

This is Start

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目录

7	微分方程	5
7.1	基本概念	5
7.2	可分离变量的微分方程	5
7.3	齐次方程	5
7.4	一阶线性微分方程	5
7.5	可降阶的高阶微分方程	5
7.6	高阶性微分方程	5
7.7	常系数齐次线性微分方程	6
7.8	常系数非齐次线性微分方程	6
7.9	欧拉方程	7
7.10	常系数线性微分方程组	7
8	向量代数与空间解析几何	1
8.1	向量及其线性运算	1
8.1.1	向量的概念	1
8.1.2	向量的运算	1
8.2	数量积 向量积 混合积	2
8.2.1	两向量的数量积	2
8.2.2	两向量的向量积	2
8.2.3	向量的混合积	2
8.3	平面及其方程	2
8.3.1	曲面方程与空间曲线方程的概念	2
8.3.2	平面的点法式方程	2
8.3.3	平面的一般方程	3
8.3.4	平面的夹角	3
8.4	空间直线及其方程	3
8.4.1	空间直线的一般方程	3
8.4.2	空间直线的对称式方程与参数方程	3
8.4.3	两直线夹角	3
8.4.4	直线与平面夹角	3
8.4.5	例	3
8.5	曲面及其方程	4
8.5.1	曲面研究的基本问题	4
8.5.2	旋转曲面	4
8.5.3	柱面	4
8.5.4	二次曲面九	4
8.6	空间曲线及其方程	4
8.6.1	空间曲线一般方程	4
8.6.2	空间曲线参数方程	4
8.6.3	空间曲线在坐标面投影	5

9 多元函数微分及其应用	6
9.1 多元函数的基本概念	6
9.1.1 平面點集 n 維空間	6
9.1.2 多元函數的概念	6
9.1.3 多元函數的極限	6
9.1.4 多元函數的連續性	7
9.2 偏導數	7
9.2.1 偏導數的定義及其計算法	7
9.2.2 高階偏導數	7
9.3 全微分	7
9.3.1 全微分定義	7
9.3.2 全微分在近似計算中的應用	8
9.4 多元復合函數的求導法則	8
9.5 隱函數求導公式	9
9.5.1 一個方程	9
9.5.2 方程組的情形	9
9.6 多元函数微分学的几何应用	9
9.6.1 一元向量值函数及其导数	9
9.6.2 空间曲线的切线与法平面	10
9.6.3 曲面的切平面与法线	11
9.7 方向导数与梯度	11
9.7.1 方向导数	11
9.7.2 梯度	11
9.8 多元函数的极值及其求法	12
9.8.1 多元函数的极值及最大值最小值	12
9.8.2 条件极值 拉格朗日数乘法	12
9.8.3 二元函数的泰勒公式	12
9.8.4 极值充分证明	13
9.8.5 最小二乘法	13
10 重积分	14
10.1 二重积分的概念与性质	14
10.1.1 二重积分的概念	14
10.1.2 二重积分的性质	14
10.2 二重积分的计算法	14
10.2.1 直角坐标计算二重积分	14
10.2.2 极坐标计算二重积分	14
10.2.3 二重积分换元法	15
10.3 三重积分	15
10.3.1 三重积分的概念	15
10.3.2 三重积分的计算	15
10.4 重积分的应用	15
10.5 含参变量的积分	16

11 曲线积分与曲面积分	17
11.1 对弧长的曲线积分	17
11.1.1 对弧长的曲线积分的概念与性质	17
11.1.2 对弧长的曲线积分的计算法	17
11.2 对坐标的曲线积分	17
11.2.1 对坐标的曲线积分的概念与性质	17
11.2.2 对坐标的曲线积分的计算法	18
11.2.3 两类曲线积分的关系	18
11.3 格林公式及其应用	18
11.3.1 格林公式	18
11.3.2 平面上曲线积分与路径无关的条件	19
11.3.3 二元函数的全微分求积	19
11.3.4 曲线积分的基本定理	19
11.4 对面积的曲面积分	19
11.4.1 对面积的曲面积分的概念与性质	19
11.4.2 对面积的曲面积分的计算法	20
11.5 对坐标的曲面积分	20
11.5.1 对坐标的曲面积分的概念与性质	20
11.5.2 对坐标的曲面积分的计算法	20
11.5.3 两类曲面积分之间的关系	20
11.6 高斯公式 通量与散度	20
11.6.1 高斯公式	20
11.6.2 沿任意闭曲面的曲面积分为零的条件	21
11.6.3 通量与散度	21
11.7 斯托克斯公式 * 环流量与旋度	22
11.7.1 斯托克斯公式	22
11.7.2 空间曲线积分与路径无关的条件	22
11.7.3 环流量与旋度	22
12 无穷级数	24
12.1 常数项级数的概念和性质	24
12.1.1 数项级数的概念	24
12.1.2 收敛级数的基本性质	24
12.1.3 柯西审敛原理	24
12.2 常数项级数审敛法	24
12.2.1 正项级数及其审敛法	24
12.2.2 交错级数审敛法	25
12.2.3 绝对收敛与条件收敛	25
12.2.4 绝对收敛级数的性质	25
12.3 幂级数	26
12.3.1 函数项级数的概念	26
12.3.2 幂级数及其收敛性	26
12.3.3 幂级数的运算	26
12.4 函数展开成幂级数	27
12.5 函数的幂级数展开应用	27

7 微分方程

7.1 基本概念

$$\begin{cases} \frac{dy}{dx} = 2x \\ x = 1, y = 2 \end{cases} \Rightarrow y = 2x + 1$$

函數，函數導數，自變量關係的方程**微分方程**

最高階導數的階數**微分方程的階**

$F(x, y', \dots, y^{(n)}) = 0$ **一般形式**

函數**微分方程的解**

函數含常數，常數個數同階數**微分方程的通解**

確定常數的通解**微分方程的通解**

7.2 可分離變量的微分方程

$$\frac{dy}{dx} = 2x \Rightarrow dy = 2x dx \Rightarrow$$

$$\frac{dy}{dx} = 2xy^2 \Rightarrow \frac{dy}{y^2} = 2x dx \Rightarrow$$

$$g(y) dy = f(x) dx \Rightarrow y = \varphi(x)$$

$$G(y) = F(x) + C \Rightarrow y = \Phi(x)$$

7.3 齊次方程

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$$

7.4 一階☐性微分方程

$\frac{dy}{dx} + P(x)y = Q(x)$ 關於函數，函數導數是一次方程**一階☐性微分方程**

$Q(x) = 0$ 時**齊次方程**

$y = Ce^{-\int P(x)dx}$ **齊次通解**，設 $C = \mu(x)$

$$\mu(x) = \int Q(x) e^{\int P(x)dx} dx + C_2$$

$y = \left(\int Q(x) e^{\int P(x)dx} dx + C_2\right) e^{-\int P(x)dx}$ **非齊次通解**

$\frac{dy}{dx} + P(x)y = Q(x)y^n$ **伯努利方程**

$\frac{dy}{dx}y^{-n} + P(x)yy^{-n} = Q(x)$ $z = y^{1-n}, z' = (1-n)y^{-n}\frac{dy}{dx} = \frac{z'}{1-n} + P(x)z = Q(x)$ **一階☐性微分方程**

7.5 可降階的高階微分方程

$$y^{(n)} = f(x) \Rightarrow y^{(n-1)} = \int f(x) dx$$

$$y'' = f(x, y') \quad \underline{y' = p, y'' = p'} \quad p' = f(x, p)$$

$$y'' = f(y, y') \quad \underline{y' = p, y'' = p \frac{dp}{dy}} \quad p \frac{dp}{dy} = f(y, p) \quad \underline{p = y' = \varphi(y, c_1)}$$

7.6 高階性微分方程

二階微分方程

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x)$$

解的結構

$y = C_1y_1(x) + C_2y_2(x)$ 是解

$y = C_1 y_1(x) + C_2 y_2(x)$ 无关特解是通解

$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$ 无关特解是通解

$y = Y(x) + y^*(x)$ 齐次通, 非齐特, 非齐通

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = f_1(x) + f_2(x)$$

$y = y_1^*(x) + y_2^*(x)$ 特解, 特解, 特解

常数变易法

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$Y(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$$

$$\begin{cases} y_1 v_1' + y_2 v_2' = 0 \\ y_1' v_1 + y_2' v_2 = f \end{cases} \Rightarrow \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & f \end{vmatrix} \xrightarrow{W = y_1 y_2' - y_1' y_2} \begin{cases} v_1' = -\frac{y_2 f}{W} \\ v_2' = \frac{y_1 f}{W} \end{cases} \Rightarrow \begin{cases} v_1 = -\int \frac{y_2 f}{W} dx + c_1 \\ v_2 = \int \frac{y_1 f}{W} dx + c_2 \end{cases}$$

$$\begin{cases} v_1 y_1 = \left(-\int \frac{y_2 f}{W} dx + c_1\right) y_1 \\ v_2 y_2 = \left(\int \frac{y_1 f}{W} dx + c_2\right) y_2 \end{cases} \Rightarrow Y(x) = c_1 y_1 + c_2 y_2 - y_1 \int \frac{y_2 f}{W} dx + y_2 \int \frac{y_1 f}{W} dx$$

7.7 常系数齐次线性微分方程

$$y'' + P(x) y' + Q(x) y = 0 \Rightarrow y'' + p y' + q y = 0$$

$$y = e^{rx}$$

$$(e^{rx})' = r e^{rx}$$

$$(r^2 + pr + q) e^{rx} = 0$$

$$\begin{cases} p^2 - 4q > 0, r_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \\ p^2 - 4q = 0, r_1 = r_2 = -\frac{p}{2}, y_2 = e^{r_1 x} \mu(x), \mu'' = 0 \\ p^2 - 4q < 0, r_{1,2} = \alpha \pm \beta i, e^{(i\theta)} = \cos(\theta) + i \sin(\theta), \overline{y_1} = \frac{1}{2}(y_1 + y_2), \overline{y_1} = \frac{1}{2i}(y_1 - y_2) \end{cases}$$

$$\begin{cases} p^2 - 4q > 0, y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x} \\ p^2 - 4q = 0, y = C_1 y_1 + C_2 \mu y_1 = (C_1 x + C_2) e^{r_1 x} \\ p^2 - 4q < 0, y = C_1 \overline{y_1} + C_2 \overline{y_2} = e^{\alpha x} (C_1 \cos(\beta x) + C_2 \sin(\beta x)) \end{cases}$$

n 阶常系数齐次微分方程

$$y^n + p_1 y^{n-1} + p_2 y^{n-2} + \cdots + p_{n-1} y' + p_n y = 0$$

$$\begin{cases} D, \frac{d}{dx} \\ Dy, \frac{dy}{dx} \\ D^n y, \frac{d^n y}{dx^n} \end{cases}$$

$$(D^n + p_1 D^{n-1} + \cdots + p_{n-1} D + p_n) y = 0 \Rightarrow L(D) y = 0, \text{微分算子 } D \text{ 的 } n \text{ 次多项式}$$

$$D e^{rx} = r e^{rx}, \dots, D^n e^{rx} = r^n e^{rx} \Rightarrow (r^n + p_1 r^{n-1} + \cdots + p_{n-1} r + p_n) e^{rx} = L(r) e^{rx} = 0 \Rightarrow L(r) = 0$$

$$\begin{cases} \text{单实根} & C e^{rx} \\ k \text{ 重实根} & (C_1 + C_2 x + \cdots + C_k x^{k-1}) e^{rx} \\ \text{单复根} & e^{\alpha x} (C_1 \cos(\beta x) + D_1 \sin(\beta x)) \\ k \text{ 重复根} & e^{\alpha x} ((C_1 + C_2 x + \cdots + C_k x^{k-1}) \cos(\beta x) + (D_1 + D_2 x + \cdots + D_k x^{k-1}) \sin(\beta x)) \end{cases}$$

7.8 常系数非齐次线性微分方程

$$y'' + p y' + q y = f(x)$$

$$p_m(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$

$$\begin{cases} e^{\theta i} = \cos(\theta) + i \sin(\theta) \\ e^{-\theta i} = \cos(\theta) - i \sin(\theta) \end{cases} \Rightarrow \begin{cases} \cos(\theta) = \frac{1}{2}(e^{\theta i} + e^{-\theta i}) \\ \sin(\theta) = \frac{1}{2i}(e^{\theta i} - e^{-\theta i}) \end{cases} \Rightarrow \begin{cases} \cos(\omega x) P = \frac{P}{2}(e^{\omega x i} + e^{-\omega x i}) \\ \sin(\omega x) Q = \frac{Q}{2i}(e^{\omega x i} - e^{-\omega x i}) \end{cases}$$

函数 (多项式) 共轭, 倒数共轭; 两对共轭, 乘积共轭; $e^{\alpha+\theta i}$ 与 $e^{\alpha-\theta i}$ 共轭

$f(x) = e^{\lambda x} P_m(x)$	$y^* = R_l(x) e^{\lambda x}$ $R''x + (2\lambda + p)R'x + (\lambda^2 + p\lambda + q)R(x) = p_m(x)$ $y^* = x^k P_m(x) e^{\lambda x}$
$f(x) = e^{\lambda x} [P_l(x) \cos(\omega x) + Q_n(x) \sin(\omega x)]$	$f(x) = e^{\lambda x} \left[\left(\frac{P}{2} + \frac{Q}{2i} \right) e^{\omega x i} + \left(\frac{P}{2} - \frac{Q}{2i} \right) e^{-\omega x i} \right]$ $f(x) = \left(\frac{P}{2} + \frac{Q}{2i} \right) e^{\lambda x + \omega x i} + \left(\frac{P}{2} - \frac{Q}{2i} \right) e^{\lambda x - \omega x i}$ $f(x) = \left(\frac{P}{2} + \frac{Q}{2i} \right) e^{(\lambda + \omega i)x} + \left(\frac{P}{2} - \frac{Q}{2i} \right) e^{(\lambda - \omega i)x}$ $f(x) = P_1 e^{(\lambda + \omega i)x} + Q_2 e^{(\lambda - \omega i)x} \quad (P_1, Q_2 \text{共轭})$ $y_1^* = x^k R_m e^{(\lambda + \omega i)x} \quad (m = \max\{P_l, Q_n\})$ $y_2^* = x^k \overline{R_m} e^{(\lambda - \omega i)x}$ $y^* = y_1^* + y_2^* = x^k e^{\lambda x} (R_m e^{\omega x i} + \overline{R_m} e^{-\omega x i})$ $y^* = x^k e^{\lambda x} [R_m (\cos(\omega x) + i \sin(\omega x)) + \overline{R_m} (\cos(\omega x) - i \sin(\omega x))]$ $y^* = x^k e^{\lambda x} \left(R_m^{(1)} \cos(\omega x) + R_m^{(2)} \sin(\omega x) \right)$

7.9 欧拉方程

$$x^n y^{(n)} + p^1 x^{n-1} y^{(n-1)} + \cdots + p^{n-1} x y' + p^n y = f(x)$$

$$x = e^t, t = \ln x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} \quad D \text{表示} \frac{d}{dt} \quad \begin{cases} xy' = Dy \\ x^2 y'' = (D^2 - D)y \\ x^3 y''' = (D^3 - 3D^2 + 2D)y \\ \cdots \\ x^k y^{(k)} = (D^n + c_1 D^{n-1} + \cdots + C_{n-1} D)y \end{cases}$$

7.10 常系数线性微分方程组

$$\begin{cases} \frac{d^2 x}{dt^2} + \frac{dy}{dt} - x = e^t, \\ \frac{d^2 y}{dt^2} + \frac{dx}{dt} + y = 0, \end{cases}$$

$$\text{记} \frac{d}{dt} \text{为} D \Rightarrow \begin{cases} (D^2 - 1)x + Dy = e^t \\ Dx + (D^2 + 1)y = 0 \end{cases}$$

8 向量代数与空间解析几何

8.1 向量及其线性运算

8.1.1 向量的概念

大小, 方向, 向量, \overrightarrow{AB}

与起点无关, 自由向量

向量的大小, 向量的模, $|\overrightarrow{AB}|$, 单位向量, 零向量

$\overrightarrow{OA} = a, \overrightarrow{OB} = b, \angle AOB < \pi$ 向量夹角, $(\widehat{a, b}) = 0 \text{ or } \pi, a \text{ 与 } b \text{ 平行, 同起点共线}$
 $= \frac{\pi}{2}, a \text{ 与 } b \text{ 垂直}$

8.1.2 向量的运算

向量的加减法

$c = a + b$ 三角形法则

$a + b = b + a$ 交换

$(a + b) + c = a + (b + c)$ 分配

$a - b = a + (-b)$ 负向量

$|a \pm b| < |a| + |b|$

向量与数的乘法

$|\lambda a| = |\lambda| |a|$

$\lambda(\mu a) = \mu(\lambda a) = (\lambda\mu)a$

$|\lambda(\mu a)| = |\mu(\lambda a)| = |(\lambda\mu)a| = |\lambda\mu| |a|$

$(\lambda + \mu)a = \lambda a + \mu a, \lambda(a + b) = \lambda a + \lambda b$

$a \neq 0, a // b \Leftrightarrow \exists \text{ 唯一实数 } \lambda, b = \lambda a$

空间直角坐标系

$[O; i, j, k]$ 右手规则, 卦限

$\overrightarrow{OM} = r = xi + yj + zk$ 坐标式分解, 分向量, $r = (x, y, z), M(x, y, z)$ 坐标, r 为 M 关于 O 向径

利用坐标向量的线性运算

$$a = (a_x, a_y, a_z), b = (b_x, b_y, b_z) \begin{cases} a + b = (a_x + b_x, a_y + b_y, a_z + b_z) \\ a - b = (a_x - b_x, a_y - b_y, a_z - b_z) \\ \lambda a = (\lambda a_x, \lambda a_y, \lambda a_z) \end{cases}$$

向量的模, 方向角, 投影

向量的模与两点间的距离

$r = (x, y, z), |r| = \sqrt{x^2 + y^2 + z^2}$

$A = (x_1, y_1, z_1), B = (x_2, y_2, z_2), |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

方向角与方向余弦

r 与三条坐标轴的夹角 α, β, γ 称为 r 的方向角

$(\cos \alpha, \cos \beta, \cos \gamma) = \frac{1}{|r|} (x, y, z), \cos *$ 方向余弦

向量在轴上的投影

$$\begin{cases} (a)_u = |a| \cos \varphi \\ (a + b)_u = (a)_u + (b)_u = |a| \cos \varphi + |b| \cos \varphi \\ (\lambda a)_u = \lambda (a)_u \end{cases}$$

8.2 数量积 向量积 混合积

8.2.1 两向量的数量积

$$\begin{aligned}
 a \cdot b &= |a| |b| \cos \theta \text{数量积} \\
 \begin{cases} a \cdot a = |a|^2 \\ a, b \neq 0, a \cdot b = 0 \Leftrightarrow a \perp b \\ a \cdot b = b \cdot a \text{交}\mathbb{F} \\ (a+b) \cdot c = a \cdot c + b \cdot c \text{分配} \\ (\lambda a) \cdot b = \lambda (a \cdot b) \text{结合} \end{cases} \\
 a &= (a_x, a_y, a_z) \Rightarrow a \cdot b = a_x b_x + a_y b_y + a_z b_z \\
 b &= (b_x, b_y, b_z) \Rightarrow \cos \theta = \frac{a \cdot b}{|a| |b|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \sqrt{b_x^2 + b_y^2 + b_z^2}}
 \end{aligned}$$

8.2.2 两向量的向量积

$$\begin{aligned}
 |c| &= |a| |b| \sin \theta \\
 c &= a \times b \text{向量积} \\
 \begin{cases} a \times a = 0 \\ a, b \neq 0, a \times b = 0 \Leftrightarrow a // b \\ a \times b = -b \times a \\ (a+b) \times c = a \times c + b \times c \\ (\lambda a) \times b = a \times (\lambda b) = \lambda (a \times b) \end{cases} \\
 a \times b &= (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x) \\
 a &= (a_x, a_y, a_z) \Rightarrow \\
 b &= (b_x, b_y, b_z) \Rightarrow \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}
 \end{aligned}$$

8.2.3 向量的混合积

$$\begin{aligned}
 (a \times b) \cdot c &= [abc], \text{向量的混合积} \\
 |[abc]| &= |a| |b| \sin(\widehat{a, b}) |c| \cos(\widehat{a \times b, c}), a, b, c \text{ 右手系模大于0} \\
 [abc] &= (a \times b) \cdot c \\
 a &= (a_x, a_y, a_z) \Rightarrow \\
 b &= (b_x, b_y, b_z) \Rightarrow \\
 c &= (c_x, c_y, c_z) \Rightarrow \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}
 \end{aligned}$$

8.3 平面及其方程

8.3.1 曲面方程与空间曲线方程的概念

$$\begin{aligned}
 F(x, y, z) = 0, \text{曲面 } S \text{ 上点满足方程, 不在曲面 } S \text{ 上点不满足方程} \Rightarrow \text{曲面 } S \text{ 的方程} \\
 \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \Rightarrow \text{曲线 } C \text{ 的方程}
 \end{aligned}$$

8.3.2 平面的点法式方程

$$\begin{aligned}
 \text{向量垂直平面, 平面的法线向量} \\
 n = (A, B, C) \quad \text{法向量} \\
 \overrightarrow{M_0 M} = (x - x_0, y - y_0, z - z_0) \quad \text{平面上点} \Rightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \text{平面的点法式方程}
 \end{aligned}$$

8.3.3 平面的一般方程

$$Ax + By + Cz + D = 0 \text{ 平面的一般方程}$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \text{ 平面的截距式方程}$$

8.3.4 平面的夹角

平面法向量的夹角, 平面的夹角 ($0 \leq \theta \leq \frac{\pi}{2}$)

$$\Pi_1, n_1 = (A_1, B_1, C_1), \Pi_2, n_2 = (A_2, B_2, C_2), \cos \theta = |\widehat{(n_1, n_2)}| = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \text{ 平面夹角}$$

$$\begin{aligned} d &= \left| \overrightarrow{P_1P_0} \right| |\cos \theta| \\ &= \frac{|\overrightarrow{P_1P_0} \times n|}{|n|} \\ &= \frac{|A(x_0 - x) + B(y_0 - y) + C(z_0 - z)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \end{aligned}$$

8.4 空间直线及其方程

8.4.1 空间直线的一般方程

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

8.4.2 空间直线的对称式方程与参数方程

向量平行直线, 直线的方向向量

$$\begin{aligned} \overrightarrow{M_oM} &= (x - x_0, y - y_0, z - z_0) \\ S &= (m, n, p) \end{aligned} \Rightarrow \begin{cases} \frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p} = t, \text{ 对称式, 点向式方程} \\ \begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases} \text{ 参数方程} \end{cases}$$

8.4.3 两直线夹角

$$L_1, s_1 = (A_1, B_1, C_1), L_2, s_2 = (A_2, B_2, C_2), \cos \theta = |\widehat{(s_1, s_2)}| = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \text{ 直线夹角}$$

8.4.4 直线与平面夹角

直线和投影直线的夹角, 直线与平面夹角

$$\begin{aligned} \varphi &= \left| \frac{\pi}{2} - (\vec{s}, \vec{n}) \right| \\ s &= (m, n, p) \\ n &= (A, B, C) \end{aligned} \Rightarrow \begin{aligned} \sin \varphi &= \frac{|Am + Bn + Cp|}{\sqrt{A^2 + B^2 + C^2}\sqrt{m^2 + n^2 + p^2}} \\ \frac{A}{m} &= \frac{B}{n} = \frac{C}{p} \text{ 垂直} \end{aligned}$$

8.4.5 例

$$\left. \begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0 & I \\ A_2x + B_2y + C_2z + D_2 &= 0 & II \end{aligned} \right\} L \Rightarrow A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0 \text{ 过 } L \text{ 平面束, 不含 } II$$

8.5 曲面及其方程

8.5.1 曲面研究的基本问题

轨迹到方程, 方程到形状

$$x^2 + y^2 + z^2 = R^2 \text{ 球面}$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2 \text{ 球面}$$

$$Ax^2 + Ay^2 + Az^2 + Dx + Ey + Fz + G = 0 \text{ 一般球面}$$

8.5.2 旋转曲面

平面上曲线绕直线旋转一周, 母线, 轴, 旋转曲面

$$f(y, z) = 0 \Rightarrow f(\pm\sqrt{x^2 + y^2}, z) = 0 \text{ 圆台}$$

相交直线 L 绕 R 旋转一周, 交点顶点, 夹角 $(0 < \alpha < \frac{\pi}{2})$ 半顶角

$$z^2 = a^2(x^2 + y^2) \text{ 圆锥}$$

$$\left. \begin{aligned} \frac{x^2}{a^2} - \frac{z^2}{b^2} = 1 \text{ 双曲线} \Rightarrow \frac{(x^2 + y^2)^2}{a^2} - \frac{z^2}{b^2} = 1 \text{ 单叶双曲面} \\ \frac{x^2}{a^2} - \frac{(z^2 + y^2)^2}{b^2} = 1 \text{ 双叶双曲面} \end{aligned} \right\} \text{ 两种二次}$$

8.5.3 柱面

$x^2 + y^2 = R^2$ 圆柱面, 准线 (定曲线), 母线

$$y^2 = ax \text{ 抛物柱面}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ 椭圆柱面}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ 双曲柱面}$$

三种二次

8.5.4 二次曲面九

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2 \text{ 椭圆锥面}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z \text{ 椭圆抛物面}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z \text{ 双曲抛物面}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ 椭球面}$$

8.6 空间曲线及其方程

8.6.1 空间曲线一般方程

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

8.6.2 空间曲线参数方程

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \Rightarrow \begin{cases} x = a \cos \theta \\ y = a \sin \theta \\ z = b\theta \end{cases} \text{ 螺旋线, } 2\pi b \text{ 螺距}$$

空间曲线参数方程

$$\Gamma \begin{cases} x = \varphi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases} \Rightarrow \begin{cases} x = \sqrt{[\varphi(t)]^2 + [\psi(t)]^2} \cos \theta \\ y = \sqrt{[\varphi(t)]^2 + [\psi(t)]^2} \sin \theta \\ z = \omega(t) \end{cases} \text{ 曲面}$$

$$\Gamma \begin{cases} x = a \sin \varphi \\ y = 0 \\ z = a \cos \varphi \end{cases} \Rightarrow \begin{cases} x = \sqrt{[a \sin \varphi]^2 + [0]^2} \cos \theta = a \sin \varphi \cos \theta \\ y = \sqrt{[a \sin \varphi]^2 + [0]^2} \sin \theta = a \sin \varphi \sin \theta \\ z = a \cos \varphi, \end{cases} \quad \text{球面}$$

8.6.3 空间曲线在坐标面投影

$H(x, y) = 0$ 投影柱面

$$\begin{cases} H(x, y) = 0 \\ z = 0 \end{cases} \quad \text{投影曲线}$$

9 多元函数微分及其应用

9.1 多元函数的基本概念

9.1.1 平面點集 n 維空間

平面點集

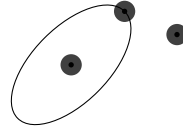
R^2 坐標平面

$C = \{(x, y) | x^2 + y^2 < r^2\}$ 平面點集

$U(P_0, \delta) = \{P | PP_0 < \delta\}$ 鄰域

$\dot{U}(P_0, \delta) = \{P | 0 < PP_0 < \delta\}$ 去心鄰域

$\exists U(P), U(P) \subset E, P \in E$ 內點
 $\exists U(P), U(P) \cap E = \emptyset, P \notin E$ 外點
 $P \in R^2, \quad P = P_1 \cap P_2$
 $E \subset R^2 \quad \left. \begin{array}{l} \exists U(P_1), U(P_1) \subset E \\ \exists U(P_2), U(P_2) \cap E = \emptyset \end{array} \right\}, P \in E$ 邊界點



E 邊界點的全體, ∂E 邊界點, ∂E

$\forall \delta > 0, \dot{U}(P, \delta) \cap E \neq \emptyset$ 總有 E 中的點, $P \in E$ 聚點 (內點和邊界)

開集, 內點; 閉集, $\partial E \subset E$; 連通集, 任意點連線仍在集合;

區域, 連通開集; 閉區域, 區域和邊界;

有界集, $\forall E \subset R^2, \text{if } \exists r > 0, E \subset U(O, r), \text{then}$ 有界集; 無界集, 不是有界集

n 維空間

$R^n = \{(x_1, x_2, \dots, x_n) | x_i \in R, i = 1, 2, \dots, n\} = x$ 集合

$x = (x_1, x_2, \dots, x_n) \in R^n$
 $y = (y_1, y_2, \dots, y_n) \in R^n$
 $a = (a_1, a_2, \dots, a_n) \in R^n$
 $\lambda \in R$

$\left. \begin{array}{l} x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ \lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_n) \end{array} \right\} R^n \text{ 中 } \text{四性運算}$

$\Rightarrow \rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$
 $\|x\| = \rho(x, 0) = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$
 $\|x - a\| \rightarrow 0$, 變元趨于固定元, 記作 $x \rightarrow a$

某正數 $\delta > 0$, 點集 $U(a, \delta) = \{x | x \in R^n, \rho(x, a) < \delta\}$, a 的 δ 鄰域

定義四性預算的集合, n 維空間

9.1.2 多元函數的概念

D 定義域

$D \subset R^2$, 映射 $f: D \rightarrow R$, 稱四二元函數, $\Rightarrow x, y$ 自變量

記四 $z = f(x, y), (x, y) \in D$ z 因變量

函數值 $f(x, y)$ 全體, $f(D) = \{z | z = f(x, y), (x, y) \in D\}$ 值域

$D \subset R^n$, 映射 $f: D \rightarrow R$, 稱四 n 元函數 ($n \geq 2$ 多元函數),

記四 $z = f(x_1, x_2, \dots, x_n) = f(\mathbf{x}), \mathbf{x}(x_1, x_2, \dots, x_n) \in D$

多元函數 $\mu = f(\mathbf{x})$, 有意義的變元 \mathbf{x} 的點集, 自然定義域

空間點集 $\{(x, y, z) | z = f(x, y), (x, y) \in D\}$, 二元函數 $z = f(x, y)$ 的圖形

9.1.3 多元函數的極限

$f(x, y)$ 定義域 D , $\text{if } \exists A, \forall \varepsilon > 0, \exists \delta > 0$, then A 稱四函數 $f(x, y)$ 在 $(x, y) \rightarrow (x_0, y_0)$ 極限,
 $P_0(x_0, y_0)$ 是 D 聚點, when 點 $P(x_0, y_0) \in D \cap \dot{U}(P_0, \delta)$, 記四 $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = A$
 $|f(P) - A| = |f(x, y) - A| < \varepsilon$

任意方向趨近

9.1.4 多元函數的連續性

$f(P) = f(x, y)$ 定義域 $D, P_0(x_0, y_0) \in D$ 聚點, $P_0 \in D$, $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$,
 then $f(x, y)$ 在 $P_0(x_0, y_0)$ 連續
 $f(x, y)$ 定義域 D, D 內每一點都是聚點, $f(x, y)$ 在 D 內每一點連續, $f(x, y)$ 在 D 內連續
 $f(x, y)$ 定義域 $D, P_0(x_0, y_0) \in D$ 聚點, if $f(x, y)$ 在 $P_0(x_0, y_0)$ 不連續, then $P_0(x_0, y_0) \in D$ 間斷點
 常數, 不同自變量的一元基本初等函數, 有限次四則運算和復合運算, **多元初等函數**
 一切多元初等函數在**定義區域** (定義域 D 的區域或閉區域) 連續 $\rightarrow \lim_{P \rightarrow P_0} f(P) = f(P_0)$

有界閉區域 D , 多元連續函數, D 上有界, 取得最大值, 最小值 **(最值)**

性質 有界閉區域 D , 多元連續函數, 能取得介於最大值和最小值間的任何值 **(介值)**

有界閉區域 D , 多元連續函數, D 上一致連續 **(一致連續)**

9.2 偏導數

9.2.1 偏導數的定義及其計算法

$f(x, y), P_0(x_0, y_0), (x, y) \in U(P_0, \delta)$, if $\exists \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = A$,
 when $y = y_0, x = x_0 + \Delta x$, then $A \in \mathbb{R}$ $f(x, y)$ 在 (x_0, y_0) 對 x 的偏導數,
 函數增量 $f(x_0 + \Delta x, y_0) - f(x_0, y_0)$ 記 $\left. \frac{\partial f}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0}}, f_x(x_0, y_0)$

$$\left\{ \begin{array}{l} \left. \frac{\partial f}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0}} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \\ \left. \frac{\partial f}{\partial y} \right|_{\substack{x=x_0 \\ y=y_0}} = \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \end{array} \right.$$

$f_x(x, y), f_y(x, y)$ 偏導函數, 偏導數

偏導數記號 \in 整體, 不能看成微分的商

(一元可導連) 各偏導存在, 不一定連續

9.2.2 高階偏導數

若偏導數的偏導仍存在, 則稱為 $f(x, y)$ 的二階偏導數

$$z = f(x, y) \text{ 的偏導數 } \frac{\partial z}{\partial x} = f_x(x, y), \frac{\partial z}{\partial y} = f_y(x, y), \Rightarrow \left\{ \begin{array}{l} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) \\ \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y) \text{ (混合偏導數)} \\ \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y) \text{ (混合偏導數)} \\ \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) \end{array} \right.$$

if $f(x, y)$ 的二階偏導數, $\frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x \partial y}$ 在 D 連續, then $(x, y) \in D, \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

$$\left. \begin{array}{l} z = \ln \sqrt{x^2 + y^2}, \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \\ u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \end{array} \right\} \text{拉普拉斯方程}$$

9.3 全微分

9.3.1 全微分定義

$$\left\{ \begin{array}{l} f(x + \Delta x, y) - f(x, y) \approx f_x(x, y) \Delta x \\ f(x, y + \Delta y) - f(x, y) \approx f_y(x, y) \Delta y \end{array} \right. \quad \text{偏增量, 偏微分}$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \quad \text{全增量}$$

$if \quad \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = A\Delta x + B\Delta y + o(\rho),$
 $then \quad \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}, A, B \text{ 不依赖 } \Delta x, \Delta y$
 $z = f(x, y), \text{ 在 } (x, y) \text{ 的某邻域有定义,}$
 $f(x, y) \text{ 在 } (x, y) \text{ 可微分,}$
 $A\Delta x + B\Delta y, \text{ 称为函数 } f(x, y) \text{ 全微分,}$
 $记作 } dz = A\Delta x + B\Delta y$

多元函数在区域 D 内个点处都可微分, 函数在 D 内可微分

多元函数在点 P 可微分, 函数在该点连续

$if \quad f(x, y) \text{ 在点 } (x, y) \text{ 可微分,}$
 $then \quad \text{函数 } f(x, y) \text{ 在点 } (x, y) \text{ 偏导数 } \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \text{ 必存在}$
 $\text{函数 } f(x, y) \text{ 在点 } (x, y) \text{ 全微分 } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$z = f(x, y) \text{ 的偏导数 } \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \text{ 在点 } (x, y) \text{ 连续, 函数在该点可微分}$

微分 \mathbb{F} 加原 $\mathbb{F}, u = (x, y, z), du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$

9.3.2 全微分在近似计算中的应用

$$g = \frac{4\pi^2 l}{T^2}, \Delta g \leq 4\pi^2 \left(\frac{1}{T^2} \delta_l + \frac{2l}{T^3} \delta_T \right)$$

9.4 多元复合函数的求导法则

$if \quad u = \varphi(t), v = \psi(t), \text{ 在 } t \text{ 可导,}$
 $z = f(u, v) \text{ 在 } (u, v) \text{ 具有连续偏导}$
 $, then \quad \text{复合函数 } z = f(\varphi(t), \psi(t)) \text{ 在 } t \text{ 点可导, 且 } \frac{dz}{dt} = \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} \text{ 全导数}$

$if \quad u = \varphi(t), v = \psi(t), w = \omega(t), \text{ 在 } t \text{ 可导, } z = f(\varphi(t), \psi(t), \omega(t)),$
 $\frac{dz}{dt} = \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} + \frac{\partial z}{\partial w} \frac{dw}{dt}$

$u = \varphi(x, y), v = \psi(x, y),$
 $if \quad \text{在 } (x, y) \text{ 具有对 } x, y \text{ 的偏导数,}$
 $z = f(u, v) \text{ 在 } (u, v) \text{ 具有连续偏导}$
 $, then \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$
 $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$

$u = \varphi(x, y), v = \psi(x, y), w = \omega(x, y),$
 $if \quad \text{在 } (x, y) \text{ 点两偏导都存在}$
 $z = f(\varphi(x, y), \psi(x, y), \omega(x, y)),$
 $, then \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial x}$
 $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial y}$

$u = \varphi(x, y) \text{ 在 } (x, y) \text{ 点两偏导都存在,}$
 $if \quad v = \psi(y) \text{ 在 } y \text{ 点可导,}$
 $z = f(u, v) \text{ 在 } (u, v) \text{ 具有连续偏导}$
 $, then \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x}$
 $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$

$if \quad z = f(u, x, y) = f[\varphi(x, y), x, y] \text{ 有 } x, y \text{ 的偏导数, then}$
 $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial x}$
 $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial y}$

$$\begin{aligned}
 f'_1 &= f'_u(u, v) \\
 f'_2 &= f'_v(u, v) \\
 f(u, v), \quad f'_{11} &= f''_{uu}(u, v) \\
 f'_{12} &= f''_{uv}(u, v) \\
 f'_{21} &= f''_{vu}(u, v) \\
 f'_{22} &= f''_{vv}(u, v)
 \end{aligned}$$

$$\begin{aligned}
 dz &= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv \\
 z = f(u, v) = f(\varphi(x, y), \psi(x, y)), &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\
 &= \left(\frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \right) dx + \left(\frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} \right) dy
 \end{aligned}$$

9.5 隱函數求導公式

9.5.1 一個方程

if $P_0(x_0, y_0), (x, y) \in U(P_0, \delta)$ 時 $F(x, y) = 0$, 在 $(x, y) \in U(P_0, \delta)$ 時,
 $F(x, y)$ 具有連續偏導, $F(x_0, y_0) = 0, F_y(x_0, y_0) \neq 0$, then 能確定唯一連續, 連續導數的函數 $y = f(x)$,
 $y_0 = f(x_0), \frac{dy}{dx} = -\frac{F_x}{F_y}$

if $P_0(x_0, y_0, z_0), (x, y, z) \in U(P_0, \delta)$ 時 $F(x, y, z) = 0$, 在 $(x, y, z) \in U(P_0, \delta)$ 時,
 $F(x, y, z)$ 具有連續偏導, $F(x_0, y_0, z_0) = 0, F_z(x_0, y_0, z_0) \neq 0$, then 能確定唯一連續, 連續導數的函數 $z = f(x, y)$,
 $z_0 = f(x_0, y_0), \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

9.5.2 方程組的情形

if $P_0(x_0, y_0, u_0, v_0), (x, y, u, v) \in U(P_0, \delta)$ 時 $F(x, y, u, v) = 0, G(x, y, u, v) = 0$,
 $F(x, y, u, v), G(x, y, u, v)$ 具有連續偏導, 在 $(x, y, u, v) \in U(P_0, \delta)$ 時,
 $F(x_0, y_0, u_0, v_0) = 0, G(x_0, y_0, u_0, v_0) = 0$, then 能確定唯一組連續, 連續偏導數的函數,
 $u = f(x, y), v = f(x, y)$
 $J = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} \neq 0 \left(\begin{array}{l} \text{雅可比式} \\ \text{偏導数组成的函数行列式} \end{array} \right) \begin{array}{l} u_0 = f(x_0, y_0), v_0 = f(x_0, y_0) \\ \frac{\partial u}{\partial x} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, v)} \\ \frac{\partial u}{\partial y} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(y, v)} \\ \frac{\partial v}{\partial x} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, u)} \\ \frac{\partial v}{\partial y} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(y, u)} \end{array}$

$$\begin{cases} F_x + F_u \frac{\partial u}{\partial x} + F_v \frac{\partial v}{\partial x} = 0 \\ G_x + G_u \frac{\partial u}{\partial x} + G_v \frac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow \begin{array}{l} \frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F, G)}{\partial(x, v)}}{\frac{\partial(F, G)}{\partial(u, v)}} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, v)} \\ \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} F_u & -F_x \\ G_u & -G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F, G)}{\partial(u, x)}}{\frac{\partial(F, G)}{\partial(u, v)}} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(u, x)} \end{array} \quad \text{证明或推导}$$

$$\begin{aligned} dudv, x = \varphi(u, v), y = \psi(u, v), \\ dx = \frac{\partial \varphi}{\partial u} du + \frac{\partial \varphi}{\partial v} dv \\ dy = \frac{\partial \psi}{\partial u} du + \frac{\partial \psi}{\partial v} dv \end{aligned} \rightarrow \begin{aligned} & |(dx, dy)|_{du=0} \times |(dx, dy)|_{dv=0}| \\ & = \left| \left(\frac{\partial \varphi}{\partial u} du, \frac{\partial \psi}{\partial u} du \right) \times \left(\frac{\partial \varphi}{\partial v} dv, \frac{\partial \psi}{\partial v} dv \right) \right| \\ & = \left| \frac{\partial \varphi}{\partial u} \cdot \frac{\partial \psi}{\partial v} - \frac{\partial \psi}{\partial u} \cdot \frac{\partial \varphi}{\partial v} \right| dudv \\ & = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} dudv \\ & = \frac{\partial(\varphi, \psi)}{\partial(u, v)} dudv \end{aligned} \quad \text{二重积分换元和雅可比}$$

9.6 多元函数微分学的几何应用

9.6.1 一元向量值函数及其导数

$$\Gamma \begin{cases} x = \varphi(t), \\ y = \psi(t), \quad t \in [\alpha, \beta] \\ z = \omega(t) \end{cases}$$

記 $r = xi + yj + zk, f(t) = \varphi(t)i + \psi(t)j + \omega(t)k$,
 $r = f(t), t \in [\alpha, \beta]$

数集 $D \subset R$, 映射 $f: D \rightarrow R^n$ 一元向量值函数, 记 $\mathbb{F}r = f(t), t \in D$

$t \in U(t_0)$, 向量值函数 $f(t)$, if $\exists r_0, \forall \varepsilon > 0, \exists \delta$, when $0 < |t - t_0| < \delta, |f(t) - r_0| < \varepsilon$, then $\lim_{t \rightarrow t_0} f(t) = r_0$

$t \in u(t_0)$, 向量值函数 $f(t)$, if $\lim_{t \rightarrow t_0} f(t) = f(t_0)$, $f(t)$ 在 t_0 连续

$t \in D$, 向量值函数 $f(t)$, if $D_1 \subset D$, D_1 内每点连续, D_1 上连续

极限向量为 $r = f(t)$

$t \in U(t_0)$, 向量值函数 $f(t)$, if $\exists \lim_{\Delta t \rightarrow 0} \frac{\Delta r}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$, then

在 t_0 导数或导向量,

记为 $f'(t_0)$, $\frac{dr}{dt}|_{t=t_0}$

$$f'(t_0) = f'_1(t_0)i + f'_2(t_0)j + f'_3(t_0)k$$

$$\frac{d}{dt}C = 0$$

$$\frac{d}{dt}[cu(t)] = cu'(t)$$

$$\frac{d}{dt}[u(t) \pm v(t)] = u'(t) \pm v'(t)$$

$$\text{向量值函数 } u(t), v(t), \text{ 数量值函数 } \varphi(t) \Rightarrow \frac{d}{dt}[\varphi(t)v(t)] = \varphi'(t)v(t) + \varphi(t)v'(t)$$

$$\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$$

$$\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$$

$$\frac{d}{dt}u[\varphi(t)] = \varphi'(t)u'[\varphi(t)]$$

$$f'(t_0) = \lim_{t \rightarrow t_0} \frac{\Delta r}{\Delta t} \text{ 与曲线相切}$$

9.6.2 空间曲线的切线与法平面

$$\Gamma \begin{cases} x = \varphi(t), \\ y = \psi(t), \\ z = \omega(t), \end{cases} \quad t \in [\alpha, \beta] \Rightarrow \frac{x-x_0}{\varphi'(t)} = \frac{y-y_0}{\psi'(t)} = \frac{z-z_0}{\omega'(t)}$$

$$\varphi'(t)(x-x_0) + \psi'(t)(y-y_0) + \omega'(t)(z-z_0) = 0$$

$$\Gamma \begin{cases} y = \psi(x), \\ z = \omega(x), \end{cases} \Rightarrow \begin{cases} x = x, \\ y = \psi(x), \\ z = \omega(x), \end{cases} \Rightarrow \frac{x-x_0}{1} = \frac{y-y_0}{\psi'(x)} = \frac{z-z_0}{\omega'(x)}$$

$$(x-x_0) + \psi'(x)(y-y_0) + \omega'(x)(z-z_0) = 0$$

$$\Gamma \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \Rightarrow \begin{cases} y = \psi(x), \\ z = \omega(x), \end{cases} \Rightarrow \begin{cases} F(x, \psi(x), \omega(x)) = 0 \\ G(x, \psi(x), \omega(x)) = 0 \\ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} + \frac{\partial F}{\partial z} \frac{dz}{dx} = 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{dy}{dx} + \frac{\partial G}{\partial z} \frac{dz}{dx} = 0 \end{cases}$$

$$\frac{dy}{dx} = \psi'(x) = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, z)} = -\frac{\frac{\partial(F, G)}{\partial(x, z)}}{\frac{\partial(F, G)}{\partial(y, z)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}}$$

$$\frac{dz}{dx} = \omega'(x) = -\frac{1}{J} \frac{\partial(F, G)}{\partial(y, x)} = -\frac{\frac{\partial(F, G)}{\partial(y, x)}}{\frac{\partial(F, G)}{\partial(y, z)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}}$$

9.6.3 曲面的切平面与法线

$$\sum F(x, y, z) = 0$$

$$\text{任意曲线} \Gamma \begin{cases} x = \varphi(t), \\ y = \psi(t), \quad (\alpha < t < \beta), \\ z = \omega(t), \end{cases}$$

$$t = t_0 \text{ 对应 } M(x_0, y_0, z_0)$$

曲线 Γ 在曲面 \sum 上, 过 M 点

$$F[\varphi(t), \psi(t), \omega(t)] = 0,$$

$$F_x|_M \varphi'(t_0) + F_y|_M \psi'(t_0) + F_z|_M \omega'(t_0)$$

$$= (F_x|_M, F_y|_M, F_z|_M) \cdot (\varphi'(t_0), \psi'(t_0), \omega'(t_0))$$

$$= n \cdot s = 0,$$

任一过 M 曲线切线垂直过 M 向量 n , 切线共面, 称为曲面切平面

$$\Rightarrow \begin{aligned} &F_x|_M(x - x_0) + F_y|_M(y - y_0) + F_z|_M(z - z_0) = 0 \\ &\frac{(x-x_0)}{F_x|_M} = \frac{(y-y_0)}{F_y|_M} = \frac{(z-z_0)}{F_z|_M} \end{aligned}$$

$$F_x|_M = f_x(x_0, y_0), F_y|_M = f_y(x_0, y_0), F_z|_M = -1$$

$$\text{记 } K = \sqrt{f_x^2 + f_y^2 + 1}, \cos \gamma > 0,$$

$$z = f(x, y) \Rightarrow F(x, y, z) = f(x, y) - z = 0 \Rightarrow \text{方向余弦 } (\cos \alpha, \cos \beta, \cos \gamma) = \left(\frac{-f_x}{K}, \frac{-f_y}{K}, \frac{1}{K} \right)$$

$$\begin{aligned} &f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0 \\ &\frac{(x-x_0)}{f_x(x_0, y_0)} = \frac{(y-y_0)}{f_y(x_0, y_0)} = \frac{(z-z_0)}{-1} \end{aligned}$$

9.7 方向导数与梯度

9.7.1 方向导数

$$\text{单位向量 } e_l = (\cos \alpha, \cos \beta),$$

$$\text{过 } (x_0, y_0) \text{ 射线 } l \begin{cases} x = x_0 + t \cos \alpha \\ y = y_0 + t \cos \beta \end{cases} (t \geq 0), \Rightarrow \frac{\partial f}{\partial l}|_{(x_0, y_0)} = \lim_{t \rightarrow 0^+} \frac{f(x_0 + t \cos \alpha, y_0 + t \cos \beta) - f(x_0, y_0)}{t}$$

方向导数

$$z = f(x, y)$$

$$f(x, y) \text{ 在 } P_0(x_0, y_0) \text{ 可微分, 该点任意方向, 方向导数存在, } \frac{\partial f}{\partial l}|_{(x_0, y_0)} = f_x(x_0, y_0) \cos \alpha + f_y(x_0, y_0) \cos \beta$$

$$f(x, y, z), \frac{\partial f}{\partial l}|_{(x_0, y_0, z_0)} = f_x(x_0, y_0, z_0) \cos \alpha + f_y(x_0, y_0, z_0) \cos \beta + f_z(x_0, y_0, z_0) \cos \gamma$$

9.7.2 梯度

$$f(x, y), D \text{ 内连续偏导数, } \forall P_0 \in D, \mathbf{grad} f(x_0, y_0) = \Delta f(x_0, y_0) = f_x|_{P_0} i + f_y|_{P_0} j, \text{ 梯度, } \Delta = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j \text{ 向量微分算子}$$

$$\begin{aligned} f(x, y) \text{ 在 } (x_0, y_0) \text{ 可微分, } \frac{\partial f}{\partial l}|_{(x_0, y_0)} &= f_x(x_0, y_0) \cos \alpha + f_y(x_0, y_0) \cos \beta \\ &= \mathbf{grad} f(x_0, y_0) \cdot e_l \\ \text{单位向量 } e_l = (\cos \alpha, \cos \beta), &= |\mathbf{grad} f(x_0, y_0)| \cos \theta, \theta = \left(\widehat{\mathbf{grad} f(x_0, y_0)}, e_l \right) \end{aligned}$$

$$\begin{aligned} \text{曲面 } \sum z = f(x, y), \text{ 等值线 } L f(x, y) = 0, P_0(x_0, y_0), & \text{ 单位法向量 } n = \frac{1}{k} (f_x|_P, f_y|_P) \\ \text{记 } k = \sqrt{f_x^2(x_0, y_0) + f_y^2(x_0, y_0)}, & = \frac{\Delta f(x_0, y_0)}{|\Delta f(x_0, y_0)|} \\ & \Delta f(x_0, y_0) = |\Delta f(x_0, y_0)| n = \frac{\partial f}{\partial n} n \\ & \text{梯度} = \text{梯度的模} \quad \text{梯度的方向} \end{aligned}$$

$$\mathbf{grad} f = \Delta f = f_x|_{P_0} i + f_y|_{P_0} j + f_z|_{P_0} k,$$

$$f(x, y, z), G \text{ 内连续偏导数, } P_0(x_0, y_0, z_0), \text{ 等值面 } f(x, y, z) = c$$

等值面法线方向 n

空间区域 $G \quad \forall M \in G \rightarrow$ 数量 $f(M)$, G 内数量场

空间区域 $G \quad \forall M \in G \rightarrow$ 向量 $f(M)$, G 内向量场

if $F(M)$ 是 $f(M)$ 的梯度 then $f(M)$ 势函数, $F(M)$ 势场

$\frac{m}{r}$ 引力势, $\mathbf{grad} \frac{m}{r}$ 引力场

9.8 多元函数的极值及其求法

9.8.1 多元函数的极值及最大值最小值

if $\exists U(P_0) \subset D, \forall (x, y) \neq P_0, f(x, y) < f(x_0, y_0)$, then 极大值 $f(x_0, y_0)$

$D, z = f(x, y), P_0(x_0, y_0) \in D$, if $\exists U(P_0) \subset D, \forall (x, y) \neq P_0, f(x, y) > f(x_0, y_0)$, then 极小值 $f(x_0, y_0)$

极大值, 极小值统称极值

if $z = f(x, y)$ 在 (x_0, y_0) 具有偏导数, 且在 (x_0, y_0) 点具有极值, then $f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$

if $f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$, then 驻点 (x_0, y_0)

$$f_x(x_0, y_0, z_0) = 0,$$

if $z = f(x, y, z)$ 在 (x_0, y_0, z_0) 具有偏导数, 且在 (x_0, y_0, z_0) 点具有极值, then $f_y(x_0, y_0, z_0) = 0,$

$$f_z(x_0, y_0, z_0) = 0$$

$z = f(x, y)$ 在 (x_0, y_0) 的某邻域内连续,

$AC - B^2 > 0$, 有极值, $A < 0$ 极大值, $A > 0$ 极小值

且有一阶及二阶偏导数, 又 $f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$ if

$AC - B^2 < 0$, 无极值

令 $f_{xx}(x_0, y_0) = A, f_{xy}(x_0, y_0) = B, f_{yy}(x_0, y_0) = C$

$AC - B^2 = 0$, 可能有极值

驻点, 偏导不存在点

9.8.2 条件极值 拉格朗日数乘法

无条件极值, 条件极值; 条件极值到无条件极值

(x_0, y_0) 的某邻域内 $z = f(x, y), \varphi(x, y)$,

有连续一阶导数 $\varphi(x, y) = 0$ (条件) $\rightarrow y = \varphi(x)$, \Rightarrow

$z = f(x, \varphi(x))$ (函数)

$$\left. \frac{dz}{dx} \right|_{x=x_0} = 0$$

$$f_x(x_0, y_0) + f_y(x_0, y_0) \left. \frac{dy}{dx} \right|_{x=x_0} = 0$$

$$f_x(x_0, y_0) - f_y(x_0, y_0) \frac{\varphi_x(x_0, y_0)}{\varphi_y(x_0, y_0)} = 0$$

$$\text{记 } \frac{f_y(x_0, y_0)}{\varphi_y(x_0, y_0)} = -\lambda$$

$$f_y(x_0, y_0) - f_y(x_0, y_0) \frac{\lambda}{\lambda} = 0$$

$$\begin{cases} f_x(x_0, y_0) + \lambda \varphi_x(x_0, y_0) = 0 \\ f_y(x_0, y_0) + \lambda \varphi_y(x_0, y_0) = 0 \\ \varphi(x_0, y_0) = 0 \end{cases}$$

$$L(x, y) = f(x, y) + \lambda \varphi(x, y) \Rightarrow \begin{cases} L_x(x_0, y_0) = 0 \\ L_y(x_0, y_0) = 0 \\ \varphi(x_0, y_0) = 0 \end{cases}$$

拉格朗日函数 L , 拉格朗日乘子 λ
函数 $f(x, y)$, 条件 $\varphi(x, y) = 0$

$$L(x, y, z, t) = f(x, y, z, t) + \lambda \varphi(x, y, z, t) + \mu \psi(x, y, z, t) \Rightarrow \begin{cases} L_x(x_0, y_0, z_0, t_0) = 0 \\ L_y(x_0, y_0, z_0, t_0) = 0 \\ L_z(x_0, y_0, z_0, t_0) = 0 \\ L_t(x_0, y_0, z_0, t_0) = 0 \\ \varphi(x_0, y_0, z_0, t_0) = 0 \\ \psi(x_0, y_0, z_0, t_0) = 0 \end{cases}$$

拉格朗日函数 L ,
拉格朗日乘子 λ, μ

函数 $f(x, y, z, t)$,

条件 $\varphi(x, y, z, t) = 0$,

$\psi(x, y, z, t) = 0$

9.8.3 二元函数的泰勒公式

$z = f(x, y)$ 在点 $P_0(x_0, y_0)$ 某邻域内连续,

有 $n+1$ 阶连续偏导数,

$(x_0 + h, y_0 + k) \in U(P_0, \delta)$

$$\begin{aligned} f(x_0 + h, y_0 + k) &= f(x_0, y_0) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x_0, y_0) \\ &\Rightarrow \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(x_0, y_0) + \cdots + \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x_0, y_0) \\ &\quad + \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x_0 + \theta h, y_0 + \theta k) \quad (0 < \theta < 1) \end{aligned}$$

$$\Phi(t) = f(x_0 + h, y_0 + k) \quad (0 \leq t \leq 1)$$

$$\Phi^{(n)}(t) = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x_0 + ht, y_0 + kt)$$

$$\text{证明: } \Phi(t) = \Phi(0) + \Phi'(0)t + \frac{1}{2}\Phi''(0)t^2 + \cdots + \frac{1}{n!}\Phi^{(n)}(0)t^n + \frac{1}{(n+1)!}\Phi^{(n+1)}(\theta t) \quad (0 < \theta < 1) \quad (0 \leq t \leq 1)$$

$$\Phi(1) = \Phi(0) + \Phi'(0) + \frac{1}{2}\Phi''(0) + \cdots + \frac{1}{n!}\Phi^{(n)}(0) + \frac{1}{(n+1)!}\Phi^{(n+1)}(\theta) \quad (0 < \theta < 1)$$

$$n=0 \text{ 时 } f(x_0 + h, y_0 + k) - f(x_0, y_0) = hf_x(x_0 + \theta h, y_0 + \theta k) + kf_y(x_0 + \theta h, y_0 + \theta k)$$

$$f(x, y), (x, y) \in D, f_x(x, y) \equiv 0, f_y(x, y) \equiv 0, f(x, y) = c$$

9.8.4 极值充分证明

$$\Delta f = f(x, y) - f(x_0, y_0)$$

$$= \left[h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right] f(x_0, y_0) + \frac{1}{2} \left[h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right]^2 f(x_0 + \theta h, y_0 + \theta k)$$

$$f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0$$

$$= \frac{1}{2} \left[h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right]^2 f(x_0 + \theta h, y_0 + \theta k)$$

$$= \frac{1}{2} (h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})$$

$$= \frac{1}{2f_{xx}} (h^2 f_{xx}^2 + 2hk f_{xy} f_{xx} + k^2 f_{xy}^2 - k^2 f_{xy}^2 + k^2 f_{yy} f_{xx})$$

$$= \frac{1}{2f_{xx}} \left[(hf_{xx} + kf_{xy})^2 + k^2 (f_{xx} f_{yy} - f_{xy}^2) \right]$$

$$A = f_{xx}, C = f_{yy}, B = f_{xy}$$

$$AC - B^2 > 0 \begin{cases} A > 0, f(x, y) > f(x_0, y_0), \text{极小值} \\ A < 0, f(x, y) < f(x_0, y_0), \text{极大值} \end{cases}$$

$$\text{有} \begin{cases} A = C = 0, k = h, \Delta f = f_{x,y} \\ A = C = 0, k = -h, \Delta f = -f_{x,y} \end{cases}$$

$$AC - B^2 < 0 \begin{cases} A \neq C = 0, k = 0, \Delta f = \frac{1}{2}h^2 f_{xx} \\ A \neq C = 0, h = -f_{xy} s, k = f_{xx} s, \end{cases}$$

$$\text{无} \begin{cases} A \neq C = 0, h = -f_{xy} s, k = f_{xx} s, \end{cases}$$

$$AC - B^2 = 0 \begin{cases} f(x, y) = x^2 + y^4 \\ \text{有可能} \quad g(x, y) = x^2 + y^3 \end{cases}$$

$$\begin{aligned} \Delta f &= \frac{1}{2} \{ f_{xy}^2 f_{xx} - 2f_{xy}^2 f_{xx} + f_{xx}^2 f_{yy} \} \\ &= \frac{1}{2} f_{xx} \{ f_{xy}^2 - 2f_{xy}^2 + f_{xx} f_{yy} \} \\ &= \frac{1}{2} f_{xx} \{ -f_{xy}^2 \} \end{aligned}$$

9.8.5 最小二乘法

y_i 数据 $\Rightarrow f(t) = at + b$ 线性经验公式

$$M = \sum [y_i - f(t_i)]^2 = \sum [y_i - (at_i + b)]^2 \text{ 偏差平方和,}$$

M 最小为条件选择常数 a, b 的方法, 最小二乘法

\sqrt{M} 均方误差

$$\text{非线性到线性 } M_{\min}(a, b) \Rightarrow \begin{cases} M_a(a, b) = 0 \\ M_b(a, b) = 0 \\ \frac{\partial M}{\partial a} = -2 \sum [y_i - (at_i + b)] = 0 \\ \frac{\partial M}{\partial b} = -2 \sum [y_i - (at_i + b)] = 0 \\ \sum t_i y_i - a \sum t_i^2 - b \sum y_i = 0 \\ \sum y_i - a \sum t_i y_i - bn = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{\begin{vmatrix} \sum t_i y_i & \sum y_i \\ \sum y_i & n \end{vmatrix}}{\begin{vmatrix} \sum t_i^2 & \sum y_i \\ \sum t_i y_i & n \end{vmatrix}} \\ b = \frac{\begin{vmatrix} \sum t_i^2 & \sum t_i y_i \\ \sum t_i y_i & \sum y_i \end{vmatrix}}{\begin{vmatrix} \sum t_i^2 & \sum y_i \\ \sum t_i y_i & n \end{vmatrix}} \end{cases}$$

10 重积分

10.1 二重积分的概念与性质

10.1.1 二重积分的概念

曲顶柱体的体积

$$V = \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i) \Delta\sigma_i$$

平面薄片的质量

$$m = \lim_{\lambda \rightarrow 0} \sum \mu(\xi_i, \eta_i) \Delta\sigma_i$$

闭区域 D 有界函数

$$\Rightarrow \iint_D f(x, y) d\sigma = \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i) \Delta\sigma_i \text{ 二重积分}$$

分割 $D, \Delta\sigma_1, \Delta\sigma_2, \dots, \Delta\sigma_n$

$$\Delta\sigma_i = \Delta x_j \cdot y_k, d\sigma \rightarrow dxdy, \iint_D f(x, y) dxdy$$

10.1.2 二重积分的性质

$$\begin{aligned} \iint_D [\alpha f + \beta g] d\sigma &= \alpha \iint_D f d\sigma + \beta \iint_D g d\sigma \\ \iint_D f d\sigma &= \iint_{D_1} f d\sigma + \iint_{D_2} f d\sigma \\ g(x, y), g &\Rightarrow \iint_D 1 \cdot d\sigma = \iint_D d\sigma = \sigma \\ f(x, y), f &\Rightarrow \iint_D f d\sigma \leq \iint_D g d\sigma, f \leq g \\ m\sigma &\leq \iint_D f d\sigma \leq M\sigma, m \leq f \leq M \\ \iint_D f d\sigma &= f(\xi, \eta) \sigma \quad (\xi, \eta) \in D \end{aligned}$$

10.2 二重积分的计算法

10.2.1 直角坐标计算二重积分

$$\begin{aligned} x\text{型} &\begin{cases} D: \varphi_1(x) \leq y \leq \varphi_2(x), a \leq x \leq b \\ A(x_0) = \int_{\varphi_1(x_0)}^{\varphi_2(x_0)} f(x_0, y) dy \\ V = \int_a^b A(x) dx = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \right] dx = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy = \iint_D f(x, y) d\sigma \end{cases} \\ y\text{型} &\begin{cases} D: \psi_1(y) \leq x \leq \psi_2(y), a \leq y \leq b \\ A(y_0) = \int_{\psi_1(y_0)}^{\psi_2(y_0)} f(x, y_0) dx \\ V = \int_a^b A(y) dy = \int_a^b \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \right] dy = \int_a^b dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx = \iint_D f(x, y) d\sigma \end{cases} \end{aligned}$$

非 x 非 y 型, 分割积分面

10.2.2 极坐标计算二重积分

$$\begin{aligned} S_{\text{扇}} &= \frac{r^2 \theta}{2} \\ \Delta\sigma &= \frac{1}{2} [(r + \Delta r)^2 - r^2] \Delta\theta = \frac{1}{2} [\Delta r^2 + 2r\Delta r] \Delta\theta = \frac{1}{2} [\Delta r + 2r] \Delta r \Delta\theta = \bar{r} \Delta r \Delta\theta \\ \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i) \Delta\sigma_i &= \lim_{\lambda \rightarrow 0} \sum f(\bar{r} \cos \bar{\theta}, \bar{r} \sin \bar{\theta}) \bar{r} \Delta r \Delta\theta \\ \iint_D f(x, y) d\sigma &= \iint_D f(x, y) dxdy = \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta \\ \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta &= \int_a^b \left[\int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r \cos \theta, r \sin \theta) r dr \right] d\theta \\ \sigma &= \iint_D r dr d\theta = \frac{1}{2} \int_a^b [\varphi_2(\theta)^2 - \varphi_1(\theta)^2] d\theta = \frac{1}{2} \int_a^b \varphi_2(\theta)^2 d\theta \end{aligned}$$

10.2.3 二重积分换元法

$f(x, y)$, D 内连续,

变换 $T: x = x(u, v), y = y(u, v)$

uOv 面 $D' \rightarrow xOy$ 面 D , 一对一 $\Rightarrow \iint_D f(x, y) dx dy = \iint_{D'} f(x(u, v), y(u, v)) |J(u, v)| du dv$

$x(u, v), y(u, v)$ 在 D' 一阶连续偏导,

D' 上雅可比式, $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} \neq 0$,

$$c^2 = a^2 + b^2 - 2ab \cos c$$

$$\begin{cases} M_1: x_1 = x(u, v), y_1 = y(u, v) \\ M_2: x_2 = x(u+h, v) = x_1 + x_u(u, v)h + o(h) \\ y_2 = y(u+h, v) = y_1 + y_u(u, v)h + o(h) \\ M_4: x_3 = x(u, v+h) = x_1 + x_v(u, v)h + o(h) \\ y_3 = y(u, v+h) = y_1 + y_v(u, v)h + o(h) \end{cases}$$

$$\begin{aligned} s_{\text{三角形}} &= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \\ &= \frac{1}{2} \begin{vmatrix} x_u(u, v)h & y_u(u, v)h \\ x_v(u, v)h & y_v(u, v)h \end{vmatrix} = \frac{1}{2} h^2 \begin{vmatrix} x_u(u, v) & y_u(u, v) \\ x_v(u, v) & y_v(u, v) \end{vmatrix} = \frac{1}{2} h^2 \begin{vmatrix} x_u(u, v) & x_v(u, v) \\ y_u(u, v) & y_v(u, v) \end{vmatrix} = \frac{1}{2} h^2 \frac{\partial(x, y)}{\partial(u, v)} \end{aligned}$$

10.3 三重积分

10.3.1 三重积分的概念

$f(x, y, z)$, 有界闭区域 Ω , $\Rightarrow \iiint_D f(x, y, z) dv = \iiint_D f(x, y, z) dx dy dz = \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i, \zeta_i) \Delta v_i$
分割成 $\Delta v_1, \Delta v_2, \dots, \Delta v_n$,

10.3.2 三重积分的计算

利用直角坐标系计算

$$\begin{aligned} F(x_0, y_0) &= \int_{z_1(x_0, y_0)}^{z_2(x_0, y_0)} f(x_0, y_0, z) dz \\ \iiint_{\Omega} f(x, y, z) dv &= \iint_{D_{xy}} F(x, y) d\sigma = \iint_{D_{xy}} \left[\int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz \right] d\sigma = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz \\ &= \int_{c_1}^{c_2} dz \iint_{D_z} f(x, y, z) dx dy \end{aligned}$$

利用柱坐标系计算

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \Rightarrow \iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega} f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

$$dv = r dr d\theta dz$$

利用球面坐标计算

$$\begin{cases} x = OP \cos \theta = r \sin \varphi \cos \theta \\ y = OP \sin \theta = r \sin \varphi \sin \theta \\ z = r \cos \varphi \end{cases} \quad \iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega} f(r \sin \varphi \cos \theta, r \sin \varphi \sin \theta, r \cos \varphi) r^2 \sin \varphi dr d\varphi d\theta$$

φ, Z 轴; θ, X 轴
 $dv = r^2 \sin \varphi dr d\varphi d\theta$

10.4 重积分的应用

曲面的面积

曲面 $S: z = f(x, y)$

$$|(f_x, f_y, -1) \cdot (0, 0, 1)| = 1 = \sqrt{f_x^2 + f_y^2 + 1} \cdot 1 \cdot \cos \theta, \quad A = \iint_D \sqrt{f_x^2 + f_y^2 + 1} d\sigma$$

$$\cos \theta = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$\frac{d\sigma}{dA} = \cos \theta, \quad \sum \frac{\sigma_k}{A_k} = \cos \theta, \quad dA = \frac{1}{\cos \theta} d\sigma$$

* 利用曲面的参数方程求曲面的面积

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \quad (u, v) \in D \quad \Rightarrow \quad \begin{aligned} A &= \iint_D \sqrt{EG - F^2} du dv \\ E &= x_u^2 + y_u^2 + z_u^2 \\ F &= x_u x_v + y_u y_v + z_u z_v \\ G &= x_v^2 + y_v^2 + z_v^2 \end{aligned}$$

x, y, z 在 D 内连续一阶

$\frac{\partial(x, y)}{\partial(u, v)}, \frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}$ 不全为零

质心

$$\begin{aligned} dM_y &= x\mu(x, y) d\sigma, \quad M_y = \iint_D x\mu(x, y) d\sigma \\ dM_x &= y\mu(x, y) d\sigma, \quad M_x = \iint_D y\mu(x, y) d\sigma \\ dM &= \mu(x, y) d\sigma, \quad M = \iint_D \mu(x, y) d\sigma \\ \bar{x} &= \frac{M_y}{M} = \frac{\iint_D x\mu(x, y) d\sigma}{\iint_D \mu(x, y) d\sigma} = \frac{\mu(x, y)=c}{A=\iint_D d\sigma} \frac{1}{A} \iint_D x\mu(x, y) d\sigma \\ \bar{y} &= \frac{M_x}{M} = \frac{\iint_D y\mu(x, y) d\sigma}{\iint_D \mu(x, y) d\sigma} = \frac{\mu(x, y)=c}{A=\iint_D d\sigma} \frac{1}{A} \iint_D y\mu(x, y) d\sigma \end{aligned}$$

M_y, M_x 净距, M 质点系总质量

$$M = \iiint_{\Omega} \rho(x, y, z) dv, \quad \begin{cases} \bar{x} = \frac{1}{M} \iiint_{\Omega} x\rho(x, y, z) dv, \\ \bar{y} = \frac{1}{M} \iiint_{\Omega} y\rho(x, y, z) dv, \\ \bar{z} = \frac{1}{M} \iiint_{\Omega} z\rho(x, y, z) dv, \end{cases}$$

转动惯量

$$\begin{aligned} dI_x &= y^2 \mu(x, y) d\sigma \\ I_x &= \sum y_i^2 m_i, \quad \Rightarrow \quad dI_y = x^2 \mu(x, y) d\sigma \\ I_y &= \sum x_i^2 m_i \quad \Rightarrow \quad \begin{aligned} I_x &= \iint_D y^2 \mu(x, y) d\sigma \\ I_y &= \iint_D x^2 \mu(x, y) d\sigma \end{aligned} \end{aligned}$$

$$I_x = \iiint_{\Omega} (y^2 + z^2) \rho(x, y, z) dv$$

$$I_y = \iiint_{\Omega} (x^2 + z^2) \rho(x, y, z) dv$$

$$I_z = \iiint_{\Omega} (y^2 + x^2) \rho(x, y, z) dv$$

引力

$$F = G \frac{mM}{r^2}$$

$$(\cos \alpha, \cos \beta, \cos \gamma) = \left(\frac{x-x_0}{r}, \frac{y-y_0}{r}, \frac{z-z_0}{r} \right)$$

$$F = (|F| \cos \alpha, |F| \cos \beta, |F| \cos \gamma) \stackrel{|dF|=G \frac{1 \cdot \rho(x, y, z) dv}{r^2}}{=} \left(\iiint_{\Omega} \frac{G\rho(x, y, z)(x-x_0)}{r^2 \cdot r} dv, \dots \right)$$

10.5 含参变量的积分

$$f(x, y), R = [a, b] \times [c, d] = \{[x, y] | x \in [a, b], y \in [c, d]\}$$

$$\varphi(x) = \int_c^d f(x, y) dy \quad (a \leq x \leq b) \text{ 含参数变量积分}$$

$f(x, y)$ 在 R 连续, $\varphi(x)$ 在 $[a, b]$ 连续

$$f(x, y) \text{ 在 } R \text{ 连续}, \int_a^b dx \int_c^d f(x, y) dy = \int_c^d dy \int_a^b f(x, y) dx$$

$$f(x, y), f_x(x, y) \text{ 在 } R \text{ 连续}, \varphi(x) \text{ 在 } [a, b] \text{ 可微}, \varphi'(x) = \frac{d}{dx} \int_c^d f(x, y) dy = \int_c^d f_x(x, y) dy$$

$$\Phi(x) = \int_{\alpha(x)}^{\beta(x)} f(x, y) dy \quad (a \leq x \leq b) \text{ 含参数变量积分}$$

$f(x, y)$ 在 R 连续, $\alpha(x), \beta(x)$ 在 $[a, b]$ 连续, $c \leq \alpha(x), \beta(x) \leq d, \Phi(x)$ 在 $[a, b]$ 连续,

$f(x, y), f_x(x, y)$ 在 R 连续, $\alpha(x), \beta(x)$ 在 $[a, b]$ 可微, $c \leq \alpha(x), \beta(x) \leq d, \Phi(x)$ 在 $[a, b]$ 可微,

$$\Phi'(x) = \frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} f(x, y) dy = \int_{\alpha(x)}^{\beta(x)} f_x(x, y) dy + f[x, \beta(x)] \beta'(x) - f[x, \alpha(x)] \alpha'(x)$$

11 曲线积分与曲面积分

11.1 对弧长的曲线积分

11.1.1 对弧长的曲线积分的概念与性质

曲柄构件的质量

$$m \approx \sum \mu(\xi_i, \eta_i) \Delta s_i$$

$$= \lim_{\lambda \rightarrow 0} \sum \mu(\xi_i, \eta_i) \Delta s_i$$

$f(x, y)$ 在直线 L 上有界,

$$\int_L f(x, y) ds = \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i) \Delta s_i,$$

L 插入点 M_1, M_2, \dots, M_{n-1} 分割弧 Δs_i ,

函数对弧长积分, 第一类曲线积分

$$\int_{\Gamma} f(x, y, z) ds = \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i, \zeta_i) \Delta s_i$$

封闭曲线 L , $\int_L f(x, y) ds = \oint_L f(x, y) ds$

分段光滑 $\int_{L_1+L_2} f(x, y) ds = \int_{L_1} f(x, y) ds + \int_{L_2} f(x, y) ds$

常数 α, β , $\int_L [\alpha f(x, y) + \beta g(x, y)] ds = \alpha \int_L f(x, y) ds + \beta \int_L g(x, y) ds$

曲线 L 上 $f(x, y) \leq g(x, y)$, $\int_L f(x, y) ds \leq \int_L g(x, y) ds$

$$\begin{aligned} -|c| \leq a \leq |c|, \quad \Rightarrow \quad -\int |c| ds \leq \int a ds \leq \int |c| ds \quad \xrightarrow{a=c=f(x,y)} \quad \left| \int f(x, y) ds \right| \leq \int |f(x, y)| ds \\ |a| \leq |c| \quad \Rightarrow \quad \left| \int a ds \right| \leq \int |c| ds \end{aligned}$$

11.1.2 对弧长的曲线积分的计算法

$f(x, y)$ 在曲线 L 连续

$$L: \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \quad t \in [\alpha, \beta] \quad \Rightarrow \quad \int_L f(x, y) ds = \lim_{\lambda \rightarrow 0} \sum f(\varphi(t_i), \psi(t_i)) \sqrt{\varphi'^2(t_i) + \psi'^2(t_i)} \Delta t_i$$

$\varphi(t), \psi(t)$, 在 $[\alpha, \beta]$ 连续一阶导

$$\varphi'^2(t) + \psi'^2(t) \neq 0$$

$$y = \psi(x) \rightarrow \begin{cases} x = \varphi(x) = x \\ y = \psi(x) \end{cases} \rightarrow \sqrt{\varphi'^2(x) + \psi'^2(x)} = \sqrt{1 + \psi'^2(x)}$$

$$x = \varphi(y) \rightarrow \begin{cases} x = \varphi(y) \\ y = \psi(y) = y \end{cases} \rightarrow \sqrt{\varphi'^2(y) + \psi'^2(y)} = \sqrt{1 + \varphi'^2(y)}$$

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases} \quad t \in [\alpha, \beta] \rightarrow \int_{\Gamma} f(x, y, z) ds = \int_{\Gamma} f(\varphi(t), \psi(t), \omega(t)) \sqrt{\varphi'^2(t) + \psi'^2(t) + \omega'^2(t)} dt$$

11.2 对坐标的曲线积分

11.2.1 对坐标的曲线积分的概念与性质

变力沿曲线所作的功

$$W = \sum \Delta W \approx \sum [P(\xi_i, \eta_i) \Delta x_i + Q(\xi_i, \eta_i) \Delta y_i]$$

$$= \lim_{\lambda \rightarrow 0} \sum [P(\xi_i, \eta_i) \Delta x_i + Q(\xi_i, \eta_i) \Delta y_i]$$

$$\mathbf{F}(x, y) = (P(x, y, z), Q(x, y, z))$$

$$\mathbf{A}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$$

$$d\mathbf{r}_2 = (dx, dy)$$

$$d\mathbf{r}_3 = (dx, dy, dz)$$

$$\int_L \alpha \mathbf{F}_1 + \beta \mathbf{F}_2 \cdot d\mathbf{r}_2 = \alpha \int_L \mathbf{F}_1 \cdot d\mathbf{r}_2 + \beta \int_L \mathbf{F}_2 \cdot d\mathbf{r}_2$$

$$\int_L \mathbf{F} \cdot d\mathbf{r}_2 = \int_{L_1} \mathbf{F} \cdot d\mathbf{r}_2 + \int_{L_2} \mathbf{F} \cdot d\mathbf{r}_2$$

$$\int_L \mathbf{F} \cdot d\mathbf{r} = - \int_{L^-} \mathbf{F} \cdot d\mathbf{r}$$

$$\int_L P(x, y) dx = \lim_{\lambda \rightarrow 0} \sum P(\xi_i, \eta_i) \Delta x_i$$

$$\int_L P(x, y) dy = \lim_{\lambda \rightarrow 0} \sum Q(\xi_i, \eta_i) \Delta y_i$$

→ 第二类曲线积分

$$W = \int_L \mathbf{F}(x, y) \cdot d\mathbf{r}_2$$

$$W_3 = \int_{\Gamma} \mathbf{A}(x, y, z) \cdot d\mathbf{r}_3$$

11.2.2 对坐标的曲线积分的计算法

$P(x, y), Q(x, y)$ 在 L 连续

$$L: \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \rightarrow \int_L P(x, y) dx + Q(x, y) dy = \int_\alpha^\beta [P(\varphi(t), \psi(t)) \varphi'(t) + Q(\varphi(t), \psi(t)) \psi'(t)] dt$$

$\varphi(t), \psi(t)$ 在 $[\alpha, \beta]$ 连续一阶导

$$\varphi'^2(t) + \psi'^2(t) \neq 0$$

$$x = \varphi(y) \rightarrow L: \begin{cases} x = \varphi(y) \\ y = y \end{cases} \rightarrow \int_L P(x, y) dx + Q(x, y) dy = \int_\alpha^\beta [P(\varphi(y), y) \varphi'(y) + Q(\varphi(y), y)] dy$$

$$y = \psi(x) \rightarrow L: \begin{cases} x = x \\ y = \psi(x) \end{cases} \rightarrow \int_L P(x, y) dx + Q(x, y) dy = \int_\alpha^\beta [P(x, \psi(x)) + Q(x, \psi(x)) \psi'(x)] dx$$

$$\Gamma: \begin{cases} x = \varphi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases} \rightarrow \int_\Gamma (P, Q, R) \cdot (dx, dy, dz) = \int_\Gamma P\varphi' + Q\psi' + R\omega' dt$$

11.2.3 两类曲线积分的关系

$$\begin{aligned} L: \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} & \quad \int_L [P \cos \alpha + Q \cos \beta] ds = \int_L \left[P \frac{\varphi'}{k} + Q \frac{\psi'}{k} \right] k dt \\ & \quad = \int_L [P\varphi' + Q\psi'] dt \\ & \quad \int_L (P, Q) \cdot (\cos \alpha, \cos \beta) ds = \int (P, Q) \cdot (dx, dy) = \int (P, Q) dr \\ \text{记 } k &= \sqrt{\varphi'^2(t) + \psi'^2(t)} \rightarrow \int_L \mathbf{F} \cdot \mathbf{n} ds = \int_L \mathbf{F}_n ds = \int_L \mathbf{F} \cdot d\mathbf{r} \\ \text{切向量 } \mathbf{n} &= (\cos \alpha, \cos \beta) = \left(\frac{\varphi'}{k}, \frac{\psi'}{k} \right) \quad \int_L (P, Q, R) \cdot (\cos \alpha, \cos \beta, \cos \gamma) ds = \int (P, Q, R) \cdot (dx, dy, dz) = \int (P, Q, R) dr \\ & \quad \int_\Gamma \mathbf{A} \cdot \mathbf{n} ds = \int_\Gamma \mathbf{A}_n ds = \int_\Gamma \mathbf{A} \cdot d\mathbf{r} \\ & \quad dr = n ds \text{ 有向曲线元} \end{aligned}$$

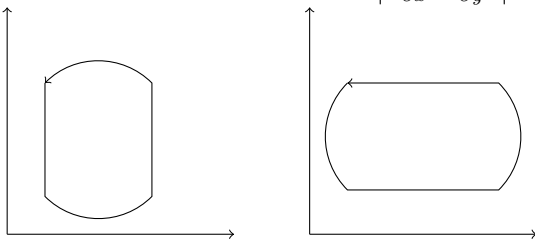
11.3 格林公式及其应用

11.3.1 格林公式

D 内任意封闭曲线所围部分都属于 D , **单连通**, 非单连通**复连通**

边界曲线正向, 左边是 D

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = - \iint_D \begin{vmatrix} P & Q \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{vmatrix} = \oint_L P dx + Q dy$$



$$\iint_D \frac{\partial P}{\partial y} dx dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial Q}{\partial y} dy \right] dx = \int_a^b P(x, \varphi_2(x)) - P(x, \varphi_1(x)) dx$$

$$\iint_D \frac{\partial Q}{\partial x} dx dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial Q}{\partial x} dx \right] dy = \int_c^d Q(\psi_2(y), y) - Q(\psi_1(y), y) dy$$

$$\begin{aligned}
\oint P dx &= \int_{L_1} P dx + \int_{L_2} P dx + \int_{L_3} P dx + \int_{L_4} P dx \\
&= \int_{L_1} P(x, \varphi_2(x)) dx + \int_{L_2} P(x, \varphi_1(x)) dx \\
&= \int_b^a P(x, \varphi_2(x)) dx + \int_a^b P(x, \varphi_1(x)) dx \\
&= \int_a^b P(x, \varphi_1(x)) - P(x, \varphi_2(x)) dx \\
&= - \left(\int_a^b P(x, \varphi_2(x)) - P(x, \varphi_1(x)) dx \right)
\end{aligned}$$

$$L_1: y = \varphi_2(x), x \in [b, a]$$

$$L_2: y = \varphi_1(x), x \in [a, b]$$

$$L_3: x = a, y \in [d, c]$$

$$L_4: x = b, y \in [a, b]$$

$$\begin{aligned}
\oint Q dy &= \int_{L_1} Q dy + \int_{L_2} Q dy + \int_{L_3} Q dy + \int_{L_4} Q dy \\
&= \int_{L_3} Q(\psi_1, y) dy + \int_{L_4} Q(\psi_2, y) dy \\
&= \int_d^c Q(\psi_1, y) dy + \int_c^d Q(\psi_2, y) dy \\
&= \int_c^d Q(\psi_2, y) - Q(\psi_1, y) dy
\end{aligned}$$

$$L_1: y = d, x \in [b, a]$$

$$L_2: y = c, x \in [a, b]$$

$$L_3: x = \psi_1(y)$$

$$L_4: x = \psi_2(y)$$

非标准分割

11.3.2 平面上曲线积分与路径无关的条件

区域 G 内任意点 A, B , 任意 A 到 B 曲线,

if $\int_{L_1} P dx + Q dy = \int_{L_2} P dx + Q dy$, then 曲线积分 $\int_L P dx + Q dy$ 在 G 内与路径无关, $\oint_{L_1-L_2} P dx + Q dy = 0$

if 单连通区域 G, P, Q 在 G 内连续一阶偏导数, then 曲线积分 $\int_L P dx + Q dy$ 在 G 内与路径无关 $\Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
奇点

11.3.3 二元函数的全微分求积

单连通区域 $G, P(x, y), Q(x, y)$ 在 G 内具有一阶连续偏,

$F dx + Q dy$ 在 G 内为某一函数 $\mu(x, y)$ 的全微分 $\Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

曲线积分 $\int_L P dx + Q dy$ 在 G 内与路径无关 $\Leftrightarrow \exists \mu(x, y), du = P dx + Q dy$

全微分方程

微分方程 $P(x, y) dx + Q(x, y) dy = 0$ 左端是某函数 $\mu(x, y)$ 全微分, 全微分方程

11.3.4 曲线积分的基本定理

曲线积分 $\int_L \mathbf{F} \cdot d\mathbf{r}$ 在区域 G 内与积分路径无关, 向量场 \mathbf{F} 为保守场

$F(x, y) = (P(x, y) + Q(x, y))$ 是区域 G 内向量场, $P(x, y), Q(x, y)$ 在 G 内连续, 且存在数量函数使得 $F = \nabla f$, 则曲线积分 $\int_L \mathbf{F} \cdot d\mathbf{r}$ 在 G 内与路径无关, 且 $\int_L \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$ 曲线积分基本公式

11.4 对面积的曲面积分

11.4.1 对面积的曲面积分的概念与性质

光滑曲面 Σ , 函数 $f(x, y, z)$ 在 Σ 有界, 任意分割 n 小块 ΔS_i , 无关面分法, 点取法, 极限 $\lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i, \zeta_i) \Delta S_i = \iint_{\Sigma} f(x, y, z) dS$ 第一类曲面积分

11.4.2 对面积的曲面积分的算法

$$\begin{aligned} \text{曲面 } \Sigma: z = z(x, y), n = (-z_x, -z_y, 1) &= \left(\frac{-z_x}{\sqrt{z_x^2 + z_y^2 + 1}}, \frac{-z_y}{\sqrt{z_x^2 + z_y^2 + 1}}, \frac{1}{\sqrt{z_x^2 + z_y^2 + 1}} \right) \\ |n \cdot (1, 0, 0)| &= \frac{z_x}{\sqrt{z_x^2 + z_y^2 + 1}} = \cos \alpha \quad \Delta S_i \cos \alpha = (\Delta \sigma_i)_{yz} \quad \Delta S_i = \frac{1}{\cos \alpha} (\Delta \sigma_i)_{yz} \\ |n \cdot (0, 1, 0)| &= \frac{z_y}{\sqrt{z_x^2 + z_y^2 + 1}} = \cos \beta \quad \Delta S_i \cos \beta = (\Delta \sigma_i)_{xz} \quad \Delta S_i = \frac{1}{\cos \beta} (\Delta \sigma_i)_{xz} \\ |n \cdot (0, 0, 1)| &= \frac{1}{\sqrt{z_x^2 + z_y^2 + 1}} = \cos \gamma \quad \Delta S_i \cos \gamma = (\Delta \sigma_i)_{xy} \quad \Delta S_i = \frac{1}{\cos \gamma} (\Delta \sigma_i)_{xy} \\ \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i, \zeta_i) \Delta S_i &= \lim_{\lambda \rightarrow 0} \sum f(\xi_i, \eta_i, \zeta_i) \frac{1}{\cos \gamma} (\Delta \sigma_i)_{xy} \\ \int_{\Sigma} f(x, y, z) dS &= \iint_{D_{xy}} f[x, y, z(x, y)] \sqrt{z_x^2 + z_y^2 + 1} dx dy \end{aligned}$$

11.5 对坐标的曲面积分

11.5.1 对坐标的曲面积分的概念与性质

$$\Delta S \text{ 单位法向量 } n, n \cdot (0, 0, 1) = \cos \gamma \Rightarrow \Delta S \text{ 在 } xOy \text{ 面投影 } (\Delta S)_{xy} = \begin{cases} (\Delta \sigma)_{xy}, & \cos \gamma > 0 \\ -(\Delta \sigma)_{xy}, & \cos \gamma < 0 \\ 0, & \cos \gamma = 0 \end{cases}$$

流向曲面一侧的流量

$$Av \cdot n$$

$$\begin{aligned} \Delta S_i \text{ 在 } xOy \text{ 面投影为 } (\Delta S_i)_{xy} \\ \text{光滑有向曲面 } \Sigma, \text{ 函数 } R(x, y, z) \text{ 在 } \Sigma \text{ 有界, 任意分割 } \Sigma \text{ 为 } \Delta S_i, \Delta S_i \text{ 在 } xOz \text{ 面投影为 } (\Delta S_i)_{xz} \\ \Delta S_i \text{ 在 } yOz \text{ 面投影为 } (\Delta S_i)_{yz} \\ \text{第二类曲面积分, 函数在有向面上对坐标轴 } x, y, z \text{ 的积分} &\begin{cases} \iint_{\Sigma} R(x, y, z) dx dy = \lim_{\lambda \rightarrow 0} R(\xi_i, \eta_i, \zeta_i) (\Delta S_i)_{xy} \\ \iint_{\Sigma} Q(x, y, z) dx dz = \lim_{\lambda \rightarrow 0} Q(\xi_i, \eta_i, \zeta_i) (\Delta S_i)_{xz} \\ \iint_{\Sigma} P(x, y, z) dy dz = \lim_{\lambda \rightarrow 0} P(\xi_i, \eta_i, \zeta_i) (\Delta S_i)_{yz} \end{cases} \\ \iint_{\Sigma} (P, Q, R) \cdot (dy dz, dx dz, dx dy) &= \iint_{\Sigma_1} (P, Q, R) \cdot (dy dz, dx dz, dx dy) + \iint_{\Sigma_2} (P, Q, R) \cdot (dy dz, dx dz, dx dy) \\ \iint_{\Sigma} (P, Q, R) \cdot (dy dz, dx dz, dx dy) &= -\iint_{\Sigma^-} (P, Q, R) \cdot (dy dz, dx dz, dx dy) \\ \iint_{\Sigma^-} P dy dz &= -\iint_{\Sigma} P dy dz \\ \iint_{\Sigma^-} P dx dz &= -\iint_{\Sigma} P dx dz \\ \iint_{\Sigma^-} P dx dy &= -\iint_{\Sigma} P dx dy \end{aligned}$$

11.5.2 对坐标的曲面积分的算法

$$\Sigma: z = z(x, y), (\Delta S_i)_{xy} = \pm (\Delta \sigma)_{xy}, \iint_{\Sigma} R(x, y, z) dx dy = \pm \iint_{D_{xy}} R(x, y, z(x, y)) dx dy$$

11.5.3 两类曲面积分之间的关系

$$\begin{aligned} \Sigma: z = z(x, y), \Sigma \text{ 法向量 } \mathbf{n} &= \left(\frac{-z_x}{|\mathbf{n}|}, \frac{-z_y}{|\mathbf{n}|}, \frac{1}{|\mathbf{n}|} \right) = (\cos \alpha, \cos \beta, \cos \gamma) \\ \iint_{\Sigma} (P, Q, R) \cdot (dy dz, dx dz, dx dy) &= \iint_{\Sigma} (P, Q, R) \cdot (\cos \alpha, \cos \beta, \cos \gamma) dS \\ \iint_{\Sigma} \mathbf{A} \cdot d\mathbf{S} &= \iint_{\Sigma} \mathbf{A} \cdot \mathbf{n} dS = \iint_{\Sigma} \mathbf{A}_n dS \\ d\mathbf{S} &= \mathbf{n} dS \text{ 有向曲面元} \end{aligned}$$

11.6 高斯公式 通量与散度

11.6.1 高斯公式

空间闭区域 Ω 由分片光滑闭曲面 (有向, 曲面外侧) Σ 组成, $P(x, y, z), Q(x, y, z), R(x, y, z)$ 在 Ω 上有一阶连续偏导数, Σ 方向余弦 $n = (\cos \alpha, \cos \beta, \cos \gamma)$ 则

$$\iint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv = \oint_{\Sigma} (P, Q, R) \cdot (dydz, dx dz, dx dy) = \oint_{\Sigma} (P, Q, R) \cdot (\cos \alpha, \cos \beta, \cos \gamma) dS$$

$$\begin{aligned} \iiint_{\Omega} \frac{\partial R}{\partial z} dv &= \iint_{D_{xy}} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z} dz \right\} dx dy \\ &= \iint_{D_{xy}} \{ R(x, y, z_2) - R(x, y, z_1) \} dx dy \\ \oint_{\Sigma} R(x, y, z) dx dy &= \iint_{\Sigma_1} R dx dy + \iint_{\Sigma_2} R dx dy + \iint_{\Sigma_3} R dx dy \\ &= \iint_{\Sigma_1} R(x, y, z) dx dy + \iint_{\Sigma_2} R(x, y, z) dx dy + 0 \\ &= - \iint_{D_{xy}} R(x, y, z_1) dx dy + \iint_{D_{xy}} R(x, y, z_2) dx dy \end{aligned} \quad \begin{aligned} \iint_{\Omega} \frac{\partial P}{\partial x} dv &= \oint_{\Sigma} P dy dz \\ \iint_{\Omega} \frac{\partial Q}{\partial y} dv &= \oint_{\Sigma} Q dx dz \\ \iint_{\Omega} \frac{\partial R}{\partial z} dv &= \oint_{\Sigma} R dx dy \end{aligned} \Rightarrow$$

不规则体积分割

函数 $u(x, y, z), v(x, y, z)$ 在闭区域 Ω 上具有一二阶连续偏导数,

则 $\iint_{\Omega} u \Delta v dx dy dz = \oint_{\Sigma} u \frac{\partial v}{\partial n} dS - \iiint_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right) dx dy dz$ 格林第一公式, Σ 为闭区域 Ω 边界曲面, $\frac{\partial v}{\partial n}$

为函数 $v(x, y, z)$ 沿 Σ 外法线方向的方向导数, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ 称为拉普拉斯算子

$$\begin{aligned} \oint_{\Sigma} u \frac{\partial v}{\partial n} dS &= \oint_{\Sigma} u \left(\frac{\partial v}{\partial x} \cos \alpha + \frac{\partial v}{\partial y} \cos \beta + \frac{\partial v}{\partial z} \cos \gamma \right) dS \\ &= \oint_{\Sigma} \left[\left(u \frac{\partial v}{\partial x} \right) \cos \alpha + \left(u \frac{\partial v}{\partial y} \right) \cos \beta + \left(u \frac{\partial v}{\partial z} \right) \cos \gamma \right] dS \\ &= \iiint_{\Omega} \left[\frac{\partial}{\partial x} \left(u \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(u \frac{\partial v}{\partial z} \right) \right] dx dy dz \\ &= \iiint_{\Omega} \left[u \frac{\partial^2 v}{\partial x^2} + u \frac{\partial^2 v}{\partial y^2} + u \frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right] dx dy dz \\ &= \iiint_{\Omega} \left[u \Delta v + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right] dx dy dz \\ &= \iiint_{\Omega} u \Delta v dx dy dz + \iiint_{\Omega} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} dx dy dz \end{aligned}$$

11.6.2 沿任意闭曲面的曲面积分为零的条件

空间区域 G , G 内任意闭曲面所围区域完全属于 G , 空间二维单连通

空间区域 G , G 内任意闭曲线总可以张成 (做曲面) 完全属于 G , 空间一维单连通

二维单连通区域 G , 若 P, Q, R , 在 G 内具有一阶连续偏导数, 则曲面积分 $\iint_{\Sigma} P dy dz + Q dx dz + R dx dy$ 在 G 内与选取曲面选取无关, 只与边界曲线有关 $\Leftrightarrow \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$

11.6.3 通量与散度

向量场 $\mathbf{A}(x, y, z) = (P, Q, R)$

有向曲面 Σ , 法向量 \mathbf{n}

通量 $\iint_{\Sigma} \mathbf{A} \cdot \mathbf{n} dS$

$$\begin{aligned} \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv &= \oint_{\Sigma} v_n dS \\ \frac{1}{V} \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv &= \frac{1}{V} \oint_{\Sigma} v_n dS \\ \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) |_{(\xi, \eta, \zeta)} &= \frac{1}{V} \oint_{\Sigma} v_n dS \quad (\xi, \eta, \zeta) \in \Omega \\ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} &= \lim_{\Omega \rightarrow M} \frac{1}{V} \oint_{\Sigma} v_n dS = \operatorname{div} v(M) = \Delta \cdot \mathbf{A} \end{aligned}$$

速度场 v 在点 M 的通量密度, 源头强度散度

$\operatorname{div} \mathbf{V}$ 处处为零, 无源场

散度体积分, 通量面积分

11.7 斯托克斯公式 * 环流量与旋度

11.7.1 斯托克斯公式

分段光滑空间有向闭曲线 Γ , Γ 张成分片光滑有向曲面 Σ , 符合右手定则若 P, Q, R 在曲面 Σ 上具有一阶连续偏导数, 则 $\iint_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dx dz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{\Gamma} P dx + Q dy + R dz$

曲面 $\Sigma: z = z(x, y)$, 法向量 $n = (\cos \alpha, \cos \beta, \cos \gamma) = \left(-\frac{z_x}{|n|}, -\frac{z_y}{|n|}, \frac{1}{|n|} \right)$

$$\begin{aligned} \iint_{\Sigma} \frac{\partial P}{\partial z} dz dx - \frac{\partial P}{\partial y} dy dx &= \iint_{\Sigma} \left(\frac{\partial P}{\partial z} \cos \beta - \frac{\partial P}{\partial y} \cos \gamma \right) dS \\ &\stackrel{\cos \beta = -f_y \cos \gamma}{=} - \iint_{\Sigma} \left(\frac{\partial P}{\partial z} f_y + \frac{\partial P}{\partial y} \right) \cos \gamma dS \\ &= - \iint_{D_{xy}} \left(\frac{\partial P}{\partial z} f_y + \frac{\partial P}{\partial y} \right) dx dy \\ &= - \iint_{D_{xy}} \frac{\partial P}{\partial y} dx dy \\ &= \oint_L P dx \end{aligned}$$

$$\iint_{\Sigma} \frac{\partial Q}{\partial x} dx dy - \frac{\partial Q}{\partial z} dz dy = \oint_{\Gamma} Q dy$$

$$\iint_{\Sigma} \frac{\partial R}{\partial y} dy dz - \frac{\partial R}{\partial x} dx dz = \oint_{\Gamma} R dz$$

$$\begin{aligned} \iint_{\Sigma} \begin{vmatrix} dydz & dx dz & dx dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} &= \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS = \oint_{\Gamma} P dx + Q dy + R dz \\ &= \begin{vmatrix} \cos \alpha & \cos \beta & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ P & Q & 0 \end{vmatrix} dS = \oint_{\Gamma} P dx + Q dy \end{aligned}$$

11.7.2 空间曲线积分与路径无关的条件

一维单连通区域 G , 若函数 P, Q, R 在 G 内具有一阶连续偏导数,

$$\text{则曲线积分 } \int_{\Gamma} P dx + Q dy + R dz \text{ 在 } G \text{ 内与路径无关} \Leftrightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \\ \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \end{cases}$$

一维单连通区域 G , 若函数 P, Q, R 在 G 内具有一阶连续偏导数,

$$\begin{aligned} \text{则存在 } u(x, y, z), du = P dx + Q dy + R dz &\Leftrightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \\ \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \end{cases} \\ u(x, y, z) &= \int_{(x_0, y_0, z_0)}^{(x, y, z)} P dx + Q dy + R dz \\ &= \int_{x_0}^x P dx + \int_{y_0}^y Q dy + \int_{z_0}^z R dz \end{aligned}$$

11.7.3 环流量与旋度

向量场 $\mathbf{A} = (P, Q, R)$, 分段光滑有向闭曲线 Γ , 单位切向量 τ , 则 $\oint_{\Gamma} \mathbf{A} \cdot \tau ds$ 称为 A 沿 τ 环流量

$$\mathbf{rot} \mathbf{A} = \Delta \times \mathbf{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{旋度}$$

旋度处处为零, 无旋场

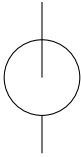
无旋, 无源, 调和场

Σ 单位法向量 $n = (\cos \alpha, \cos \beta, \cos \gamma)$

$$\mathbf{rot} \mathbf{A} \cdot n = \Delta \times \mathbf{A} \cdot n = \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\iint_{\Sigma} \mathbf{rot} \mathbf{A} \cdot n dS = \oint_{\Gamma} \mathbf{A} \cdot \tau ds$$

旋度通量，环流量



$$r = \vec{OM}, \omega = (0, 0, w), v = \omega \times r, v = \begin{vmatrix} i & j & k \\ 0 & 0 & w \\ x & y & z \end{vmatrix} = (-wy, wx, 0), \mathbf{rot} \mathbf{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix} = (0, 0, 2w) = 2\omega$$

12 无穷级数

12.1 常数项级数的概念和性质

12.1.1 数项级数的概念

数列 $\mu_1, \mu_2, \mu_3, \dots + \mu_n, \dots$, 表达式 $\mu_1 + \mu_2 + \mu_3 + \dots + \mu_n + \dots$ 称为 (常数项) 无穷级数, 简称 (常数项) 级数, 记为 $\sum_{i=1}^{\infty} \mu_i = \mu_1 + \mu_2 + \mu_3 + \dots + \mu_n + \dots$, 第 n 项 μ_n 叫做级数的一般项

$$S_n = \sum_{i=1}^n \mu_i = \mu_1 + \mu_2 + \mu_3 + \dots + \mu_n \text{ 级数部分和}$$

新数列 $\{S_n\}$ $S_1, S_2, \dots, S_n, \dots$

无穷级数 $\sum_{i=1}^{\infty} \mu_i$, 无穷级数的部分和数列 $\{S_n\}$, $\lim_{n \rightarrow \infty} S_n = s \Rightarrow \sum_{i=1}^{\infty} \mu_i$ 收敛, $s = \mu_1 + \mu_2 + \mu_3 + \dots + \mu_n + \dots$
 $\lim_{n \rightarrow \infty} S_n = \infty$ (不存在) $\Rightarrow \sum_{i=1}^{\infty} \mu_i$ 发散

$$r_n = s - S_n = S_{n+1} + S_{n+2} + \dots \text{ 余项}$$

$$\sum_{i=1}^{\infty} \mu_i = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu_i$$

$$\sum_{i=1}^{\infty} aq^i = a + aq + aq^2 + \dots + aq^i + \dots \text{ 等比级数, } S_n = a + aq + \dots + aq^{n-1} = \frac{a-aq^n}{1-q} \text{ 部分和数列}$$

$$|q| \geq 1, \text{ 发散, } |q| \leq 1, \text{ 收敛}$$

$$S_n = S_1 + (S_2 - S_1) + \dots + (S_n - S_{n-1}) + \dots = S_1 + \sum_{i=2}^{\infty} (S_i - S_{i-1}) = \sum_{i=1}^{\infty} \mu_i$$

12.1.2 收敛级数的基本性质

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s , 则级数 $\sum_{n=1}^{\infty} k\mu_n$ 收敛于 s , 和为 ks , 数乘同敛散

级数 $\sum_{n=1}^{\infty} \mu_n$ 与 $\sum_{n=1}^{\infty} v_n$ 分别收敛于 s 与 σ , 级数 $\sum_{n=1}^{\infty} (\mu_n \pm v_n)$ 也收敛, 和为 $s \pm \sigma$, 收敛和收敛

级数中去掉, 加上, 改变有限项, 不会改变级数的收敛性

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s , 任意项加括号组成的级数仍收敛, 和为 s , 括号发散源发散

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s (必要条件), 一般项趋近 0, 即 $\lim_{n \rightarrow \infty} \mu_n = 0$

调和级数 $\sum_{n=1}^{\infty} \frac{1}{n}$ 发散

12.1.3 柯西审敛原理

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛 $\Leftrightarrow \forall \varepsilon > 0, \exists N > 0, \text{ when } n > N, \forall p \in \mathbb{Z}^+, |\mu_{n+1} + \mu_{n+2} + \dots + \mu_{n+p}| < \varepsilon$

12.2 常数项级数审敛法

12.2.1 正项级数及其审敛法

各项是正数或零的级数 **正项级数**

正项级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 $s \Leftrightarrow$ 部分和数列 $\{S_n\}$ 有界

正项级数 $\sum_{n=1}^{\infty} \mu_n, \sum_{n=1}^{\infty} v_n, \mu_n \leq v_n, \sum_{n=1}^{\infty} v_n$ 收敛, $\sum_{n=1}^{\infty} \mu_n$ 收敛; $\sum_{n=1}^{\infty} \mu_n$ 发散, $\sum_{n=1}^{\infty} v_n$ 发散

正项级数 $\sum_{n=1}^{\infty} \mu_n, \sum_{n=1}^{\infty} v_n$, 存在正整数 N , $\mu_n \leq kv_n (n > N, k > 0)$, $\sum_{n=1}^{\infty} v_n$ 收敛, $\sum_{n=1}^{\infty} \mu_n$ 收敛; $\sum_{n=1}^{\infty} \mu_n$ 发散, $\sum_{n=1}^{\infty} v_n$ 发散

正项级数 $\sum_{n=1}^{\infty} \mu_n$, $\text{if } \lim_{n \rightarrow \infty} \frac{\mu_n}{v_n} = l, \{0 \leq l < \infty\}, \sum_{n=1}^{\infty} v_n$ 收敛, 则 $\sum_{n=1}^{\infty} \mu_n$ 收敛
 $\text{if } \lim_{n \rightarrow \infty} \frac{\mu_n}{v_n} = l, \{l > 0, +\infty\}, \sum_{n=1}^{\infty} v_n$ 发散, 则, $\sum_{n=1}^{\infty} \mu_n$ 发散
 $\text{if } \rho > 1 (= \infty)$ 发散 (不可能收敛)

正项级数 $\sum_{n=1}^{\infty} \mu_n, \lim_{n \rightarrow \infty} \frac{\mu_{n+1}}{\mu_n} = \rho$, $\text{if } \rho < 1$ 收敛
 $\text{if } \rho = 1$ 可能收敛

when $\rho < 1, \forall \rho + \varepsilon = r < 1, \exists m$, when $n \geq m, \frac{\mu_{n+1}}{\mu_n} = \rho < \rho + \varepsilon = r < 1, \mu_{n+1} < r\mu_n, \mu_{n+k} < r^k \mu_n$

级数 $\sum_{n=1}^{\infty} r^k \mu_n$ 收敛, 级数 $\sum_{n=1}^{\infty} \mu_{n+k}$ 收敛

$\sum_{n=1}^{\infty} \mu_n = \sum_{n=1}^{\infty} \mu_{n+k} + \sum_{n=1}^k \mu_n$ 则级数 $\sum_{n=1}^{\infty} \mu_n, \sum_{n=1}^{\infty} \mu_{n+k}$ 同敛散, 级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛

$\text{if } \rho > 1 (= \infty)$ 发散 (不可能收敛)
 正项级数 $\sum_{n=1}^{\infty} \mu_n, \lim_{n \rightarrow \infty} \sqrt[n]{\mu_n} = \rho$, $\text{if } \rho < 1$ 收敛
 $\text{if } \rho = 1$ 可能收敛

when $\rho < 1, \mu_n < r^n (r < 1)$, 级数 $\sum_{n=1}^{\infty} r^n$ 收敛, 则 $\sum_{n=1}^{\infty} \mu_n$ 收敛

正项级数 $\sum_{n=1}^{\infty} \mu_n$, $\text{if } \lim_{n \rightarrow \infty} n^p \mu_n = \frac{\mu_n}{\frac{1}{n^p}} = l, \{0 \leq l < \infty, p > 1\}, \sum_{n=1}^{\infty} \frac{1}{n^p}$ 收敛, 则级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛
 $\text{if } \lim_{n \rightarrow \infty} n \mu_n = \frac{\mu_n}{\frac{1}{n}} = l, \{l > 0, \infty\}, \sum_{n=1}^{\infty} \frac{1}{n}$ 发散, 则级数 $\sum_{n=1}^{\infty} \mu_n$ 发散

12.2.2 交错级数审敛法

各项正负交错, 可以写成 $\mu_1 - \mu_2 + \mu_3 - \mu_4 + \dots$, 其中 $\mu_n > 0$
 $-\mu_1 + \mu_2 - \mu_3 + \mu_4 - \dots$,

交错级数 $\sum_{n=1}^{\infty} \mu_n$ 满足条件: (1) $\mu_n \geq \mu_{n+1} (n = 1, 2, 3, \dots)$, 那么级数收敛, 其和 $s \leq \mu_1$, 余项绝对值 $|r_n| \leq \mu_{n+1}$
 (2) $\lim_{n \rightarrow \infty} \mu_n = 0$

12.2.3 绝对收敛与条件收敛

级数 $\sum_{n=1}^{\infty} \mu_n$, 正项级数 $\sum_{n=1}^{\infty} |\mu_n|$ 收敛, 则级数 $\sum_{n=1}^{\infty} \mu_n$ 绝对收敛
 正项级数 $\sum_{n=1}^{\infty} |\mu_n|$ 发散, 级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛, 则级数 $\sum_{n=1}^{\infty} \mu_n$ 条件收敛
 级数 $\sum_{n=1}^{\infty} \mu_n$ 绝对收敛, 则级数 $\sum_{n=1}^{\infty} \mu_n$ 必定收敛 $v_n = \frac{1}{2} (\mu_n + |\mu_n|)$

12.2.4 绝对收敛级数的性质

绝对收敛级数经改变项位置后构成的级数也收敛, 且与原级数有相同的和 $S_n^* \leq S_m \leq s, \mu_n = 2v_n - |\mu_n|$

级数 $\sum \mu_n, \sum v_n$ 绝对收敛, 和分别为 s, σ , 所有项的可能乘积 $\mu_i v_i$

$$\begin{array}{ccccccc} \mu_1 v_1 & \mu_1 v_2 & \mu_1 v_3 & \cdots & \mu_1 v_1 & \mu_1 v_2 & \mu_1 v_3 & \cdots \\ \mu_2 v_1 & \mu_2 v_2 & \mu_2 v_3 & \cdots & \mu_2 v_1 & \mu_2 v_2 & \mu_2 v_3 & \cdots \\ \mu_3 v_1 & \mu_3 v_2 & \mu_3 v_3 & \cdots & \mu_3 v_1 & \mu_3 v_2 & \mu_3 v_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

柯西乘积 (级数) $\mu_1 v_1 + (\mu_1 v_2 + \mu_2 v_1) + \cdots + (\mu_1 v_n + \mu_2 v_{n-1} + \cdots + \mu_n v_1) + \dots$ 绝对收敛, 和为 $s\sigma$

$\mu_1 v_1 + (\mu_1 v_2 + \mu_2 v_2 + \mu_2 v_1) + (\mu_1 v_3 + \mu_2 v_3 + \mu_3 v_3 + \mu_3 v_2 + \mu_3 v_1) + \cdots = (\mu_1 + \mu_2 + \cdots + \mu_n) \cdot (v_1 + v_2 + \cdots + v_n)$

12.3 幂级数

12.3.1 函数项级数的概念

区间 I 上函数列 $\mu_1(x), \mu_2(x), \mu_3(x), \dots, \mu_n(x), \dots$,
 表达式 $\mu_1(x) + \mu_2(x) + \mu_3(x) + \dots + \mu_n(x) + \dots$, 区间 I 上的 (函数项) 无穷级数, (函数项) 级数
 对于每个确定值 $x_0 \in I, \mu_1(x_0) + \mu_2(x_0) + \mu_3(x_0) + \dots + \mu_n(x_0) + \dots$ 常数项级数, 收敛点 x_0
 $S(x) = \mu_1(x) + \mu_2(x) + \mu_3(x) + \dots + \mu_n(x) + \dots$, 和函数
 部分和 $S_n(x), \lim_{n \rightarrow \infty} S_n(x) = S(x)$, 余项 $r_n(x) = S(x) - S_n(x), \lim_{n \rightarrow \infty} r_n(x) = 0$

12.3.2 幂级数及其收敛性

$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$ 幂级数

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ $x = x_0$, 幂级数收敛, if $|x| < |x_0|$, 幂级数绝对收敛
 $x = x_0$, 幂级数发散, if $|x| > |x_0|$, 幂级数发散

$$\lim_{n \rightarrow \infty} a_n x_0^n = 0 \Rightarrow |a_n x_0^n| \leq M \Rightarrow |a_n x^n| = |a_n x_0^n| \cdot \left| \frac{x}{x_0} \right|^n$$

$|x| < R$, 幂级数绝对收敛

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 不是一点收敛, 也不是整个数轴收敛, $\exists R > 0, |x| > R$, 幂级数发散

$|x| = \pm R$, 幂级数可能收敛

幂级数 $\sum_{n=0}^{\infty} a_n x^n, \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$, 收敛半径 $R = \begin{cases} \frac{1}{\rho}, & \rho \neq 0 \\ +\infty, & \rho = 0 \\ 0, & \rho = +\infty \end{cases}$

$$\left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |x| = \rho |x| \leq 1$$

12.3.3 幂级数的运算

幂级数 $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$
 $b_0 + a_1 x + b_2 x^2 + \dots + b_n x^n + \dots$

加减 $(a_0 \pm b_0) + (a_1 \pm b_1)x + (a_2 \pm b_2)x^2 + \dots + (a_n \pm b_n)x^n + \dots$ 区间取小

乘 $(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots) \cdot (b_0 + a_1 x + b_2 x^2 + \dots + b_n x^n + \dots)$

$a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \dots + (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0)x^n + \dots$ 区间取小

幂级数的柯西乘积

除 $\frac{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots}{b_0 + a_1 x + b_2 x^2 + \dots + b_n x^n + \dots} = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$ 区间取小

$$a_0 = c_0 b_0$$

$$a_1 = c_0 b_1 + c_1 b_0$$

$$a_2 = c_0 b_2 + c_1 b_1 + c_2 b_0$$

\dots

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $S_n(x)$ 在收敛域 I 上连续

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $S_n(x)$ 在收敛域 I 上可积, 所得收敛同半径

$$\int_0^x S(t) dt = \int_0^x \left[\sum_{n=0}^{\infty} a_n t^n \right] dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \quad (x \in I)$$

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $S_n(x)$ 在收敛域 I 上可导, 所得收敛同半径

$$S'(x) = \left(\sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad (x \in I)$$

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $S_n(x)$ 在收敛域 I 上具有任意阶导数

12.4 函数展开成幂级数

$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$, $x \in U(x_0)$ 泰勒级数 (极限), 泰勒展开式

函数 $f(x)$ 在 $U(x_0)$ 内具有各阶导数, $f(x)$ 能展开成泰勒级数 \Leftrightarrow 余项满足 $\sum_{n=0}^{\infty} R_n(x) = 0$

$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) (x - 0)^n$, $x \in U(0)$ 麦克劳林级数 (极限), 麦克劳林展开式

$$\sin^{(n)} x = \sin(x + n\pi)$$

$$\begin{cases} e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, (-\infty < x < \infty) \\ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, (-\infty < x < \infty) \\ \frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n, (-1 < x < 1) \end{cases}$$

已知展开, 运算得未知

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!} x^n + \dots, (-1 < x < 1) \text{ 二项展开式}$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2} \cdot -\frac{1}{2}}{2!} x^2 + \dots + \frac{\frac{1}{2} \cdot -\frac{1}{2} \dots \frac{3-2n}{2}}{n!} x^n + \dots = \frac{\prod(3-2n)}{(2n)!!} x^n, (-1 < x < 1) \text{ 二项展开式}$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{-\frac{1}{2} \cdot -\frac{3}{2}}{2!} x^2 + \dots + \frac{-\frac{1}{2} \cdot -\frac{1}{2} \dots \frac{1-2n}{2}}{n!} x^n + \dots = \frac{\prod(1-2n)}{(2n)!!} x^n, (-1 < x < 1) \text{ 二项展开式}$$

$$F(x) = (1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!} x^n + \dots$$

$$F'(x) = m \left[1 + \frac{m-1}{1} x + \dots + \frac{(m-1)\dots(m-n+1)}{(n-1)!} x^{n-1} \dots \right]$$

$$xF'(x) = m \left[x + \frac{m-1}{1} x^2 + \dots + \frac{(m-1)\dots(m-n+1)}{(n-1)!} x^n \dots \right]$$

证明: $(1+x)F'(x) = m \left[1 + mx + \frac{* (m-n+1)}{(n-1)!} x^{n-1} + \frac{*}{(n-2)!} x^{n-1} + \frac{*1(m-n)}{(n)!} x^n + \frac{*1}{(n-1)!} x^n + \dots \right]$

$$(1+x)F'(x) = m \left[1 + mx + \frac{m*}{(n-1)!} x^{n-1} + \frac{m*1}{n!} x^n + \dots \right] = mF(x)$$

$$\varphi(x) = \frac{F(x)}{(1+x)^m}, \varphi(0) = F(0) = 1$$

$$\varphi'(x) = \frac{F'(x)(1+x)^m - m(1+x)^{m-1}F(x)}{(1+x)^{2m}} = \frac{(1+x)^{m-1}[(1+x)F'(x) - mF(x)]}{(1+x)^{2m}} = 0$$

$$\varphi(x) = 1, F(x) = (1+x)^m$$

12.5 函数的幂级数展开应用

12.5.1 近似计算

12.5.2 微分方程的幂级数解法

$$\frac{dy}{dx} = f(x, y), y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' + P(x)y' + Q(x)y = 0, P, Q \text{ 定义域内可展开为幂级数, 方程解形如 } y = \sum_{n=0}^{\infty} a_n x^n$$