

This is Start

Vine

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## 7 微分方程

### 7.1 基本概念

$$\begin{cases} \frac{dy}{dx} = 2x \\ x = 1, y = 2 \end{cases} \Rightarrow y = 2x + 1$$

函數，函數導數，自變量關係的方程**微分方程**

最高階導數的階數**微分方程的階**

$F(x, y', \dots, y^{(n)}) = 0$  **一般形式**

函數**微分方程的解**

函數含常數，常數個數同階數**微分方程的通解**

確定常數的通解**微分方程的通解**

### 7.2 可分離變量的微分方程

$$\frac{dy}{dx} = 2x \Rightarrow dy = 2x dx \Rightarrow$$

$$\frac{dy}{dx} = 2xy^2 \Rightarrow \frac{dy}{y^2} = 2x dx \Rightarrow$$

$$g(y) dy = f(x) dx \Rightarrow y = \varphi(x)$$

$$G(y) = F(x) + C \Rightarrow y = \Phi(x)$$

### 7.3 齊次方程

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$$

### 7.4 一階 $\mathbb{R}$ 性微分方程

$\frac{dy}{dx} + P(x)y = Q(x)$  關於函數，函數導數是一次方程一階 $\mathbb{R}$ 性微分方程

$Q(x) = 0$  時**齊次方程**

$y = Ce^{-\int P(x)dx}$  **齊次通解**，設  $C = \mu(x)$

$$\mu(x) = \int Q(x) e^{\int P(x) dx} dx + C_2$$

$y = \left(\int Q(x) e^{\int P(x) dx} dx + C_2\right) e^{-\int P(x) dx}$  **非齊次通解**

$\frac{dy}{dx} + P(x)y = Q(x)y^n$  **伯努利方程**

$\frac{dy}{dx}y^{-n} + P(x)yy^{-n} = Q(x)$   $z = y^{1-n}, z' = (1-n)y^{-n}\frac{dy}{dx} = \frac{z'}{1-n} + P(x)z = Q(x)$  一階 $\mathbb{R}$ 性微分方程

### 7.5 可降階的高階微分方程

$$y^{(n)} = f(x) \Rightarrow y^{(n-1)} = \int f(x) dx$$

$$y'' = f(x, y') \quad \underbrace{y' = p, y'' = p'}_{p' = f(x, p)}$$

$$y'' = f(y, y') \quad \underbrace{y' = p, y'' = p \frac{dp}{dy}}_{p \frac{dp}{dy} = f(y, p)} \quad \underbrace{p = y' = \varphi(y, c_1)}_{p = y' = \varphi(y, c_1)}$$

### 7.6 高階性微分方程

二階微分方程

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x)$$

解的結構

$y = C_1y_1(x) + C_2y_2(x)$  是解

$y = C_1 y_1(x) + C_2 y_2(x)$  无关特解是通解

$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$  无关特解是通解

$y = Y(x) + y^*(x)$  齐次通, 非齐特, 非齐通

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = f_1(x) + f_2(x)$$

$y = y_1^*(x) + y_2^*(x)$  特解, 特解, 特解

常数变易法

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$Y(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$$

$$\begin{cases} y_1 v_1' + y_2 v_2' = 0 \\ y_1' v_1 + y_2' v_2 = f \end{cases} \Rightarrow \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & f \end{vmatrix} \xrightarrow{W = y_1 y_2' - y_1' y_2} \begin{cases} v_1' = -\frac{y_2 f}{W} \\ v_2' = \frac{y_1 f}{W} \end{cases} \Rightarrow \begin{cases} v_1 = -\int \frac{y_2 f}{W} dx + c_1 \\ v_2 = \int \frac{y_1 f}{W} dx + c_2 \end{cases}$$

$$\begin{cases} v_1 y_1 = \left(-\int \frac{y_2 f}{W} dx + c_1\right) y_1 \\ v_2 y_2 = \left(\int \frac{y_1 f}{W} dx + c_2\right) y_2 \end{cases} \Rightarrow Y(x) = c_1 y_1 + c_2 y_2 - y_1 \int \frac{y_2 f}{W} dx + y_2 \int \frac{y_1 f}{W} dx$$

## 7.7 常系数齐次线性微分方程

$$y'' + P(x) y' + Q(x) y = 0 \Rightarrow y'' + p y' + q y = 0$$

$$y = e^{rx}$$

$$(e^{rx})' = r e^{rx}$$

$$(r^2 + pr + q) e^{rx} = 0$$

$$\begin{cases} p^2 - 4q > 0, r_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \\ p^2 - 4q = 0, r_1 = r_2 = -\frac{p}{2}, y_2 = e^{r_1 x} \mu(x), \mu'' = 0 \\ p^2 - 4q < 0, r_{1,2} = \alpha \pm \beta i, e^{(i\theta)} = \cos(\theta) + i \sin(\theta), \overline{y_1} = \frac{1}{2}(y_1 + y_2), \overline{y_1} = \frac{1}{2i}(y_1 - y_2) \end{cases}$$

$$\begin{cases} p^2 - 4q > 0, y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x} \\ p^2 - 4q = 0, y = C_1 y_1 + C_2 \mu y_1 = (C_1 x + C_2) e^{r_1 x} \\ p^2 - 4q < 0, y = C_1 \overline{y_1} + C_2 \overline{y_2} = e^{\alpha x} (C_1 \cos(\beta x) + C_2 \sin(\beta x)) \end{cases}$$

n 阶常系数齐次微分方程

$$y^n + p_1 y^{n-1} + p_2 y^{n-2} + \cdots + p_{n-1} y' + p_n y = 0$$

$$\begin{cases} D, \frac{d}{dx} \\ Dy, \frac{dy}{dx} \\ D^n y, \frac{d^n y}{dx^n} \end{cases}$$

$$(D^n + p_1 D^{n-1} + \cdots + p_{n-1} D + p_n) y = 0 \Rightarrow L(D) y = 0, \text{微分算子 } D \text{ 的 } n \text{ 次多项式}$$

$$D e^{rx} = r e^{rx}, \dots, D^n e^{rx} = r^n e^{rx} \Rightarrow (r^n + p_1 r^{n-1} + \cdots + p_{n-1} r + p_n) e^{rx} = L(r) e^{rx} = 0 \Rightarrow L(r) = 0$$

$$\begin{cases} \text{单实根} & C e^{rx} \\ k \text{ 重实根} & (C_1 + C_2 x + \cdots + C_k x^{k-1}) e^{rx} \\ \text{单复根} & e^{\alpha x} (C_1 \cos(\beta x) + D_1 \sin(\beta x)) \\ k \text{ 重复根} & e^{\alpha x} ((C_1 + C_2 x + \cdots + C_k x^{k-1}) \cos(\beta x) + (D_1 + D_2 x + \cdots + D_k x^{k-1}) \sin(\beta x)) \end{cases}$$

## 7.8 常系数非齐次线性微分方程

$$y'' + p y' + q y = f(x)$$

$$p_m(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$

$$\begin{cases} e^{\theta i} = \cos(\theta) + i \sin(\theta) \\ e^{-\theta i} = \cos(\theta) - i \sin(\theta) \end{cases} \Rightarrow \begin{cases} \cos(\theta) = \frac{1}{2}(e^{\theta i} + e^{-\theta i}) \\ \sin(\theta) = \frac{1}{2i}(e^{\theta i} - e^{-\theta i}) \end{cases} \Rightarrow \begin{cases} \cos(\omega x) P = \frac{P}{2}(e^{\omega x i} + e^{-\omega x i}) \\ \sin(\omega x) Q = \frac{Q}{2i}(e^{\omega x i} - e^{-\omega x i}) \end{cases}$$

函数 (多项式) 共轭, 倒数共轭; 两对共轭, 乘积共轭;  $e^{\alpha + \theta i}$  与  $e^{\alpha - \theta i}$  共轭

$f(x) = e^{\lambda x} P_m(x)$	$y^* = R_{\lambda}(x) e^{\lambda x}$ $R''x + (2\lambda + p)R'x + (\lambda^2 + p\lambda + q)R(x) = p_m(x)$ $y^* = x^k P_m(x) e^{\lambda x}$
$f(x) = e^{\lambda x} [P_l(x) \cos(\omega x) + Q_n(x) \sin(\omega x)]$	$f(x) = e^{\lambda x} \left[ \left( \frac{P}{2} + \frac{Q}{2i} \right) e^{\omega x i} + \left( \frac{P}{2} - \frac{Q}{2i} \right) e^{-\omega x i} \right]$ $f(x) = \left( \frac{P}{2} + \frac{Q}{2i} \right) e^{\lambda x + \omega x i} + \left( \frac{P}{2} - \frac{Q}{2i} \right) e^{\lambda x - \omega x i}$ $f(x) = \left( \frac{P}{2} + \frac{Q}{2i} \right) e^{(\lambda + \omega i)x} + \left( \frac{P}{2} - \frac{Q}{2i} \right) e^{(\lambda - \omega i)x}$ $f(x) = P_1 e^{(\lambda + \omega i)x} + Q_2 e^{(\lambda - \omega i)x} \quad (P_1, Q_2 \text{共轭})$ $y_1^* = x^k R_m e^{(\lambda + \omega i)x} \quad (m = \max\{P_l, Q_n\})$ $y_2^* = x^k \overline{R_m} e^{(\lambda - \omega i)x}$ $y^* = y_1^* + y_2^* = x^k e^{\lambda x} (R_m e^{\omega x i} + \overline{R_m} e^{-\omega x i})$ $y^* = x^k e^{\lambda x} [R_m (\cos(\omega x) + i \sin(\omega x)) + \overline{R_m} (\cos(\omega x) - i \sin(\omega x))]$ $y^* = x^k e^{\lambda x} \left( R_m^{(1)} \cos(\omega x) + R_m^{(2)} \sin(\omega x) \right)$

## 7.9 欧拉方程

$$x^n y^{(n)} + p^1 x^{n-1} y^{(n-1)} + \cdots + p^{n-1} x y' + p^n y = f(x)$$

$$x = e^t, t = \ln x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} \quad D \text{表示} \frac{d}{dt} \quad \begin{cases} xy' = Dy \\ x^2 y'' = (D^2 - D)y \\ x^3 y''' = (D^3 - 3D^2 + 2D)y \\ \cdots \\ x^k y^{(k)} = (D^n + c_1 D^{n-1} + \cdots + C_{n-1} D)y \end{cases}$$

## 7.10 常系数线性微分方程组

$$\begin{cases} \frac{d^2 x}{dt^2} + \frac{dy}{dt} - x = e^t, \\ \frac{d^2 y}{dt^2} + \frac{dx}{dt} + y = 0, \end{cases}$$

$$\text{记} \frac{d}{dt} \text{为} D \Rightarrow \begin{cases} (D^2 - 1)x + Dy = e^t \\ Dx + (D^2 + 1)y = 0 \end{cases}$$

## 8 向量代数与空间解析几何

### 8.1 向量及其线性运算

#### 8.1.1 向量的概念

大小, 方向, 向量,  $\overrightarrow{AB}$

与起点无关, 自由向量

向量的大小, 向量的模,  $|\overrightarrow{AB}|$ , 单位向量, 零向量

$\overrightarrow{OA} = a, \overrightarrow{OB} = b, \angle AOB < \pi$  向量夹角,  $(\widehat{a, b}) = 0 \text{ or } \pi, a \text{ 与 } b \text{ 平行, 同起点共线}$   
 $= \frac{\pi}{2}, a \text{ 与 } b \text{ 垂直}$

#### 8.1.2 向量的运算

##### 向量的加减法

$c = a + b$  三角形法则

$a + b = b + a$  交换

$(a + b) + c = a + (b + c)$  分配

$a - b = a + (-b)$  负向量

$|a \pm b| < |a| + |b|$

##### 向量与数的乘法

$|\lambda a| = |\lambda| |a|$

$\lambda(\mu a) = \mu(\lambda a) = (\lambda\mu)a$

$|\lambda(\mu a)| = |\mu(\lambda a)| = |(\lambda\mu)a| = |\lambda\mu| |a|$

$(\lambda + \mu)a = \lambda a + \mu a, \lambda(a + b) = \lambda a + \lambda b$

$a \neq 0, a // b \Leftrightarrow \exists \text{ 唯一实数 } \lambda, b = \lambda a$

##### 空间直角坐标系

$[O; i, j, k]$  右手规则, 卦限

$\overrightarrow{OM} = r = xi + yj + zk$  坐标式分解, 分向量,  $r = (x, y, z), M(x, y, z)$  坐标,  $r$  为  $M$  关于  $O$  向径

##### 利用坐标向量的线性运算

$$a = (a_x, a_y, a_z), b = (b_x, b_y, b_z) \begin{cases} a + b = (a_x + b_x, a_y + b_y, a_z + b_z) \\ a - b = (a_x - b_x, a_y - b_y, a_z - b_z) \\ \lambda a = (\lambda a_x, \lambda a_y, \lambda a_z) \end{cases}$$

##### 向量的模, 方向角, 投影

###### 向量的模与两点间的距离

$r = (x, y, z), |r| = \sqrt{x^2 + y^2 + z^2}$

$A = (x_1, y_1, z_1), B = (x_2, y_2, z_2), |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

###### 方向角与方向余弦

$r$  与三条坐标轴的夹角  $\alpha, \beta, \gamma$  称为  $r$  的方向角

$(\cos \alpha, \cos \beta, \cos \gamma) = \frac{1}{|r|} (x, y, z), \cos *$  方向余弦

###### 向量在轴上的投影

$$\begin{cases} (a)_u = |a| \cos \varphi \\ (a + b)_u = (a)_u + (b)_u = |a| \cos \varphi + |b| \cos \varphi \\ (\lambda a)_u = \lambda (a)_u \end{cases}$$

## 8.2 数量积 向量积 混合积

### 8.2.1 两向量的数量积

$$\begin{aligned}
 a \cdot b &= |a| |b| \cos \theta \text{数量积} \\
 \begin{cases} a \cdot a = |a|^2 \\ a, b \neq 0, a \cdot b = 0 \Leftrightarrow a \perp b \\ a \cdot b = b \cdot a \text{交}\mathbb{F} \\ (a+b) \cdot c = a \cdot c + b \cdot c \text{分配} \\ (\lambda a) \cdot b = \lambda (a \cdot b) \text{结合} \end{cases} \\
 \begin{aligned} a &= (a_x, a_y, a_z) \\ b &= (b_x, b_y, b_z) \end{aligned} \Rightarrow \begin{aligned} a \cdot b &= a_x b_x + a_y b_y + a_z b_z \\ \cos \theta &= \frac{a \cdot b}{|a| |b|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \sqrt{b_x^2 + b_y^2 + b_z^2}} \end{aligned}
 \end{aligned}$$

### 8.2.2 两向量的向量积

$$\begin{aligned}
 |c| &= |a| |b| \sin \theta \\
 c &= a \times b \text{向量积} \\
 \begin{cases} a \times a = 0 \\ a, b \neq 0, a \times b = 0 \Leftrightarrow a // b \\ a \times b = -b \times a \\ (a+b) \times c = a \times c + b \times c \\ (\lambda a) \times b = a \times (\lambda b) = \lambda (a \times b) \end{cases} \\
 a \times b &= (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x) \\
 \begin{aligned} a &= (a_x, a_y, a_z) \\ b &= (b_x, b_y, b_z) \end{aligned} \Rightarrow \begin{aligned} &= \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \end{aligned}
 \end{aligned}$$

### 8.2.3 向量的混合积

$$\begin{aligned}
 (a \times b) \cdot c &= [abc], \text{向量的混合积} \\
 |[abc]| &= |a| |b| \sin(\widehat{a, b}) |c| \cos(\widehat{a \times b, c}), a, b, c \text{ 右手系模大于0} \\
 [abc] &= (a \times b) \cdot c \\
 \begin{aligned} a &= (a_x, a_y, a_z) \\ b &= (b_x, b_y, b_z) \\ c &= (c_x, c_y, c_z) \end{aligned} \Rightarrow \begin{aligned} &= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \end{aligned}
 \end{aligned}$$

## 8.3 平面及其方程

### 8.3.1 曲面方程与空间曲线方程的概念

$$\begin{aligned}
 F(x, y, z) = 0, \text{曲面 } S \text{ 上点满足方程, 不在曲面 } S \text{ 上点不满足方程} \Rightarrow \text{曲面 } S \text{ 的方程} \\
 \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \Rightarrow \text{曲线 } C \text{ 的方程}
 \end{aligned}$$

### 8.3.2 平面的点法式方程

$$\begin{aligned}
 &\text{向量垂直平面, 平面的法线向量} \\
 \begin{aligned} n &= (A, B, C) \\ \overrightarrow{M_0 M} &= (x - x_0, y - y_0, z - z_0) \end{aligned} \begin{aligned} &\text{法向量} \\ &\text{平面上点} \end{aligned} \Rightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \text{平面的点法式方程}
 \end{aligned}$$



### 8.3.3 平面的一般方程

$Ax + By + Cz + D = 0$  平面的一般方程

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$  平面的截距式方程

### 8.3.4 平面的夹角

平面法向量的夹角, 平面的夹角 ( $0 \leq \theta \leq \frac{\pi}{2}$ )

$$\Pi_1, n_1 = (A_1, B_1, C_1), \Pi_2, n_2 = (A_2, B_2, C_2), \cos \theta = |\widehat{(n_1, n_2)}| = \frac{|A_1 A_2 + B_1 B_2 + C_1 C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}} \text{平面夹角}$$

$$\begin{aligned} d &= \left| \overrightarrow{P_1 P_0} \right| |\cos \theta| \\ &= \frac{|\overrightarrow{P_1 P_0} \times n|}{|n|} \\ &= \frac{|A(x_0 - x) + B(y_0 - y) + C(z_0 - z)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \end{aligned}$$

## 8.4 空间直线及其方程

### 8.4.1 空间直线的一般方程

$$\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$

### 8.4.2 空间直线的对称式方程与参数方程

向量平行直线, 直线的方向向量

$$\begin{aligned} \overrightarrow{M_o M} &= (x - x_0, y - y_0, z - z_0) \\ S &= (m, n, p) \end{aligned} \Rightarrow \begin{cases} \frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p} = t, \text{对称式, 点向式方程} \\ \begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases} \text{参数方程} \end{cases}$$

### 8.4.3 两直线夹角

$$L_1, s_1 = (A_1, B_1, C_1), L_2, s_2 = (A_2, B_2, C_2), \cos \theta = |\widehat{(s_1, s_2)}| = \frac{|A_1 A_2 + B_1 B_2 + C_1 C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}} \text{直线夹角}$$

### 8.4.4 直线与平面夹角

直线和投影直线的夹角, 直线与平面夹角

$$\begin{aligned} \varphi &= \left| \frac{\pi}{2} - (\vec{s}, \vec{n}) \right| \\ s &= (m, n, p) \\ n &= (A, B, C) \end{aligned} \Rightarrow \begin{aligned} \sin \varphi &= \frac{|Am + Bn + Cp|}{\sqrt{A^2 + B^2 + C^2} \sqrt{m^2 + n^2 + p^2}} \\ \frac{A}{m} &= \frac{B}{n} = \frac{C}{p} \text{垂直} \end{aligned}$$

### 8.4.5 例

$$\left. \begin{aligned} A_1 x + B_1 y + C_1 z + D_1 &= 0 & I \\ A_2 x + B_2 y + C_2 z + D_2 &= 0 & II \end{aligned} \right\} L \Rightarrow A_1 x + B_1 y + C_1 z + D_1 + \lambda (A_2 x + B_2 y + C_2 z + D_2) = 0 \text{过 } L \text{ 平面束, 不含 } II$$

## 8.5 曲面及其方程

### 8.5.1 曲面研究的基本问题

轨迹到方程, 方程到形状

$$x^2 + y^2 + z^2 = R^2 \text{ 球面}$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2 \text{ 球面}$$

$$Ax^2 + Ay^2 + Az^2 + Dx + Ey + Fz + G = 0 \text{ 一般球面}$$

### 8.5.2 旋转曲面

平面上曲线绕直线旋转一周, 母线, 轴, 旋转曲面

$$f(y, z) = 0 \Rightarrow f(\pm\sqrt{x^2 + y^2}, z) = 0 \text{ 圆台}$$

相交直线 L 绕 R 旋转一周, 交点顶点, 夹角  $(0 < \alpha < \frac{\pi}{2})$  半顶角

$$z^2 = a^2(x^2 + y^2) \text{ 圆锥}$$

$$\left. \begin{aligned} \frac{x^2}{a^2} - \frac{z^2}{b^2} = 1 \text{ 双曲线} &\Rightarrow \frac{\frac{(x^2+y^2)^2}{a^2} - \frac{z^2}{b^2} = 1 \text{ 单叶双曲面}}{\frac{x^2}{a^2} - \frac{(z^2+y^2)^2}{b^2} = 1 \text{ 双叶双曲面}} \end{aligned} \right\} \text{ 两种二次}$$

### 8.5.3 柱面

$x^2 + y^2 = R^2$  圆柱面, 准线 (定曲线), 母线

$$y^2 = ax \text{ 抛物柱面}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ 椭圆柱面}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ 双曲柱面}$$

三种二次

### 8.5.4 二次曲面九

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2 \text{ 椭圆锥面}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z \text{ 椭圆抛物面}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z \text{ 双曲抛物面}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ 椭球面}$$

## 8.6 空间曲线及其方程

### 8.6.1 空间曲线一般方程

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

### 8.6.2 空间曲线参数方程

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \Rightarrow \begin{cases} x = a \cos \theta \\ y = a \sin \theta \\ z = b\theta \end{cases} \text{ 螺旋线, } 2\pi b \text{ 螺距}$$

### 空间曲线参数方程

$$\Gamma \begin{cases} x = \varphi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases} \Rightarrow \begin{cases} x = \sqrt{[\varphi(t)]^2 + [\psi(t)]^2} \cos \theta \\ y = \sqrt{[\varphi(t)]^2 + [\psi(t)]^2} \sin \theta \\ z = \omega(t) \end{cases} \text{ 曲面}$$

$$\Gamma \begin{cases} x = a \sin \varphi \\ y = 0 \\ z = a \cos \varphi \end{cases} \Rightarrow \begin{cases} x = \sqrt{[a \sin \varphi]^2 + [0]^2} \cos \theta = a \sin \varphi \cos \theta \\ y = \sqrt{[a \sin \varphi]^2 + [0]^2} \sin \theta = a \sin \varphi \sin \theta \\ z = a \cos \varphi, \end{cases} \quad \text{球面}$$

### 8.6.3 空间曲线在坐标面投影

$H(x, y) = 0$  投影柱面

$$\begin{cases} H(x, y) = 0 \\ z = 0 \end{cases} \quad \text{投影曲线}$$

## 9 多元函数微分及其应用

### 9.1 多元函数的基本概念

#### 9.1.1 平面點集 $n$ 維空間

平面點集

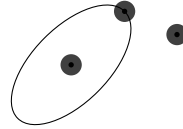
$R^2$  坐標平面

$C = \{(x, y) | x^2 + y^2 < r^2\}$  平面點集

$U(P_0, \delta) = \{P | PP_0 < \delta\}$  鄰域

$\dot{U}(P_0, \delta) = \{P | 0 < PP_0 < \delta\}$  去心鄰域

$\exists U(P), U(P) \subset E, P \in E$  內點  
 $\exists U(P), U(P) \cap E = \emptyset, P \notin E$  外點  
 $P \in R^2, P = P_1 \cap P_2$   
 $E \subset R^2 \quad \left. \begin{array}{l} \exists U(P_1), U(P_1) \subset E \\ \exists U(P_2), U(P_2) \cap E = \emptyset \end{array} \right\}, P \in E$  邊界點



$E$  邊界點的全體,  $\partial E$  邊界點,  $\partial E$

$\forall \delta > 0, \dot{U}(P, \delta) \cap E \neq \emptyset$  總有  $E$  中的點,  $P \in E$  聚點 (內點和邊界)

開集, 內點; 閉集,  $\partial E \subset E$ ; 連通集, 任意點連線仍在集合;

區域, 連通開集; 閉區域, 區域和邊界;

有界集,  $\forall E \subset R^2, \text{if } \exists r > 0, E \subset U(O, r), \text{then } E$  有界集; 無界集, 不是有界集

$n$  維空間

$R^n = \{(x_1, x_2, \dots, x_n) | x_i \in R, i = 1, 2, \dots, n\}$  集合

$x = (x_1, x_2, \dots, x_n) \in R^n$   
 $y = (y_1, y_2, \dots, y_n) \in R^n$   
 $a = (a_1, a_2, \dots, a_n) \in R^n$   
 $\lambda \in R$

$\left. \begin{array}{l} x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ \lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_n) \end{array} \right\} R^n \text{ 中 } \text{四} \text{性運算}$

$\Rightarrow \rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$   
 $\|x\| = \rho(x, 0) = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$   
 $\|x - a\| \rightarrow 0$ , 變元趨于固定元, 記作  $x \rightarrow a$

某正數  $\delta > 0$ , 點集  $U(a, \delta) = \{x | x \in R^n, \rho(x, a) < \delta\}$ ,  $a$  的  $\delta$  鄰域

定義  $\text{四} \text{性運算}$  的集合,  $n$  維空間

#### 9.1.2 多元函數的概念

$D$  定義域

$D \subset R^2$ , 映射  $f: D \rightarrow R$ , 稱  $\text{四} \text{二元函數}$ ,  $\Rightarrow x, y$  自變量

記  $\text{四} z = f(x, y), (x, y) \in D$   $z$  因變量

函數值  $f(x, y)$  全體,  $f(D) = \{z | z = f(x, y), (x, y) \in D\}$  值域

$D \subset R^n$ , 映射  $f: D \rightarrow R$ , 稱  $\text{四} n$  元函數 ( $n \geq 2$  多元函數),

記  $\text{四} z = f(x_1, x_2, \dots, x_n) = f(\mathbf{x}), \mathbf{x}(x_1, x_2, \dots, x_n) \in D$

多元函數  $\mu = f(\mathbf{x})$ , 有意義的變元  $\mathbf{x}$  的點集, 自然定義域

空間點集  $\{(x, y, z) | z = f(x, y), (x, y) \in D\}$ , 二元函數  $z = f(x, y)$  的圖形

#### 9.1.3 多元函數的極限

$f(x, y)$  定義域  $D$ ,  $\text{if } \exists A, \forall \varepsilon > 0, \exists \delta > 0$ ,  $\text{then } A$  稱  $\text{四} \text{函數 } f(x, y) \text{ 在 } (x, y) \rightarrow (x_0, y_0) \text{ 極限,}$   
 $P_0(x_0, y_0)$  是  $D$  聚點,  $\text{when } \text{點 } P(x_0, y_0) \in D \cap \dot{U}(P_0, \delta),$  記  $\text{四} \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = A$   
 $|f(P) - A| = |f(x, y) - A| < \varepsilon$

任意方向趨近

### 9.1.4 多元函數的連續性

$f(P) = f(x, y)$  定義域  $D, P_0(x_0, y_0) \in D$  聚點,  $P_0 \in D$ ,  $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$ ,  
 then  $f(x, y)$  在  $P_0(x_0, y_0)$  連續  
 $f(x, y)$  定義域  $D, D$  內每一點都是聚點,  $f(x, y)$  在  $D$  內每一點連續,  $f(x, y)$  在  $D$  內連續  
 $f(x, y)$  定義域  $D, P_0(x_0, y_0) \in D$  聚點, if  $f(x, y)$  在  $P_0(x_0, y_0)$  不連續, then  $P_0(x_0, y_0) \in D$  是  $f(x, y)$  的間斷點  
 常數, 不同自變量的一元基本初等函數, 有限次四則運算和複合運算, **多元初等函數**  
 一切多元初等函數在**定義區域** (定義域  $D$  的區域或閉區域) 連續  $\rightarrow \lim_{P \rightarrow P_0} f(P) = f(P_0)$   
 有界閉區域  $D$ , 多元連續函數,  $D$  上有界, 取得最大值, 最小值 (**最值**)  
 性質 有界閉區域  $D$ , 多元連續函數, 能取得介於最大值和最小值間的任何值 (**介值**)  
 有界閉區域  $D$ , 多元連續函數,  $D$  上一致連續 (**一致連續**)

## 9.2 偏導數

### 9.2.1 偏導數的定義及其計算法

$f(x, y), P_0(x_0, y_0), (x, y) \in U(P_0, \delta)$ ,  $\text{if } \exists \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = A$ ,  
 when  $y = y_0, x = x_0 + \Delta x$ , then  $A$  是  $f(x, y)$  在  $(x_0, y_0)$  對  $x$  的偏導數,  
 函數增量  $f(x_0 + \Delta x, y_0) - f(x_0, y_0)$  記作  $\left. \frac{\partial f}{\partial x} \right|_{x=x_0, y=y_0}, f_x(x_0, y_0)$   
 $\left\{ \begin{array}{l} \left. \frac{\partial f}{\partial x} \right|_{x=x_0, y=y_0} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \\ \left. \frac{\partial f}{\partial y} \right|_{x=x_0, y=y_0} = \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \end{array} \right.$   
 $f_x(x, y), f_y(x, y)$  偏導函數, 偏導數  
 偏導數記號  $\frac{\partial}{\partial x}$  整體, 不能看成微分的商  
 (一元可導連) 各偏導存在, 不一定連續

### 9.2.2 高階偏導數

若偏導數的偏導仍存在, 則稱為  $f(x, y)$  的二階偏導數  
 $z = f(x, y)$  的偏導數  $\frac{\partial z}{\partial x} = f_x(x, y), \frac{\partial z}{\partial y} = f_y(x, y)$ ,  $\Rightarrow$   $\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y)$   
 都是  $x, y$  的函數,  $\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y)$  (混合偏導數)  
 $\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y)$  (混合偏導數)  
 $\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y)$

if  $f(x, y)$  的二階偏導數,  $\frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x \partial y}$  在  $D$  連續, then  $(x, y) \in D, \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$   
 $z = \ln \sqrt{x^2 + y^2}, \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$   
 $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$  } 拉普拉斯方程

## 9.3 全微分

### 9.3.1 全微分定義

$\left\{ \begin{array}{l} f(x + \Delta x, y) - f(x, y) \approx f_x(x, y) \Delta x \\ f(x, y + \Delta y) - f(x, y) \approx f_y(x, y) \Delta y \end{array} \right.$  偏增量, 偏微分  
 $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$  全增量

$if \quad \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = A\Delta x + B\Delta y + o(\rho),$   
 $then \quad \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}, A, B \text{ 不依赖 } \Delta x, \Delta y$   
 $z = f(x, y), \text{ 在 } (x, y) \text{ 的某邻域有定义,}$   
 $f(x, y) \text{ 在 } (x, y) \text{ 可微分,}$   
 $A\Delta x + B\Delta y, \text{ 称为函数 } f(x, y) \text{ 全微分,}$   
 $记作 } dz = A\Delta x + B\Delta y$

多元函数在区域  $D$  内个点处都可微分, 函数在  $D$  内可微分

多元函数在点  $P$  可微分, 函数在该点连续

$if \quad f(x, y) \text{ 在点 } (x, y) \text{ 可微分,}$   
 $then \quad \text{函数 } f(x, y) \text{ 在点 } (x, y) \text{ 偏导数 } \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \text{ 必存在}$   
 $\text{函数 } f(x, y) \text{ 在点 } (x, y) \text{ 全微分 } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$z = f(x, y) \text{ 的偏导数 } \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \text{ 在点 } (x, y) \text{ 连续, 函数在该点可微分}$

微分  $\mathbb{F}$  加原  $\mathbb{F}, u = (x, y, z), du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$

### 9.3.2 全微分在近似计算中的应用

$$g = \frac{4\pi^2 l}{T^2}, \Delta g \leq 4\pi^2 \left( \frac{1}{T^2} \delta_l + \frac{2l}{T^3} \delta_T \right)$$

## 9.4 多元复合函数的求导法则

$if \quad u = \varphi(t), v = \psi(t), \text{ 在 } t \text{ 可导,}$   
 $z = f(u, v) \text{ 在 } (u, v) \text{ 具有连续偏导}$   
 $, then \quad \text{复合函数 } z = f(\varphi(t), \psi(t)) \text{ 在 } t \text{ 点可导, 且 } \frac{dz}{dt} = \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} \text{ 全导数}$

$if \quad u = \varphi(t), v = \psi(t), w = \omega(t), \text{ 在 } t \text{ 可导, } z = f(\varphi(t), \psi(t), \omega(t)),$   
 $\frac{dz}{dt} = \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} + \frac{\partial z}{\partial w} \frac{dw}{dt}$

$u = \varphi(x, y), v = \psi(x, y),$   
 $if \quad \text{在 } (x, y) \text{ 具有对 } x, y \text{ 的偏导数,}$   
 $z = f(u, v) \text{ 在 } (u, v) \text{ 具有连续偏导}$   
 $, then \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$   
 $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$

$u = \varphi(x, y), v = \psi(x, y), w = \omega(x, y),$   
 $if \quad \text{在 } (x, y) \text{ 点两偏导都存在}$   
 $z = f(\varphi(x, y), \psi(x, y), \omega(x, y)),$   
 $, then \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial x}$   
 $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial y}$

$u = \varphi(x, y) \text{ 在 } (x, y) \text{ 点两偏导都存在,}$   
 $if \quad v = \psi(y) \text{ 在 } y \text{ 点可导,}$   
 $z = f(u, v) \text{ 在 } (u, v) \text{ 具有连续偏导}$   
 $, then \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x}$   
 $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$

$if \quad z = f(u, x, y) = f[\varphi(x, y), x, y] \text{ 有 } x, y \text{ 的偏导数, then}$   
 $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial x}$   
 $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial y}$

$$\begin{aligned}
 f'_1 &= f'_u(u, v) \\
 f'_2 &= f'_v(u, v) \\
 f(u, v), \quad f'_{11} &= f''_{uu}(u, v) \\
 f'_{12} &= f''_{uv}(u, v) \\
 f'_{21} &= f''_{vu}(u, v) \\
 f'_{22} &= f''_{vv}(u, v)
 \end{aligned}$$

$$\begin{aligned}
 dz &= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv \\
 z = f(u, v) &= f(\varphi(x, y), \psi(x, y)), \\
 &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\
 &= \left( \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \right) dx + \left( \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} \right) dy
 \end{aligned}$$

## 9.5 隱函數求導公式

### 9.5.1 一個方程

$P_0(x_0, y_0), (x, y) \in U(P_0, \delta)$  時  $F(x, y) = 0$ , 在  $(x, y) \in U(P_0, \delta)$  時,  
 if  $F(x, y)$  具有連續偏導,  $F(x_0, y_0) = 0, F_y(x_0, y_0) \neq 0$ , then 能確定唯一連續, 連續導數的函數  $y = f(x)$ ,  
 $y_0 = f(x_0), \frac{dy}{dx} = -\frac{F_x}{F_y}$

$P_0(x_0, y_0, z_0), (x, y, z) \in U(P_0, \delta)$  時  $F(x, y, z) = 0$ , 在  $(x, y, z) \in U(P_0, \delta)$  時,  
 if  $F(x, y, z)$  具有連續偏導,  $F(x_0, y_0, z_0) = 0, F_z(x_0, y_0, z_0) \neq 0$ , then 能確定唯一連續, 連續導數的函數  $z = f(x, y)$ ,  
 $z_0 = f(x_0, y_0), \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

### 9.5.2 方程組的情形

$P_0(x_0, y_0, u_0, v_0), (x, y, u, v) \in U(P_0, \delta)$  時  
 $F(x, y, u, v), G(x, y, u, v)$  具有連續偏導,  
 if  $F(x_0, y_0, u_0, v_0) = 0, G(x_0, y_0, u_0, v_0) = 0$ , then  $u = f(x, y), v = f(x, y)$   
 $J = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} \neq 0$  (雅可比式 偏導数组成的函数行列式)  
 $F(x, y, u, v) = 0, G(x, y, u, v) = 0$ ,  
 在  $(x, y, u, v) \in U(P_0, \delta)$  時,  
 能確定唯一組連續, 連續偏導數的函數,  
 $u = f(x, y), v = f(x, y)$   
 $u_0 = f(x_0, y_0), v_0 = f(x_0, y_0)$   
 $\frac{\partial u}{\partial x} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, v)}$   
 $\frac{\partial u}{\partial y} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(y, v)}$   
 $\frac{\partial v}{\partial x} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(u, x)}$   
 $\frac{\partial v}{\partial y} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(u, y)}$

$$\begin{cases} F_x + F_u \frac{\partial u}{\partial x} + F_v \frac{\partial v}{\partial x} = 0 \\ G_x + G_u \frac{\partial u}{\partial x} + G_v \frac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F, G)}{\partial(x, v)}}{\frac{\partial(F, G)}{\partial(u, v)}} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, v)} \\ \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} F_u & -F_x \\ G_u & -G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F, G)}{\partial(u, x)}}{\frac{\partial(F, G)}{\partial(u, v)}} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(u, x)} \end{cases} \quad \text{证明或推导}$$

$$\begin{aligned} dudv, x = \varphi(u, v), y = \psi(u, v), \\ dx = \frac{\partial \varphi}{\partial u} du + \frac{\partial \varphi}{\partial v} dv \\ dy = \frac{\partial \psi}{\partial u} du + \frac{\partial \psi}{\partial v} dv \end{aligned} \rightarrow \begin{aligned} & |(dx, dy)|_{du=0} \times |(dx, dy)|_{dv=0}| \\ & = \left| \left( \frac{\partial \varphi}{\partial u} du, \frac{\partial \psi}{\partial u} du \right) \times \left( \frac{\partial \varphi}{\partial v} dv, \frac{\partial \psi}{\partial v} dv \right) \right| \\ & = \left| \frac{\partial \varphi}{\partial u} \cdot \frac{\partial \psi}{\partial v} - \frac{\partial \psi}{\partial u} \cdot \frac{\partial \varphi}{\partial v} \right| dudv \\ & = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} dudv \\ & = \frac{\partial(\varphi, \psi)}{\partial(u, v)} dudv \end{aligned} \quad \text{二重积分换元和雅可比}$$

## 9.6 多元函数微分学的几何应用

### 9.6.1 一元向量值函数及其导数

$$\Gamma \begin{cases} x = \varphi(t), \\ y = \psi(t), \quad t \in [\alpha, \beta] \\ z = \omega(t) \end{cases}$$

$$\text{記 } r = xi + yj + zk, f(t) = \varphi(t)i + \psi(t)j + \omega(t)k, \\ r = f(t), t \in [\alpha, \beta]$$

数集  $D \subset R$ , 映射  $f: D \rightarrow R^n$  一元向量值函数, 记  $\mathbb{F}r = f(t), t \in D$

$t \in U(t_0)$ , 向量值函数  $f(t)$ , if  $\exists r_0, \forall \varepsilon > 0, \exists \delta$ , when  $0 < |t - t_0| < \delta, |f(t) - r_0| < \varepsilon$ , then  $\lim_{t \rightarrow t_0} f(t) = r_0$

$t \in u(t_0)$ , 向量值函数  $f(t)$ , if  $\lim_{t \rightarrow t_0} f(t) = f(t_0)$ ,  $f(t)$  在  $t_0$  连续

$t \in D$ , 向量值函数  $f(t)$ , if  $D_1 \subset D$ ,  $D_1$  内每点连续,  $D_1$  上连续

极限向量为  $r = f(t)$

$t \in U(t_0)$ , 向量值函数  $f(t)$ , if  $\exists \lim_{\Delta t \rightarrow 0} \frac{\Delta r}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$ , then

在  $t_0$  导数或导向量,

记为  $f'(t_0)$ ,  $\frac{dr}{dt}|_{t=t_0}$

$$f'(t_0) = f'_1(t_0)i + f'_2(t_0)j + f'_3(t_0)k$$

$$\frac{d}{dt}C = 0$$

$$\frac{d}{dt}[cu(t)] = cu'(t)$$

$$\frac{d}{dt}[u(t) \pm v(t)] = u'(t) \pm v'(t)$$

$$\text{向量值函数 } u(t), v(t), \text{ 数量值函数 } \varphi(t) \Rightarrow \frac{d}{dt}[\varphi(t)v(t)] = \varphi'(t)v(t) + \varphi(t)v'(t)$$

$$\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$$

$$\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$$

$$\frac{d}{dt}u[\varphi(t)] = \varphi'(t)u'[\varphi(t)]$$

$$f'(t_0) = \lim_{t \rightarrow t_0} \frac{\Delta r}{\Delta t} \text{ 与曲线相切}$$

### 9.6.2 空间曲线的切线与法平面

$$\Gamma \begin{cases} x = \varphi(t), \\ y = \psi(t), \\ z = \omega(t), \end{cases} \quad t \in [\alpha, \beta] \Rightarrow \frac{x-x_0}{\varphi'(t)} = \frac{y-y_0}{\psi'(t)} = \frac{z-z_0}{\omega'(t)}$$

$$\varphi'(t)(x-x_0) + \psi'(t)(y-y_0) + \omega'(t)(z-z_0) = 0$$

$$\Gamma \begin{cases} y = \psi(x), \\ z = \omega(x), \end{cases} \Rightarrow \begin{cases} x = x, \\ y = \psi(x), \\ z = \omega(x), \end{cases} \Rightarrow \frac{x-x_0}{1} = \frac{y-y_0}{\psi'(x)} = \frac{z-z_0}{\omega'(x)}$$

$$(x-x_0) + \psi'(x)(y-y_0) + \omega'(x)(z-z_0) = 0$$

$$\Gamma \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \Rightarrow \begin{cases} y = \psi(x), \\ z = \omega(x), \end{cases} \Rightarrow \begin{cases} F(x, \psi(x), \omega(x)) = 0 \\ G(x, \psi(x), \omega(x)) = 0 \\ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} + \frac{\partial F}{\partial z} \frac{dz}{dx} = 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{dy}{dx} + \frac{\partial G}{\partial z} \frac{dz}{dx} = 0 \end{cases}$$

$$\frac{dy}{dx} = \psi'(x) = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, z)} = -\frac{\frac{\partial(F, G)}{\partial(x, z)}}{\frac{\partial(F, G)}{\partial(y, z)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}}$$

$$\frac{dz}{dx} = \omega'(x) = -\frac{1}{J} \frac{\partial(F, G)}{\partial(y, x)} = -\frac{\frac{\partial(F, G)}{\partial(y, x)}}{\frac{\partial(F, G)}{\partial(y, z)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}}$$



### 9.6.3 曲面的切平面与法线

$$\sum F(x, y, z) = 0$$

$$\text{任意曲线} \Gamma \begin{cases} x = \varphi(t), \\ y = \psi(t), \quad (\alpha < t < \beta), \\ z = \omega(t), \end{cases}$$

$$t = t_0 \text{ 对应 } M(x_0, y_0, z_0)$$

曲线  $\Gamma$  在曲面  $\sum$  上, 过  $M$  点

$$F[\varphi(t), \psi(t), \omega(t)] = 0,$$

$$F_x|_M \varphi'(t_0) + F_y|_M \psi'(t_0) + F_z|_M \omega'(t_0)$$

$$= (F_x|_M, F_y|_M, F_z|_M) \cdot (\varphi'(t_0), \psi'(t_0), \omega'(t_0))$$

$$= n \cdot s = 0,$$

任一过  $M$  曲线切线垂直过  $M$  向量  $n$ , 切线共面, 称为曲面切平面

$$\Rightarrow \begin{aligned} & F_x|_M(x - x_0) + F_y|_M(y - y_0) + F_z|_M(z - z_0) = 0 \\ & \frac{(x-x_0)}{F_x|_M} = \frac{(y-y_0)}{F_y|_M} = \frac{(z-z_0)}{F_z|_M} \end{aligned}$$

$$F_x|_M = f_x(x_0, y_0), F_y|_M = f_y(x_0, y_0), F_z|_M = -1$$

$$\text{记 } K = \sqrt{f_x^2 + f_y^2 + 1}, \cos \gamma > 0,$$

$$z = f(x, y) \Rightarrow F(x, y, z) = f(x, y) - z = 0 \Rightarrow \text{方向余弦 } (\cos \alpha, \cos \beta, \cos \gamma) = \left( \frac{-f_x}{K}, \frac{-f_y}{K}, \frac{1}{K} \right)$$

$$\begin{aligned} & f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0 \\ & \frac{(x-x_0)}{f_x(x_0, y_0)} = \frac{(y-y_0)}{f_y(x_0, y_0)} = \frac{(z-z_0)}{-1} \end{aligned}$$

## 9.7 方向导数与梯度

### 9.7.1 方向导数

$$\text{单位向量 } e_l = (\cos \alpha, \cos \beta),$$

$$\text{过 } (x_0, y_0) \text{ 射线 } l \begin{cases} x = x_0 + t \cos \alpha \\ y = y_0 + t \cos \beta \end{cases} (t \geq 0), \Rightarrow \frac{\partial f}{\partial l}|_{(x_0, y_0)} = \lim_{t \rightarrow 0^+} \frac{f(x_0 + t \cos \alpha, y_0 + t \cos \beta) - f(x_0, y_0)}{t}$$

方向导数

$$z = f(x, y)$$

$$f(x, y) \text{ 在 } P_0(x_0, y_0) \text{ 可微分, 该点任意方向, 方向导数存在, } \frac{\partial f}{\partial l}|_{(x_0, y_0)} = f_x(x_0, y_0) \cos \alpha + f_y(x_0, y_0) \cos \beta$$

$$f(x, y, z), \frac{\partial f}{\partial l}|_{(x_0, y_0, z_0)} = f_x(x_0, y_0, z_0) \cos \alpha + f_y(x_0, y_0, z_0) \cos \beta + f_z(x_0, y_0, z_0) \cos \gamma$$

### 9.7.2 梯度

$$f(x, y), D \text{ 内连续偏导数, } \forall P_0 \in D, \mathbf{grad} f(x_0, y_0) = \Delta f(x_0, y_0) = f_x|_{P_0} i + f_y|_{P_0} j, \text{ 梯度, } \Delta = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j \text{ 向量微分算子}$$

$$\begin{aligned} f(x, y) \text{ 在 } (x_0, y_0) \text{ 可微分, } \frac{\partial f}{\partial l}|_{(x_0, y_0)} &= f_x(x_0, y_0) \cos \alpha + f_y(x_0, y_0) \cos \beta \\ &= \mathbf{grad} f(x_0, y_0) \cdot e_l \\ \text{单位向量 } e_l = (\cos \alpha, \cos \beta), &= |\mathbf{grad} f(x_0, y_0)| \cos \theta, \theta = \left( \mathbf{grad} f(x_0, y_0), e_l \right) \end{aligned}$$

$$\begin{aligned} \text{曲面 } \sum z = f(x, y), \text{ 等值线 } L f(x, y) = 0, P_0(x_0, y_0), & \text{ 单位法向量 } n = \frac{1}{k} (f_x|_P, f_y|_P) \\ \text{记 } k = \sqrt{f_x^2(x_0, y_0) + f_y^2(x_0, y_0)}, & = \frac{\Delta f(x_0, y_0)}{|\Delta f(x_0, y_0)|} \\ & \Delta f(x_0, y_0) = |\Delta f(x_0, y_0)| n = \frac{\partial f}{\partial n} n \\ & \text{梯度} = \text{梯度的模} \quad \text{梯度的方向} \end{aligned}$$

$$\mathbf{grad} f = \Delta f = f_x|_{P_0} i + f_y|_{P_0} j + f_z|_{P_0} k,$$

$$f(x, y, z), G \text{ 内连续偏导数, } P_0(x_0, y_0, z_0), \text{ 等值面 } f(x, y, z) = c$$

等值面法线方向  $n$

空间区域  $G \quad \forall M \in G \rightarrow$  数量  $f(M)$ ,  $G$  内数量场

空间区域  $G \quad \forall M \in G \rightarrow$  向量  $f(M)$ ,  $G$  内向量场

if  $F(M)$  是  $f(M)$  的梯度 then  $f(M)$  势函数,  $F(M)$  势场

$\frac{m}{r}$  引力势,  $\mathbf{grad} \frac{m}{r}$  引力场

## 9.8 多元函数的极值及其求法

### 9.8.1 多元函数的极值及最大值最小值

if  $\exists U(P_0) \subset D, \forall (x, y) \neq P_0, f(x, y) < f(x_0, y_0)$ , then 极大值  $f(x_0, y_0)$

$D, z = f(x, y), P_0(x_0, y_0) \in D$ , if  $\exists U(P_0) \subset D, \forall (x, y) \neq P_0, f(x, y) > f(x_0, y_0)$ , then 极小值  $f(x_0, y_0)$

极大值, 极小值统称极值

if  $z = f(x, y)$  在  $(x_0, y_0)$  具有偏导数, 且在  $(x_0, y_0)$  点具有极值, then  $f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$

if  $f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$ , then 驻点  $(x_0, y_0)$

$$f_x(x_0, y_0, z_0) = 0,$$

if  $z = f(x, y, z)$  在  $(x_0, y_0, z_0)$  具有偏导数, 且在  $(x_0, y_0, z_0)$  点具有极值, then  $f_y(x_0, y_0, z_0) = 0,$

$$f_z(x_0, y_0, z_0) = 0$$

$z = f(x, y)$  在  $(x_0, y_0)$  的某邻域内连续,

$AC - B^2 > 0$ , 有极值,  $A < 0$  极大值,  $A > 0$  极小值

且有一阶及二阶偏导数, 又  $f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$  if

$AC - B^2 < 0$ , 无极值

令  $f_{xx}(x_0, y_0) = A, f_{xy}(x_0, y_0) = B, f_{yy}(x_0, y_0) = C$

$AC - B^2 = 0$ , 可能有极值

驻点, 偏导不存在点

### 9.8.2 条件极值 拉格朗日数乘法

无条件极值, 条件极值; 条件极值到无条件极值

$(x_0, y_0)$  的某邻域内  $z = f(x, y), \varphi(x, y)$ ,

有连续一阶导数  $\varphi(x, y) = 0$  (条件)  $\rightarrow y = \varphi(x)$ ,  $\Rightarrow$

$z = f(x, \varphi(x))$  (函数)

$$\left. \frac{dz}{dx} \right|_{x=x_0} = 0$$

$$f_x(x_0, y_0) + f_y(x_0, y_0) \left. \frac{dy}{dx} \right|_{x=x_0} = 0$$

$$f_x(x_0, y_0) - f_y(x_0, y_0) \frac{\varphi_x(x_0, y_0)}{\varphi_y(x_0, y_0)} = 0$$

$$\text{记 } \frac{f_y(x_0, y_0)}{\varphi_y(x_0, y_0)} = -\lambda$$

$$f_y(x_0, y_0) - f_y(x_0, y_0) \frac{\lambda}{\lambda} = 0$$

$$\begin{cases} f_x(x_0, y_0) + \lambda \varphi_x(x_0, y_0) = 0 \\ f_y(x_0, y_0) + \lambda \varphi_y(x_0, y_0) = 0 \\ \varphi(x_0, y_0) = 0 \end{cases}$$

$$L(x, y) = f(x, y) + \lambda \varphi(x, y) \Rightarrow \begin{cases} L_x(x_0, y_0) = 0 \\ L_y(x_0, y_0) = 0 \\ \varphi(x_0, y_0) = 0 \end{cases}$$

拉格朗日函数  $L$ , 拉格朗日乘子  $\lambda$   
函数  $f(x, y)$ , 条件  $\varphi(x, y) = 0$

$$L(x, y, z, t) = f(x, y, z, t) + \lambda \varphi(x, y, z, t) + \mu \psi(x, y, z, t) \Rightarrow \begin{cases} L_x(x_0, y_0, z_0, t_0) = 0 \\ L_y(x_0, y_0, z_0, t_0) = 0 \\ L_z(x_0, y_0, z_0, t_0) = 0 \\ L_t(x_0, y_0, z_0, t_0) = 0 \\ \varphi(x_0, y_0, z_0, t_0) = 0 \\ \psi(x_0, y_0, z_0, t_0) = 0 \end{cases}$$

拉格朗日函数  $L$ ,  
拉格朗日乘子  $\lambda, \mu$

函数  $f(x, y, z, t)$ ,

条件  $\varphi(x, y, z, t) = 0$ ,

$\psi(x, y, z, t) = 0$

### 9.8.3 二元函数的泰勒公式

$z = f(x, y)$  在点  $P_0(x_0, y_0)$  某邻域内连续,

有  $n+1$  阶连续偏导数,

$(x_0 + h, y_0 + k) \in U(P_0, \delta)$

$$\begin{aligned} f(x_0 + h, y_0 + k) &= f(x_0, y_0) + \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x_0, y_0) \\ &\Rightarrow \frac{1}{2!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(x_0, y_0) + \cdots + \frac{1}{n!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x_0, y_0) \\ &\quad + \frac{1}{(n+1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x_0 + \theta h, y_0 + \theta k) \quad (0 < \theta < 1) \end{aligned}$$

$$\Phi(t) = f(x_0 + h, y_0 + k) \quad (0 \leq t \leq 1)$$

$$\Phi^{(n)}(t) = \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x_0 + ht, y_0 + kt)$$

$$\Phi(t) = \Phi(0) + \Phi'(0) + \frac{1}{2}\Phi''(0) + \cdots + \frac{1}{n!}\Phi^{(n)}(0) + \frac{1}{(n+1)!}\Phi^{(n+1)}(\theta t) \quad (0 < \theta < 1) \quad (0)$$

$$\Phi(1) = \Phi(0) + \Phi'(0) + \frac{1}{2}\Phi''(0) + \cdots + \frac{1}{n!}\Phi^{(n)}(0) + \frac{1}{(n+1)!}\Phi^{(n+1)}(\theta) \quad (0 < \theta < 1) \quad ( )$$

$$n=0 \text{ 时 } f(x_0 + h, y_0 + k) - f(x_0, y_0) = hf_x(x_0 + \theta h, y_0 + \theta k) + kf_y(x_0 + \theta h, y_0 + \theta k)$$

$$f(x, y), (x, y) \in D, f_x(x, y) \equiv 0, f_y(x, y) \equiv 0, f(x, y) = c$$

#### 9.8.4 极值充分证明

$$\Delta f = f(x, y) - f(x_0, y_0)$$

$$= \left[ h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right] f(x_0, y_0) + \frac{1}{2} \left[ h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right]^2 f(x_0 + \theta h, y_0 + \theta k)$$

$$f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$$

$$= \frac{1}{2} \left[ h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right]^2 f(x_0 + \theta h, y_0 + \theta k)$$

$$= \frac{1}{2} (h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})$$

$$= \frac{1}{2f_{xx}} (h^2 f_{xx}^2 + 2hk f_{xy} f_{xx} + k^2 f_{xy}^2 - k^2 f_{xy}^2 + k^2 f_{yy} f_{xx})$$

$$= \frac{1}{2f_{xx}} \left[ (hf_{xx} + kf_{xy})^2 + k^2 (f_{xx} f_{yy} - f_{xy}^2) \right]$$

$$A = f_{xx}, C = f_{yy}, B = f_{xy}$$

$$AC - B^2 > 0 \quad \begin{cases} A > 0, f(x, y) > f(x_0, y_0), \text{极小值} \\ A < 0, f(x, y) < f(x_0, y_0), \text{极大值} \end{cases}$$

$$AC - B^2 < 0 \quad \begin{cases} A = C = 0, k = h, \Delta f = f_{x,y} \\ A = C = 0, k = -h, \Delta f = -f_{x,y} \end{cases}$$

$$AC - B^2 = 0 \quad \begin{cases} A \neq C = 0, k = 0, \Delta f = \frac{1}{2}h^2 f_{xx} \\ A \neq C = 0, h = -f_{xy} s, k = f_{xx} s, \end{cases}$$

$$AC - B^2 = 0 \quad \begin{cases} f(x, y) = x^2 + y^4 \\ g(x, y) = x^2 + y^3 \end{cases}$$

$$\begin{aligned} \Delta f &= \frac{1}{2} \{ f_{xy}^2 f_{xx} - 2f_{xy}^2 f_{xx} + f_{xx}^2 f_{yy} \} \\ &= \frac{1}{2} f_{xx} \{ f_{xy}^2 - 2f_{xy}^2 + f_{xx} f_{yy} \} \\ &= \frac{1}{2} f_{xx} \{ -f_{xy}^2 \} \end{aligned}$$

#### 9.8.5 最小二乘法

$y_i$  数据  $\Rightarrow f(t) = at + b$  线性经验公式

$$M = \sum [y_i - f(t_i)]^2 = \sum [y_i - (at_i + b)]^2 \text{ 偏差平方和,}$$

M 最小为条件选择常数 a, b 的方法, 最小二乘法

$\sqrt{M}$  均方误差

$$\text{非线性到线性 } M_{\min}(a, b) \Rightarrow \begin{cases} M_a(a, b) = 0 \\ M_b(a, b) = 0 \\ \frac{\partial M}{\partial a} = -2 \sum [y_i - (at_i + b)] = 0 \\ \frac{\partial M}{\partial b} = -2 \sum [y_i - (at_i + b)] = 0 \\ \sum t_i y_i - a \sum t_i^2 - b \sum y_i = 0 \\ \sum y_i - a \sum t_i y_i - bn = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{\left| \begin{array}{cc} \sum t_i y_i & \sum y_i \\ \sum y_i & n \end{array} \right|}{\left| \begin{array}{cc} \sum t_i^2 & \sum y_i \\ \sum t_i y_i & n \end{array} \right|} \\ b = \frac{\left| \begin{array}{cc} \sum t_i^2 & \sum t_i y_i \\ \sum t_i y_i & \sum y_i \end{array} \right|}{\left| \begin{array}{cc} \sum t_i^2 & \sum y_i \\ \sum t_i y_i & n \end{array} \right|} \end{cases}$$

## 10 重积分

### 10.1 二重积分的概念与性质

#### 10.1.1 二重积分的概念

曲顶柱体的体积

$$V = \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i) \Delta\sigma_i$$

平面薄片的质量

$$m = \lim_{\lambda \rightarrow 0} \sum \mu(\xi_i, \eta_i) \Delta\sigma_i$$

闭区域 $D$ 有界函数

$$\Rightarrow \iint_D f(x, y) d\sigma = \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i) \Delta\sigma_i \text{ 二重积分}$$

分割 $D, \Delta\sigma_1, \Delta\sigma_2, \dots, \Delta\sigma_n$

$$\Delta\sigma_i = \Delta x_j \cdot y_k, d\sigma \rightarrow dxdy, \iint_D f(x, y) dxdy$$

#### 10.1.2 二重积分的性质

$$\begin{aligned} \iint_D [\alpha f + \beta g] d\sigma &= \alpha \iint_D f d\sigma + \beta \iint_D g d\sigma \\ \iint_D f d\sigma &= \iint_{D_1} f d\sigma + \iint_{D_2} f d\sigma \\ g(x, y), g &\Rightarrow \iint_D 1 \cdot d\sigma = \iint_D d\sigma = \sigma \\ f(x, y), f &\Rightarrow \iint_D f d\sigma \leq \iint_D g d\sigma, f \leq g \\ m\sigma &\leq \iint_D f d\sigma \leq M\sigma, m \leq f \leq M \\ \iint_D f d\sigma &= f(\xi, \eta) \sigma \quad (\xi, \eta) \in D \end{aligned}$$

### 10.2 二重积分的计算法

#### 10.2.1 直角坐标计算二重积分

$$\begin{aligned} x\text{型} &\begin{cases} D: \varphi_1(x) \leq y \leq \varphi_2(x), a \leq x \leq b \\ A(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \\ V = \int_a^b A(x) dx = \int_a^b \left[ \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \right] dx = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy = \iint_D f(x, y) d\sigma \end{cases} \\ y\text{型} &\begin{cases} D: \psi_1(y) \leq x \leq \psi_2(y), a \leq y \leq b \\ A(y) = \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \\ V = \int_a^b A(y) dy = \int_a^b \left[ \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \right] dy = \int_a^b dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx = \iint_D f(x, y) d\sigma \end{cases} \end{aligned}$$

非 x 非 y 型, 分割积分面

#### 10.2.2 极坐标计算二重积分

$$\begin{aligned} S_{\text{扇}} &= \frac{r^2 \theta}{2} \\ \Delta\sigma &= \frac{1}{2} [(r + \Delta r)^2 - r^2] \Delta\theta = \frac{1}{2} [\Delta r^2 + 2r\Delta r] \Delta\theta = \frac{1}{2} [\Delta r + 2r] \Delta r \Delta\theta = \bar{r} \Delta r \Delta\theta \\ \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i) \Delta\sigma_i &= \lim_{\lambda \rightarrow 0} \sum f(\bar{r} \cos \bar{\theta}, \bar{r} \sin \bar{\theta}) \bar{r} \Delta r \Delta\theta \\ \iint_D f(x, y) d\sigma &= \iint_D f(x, y) dxdy = \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta \\ \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta &= \int_a^b \left[ \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r \cos \theta, r \sin \theta) r dr \right] d\theta \\ \sigma &= \iint_D r dr d\theta = \frac{1}{2} \int_a^b [\varphi_2(\theta)^2 - \varphi_1(\theta)^2] d\theta = \frac{1}{2} \int_a^b \varphi_2(\theta)^2 d\theta \end{aligned}$$

### 10.2.3 二重积分换元法

$f(x, y)$ ,  $D$  内连续,

变换  $T: x = x(u, v), y = y(u, v)$

$uOv$  面  $D' \rightarrow xOy$  面  $D$ , 一对一  $\Rightarrow \iint_D f(x, y) dx dy = \iint_{D'} f(x(u, v), y(u, v)) |J(u, v)| du dv$

$x(u, v), y(u, v)$  在  $D'$  一阶连续偏导,

$D'$  上雅可比式,  $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} \neq 0$ ,

$$c^2 = a^2 + b^2 - 2ab \cos c$$

$$\begin{cases} M_1: x_1 = x(u, v), y_1 = y(u, v) \\ M_2: x_2 = x(u+h, v) = x_1 + x_u(u, v)h + o(h) \\ y_2 = y(u+h, v) = y_1 + y_u(u, v)h + o(h) \\ M_4: x_3 = x(u, v+h) = x_1 + x_v(u, v)h + o(h) \\ y_3 = y(u, v+h) = y_1 + y_v(u, v)h + o(h) \end{cases}$$

$$\begin{aligned} s_{\text{三角形}} &= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \\ &= \frac{1}{2} \begin{vmatrix} x_u(u, v)h & y_u(u, v)h \\ x_v(u, v)h & y_v(u, v)h \end{vmatrix} = \frac{1}{2} h^2 \begin{vmatrix} x_u(u, v) & y_u(u, v) \\ x_v(u, v) & y_v(u, v) \end{vmatrix} = \frac{1}{2} h^2 \begin{vmatrix} x_u(u, v) & x_v(u, v) \\ y_u(u, v) & y_v(u, v) \end{vmatrix} = \frac{1}{2} h^2 \frac{\partial(x, y)}{\partial(u, v)} \end{aligned}$$

## 10.3 三重积分

### 10.3.1 三重积分的概念

$f(x, y, z)$ , 有界闭区域  $\Omega$ ,  $\Rightarrow \iiint_D f(x, y, z) dv = \iiint_D f(x, y, z) dx dy dz = \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i, \zeta_i) \Delta v_i$   
分割成  $\Delta v_1, \Delta v_2, \dots, \Delta v_n$ ,

### 10.3.2 三重积分的计算

利用直角坐标系计算

$$\begin{aligned} F(x_0, y_0) &= \int_{z_1(x_0, y_0)}^{z_2(x_0, y_0)} f(x_0, y_0, z) dz \\ \iiint_{\Omega} f(x, y, z) dv &= \iint_{D_{xy}} F(x, y) d\sigma = \iint_{D_{xy}} \left[ \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz \right] d\sigma = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz \\ &= \int_{c_1}^{c_2} dz \iint_{D_z} f(x, y, z) dx dy \end{aligned}$$

利用柱坐标系计算

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \Rightarrow \iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega} f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

$$dv = r dr d\theta dz$$

利用球面坐标计算

$$\begin{cases} x = OP \cos \theta = r \sin \varphi \cos \theta \\ y = OP \sin \theta = r \sin \varphi \sin \theta \\ z = r \cos \varphi \end{cases} \iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega} f(r \sin \varphi \cos \theta, r \sin \varphi \sin \theta, r \cos \varphi) r^2 \sin \varphi dr d\varphi d\theta$$

$\varphi, Z$  轴;  $\theta, X$  轴  
 $dv = r^2 \sin \varphi dr d\varphi d\theta$

## 10.4 重积分的应用

曲面的面积

曲面  $S: z = f(x, y)$

$$|(f_x, f_y, -1) \cdot (0, 0, 1)| = 1 = \sqrt{f_x^2 + f_y^2 + 1} \cdot 1 \cdot \cos \theta, \quad A = \iint_D \sqrt{f_x^2 + f_y^2 + 1} d\sigma$$

$$\cos \theta = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$\frac{d\sigma}{dA} = \cos \theta, \quad \sum \frac{\sigma_k}{A_k} = \cos \theta, \quad dA = \frac{1}{\cos \theta} d\sigma$$

\* 利用曲面的参数方程求曲面的面积

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \quad (u, v) \in D \quad \Rightarrow \quad \begin{aligned} A &= \iint_D \sqrt{EG - F^2} du dv \\ E &= x_u^2 + y_u^2 + z_u^2 \\ F &= x_u x_v + y_u y_v + z_u z_v \\ G &= x_v^2 + y_v^2 + z_v^2 \end{aligned}$$

$x, y, z$  在  $D$  内连续一阶

$\frac{\partial(x, y)}{\partial(u, v)}, \frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}$  不全为零

质心

$$\begin{aligned} \bar{x} &= \frac{M_y}{M} = \frac{\sum m_i x_i}{\sum m_i}, & dM_y &= x \mu(x, y) d\sigma, \quad M_y = \iint_D x \mu(x, y) d\sigma \\ \bar{y} &= \frac{M_x}{M} = \frac{\sum m_i y_i}{\sum m_i}, & dM_x &= y \mu(x, y) d\sigma, \quad M_x = \iint_D y \mu(x, y) d\sigma \\ M_y, M_x &\text{净距, } M \text{ 质点系总质量} & dM &= \mu(x, y) d\sigma, \quad M = \iint_D \mu(x, y) d\sigma \end{aligned}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\iint_D x \mu(x, y) d\sigma}{\iint_D \mu(x, y) d\sigma} = \frac{\mu(x, y)=c}{A=\iint_D d\sigma} \frac{1}{A} \iint_D x \mu(x, y) d\sigma$$

$$\bar{y} = \frac{M_x}{M} = \frac{\iint_D y \mu(x, y) d\sigma}{\iint_D \mu(x, y) d\sigma} = \frac{\mu(x, y)=c}{A=\iint_D d\sigma} \frac{1}{A} \iint_D y \mu(x, y) d\sigma$$

$$M = \iiint_{\Omega} \rho(x, y, z) dv, \quad \begin{cases} \bar{x} = \frac{1}{M} \iiint_{\Omega} x \rho(x, y, z) dv, \\ \bar{y} = \frac{1}{M} \iiint_{\Omega} y \rho(x, y, z) dv, \\ \bar{z} = \frac{1}{M} \iiint_{\Omega} z \rho(x, y, z) dv, \end{cases}$$

转动惯量

$$\begin{aligned} I_x &= \sum y_i^2 m_i, & dI_x &= y^2 \mu(x, y) d\sigma \\ I_y &= \sum x_i^2 m_i, & dI_y &= x^2 \mu(x, y) d\sigma \\ & \Rightarrow & I_x &= \iint_D y^2 \mu(x, y) d\sigma \\ & & I_y &= \iint_D x^2 \mu(x, y) d\sigma \end{aligned}$$

$$I_x = \iiint_{\Omega} (y^2 + z^2) \rho(x, y, z) dv$$

$$I_y = \iiint_{\Omega} (x^2 + z^2) \rho(x, y, z) dv$$

$$I_z = \iiint_{\Omega} (y^2 + x^2) \rho(x, y, z) dv$$

引力

$$F = G \frac{mM}{r^2}$$

$$(\cos \alpha, \cos \beta, \cos \gamma) = \left( \frac{x-x_0}{r}, \frac{y-y_0}{r}, \frac{z-z_0}{r} \right)$$

$$F = (|F| \cos \alpha, |F| \cos \beta, |F| \cos \gamma) \stackrel{|dF|=G \frac{1 \cdot \rho(x, y, z) dv}{r^2}}{=} \left( \iiint_{\Omega} \frac{G \rho(x, y, z)(x-x_0)}{r^2 \cdot r} dv, , \right)$$

## 10.5 含参变量的积分

$$f(x, y), R = [a, b] \times [c, d] = \{[x, y] | x \in [a, b], y \in [c, d]\}$$

$$\varphi(x) = \int_c^d f(x, y) dy \quad (a \leq x \leq b) \text{ 含参数变量积分}$$

$f(x, y)$  在  $R$  连续,  $\varphi(x)$  在  $[a, b]$  连续

$$f(x, y) \text{ 在 } R \text{ 连续, } \int_a^b dx \int_c^d f(x, y) dy = \int_c^d dy \int_a^b f(x, y) dx$$

$$f(x, y), f_x(x, y) \text{ 在 } R \text{ 连续, } \varphi(x) \text{ 在 } [a, b] \text{ 可微, } \varphi'(x) = \frac{d}{dx} \int_c^d f(x, y) dy = \int_c^d f_x(x, y) dy$$

tikz test

