This is Start

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2022 年 6 月 28 日

目录

7 微分方程

7.1 基本概念

$\begin{cases} \frac{dy}{dx} = 2x \\ x = 1, y = 2 \end{cases} \Rightarrow y = 2x + 1$

函數，函數導數，自變量關🄵的方程**微分方程**

最高階導數的階數**微分方程的階**

$F\left(x, y^{'}, \ldots, y^{(n)}\right) = 0$ **一般形式**

函數**微分方程的解**

函數含常數，常數個數同階數**微分方程的通解**

確定常數的通解**微分方程的通解**

7.2 可分離變量的微分方程

$\frac{dy}{dx} = 2x \Rightarrow dy = 2xdx \Rightarrow$

$\frac{dy}{dx} = 2xy^2 \Rightarrow \frac{dy}{y^2} = 2xdx \Rightarrow$

$g(y)\,dy = f(x)\,dx \Rightarrow y = \varphi(x)$

$G(y) = F(x) + C \Rightarrow y = \varPhi(x)$

7.3 齊次方程

$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$

$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$

7.4 一階🄵性微分方程

$\frac{dy}{dx} + P(x)\,y = Q(x)$ 關於函數，函數導數是一次方程**一階🄵性微分方程**

$Q(x) = 0$ 時**齊次方程**

$y = Ce^{-\int P(x)dx}$ **齊次通解**, 設 $C = \mu(x)$

$\mu(x) = \int Q(x)\,e^{\int P(x)\,dx}\,dx + C_2$

$y = \left(\int Q(x)\,e^{\int P(x)\,dx}\,dx + C_2\right)e^{-\int P(x)dx}$ **非齊次通解**

$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$ **伯努利方程**

$\frac{dy}{dx}y^{-n} + P(x)\,yy^{-n} = Q(x) \xrightarrow{z = y^{1-n}, z^{'} = (1-n)\,y^{-n}\frac{dy}{dx}} \frac{z^{'}}{(1-n)} + P(x)\,z = Q(x)$ **一階🄵性微分方程**

7.5 可降階的高階微分方程

$y^{(n)} = f(x) \quad \Rightarrow \quad y^{(n-1)} = \int f(x)\,dx$

$y^{''} = f\left(x, y^{'}\right) \quad \xrightarrow{y^{'} = p, y^{''} = p^{'}} \quad p^{'} = f(x, p)$

$y^{''} = f\left(y, y^{'}\right) \quad \xrightarrow{y^{'} = p, y^{''} = p\frac{dp}{dy}} \quad p\frac{dp}{dy} = f(y, p) \quad \xrightarrow{p = y^{`} = \varphi(y, c_1)}$

7.6 高階性微分方程

二阶微分方程

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = f(x)$$

解的结构

$y = C_1y_1(x) + C_2y_2(x)$ 是解

$y=C_1y_1(x)+C_2y_2(x)$ 无关特解是通解
$y=C_1y_1(x)+C_2y_2(x)+\cdots+C_ny_n(x)$ 无关特解是通解
$y=Y(x)+y^*(x)$ 齐次通，非齐特，非齐通

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=f_1(x)+f_2(x)$$

$y=y_1^*(x)+y_2^*(x)$ 特解，特解，特解
常数变易法
$Y(x)=C_1y_1(x)+C_2y_2(x)$
$Y(x)=v_1(x)y_1(x)+v_2(x)y_2(x)$
$\begin{cases} y_1v_1^{'}+y_2v_2^{'}=0 \\ y_1^{'}v_1^{'}+y_2^{'}v_2^{'}=f \end{cases} \Rightarrow \begin{vmatrix} y_1 & y_2 & 0 \\ y_1^{'} & y_2^{'} & f \end{vmatrix} \xrightarrow{W=y_1y_2^{'}-y_1^{'}y_2} \begin{cases} v_1^{'}=-\frac{y_2f}{W} \\ v_2^{'}=\frac{y_1f}{W} \end{cases} \Rightarrow \begin{cases} v_1=-\int\frac{y_2f}{W}dx+c_1 \\ v_2=\int\frac{y_1f}{W}dx+c_2 \end{cases}$
$\begin{cases} v_1y_1=\left(-\int\frac{y_2f}{W}dx+c_1\right)y_1 \\ v_2y_2=\left(\int\frac{y_1f}{W}dx+c_2\right)y_2 \end{cases} \Rightarrow Y(x)=c_1y_1+c_2y_2-y_1\int\frac{y_2f}{W}dx+y_2\int\frac{y_1f}{W}dx$

7.7 常系数齐次线性微分方程

$y^{''}+P(x)y^{'}+Q(x)y=0\Rightarrow y^{''}+py^{'}+qy=0$
$y=e^{rx}$
$(e^{rx})^{'}=re^{rx}$
$(r^2+pr+q)e^{rx}=0$
$\begin{cases} p^2-4q>0, r_{1,2}=\frac{-p\pm\sqrt{p^2-4q}}{2} \\ p^2-4q=0, r_1=r_2=\frac{-p}{2}, y_2=e^{r_1x}\mu(x), \mu^{''}=0 \\ p^2-4q<0, r_{1,2}=\alpha\pm\beta i, e^{(i\theta)}=\cos(\theta)+i\sin(\theta), \overline{y_1}=\frac{1}{2}(y_1+y_2), \overline{y_1}=\frac{1}{2i}(y_1-y_2) \end{cases}$
$\begin{cases} p^2-4q>0, y=C_1y_1+C_2y_2=C_1e^{r_1x}+C_2e^{r_2x} \\ p^2-4q=0, y=C_1y_1+C_2\mu y_1=(C_1x+C_2)e^{r_1x} \\ p^2-4q<0, y=C_1\overline{y_1}+C_2\overline{y_2}=e^{\alpha x}(C_1\cos(\beta x)+C_2\sin(\beta x)) \end{cases}$
n 阶常系数齐次微分方程
$y^n+p_1y^{n-1}+p_2y^{n-2}+\cdots+p_{n-1}y^{'}+p_ny=0$
$\begin{cases} D, \frac{d}{dx} \\ Dy, \frac{dy}{dx} \\ D^ny, \frac{d^ny}{dx^n} \end{cases}$
$(D^n+p_1D^{n-1}+\cdots+p_{n-1}D+p_n)y=0\Rightarrow L(D)y=0,$ 微分算子 D 的 n 次多项式
$De^{rx}=re^{rx},\ldots,D^ne^{rx}=r^ne^{rx}\Rightarrow(r^n+p_1r^{n-1}+\cdots+p_{n-1}r+p_n)e^{rx}=L(r)e^{rx}=0\Rightarrow L(r)=0$
$\begin{cases} \text{单实根} & Ce^{rx} \\ \text{k 重实根} & \left(C_1+C_2x+\cdots+C_kx^{k-1}\right)e^{rx} \\ \text{单复根} & e^{\alpha x}(C_1\cos(\beta x)+D_1\sin(\beta x)) \\ \text{k 重复根} & e^{\alpha x}\left(\left(C_1+C_2x+\cdots+C_kx^{k-1}\right)\cos(\beta x)+\left(D_1+D_2x+\cdots+D_kx^{k-1}\right)\sin(\beta x)\right) \end{cases}$

7.8 常系数非齐次线性微分方程

$y^{''}+py^{'}+qy=f(x)$
$p_m(x)=a_0x^m+a_1x^{m-1}+\cdots+a_m$
$\begin{cases} e^{\theta i}=\cos(\theta)+i\sin(\theta) \\ e^{-\theta i}=\cos(\theta)-i\sin(\theta) \end{cases} \Rightarrow \begin{cases} \cos(\theta)=\frac{1}{2}\left(e^{\theta i}+e^{-\theta i}\right) \\ \sin(\theta)=\frac{1}{2i}\left(e^{\theta i}-e^{-\theta i}\right) \end{cases} \Rightarrow \begin{cases} \cos(\omega x)P=\frac{P}{2}\left(e^{\omega xi}+e^{-\omega xi}\right) \\ \sin(\omega x)Q=\frac{Q}{2i}\left(e^{\omega xi}-e^{-\omega xi}\right) \end{cases}$
函数 (多项式) 共轭，倒数共轭；　两对共轭，乘积共轭；　$e^{\alpha+\theta i}$与$e^{\alpha-\theta i}$ 共轭

$f(x)=e^{\lambda x}P_m(x)$	$y^*=R_?(x)e^{\lambda x}$ $R^{''}x+(2\lambda+p)R^{'}x+(\lambda^2+p\lambda+q)R(x)=p_m(x)$ $y^*=x^kP_m(x)e^{\lambda x}$
$f(x)=e^{\lambda x}[P_l(x)\cos(\omega x)+Q_n(x)\sin(\omega x)]$	$f(x)=e^{\lambda x}\left[\left(\frac{P}{2}+\frac{Q}{2i}\right)e^{\omega xi}+\left(\frac{P}{2}-\frac{Q}{2i}\right)e^{-\omega xi}\right]$ $f(x)=\left(\frac{P}{2}+\frac{Q}{2i}\right)e^{\lambda x+\omega xi}+\left(\frac{P}{2}-\frac{Q}{2i}\right)e^{\lambda x-\omega xi}$ $f(x)=\left(\frac{P}{2}+\frac{Q}{2i}\right)e^{(\lambda+\omega i)x}+\left(\frac{P}{2}-\frac{Q}{2i}\right)e^{(\lambda-\omega i)x}$ $f(x)=P_1e^{(\lambda+\omega i)x}+Q_2e^{(\lambda-\omega i)x}\,(P_1,Q_2\text{共轭})$ $y_1^*=x^kR_me^{(\lambda+\omega i)x}\,(m=max\{P_l,Q_n\})$ $y_2^*=x^k\overline{R_m}e^{(\lambda-\omega i)x}$ $y^*=y_1^*+y_2^*=x^ke^{\lambda x}\left(R_me^{\omega xi}+\overline{R_m}e^{-\omega xi}\right)$ $y^*=x^ke^{\lambda x}\left[R_m(\cos(\omega x)+i\sin(\omega x))+\overline{R_m}(\cos(\omega x)-i\sin(\omega x))\right]$ $y^*=x^ke^{\lambda x}\left(R_m^{(1)}\cos(\omega x)+R_m^{(2)}\sin(\omega x)\right)$

7.9 欧拉方程

$x^ny^{(n)}+p^1x^{n-1}y^{(n)}+\cdots+p^{n-1}xy^{'}+p^ny=f(x)$

$x=e^t,t=\ln x$

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{1}{x}\frac{dy}{dt}\ D\text{表示}\frac{d}{dt}\begin{cases}xy^{'}=Dy\\x^2y^{''}=(D^2-D)y\\x^3y^{'''}=(D^3-3D^2+2D)y\\\cdots\\x^ky^{(k)}=(D^n+c_1D^{n-1}+\cdots+C_{n-1}D)y\end{cases}$$

7.10 常系数线性微分方程组

$$\begin{cases}\frac{d^2x}{dt^2}+\frac{dy}{dt}-x=e^t,\\\frac{d^2y}{dt^2}+\frac{dx}{dt}+y=0,\end{cases}$$

$$\text{记}\frac{d}{dt}\text{为}D\Rightarrow\begin{cases}(D^2-1)x+Dy=e^t\\Dx+(D^2+1)y=0\end{cases}$$

8 向量代数与空间解析几何

8.1 向量及其线性运算

8.1.1 向量的概念

大小, 方向, **向量**,$\overrightarrow{AB}$

与起点无关, **自由向量**

向量的大小, **向量的模**,$\left|\overrightarrow{AB}\right|$**, 单位向量, 零向量**

$\overrightarrow{OA}=a,\overrightarrow{OB}=b,\angle AOB<\pi$向量夹角, $\left(\widehat{a,b}\right)\begin{array}{l} =0 \text{ or } \pi, a\text{与}b\text{平行, 同起点共线} \\ =\frac{\pi}{2}, a\text{与}b\text{垂直} \end{array}$

8.1.2 向量的运算

向量的加减法

$c=a+b$三角形法则

$a+b=b+a$交换

$(a+b)+c=a+(b+c)$ 分配

$a-b=a+(-b)$ 负向量

$|a\pm b|<|a|+|b|$

向量与数的乘法

$|\lambda a|=|\lambda|\ |a|$

$\lambda(\mu a)=\mu(\lambda a)=(\lambda\mu)a$

$|\lambda(\mu a)|=|\mu(\lambda a)|=|(\lambda\mu)a|=|\lambda\mu|\ |a|$

$(\lambda+\mu)a=\lambda a+\mu a,\lambda(a+b)=\lambda a+\lambda b$

$a\neq 0,a//b\Leftrightarrow\exists$唯一实数$\lambda,b=\lambda a$

空间直角坐标系

$[O;i,j,k]$ 右手规则, 卦限

$\overrightarrow{OM}=r=xi+yj+zk$坐标式分解, 分向量,$r=(x,y,z)$,$M(x,y,z)$ 坐标,r为M关于O向径

利用坐标向量的线性运算

$$a=(a_x,a_y,a_z),b=(b_x,b_y,b_z)\left\{\begin{array}{ll} a+b & =(a_x+b_x,a_y+b_y,a_z+b_z) \\ a-b & =(a_x-b_x,a_y-b_y,a_z-b_z) \\ \lambda a & =(\lambda a_x,\lambda a_y,\lambda a_z) \end{array}\right.$$

向量的模, 方向角, 投影

向量的模与两点间的距离

$r=(x,y,z),|r|=\sqrt{x^2+y^2+z^2}$

$A=(x_1,y_1,z_1),B=(x_2,y_2,z_2),|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

方向角与方向余弦

r与三条坐标轴的夹角α,β,γ称为r的方向角

$(\cos\alpha,\cos\beta,\cos\gamma)=\frac{1}{|r|}(x,y,z)$,$\cos*$方向余弦

向量在轴上的投影

$$\left\{\begin{array}{l} (a)_u=|a|\cos\varphi \\ (a+b)_u=(a)_u+(b)_u=|a|\cos\varphi+|b|\cos\varphi \\ (\lambda a)_u=\lambda(a)_u \end{array}\right.$$

8.2 数量积 向量积 混合积

8.2.1 兩向量的数量积

$a \cdot b = |a|\ |b| \cos\theta$数量積

$$\begin{cases} a \cdot a = |a|^2 \\ a, b \neq 0,\ a \cdot b = 0 \Leftrightarrow a \perp b \\ a \cdot b = b \cdot a \text{交}\boxed{\text{F}} \\ (a+b) \cdot c = a \cdot c + b \cdot c \text{分配} \\ (\lambda a) \cdot b = \lambda (a \cdot b) \text{結合} \end{cases}$$

$$\begin{matrix} a = (a_x, a_y, a_z) \\ b = (b_x, b_y, b_z) \end{matrix} \Rightarrow \begin{matrix} a \cdot b = a_x b_x + a_y b_y + a_z b_z \\ \cos\theta = \frac{a \cdot b}{|a|\ |b|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} + \sqrt{{b_x}^2 + {b_y}^2 + {b_z}^2}} \end{matrix}$$

8.2.2 兩向量的向量积

$|c| = |a|\ |b| \sin\theta$

$c = a \times b$向量積

$$\begin{cases} a \times a = 0 \\ a, b \neq 0, a \times b = 0 \Leftrightarrow a // b \\ a \times b = -b \times a \\ (a+b) \times c = a \times c + b \times c \\ (\lambda a) \times b = a \times (\lambda b) = \lambda (a \times b) \end{cases}$$

$$\begin{matrix} a = (a_x, a_y, a_z) \\ b = (b_x, b_y, b_z) \end{matrix} \Rightarrow \begin{aligned} a \times b &= (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x) \\ &= \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \end{aligned}$$

8.2.3 向量的混合积

$(a \times b) \cdot c = [abc]$, 向量的混合积

$|[abc]| = |a|\ |b| \sin\left(\widehat{a, b}\right) |c| \cos\left(\widehat{a \times b, c}\right)$, a,b,c 右手系模大于0

$$\begin{matrix} a = (a_x, a_y, a_z) \\ b = (b_x, b_y, b_z) \\ c = (c_x, c_y, c_z) \end{matrix} \Rightarrow \begin{aligned} [abc] &= (a \times b) \cdot c \\ &= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \end{aligned}$$

8.3 平面及其方程

8.3.1 曲面方程与空间曲线方程的概念

$F(x, y, z) = 0$, 曲面 S 上点满足方程, 不在曲面 S 上点不满足方程 $\Rightarrow$ 曲面 S 的方程

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \Rightarrow \text{曲线 C 的方程}$$

8.3.2 平面的点法式方程

向量垂直平面, 平面的法线向量

$$\begin{matrix} n = (A, B, C) & \text{法向量} \\ \overrightarrow{M_0 M} = (x - x_0, y - y_0, z - z_0) & \text{平面上点} \end{matrix} \Rightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \text{平面的点法式方程}$$

8.3.3 平面的一般方程

$Ax+By+Cz+D=0$平面的一般方程

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$平面的截距式方程

8.3.4 平面的夹角

平面法向量的夹角, 平面的夹角 $\left(0\leqslant\theta\leqslant\frac{\pi}{2}\right)$

$\mathit{\Pi}_1, n_1=\left(A_1,B_1,C_1\right), \mathit{\Pi}_2, n_2=\left(A_2,B_2,C_2\right), \cos\theta=\left|\left(\widehat{n_1,n_2}\right)\right|=\dfrac{\left|A_1A_2+B_1B_2+C_1C_2\right|}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$平面夹角

$$\begin{aligned} d &= \left|\overrightarrow{P_1P_0}\right|\left|\cos\theta\right| \\ &= \tfrac{\left|\overrightarrow{P_1P_0}\times n\right|}{|n|} \\ &= \tfrac{|A(x_0-x)+B(y_0-y)+C(z_0-z)|}{\sqrt{A^2+B^2+C^2}} \\ &= \tfrac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}} \end{aligned}$$

8.4 空间直线及其方程

8.4.1 空间直线的一般方程

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$$

8.4.2 空间直线的对称式方程与参数方程

向量平行直线, 直线的方向向量

$$\begin{aligned} \overrightarrow{M_oM} &= (x-x_0,y-y_0,z-z_0) \\ S &= (m,n,p) \end{aligned} \Rightarrow \begin{array}{l} \dfrac{x-x_0}{m}=\dfrac{y-y_0}{n}=\dfrac{z-z_0}{p}=t, \text{对称式, 点向式方程} \\ \begin{cases} x=x_0+mt \\ y=y_0+nt \quad \text{参数方程} \\ z=z_0+pt \end{cases} \end{array}$$

8.4.3 两直线夹角

$L_1, s_1=\left(A_1,B_1,C_1\right), L_2, s_2=\left(A_2,B_2,C_2\right), \cos\theta=\left|\left(\widehat{s_1,s_2}\right)\right|=\dfrac{\left|A_1A_2+B_1B_2+C_1C_2\right|}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$直线夹角

8.4.4 直线与平面夹角

直线和投影直线的夹角, 直线与平面夹角

$$\begin{array}{l} \varphi=\left|\frac{\pi}{2}-(\overrightarrow{s,n})\right| \\ s=(m,n,p) \\ n=(A,B,C) \end{array} \Rightarrow \begin{aligned} \sin\varphi &= \tfrac{|Am+Bn+Cp|}{\sqrt{A^2+B^2+C^2}\sqrt{m^2+n^2+p^2}} \\ \tfrac{A}{m} &= \tfrac{B}{n}=\tfrac{C}{p}\text{垂直} \end{aligned}$$

8.4.5 例

$$\left.\begin{array}{ll} A_1x+B_1y+C_1z+D_1=0 & I \\ A_2x+B_2y+C_2z+D_2=0 & II \end{array}\right\} L \Rightarrow A_1x+B_1y+C_1z+D_1+\lambda\left(A_2x+B_2y+C_2z+D_2\right)=0\text{过 L 平面束, 不含 } II$$

8.5 曲面及其方程

8.5.1 曲面研究的基本问题

轨迹到方程, 方程到形状

$x^2+y^2+z^2=R^2$球面

$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$球面

$Ax^2+Ay^2+Az^2+Dx+Ey+Fz+G=0$一般球面

8.5.2 旋转曲面

平面上曲线绕直线旋转一周, 母线, 轴, 旋转曲面

$f(y,z)=0\Rightarrow f\left(\pm\sqrt{x^2+y^2},z\right)=0$圆台

相交直线 L 绕 R 旋转一周, 交点顶点, 夹角 $\left(0<\alpha<\frac{\pi}{2}\right)$ 半顶角

$z^2=a^2\left(x^2+y^2\right)$ 圆锥

$$\frac{x^2}{a^2}-\frac{z^2}{b^2}=1\text{双曲线}\Rightarrow\left.\begin{array}{l}\frac{\left(x^2+y^2\right)^2}{a^2}-\frac{z^2}{b^2}=1\text{单叶双曲面}\\ \frac{x^2}{a^2}-\frac{\left(z^2+y^2\right)^2}{b^2}=1\text{双叶双曲面}\end{array}\right\}\text{两种二次}$$

8.5.3 柱面

$x^2+y^2=R^2$圆柱面, 准线 (定曲线), 母线

$$\left.\begin{array}{l}y^2=ax\text{抛物柱面}\\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\text{椭圆柱面}\\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\text{双曲柱面}\end{array}\right\}\text{三种二次}$$

8.5.4 二次曲面九

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$椭圆锥面

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$椭圆抛物面

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$双曲抛物面

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$椭球面

8.6 空间曲线及其方程

8.6.1 空间曲线一般方程

$$\begin{cases}F(x,y,z)=0\\ G(x,y,z)=0\end{cases}$$

8.6.2 空间曲线参数方程

$$\begin{cases}x=x(t)\\ y=y(t)\\ z=z(t)\end{cases}\Rightarrow\begin{cases}x=a\cos\theta\\ y=a\cos\theta\quad\text{螺旋线},2\pi b\text{螺距}\\ z=b\theta\end{cases}$$

空间曲线参数方程

$$\Gamma\begin{cases}x=\varphi(t)\\ y=\psi(t)\\ z=\omega(t)\end{cases}\Rightarrow\begin{cases}x=\sqrt{[\varphi(t)]^2+[\varphi(t)]^2}\cos\theta\\ y=\sqrt{[\varphi(t)]^2+[\varphi(t)]^2}\sin\theta z=\omega(t),\end{cases}\quad\text{曲面}$$

$$\Gamma\begin{cases} x = a\sin\varphi \\ y = 0 \\ z = a\cos\varphi \end{cases} \Rightarrow \begin{cases} x = \sqrt{[a\sin\varphi]^2 + [0]^2}\cos\theta = a\sin\varphi\cos\theta \\ y = \sqrt{[a\sin\varphi]^2 + [0]^2}\sin\theta = a\sin\varphi\sin\theta \\ z = a\cos\varphi, \end{cases} \text{球面}$$

8.6.3 空间曲线在坐标面投影

$H(x,y) = 0$ 投影柱面

$$\begin{cases} H(x,y) = 0 \\ z = 0 \end{cases} \text{投影曲线}$$

9　多元函数微分及其应用

9.1　多元函数的基本概念

9.1.1　平面點集　*n 維空間

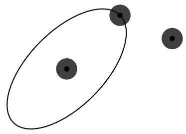
平面點集

R^2 坐標平面

$C = \{(x,y)\,|x^2+y^2<r^2\}$ 平面點集

$U(P_0,\delta) = \{P|PP_0<\delta\}$ 鄰域

$\mathring{U}(P_0,\delta) = \{P|0<PP_0<\delta\}$ 去心鄰域



$$P\in R^2,\ E\subset R^2\quad \begin{array}{l} \exists U(P),U(P)\subset E, P□E□\text{點} \\ \exists U(P),U(P)\cap E=\varnothing, P□E\text{外點} \\ \left.\begin{array}{l} P=P_1\cap P_2 \\ \exists U(P_1),U(P_1)\subset E \\ \exists U(P_2),U(P_2)\cap E=\varnothing \end{array}\right\}, P□E\text{邊界點} \end{array}$$

E邊界點的全體, □E邊界點,∂E

$\forall\delta>0,\mathring{U}(P,\delta)$ □總有E中的點,P□E聚點 (□點和邊界)

開集, □點; 閉集,$\partial E\subset E$; 連通集, 任意點連□仍在集合;

區域, 連通開集；閉區域，區域和邊界；

有界集，$\forall E\subset R^2, if\quad \exists r>0, E\subset U(O,r), then$　有界集；無界集，不是有界集

n 維空間

$R^n = \{(x_1,x_2,\cdots,x_n)\,|x_i\in R, i=1,2,\cdots,n\} = x$集合

$$\begin{array}{l} x+y \quad =(x_1+y_1,x_2+y_2,\cdots,x_n+y_n) \\ \lambda x \qquad =(\lambda x_1,\lambda x_2,\cdots,\lambda x_n) \end{array}\Bigg\} R^n\text{中□性運算}$$

$$\left.\begin{array}{l} x=(x_1,x_2,\cdots,x_n)\in R^n \\ y=(y_1,y_2,\cdots,y_n)\in R^n \\ a=(a_1,a_2,\cdots,a_n)\in R^n \\ \lambda\in R \end{array}\right\} \Rightarrow \begin{array}{l} \rho(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2} \\ ||x||=\rho(x,0)=\sqrt{{x_1}^2+{x_2}^2+\cdots+{x_n}^2} \\ ||x-a||\to 0\text{，變元趨于固定元，記作}x\to a \\ \text{某正數}\delta>0,\text{點集}U(a,\delta)=\{x|x\in R^n,\rho(x,a)<\delta\}, a\text{的}\delta\text{鄰域} \end{array}$$

定義□性預算的集合，n 維空間

9.1.2　多元函數的概念

$$\begin{array}{l} D\subset R^2,\text{映射}f:D\to R,\text{稱□二元函數}, \\ \text{記□}z=f(x,y),(x,y)\in D \end{array} \Rightarrow \begin{array}{l} D\text{定義域} \\ x,y\text{自變量} \\ z\text{因變量} \\ \text{函數值}f(x,y)\text{全體}, f(D)=\{z|z=f(x,y),(x,y)\in D\}\text{值域} \end{array}$$

$D\subset R^n$,映射$f:D\to R$,稱□n元函數 ($n\geqslant 2$多元函數),

記□$z=f(x_1,x_2,\cdots,x_n)=f(\mathbf{x}),\mathbf{x}(x_1,x_2,\cdots,x_n)\in D$

多元函數$\mu=f(\mathbf{x})$,有意義的變元$\mathbf{x}$的點集，自然定義域

空間點集$\{(x,y,z)\,|z=f(x,y),(x,y)\in D\}$,二元函數$z=f(x,y)$的圖形

9.1.3　多元函數的極限

$$\begin{array}{l} f(x,y)\text{定義域}D, \\ P_0(x_0,y_0)\text{是}D\text{聚點} \end{array}\quad if\quad \begin{array}{l} \exists A,\forall\varepsilon>0,\exists\delta>0, \\ when\ \text{點}P(x_0,y_0)\in D\cap\mathring{U}(P_0,\delta), \\ |f(P)-A|=|f(x,y)-A|<\varepsilon \end{array}\quad then\quad \begin{array}{l} A\text{稱□函數}f(x,y)\text{在}(x,y)\to(x_0,y_0)\text{極限}, \\ \text{記□}\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=A \end{array}$$

任意方向趨近

9.1.4 多元函數的連續性

$f(P) = f(x,y)$ 定義域$D, P_0(x_0,y_0)$ □D聚點$, P_0 \in D,$ $\begin{array}{l} if \quad \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0), \\ then \quad f(x,y) \text{ 在} P_0(x_0,y_0) \text{ 連續} \end{array}$

$f(x,y)$ 定義域D, D□每一點都是聚點$, f(x,y)$ 在D□每一點連續$, f(x,y)$ 在D□連續

$f(x,y)$ 定義域$D, P_0(x_0,y_0)$ □D聚點$, if \quad f(x,y)$ 在$P_0(x_0,y_0)$ 不連續$, then \quad P_0(x_0,y_0)$ □$f(x,y)$ 間斷點

常數，不同自變量的一元基本初等函數，有限次四則運算和復合運算，**多元初等函數**

一切多元初等函數在**定義區域**（定義域□的區域或閉區域）連續 $\rightarrow \lim\limits_{P\to P_0} f(P) = f(P_0)$

	有界閉區域 D，多元連續函數，D 上有界，取得最大值，最小值	(**最值**)
性質	有界閉區域 D，多元連續函數，能取得介於最大值和最小值間的任何值	(**介值**)
	有界閉區域 D，多元連續函數，D 上一致連續	(**一致連續**)

9.2 偏導數

9.2.1 偏導數的定義及其計算法

$f(x,y), P_0(x_0,y_0), (x,y) \in U(P_0,\delta),$
$when \quad y = y_0, x = x_0 + \Delta x,$
函數增量$f(x_0+\Delta x, y_0) - f(x_0,y_0)$

$if \quad \exists \lim\limits_{\Delta x\to 0} \dfrac{f(x_0+\Delta x,y_0) - f(x_0,y_0)}{\Delta x} = A,$
$then \quad A$□$f(x,y)$ 在 (x_0,y_0) 對x的偏導數，
記□ $\left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}, f_x(x_0,y_0)$

$$\begin{cases} \left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} = \lim\limits_{\Delta x\to 0} \dfrac{f(x_0+\Delta x,y_0) - f(x_0,y_0)}{\Delta x} \\ \left.\dfrac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}} = \lim\limits_{\Delta y\to 0} \dfrac{f(x_0,y_0+\Delta x) - f(x_0,y_0)}{\Delta y} \end{cases}$$

$f_x(x,y), f_y(x,y)$ 偏導函數, 偏導數

偏導數記號□整體，不能看成微分的商

(一元可導連) 各偏導存在，不一定連續

9.2.2 高階偏導數

$$z = \begin{array}{l} f(x,y) \text{ 的偏导数} \frac{\partial z}{\partial x} = f_x(x,y), \frac{\partial z}{\partial y} = f_y(x,y), \\ \text{都是} x, y \text{的函数,} \end{array} \Rightarrow \begin{array}{l} \text{若偏导数的偏导仍存在，则称为} f(x,y) \text{ 的二阶偏导数} \\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y) \\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x,y) \text{ (混合偏导数)} \\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x,y) \text{ (混合偏导数)} \\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y) \end{array}$$

$if \quad f(x,y)$ 的二阶偏导数$, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x \partial y}$在 D 连续$, then \quad (x,y) \in D \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

$$\left.\begin{array}{l} z = \ln\sqrt{x^2+y^2}, \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \\ u = \frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \end{array}\right\} \text{拉普拉斯方程}$$

9.3 全微分

9.3.1 全微分定義

$$\begin{cases} f(x+\Delta x, y) - f(x,y) \approx f_x(x,y)\Delta x \\ f(x, y+\Delta y) - f(x,y) \approx f_y(x,y)\Delta y \end{cases} \text{偏增量，偏微分}$$

$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) \quad$ 全增量

$$z=f(x,y),\text{在}(x,y)\text{的某邻域有定义},\quad \begin{array}{ll} if & \Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)=A\Delta x+B\Delta y+o(\rho), \\ then & \rho=\sqrt{(\Delta x)^2+(\Delta y)^2},A,B\text{不依赖}\Delta x,\Delta y \\ & f(x,y)\text{在}(x,y)\text{可微分}, \\ & A\Delta x+B\Delta y,\text{称为函数}f(x,y)\text{全微分}, \\ & \text{记作}dz=A\Delta x+B\Delta y \end{array}$$

多元函数在区域 D 内个点处都可微分，函数在 D 内可微分

多元函数在**点 P** 可微分，函数在该点连续

$$if\quad f(x,y)\text{在点}(x,y)\text{可微分},\quad \begin{array}{l} then\quad \text{函数}f\text{x,y在点}(x,y)\text{偏导数}\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\text{必存在} \\ \text{函数}f(x,y)\text{在点}(x,y)\text{全微分}dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \end{array}$$

$z=f(x,y)$的偏導數$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在點$(x,y)$連續,函數在該點可微分

微分F加原F, $u=(x,y,z),du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

9.3.2 全微分在近似計算中的應用

$g=\frac{4\pi^2 l}{T^2},\Delta g\leqslant 4\pi^2\left(\frac{1}{T^2}\delta_l+\frac{2l}{T^3}\delta_T\right)$

9.4 多元復合函數的求導法則

$$if\ \begin{array}{l} u=\varphi(t),v=\psi(t),\text{在}t\text{可導}, \\ z=f(u,v)\text{在}(u,v)\text{具有連續偏導} \end{array},then\quad \text{復合函數}z=f(\varphi(t),\psi(t))\text{在}t\text{點可導, 且}\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}\text{全導數}$$

$$if\quad u=\varphi(t),v=\psi(t),w=\omega(t),\text{在}t\text{可導},z=f(\varphi(t),then\quad \psi(t)+\omega(t)),\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}$$

$$if\ \begin{array}{l} u=\varphi(x,y),v=\psi(x,y), \\ \text{在}(x,y)\text{具有對}x,y\text{的偏導數}, \\ z=f(u,v)\text{在}(u,v)\text{具有連續偏導} \end{array},then\quad \begin{array}{l} \text{復合函數}z=f(\varphi(x,y),\psi(x,y))\text{在}(x,y)\text{點兩偏導都存在, 且} \\ \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \end{array}$$

$$if\quad \begin{array}{l} u=\varphi(x,y),v=\psi(x,y),w=\omega(x,y), \\ \text{在}(x,y)\text{點兩偏導都存在} \\ z=f(\varphi(x,y),\psi(x,y)+\omega(x,y)), \end{array},then\quad \begin{array}{l} \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y} \end{array}$$

$$if\quad \begin{array}{ll} u & =\varphi(x,y)\text{在}(x,y)\text{點兩偏導都存在}, \\ v & =\psi(y)\text{在}y\text{點可導}, \\ z & =f(u,v)\text{在}(u,v)\text{具有連續偏導} \end{array}\quad then\quad \begin{array}{l} z=f(\varphi(x,y),\psi(y))\text{在}(x,y)\text{點兩偏導都存在, 且} \\ \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \end{array}$$

$$ifz=f(u,x,y)=f[\varphi(x,y),x,y]\text{有}x,y\text{的偏導數},then\quad \begin{array}{l} \frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y} \end{array}$$

$$f(u,v),\quad \begin{array}{l} f_1^{'}=f_u(u,v) \\ f_2^{'}=f_v(u,v) \\ f_{11}^{'}=f_{uu}(u,v) \\ f_{12}^{'}=f_{uv}(u,v) \\ f_{21}^{'}=f_{vu}(u,v) \\ f_{22}^{'}=f_{vv}(u,v) \end{array}$$

$$z=f(u,v)=f(\varphi(x,y),\psi(x,y)),\quad \begin{array}{ll} dz & =\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv \\ & =\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \\ & =\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)dy \end{array}$$

9.5 隱函數求導公式

9.5.1 一個方程

$$
if\quad \begin{array}{l} P_0(x_0,y_0),(x,y)\in U(P_0,\delta)\text{時}\\ F(x,y)\text{具有連續偏導,}\\ F(x_0,y_0)=0,F_y(x_0,y_0)\neq 0\end{array},then\quad \begin{array}{l} F(x,y)=0,\text{在}(x,y)\in U(P_0,\delta)\text{時,}\\ \text{能確定唯一連續，連續導數的函數}y=f(x),\\ y_0=f(x_0),\frac{dy}{dx}=-\frac{F(x)}{F(y)}\end{array}
$$

$$
if\quad \begin{array}{l} P_0(x_0,y_0,z_0),(x,y,z)\in U(P_0,\delta)\text{時}\\ F(x,y,z)\text{具有連續偏導,}\\ F(x_0,y_0,z_0)=0,F_z(x_0,y_0,z_0)\neq 0\end{array},then\quad \begin{array}{l} F(x,y,z)=0,\text{在}(x,y,z)\in U(P_0,\delta)\text{時,}\\ \text{能確定唯一連續，連續導數的函數}z=f(x,y),\\ z_0=f(x_0,y_0),\frac{\partial z}{\partial x}=-\frac{F(x)}{F(z)},\frac{\partial z}{\partial y}=-\frac{F(y)}{F(z)}\end{array}
$$

9.5.2 方程組的情形

$$
if\quad \begin{array}{l} P_0(x_0,y_0,u_0,v_0),(x,y,u,v)\in U(P_0,\delta)\text{時}\\ F(x,y,u,v),G(x,y,u,v)\text{具有連續偏導,}\\ F(x_0,y_0,u_0,v_0)=0,G(x_0,y_0,u_0,v_0)=0,\\ J=\begin{vmatrix}\frac{\partial F}{\partial u} & \frac{\partial F}{\partial v}\\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v}\end{vmatrix}\neq 0\left(\begin{array}{l}\text{雅可比式}\\ \text{偏导数组成的函数行列式}\end{array}\right)\end{array}\quad then\quad \begin{array}{l} F(x,y,u,v)=0,G(x,y,u,v)=0,\\ \text{在}(x,y,u,v)\in U(P_0,\delta)\text{時,}\\ \text{能確定唯一組連續，連續偏導數的函數,}\\ u=f(x,y),v=f(x,y)\\ u_0=f(x_0,y_0),v_0=f(x_0,y_0)\\ \frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}\\ \frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}\\ \frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}\\ \frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}\end{array}
$$

$$
\left\{\begin{array}{l} F_x+F_u\frac{\partial u}{\partial x}+F_v\frac{\partial v}{\partial x}=0\\ G_x+G_u\frac{\partial u}{\partial x}+G_v\frac{\partial v}{\partial x}=0\end{array}\right.\Rightarrow \begin{array}{l}\frac{\partial u}{\partial x}=\frac{\begin{vmatrix}-F_x & F_v\\ -G_x & G_v\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}=-\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}\\ \frac{\partial v}{\partial x}=\frac{\begin{vmatrix}F_u & -F_x\\ G_u & -G_x\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}=-\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}\end{array}\quad\text{证明或推导}
$$

$$
\begin{array}{l} dudv,x=\varphi(u,v),y=\psi(u,v),\\ dx=\frac{\partial\varphi}{\partial u}du+\frac{\partial\varphi}{\partial v}dv\\ dy=\frac{\partial\psi}{\partial u}du+\frac{\partial\psi}{\partial v}dv\end{array}\rightarrow\begin{array}{l}|(dx,dy)|_{du=0}\times(dx,dy)|_{dv=0}|\\ =\left|\left(\frac{\partial\varphi}{\partial u}du,\frac{\partial\psi}{\partial u}du\right)\times\left(\frac{\partial\varphi}{\partial v}dv,\frac{\partial\psi}{\partial v}dv\right)\right|\\ =\left|\frac{\partial\varphi}{\partial u}\cdot\frac{\partial\psi}{\partial v}-\frac{\partial\psi}{\partial u}\cdot\frac{\partial\varphi}{\partial v}\right|dudv\\ =\begin{vmatrix}\frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v}\\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v}\end{vmatrix}dudv\\ =\frac{\partial(\varphi,\psi)}{\partial(u,v)}dudv\end{array}\quad\text{二重积分换元和雅可比}
$$

9.6 多元函数微分学的几何应用

9.6.1 一元向量值函数及其导数

$$
\Gamma\left\{\begin{array}{l} x=\varphi(t),\\ y=\psi(t),\quad t\in[\alpha,\beta]\\ z=\omega(t)\end{array}\right.
$$

記$r=xi+yj+zk,f(t)=\varphi(t)i+\psi(t)j+\omega(t)k,$

$r=f(t),t\in[\alpha,\beta]$

数集$D\subset R$,映射$f:D\to R^n$$\boxed{\text{F}}$一元向量值函数，記$\boxed{\text{F}}$$r=f\left(t\right),t\in D$

$t\in U\left(t_0\right)$,向量值函数$f\left(t\right),if\quad \exists r_0,\forall\varepsilon>0,\exists\delta,when\quad 0<\left|t-t_0\right|<\delta,\left|f\left(t\right)-r_0\right|<\varepsilon,then\quad \lim_{t\to t_0}f\left(t\right)=r_0$

$t\in u\left(t_0\right)$,向量值函数$f\left(t\right),if\quad \lim_{t\to t_0}f\left(t\right)=f\left(t_0\right),f\left(t\right)$在$t_0$连续

$t\in D,$,向量值函数$f\left(t\right),if\quad D_1\subset D,D_1$内每点连续,$D_1$上连续

$t\in U\left(t_0\right)$,向量值函数$f\left(t\right),if\quad \exists\lim_{\Delta t\to 0}\frac{\Delta r}{\Delta t}=\lim_{\Delta t\to 0}\frac{f\left(t_0+\Delta t\right)-f\left(t_0\right)}{\Delta t},then$
极限向量为$r=f\left(t\right)$
在t_0导数或导向量,
记为$f^{'}\left(t_0\right),\frac{dr}{dt}|_{t=t_0}$

$f^{'}\left(t_0\right)=f_1^{'}\left(t_0\right)i+f_2^{'}\left(t_0\right)j+f_3^{'}\left(t_0\right)k$

向量值函数$u\left(t\right),v\left(t\right)$,数量值函数$\varphi\left(t\right)\Rightarrow$

$$\begin{array}{l}
\frac{d}{dt}C=0\\
\frac{d}{dt}\left[cu\left(t\right)\right]=cu^{'}\left(t\right)\\
\frac{d}{dt}\left[u\left(t\right)\pm v\left(t\right)\right]=u^{'}\left(t\right)+v^{'}\left(t\right)\\
\frac{d}{dt}\left[\varphi\left(t\right)v\left(t\right)\right]=\varphi^{'}\left(t\right)v\left(t\right)+\varphi\left(t\right)v^{'}\left(t\right)\\
\frac{d}{dt}\left[u\left(t\right)\cdot v\left(t\right)\right]=u^{'}\left(t\right)\cdot v\left(t\right)+u\left(t\right)\cdot v^{'}\left(t\right)\\
\frac{d}{dt}\left[u\left(t\right)\times v\left(t\right)\right]=u^{'}\left(t\right)\times v\left(t\right)+u\left(t\right)\times v^{'}\left(t\right)\\
\frac{d}{dt}u\left[\varphi\left(t\right)\right]=\varphi^{'}\left(t\right)u^{'}\left[\varphi\left(t\right)\right]
\end{array}$$

$f^{'}\left(t_0\right)=\lim_{t\to t_0}\frac{\Delta r}{\Delta t}$与曲线相切

9.6.2 空间曲线的切线与法平面

$$\Gamma\left\{\begin{array}{l}x=\varphi\left(t\right),\\ y=\psi\left(t\right),\\ z=\omega\left(t\right),\end{array}\right.\quad t\in\left[\alpha,\beta\right]\Rightarrow\begin{array}{l}\frac{x-x_0}{\varphi^{'}\left(t\right)}=\frac{y-y_0}{\psi^{'}\left(t\right)}=\frac{z-z_0}{\omega^{'}\left(t\right)}\\ \varphi^{'}\left(t\right)\left(x-x_0\right)+\psi^{'}\left(t\right)\left(y-y_0\right)+\omega^{'}\left(t\right)\left(z-z_0\right)=0\end{array}$$

$$\Gamma\left\{\begin{array}{l}y=\psi\left(x\right),\\ z=\omega\left(x\right),\end{array}\right.\Rightarrow\left\{\begin{array}{l}x=x,\\ y=\psi\left(x\right),\\ z=\omega\left(x\right),\end{array}\right.\Rightarrow\begin{array}{l}\frac{x-x_0}{1}=\frac{y-y_0}{\psi^{'}\left(t\right)}=\frac{z-z_0}{\omega^{'}\left(t\right)}\\ \left(x-x_0\right)+\psi^{'}\left(t\right)\left(y-y_0\right)+\omega^{'}\left(t\right)\left(z-z_0\right)=0\end{array}$$

$$\begin{array}{l}\left\{\begin{array}{l}F\left(x,\psi\left(x\right),\omega\left(x\right)\right)=0\\ G\left(x,\psi\left(x\right),\omega\left(x\right)\right)=0\end{array}\right.\\ \left\{\begin{array}{l}\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}+\frac{\partial F}{\partial z}\frac{dz}{dx}=0\\ \frac{\partial G}{\partial x}+\frac{\partial G}{\partial y}\frac{dy}{dx}+\frac{\partial G}{\partial z}\frac{dz}{dx}=0\end{array}\right.\end{array}$$

$$\begin{array}{l}\Gamma\left\{\begin{array}{l}F\left(x,y,z\right)=0\\ G\left(x,y,z\right)=0\end{array}\right.\\ \frac{\partial\left(F,G\right)}{\partial\left(y,z\right)}|_{\left(x_0,y_0,z_0\right)}\neq 0\end{array}\Rightarrow\left\{\begin{array}{l}y=\psi\left(x\right),\\ z=\omega\left(x\right),\end{array}\right.\Rightarrow\left\{\begin{array}{l}\frac{dy}{dx}=\psi^{'}\left(x\right)=-\frac{1}{J}\frac{\partial\left(F,G\right)}{\partial\left(x,z\right)}=-\frac{\frac{\partial\left(F,G\right)}{\partial\left(x,z\right)}}{\frac{\partial\left(F,G\right)}{\partial\left(y,z\right)}}=-\frac{\left|\begin{array}{cc}\frac{\partial F}{\partial x}&\frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial x}&\frac{\partial G}{\partial z}\end{array}\right|}{\left|\begin{array}{cc}\frac{\partial F}{\partial y}&\frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial y}&\frac{\partial G}{\partial z}\end{array}\right|}\\ \frac{dz}{dx}=\omega^{'}\left(x\right)=-\frac{1}{J}\frac{\partial\left(F,G\right)}{\partial\left(y,x\right)}=-\frac{\frac{\partial\left(F,G\right)}{\partial\left(y,x\right)}}{\frac{\partial\left(F,G\right)}{\partial\left(y,z\right)}}=-\frac{\left|\begin{array}{cc}\frac{\partial F}{\partial y}&\frac{\partial F}{\partial x}\\ \frac{\partial G}{\partial y}&\frac{\partial G}{\partial x}\end{array}\right|}{\left|\begin{array}{cc}\frac{\partial F}{\partial y}&\frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial y}&\frac{\partial G}{\partial z}\end{array}\right|}\end{array}\right.$$

9.6.3 曲面的切平面与法线

$$
\begin{array}{l}
\sum F(x,y,z)=0 \\
\text{任意曲线}\Gamma\left\{\begin{array}{l} x=\varphi(t), \\ y=\psi(t),\quad (\alpha<t<\beta), \\ z=\omega(t), \end{array}\right. \\
t=t_0\text{对应}M(x_0,y_0,z_0) \\
\text{曲线}\Gamma\text{在曲面}\sum\text{上，过}M\text{点} \\
F[\varphi(t),\psi(t),\omega(t)]=0, \\
F_x|_M\varphi'(t_0)+F_y|_M\psi'(t_0)+F_z|_M\omega'(t_0) \\
=(F_x|_M,F_z|_M,F_z|_M)\cdot\left(\varphi'(t_0),\psi'(t_0),\omega'(t_0)\right) \\
=n\cdot s=0, \\
\text{任一过}M\text{曲线切线垂直过}M\text{向量}n\text{,切线共面, 称为曲面切平面}
\end{array}
\Rightarrow
\begin{array}{l}
F_x|_M(x-x_0)+F_z|_M(y-y_0)+F_z|_M(z-z_0)=0 \\
\frac{(x-x_0)}{F_x|_M}=\frac{(y-y_0)}{F_y|_M}=\frac{(z-z_0)}{F_z|_M}
\end{array}
$$

$$
z=f(x,y)\Rightarrow F(x,y,z)=f(x,y)-z=0\Rightarrow
\begin{array}{l}
F_x|_M=f_x(x_0,y_0),F_y|_M=f_y(x_0,y_0),F_z|_M=-1 \\
\text{记}K=\sqrt{f_x^2+f_y^2+1},\cos\gamma>0, \\
\text{方向余弦}(\cos\alpha,\cos\beta,\cos\gamma)=\left(\frac{-f_x}{K},\frac{-f_y}{K},\frac{1}{K}\right) \\
f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0 \\
\frac{(x-x_0)}{f_x(x_0,y_0)}=\frac{(y-y_0)}{f_y(x_0,y_0)}=\frac{(z-z_0)}{-1}
\end{array}
$$

9.7 方向导数与梯度

9.7.1 方向导数

$$
\begin{array}{l}
\text{单位向量}e_l=(\cos\alpha,\cos\beta), \\
\text{过}(x_0,y_0)\text{射线}l\left\{\begin{array}{l} x=x_0+t\cos\alpha \\ y=y_0+t\cos\beta \end{array}\right.\quad(t\geqslant 0),\quad\Rightarrow
\begin{array}{l}
\frac{\partial f}{\partial l}|_{(x_0,y_0)}=\lim\limits_{t\to 0^+}\dfrac{f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)}{t} \\
\text{方向导数}
\end{array} \\
z=f(x,y)
\end{array}
$$

$f(x,y)$在$P_0(x_0,y_0)$可微分，该点任意方向，方向导数存在,$\frac{\partial f}{\partial l}|_{(x_0,y_0)}=f_x(x_0,y_0)\cos\alpha+f_y(x_0,y_0)\cos\beta$

$f(x,y,z)$, $\frac{\partial f}{\partial l}|_{(x_0,y_0,z_0)}=f_x(x_0,y_0,z_0)\cos\alpha+f_y(x_0,y_0,z_0)\cos\beta+f_z(x_0,y_0,z_0)\cos\gamma$

9.7.2 梯度

$f(x,y)$, D内连续偏导数,$\forall P_0\in D$, $\mathbf{grad}f(x_0,y_0)=\Delta f(x_0,y_0)=f_x|_{P_0}i+f_y|_{p_0}j$,梯度,$\Delta=\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j$向量微分算子

$$
\begin{array}{l}
f(x,y)\text{在}(x_0,y_0)\text{可微分}, \\
\text{单位向量}e_l=(\cos\alpha,\cos\beta)
\end{array},
\begin{array}{ll}
\frac{\partial f}{\partial l}|_{(x_0,y_0)} & =f_x(x_0,y_0)\cos\alpha+f_y(x_0,y_0)\cos\beta \\
 & =\mathbf{grad}f(x_0,y_0)\cdot e_l \\
 & =|\mathbf{grad}f(x_0,y_0)|\cos\theta,\theta=\left(\mathbf{grad}\widehat{f(x_0,y_0)},e_l\right)
\end{array}
$$

$$
\begin{array}{l}
\text{曲面}\sum z=f(x,y),\text{等值线}L\,f(x,y)=0,P_0(x_0,y_0), \\
\text{记}k=\sqrt{f_x^2(x_0,t_0)+f_y^2(x_0,y_0)},
\end{array}
\Rightarrow
\begin{array}{l}
\begin{array}{rl}\text{单位法向量}n & =\frac{1}{k}(f_x|_p,f_y|_p) \\ & =\frac{\Delta f(x_0,y_0)}{|\Delta f(x_0,y_0)|}\end{array} \\
\Delta f(x_0,y_0)=|\Delta f(x_0,y_0)|\,n=\frac{\partial f}{\partial n}n \\
\text{梯度}=\text{梯度的模 梯度的方向}
\end{array}
$$

$$
f(x,y,z),G\text{内连续偏导数},P_0(x_0,y_0,z_0),
\begin{array}{l}
\mathbf{grad}f=\Delta f=f_x|_{P_0}i+f_y|_{P_0}j+f_z|_{P_0}k, \\
\text{等值面}f(x,y,z)=c \\
\text{等值面法线方向}n
\end{array}
$$

空间区域$G\quad \forall M\in G\rightarrow$ 数量$f\left(M\right)$, G内数量场
空间区域$G\quad \forall M\in G\rightarrow$ 向量$f\left(M\right)$, G内向量场
$if\quad F\left(M\right)$ 是$f\left(M\right)$ 的梯度 $then\quad f\left(M\right)$ 势函数, $F\left(M\right)$ 势场
$\frac{m}{r}$引力势, $\mathbf{grad}\frac{m}{r}$引力场

9.8 多元函数的极值及其求法

9.8.1 多元函数的极值及最大值最小值

$$D,z=f\left(x,y\right),P_0\left(x_0,y_0\right)\in D,\begin{array}{l} if\quad \exists U\left(P_0\right)\subset D,\forall\left(x,y\right)\neq P_0,f\left(x,y\right)<f\left(x_0,y_0\right),then\quad \text{极大值}f\left(x_0,y_0\right)\\ if\quad \exists U\left(P_0\right)\subset D,\forall\left(x,y\right)\neq P_0,f\left(x,y\right)>f\left(x_0,y_0\right),then\quad \text{极小值}f\left(x_0,y_0\right)\\ \text{极大值，极小值统称极值}\end{array}$$

$if\quad z=f\left(x,y\right)$ 在 $\left(x_0,y_0\right)$ 具有偏导数, 且在 $\left(x_0,y_0\right)$ 点具有极值, $then\quad f_x\left(x_0,y_0\right)=0,f_y\left(x_0,y_0\right)=0$
$if\quad f_x\left(x_0,y_0\right)=0,f_y\left(x_0,y_0\right)=0,then$ 驻点 $\left(x_0,y_0\right)$

$$if\quad z=f\left(x,y,z\right)\text{ 在 }\left(x_0,y_0,z_0\right)\text{ 具有偏导数, 且在 }\left(x_0,y_0,z_0\right)\text{ 点具有极值,}then\quad \begin{array}{l} f_x\left(x_0,y_0,z_0\right)=0,\\ f_y\left(x_0,y_0,z_0\right)=0,\\ f_z\left(x_0,y_0,z_0\right)=0\end{array}$$

$$\begin{array}{ll} z=f\left(x,y\right)\text{ 在 }\left(x_0,y_0\right)\text{ 的某邻域内连续，} & AC-B^2>0,\text{有极值},A<0\text{极大值},A>0\text{极小值}\\ \text{且有一阶及二阶偏导数，又}f_x\left(x_0,y_0\right)=0,f_y\left(x_0,y_0\right)=0\ \ if & AC-B^2<0,\text{无极值}\\ \text{令}f_{xx}\left(x_0,y_0\right)=A,f_{xy}\left(x_0,y_0\right)=B,f_{yy}\left(x_0,y_0\right)=C & AC-B^2=0,\text{可能有极值}\end{array}$$

驻点，偏导不存在点

9.8.2 条件极值　　拉格朗日数乘法

无条件极值，条件极值；条件极值到无条件极值

$$\begin{array}{l} \left(x_o,y_0\right)\text{ 的某邻域内}z=f\left(x,y\right),\varphi\left(x,y\right),\\ \text{有连续一阶导数 }\varphi\left(x,y\right)=0\left(\text{条件}\right)\rightarrow y=\varphi\left(x\right),\\ z=f\left(x,\varphi\left(x\right)\right)\left(\text{函数}\right)\end{array}\Rightarrow\begin{array}{l} \frac{dz}{dx}|_{x=x_0}=0\\ f_x\left(x_0,y_0\right)+f_y\left(x_0,y_0\right)\frac{dy}{dx}|_{x=x_0}=0\\ f_x\left(x_0,y_0\right)-f_y\left(x_0,y_0\right)\frac{\varphi_x\left(x_0,y_0\right)}{\varphi_y\left(x_0,y_0\right)}=0\\ \text{记}\frac{f_y\left(x_0,y_0\right)}{\varphi_y\left(x_0,y_0\right)}=-\lambda\\ f_y\left(x_0,y_0\right)-f_y\left(x_0,y_0\right)\frac{\lambda}{\lambda}=0\\ \left\{\begin{array}{l} f_x\left(x_0,y_0\right)+\lambda\varphi_x\left(x_0,y_0\right)=0\\ f_y\left(x_0,y_0\right)+\lambda\varphi_y\left(x_0,y_0\right)=0\\ \varphi\left(x_0,y_0\right)=0\end{array}\right.\end{array}$$

$$L\left(x,y\right)=f\left(x,y\right)+\lambda\varphi\left(x,y\right)\Rightarrow\left\{\begin{array}{l} L_x\left(x_0,y_0\right)=0\\ L_y\left(x_0,y_0\right)=0\\ \varphi\left(x_0,y_0\right)=0\end{array}\right.\quad\begin{array}{l}\text{拉格朗日函数}L,\text{拉格朗日乘子}\lambda\\ \text{函数}f\left(x,y\right),\text{条件}\varphi\left(x,y\right)=0\end{array}$$

$$L\left(x,y,z,t\right)=f\left(x,y,z,t\right)+\lambda\varphi\left(x,y,z,t\right)+\mu\psi\left(x,y,z,t\right)\Rightarrow\left\{\begin{array}{l} L_x\left(x_0,y_0,z_0,t_0\right)=0\\ L_y\left(x_0,y_0,z_0,t_0\right)=0\\ L_z\left(x_0,y_0,z_0,t_0\right)=0\\ L_t\left(x_0,y_0,z_0,t_0\right)=0\\ \varphi\left(x_0,y_0,z_0,t_0\right)=0\\ \psi\left(x_0,y_0,z_0,t_0\right)=0\end{array}\right.\quad\begin{array}{l}\text{拉格朗日函数}L,\\ \text{拉格朗日乘子}\lambda,\mu\\ \text{函数}f\left(x,y,z,t\right),\\ \text{条件}\varphi\left(x,y,z,t\right)=0,\\ \psi\left(x,y,z,t\right)=0\end{array}$$

9.8.3 二元函数的泰勒公式

$$\begin{array}{l} z=f\left(x,y\right)\text{ 在点}P_0\left(x_0,y_0\right)\text{ 某邻域内连续，}\\ \text{有 n+1 阶连续偏导数,}\\ \left(x_0+h,y_0+k\right)\in U\left(P_0,\delta\right)\end{array}\Rightarrow\begin{array}{l} f\left(x_0+h,y_0+k\right)=f\left(x_0,y_0\right)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f\left(x_0,y_0\right)+\\ \frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f\left(x_0,y_0\right)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f\left(x_0,y_0\right)\\ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f\left(x_0+\theta h,y_0+\theta k\right)\quad\left(0<\theta<1\right)\end{array}$$

证明：
$$\begin{array}{l}
\Phi(t) = f(x_0+h, y_0+k)\ (0 \leqslant t \leqslant 1)\\
\Phi^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0+ht, y_0+kt)\\
\Phi(t) = \Phi(0) + \Phi'(0) + \frac{1}{2}\Phi''(0) + \cdots + \frac{1}{n!}\Phi^{(n)}(0) + \frac{1}{(n+1)!}\Phi^{(n+1)}(\theta t)\ (0<\theta<1)\,(0\quad)\\
\Phi(1) = \Phi(0) + \Phi'(0) + \frac{1}{2}\Phi''(0) + \cdots + \frac{1}{n!}\Phi^{(n)}(0) + \frac{1}{(n+1)!}\Phi^{(n+1)}(\theta)\ (0<\theta<1)\,(\quad)
\end{array}$$

$n=0$时$f(x_0+h, y_0+k) - f(x_0, y_0) = hf_x(x_0+\theta h, y_0+\theta k) + kf_y(x_0+\theta h, y_0+\theta k)$

$f(x,y), (x,y) \in D, f_x(x,y) \equiv 0, f_y(x,y) \equiv 0, f(x,y) = c$

9.8.4 极值充分证明

$$\begin{aligned}
\Delta f &= f(x,y) - f(x_0,y_0)\\
&= \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(x_0,y_0) + \frac{1}{2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(x_0+\theta h, y_0+\theta k)\\
&f_x(x_0,y_0) = 0, f_y(x_0,y_0) = 0\\
&= \frac{1}{2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(x_0+\theta h, y_0+\theta k)\\
&= \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)\\
&= \frac{1}{2f_{xx}}\left(h^2 f_{xx}^2 + 2hkf_{xy}f_{xx} + k^2 f_{xy}^2 - k^2 f_{xy}^2 + k^2 f_{yy}f_{xx}\right)\\
&= \frac{1}{2f_{xx}}\left[\left(hf_{xx} + kf_{xy}\right)^2 + k^2\left(f_{xx}f_{yy} - f_{xy}^2\right)\right]\\
&A = f_{xx}, C = f_{yy}, B = f_{xy}
\end{aligned}$$

$$\begin{array}{ll}
\begin{array}{l}AC - B^2 > 0\\ \text{有}\end{array} & \left\{\begin{array}{l} A > 0, f(x,y) > f(x_0,y_0),\text{极小值}\\ A < 0, f(x,y) < f(x_0,y_0),\text{极大值}\end{array}\right.\\
\begin{array}{l}AC - B^2 < 0\\ \text{无}\end{array} & \left\{\begin{array}{ll} A = C = 0, k = h, \Delta f = f_{x,y} & \\ A = C = 0, k = -h, \Delta f = -f_{x,y} & \\ A \neq C = 0, k = 0, \Delta f = \frac{1}{2}h^2 f_{xx} & \\ A \neq C = 0, h = -f_{xy}s, k = f_{xx}s, & \begin{aligned}\Delta f &= \frac{1}{2}\left\{f_{xy}^2 f_{xx} - 2f_{xy}^2 f_{xx} + f_{xx}^2 f_{yy}\right\}\\ &= \frac{1}{2}f_{xx}\left\{f_{xy}^2 - 2f_{xy}^2 + f_{xx}f_{yy}\right\}\\ &= \frac{1}{2}f_{xx}\left\{-f_{xy}^2\right\}\end{aligned}\end{array}\right.\\
\begin{array}{l}AC - B^2 = 0\\ \text{有可能}\end{array} & \left\{\begin{array}{l} f(x,y) = x^2 + y^4\\ g(x,y) = x^2 + y^3\end{array}\right.
\end{array}$$

9.8.5 最小二乘法

y_i数据$\Rightarrow f(t) = at + b$线性经验公式

$M = \sum\left[y_i - f(t_i)\right]^2 = \sum\left[y_i - (at_i + b)\right]^2$ 偏差平方和,

M 最小为条件选择常数 a,b 的方法, 最小二乘法

$\sqrt{M}$均方误差

$$\text{非线性到线性}M_{min}(a,b) \Rightarrow \left\{\begin{array}{l} M_a(a,b) = 0\\ M_b(a,b) = 0\\ \frac{\partial M}{\partial a} = -2t_i\sum\left[y_i - (at_i + b)\right] = 0\\ \frac{\partial M}{\partial b} = -2\sum\left[y_i - (at_i + b)\right] = 0\\ \sum t_i y_i - a\sum {t_i}^2 - b\sum y_i = 0\\ \sum y_i - a\sum t_i y_i - bn = 0\end{array}\right. \Rightarrow \left\{\begin{array}{l} a = \dfrac{\begin{vmatrix}\sum t_i y_i & \sum y_i\\ \sum y_i & n\end{vmatrix}}{\begin{vmatrix}\sum {t_i}^2 & \sum y_i\\ \sum t_i y_i & n\end{vmatrix}}\\ b = \dfrac{\begin{vmatrix}\sum {t_i}^2 & \sum t_i y_i\\ \sum t_i y_i & \sum y_i\end{vmatrix}}{\begin{vmatrix}\sum {t_i}^2 & \sum y_i\\ \sum t_i y_i & n\end{vmatrix}}\end{array}\right.$$

10 重积分

10.1 二重积分的概念与性质

10.1.1 二重积分的概念

曲顶柱体的体积

$V=\lim_{\lambda\to 0}\sum f\left(\xi_i,\eta_i\right)\Delta\sigma_i$

平面薄片的质量

$m=\lim_{\lambda\to 0}\sum \mu\left(\xi_i,\eta_i\right)\Delta\sigma_i$

$$\begin{array}{l}\text{闭区域}D\text{有界函数}\\ \text{分割}D,\Delta\sigma_1,\Delta\sigma_2,\ldots,\Delta\sigma_n\end{array}\Rightarrow \iint_D f\left(x,y\right)d\sigma=\lim_{\lambda\to 0}f\left(\xi_i,\eta_i\right)\Delta\sigma\text{二重积分}$$

$\Delta\sigma_i=\Delta x_j\cdot y_k, d\sigma\to dxdy, \iint_D f\left(x,y\right)dxdy$

10.1.2 二重积分的性质

$$\begin{array}{l}g\left(x,y\right),g\\ f\left(x,y\right),f\end{array}\Rightarrow\begin{array}{l}\iint_D\left[\alpha f+\beta g\right]d\sigma=\alpha\iint_D fd\sigma+\beta\iint_D gd\sigma\\ \iint_D fd\sigma=\iint_{D_1}fd\sigma+\iint_{D_2}fd\sigma\\ \iint_D 1\cdot d\sigma=\iint_D\cdot d\sigma=\sigma\\ \iint_D fd\sigma\leqslant\iint_D gd\sigma, f\leqslant g\\ m\sigma\leqslant\iint_D fd\sigma\leqslant M\sigma, m\leqslant f\leqslant M\\ \iint_D fd\sigma=f\left(\xi,\eta\right)\sigma\quad\left(\xi,\eta\right)\in D\end{array}$$

10.2 二重积分的计算法

10.2.1 直角坐标计算二重积分

$$x\text{型}\begin{cases}D:\varphi_1\left(x\right)\leqslant y\leqslant\varphi_1\left(x\right),a\leqslant x\leqslant b\\ A\left(x_0\right)=\int_{\varphi_1\left(x_0\right)}^{\varphi_2\left(x_0\right)}f\left(x_0,y\right)dy\\ V=\int_a^b A\left(x\right)dx=\int_a^b\left[\int_{\varphi_1\left(x\right)}^{\varphi_2\left(x\right)}f\left(x,y\right)dy\right]dx=\int_a^b dx\int_{\varphi_1\left(x\right)}^{\varphi_2\left(x\right)}f\left(x,y\right)dy=\iint_D f\left(x,y\right)d\sigma\end{cases}$$

$$y\text{型}\begin{cases}D:\psi_1\left(y\right)\leqslant x\leqslant\psi_1\left(y\right),a\leqslant y\leqslant b\\ A\left(y_0\right)=\int_{\psi_1\left(y_0\right)}^{\psi_2\left(y_0\right)}f\left(x,y_0\right)dx\\ V=\int_a^b A\left(y\right)dy=\int_a^b\left[\int_{\psi_1\left(y\right)}^{\psi_2\left(y\right)}f\left(x,y\right)dx\right]dy=\int_a^b dy\int_{\psi_1\left(y\right)}^{\psi_2\left(y\right)}f\left(x,y\right)dx=\iint_D f\left(x,y\right)d\sigma\end{cases}$$

非 x 非 y 型，分割积分面

10.2.2 极坐标计算二重积分

$$S_{\text{扇}}=\frac{r^2\theta}{2}$$
$$\Delta\sigma=\frac{1}{2}\left[\left(r+\Delta r\right)^2-r^2\right]\Delta\theta=\frac{1}{2}\left[\Delta r^2+2r\Delta r\right]\Delta\theta=\frac{1}{2}\left[\Delta r+2r\right]\Delta r\Delta\theta=\overline{r}\Delta r\Delta\theta$$
$$\lim_{\lambda\to 0}\sum f\left(\xi_i,\eta_i\right)\Delta\sigma_i=\lim_{\lambda\to 0}\sum f\left(\overline{r}\cos\overline{\theta},\overline{r}\sin\overline{\theta}\right)\overline{r}\Delta r\Delta\theta$$
$$\iint_D f\left(x,y\right)d\sigma=\iint_D f\left(x,y\right)dxdy=\iint_D f\left(r\cos\theta,r\sin\theta\right)rdrd\theta$$
$$\iint_D f\left(r\cos\theta,r\sin\theta\right)rdrd\theta=\int_a^b\left[\int_{\varphi_1\left(\theta\right)}^{\varphi_2\left(\theta\right)}f\left(r\cos\theta,r\sin\theta\right)rdr\right]d\theta$$
$$\sigma=\iint_D rdrd\theta=\frac{1}{2}\int_a^b\left[\varphi_2\left(\theta\right)^2-\varphi_1\left(\theta\right)^2\right]d\theta=\frac{1}{2}\int_a^b\varphi_2\left(\theta\right)^2d\theta$$

10.2.3 二重积分换元法

$f(x,y)$, D内连续,
变换$T: x=x(u,v), y=y(u,v)$
uOv面$D' \to xOy$面D, 一对一 $\Rightarrow \iint_D f(x,y)\,dxdy = \iint_{D'} f(x(u,v), y(u,v))\,|J(u,v)|\,dudv$
$x(u,v), y(u,v)$在D'一阶连续偏导,
D'上雅可比式, $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$,

$$c^2 = a^2 + b^2 - 2ab\cos c$$

$$\begin{cases} M_1: x_1 = x(u,v), y_1 = y(u,v) \\ M_2: x_2 = x(u+h,v) = x_1 + x_u(u,v)h + o(h) \\ y_2 = y(u+h,v) = y_1 + y_u(u,v)h + o(h) \\ M_4: x_3 = x(u,v+h) = x_1 + x_v(u,v)h + o(h) \\ y_3 = y(u,v+h) = y_1 + y_v(u,v)h + o(h) \end{cases}$$

$$\begin{aligned} s_{\text{三角形}} &= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \\ &= \frac{1}{2}\begin{vmatrix} x_u(u,v)h & y_u(u,v)h \\ x_v(u,v)h & y_v(u,v)h \end{vmatrix} = \frac{1}{2}h^2\begin{vmatrix} x_u(u,v) & y_u(u,v) \\ x_v(u,v) & y_v(u,v) \end{vmatrix} = \frac{1}{2}h^2\begin{vmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \end{vmatrix} = \frac{1}{2}h^2\frac{\partial(x,y)}{\partial(u,v)} \end{aligned}$$

10.3 三重积分

10.3.1 三重积分的概念

$f(x,y,z)$, 有界闭区域Ω,
分割成$\Delta v_1, \Delta v_2, \cdots, \Delta v_n$, $\Rightarrow \iiint_D f(x,y,z)\,dv = \iiint_D f(x,y,z)\,dxdydz = \lim_{\lambda\to 0}\sum f(\xi_i,\eta_i,\zeta_i)\,\Delta v_i$

10.3.2 三重积分的计算

利用直角坐标系计算

$$F(x_0,y_0) = \int_{z_1(x_0,y_0)}^{z_2(x_0,y_0)} f(x_0,y_0,z)\,dz$$

$$\iiint_\Omega f(x,y,z)\,dv \begin{aligned} &= \iint_{D_{xy}} F(x,y)\,d\sigma = \iint_{D_{xy}}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right]d\sigma = \int_a^b dx\int_{y_1(x)}^{y_2(x)} dy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \\ &= \int_{c_1}^{c_2} dz\iint_{D_z} f(x,y,z)\,dxdy \end{aligned}$$

利用柱坐标系计算

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \iiint_\Omega f(x,y,z)\,dxdydz = \iiint_\Omega f(r\cos\theta, r\sin\theta, z)\,rdrd\theta dz$$

$dv = rdrd\theta dz$

利用球面坐标计算

$$\begin{cases} x = OP\cos\theta = r\sin\varphi\cos\theta \\ y = OP\sin\theta = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases} \quad \iiint_\Omega f(x,y,z)\,dxdydz = \iiint_\Omega f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)\,r^2\sin\varphi drd\varphi d\theta$$

φ,Z 轴; θ,X 轴
$dv = r^2\sin\varphi drd\varphi d\theta$

10.4 重积分的应用

曲面的面积

曲面$S:z=f\left(x,y\right)$

$\left|\left(f_x,f_y,-1\right)\cdot\left(0,0,1\right)\right|=1=\sqrt{{f_x}^2+{f_y}^2+-1^2}\cdot1\cdot\cos\theta,$ $A=\iint_D\sqrt{{f_x}^2+{f_y}^2+1}d\sigma$

$\cos\theta=\frac{1}{\sqrt{{f_x}^2+{f_y}^2+(-1)^2}}$

$\frac{d\sigma}{dA}=\cos\theta,\frac{\sum\sigma_k}{\sum A_k}=\cos\theta,dA=\frac{1}{\cos\theta}d\sigma$

*** 利用曲面的参数方程求曲面的面积**

$$\begin{cases}x=x\left(u,v\right)\\y=y\left(u,v\right)\quad\left(u,v\right)\in D\\z=z\left(u,v\right)\end{cases}\Rightarrow\begin{array}{l}A=\iint_D\sqrt{EG-F^2}dudv\\E=x_u^2+y_u^2+z_u^2\\F=x_ux_v+y_uy_v+z_uz_v\\G=x_v^2+y_v^2+z_v^2\end{array}$$

x,y,z在D内连续一阶

$\frac{\partial(x,y)}{\partial(u,v)},\frac{\partial(y,z)}{\partial(u,v)},\frac{\partial(z,x)}{\partial(u,v)}$不全为零

质心

$$\begin{array}{l}\overline{x}=\frac{M_y}{M}=\frac{\sum m_ix_i}{\sum m_i},\\\overline{y}=\frac{M_x}{M}=\frac{\sum m_iy_i}{\sum m_i},\\M_y,M_x\text{净距},M\text{质点系总质量}\end{array}\Rightarrow\begin{array}{l}dM_y=x\mu\left(x,y\right)d\sigma,M_y=\iint_Dx\mu\left(x,y\right)d\sigma\\dM_x=y\mu\left(x,y\right)d\sigma,M_x=\iint_Dy\mu\left(x,y\right)d\sigma\\dM=\mu\left(x,y\right)d\sigma,M=\iint_D\mu\left(x,y\right)d\sigma\\\overline{x}=\frac{M_y}{M}=\frac{\iint_Dx\mu(x,y)d\sigma}{\iint_D\mu(x,y)d\sigma}\xlongequal[A=\iint_Dd\sigma]{\mu(x,y)=c}\frac{1}{A}\iint_Dx\mu\left(x,y\right)d\sigma\\\overline{y}=\frac{M_x}{M}=\frac{\iint_Dy\mu(x,y)d\sigma}{\iint_D\mu(x,y)d\sigma}\xlongequal[A=\iint_Dd\sigma]{\mu(x,y)=c}\frac{1}{A}\iint_Dy\mu\left(x,y\right)d\sigma\end{array}$$

$$M=\iiint_\Omega\rho\left(x,y,z\right)dv,\begin{cases}\overline{x}=\frac{1}{M}\iiint_\Omega x\rho\left(x,y,z\right)dv,\\\overline{y}=\frac{1}{M}\iiint_\Omega y\rho\left(x,y,z\right)dv,\\\overline{z}=\frac{1}{M}\iiint_\Omega z\rho\left(x,y,z\right)dv,\end{cases}$$

转动惯量

$$\begin{array}{l}I_x=\sum y_i^2m_i,\\I_y=\sum x_i^2m_i\end{array}\Rightarrow\begin{array}{l}dI_x=y^2\mu\left(x,y\right)d\sigma\\dI_y=x^2\mu\left(x,y\right)d\sigma\\I_x=\iint_Dy^2\mu\left(x,y\right)d\sigma\\I_y=\iint_Dx^2\mu\left(x,y\right)d\sigma\end{array}$$

$I_x=\iiint_\Omega\left(y^2+z^2\right)\rho\left(x,y,z\right)dv$

$I_y=\iiint_\Omega\left(x^2+z^2\right)\rho\left(x,y,z\right)dv$

$I_z=\iiint_\Omega\left(y^2+x^2\right)\rho\left(x,y,z\right)dv$

引力

$F=G\frac{mM}{r^2}$

$\left(\cos\alpha,\cos\beta,\cos\gamma\right)=\left(\frac{x-x_0}{r},\frac{y-y_0}{r},\frac{z-z_0}{r}\right)$

$F=\left(\left|F\right|\cos\alpha,\left|F\right|\cos\beta,\left|F\right|\cos\gamma\right)\xlongequal{\left|dF\right|=G\frac{1\cdot\rho(x,y,z)dv}{r^2}}\left(\iiint_\Omega\frac{G\rho(x,y,z)(x-x_0)}{r^2\cdot r}dv,,\right)$

10.5　含参变量的积分

$f\left(x,y\right),R=\left[a,b\right]\times\left[c,d\right]=\left\{\left[x,y\right]\left|x\in\left[a,b\right],y\in\left[c,d\right]\right.\right\}$

$\varphi\left(x\right)=\int_c^df\left(x,y\right)dy\quad\left(a\leqslant x\leqslant b\right)$ **含参数变量积分**

$f\left(x,y\right)$在R连续,$\varphi\left(x\right)$在$\left[a,b\right]$连续

$f\left(x,y\right)$在R连续,$\int_a^bdx\int_c^df\left(x,y\right)dy=\int_c^ddy\int_a^bf\left(x,y\right)dx$

$f\left(x,y\right),f_x\left(x,y\right)$在$R$连续,$\varphi\left(x\right)$在$\left[a,b\right]$可微,$\varphi^{'}\left(x\right)=\frac{d}{dx}\int_c^df\left(x,y\right)dy=\int_c^df_x\left(x,y\right)dy$

$\Phi\left(x\right)=\int_{\alpha(x)}^{\beta(x)}f\left(x,y\right)dy\quad\left(a\leqslant x\leqslant b\right)$ **含参数变量积分**

$f\left(x,y\right)$在R连续,$\alpha\left(x\right),\beta\left(x\right)$在$\left[a,b\right]$连续,$c\leqslant\alpha\left(x\right),\beta\left(x\right)\leqslant d,\Phi\left(x\right)$在$\left[a,b\right]$连续,

$f\left(x,y\right),f_x\left(x,y\right)$在$R$连续,$\alpha\left(x\right),\beta\left(x\right)$在$\left[a,b\right]$可微,$c\leqslant\alpha\left(x\right),\beta\left(x\right)\leqslant d\ \Phi\left(x\right)$在$\left[a,b\right]$可微,

$$\Phi^{'}\left(x\right)=\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)}f\left(x,y\right)dy=\int_{\alpha(x)}^{\beta(x)}f_x\left(x,y\right)dy+f\left[x,\beta\left(x\right)\right]\beta^{'}\left(x\right)-f\left[x,\alpha\left(x\right)\right]\alpha^{'}\left(x\right)$$

11 曲线积分与曲面积分

11.1 对弧长的曲线积分

11.1.1 对弧长的曲线积分的概念与性质

曲柄构件的质量

$$\begin{aligned} m &\approx \sum \mu\left(\xi_i,\eta_i\right)\Delta s_i \\ &= \lim_{\lambda\to 0}\sum \mu\left(\xi_i,\eta_i\right)\Delta s_i \end{aligned}$$

$$\begin{array}{l} f\left(x,y\right)\text{在直线}L\text{上有界}, \\ L\text{插入点}M_1,M_2,\cdots,M_{n-1}\text{分割弧}\Delta s_i, \end{array} \Rightarrow \begin{array}{l} \int_L f\left(x,y\right)ds = \lim_{\lambda\to 0}\sum f\left(\xi_i,\eta_i\right)\Delta s_i, \\ \textbf{函数对弧长积分，第一类曲线积分} \end{array}$$

$\int_\Gamma f\left(x,y,z\right)ds = \lim_{\lambda\to 0}\sum f\left(\xi_i,\eta_i,\zeta_i\right)\Delta s_i$

封闭曲线L, $\int_L f\left(x,y\right)ds = \oint_L f\left(x,y\right)ds$

分段光滑 $\int_{L_1+L_2} f\left(x,y\right)ds = \int_{L_1} f\left(x,y\right)ds + \int_{L_2} f\left(x,y\right)ds$

常数α,β, $\int_L \left[\alpha f\left(x,y\right)+\beta g\left(x,y\right)\right]ds = \alpha\int_L f\left(x,y\right)ds + \beta\int_L g\left(x,y\right)ds$

曲线L上$f\left(x,y\right)\leqslant g\left(x,y\right)$, $\int_L f\left(x,y\right)ds \leqslant \int_L g\left(x,y\right)ds$

$$\begin{array}{l} -\left|c\right|\leqslant a\leqslant\left|c\right|, \\ \left|a\right|\leqslant\left|c\right| \end{array} \Rightarrow \begin{array}{l} -\int\left|c\right|ds\leqslant\int ads\leqslant\int\left|c\right|ds \\ \left|\int ads\right|\leqslant\int\left|c\right|ds \end{array} \xrightarrow{a=c=f(x,y)} \left|\int f\left(x,y\right)ds\right|\leqslant\int\left|f\left(x,y\right)\right|ds$$

11.1.2 对弧长的曲线积分的计算法

$$\begin{array}{l} f\left(x,y\right)\text{在曲线}L\text{连续} \\ L:\begin{cases} x=\varphi\left(t\right) \\ y=\psi\left(t\right)\end{cases} t\in\left[\alpha,\beta\right] \\ \varphi\left(t\right),\psi\left(t\right),\text{在}\left[\alpha,\beta\right]\text{连续一阶导} \\ {\varphi'}^2\left(t\right)+{\psi'}^2\left(t\right)\neq 0 \end{array} \Rightarrow \begin{aligned} \int_L f\left(x,y\right)ds &= \lim_{\lambda\to 0}\sum f\left(\varphi\left(t\right),\psi\left(t\right)\right)\sqrt{{\varphi'}^2\left(t\right)+{\psi'}^2\left(t\right)}\Delta t_i \\ &= \int_\alpha^\beta f\left(\varphi\left(t\right),\psi\left(t\right)\right)\sqrt{{\varphi'}^2\left(t\right)+{\psi'}^2\left(t\right)}dt \end{aligned}$$

$$y=\psi\left(x\right)\to\begin{cases} x=\varphi\left(x\right)=x \\ y=\psi\left(x\right)\end{cases}\to\sqrt{{\varphi'}^2\left(x\right)+{\psi'}^2\left(x\right)}=\sqrt{1+{\psi'}^2\left(x\right)}$$

$$x=\varphi\left(y\right)\to\begin{cases} x=\varphi\left(y\right) \\ y=\psi\left(y\right)=y\end{cases}\to\sqrt{{\varphi'}^2\left(y\right)+{\psi'}^2\left(y\right)}=\sqrt{1+{\varphi'}^2\left(y\right)}$$

$$\begin{cases} x=\varphi\left(t\right) \\ y=\psi\left(t\right) \\ z=\omega\left(t\right)\end{cases} t\in\left[\alpha,\beta\right]\to\int_\Gamma f\left(x,y,z\right)ds=\int_\Gamma f\left(\varphi\left(t\right),\psi\left(t\right),\omega\left(t\right)\right)\sqrt{{\varphi'}^2\left(t\right)+{\psi'}^2\left(t\right)+{\omega'}^2\left(t\right)}dt$$

11.2 对坐标的曲线积分

11.2.1 对坐标的曲线积分的概念与性质

变力沿曲线所作的功

$$\begin{aligned} W &= \sum\Delta W\approx\sum\left[P\left(\xi_i,\eta_i\right)\Delta x_i+Q\left(\xi_i,\eta_i\right)\Delta y_i\right] \\ &= \lim_{\lambda\to 0}\sum\left[P\left(\xi_i,\eta_i\right)\Delta x_i+Q\left(\xi_i,\eta_i\right)\Delta y_i\right] \\ &\mathbf{F}\left(x,y\right)=\left(P\left(x,y,z\right),Q\left(x,y,z\right)\right) \\ &\mathbf{A}\left(x,y,z\right)=\left(P\left(x,y,z\right),Q\left(x,y,z\right),R\left(x,y,z\right)\right) \\ &d\mathbf{r_2}=\left(dx,dy\right) \\ &d\mathbf{r_3}=\left(dx,dy,dz\right) \end{aligned} \to \begin{array}{l} \int_L P\left(x,y\right)dx=\lim_{\lambda\to 0}\sum P\left(\xi_i,\eta_i\right)\Delta x_i \\ \int_L P\left(x,y\right)dy=\lim_{\lambda\to 0}\sum Q\left(\xi_i,\eta_i\right)\Delta y_i \\ \textbf{第二类曲线积分} \\ W=\int_L\mathbf{F}\left(x,y\right)\cdot d\mathbf{r_2} \\ W_3=\int_\Gamma\mathbf{A}\left(x,y,z\right)\cdot d\mathbf{r_3} \end{array}$$

$\int_L \alpha\mathbf{F}_1+\beta\mathbf{F}_2\cdot d\mathbf{r}_2=\alpha\int_L\mathbf{F}_1\cdot d\mathbf{r}_2+\beta\int_L\mathbf{F}_2\cdot d\mathbf{r}_2$

$\int_L\mathbf{F}\cdot d\mathbf{r}_2=\int_{L_1}\mathbf{F}\cdot d\mathbf{r}_2+\int_{L_2}\mathbf{F}\cdot d\mathbf{r}_2$

$\int_L\mathbf{F}\cdot d\mathbf{r}=-\int_{L^-}\mathbf{F}\cdot d\mathbf{r}$

11.2.2 对坐标的曲线积分的计算法

$P(x,y), Q(x,y)$ 在L连续

$L:\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \rightarrow \int_L P(x,y)\,dx+Q(x,y)\,dy=\int_\alpha^\beta \left[P(\varphi(t),\psi(t))\varphi'+Q(\varphi(t),\psi(t))\psi'\right]dt$

$\varphi(t), \psi(t)$ 在 $[\alpha,\beta]$ 连续一阶导

$\varphi'^2(t)+\psi'^2(t)\neq 0$

$x=\varphi(y)\rightarrow L:\begin{cases} x=\varphi(y) \\ y=y \end{cases} \rightarrow \int_L P(x,y)\,dx+Q(x,y)\,dy=\int_\alpha^\beta \left[P(\varphi(y),y)\varphi'(y)+Q(\varphi(y),y)\right]dy$

$y=\psi(x)\rightarrow L:\begin{cases} x=x \\ y=\psi(x) \end{cases} \rightarrow \int_L P(x,y)\,dx+Q(x,y)\,dy=\int_\alpha^\beta \left[P(x,\psi(x))+Q(x,\psi(x))\psi'(x)\right]dx$

$\Gamma:\begin{cases} x=\varphi(t) \\ y=\psi(t) \\ z=\omega(t) \end{cases} \rightarrow \int_\Gamma (P,Q,R)\cdot(dx,dy,dz)=\int_\Gamma P\varphi'+Q\psi'+R\omega'\,dt$

11.2.3 两类曲线积分的关系

$L:\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$

记$k=\sqrt{\varphi'^2(t)+\psi'^2(t)}$

切向量$n=(\cos\alpha,\cos\beta)=\left(\frac{\varphi'}{k},\frac{\psi'}{k}\right)$

$\rightarrow$

$$\begin{aligned}\int_L[P\cos\alpha+Q\cos\beta]\,ds &= \int_L\left[P\frac{\varphi'}{k}+Q\frac{\psi'}{k}\right]kdt \\ &= \int_L\left[P\varphi'+Q\psi'\right]dt\end{aligned}$$

$\int_L (P,Q)\cdot(\cos\alpha,\cos\beta)\,ds=\int(P,Q)\cdot(dx,dy)=\int(P,Q)\,dr$

$\int_L \mathbf{F}\cdot nds=\int_L \mathbf{F}_n ds=\int_L \mathbf{F}\cdot dr$

$\int_L (P,Q,R)\cdot(\cos\alpha,\cos\beta,\cos\gamma)\,ds=\int(P,Q,R)\cdot(dx,dy,dz)=\int(P,Q,R)\,dr$

$\int_\Gamma \mathbf{A}\cdot nds=\int_\Gamma \mathbf{A}_n ds=\int_\Gamma \mathbf{A}\cdot dr$

$dr=nds$有向曲线元

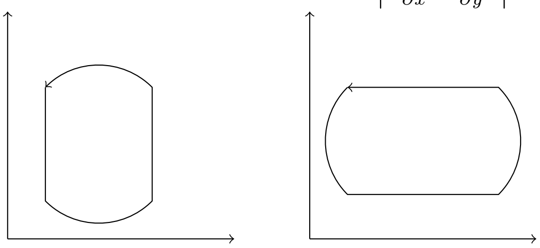
11.3 格林公式及其应用

11.3.1 格林公式

D 内任意封闭曲线所围部分都属于 D, **单连通**，非单连通**复连通**

边界曲线正向，左边是 D

$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=-\iint_D\begin{vmatrix} P & Q \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{vmatrix}=\oint_L Pdx+Qdy$



$\iint_D \frac{\partial P}{\partial y}dxdy=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\frac{\partial Q}{\partial y}dy\right]dx=\int_a^b P(x,\varphi_2(x))-P(x,\varphi_1(x))\,dx$

$\iint_D \frac{\partial Q}{\partial x}dxdy=\int_c^d\left[\int_{\psi_1(y)}^{\psi_2(y)}\frac{\partial Q}{\partial x}dx\right]dy=\int_c^d Q(\psi_2(y),y)-Q(\psi_1(y),y)\,dy$

$\oint Pdx = \int_{L_1} Pdx + \int_{L_2} Pdx + \int_{L_3} Pdx + \int_{L_4} Pdx$
$= \int_{L_1} P\left(x, \varphi_2\left(x\right)\right) dx + \int_{L_2} P\left(x, \varphi_1\left(x\right)\right) dx$
$= \int_b^a P\left(x, \varphi_2\left(x\right)\right) dx + \int_a^b P\left(x, \varphi_1\left(x\right)\right) dx$
$= \int_a^b P\left(x, \varphi_1\left(x\right)\right) - P\left(x, \varphi_2\left(x\right)\right) dx$
$= -\left(\int_a^b P\left(x, \varphi_2\left(x\right)\right) - P\left(x, \varphi_1\left(x\right)\right) dx\right)$
$L_1 : y = \varphi_2\left(x\right), x \in [b, a]$
$L_2 : y = \varphi_1\left(x\right), x \in [a, b]$
$L_3 : x = a, y \in [d, c]$
$L_4 : x = b, y \in [a, b]$
$\oint Qdy = \int_{L1} Qdy + \int_{L2} Qdy + \int_{L3} Qdy + \int_{L4} Qdy$
$= \int_{L_3} Q\left(\psi_1, y\right) dy + \int_{L_4} Q\left(\psi_2, y\right) dy$
$= \int_d^c Q\left(\psi_1, y\right) dy + \int_c^d Q\left(\psi_2, y\right) dy$
$= \int_c^d Q\left(\psi_2, y\right) - Q\left(\psi_1, y\right) dy$
$L_1 : y = d, x \in [b, a]$
$L_2 : y = c, x \in [a, b]$
$L_3 : x = \psi_1\left(y\right)$
$L_4 : x = \psi_2\left(y\right)$
非标准分割

11.3.2 平面上曲线积分与路径无关的条件

区域 G 内任意点 A,B, 任意 A 到 B 曲线,
$if \quad \int_{L_1} Pdx + Qdy = \int_{L_2} Pdx + Qdy, then$ 曲线积分 $\int_L Pdx + Qdy$在 G 内与路径无关, $\oint_{L_1 - L_2} Pdx + Qdy = 0$
if 单连通区域G, P, Q在 G 内连续一阶偏导数, $then$ 曲线积分 $\int_L Pdx + Qdy$在 G 内与**路径无关** $\Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
奇点

11.3.3 二元函数的全微分求积

单连通区域 $G, P\left(x, y\right), Q\left(x, y\right)$ 在 G 内具有一阶连续偏,
$Fdx + Qdy$ 在 G 内为某一函数 $\mu\left(x, y\right)$ 的全微分 $\Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
曲线积分 $\int_L Pdx + Qdy$ 在 G 内与路径无关 $\Leftrightarrow \exists \mu\left(x, y\right), du = Pdx + Qdy$

全微分方程

微分方程 $P\left(x, y\right) dx + Q\left(x, y\right) dy = 0$ 左端是某函数 $\mu(x, y)$ 全微分, **全微分方程**

11.3.4 曲线积分的基本定理

曲线积分 $\int_L \mathbf{F} \cdot d\mathbf{r}$ 在区域 G 内与积分路径无关，向量场 $\mathbf{F}$ 为保守场

$F\left(x, y\right) = \left(P\left(x, y\right) + Q\left(x, y\right)\right)$ 是区域 G 内向量场,$P\left(x, y\right), Q\left(x, y\right)$ 在 G 内连续，且存在数量函数使得 $F = \nabla f$, 则曲线积分 $\int_L \mathbf{F} \cdot d\mathbf{r}$ 在 G 内与路径无关，且 $\int_L \mathbf{F} \cdot d\mathbf{r} = f\left(B\right) - f\left(A\right)$ **曲线积分基本公式**

11.4 对面积的曲面积分

11.4.1 对面积的曲面积分的概念与性质

光滑曲面 Σ, 函数 $f\left(x, y, z\right)$ 在 Σ 有界,任意分割 n 小块 ΔS_i, 无关面分法,点取法,极限 $\lim_{\lambda \to 0} \sum f\left(\xi_i, \eta_i, \zeta_i\right) \Delta S_i = \iint_\Sigma f\left(x, y, z\right) dS$ **第一类曲面积分**

11.4.2 对面积的曲面积分的计算法

曲面 $\Sigma : z = z(x,y)\,, n = (-z_x, -z_y, 1) = \left(\frac{-z_x}{\sqrt{{z_x}^2+{z_y}^2+1}}, \frac{-z_y}{\sqrt{{z_x}^2+{z_y}^2+1}}, \frac{1}{\sqrt{{z_x}^2+{z_y}^2+1}}\right)$

$|n\cdot(1,0,0)| = \frac{z_x}{\sqrt{{z_x}^2+{z_y}^2+1}} = \cos\alpha \quad \Delta S_i\cos\alpha = (\Delta\sigma_i)_{yz} \quad \Delta S_i = \frac{1}{\cos\alpha}(\Delta\sigma_i)_{yz}$

$|n\cdot(0,1,0)| = \frac{z_y}{\sqrt{{z_x}^2+{z_y}^2+1}} = \cos\beta \quad \Delta S_i\cos\beta = (\Delta\sigma_i)_{xz} \quad \Delta S_i = \frac{1}{\cos\beta}(\Delta\sigma_i)_{xz}$

$|n\cdot(0,0,1)| = \frac{1}{\sqrt{{z_x}^2+{z_y}^2+1}} = \cos\gamma \quad \Delta S_i\cos\gamma = (\Delta\sigma_i)_{xy} \quad \Delta S_i = \frac{1}{\cos\gamma}(\Delta\sigma_i)_{xy}$

$\lim_{\lambda\to 0}\sum f(\xi_i,\eta_i,\zeta_i)\,\Delta S_i = \lim_{\lambda\to 0}\sum f(\xi_i,\eta_i,\zeta_i)\frac{1}{\cos\gamma}(\Delta\sigma_i)_{xy}$

$\int_\Sigma f(x,y,z)\,dS = \iint_{D_{xy}} f[x,y,(x,y)]\sqrt{{z_x}^2+{z_y}^2+1}dxdy$

11.5 对坐标的曲面积分

11.5.1 对坐标的曲面积分的概念与性质

ΔS 单位法向量 n ,$n\cdot(0,0,1) = \cos\gamma \Rightarrow \Delta S$在$xOy$面投影 $(\Delta S)_{xy} = \begin{cases} (\Delta\sigma)_{xy}\,, & \cos\gamma > 0 \\ -(\Delta\sigma)_{xy}\,, & \cos\gamma < 0 \\ 0, & \cos\gamma = 0 \end{cases}$

流向曲面一侧的流量

$Av\cdot n$

光滑有向曲面 Σ, 函数 $R(x,y,z)$ 在 Σ 有界，任意分割 Σ 为 ΔS_i, $\begin{matrix} \Delta S_i在xOy面投影为(\Delta S_i)_{xy} \\ \Delta S_i在xOz面投影为(\Delta S_i)_{xz} \\ \Delta S_i在yOz面投影为(\Delta S_i)_{yz} \end{matrix}$

第二类曲面积分,函数在有向面上对坐标轴 x,y(x,z/y,z) 的积分 $\begin{cases} \iint_\Sigma R(x,y,z)\,dxdy = \lim_{\lambda\to 0} R(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{xy} \\ \iint_\Sigma Q(x,y,z)\,dxdz = \lim_{\lambda\to 0} Q(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{xz} \\ \iint_\Sigma P(x,y,z)\,dydz = \lim_{\lambda\to 0} P(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{yz} \end{cases}$

$\iint_\Sigma (P,Q,R)\cdot(dydz,dxdz,dxdy) = \iint_{\Sigma_1}(P,Q,R)\cdot(dydz,dxdz,dxdy) + \iint_{\Sigma_2}(P,Q,R)\cdot(dydz,dxdz,dxdy)$

$\iint_\Sigma (P,Q,R)\cdot(dydz,dxdz,dxdy) = -\iint_{\Sigma^-}(P,Q,R)\cdot(dydz,dxdz,dxdy)$

$\iint_{\Sigma^-} Pdydz = -\iint_\Sigma Pdydz$

$\iint_{\Sigma^-} Pdxdz = -\iint_\Sigma Pdxdz$

$\iint_{\Sigma^-} Pdxdy = -\iint_\Sigma Pdxdy$

11.5.2 对坐标的曲面积分的计算法

$\Sigma : z = z(x,y)\,, (\Delta S_i)_{xy} = \pm(\Delta\sigma)_{xy}\,, \iint_\Sigma R(x,y,z)\,dxdy = \pm\iint_{D_{xy}} R(x,y,z(x,y))\,dxdy$

11.5.3 两类曲面积分之间的关系

$\Sigma : z = z(x,y)\,, \Sigma$ 法向量 $\mathbf{n} = \left(\frac{-z_x}{|n|}, \frac{-z_y}{|n|}, \frac{1}{|n|}\right) = (\cos\alpha, \cos\beta, \cos\gamma)$

$\iint_\Sigma (P,Q,R)\cdot(dydz,dxdz,dxdy) = \iint_\Sigma (P,Q,R)\cdot(\cos\alpha,\cos\beta,\cos\gamma)\,dS$

$\iint_\Sigma \mathbf{A}\cdot d\mathbf{S} = \iint_\Sigma \mathbf{A}\cdot\mathbf{n}dS = \iint_\Sigma \mathbf{A}_n dS$

$d\mathbf{S} = \mathbf{n}dS$ 有向曲面元

11.6 高斯公式　通量与散度

11.6.1 高斯公式

空间闭区域 Ω 由分片光滑闭曲面 (有向，曲面外侧)Σ 组成，$P(x,y,z)\,, Q(x,y,z)\,, R(x,y,z)$ 在 Ω 上有一阶连续偏导数，Σ 方向余弦 $n = (\cos\alpha, \cos\beta, \cos\gamma)$ 则

$$\iint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dv=\oiint_{\Sigma}(P,Q,R)\cdot(dydz,dxdz,dxdy)=\oiint_{\Sigma}(P,Q,R)\cdot(\cos\alpha,\cos\beta,\cos\gamma)\,dS$$

$$\begin{aligned}\iiint_{\Omega}\tfrac{\partial R}{\partial z}dv &=\iint_{D_{xy}}\left\{\int_{z_1(x,y)}^{z_2(x,y)}\tfrac{\partial R}{\partial z}dz\right\}dxdy\\ &=\iint_{D_{xy}}\left\{R\left(x,y,z_2\right)-R\left(x,y,z_1\right)\right\}dxdy\\ \oiint_{\Sigma}R\left(x,y,z\right)dxdy &=\iint_{\Sigma_1}Rdxdy+\iint_{\Sigma_2}Rdxdy+\iint_{\Sigma_3}Rdxdy\\ &=\iint_{\Sigma_1}R\left(x,y,z\right)dxdy+\iint_{\Sigma_2}R\left(x,y,z\right)dxdy+0\\ &=-\iint_{D_{xy}}R\left(x,y,z_1\right)dxdy+\iint_{D_{xy}}R\left(x,y,z_2\right)dxdy\end{aligned}\quad\Rightarrow\quad\begin{aligned}\iint_{\Omega}\tfrac{\partial P}{\partial x}dv&=\oiint_{\Sigma}Pdydz\\ \iint_{\Omega}\tfrac{\partial Q}{\partial y}dv&=\oiint_{\Sigma}Qdxdz\\ \iint_{\Omega}\tfrac{\partial R}{\partial z}dv&=\oiint_{\Sigma}Rdxdy\end{aligned}$$

不规则体积分割

函数 $u\left(x,y,z\right),v\left(x,y,z\right)$ 在闭区域 Ω 上具有一二阶连续偏导数，

则 $\iint_{\Omega}u\Delta vdxdydz=\oiint_{\Sigma}u\dfrac{\partial v}{\partial n}dS-\iiint_{\Omega}\left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y}+\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y}\right)$ **格林第一公式**,Σ 为闭区域 Ω 边界曲面,$\frac{\partial v}{\partial n}$ 为函数 $v\left(x,y,z\right)$ 沿 Σ 外法线方向的方向导数,$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ 称为**拉普拉斯算子**

$$\begin{aligned}\oiint_{\Sigma}u\frac{\partial v}{\partial n}dS &=\oiint_{\Sigma}u\left(\frac{\partial v}{\partial x}\cos\alpha+\frac{\partial v}{\partial y}\cos\beta+\frac{\partial v}{\partial z}\cos\gamma\right)dS\\ &=\oiint_{\Sigma}\left[\left(u\frac{\partial v}{\partial x}\right)\cos\alpha+\left(u\frac{\partial v}{\partial y}\right)\cos\beta+\left(u\frac{\partial v}{\partial z}\right)\cos\gamma\right]dS\\ &=\iiint_{\Omega}\left[\tfrac{\partial}{\partial x}\left(u\tfrac{\partial v}{\partial x}\right)+\tfrac{\partial}{\partial y}\left(u\tfrac{\partial v}{\partial y}\right)+\tfrac{\partial}{\partial z}\left(u\tfrac{\partial v}{\partial z}\right)\right]dxdydz\\ &=\iiint_{\Omega}\left[u\tfrac{\partial^2 v}{\partial x^2}+u\tfrac{\partial^2 v}{\partial y^2}+u\tfrac{\partial^2 v}{\partial z^2}+\tfrac{\partial u}{\partial x}\tfrac{\partial v}{\partial x}+\tfrac{\partial u}{\partial y}\tfrac{\partial v}{\partial y}+\tfrac{\partial u}{\partial z}\tfrac{\partial v}{\partial z}\right]dxdydz\\ &=\iiint_{\Omega}\left[u\Delta v+\tfrac{\partial u}{\partial x}\tfrac{\partial v}{\partial x}+\tfrac{\partial u}{\partial y}\tfrac{\partial v}{\partial y}+\tfrac{\partial u}{\partial z}\tfrac{\partial v}{\partial z}\right]dxdydz\\ &=\iiint_{\Omega}u\Delta vdxdydz+\iiint_{\Omega}\tfrac{\partial u}{\partial x}\tfrac{\partial v}{\partial x}+\tfrac{\partial u}{\partial y}\tfrac{\partial v}{\partial y}+\tfrac{\partial u}{\partial z}\tfrac{\partial v}{\partial z}dxdydz\end{aligned}$$

11.6.2 沿任意闭曲面的曲面积分为零的条件

空间区域 G,G 内任意闭曲面所围区域完全属于 G, **空间二维单连通**

空间区域 G,G 内任意闭曲线总可以张成 (做曲面) 完全属于 G, **空间一维单连通**

二维单连通区域 G，若 P,Q,R, 在 G 内具有一阶连续偏导数，则曲面积分 $\iint_{\Sigma}Pdydz+Qdxdz+Rdxdy$ 在 G 内与选取曲面选取无关，只与边界曲线有关 $\Leftrightarrow\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$

11.6.3 通量与散度

向量场$\mathbf{A}\left(x,y,z\right)=\left(P,Q,R\right)$

有向曲面Σ, 法向量$\mathbf{n}$

通量 $\iint_{\Sigma}\mathbf{A}\cdot\mathbf{n}dS$

速度场$v\left(x,y,z\right)=\left(P,Q,R\right)$

$$\begin{aligned}\iiint_{\Omega}\left(\tfrac{\partial P}{\partial x}+\tfrac{\partial Q}{\partial y}+\tfrac{\partial P}{\partial y}\right)dv &=\oiint_{\Sigma}v_ndS\\ \tfrac{1}{V}\iiint_{\Omega}\left(\tfrac{\partial P}{\partial x}+\tfrac{\partial Q}{\partial y}+\tfrac{\partial P}{\partial y}\right)dv &=\tfrac{1}{V}\oiint_{\Sigma}v_ndS\\ \left(\tfrac{\partial P}{\partial x}+\tfrac{\partial Q}{\partial y}+\tfrac{\partial P}{\partial y}\right)|_{(\xi,\eta,\zeta)} &=\tfrac{1}{V}\oiint_{\Sigma}v_ndS\quad(\xi,\eta,\zeta)\in\Omega\\ \tfrac{\partial P}{\partial x}+\tfrac{\partial Q}{\partial y}+\tfrac{\partial P}{\partial y} &=\lim_{\Omega\to M}\tfrac{1}{V}\oiint_{\Sigma}v_ndS=\mathrm{div}v\left(M\right)=\Delta\cdot\mathbf{A}\end{aligned}$$

速度场v在点M的通量密度, 源头强度**散度**

div$\mathbf{V}$ 处处为零，**无源场**

散度体积分，通量面积分

11.7 斯托克斯公式 * 环流量与旋度

11.7.1 斯托克斯公式

分段光滑空间有向闭曲线 Γ,Γ 张成分片光滑有向曲面 Σ，符合右手定则若 P,Q,R 在曲面 Σ 上具有一阶连续偏导数，则 $\iint_\Sigma \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dxdz+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\oint_\Gamma Pdx+Qdy+Rdz$

曲面$\Sigma:z=z\left(x,y\right)$,法向量$n=(\cos\alpha,\cos\beta,\cos\gamma)=\left(-\frac{z_x}{|n|},-\frac{z_y}{|n|}\ \frac{1}{|n|}\right)$

$$\begin{aligned}\iint_\Sigma \frac{\partial P}{\partial z}dzdx-\frac{\partial P}{\partial y}dydx &= \iint_\Sigma \left(\frac{\partial P}{\partial z}\cos\beta-\frac{\partial P}{\partial y}\cos\gamma\right)dS\\ &\xlongequal{\cos\beta=-f_y\cos\gamma}-\iint_\Sigma\left(\frac{\partial P}{\partial z}f_y+\frac{\partial P}{\partial y}\right)\cos\gamma dS\\ &=-\iint_{D_{xy}}\left(\frac{\partial P}{\partial z}f_y+\frac{\partial P}{\partial y}\right)dxdy\\ &=-\iint_{D_{xy}}\frac{\partial P}{\partial y}dxdy\\ &=\oint_L Pdx\end{aligned}$$

$$\iint_\Sigma \frac{\partial Q}{\partial x}dxdy-\frac{\partial Q}{\partial z}dzdy=\oint_\Gamma Qdy$$
$$\iint_\Sigma \frac{\partial R}{\partial y}dydz-\frac{\partial R}{\partial x}dxdz=\oint_\Gamma Rdz$$

$$\iint_\Sigma\begin{vmatrix} dydz & dxdz & dxdy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}\begin{aligned}&=\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}dS=\oint_\Gamma Pdx+Qdy+Rdz\\ &=\begin{vmatrix}\cos\alpha & \cos\beta & 1\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0\\ P & Q & 0\end{vmatrix}dS=\oint_\Gamma Pdx+Qdy\end{aligned}$$

11.7.2 空间曲线积分与路径无关的条件

一维单连通区域 G，若函数 P,Q,R 在 G 内具有一阶连续偏导数，

则曲线积分 $\int_\Gamma Pdx+Qdy+Rdz$ 在 G 内与路径无关 $\Leftrightarrow\begin{cases}\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\\ \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}\\ \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}\end{cases}$

一维单连通区域 G，若函数 P,Q,R 在 G 内具有一阶连续偏导数，

则存在 $u\left(x,y,z\right),du=Pdx+Qdy+Rdz\Leftrightarrow\begin{cases}\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\\ \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}\\ \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}\end{cases}$

$$\begin{aligned}u\left(x,y,z\right)&=\int_{(x_0,y_0,z_0)}^{(x,y,z)}Pdx+Qdy+Rdz\\ &=\int_{x_0}^{x}Pdx+\int_{y_0}^{y}Qdy+\int_{z_0}^{z}Rdz\end{aligned}$$

11.7.3 环流量与旋度

向量场 $\mathbf{A}=(P,Q,R)$, 分段光滑有向闭曲线 Γ, 单位切向量 τ , 则 $\oint_\Gamma \mathbf{A}\cdot\tau ds$ 称为 A 沿 τ **环流量**

$\mathbf{rot}\ \mathbf{A}=\Delta\times\mathbf{A}=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$ **旋度**

旋度处处为零，**无旋场**

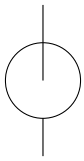
无旋，无源，**调和场**

Σ 单位法向量 $n=(\cos\alpha,\cos\beta,\cos\gamma)$

$\mathbf{rot}\ \mathbf{A}\cdot n=\Delta\times\mathbf{A}\cdot n=\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$

$\iint_\Sigma \mathbf{rot}\ \mathbf{A}\cdot\mathbf{n}dS=\oint_\Gamma \mathbf{A}\cdot\tau ds$

旋度通量，环流量



$$r=\vec{OM},\omega=(0,0,w)\,,v=\omega\times r,v=\begin{vmatrix} i & j & k \\ 0 & 0 & w \\ x & y & z \end{vmatrix}=(-wy,wx,0)\,,\mathbf{rot}\ \mathbf{v}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix}=(0,0,2w)=2\omega$$

12 无穷级数

12.1 常数项级数的概念和性质

12.1.1 数项级数的概念

数列 $\mu_1, \mu_2, \mu_3, \cdots + \mu_n, \ldots$，表达式 $\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n + \ldots$ 称为 (常数项) 无穷级数, 简称 **(常数项) 级数**, 记为 $\sum_{i=1}^{\infty} \mu_i = \mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n + \ldots$, 第 n 项 μ_n 叫做级数的一般项

$S_n = \sum_{i=1}^{n} \mu_i = \mu_1, \mu_2, \mu_3, \cdots + \mu_n$ 级数部分和

新数列 $\{S_n\}$ $\quad S_1, S_2, \ldots, S_n, \ldots$

无穷级数 $\sum_{i=1}^{\infty} \mu_i$，无穷级数的部分和数列 $\{S_n\}$， $\begin{array}{l} \lim_{n\to\infty} S_n = s \Rightarrow \sum_{i=1}^{\infty} \mu_i \text{收敛}, s = \mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n + \ldots \\ \lim_{n\to\infty} S_n = \infty(\text{不存在}) \Rightarrow \sum_{i=1}^{\infty} \mu_i \text{发散} \end{array}$

$r_n = s - S_n = S_{n+1} + S_{n+2} + \ldots$ 余项

$$\sum_{i=1}^{\infty} \mu_i = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_i$$

$\sum_{i=1}^{\infty} aq^i = a + aq + aq^2 + \cdots + aq^i + \ldots$ 等比级数, $\begin{array}{l} S_n = a + aq + \cdots + aq^{n-1} = \frac{a-aq^n}{1-q} \text{部分和数列} \\ |q| \geqslant 1, \text{发散}, |q| \leqslant 1, \text{收敛} \end{array}$

$$S_n = S_1 + (S_2 - S_1) + \cdots + (S_n - S_{n-1}) + ... = S_1 + \sum_{i=2}^{\infty} = \sum_{i=1}^{\infty} \mu_i$$

12.1.2 收敛级数的基本性质

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s, 则级数 $\sum_{n=1}^{\infty} k\mu_n$ 收敛于 s, 和为 ks, 数乘同敛散

级数 $\sum_{n=1}^{\infty} \mu_n$ 与 $\sum_{n=1}^{\infty} v_n$ 分别收敛于 s 与 σ, 级数 $\sum_{n=1}^{\infty} (\mu_n \pm v_n)$ 也收敛, 和为 $s \pm \sigma$, 收敛和收敛

级数中去掉，加上，改变有限项，不会改变级数的收敛性

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s，任意项加括号组成的级数仍收敛，和为 s, 括号发散源发散

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s(必要条件), 一般项趋近 0，即 $\lim_{n\to\infty} \mu_n = 0$

调和级数 $\sum_{n=1}^{\infty} \frac{1}{n}$ 发散

12.1.3 柯西审敛原理

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛 $\Leftrightarrow \forall \varepsilon > 0, \exists N > 0, when \quad n > N, \forall p \in Z^+, |\mu_{n+1} + \mu_{n+2} + \cdots + \mu_{n+p}| < \varepsilon$

12.2 常数项级数审敛法

各项是正数或零的级数**正项级数**

正项级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s $\Leftrightarrow$ 部分和数列 $\{S_n\}$ 有界

正项级数 $\sum_{n=1}^{\infty} \mu_n, \sum_{n=1}^{\infty} v_n, \mu_n \leqslant v_n, \sum_{n=1}^{\infty} v_n$ 收敛，$\sum_{n=1}^{\infty} \mu_n$ 收敛；$\sum_{n=1}^{\infty} \mu_n$ 发散,$\sum_{n=1}^{\infty} v_n$ 发散

正项级数 $\sum_{n=1}^{\infty} \mu_n, \sum_{n=1}^{\infty} v_n$, 存在正整数 N， $\mu_n \leqslant kv_n\, (n > N, k > 0), \sum_{n=1}^{\infty} v_n$ 收敛，$\sum_{n=1}^{\infty} \mu_n$ 收敛；$\sum_{n=1}^{\infty} \mu_n$ 发散,$\sum_{n=1}^{\infty} v_n$ 发散

$$\text{正项级数}\sum_{n=1}^{\infty}\mu_n,\begin{array}{ll} if & \lim_{n\to\infty}\frac{\mu_n}{v_n}=l,\{0\leqslant l<\infty\},\sum_{n=1}^{\infty}v_n\text{收敛, 则}\sum_{n=1}^{\infty}\mu_n\text{收敛}\\ if & \lim_{n\to\infty}\frac{\mu_n}{v_n}=l,\{l>0,+\infty\},\sum_{n=1}^{\infty}v_n\text{发散, 则, }\sum_{n=1}^{\infty}\mu_n\text{发散}\end{array}$$

$$\text{正项级数}\sum_{n=1}^{\infty}\mu_n,\lim_{n\to\infty}\frac{\mu_{n+1}}{\mu_n}=\rho,\begin{array}{ll} if & \rho>1\ (=\infty)\text{ 发散 (不可能收敛)}\\ if & \rho<1\text{收敛}\\ if & \rho=1\text{可能收敛}\end{array}$$

$$when\quad \rho<1,\forall\rho+\varepsilon=r<1,\exists m,when\quad n\geqslant m,\frac{\mu_{n+1}}{\mu_n}=\rho<\rho+\varepsilon=r<1,\mu_{n+1}<r\mu_n,\mu_{n+k}<r^k\mu_n$$

级数 $\sum_{n=1}^{\infty}r^k\mu_n$ 收敛，级数 $\sum_{n=1}^{\infty}\mu_{n+k}$ 收敛

$$\sum_{n=1}^{\infty}\mu_n=\sum_{n=1}^{\infty}\mu_{n+k}+\sum_{n=1}^{k}\mu_n\text{ 则级数 }\sum_{n=1}^{\infty}\mu_n,\sum_{n=1}^{\infty}\mu_{n+k}\text{ 同敛散，级数 }\sum_{n=1}^{\infty}\mu_n\text{ 收敛}$$

$$\text{正项级数}\sum_{n=1}^{\infty}\mu_n,\lim_{n\to\infty}\sqrt[n]{\mu_n}=\rho,\begin{array}{ll} if & \rho>1\ (=\infty)\text{ 发散 (不可能收敛)}\\ if & \rho<1\text{收敛}\\ if & \rho=1\text{可能收敛}\end{array}$$

$$when\quad \rho<1,\mu_n<r^n\,(r<1)\,,\text{级数}\sum_{n=1}^{\infty}r^n\text{ 收敛, 则 }\sum_{n=1}^{\infty}\mu_n\text{ 收敛}$$

$$\text{正项级数}\sum_{n=1}^{\infty}\mu_n,\begin{array}{ll} if & \lim_{n\to\infty}n^p\mu_n=\frac{\mu_n}{\frac{1}{n^p}}=l,\{0\leqslant l<\infty,p>1\},\sum_{n=1}^{\infty}\frac{1}{n^p}\text{收敛, 则级数}\sum_{n=1}^{\infty}\mu_n\text{收敛}\\ if & \lim_{n\to\infty}n\mu_n=\frac{\mu_n}{\frac{1}{n}}=l,\{l>0,\infty\},\sum_{n=1}^{\infty}\frac{1}{n}\text{发散, 则级数}\sum_{n=1}^{\infty}\text{ 发散}\mu_n\end{array}$$

12.3 交错级数审敛法

$$\text{各项正负交错，可以写成 }\begin{array}{l}\mu_1-\mu_2+\mu_3-\mu_4+\dots,\\ -\mu_1+\mu_2-\mu_3+\mu_4-\dots,\end{array}\text{ 其中 }\mu_n>0$$

$$\text{交错级数}\sum_{n=1}^{\infty}\mu_n\text{ 满足条件：}\begin{array}{ll}(1) & \mu_n\geqslant\mu_{n+1}\,(n=1,2,3,\dots)\\ (2) & \lim_{n\to\infty}\mu_n=0\end{array},\text{那么级数收敛,其和 }s\leqslant\mu_1,\text{余项绝对值 }|r_n|\leqslant\mu_{n+1}$$

12.4 绝对收敛与条件收敛

$$\text{级数}\sum_{n=1}^{\infty}\mu_n,\begin{array}{l}\text{正项级数}\sum_{n=1}^{\infty}|\mu_n|\text{ 收敛, 则级数}\sum_{n=1}^{\infty}\mu_n\text{绝对收敛}\\ \text{正项级数}\sum_{n=1}^{\infty}|\mu_n|\text{ 发散,, 级数}\sum_{n=1}^{\infty}\mu_n\text{收敛, 则级数}\sum_{n=1}^{\infty}\mu_n\text{条件收敛}\end{array}$$

级数 $\sum_{n=1}^{\infty}\mu_n$绝对收敛, 则级数 $\sum_{n=1}^{\infty}\mu_n$必定收敛 $\quad v_n=\frac{1}{2}\left(\mu_n+|\mu_n|\right)$