1v_1^{'}+y_2v_2^{'}=0 \\ y_1^{'}v_1^{'}+y_2^{'}v_2^{'}=f \end{cases} \Rightarrow \begin{vmatrix} y_1 & y_2 & 0 \\ y_1^{'} & y_2^{'} & f \end{vmatrix} \xrightarrow{W=y_1y_2^{'}-y_1^{'}y_2} \begin{cases} v_1^{'}=-\frac{y_2f}{W} \\ v_2^{'}=\frac{y_1f}{W} \end{cases} \Rightarrow \begin{cases} v_1=-\int\frac{y_2f}{W}dx+c_1 \\ v_2=\int\frac{y_1f}{W}dx+c_2 \end{cases}$
$\begin{cases} v_1y_1=\left(-\int\frac{y_2f}{W}dx+c_1\right)y_1 \\ v_2y_2=\left(\int\frac{y_1f}{W}dx+c_2\right)y_2 \end{cases} \Rightarrow Y(x)=c_1y_1+c_2y_2-y_1\int\frac{y_2f}{W}dx+y_2\int\frac{y_1f}{W}dx$

7.7 常系数齐次线性微分方程

$y^{''}+P(x)y^{'}+Q(x)y=0 \Rightarrow y^{''}+py^{'}+qy=0$
$y=e^{rx}$
$(e^{rx})^{'}=re^{rx}$
$(r^2+pr+q)e^{rx}=0$
$\begin{cases} p^2-4q>0, r_{1,2}=\frac{-p\pm\sqrt{p^2-4q}}{2} \\ p^2-4q=0, r_1=r_2=\frac{-p}{2}, y_2=e^{r_1x}\mu(x), \mu^{''}=0 \\ p^2-4q<0, r_{1,2}=\alpha\pm\beta i, e^{(i\theta)}=\cos(\theta)+i\sin(\theta), \overline{y_1}=\frac{1}{2}(y_1+y_2), \overline{y_1}=\frac{1}{2i}(y_1-y_2) \end{cases}$
$\begin{cases} p^2-4q>0, y=C_1y_1+C_2y_2=C_1e^{r_1x}+C_2e^{r_2x} \\ p^2-4q=0, y=C_1y_1+C_2\mu y_1=(C_1x+C_2)e^{r_1x} \\ p^2-4q<0, y=C_1\overline{y_1}+C_2\overline{y_2}=e^{\alpha x}(C_1\cos(\beta x)+C_2\sin(\beta x)) \end{cases}$
n 阶常系数齐次微分方程
$y^n+p_1y^{n-1}+p_2y^{n-2}+\cdots+p_{n-1}y^{'}+p_ny=0$
$\begin{cases} D, \frac{d}{dx} \\ Dy, \frac{dy}{dx} \\ D^ny, \frac{d^ny}{dx^n} \end{cases}$
$(D^n+p_1D^{n-1}+\cdots+p_{n-1}D+p_n)y=0 \Rightarrow L(D)y=0,$ 微分算子 D 的 n 次多项式
$De^{rx}=re^{rx},\ldots,D^ne^{rx}=r^ne^{rx} \Rightarrow (r^n+p_1r^{n-1}+\cdots+p_{n-1}r+p_n)e^{rx}=L(r)e^{rx}=0 \Rightarrow L(r)=0$
$\begin{cases} \text{单实根} & Ce^{rx} \\ \text{k 重实根} & \left(C_1+C_2x+\cdots+C_kx^{k-1}\right)e^{rx} \\ \text{单复根} & e^{\alpha x}(C_1\cos(\beta x)+D_1\sin(\beta x)) \\ \text{k 重复根} & e^{\alpha x}\left(\left(C_1+C_2x+\cdots+C_kx^{k-1}\right)\cos(\beta x)+\left(D_1+D_2x+\cdots+D_kx^{k-1}\right)\sin(\beta x)\right) \end{cases}$

7.8 常系数非齐次线性微分方程

$y^{''}+py^{'}+qy=f(x)$
$p_m(x)=a_0x^m+a_1x^{m-1}+\cdots+a_m$
$\begin{cases} e^{\theta i}=\cos(\theta)+i\sin(\theta) \\ e^{-\theta i}=\cos(\theta)-i\sin(\theta) \end{cases} \Rightarrow \begin{cases} \cos(\theta)=\frac{1}{2}\left(e^{\theta i}+e^{-\theta i}\right) \\ \sin(\theta)=\frac{1}{2i}\left(e^{\theta i}-e^{-\theta i}\right) \end{cases} \Rightarrow \begin{cases} \cos(\omega x)P=\frac{P}{2}\left(e^{\omega xi}+e^{-\omega xi}\right) \\ \sin(\omega x)Q=\frac{Q}{2i}\left(e^{\omega xi}-e^{-\omega xi}\right) \end{cases}$
函数 (多项式) 共轭，倒数共轭；　两对共轭，乘积共轭；　$e^{\alpha+\theta i}$与$e^{\alpha-\theta i}$ 共轭

$f(x)=e^{\lambda x}P_m(x)$	$y^*=R_?(x)\,e^{\lambda x}$ $R^{''}x+(2\lambda+p)\,R^{'}x+(\lambda^2+p\lambda+q)\,R(x)=p_m(x)$ $y^*=x^kP_m(x)\,e^{\lambda x}$
$f(x)=e^{\lambda x}\left[P_l(x)\cos(\omega x)+Q_n(x)\sin(\omega x)\right]$	$f(x)=e^{\lambda x}\left[\left(\frac{P}{2}+\frac{Q}{2i}\right)e^{\omega xi}+\left(\frac{P}{2}-\frac{Q}{2i}\right)e^{-\omega xi}\right]$ $f(x)=\left(\frac{P}{2}+\frac{Q}{2i}\right)e^{\lambda x+\omega xi}+\left(\frac{P}{2}-\frac{Q}{2i}\right)e^{\lambda x-\omega xi}$ $f(x)=\left(\frac{P}{2}+\frac{Q}{2i}\right)e^{(\lambda+\omega i)x}+\left(\frac{P}{2}-\frac{Q}{2i}\right)e^{(\lambda-\omega i)x}$ $f(x)=P_1e^{(\lambda+\omega i)x}+Q_2e^{(\lambda-\omega i)x}\,(P_1,Q_2$共轭$)$ $y_1^*=x^kR_me^{(\lambda+\omega i)x}\,(m=max\{P_l,Q_n\})$ $y_2^*=x^k\overline{R_m}e^{(\lambda-\omega i)x}$ $y^*=y_1^*+y_2^*=x^ke^{\lambda x}\left(R_me^{\omega xi}+\overline{R_m}e^{-\omega xi}\right)$ $y^*=x^ke^{\lambda x}\left[R_m(\cos(\omega x)+i\sin(\omega x))+\overline{R_m}(\cos(\omega x)-i\sin(\omega x))\right]$ $y^*=x^ke^{\lambda x}\left(R_m^{(1)}\cos(\omega x)+R_m^{(2)}\sin(\omega x)\right)$

7.9 欧拉方程

$$x^ny^{(n)}+p^1x^{n-1}y^{(n)}+\cdots+p^{n-1}xy^{'}+p^ny=f(x)$$

$$x=e^t,t=\ln x$$

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{1}{x}\frac{dy}{dt}\ D\text{表示}\frac{d}{dt}\begin{cases}xy^{'}=Dy\\x^2y^{''}=(D^2-D)\,y\\x^3y^{'''}=(D^3-3D^2+2D)\,y\\\ldots\\x^ky^{(k)}=(D^n+c_1D^{n-1}+\cdots+C_{n-1}D)\,y\end{cases}$$

7.10 常系数线性微分方程组

$$\begin{cases}\frac{d^2x}{dt^2}+\frac{dy}{dt}-x=e^t,\\\frac{d^2y}{dt^2}+\frac{dx}{dt}+y=0,\end{cases}$$

$$\text{记}\frac{d}{dt}\text{为}D\Rightarrow\begin{cases}(D^2-1)\,x+Dy=e^t\\Dx+(D^2+1)\,y=0\end{cases}$$

8 向量代数与空间解析几何

8.1 向量及其线性运算

8.1.1 向量的概念

大小, 方向, **向量**,$\overrightarrow{AB}$

与起点无关, **自由向量**

向量的大小, **向量的模**,$\left|\overrightarrow{AB}\right|$**, 单位向量, 零向量**

$\overrightarrow{OA}=a,\overrightarrow{OB}=b,\angle AOB<\pi$向量夹角, $\left(\widehat{a,b}\right)\begin{aligned}&=0 \text{ or } \pi, a\text{与}b\text{平行, 同起点共线}\\&=\frac{\pi}{2}, a\text{与}b\text{垂直}\end{aligned}$

8.1.2 向量的运算

向量的加减法

$c=a+b$三角形法则

$a+b=b+a$交换

$(a+b)+c=a+(b+c)$ 分配

$a-b=a+(-b)$ 负向量

$|a\pm b|<|a|+|b|$

向量与数的乘法

$|\lambda a|=|\lambda|\ |a|$

$\lambda(\mu a)=\mu(\lambda a)=(\lambda\mu)a$

$|\lambda(\mu a)|=|\mu(\lambda a)|=|(\lambda\mu)a|=|\lambda\mu|\ |a|$

$(\lambda+\mu)a=\lambda a+\mu a,\lambda(a+b)=\lambda a+\lambda b$

$a\neq 0, a//b\Leftrightarrow\exists$唯一实数$\lambda, b=\lambda a$

空间直角坐标系

$[O;i,j,k]$ 右手规则, 卦限

$\overrightarrow{OM}=r=xi+yj+zk$坐标式分解, 分向量,$r=(x,y,z)$, $M(x,y,z)$ 坐标,r为M关于O向径

利用坐标向量的线性运算

$$a=(a_x,a_y,a_z),b=(b_x,b_y,b_z)\begin{cases}a+b&=(a_x+b_x,a_y+b_y,a_z+b_z)\\a-b&=(a_x-b_x,a_y-b_y,a_z-b_z)\\\lambda a&=(\lambda a_x,\lambda a_y,\lambda a_z)\end{cases}$$

向量的模, 方向角, 投影

向量的模与两点间的距离

$r=(x,y,z),|r|=\sqrt{x^2+y^2+z^2}$

$A=(x_1,y_1,z_1),B=(x_2,y_2,z_2),|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

方向角与方向余弦

r与三条坐标轴的夹角α,β,γ称为r的方向角

$(\cos\alpha,\cos\beta,\cos\gamma)=\frac{1}{|r|}(x,y,z)$, $\cos*$方向余弦

向量在轴上的投影

$$\begin{cases}(a)_u=|a|\cos\varphi\\(a+b)_u=(a)_u+(b)_u=|a|\cos\varphi+|b|\cos\varphi\\(\lambda a)_u=\lambda(a)_u\end{cases}$$

8.2 数量积 向量积 混合积

8.2.1 兩向量的数量积

$a \cdot b = |a|\ |b| \cos\theta$數量積

$$\begin{cases} a \cdot a = |a|^2 \\ a, b \neq 0,\ a \cdot b = 0 \Leftrightarrow a \perp b \\ a \cdot b = b \cdot a\text{交}\boxed{\text{F}} \\ (a+b) \cdot c = a \cdot c + b \cdot c\text{分配} \\ (\lambda a) \cdot b = \lambda (a \cdot b)\ \text{結合} \end{cases}$$

$$\begin{array}{l} a = (a_x, a_y, a_z) \\ b = (b_x, b_y, b_z) \end{array} \Rightarrow \begin{array}{l} a \cdot b = a_x b_x + a_y b_y + a_z b_z \\ \cos\theta = \frac{a \cdot b}{|a|\ |b|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} + \sqrt{{b_x}^2 + {b_y}^2 + {b_z}^2}} \end{array}$$

8.2.2 兩向量的向量积

$|c| = |a|\ |b| \sin\theta$

$c = a \times b$向量積

$$\begin{cases} a \times a = 0 \\ a, b \neq 0, a \times b = 0 \Leftrightarrow a // b \\ a \times b = -b \times a \\ (a+b) \times c = a \times c + b \times c \\ (\lambda a) \times b = a \times (\lambda b) = \lambda (a \times b) \end{cases}$$

$$\begin{array}{l} a = (a_x, a_y, a_z) \\ b = (b_x, b_y, b_z) \end{array} \Rightarrow \begin{aligned} a \times b &= (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x) \\ &= \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \end{aligned}$$

8.2.3 向量的混合积

$(a \times b) \cdot c = [abc]$, 向量的混合积

$|[abc]| = |a|\ |b|\ \sin\left(\widehat{a, b}\right) |c| \cos\left(\widehat{a \times b, c}\right)$, a,b,c 右手系模大于0

$$\begin{array}{l} a = (a_x, a_y, a_z) \\ b = (b_x, b_y, b_z) \\ c = (c_x, c_y, c_z) \end{array} \Rightarrow \begin{aligned} [abc] &= (a \times b) \cdot c \\ &= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \end{aligned}$$

8.3 平面及其方程

8.3.1 曲面方程与空间曲线方程的概念

$F(x, y, z) = 0$, 曲面 S 上点满足方程, 不在曲面 S 上点不满足方程 $\Rightarrow$ 曲面 S 的方程

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \Rightarrow \text{曲线 C 的方程}$$

8.3.2 平面的点法式方程

向量垂直平面, 平面的法线向量

$$\begin{array}{ll} n = (A, B, C) & \text{法向量} \\ \overrightarrow{M_0 M} = (x - x_0, y - y_0, z - z_0) & \text{平面上点} \end{array} \Rightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\text{平面的点法式方程}$$

8.3.3 平面的一般方程

$$Ax+By+Cz+D=0\text{平面的一般方程}$$
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\text{平面的截距式方程}$$

8.3.4 平面的夹角

平面法向量的夹角, 平面的夹角 $\left(0\leqslant\theta\leqslant\frac{\pi}{2}\right)$

$$\Pi_1,n_1=(A_1,B_1,C_1),\Pi_2,n_2=(A_2,B_2,C_2),\cos\theta=\left|\left(\widehat{n_1,n_2}\right)\right|=\frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}\text{平面夹角}$$

$$\begin{aligned}d&=\left|\overrightarrow{P_1P_0}\right||\cos\theta|\\&=\tfrac{\left|\overrightarrow{P_1P_0}\times n\right|}{|n|}\\&=\tfrac{|A(x_0-x)+B(y_0-y)+C(z_0-z)|}{\sqrt{A^2+B^2+C^2}}\\&=\tfrac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}\end{aligned}$$

8.4 空间直线及其方程

8.4.1 空间直线的一般方程

$$\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}$$

8.4.2 空间直线的对称式方程与参数方程

向量平行直线, 直线的方向向量

$$\begin{aligned}\overrightarrow{M_oM}&=(x-x_0,y-y_0,z-z_0)\\S&=(m,n,p)\end{aligned}\Rightarrow\begin{aligned}&\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}=t,\text{对称式, 点向式方程}\\&\begin{cases}x=x_0+mt\\y=y_0+nt\quad\text{参数方程}\\z=z_0+pt\end{cases}\end{aligned}$$

8.4.3 两直线夹角

$$L_1,s_1=(A_1,B_1,C_1),L_2,s_2=(A_2,B_2,C_2),\cos\theta=\left|\left(\widehat{s_1,s_2}\right)\right|=\frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}\text{直线夹角}$$

8.4.4 直线与平面夹角

直线和投影直线的夹角, 直线与平面夹角

$$\begin{aligned}&\varphi=\left|\tfrac{\pi}{2}-(\overrightarrow{s,n})\right|\\&s=(m,n,p)\\&n=(A,B,C)\end{aligned}\quad\Rightarrow\quad\begin{aligned}\sin\varphi&=\tfrac{|Am+Bn+Cp|}{\sqrt{A^2+B^2+C^2}\sqrt{m^2+n^2+p^2}}\\\tfrac{A}{m}&=\tfrac{B}{n}=\tfrac{C}{p}\text{垂直}\end{aligned}$$

8.4.5 例

$$\left.\begin{aligned}A_1x+B_1y+C_1z+D_1=0\quad I\\A_2x+B_2y+C_2z+D_2=0\quad II\end{aligned}\right\}L\Rightarrow A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0\text{过 L 平面束, 不含 }II$$

8.5 曲面及其方程

8.5.1 曲面研究的基本问题

轨迹到方程, 方程到形状
$x^2+y^2+z^2=R^2$球面
$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$球面
$Ax^2+Ay^2+Az^2+Dx+Ey+Fz+G=0$一般球面

8.5.2 旋转曲面

平面上曲线绕直线旋转一周, 母线, 轴, 旋转曲面
$f(y,z)=0 \Rightarrow f\left(\pm\sqrt{x^2+y^2},z\right)=0$圆台
相交直线 L 绕 R 旋转一周, 交点顶点, 夹角 $\left(0<\alpha<\frac{\pi}{2}\right)$ 半顶角
$z^2=a^2(x^2+y^2)$ 圆锥
$\frac{x^2}{a^2}-\frac{z^2}{b^2}=1$双曲线 $\Rightarrow$ $\left.\begin{array}{l} \frac{(x^2+y^2)^2}{a^2}-\frac{z^2}{b^2}=1\text{单叶双曲面} \\ \frac{x^2}{a^2}-\frac{(z^2+y^2)^2}{b^2}=1\text{双叶双曲面} \end{array}\right\}$两种二次

8.5.3 柱面

$x^2+y^2=R^2$圆柱面, 准线 (定曲线), 母线
$\left.\begin{array}{l} y^2=ax\text{抛物柱面} \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\text{椭圆柱面} \\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\text{双曲柱面} \end{array}\right\}$三种二次

8.5.4 二次曲面九

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$椭圆锥面
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$椭圆抛物面
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$双曲抛物面
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$椭球面

8.6 空间曲线及其方程

8.6.1 空间曲线一般方程

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$$

8.6.2 空间曲线参数方程

$$\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases} \Rightarrow \begin{cases} x=a\cos\theta \\ y=a\cos\theta \quad \text{螺旋线}, 2\pi b\text{螺距} \\ z=b\theta \end{cases}$$

空间曲线参数方程

$$\Gamma\begin{cases} x=\varphi(t) \\ y=\psi(t) \\ z=\omega(t) \end{cases} \Rightarrow \begin{cases} x=\sqrt{[\varphi(t)]^2+[\varphi(t)]^2}\cos\theta \\ y=\sqrt{[\varphi(t)]^2+[\varphi(t)]^2}\sin\theta z=\omega(t), \end{cases} \quad \text{曲面}$$

$$\Gamma\begin{cases} x = a\sin\varphi \\ y = 0 \\ z = a\cos\varphi \end{cases} \Rightarrow \begin{cases} x = \sqrt{[a\sin\varphi]^2 + [0]^2}\cos\theta = a\sin\varphi\cos\theta \\ y = \sqrt{[a\sin\varphi]^2 + [0]^2}\sin\theta = a\sin\varphi\sin\theta \\ z = a\cos\varphi, \end{cases} \text{球面}$$

8.6.3 空间曲线在坐标面投影

$H(x,y) = 0$ 投影柱面

$$\begin{cases} H(x,y) = 0 \\ z = 0 \end{cases} \text{投影曲线}$$

9 多元函数微分及其应用

9.1 多元函数的基本概念

9.1.1 平面點集 *n 維空間

平面點集

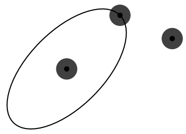
R^2 坐標平面

$C = \{(x,y)\,|x^2+y^2<r^2\}$ 平面點集

$U(P_0,\delta)=\{P|PP_0<\delta\}$ 鄰域

$\mathring{U}(P_0,\delta)=\{P|0<PP_0<\delta\}$ 去心鄰域

$$\begin{array}{ll} & \exists U(P), U(P)\subset E, P\text{□}E\text{□點} \\ & \exists U(P), U(P)\cap E=\varnothing, P\text{□}E\text{外點} \\ \begin{array}{l} P\in R^2, \\ E\subset R^2 \end{array} & \left.\begin{array}{l} P=P_1\cap P_2 \\ \exists U(P_1), U(P_1)\subset E \\ \exists U(P_2), U(P_2)\cap E=\varnothing \end{array}\right\}, P\text{□}E\text{邊界點} \end{array}$$



E邊界點的全體, □E邊界點, ∂E

$\forall\delta>0, \mathring{U}(P,\delta)$ □總有E中的點, P□E聚點 (□點和邊界)

開集, □點; 閉集, $\partial E\subset E$; 連通集, 任意點連□仍在集合;

區域, 連通開集；閉區域，區域和邊界；

有界集，$\forall E\subset R^2, if \quad \exists r>0, E\subset U(O,r), then$ 有界集；無界集，不是有界集

n 維空間

$R^n=\{(x_1,x_2,\cdots,x_n)\,|x_i\in R, i=1,2,\cdots,n\}=x$集合

$$\left.\begin{array}{ll} x+y & =(x_1+y_1,x_2+y_2,\cdots,x_n+y_n) \\ \lambda x & =(\lambda x_1,\lambda x_2,\cdots,\lambda x_n) \end{array}\right\} R^n\text{中□性運算}$$

$$\left.\begin{array}{l} x=(x_1,x_2,\cdots,x_n)\in R^n \\ y=(y_1,y_2,\cdots,y_n)\in R^n \\ a=(a_1,a_2,\cdots,a_n)\in R^n \\ \lambda\in R \end{array}\right\} \Rightarrow \begin{array}{l} \rho(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2} \\ ||x||=\rho(x,0)=\sqrt{{x_1}^2+{x_2}^2+\cdots+{x_n}^2} \\ ||x-a||\to 0\text{，變元趨于固定元，記作}x\to a \\ \text{某正數}\delta>0\text{, 點集}U(a,\delta)=\{x|x\in R^n,\rho(x,a)<\delta\}, a\text{的}\delta\text{鄰域} \end{array}$$

定義□性預算的集合，n 維空間

9.1.2 多元函數的概念

$$\begin{array}{l} D\subset R^2\text{, 映射}f:D\to R\text{, 稱□二元函數,} \\ \text{記□}z=f(x,y), (x,y)\in D \end{array} \Rightarrow \begin{array}{l} D\text{定義域} \\ x,y\text{自變量} \\ z\text{因變量} \\ \text{函數值}f(x,y)\text{全體}, f(D)=\{z|z=f(x,y),(x,y)\in D\}\text{值域} \end{array}$$

$D\subset R^n$, 映射$f:D\to R$, 稱□n元函數 ($n\geqslant 2$多元函數),

記□$z=f(x_1,x_2,\cdots,x_n)=f(\mathbf{x}), \mathbf{x}(x_1,x_2,\cdots,x_n)\in D$

多元函數$\mu=f(\mathbf{x})$, 有意義的變元$\mathbf{x}$的點集，自然定義域

空間點集 $\{(x,y,z)\,|z=f(x,y),(x,y)\in D\}$, 二元函數$z=f(x,y)$ 的圖形

9.1.3 多元函數的極限

$$\begin{array}{l} f(x,y)\text{定義域}D, \\ P_0(x_0,y_0)\text{是}D\text{聚點} \end{array} \quad if \quad \begin{array}{l} \exists A,\forall\varepsilon>0,\exists\delta>0, \\ when\text{ 點}P(x_0,y_0)\in D\cap\mathring{U}(P_0,\delta), \\ |f(P)-A|=|f(x,y)-A|<\varepsilon \end{array} \quad then \quad \begin{array}{l} A\text{稱□函數}f(x,y)\text{在}(x,y)\to(x_0,y_0)\text{極限,} \\ \text{記□}\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=A \end{array}$$

任意方向趨近

9.1.4 多元函數的連續性

$f(P)=f(x,y)$ 定義域$D, P_0(x_0,y_0)$ □D聚點$, P_0\in D,$ $\begin{array}{l} if \quad \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0), \\ then \quad f(x,y) \text{ 在} P_0(x_0,y_0) \text{ 連續} \end{array}$

$f(x,y)$ 定義域D, D□每一點都是聚點, $f(x,y)$ 在D□每一點連續, $f(x,y)$ 在D□連續

$f(x,y)$ 定義域$D, P_0(x_0,y_0)$ □D聚點$, if \quad f(x,y)$ 在$P_0(x_0,y_0)$ 不連續$, then \quad P_0(x_0,y_0)$ □$f(x,y)$ 間斷點

常數，不同自變量的一元基本初等函數，有限次四則運算和復合運算，**多元初等函數**

一切多元初等函數在**定義區域**（定義域□的區域或閉區域）連續 $\to \lim\limits_{P\to P_0} f(P)=f(P_0)$

	有界閉區域 D，多元連續函數，D 上有界，取得最大值，最小值	(**最值**)
性質	有界閉區域 D，多元連續函數，能取得介於最大值和最小值間的任何值	(**介值**)
	有界閉區域 D，多元連續函數，D 上一致連續	(**一致連續**)

9.2 偏導數

9.2.1 偏導數的定義及其計算法

$f(x,y), P_0(x_0,y_0), (x,y)\in U(P_0,\delta),$
$when \quad y=y_0, x=x_0+\Delta x,$
函數增量$f(x_0+\Delta x,y_0)-f(x_0,y_0)$

$if \quad \exists \lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=A,$
$then \quad A$□$f(x,y)$ 在 (x_0,y_0) 對x的偏導數,
記□ $\left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}}, f_x(x_0,y_0)$

$$\begin{cases} \left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} = \lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x} \\ \left.\dfrac{\partial f}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} = \lim\limits_{\Delta y\to 0}\dfrac{f(x_0,y_0+\Delta x)-f(x_0,y_0)}{\Delta y} \end{cases}$$

$f_x(x,y), f_y(x,y)$ 偏導函數, 偏導數

偏導數記號□整體，不能看成微分的商

(一元可導連) 各偏導存在，不一定連續

9.2.2 高階偏導數

$z=$ $\begin{array}{l} f(x,y) \text{ 的偏导数} \frac{\partial z}{\partial x}=f_x(x,y), \frac{\partial z}{\partial y}=f_y(x,y), \\ \text{都是} x,y \text{的函数,} \end{array}$ $\Rightarrow$

若偏导数的偏导仍存在，则称为$f(x,y)$ 的二阶偏导数

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y)$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f_{yx}(x,y)$ (混合偏导数)

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f_{xy}(x,y)$ (混合偏导数)

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)$

$if \quad f(x,y)$ 的二阶偏导数, $\frac{\partial^2 z}{\partial y\partial x}, \frac{\partial^2 z}{\partial x\partial y}$ 在 D 连续$, then \quad (x,y)\in D \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$

$$\left.\begin{array}{l} z=\ln\sqrt{x^2+y^2}, \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \\ u=\frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \end{array}\right\} \text{拉普拉斯方程}$$

9.3 全微分

9.3.1 全微分定義

$$\begin{cases} f(x+\Delta x,y)-f(x,y)\approx f_x(x,y)\Delta x \\ f(x,y+\Delta y)-f(x,y)\approx f_y(x,y)\Delta y \end{cases} \text{偏增量，偏微分}$$

$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y) \quad$ 全增量

$$z=f\left(x,y\right),\text{在}\left(x,y\right)\text{的某邻域有定义,}\quad \begin{array}{ll} if & \Delta z=f\left(x+\Delta x,y+\Delta y\right)-f\left(x,y\right)=A\Delta x+B\Delta y+o\left(\rho\right), \\ then & \rho=\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2},A,B\text{不依赖}\Delta x,\Delta y \\ & f\left(x,y\right)\text{在}\left(x,y\right)\text{可微分,} \\ & A\Delta x+B\Delta y,\text{称为函数}f\left(x,y\right)\text{全微分,} \\ & \text{记作}dz=A\Delta x+B\Delta y \end{array}$$

多元函数在区域 D 内个点处都可微分，函数在 D 内可微分

多元函数在**点 P** 可微分，函数在该点连续

$$if\quad f\left(x,y\right)\text{在点}\left(x,y\right)\text{可微分,}\quad \begin{array}{l} then\quad \text{函数}f\text{x,y在点}\left(x,y\right)\text{偏导数}\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\text{必存在} \\ \text{函数}f\left(x,y\right)\text{在点}\left(x,y\right)\text{全微分}dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \end{array}$$

$z=f\left(x,y\right)$的偏導數$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在點$\left(x,y\right)$連續,函數在該點可微分

微分F加原F,$u=\left(x,y,z\right),du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

9.3.2 全微分在近似計算中的應用

$g=\frac{4\pi^2 l}{T^2},\Delta g\leqslant 4\pi^2\left(\frac{1}{T^2}\delta_l+\frac{2l}{T^3}\delta_T\right)$

9.4 多元復合函數的求導法則

$$if\ \begin{array}{l} u=\varphi\left(t\right),v=\psi\left(t\right),\text{在}t\text{可導,} \\ z=f\left(u,v\right)\text{在}\left(u,v\right)\text{具有連續偏導} \end{array},then\quad \text{復合函數}z=f\left(\varphi\left(t\right),\psi\left(t\right)\right)\text{在}t\text{點可導, 且}\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}\text{全導數}$$

$$if\quad u=\varphi\left(t\right),v=\psi\left(t\right),w=\omega\left(t\right),\text{在}t\text{可導,}z=f\left(\varphi\left(t\right),then\quad \psi\left(t\right)+\omega\left(t\right)\right),\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}$$

$$if\ \begin{array}{l} u=\varphi\left(x,y\right),v=\psi\left(x,y\right), \\ \text{在}\left(x,y\right)\text{具有對}x,y\text{的偏導數,} \\ z=f\left(u,v\right)\text{在}\left(u,v\right)\text{具有連續偏導} \end{array},then\quad \begin{array}{l} \text{復合函數}z=f\left(\varphi\left(x,y\right),\psi\left(x,y\right)\right)\text{在}(x,y)\text{點兩偏導都存在, 且} \\ \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \end{array}$$

$$if\quad \begin{array}{l} u=\varphi\left(x,y\right),v=\psi\left(x,y\right),w=\omega\left(x,y\right), \\ \text{在}\left(x,y\right)\text{點兩偏導都存在} \\ z=f\left(\varphi\left(x,y\right),\psi\left(x,y\right)+\omega\left(x,y\right)\right), \end{array},then\quad \begin{array}{l} \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y} \end{array}$$

$$if\quad \begin{array}{ll} u & =\varphi\left(x,y\right)\text{在}\left(x,y\right)\text{點兩偏導都存在,} \\ v & =\psi\left(y\right)\text{在}y\text{點可導,} \\ z & =f\left(u,v\right)\text{在}\left(u,v\right)\text{具有連續偏導} \end{array}\quad then\quad \begin{array}{l} z=f\left(\varphi\left(x,y\right),\psi\left(y\right)\right)\text{在}\left(x,y\right)\text{點兩偏導都存在, 且} \\ \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \end{array}$$

$$ifz=f\left(u,x,y\right)=f\left[\varphi\left(x,y\right),x,y\right]\text{有}x,y\text{的偏導數,}then\quad \begin{array}{l} \frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y} \end{array}$$

$$f\left(u,v\right),\quad \begin{array}{l} f_1^{'}=f_u\left(u,v\right) \\ f_2^{'}=f_v\left(u,v\right) \\ f_{11}^{'}=f_{uu}\left(u,v\right) \\ f_{12}^{'}=f_{uv}\left(u,v\right) \\ f_{21}^{'}=f_{vu}\left(u,v\right) \\ f_{22}^{'}=f_{vv}\left(u,v\right) \end{array}$$

$$z=f\left(u,v\right)=f\left(\varphi\left(x,y\right),\psi\left(x,y\right)\right),\quad \begin{array}{ll} dz & =\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv \\ & =\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \\ & =\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)dy \end{array}$$

9.5 隱函數求導公式

9.5.1 一個方程

$$
if\quad \begin{array}{l} P_0(x_0,y_0),(x,y)\in U(P_0,\delta)\text{時} \\ F(x,y)\text{具有連續偏導}, \\ F(x_0,y_0)=0,F_y(x_0,y_0)\neq 0 \end{array},then\quad \begin{array}{l} F(x,y)=0,\text{在}(x,y)\in U(P_0,\delta)\text{時}, \\ \text{能確定唯一連續，連續導數的函數}y=f(x), \\ y_0=f(x_0),\frac{dy}{dx}=-\frac{F(x)}{F(y)} \end{array}
$$

$$
if\quad \begin{array}{l} P_0(x_0,y_0,z_0),(x,y,z)\in U(P_0,\delta)\text{時} \\ F(x,y,z)\text{具有連續偏導}, \\ F(x_0,y_0,z_0)=0,F_z(x_0,y_0,z_0)\neq 0 \end{array},then\quad \begin{array}{l} F(x,y,z)=0,\text{在}(x,y,z)\in U(P_0,\delta)\text{時}, \\ \text{能確定唯一連續，連續導數的函數}z=f(x,y), \\ z_0=f(x_0,y_0),\frac{\partial z}{\partial x}=-\frac{F(x)}{F(z)},\frac{\partial z}{\partial y}=-\frac{F(y)}{F(z)} \end{array}
$$

9.5.2 方程組的情形

$$
if\quad \begin{array}{l} P_0(x_0,y_0,u_0,v_0),(x,y,u,v)\in U(P_0,\delta)\text{時} \\ F(x,y,u,v),G(x,y,u,v)\text{具有連續偏導}, \\ F(x_0,y_0,u_0,v_0)=0,G(x_0,y_0,u_0,v_0)=0, \\ J=\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}\neq 0\left(\begin{array}{l}\text{雅可比式} \\ \text{偏导数组成的函数行列式}\end{array}\right) \end{array}\quad then\quad \begin{array}{l} F(x,y,u,v)=0,G(x,y,u,v)=0, \\ \text{在}(x,y,u,v)\in U(P_0,\delta)\text{時}, \\ \text{能確定唯一組連續，連續偏導數的函數}, \\ u=f(x,y),v=f(x,y) \\ u_0=f(x_0,y_0),v_0=f(x_0,y_0) \\ \frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)} \\ \frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)} \\ \frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)} \\ \frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)} \end{array}
$$

$$
\left\{\begin{array}{l} F_x+F_u\frac{\partial u}{\partial x}+F_v\frac{\partial v}{\partial x}=0 \\ G_x+G_u\frac{\partial u}{\partial x}+G_v\frac{\partial v}{\partial x}=0 \end{array}\right.\Rightarrow \begin{array}{l} \frac{\partial u}{\partial x}=\frac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}=-\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)} \\ \frac{\partial v}{\partial x}=\frac{\begin{vmatrix} F_u & -F_x \\ G_u & -G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}=-\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)} \end{array}\quad\text{证明或推导}
$$

$$
\begin{array}{l} dudv,x=\varphi(u,v),y=\psi(u,v), \\ dx=\frac{\partial\varphi}{\partial u}du+\frac{\partial\varphi}{\partial v}dv \\ dy=\frac{\partial\psi}{\partial u}du+\frac{\partial\psi}{\partial v}dv \end{array}\rightarrow \begin{array}{l} |(dx,dy)|_{du=0}\times(dx,dy)|_{dv=0}| \\ =\left|\left(\frac{\partial\varphi}{\partial u}du,\frac{\partial\psi}{\partial u}du\right)\times\left(\frac{\partial\varphi}{\partial v}dv,\frac{\partial\psi}{\partial v}dv\right)\right| \\ =\left|\frac{\partial\varphi}{\partial u}\cdot\frac{\partial\psi}{\partial v}-\frac{\partial\psi}{\partial u}\cdot\frac{\partial\varphi}{\partial v}\right|dudv \\ =\begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{vmatrix}dudv \\ =\frac{\partial(\varphi,\psi)}{\partial(u,v)}dudv \end{array}\quad\text{二重积分换元和雅可比}
$$

9.6 多元函数微分学的几何应用

9.6.1 一元向量值函数及其导数

$$
\Gamma\left\{\begin{array}{l} x=\varphi(t), \\ y=\psi(t),\quad t\in[\alpha,\beta] \\ z=\omega(t) \end{array}\right.
$$

記$r=xi+yj+zk,f(t)=\varphi(t)i+\psi(t)j+\omega(t)k,$

$r=f(t),t\in[\alpha,\beta]$

数集$D \subset R$,映射$f: D \rightarrow R^n$$\boxed{\text{F}}$一元向量值函數，記$\boxed{\text{F}}$$r = f(t), t \in D$

$t \in U(t_0)$,向量值函数$f(t), if \quad \exists r_0, \forall \varepsilon > 0, \exists \delta, when \quad 0 < |t - t_0| < \delta, |f(t) - r_0| < \varepsilon, then \quad \lim_{t \to t_0} f(t) = r_0$

$t \in u(t_0)$,向量值函数$f(t), if \quad \lim_{t \to t_0} f(t) = f(t_0), f(t)$在$t_0$连续

$t \in D$,,向量值函数$f(t), if \quad D_1 \subset D, D_1$内每点连续,$D_1$上连续

$t \in U(t_0)$,向量值函数$f(t), if \quad \exists \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}, then$ 极限向量为$r = f(t)$在t_0导数或导向量,记为$f'(t_0), \frac{dr}{dt}|_{t=t_0}$

$f'(t_0) = f_1'(t_0)i + f_2'(t_0)j + f_3'(t_0)k$

向量值函数$u(t), v(t)$,数量值函数$\varphi(t) \Rightarrow$
$$
\begin{array}{l}
\frac{d}{dt}C = 0 \\
\frac{d}{dt}[cu(t)] = cu'(t) \\
\frac{d}{dt}[u(t) \pm v(t)] = u'(t) + v'(t) \\
\frac{d}{dt}[\varphi(t) v(t)] = \varphi'(t) v(t) + \varphi(t) v'(t) \\
\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t) \\
\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t) \\
\frac{d}{dt}u[\varphi(t)] = \varphi'(t) u'[\varphi(t)]
\end{array}
$$

$f'(t_0) = \lim_{t \to t_0} \frac{\Delta r}{\Delta t}$与曲线相切

9.6.2 空间曲线的切线与法平面

$$
\Gamma \left\{ \begin{array}{l} x = \varphi(t), \\ y = \psi(t), \\ z = \omega(t), \end{array} \right. \quad t \in [\alpha, \beta] \Rightarrow \begin{array}{l} \frac{x - x_0}{\varphi'(t)} = \frac{y - y_0}{\psi'(t)} = \frac{z - z_0}{\omega'(t)} \\ \varphi'(t)(x - x_0) + \psi'(t)(y - y_0) + \omega'(t)(z - z_0) = 0 \end{array}
$$

$$
\Gamma \left\{ \begin{array}{l} y = \psi(x), \\ z = \omega(x), \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x = x, \\ y = \psi(x), \\ z = \omega(x), \end{array} \right. \Rightarrow \begin{array}{l} \frac{x - x_0}{1} = \frac{y - y_0}{\psi'(t)} = \frac{z - z_0}{\omega'(t)} \\ (x - x_0) + \psi'(t)(y - y_0) + \omega'(t)(z - z_0) = 0 \end{array}
$$

$$
\begin{array}{l}
\Gamma \left\{ \begin{array}{l} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{array} \right. \\
\frac{\partial(F,G)}{\partial(y,z)}|_{(x_0, y_0, z_0)} \neq 0
\end{array}
\Rightarrow \left\{ \begin{array}{l} y = \psi(x), \\ z = \omega(x), \end{array} \right. \Rightarrow
\begin{array}{l}
\left\{ \begin{array}{l} F(x, \psi(x), \omega(x)) = 0 \\ G(x, \psi(x), \omega(x)) = 0 \end{array} \right. \\
\left\{ \begin{array}{l} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx} = 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} + \frac{\partial G}{\partial z}\frac{dz}{dx} = 0 \end{array} \right. \\
\left\{ \begin{array}{l}
\frac{dy}{dx} = \psi'(x) = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,z)} = -\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} \\
\frac{dz}{dx} = \omega'(x) = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,x)} = -\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}}
\end{array} \right.
\end{array}
$$

9.6.3 曲面的切平面与法线

$$\begin{array}{l}\sum F(x,y,z)=0\\ 任意曲线\Gamma\left\{\begin{array}{l}x=\varphi(t),\\ y=\psi(t),\quad(\alpha<t<\beta),\\ z=\omega(t),\end{array}\right.\\ t=t_0对应M(x_0,y_0,z_0)\\ 曲线\Gamma在曲面\sum上，过M点\\ F[\varphi(t),\psi(t),\omega(t)]=0,\\ F_x|_M\varphi'(t_0)+F_y|_M\psi'(t_0)+F_z|_M\omega'(t_0)\\ =(F_x|_M,F_z|_M,F_z|_M)\cdot\left(\varphi'(t_0),\psi'(t_0),\omega'(t_0)\right)\\ =n\cdot s=0,\\ 任一过M曲线切线垂直过M向量n,切线共面,称为曲面切平面\end{array}\Rightarrow\begin{array}{l}F_x|_M(x-x_0)+F_z|_M(y-y_0)+F_z|_M(z-z_0)=0\\ \frac{(x-x_0)}{F_x|_M}=\frac{(y-y_0)}{F_y|_M}=\frac{(z-z_0)}{F_z|_M}\end{array}$$

$$z=f(x,y)\Rightarrow F(x,y,z)=f(x,y)-z=0\Rightarrow\begin{array}{l}F_x|_M=f_x(x_0,y_0),F_y|_M=f_y(x_0,y_0),F_z|_M=-1\\ 记K=\sqrt{f_x^2+f_y^2+1},\cos\gamma>0,\\ 方向余弦(\cos\alpha,\cos\beta,\cos\gamma)=\left(\frac{-f_x}{K},\frac{-f_y}{K},\frac{1}{K}\right)\\ f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0\\ \frac{(x-x_0)}{f_x(x_0,y_0)}=\frac{(y-y_0)}{f_y(x_0,y_0)}=\frac{(z-z_0)}{-1}\end{array}$$

9.7 方向导数与梯度

9.7.1 方向导数

$$\begin{array}{l}单位向量e_l=(\cos\alpha,\cos\beta),\\ 过(x_0,y_0)射线l\left\{\begin{array}{l}x=x_0+t\cos\alpha\\ y=y_0+t\cos\beta\end{array}\right.\quad(t\geqslant0),\\ z=f(x,y)\end{array}\Rightarrow\begin{array}{l}\frac{\partial f}{\partial l}|_{(x_0,y_0)}=\lim\limits_{t\to0^+}\frac{f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)}{t}\\ 方向导数\end{array}$$

$f(x,y)$ 在 $P_0(x_0,y_0)$ 可微分，该点任意方向，方向导数存在, $\frac{\partial f}{\partial l}|_{(x_0,y_0)}=f_x(x_0,y_0)\cos\alpha+f_y(x_0,y_0)\cos\beta$

$f(x,y,z),\frac{\partial f}{\partial l}|_{(x_0,y_0,z_0)}=f_x(x_0,y_0,z_0)\cos\alpha+f_y(x_0,y_0,z_0)\cos\beta+f_z(x_0,y_0,z_0)\cos\gamma$

9.7.2 梯度

$f(x,y),D$ 内连续偏导数, $\forall P_0\in D,\mathbf{grad}f(x_0,y_0)=\Delta f(x_0,y_0)=f_x|_{P_0}i+f_y|_{p_0}j$, 梯度, $\Delta=\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j$ 向量微分算子

$$\begin{array}{l}f(x,y)在(x_0,y_0)可微分,\\ 单位向量e_l=(\cos\alpha,\cos\beta)\end{array},\begin{array}{ll}\frac{\partial f}{\partial l}|_{(x_0,y_0)}&=f_x(x_0,y_0)\cos\alpha+f_y(x_0,y_0)\cos\beta\\ &=\mathbf{grad}f(x_0,y_0)\cdot e_l\\ &=|\mathbf{grad}f(x_0,y_0)|\cos\theta,\theta=\left(\mathbf{grad}\widehat{f(x_0,y_0),e_l}\right)\end{array}$$

$$\begin{array}{l}曲面\sum z=f(x,y),等值线L\,f(x,y)=0,P_0(x_0,y_0),\\ 记k=\sqrt{f_x^2(x_0,t_0)+f_y^2(x_0,y_0)},\end{array}\Rightarrow\begin{array}{l}\begin{array}{rl}单位法向量n&=\frac{1}{k}(f_x|_p,f_y|_p)\\ &=\frac{\Delta f(x_0,y_0)}{|\Delta f(x_0,y_0)|}\end{array}\\ \Delta f(x_0,y_0)=|\Delta f(x_0,y_0)|\,n=\frac{\partial f}{\partial n}n\\ 梯度=梯度的模\ 梯度的方向\end{array}$$

$$f(x,y,z),G内连续偏导数,P_0(x_0,y_0,z_0),\begin{array}{l}\mathbf{grad}f=\Delta f=f_x|_{P_0}i+f_y|_{P_0}j+f_z|_{P_0}k,\\ 等值面f(x,y,z)=c\\ 等值面法线方向n\end{array}$$

空间区域$G \quad \forall M \in G \to$ 数量$f(M)$, G内数量场
空间区域$G \quad \forall M \in G \to$ 向量$f(M)$, G内向量场
$if \quad F(M)$ 是$f(M)$的梯度 $then \quad f(M)$ 势函数, $F(M)$ 势场
$\frac{m}{r}$引力势, $\mathbf{grad}\frac{m}{r}$引力场

9.8 多元函数的极值及其求法

9.8.1 多元函数的极值及最大值最小值

$$D, z = f(x,y), P_0(x_0,y_0) \in D, \begin{array}{l} if \quad \exists U(P_0) \subset D, \forall (x,y) \neq P_0, f(x,y) < f(x_0,y_0), then \quad \text{极大值} f(x_0,y_0) \\ if \quad \exists U(P_0) \subset D, \forall (x,y) \neq P_0, f(x,y) > f(x_0,y_0), then \quad \text{极小值} f(x_0,y_0) \\ \text{极大值，极小值统称极值} \end{array}$$

$if \quad z = f(x,y)$ 在 (x_0,y_0) 具有偏导数, 且在 (x_0,y_0) 点具有极值, $then \quad f_x(x_0,y_0) = 0, f_y(x_0,y_0) = 0$
$if \quad f_x(x_0,y_0) = 0, f_y(x_0,y_0) = 0, then$ 驻点 (x_0,y_0)

$$if \quad z = f(x,y,z) \text{ 在 } (x_0,y_0,z_0) \text{ 具有偏导数, 且在 } (x_0,y_0,z_0) \text{ 点具有极值}, then \quad \begin{array}{l} f_x(x_0,y_0,z_0) = 0, \\ f_y(x_0,y_0,z_0) = 0, \\ f_z(x_0,y_0,z_0) = 0 \end{array}$$

$$\begin{array}{l} z = f(x,y) \text{ 在 } (x_0,y_0) \text{ 的某邻域内连续，} \\ \text{且有一阶及二阶偏导数，又} f_x(x_0,y_0) = 0, f_y(x_0,y_0) = 0 \\ \text{令} f_{xx}(x_0,y_0) = A, f_{xy}(x_0,y_0) = B, f_{yy}(x_0,y_0) = C \end{array} \; if \quad \begin{array}{l} AC - B^2 > 0, \text{有极值}, A < 0 \text{极大值}, A > 0 \text{极小值} \\ AC - B^2 < 0, \text{无极值} \\ AC - B^2 = 0, \text{可能有极值} \end{array}$$

驻点，偏导不存在点

9.8.2 条件极值　拉格朗日数乘法

无条件极值，条件极值；条件极值到无条件极值

$$\begin{array}{l} (x_o,y_0) \text{ 的某邻域内} z = f(x,y), \varphi(x,y), \\ \text{有连续一阶导数 } \varphi(x,y) = 0 \text{ (条件)} \to y = \varphi(x), \\ z = f(x,\varphi(x)) \text{ (函数)} \end{array} \Rightarrow \begin{array}{l} \frac{dz}{dx}|_{x=x_0} = 0 \\ f_x(x_0,y_0) + f_y(x_0,y_0)\frac{dy}{dx}|_{x=x_0} = 0 \\ f_x(x_0,y_0) - f_y(x_0,y_0)\frac{\varphi_x(x_0,y_0)}{\varphi_y(x_0,y_0)} = 0 \\ \text{记} \frac{f_y(x_0,y_0)}{\varphi_y(x_0,y_0)} = -\lambda \\ f_y(x_0,y_0) - f_y(x_0,y_0)\frac{\lambda}{\lambda} = 0 \\ \begin{cases} f_x(x_0,y_0) + \lambda\varphi_x(x_0,y_0) = 0 \\ f_y(x_0,y_0) + \lambda\varphi_y(x_0,y_0) = 0 \\ \varphi(x_0,y_0) = 0 \end{cases} \end{array}$$

$$L(x,y) = f(x,y) + \lambda\varphi(x,y) \Rightarrow \begin{cases} L_x(x_0,y_0) = 0 \\ L_y(x_0,y_0) = 0 \\ \varphi(x_0,y_0) = 0 \end{cases} \begin{array}{l} \text{拉格朗日函数} L, \text{拉格朗日乘子} \lambda \\ \text{函数} f(x,y), \text{条件} \varphi(x,y) = 0 \end{array}$$

$$L(x,y,z,t) = f(x,y,z,t) + \lambda\varphi(x,y,z,t) + \mu\psi(x,y,z,t) \Rightarrow \begin{cases} L_x(x_0,y_0,z_0,t_0) = 0 \\ L_y(x_0,y_0,z_0,t_0) = 0 \\ L_z(x_0,y_0,z_0,t_0) = 0 \\ L_t(x_0,y_0,z_0,t_0) = 0 \\ \varphi(x_0,y_0,z_0,t_0) = 0 \\ \psi(x_0,y_0,z_0,t_0) = 0 \end{cases} \begin{array}{l} \text{拉格朗日函数} L, \\ \text{拉格朗日乘子} \lambda, \mu \\ \text{函数} f(x,y,z,t), \\ \text{条件} \varphi(x,y,z,t) = 0, \\ \psi(x,y,z,t) = 0 \end{array}$$

9.8.3 二元函数的泰勒公式

$$\begin{array}{l} z = f(x,y) \text{ 在点} P_0(x_0,y_0) \text{ 某邻域内连续,} \\ \text{有 n+1 阶连续偏导数,} \\ (x_0+h, y_0+k) \in U(P_0,\delta) \end{array} \Rightarrow \begin{array}{l} f(x_0+h,y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0,y_0) + \\ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(x_0,y_0) \\ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k) \quad (0 < \theta < 1) \end{array}$$

证明:
$$
\begin{aligned}
&\Phi(t)=f(x_0+h,y_0+k)\ (0\leqslant t\leqslant 1)\\
&\Phi^{(n)}(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0+ht,y_0+kt)\\
&\Phi(t)=\Phi(0)+\Phi'(0)+\frac{1}{2}\Phi''(0)+\cdots+\frac{1}{n!}\Phi^{(n)}(0)+\frac{1}{(n+1)!}\Phi^{(n+1)}(\theta t)\ (0<\theta<1)\,(0\quad)\\
&\Phi(1)=\Phi(0)+\Phi'(0)+\frac{1}{2}\Phi''(0)+\cdots+\frac{1}{n!}\Phi^{(n)}(0)+\frac{1}{(n+1)!}\Phi^{(n+1)}(\theta)\ (0<\theta<1)\,(\quad)
\end{aligned}
$$

$n=0$时$f(x_0+h,y_0+k)-f(x_0,y_0)=hf_x(x_0+\theta h,y_0+\theta k)+kf_y(x_0+\theta h,y_0+\theta k)$

$f(x,y),(x,y)\in D,f_x(x,y)\equiv 0,f_y(x,y)\equiv 0,f(x,y)=c$

9.8.4 极值充分证明

$$
\begin{aligned}
\Delta f\ &=f(x,y)-f(x_0,y_0)\\
&=\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]f(x_0,y_0)+\frac{1}{2}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^2 f(x_0+\theta h,y_0+\theta k)\\
&f_x(x_0,y_0)=0,f_y(x_0,y_0)=0\\
&=\frac{1}{2}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^2 f(x_0+\theta h,y_0+\theta k)\\
&=\frac{1}{2}\left(h^2f_{xx}+2hkf_{xy}+k^2f_{yy}\right)\\
&=\frac{1}{2f_{xx}}\left(h^2f_{xx}^2+2hkf_{xy}f_{xx}+k^2f_{xy}^2-k^2f_{xy}^2+k^2f_{yy}f_{xx}\right)\\
&=\frac{1}{2f_{xx}}\left[\left(hf_{xx}+kf_{xy}\right)^2+k^2\left(f_{xx}f_{yy}-f_{xy}^2\right)\right]\\
&A=f_{xx},C=f_{yy},B=f_{xy}\\
&\begin{matrix}AC-B^2>0\\ \text{有}\end{matrix}\left\{\begin{matrix}A>0,f(x,y)>f(x_0,y_0),\text{极小值}\\ A<0,f(x,y)<f(x_0,y_0),\text{极大值}\end{matrix}\right.\\
&\begin{matrix}AC-B^2<0\\ \text{无}\end{matrix}\left\{\begin{matrix}A=C=0,k=h,\Delta f=f_{x,y}\\ A=C=0,k=-h,\Delta f=-f_{x,y}\\ A\neq C=0,k=0,\Delta f=\frac{1}{2}h^2f_{xx}\\ A\neq C=0,h=-f_{xy}s,k=f_{xx}s,\quad \begin{aligned}\Delta f\ &=\frac{1}{2}\left\{f_{xy}^2f_{xx}-2f_{xy}^2f_{xx}+f_{xx}^2f_{yy}\right\}\\ &=\frac{1}{2}f_{xx}\left\{f_{xy}^2-2f_{xy}^2+f_{xx}f_{yy}\right\}\\ &=\frac{1}{2}f_{xx}\left\{-f_{xy}^2\right\}\end{aligned}\end{matrix}\right.\\
&\begin{matrix}AC-B^2=0\\ \text{有可能}\end{matrix}\left\{\begin{matrix}f(x,y)=x^2+y^4\\ g(x,y)=x^2+y^3\end{matrix}\right.
\end{aligned}
$$

9.8.5 最小二乘法

y_i数据$\Rightarrow f(t)=at+b$线性经验公式
$M=\sum\left[y_i-f(t_i)\right]^2=\sum\left[y_i-(at_i+b)\right]^2$ 偏差平方和,
M 最小为条件选择常数 a,b 的方法, 最小二乘法
$\sqrt{M}$均方误差

$$
\text{非线性到线性}M_{min}(a,b)\Rightarrow\left\{\begin{matrix}M_a(a,b)=0\\ M_b(a,b)=0\end{matrix}\right.\left\{\begin{matrix}\frac{\partial M}{\partial a}=-2t_i\sum\left[y_i-(at_i+b)\right]=0\\ \frac{\partial M}{\partial b}=-2\sum\left[y_i-(at_i+b)\right]=0\end{matrix}\right.\left\{\begin{matrix}\sum t_iy_i-a\sum {t_i}^2-b\sum y_i=0\\ \sum y_i-a\sum t_iy_i-bn=0\end{matrix}\right.\Rightarrow\left\{\begin{matrix}a=\dfrac{\begin{vmatrix}\sum t_iy_i & \sum y_i\\ \sum y_i & n\end{vmatrix}}{\begin{vmatrix}\sum {t_i}^2 & \sum y_i\\ \sum t_iy_i & n\end{vmatrix}}\\ b=\dfrac{\begin{vmatrix}\sum {t_i}^2 & \sum t_iy_i\\ \sum t_iy_i & \sum y_i\end{vmatrix}}{\begin{vmatrix}\sum {t_i}^2 & \sum y_i\\ \sum t_iy_i & n\end{vmatrix}}\end{matrix}\right.
$$

10 重积分

10.1 二重积分的概念与性质

10.1.1 二重积分的概念

曲顶柱体的体积

$V=\lim_{\lambda\to 0}\sum f\left(\xi_i,\eta_i\right)\Delta\sigma_i$

平面薄片的质量

$m=\lim_{\lambda\to 0}\sum \mu\left(\xi_i,\eta_i\right)\Delta\sigma_i$

$$\begin{array}{l}\text{闭区域}D\text{有界函数}\\ \text{分割}D,\Delta\sigma_1,\Delta\sigma_2,\ldots,\Delta\sigma_n\end{array}\Rightarrow \iint_D f\left(x,y\right)d\sigma=\lim_{\lambda\to 0}f\left(\xi_i,\eta_i\right)\Delta\sigma\text{二重积分}$$

$\Delta\sigma_i=\Delta x_j\cdot y_k, d\sigma\to dxdy, \iint_D f\left(x,y\right)dxdy$

10.1.2 二重积分的性质

$$\begin{array}{l}g\left(x,y\right),g\\ f\left(x,y\right),f\end{array}\Rightarrow\begin{array}{l}\iint_D\left[\alpha f+\beta g\right]d\sigma=\alpha\iint_D fd\sigma+\beta\iint_D gd\sigma\\ \iint_D fd\sigma=\iint_{D_1}fd\sigma+\iint_{D_2}fd\sigma\\ \iint_D 1\cdot d\sigma=\iint_D\cdot d\sigma=\sigma\\ \iint_D fd\sigma\leqslant\iint_D gd\sigma, f\leqslant g\\ m\sigma\leqslant\iint_D fd\sigma\leqslant M\sigma, m\leqslant f\leqslant M\\ \iint_D fd\sigma=f\left(\xi,\eta\right)\sigma\quad\left(\xi,\eta\right)\in D\end{array}$$

10.2 二重积分的计算法

10.2.1 直角坐标计算二重积分

$$x\text{型}\begin{cases}D:\varphi_1\left(x\right)\leqslant y\leqslant\varphi_1\left(x\right),a\leqslant x\leqslant b\\ A\left(x_0\right)=\int_{\varphi_1(x_0)}^{\varphi_2(x_0)}f\left(x_0,y\right)dy\\ V=\int_a^b A\left(x\right)dx=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}f\left(x,y\right)dy\right]dx=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)}f\left(x,y\right)dy=\iint_D f\left(x,y\right)d\sigma\end{cases}$$

$$y\text{型}\begin{cases}D:\psi_1\left(y\right)\leqslant x\leqslant\psi_1\left(y\right),a\leqslant y\leqslant b\\ A\left(y_0\right)=\int_{\psi_1(y_0)}^{\psi_2(y_0)}f\left(x,y_0\right)dx\\ V=\int_a^b A\left(y\right)dy=\int_a^b\left[\int_{\psi_1(y)}^{\psi_2(y)}f\left(x,y\right)dx\right]dy=\int_a^b dy\int_{\psi_1(y)}^{\psi_2(y)}f\left(x,y\right)dx=\iint_D f\left(x,y\right)d\sigma\end{cases}$$

非 x 非 y 型，分割积分面

10.2.2 极坐标计算二重积分

$S_{\text{扇}}=\frac{r^2\theta}{2}$

$\Delta\sigma=\frac{1}{2}\left[\left(r+\Delta r\right)^2-r^2\right]\Delta\theta=\frac{1}{2}\left[\Delta r^2+2r\Delta r\right]\Delta\theta=\frac{1}{2}\left[\Delta r+2r\right]\Delta r\Delta\theta=\overline{r}\Delta r\Delta\theta$

$\lim_{\lambda\to 0}\sum f\left(\xi_i,\eta_i\right)\Delta\sigma_i=\lim_{\lambda\to 0}\sum f\left(\overline{r}\cos\overline{\theta},\overline{r}\sin\overline{\theta}\right)\overline{r}\Delta r\Delta\theta$

$\iint_D f\left(x,y\right)d\sigma=\iint_D f\left(x,y\right)dxdy=\iint_D f\left(r\cos\theta,r\sin\theta\right)rdrd\theta$

$\iint_D f\left(r\cos\theta,r\sin\theta\right)rdrd\theta=\int_a^b\left[\int_{\varphi_1(\theta)}^{\varphi_2(\theta)}f\left(r\cos\theta,r\sin\theta\right)rdr\right]d\theta$

$\sigma=\iint_D rdrd\theta=\frac{1}{2}\int_a^b\left[\varphi_2\left(\theta\right)^2-\varphi_1\left(\theta\right)^2\right]d\theta=\frac{1}{2}\int_a^b\varphi_2\left(\theta\right)^2d\theta$

10.2.3 二重积分换元法

$f(x,y)$, D内连续,
变换$T: x = x(u,v), y = y(u,v)$
uOv面$D^{'} \to xOy$面D, 一对一 $\Rightarrow \iint_D f(x,y)\,dxdy = \iint_{D'} f(x(u,v), y(u,v))\,|J(u,v)|\,dudv$
$x(u,v), y(u,v)$ 在$D^{'}$一阶连续偏导,
$D^{'}$上雅可比式,$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$,
$c^2 = a^2 + b^2 - 2ab\cos c$

$$\begin{cases} M_1: x_1 = x(u,v), y_1 = y(u,v) \\ M_2: x_2 = x(u+h,v) = x_1 + x_u(u,v)h + o(h) \\ y_2 = y(u+h,v) = y_1 + y_u(u,v)h + o(h) \\ M_4: x_3 = x(u,v+h) = x_1 + x_v(u,v)h + o(h) \\ y_3 = y(u,v+h) = y_1 + y_v(u,v)h + o(h) \end{cases}$$

$$\begin{aligned} s_{\text{三角形}} &= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \\ &= \frac{1}{2}\begin{vmatrix} x_u(u,v)h & y_u(u,v)h \\ x_v(u,v)h & y_v(u,v)h \end{vmatrix} = \frac{1}{2}h^2\begin{vmatrix} x_u(u,v) & y_u(u,v) \\ x_v(u,v) & y_v(u,v) \end{vmatrix} = \frac{1}{2}h^2\begin{vmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \end{vmatrix} = \frac{1}{2}h^2\frac{\partial(x,y)}{\partial(u,v)} \end{aligned}$$

10.3 三重积分

10.3.1 三重积分的概念

$f(x,y,z)$, 有界闭区域Ω,
分割成$\Delta v_1, \Delta v_2, \cdots, \Delta v_n$, $\Rightarrow \iiint_D f(x,y,z)\,dv = \iiint_D f(x,y,z)\,dxdydz = \lim_{\lambda\to 0}\sum f(\xi_i,\eta_i,\zeta_i)\,\Delta v_i$

10.3.2 三重积分的计算

利用直角坐标系计算

$F(x_0,y_0) = \int_{z_1(x_0,y_0)}^{z_2(x_0,y_0)} f(x_0,y_0,z)\,dz$

$$\iiint_\Omega f(x,y,z)\,dv \begin{aligned} &= \iint_{D_{xy}} F(x,y)\,d\sigma = \iint_{D_{xy}}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right]d\sigma = \int_a^b dx\int_{y_1(x)}^{y_2(x)} dy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \\ &= \int_{c_1}^{c_2} dz\iint_{D_z} f(x,y,z)\,dxdy \end{aligned}$$

利用柱坐标系计算

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \iiint_\Omega f(x,y,z)\,dxdydz = \iiint_\Omega f(r\cos\theta, r\sin\theta, z)\,rdrd\theta dz$$

$dv = rdrd\theta dz$

利用球面坐标计算

$$\begin{cases} x = OP\cos\theta = r\sin\varphi\cos\theta \\ y = OP\sin\theta = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases} \qquad \iiint_\Omega f(x,y,z)\,dxdydz = \iiint_\Omega f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)\,r^2\sin\varphi drd\varphi d\theta$$

φ,Z 轴;θ,X 轴
$dv = r^2\sin\varphi drd\varphi d\theta$

10.4 重积分的应用

曲面的面积

曲面$S: z = f\left(x, y\right)$

$\left|\left(f_x, f_y, -1\right)\cdot\left(0,0,1\right)\right| = 1 = \sqrt{{f_x}^2 + {f_y}^2 + -1^2}\cdot 1\cdot\cos\theta,$ $A = \iint_D \sqrt{{f_x}^2 + {f_y}^2 + 1} d\sigma$

$\cos\theta = \frac{1}{\sqrt{{f_x}^2 + {f_y}^2 + (-1)^2}}$

$\frac{d\sigma}{dA} = \cos\theta, \frac{\sum\sigma_k}{\sum A_k} = \cos\theta, dA = \frac{1}{\cos\theta}d\sigma$

*** 利用曲面的参数方程求曲面的面积**

$$\begin{cases} x = \\ y = \\ z = \end{cases}$$

tikz test