This is Start

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12.1 常数项级数的概念和性质

12.1.1 数项级数的概念

数列 $\mu_1,\mu_2,\mu_3,\dots+\mu_n,\dots$，表达式 $\mu_1+\mu_2+\mu_3+\dots+\mu_n+\dots$ 称为(常数项)无穷级数,简称**(常数项)级数**，记为$\sum_{i=1}^{\infty}\mu_i=\mu_1+\mu_2+\mu_3+\dots+\mu_n+\dots$,第n项$\mu_n$叫做级数的一般项

$S_n=\sum_{i=1}^{n}\mu_i=\mu_1,\mu_2,\mu_3,\dots+\mu_n$ 级数部分和

新数列 $\{S_n\}$ $\quad S_1,S_2,\dots,S_n,\dots$

无穷级数$\sum_{i=1}^{\infty}\mu_i$，无穷级数的部分和数列$\{S_n\}$， $\lim_{n\to\infty}S_n=s\Rightarrow\sum_{i=1}^{\infty}\mu_i$收敛$,s=\mu_1+\mu_2+\mu_3+\dots+\mu_n+\dots$
$\lim_{n\to\infty}S_n=\infty$(不存在) $\Rightarrow\sum_{i=1}^{\infty}\mu_i$发散

$r_n=s-S_n=S_{n+1}+S_{n+2}+\dots$ 余项

$$\sum_{i=1}^{\infty}\mu_i=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\sum_{i=1}^{\infty}\mu_i$$

$\sum\limits_{i=1}^{\infty}aq^i=a+aq+aq^2+\dots+aq^i+\dots$等比级数, $S_n=a+aq+\dots+aq^{n-1}=\frac{a-aq^n}{1-q}$部分和数列
$|q|\geqslant 1$, 发散, $|q|\leqslant 1$, 收敛

$$S_n=S_1+(S_2-S_1)+\dots+(S_n-S_{n-1})+...=S_1+\sum_{i=2}^{\infty}=\sum_{i=1}^{\infty}\mu_i$$

12.1.2 收敛级数的基本性质

级数$\sum\limits_{n=1}^{\infty}\mu_n$收敛于s,则级数$\sum\limits_{n=1}^{\infty}k\mu_n$收敛于s,和为ks,数乘同敛散

级数$\sum\limits_{n=1}^{\infty}\mu_n$与$\sum\limits_{n=1}^{\infty}v_n$分别收敛于$s$与$\sigma$,级数$\sum\limits_{n=1}^{\infty}(\mu_n\pm v_n)$也收敛,和为$s\pm\sigma$,收敛和收敛

级数中去掉，加上，改变有限项，不会改变级数的收敛性

级数$\sum\limits_{n=1}^{\infty}\mu_n$收敛于s，任意项加括号组成的级数仍收敛，和为s,括号发散源发散

级数$\sum\limits_{n=1}^{\infty}\mu_n$收敛于s(必要条件),一般项趋近0，即$\lim\limits_{n\to\infty}\mu_n=0$

调和级数$\sum\limits_{n=1}^{\infty}\frac{1}{n}$发散

12.1.3 柯西审敛原理

级数$\sum\limits_{n=1}^{\infty}\mu_n$收敛 $\Leftrightarrow\forall\varepsilon>0,\exists N>0,when\quad n>N,\forall p\in Z^+,|\mu_{n+1}+\mu_{n+2}+\dots+\mu_{n+p}|<\varepsilon$

12.2 常数项级数审敛法

12.2.1 正项级数及其审敛法

各项是正数或零的级数**正项级数**

正项级数$\sum\limits_{n=1}^{\infty}\mu_n$收敛于s $\Leftrightarrow$ 部分和数列 $\{S_n\}$有界

正项级数$\sum\limits_{n=1}^{\infty}\mu_n,\sum\limits_{n=1}^{\infty}v_n,\mu_n\leqslant v_n,\sum\limits_{n=1}^{\infty}v_n$收敛，$\sum\limits_{n=1}^{\infty}\mu_n$收敛；$\sum\limits_{n=1}^{\infty}\mu_n$发散,$\sum\limits_{n=1}^{\infty}v_n$发散

正项级数$\sum\limits_{n=1}^{\infty}\mu_n,\sum\limits_{n=1}^{\infty}v_n$,存在正整数N, $\quad\mu_n\leqslant kv_n\,(n>N,k>0),\sum\limits_{n=1}^{\infty}v_n$收敛，$\sum\limits_{n=1}^{\infty}\mu_n$收敛；$\sum\limits_{n=1}^{\infty}\mu_n$发散,$\sum\limits_{n=1}^{\infty}v_n$发散

$$\text{正项级数}\sum_{n=1}^{\infty}\mu_n,\begin{array}{ll} if & \lim_{n\to\infty}\frac{\mu_n}{v_n}=l,\{0\leqslant l<\infty\},\sum_{n=1}^{\infty}v_n\text{收敛,则}\sum_{n=1}^{\infty}\mu_n\text{收敛}\\ if & \lim_{n\to\infty}\frac{\mu_n}{v_n}=l,\{l>0,+\infty\},\sum_{n=1}^{\infty}v_n\text{发散,则,ß}\sum_{n=1}^{\infty}\mu_n\text{发散}\end{array}$$

$$\text{正项级数}\sum_{n=1}^{\infty}\mu_n,\lim_{n\to\infty}\frac{\mu_{n+1}}{\mu_n}=\rho,\begin{array}{ll} if & \rho>1\ (=\infty)\ \text{发散(不可能收敛)}\\ if & \rho<1\text{收敛}\\ if & \rho=1\text{可能收敛}\end{array}$$

$$when\quad \rho<1,\forall\rho+\varepsilon=r<1,\exists m,when\quad n\geqslant m,\frac{\mu_{n+1}}{\mu_n}=\rho<\rho+\varepsilon=r<1,\mu_{n+1}<r\mu_n,\mu_{n+k}<r^k\mu_n$$

级数$\sum_{n=1}^{\infty}r^k\mu_n$收敛，级数$\sum_{n=1}^{\infty}\mu_{n+k}$收敛

$$\sum_{n=1}^{\infty}\mu_n=\sum_{n=1}^{\infty}\mu_{n+k}+\sum_{n=1}^{k}\mu_n\text{则级数}\sum_{n=1}^{\infty}\mu_n,\sum_{n=1}^{\infty}\mu_{n+k}\text{同敛散，级数}\sum_{n=1}^{\infty}\mu_n\text{收敛}$$

$$\text{正项级数}\sum_{n=1}^{\infty}\mu_n,\lim_{n\to\infty}\sqrt[n]{\mu_n}=\rho,\begin{array}{ll} if & \rho>1\ (=\infty)\ \text{发散(不可能收敛)}\\ if & \rho<1\text{收敛}\\ if & \rho=1\text{可能收敛}\end{array}$$

$$when\quad \rho<1,\mu_n<r^n\,(r<1)\,,\text{级数}\sum_{n=1}^{\infty}r^n\text{收敛,则}\sum_{n=1}^{\infty}\mu_n\text{收敛}$$

$$\text{正项级数}\sum_{n=1}^{\infty}\mu_n,\begin{array}{ll} if & \lim_{n\to\infty}n^p\mu_n=\frac{\mu_n}{\frac{1}{n^p}}=l,\{0\leqslant l<\infty,p>1\},\sum_{n=1}^{\infty}\frac{1}{n^p}\text{收敛,则级数}\sum_{n=1}^{\infty}\mu_n\text{收敛}\\ if & \lim_{n\to\infty}n\mu_n=\frac{\mu_n}{\frac{1}{n}}=l,\{l>0,\infty\},\sum_{n=1}^{\infty}\frac{1}{n}\text{发散,则级数}\sum_{n=1}^{\infty}\text{发散}\mu_n\end{array}$$

12.2.2 交错级数审敛法

$$\text{各项正负交错，可以写成}\begin{array}{l}\mu_1-\mu_2+\mu_3-\mu_4+\dots,\\ -\mu_1+\mu_2-\mu_3+\mu_4-\dots,\end{array}\text{其中}\mu_n>0$$

$$\text{交错级数}\sum_{n=1}^{\infty}\mu_n\text{满足条件：}\begin{array}{ll}(1) & \mu_n\geqslant\mu_{n+1}\,(n=1,2,3,\dots)\\ (2) & \lim_{n\to\infty}\mu_n=0\end{array}\text{,那么级数收敛，其和}s\leqslant\mu_1\text{,余项绝对值}|r_n|\leqslant\mu_{n+1}$$

12.2.3 绝对收敛与条件收敛

$$\text{级数}\sum_{n=1}^{\infty}\mu_n,\begin{array}{l}\text{正项级数}\sum_{n=1}^{\infty}|\mu_n|\text{收敛,则级数}\sum_{n=1}^{\infty}\mu_n\text{绝对收敛}\\ \text{正项级数}\sum_{n=1}^{\infty}|\mu_n|\text{发散,,级数}\sum_{n=1}^{\infty}\mu_n\text{收敛,则级数}\sum_{n=1}^{\infty}\mu_n\text{条件收敛}\end{array}$$

级数 $\sum_{n=1}^{\infty}\mu_n$绝对收敛,则级数 $\sum_{n=1}^{\infty}\mu_n$必定收敛 $\quad v_n=\frac{1}{2}(\mu_n+|\mu_n|)$

12.2.4 绝对收敛级数的性质

绝对收敛级数经改变项位置后构成的级数也收敛，且与原级数有相同的和$S_n{}^*\leqslant S_m\leqslant s,\mu_n=2v_n-|\mu_n|$

级数$\sum\mu_n,\sum v_n$绝对收敛，和分别为s,σ,所有项的可能乘积μ_iv_i

$$\begin{array}{cccc}\mu_1v_1 & \mu_1v_2 & \mu_1v_3 & \cdots\\ \mu_2v_1 & \mu_2v_2 & \mu_2v_3 & \cdots\\ \mu_3v_1 & \mu_3v_2 & \mu_3v_3 & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{array}\qquad\begin{array}{cccc}\mu_1v_1 & \mu_1v_2 & \mu_1v_3 & \cdots\\ \mu_2v_1 & \mu_2v_2 & \mu_2v_3 & \cdots\\ \mu_3v_1 & \mu_3v_2 & \mu_3v_3 & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{array}$$

柯西乘积(级数) $\mu_1v_1+(\mu_1v_2+\mu_2v_1)+\cdots+(\mu_1v_n+\mu_2v_{n-1}+\cdots+\mu_nv_1)+\dots$绝对收敛，和为$s\sigma$

$$\mu_1v_1+(\mu_1v_2+\mu_2v_2+\mu_2v_1)+(\mu_1v_3+\mu_2v_3+\mu_3v_3+\mu_3v_2+\mu_3v_1)+\cdots=(\mu_1+\mu_2+\cdots+\mu_n)\cdot(v_1+v_2+\cdots+v_n)$$

12.3 幂级数

12.3.1 函数项级数的概念

区间I上函数列 $\mu_1(x), \mu_2(x), \mu_3(x), \ldots, \mu_n(x), \ldots,$

表达式 $\mu_1(x) + \mu_2(x) + \mu_3(x) + \cdots + \mu_n(x) + \ldots,$区间I上的(函数项)无穷级数，(函数项)级数

对于每个确定值$x_0 \in I, \mu_1(x_0) + \mu_2(x_0) + \mu_3(x_0) + \cdots + \mu_n(x_0) + \ldots$常数项级数，收敛点$x_0$

$S(x) = \mu_1(x) + \mu_2(x) + \mu_3(x) + \cdots + \mu_n(x) + \ldots,$和函数

部分和$S_n(x), \lim_{n\to\infty} S_n(x) = S(x)$,余项$r_n(x) = S(x) - S_n(x), \lim_{n\to\infty} r_n(x) = 0$

12.3.2 幂级数及其收敛性

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \text{ 幂级数}$$

$$\text{幂级数} \sum_{n=0}^{\infty} a_n x^n \begin{array}{l} x = x_0, \text{幂级数收敛}, if \quad |x| < |x_0|, \text{幂级数绝对收敛} \\ x = x_0, \text{幂级数发散}, if \quad |x| > |x_0|, \text{幂级数发散} \end{array}$$

$$\lim_{n\to\infty} a_n x_0^n = 0 \Rightarrow |a_n x_0^n| \leqslant M \Rightarrow |a_n x^n| = |a_n x_0^n| \cdot \left|\frac{x}{x_0}\right|^n$$

$$\text{幂级数} \textstyle\sum_{n=0}^{\infty} a_n x^n \text{不是一点收敛，也不是整个数轴收敛}, \exists R > 0, \begin{array}{l} |x| < R, \text{幂级数绝对收敛} \\ |x| > R, \text{幂级数发散} \\ |x| = \pm R, \text{幂级数可能收敛} \end{array}$$

$$\text{幂级数} \textstyle\sum_{n=0}^{\infty} a_n x^n, \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho, \text{收敛半径} R = \begin{cases} \frac{1}{\rho}, & \rho \neq 0 \\ +\infty, & \rho = 0 \\ 0, & \rho = +\infty \end{cases}$$

$$\frac{|a_{n+1} x^{n+1}|}{|a_n x^n|} = \frac{|a_{n+1}|}{|a_n|} |x| = \rho |x| \overset{?}{<} 1$$

12.3.3 幂级数的运算

$$\text{幂级数} \begin{array}{l} a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \\ b_0 + a_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots \end{array}$$

加减 $(a_0 \pm b_0) + (a_1 \pm b_1) x + (a_2 \pm b_2) x^2 + \cdots + (a_n \pm b_n) x^n + \cdots$ 区间取小

乘 $(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots) \cdot (b_0 + a_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots)$

$a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) x^n + \cdots$ 区间取小

幂级数的柯西乘积

除 $\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots}{b_0 + a_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots} = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$区间取小

$$\begin{array}{l} a_0 = c_0 b_0 \\ a_1 = c_0 b_1 + c_1 b_0 \\ a_2 = c_0 b_2 + c_1 b_1 + c_2 b_0 \\ \ldots \end{array} \quad \text{解得} c_0, c_1, c_2, \cdots, c_n, \cdots$$

幂级数$\sum_{n=0}^{\infty} a_n x^n$的和函数$S_n(x)$在收敛域I上连续

幂级数$\sum_{n=0}^{\infty} a_n x^n$的和函数$S_n(x)$在收敛域I上可积，所得收敛同半径

$$\int_0^x S(t)\,dt = \int_0^x \left[\sum_{n=0}^{\infty} a_n t^n\right] dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \, (x \in I)$$

幂级数$\sum_{n=0}^{\infty} a_n x^n$的和函数$S_n(x)$在收敛域I上可导，所得收敛同半径

$$S'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=0}^{\infty} n a_n x^{n-1} \, (x \in I)$$

幂级数$\sum_{n=0}^{\infty} a_n x^n$的和函数$S_n(x)$在收敛域I上具有任意阶导数

12.4 函数展开成幂级数

$f\left(x\right)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{\left(n\right)}\left(x_{0}\right)\left(x-x_{0}\right)^{n},x\in U\left(x_{0}\right)$ 泰勒级数（极限），泰勒展开式
函数$f\left(x\right)$在$U\left(x_{0}\right)$内具有各阶导数，$f\left(x\right)$能展开成泰勒级数$\Leftrightarrow$ 余项满足$\sum_{n=0}^{\infty}R_{n}\left(x\right)=0$
$f\left(x\right)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{\left(n\right)}\left(0\right)\left(x-0\right)^{n},x\in U\left(0\right)$ 麦克劳林级数（极限），麦克劳林展开式
$\sin^{\left(n\right)}x=\sin\left(x+n\pi\right)$

$$\begin{cases}e^{x}=\sum_{n=0}^{\infty}\frac{1}{n!}x^{n},\left(-\infty<x<\infty\right)\\ \sin x=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{\left(2n+1\right)!}x^{2n+1},\left(-\infty<x<\infty\right)\\ \frac{1}{x+1}=\sum_{n=0}^{\infty}\left(-1\right)^{n}x^{n},\left(-1<x<1\right)\end{cases}$$

已知展开，运算得未知
$\left(1+x\right)^{m}=1+mx+\frac{m\left(m-1\right)}{2!}x^{2}+\cdots+\frac{m\left(m-1\right)\cdots\left(m-n+1\right)}{n!}x^{n}+\ldots,\left(-1<x<1\right)$二项展开式
$\left(1+x\right)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\frac{1}{2}\cdot-\frac{1}{2}}{2!}x^{2}+\cdots+\frac{\frac{1}{2}\cdot-\frac{1}{2}\cdots\frac{3-2n}{2}}{n!}x^{n}+\cdots=\frac{\prod\left(3-2n\right)}{\left(2n\right)!!}x^{n},\left(-1<x<1\right)$二项展开式
$\left(1+x\right)^{-\frac{1}{2}}=1-\frac{1}{2}x+\frac{-\frac{1}{2}\cdot-\frac{3}{2}}{2!}x^{2}+\cdots+\frac{-\frac{1}{2}\cdot-\frac{1}{2}\cdots\frac{1-2n}{2}}{n!}x^{n}+\cdots=\frac{\prod\left(1-2n\right)}{\left(2n\right)!!}x^{n},\left(-1<x<1\right)$二项展开式

证明：

$$\begin{aligned}&F\left(x\right)=\left(1+x\right)^{m}=1+mx+\frac{m\left(m-1\right)}{2!}x^{2}+\cdots+\frac{m\left(m-1\right)\cdots\left(m-n+1\right)}{n!}x^{n}+\ldots\\&F^{'}\left(x\right)=m\left[1+\frac{m-1}{1}x+\cdots+\frac{\left(m-1\right)\cdots\left(m-n+1\right)}{\left(n-1\right)!}x^{n-1}\ldots\right]\\&xF^{'}\left(x\right)=m\left[x+\frac{m-1}{1}x^{2}+\cdots+\frac{\left(m-1\right)\cdots\left(m-n+1\right)}{\left(n-1\right)!}x^{n}\ldots\right]\\&\left(1+x\right)F^{'}\left(x\right)=m\left[1+mx+\frac{*\left(m-n+1\right)}{\left(n-1\right)!}x^{n-1}+\frac{*}{\left(n-2\right)!}x^{n-1}+\frac{*1\left(m-n\right)}{\left(n\right)!}x^{n}+\frac{*1}{\left(n-1\right)!}x^{n}+\ldots\right]\\&\left(1+x\right)F^{'}\left(x\right)=m\left[1+mx+\frac{m*}{\left(n-1\right)!}x^{n-1}+\frac{m*1}{n!}x^{n}+\ldots\right]=mF\left(x\right)\\&\varphi\left(x\right)=\frac{F\left(x\right)}{\left(1+x\right)^{m}},\varphi\left(0\right)=F\left(0\right)=1\\&\varphi^{'}\left(x\right)=\frac{F^{'}\left(x\right)\left(1+x\right)^{m}-m\left(1+x\right)^{m-1}F\left(x\right)}{\left(1+x\right)^{2m}}=\frac{\left(1+x\right)^{m-1}\left[\left(1+x\right)F^{'}\left(x\right)-mF\left(x\right)\right]}{\left(1+x\right)^{2m}}=0\\&\varphi\left(x\right)=1,F\left(x\right)=\left(1+x\right)^{m}\end{aligned}$$

12.5 函数的幂级数展开应用

12.5.1 近似计算

12.5.2 微分方程的幂级数解法

$\frac{dy}{dx}=f\left(x,y\right),y=\sum_{n=0}^{\infty}a_{n}x^{n}$
$y^{''}+P\left(x\right)y^{'}+Q\left(x\right)y=0,P,Q$定义域内可展开为幂级数，方程解形如$y=\sum_{n=0}^{\infty}a_{n}x^{n}$

12.6 欧拉公式

$u_{1}+u_{2}+\cdots+u_{n}+\ldots$ (1)實部組成的級數收斂于u
$v_{1}+v_{2}+\cdots+v_{n}+\ldots$ (2)虛部組成的級數收斂于v
$\left(u_{1}+v_{1}i\right)+\left(u_{2}+v_{2}i\right)+\cdots+\left(u_{n}+v_{n}i\right)+\ldots$ (3)復數項級數收斂於$u+vi$
$\sqrt{u_{1}^{2}+v_{1}^{2}}+\sqrt{u_{2}^{2}+v_{2}^{2}}+\cdots+\sqrt{u_{n}^{2}+v_{n}^{2}}+\cdots$ (4) (3)各項的模組成的級數收斂,$\left(3\right),\left(2\right),\left(1\right)$絕對收斂
$e^{z}=1+z+\frac{1}{2!}z^{2}+\cdots+\frac{1}{n!}z^{n}+\ldots,z=x+yi$復變量指數函數

$$\begin{aligned}&e^{yi}=1+\left(yi\right)+\frac{1}{2!}\left(yi\right)^{2}+\cdots+\frac{1}{n!}\left(yi\right)^{n}+\ldots\\&=\left(1+\frac{1}{2!}\left(yi\right)^{2}+\cdots+\frac{1}{\left(2n\right)!}\left(yi\right)^{2n}\right)+\left(\left(yi\right)+\frac{1}{3!}\left(yi\right)^{3}+\cdots+\frac{1}{\left(2n-1\right)!}\left(yi\right)^{2n-1}\right)+\ldots\\&=\left(1-\frac{1}{2!}y^{2}+\cdots+\left(-1\right)^{n}\frac{1}{\left(2n\right)!}y^{2n}\right)+i\left(y+\frac{1}{3!}\left(yi\right)^{2}+\cdots+\frac{1}{\left(2n-1\right)!}y^{2n-1}i^{2n-2}\right)+\ldots\\&=\left(1-\frac{1}{2!}y^{2}+\cdots+\left(-1\right)^{n}\frac{1}{\left(2n\right)!}y^{2n}\right)+i\left(y+\frac{1}{3!}\left(yi\right)^{2}+\cdots+\left(-1\right)^{n-1}\frac{1}{\left(2n-1\right)!}y^{2n-1}\right)+\ldots\\&=\left(1-\frac{1}{2!}y^{2}+\cdots+\left(-1\right)^{n}\frac{1}{\left(2n\right)!}y^{2n}\right)+i\left(y+\frac{1}{3!}\left(yi\right)^{2}+\cdots+\left(-1\right)^{n-1}\frac{1}{\left(2n-1\right)!}y^{2n-1}+\left(-1\right)^{n}\frac{1}{\left(2n+1\right)!}y^{2n+1}\right)+\ldots\\&=\cos y+i\sin y\end{aligned}$$