This is Start

Vine

2022 年 7 月 2 日

目录

1 函數與極限

1.1 映射與函數

1.1.1 映射

概念

非空集合 X, Y 法則 f $f: X \to Y$

$D_f = X$ $R_f = f(X) \subset Y$

$R_f = Y$ 滿射 $x_1 \neq x_2, f(x_1) \neq f(x_2)$ 單射

算子 非空集，數集 (泛函) 非空集，非空集 (變換) 實數集/子集，實數集 (函數)

逆映射與復合映射

$g: R_f \to X$

對每個 $y \in R_f$ $g(y) = x$ g 稱為 f 的逆映射，記作 f^{-1}

$g: X \to Y_1,$ $f: Y_2 \to Z$ $Y_1 \subset Y_2$

$f \circ g: X \to Z$ $f \circ g(x) = g[f(x)]$ $x \in X$

1.1.2 函數

概念

數集 $D \in R,$ $f: D \to R$ 為定義在 D 上的函數，簡記 $y = f(x), x \in D$

自然定義域 $y = |x|$ $y = sgn(x)$ $y = [x]$ $y = f(x) \begin{cases} 2\sqrt{x}, & 0 < x <= 1 \\ 1 + x, & 1 < x \end{cases}$

特性

有界性 $f(x) <= K_1$ $f(x) >= K_2$ $|f(x)| <= M$

單調性 $x_1 < x_2,$ $f(x_1) < f(x_2)$

奇偶性 $f(-x) = f(x)$ $f(-x) = -f(x)$

週期性 $f(x + l) = f(x)$

狄利克雷函數 $D(x) = \begin{cases} 1, x \in Q \\ 0, x \in Q^c \end{cases}$

反函數與復合函數

函數 $f: D \to f(D)$ 是單射 逆映射 $f^{-1}: f(D) \to D$ f^{-1} 稱為 f 的反函數

$y = f(\mu), D_f$ $\mu = g(x), D_g, R_g, R_g \in D_f$ $y = f[g(x)], x \in D_f$

函数的运算

$$f(x), g(x), D = D_f \cap D_g \Rightarrow \begin{cases} (f \pm g)(x) = f(x) \pm g(x), & x \in D \\ (f.g)(x) = f(x).g(x), & x \in D \\ \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, & x \in D \end{cases}$$

初等函数

$$\begin{cases} \text{幂函数} y = x^{\mu} & (\mu \in R) \\ \text{指数函数} y = a^x & (a > 0 \quad a \neq 1) \\ \text{对数函数} y = \log_a x & (a > 0 \quad a \neq 1, \quad \log_e x \to \ln x) \\ \text{三角函数} y = \sin(x) & \\ \text{反函数} y = \arcsin(x) & \end{cases}$$

基本初等函数 $\to$ 初等函数

$$\begin{cases} \text{双曲正弦} & sh\,x = \frac{e^x - e^{-x}}{2} & \text{奇函数} \\ \text{双曲余弦} & ch\,x = \frac{e^x + e^{-x}}{2} & \text{偶函数} \\ \text{双曲正切} & th\,x = \frac{sh\,x}{ch\,x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} & \text{奇函数} \end{cases} \Rightarrow \begin{cases} \text{反双曲正弦} & arsh\,x = \ln\left(x + \sqrt{x_2 + 1}\right) & \text{奇函数} \\ \text{反双曲余弦} & arch\,x = \ln\left(x + \sqrt{x_2 - 1}\right) & \text{非奇非偶函数} \\ \text{反双曲正切} & arth\,x = \frac{1}{2} \ln \frac{1+x}{1-x} & \text{奇函数} \end{cases}$$

1.2 數列的極限

$\forall\varepsilon>0,\exists Z^{+},|x_n-a|<\varepsilon \qquad (n>Z^{+})$

1.2.1 定义

法则, 每个 $n\in N_{+},x_n\to x_1,x_2,x_3,\ldots,x_n,\cdots\to$ 数列 $\{x_n\}$
x_n 一般项 $\left(\frac{n}{n+1},2^n,\frac{1}{2^n}\right)$
$\forall\varepsilon>0,\exists$正整数$N,n>N$时$,|x_n-a|<\varepsilon$

1.2.2 收敛数列的性质

唯一性
有界性 $|x_n|<=M$
保号性 $lim_{n\to\infty}=a>0\ (a<0),\exists$正整数$N,$当$n>N$有$x_n>0\ (x_n<0)$
子数列收敛 $\{x_n\},\{x_{n_k}\}$

1.3 函數的極限

1.3.1 定义

$x_n=f(n),n\in N^{+},n\to\infty$ 数列极限
$x\to x_0,x\to\infty$ 函数极限
自变量趋于有限值
邻域开区间 $U(x_0,\delta),U^{\circ}(x_0,\delta)$
$\forall\varepsilon>0,\exists\delta,$当$0<|x-x_0|<\delta,|f(x)-a|<\varepsilon\Leftrightarrow\lim_{x\to x_0}f(x)=a$
$\forall\varepsilon>0,\exists\delta,$当$x_0-\delta<x<x_0,|f(x)-a|<\varepsilon\Leftrightarrow\lim_{x\to x_0^-}f(x)=a$
$\forall\varepsilon>0,\exists\delta,$当$x_0<x<x_0+\delta,|f(x)-a|<\varepsilon\Leftrightarrow\lim_{x\to x_0^+}f(x)=a$
自变量趋于无穷大
$\forall\varepsilon>0,\exists\delta,$当 $|x|>\delta,|f(x)-a|<\varepsilon\Leftrightarrow\lim_{x\to\infty}f(x)=a$

1.3.2 性质

唯一性
局部有界
$\lim_{x\to x_0}f(x)=a\Rightarrow\exists M>0,\delta>0,0<|x-x_0|<\delta$时$,|f(x)|<=M$
局部保号
$\lim_{x\to x_0}f(x)=a>0\Rightarrow\exists\delta>0,0<|x-x_0|<\delta$时$,f(x)>0$
$\lim_{x\to x_0}f(x)=a\neq0\Rightarrow\exists\delta>0,0<|x-x_0|<\delta$时$,|f(x)|>\frac{|a|}{2}$
$\exists\delta>0,0<|x-x_0|<\delta$时$f(x)>=0,$且 $\lim_{x\to x_0}f(x)=a\Rightarrow a>=0$
函数极限与数列极限
$\lim_{x\to x_0}f(x)=a\Leftrightarrow$ 所有的 $\lim_{n\to\infty}x_n=x_0,$有 $\lim_{n\to\infty}f(x_n)=a$

1.4 無窮小與無窮大

1.4.1 无穷小

$\lim_{x\to x_0}f(x)=0\Rightarrow f(x)$ 为$x\to x_0$时的无穷小
$\lim_{x\to x_0}f(x)=A\Rightarrow f(x)=A+\alpha$

1.4.2 无穷大

$\lim_{x\to x_0}|f(x)|>$ 任意正数$M \Rightarrow f(x)$ 为$x\to x_0$时的无穷大 记为 $\lim_{x\to x_0}f(x)=\infty$
$\lim_{x\to x_0}f(x)=0\Rightarrow \lim_{x\to x_0}\frac{1}{f(x)}=\infty$

1.5 極限運算法則

1.5.1 兩個無窮小的和

有限個無窮小的和

1.5.2 有界函數與無窮小的乘積

常數與無窮小的乘積
有限個無窮小的乘積

1.5.3 $\lim f(x)=A,\lim g(x)=B$

$\lim[f(x)\pm g(x)]=\lim f(x)\pm\lim g(x)=A\pm B$
$\lim[f(x)\cdot g(x)]=\lim f(x)\cdot\lim g(x)=A\cdot B$
$\lim\frac{f(x)}{g(x)}=\frac{\lim f(x)}{\lim g(x)}=\frac{A}{B}\quad(B\neq 0)$
$\lim[cf(x)]=c\lim(x)$
$\lim[x]^n=[f(x)]^n$

1.5.4 數列 $\{x_n\},\{y_n\},\lim_{n\to\infty}x_n=A,\lim_{n\to\infty}y_n=B$

$\lim_{n\to\infty}(x_n\pm y_n)=A\pm B$
$\lim_{n\to\infty}(x_n\cdot y_n)=A\cdot B$
$\lim_{n\to\infty}\frac{x_n}{y_n}=\frac{A}{B}$

1.5.5 $\psi(x)>=\Psi(x),\lim\limits_{n\to\infty}\psi(x)=A,\lim\limits_{n\to\infty}\Psi(x)=B\Rightarrow A>=B$

1.5.6 復合函數極限

$\lim\limits_{x\to x_0}g(x)=\mu_0,\lim\limits_{\mu\to\mu_0}f(\mu)=A,\exists\delta_0>0,x\in\mathring{U}(x_0,\delta_0),g(x)\neq\mu_0\Rightarrow\lim\limits_{x\to x_0}f[g(x)]=\lim\limits_{\mu\to\mu_0}f(\mu)=A$

1.6 极限存在准则 两个重要极限

I 数列 $\{x_n\},\{y_n\},\{z_n\}$ 满足

$$\left.\begin{array}{l}\exists n_0\in\mathbf{N}^+,n>n_0\text{时},y_n\leqslant x_n\leqslant z_n\\ \lim\limits_{n\to\infty}y_n=a,\lim\limits_{n\to\infty}x_n=a\end{array}\right.\Rightarrow\{x_n\}\text{ 极限存在, 且 }\lim\limits_{n\to\infty}x_n=a$$

$$\left.\begin{array}{l}x\in\mathring{U}(x_0,r)\text{ 时},g(x)\leqslant f(x)\leqslant h(x)\\ \lim\limits_{x\to x_0}g(x)=a,\lim\limits_{x\to x_0}h(x)=a\end{array}\right.\Rightarrow\text{极限}\lim\limits_{x\to x_0}f(x)\text{ 存在, 且等于}a$$

II 单调有界数列必有极限
$x_1\leqslant x_2\leqslant x_3\leqslant\cdots\leqslant x_n\leqslant x_{n+1}\leqslant\ldots$ 单调递增数列
x_0的某个左邻域内, $f(x)$ 单调有界, 则$f\left(x_0^-\right)$ 必定存在
柯西极限存在准则 $\{x_n\}$ 收敛 $\Leftrightarrow\forall\varepsilon>0,\exists N^+$, 当$m>N,n>N$时, $|x_n-x_m|<\varepsilon$

1.7 無窮小的比較

$$\begin{array}{ll} \lim\frac{\beta}{\alpha}=0 & \beta\text{是}\alpha\text{的高階無窮小, 記為}\beta=o\left(\alpha\right) \\ \lim\frac{\beta}{\alpha}=\infty & \beta\text{是}\alpha\text{的低階無窮小} \\ \lim\frac{\beta}{\alpha}=c\neq 0 & \beta\text{是}\alpha\text{的同階無窮小} \\ \lim\frac{\beta}{\alpha^k}=c\neq 0 & \beta\text{是}\alpha\text{的 k 階無窮小} \\ \lim\frac{\beta}{\alpha}=1 & \beta\text{是}\alpha\text{的等價無窮小, 記為}\alpha\sim\beta \end{array}$$

α,β是等價無窮小 $\Leftrightarrow \beta=\alpha+o\left(\alpha\right)$

$\alpha\sim\widetilde{\alpha},\beta\sim\widetilde{\beta},\exists\lim\frac{\widetilde{\alpha}}{\widetilde{\beta}}\Rightarrow\lim\frac{\alpha}{\beta}=\lim\frac{\widetilde{\alpha}}{\widetilde{\beta}}$

1.8 函數的連續性與間斷點

1.8.1 函數的連續性

$\Delta\mu=\mu_2-\mu_1,x\in U\left(x_0\right),\Delta y=f\left(x_0+\Delta x\right)-f\left(x_0\right)$

$$\begin{array}{ll} f\left(x\right),x\in U\left(x_0\right),\lim\limits_{\Delta x\to 0}\Delta y=\lim\limits_{\Delta x\to 0}\left[f\left(x_0+\Delta x\right)-f\left(x_0\right)\right]=0 & ,\text{稱}f\left(x\right)\text{在}x_0\text{連續} \\ f\left(x\right),x\in U\left(x_0\right),\lim\limits_{x\to x_0}f\left(x\right)=f\left(x_0\right) & ,\text{稱}f\left(x\right)\text{在}x_0\text{連續} \\ \forall\varepsilon>0,\exists\delta>0,\text{當}\left|x-x_0\right|<\delta\text{時, 有}\left|f\left(x\right)-f\left(x_0\right)\right|<\varepsilon & ,\text{稱}f\left(x\right)\text{在}x_0\text{連續} \end{array}$$

$f\left(x_0^-\right)=f\left(x_0\right)$, 左連續 $\quad f\left(x_0^+\right)=f\left(x_0\right)$, 右連續

區間上每一點都連續的函數, 函數在該區間上連續, 包含端點時右端點左連續, 左端點右連續

$$\begin{cases} \text{有理整函數}\left(\text{多項式}\right), \\ \text{有理分式函數},F\left(x\right)=\frac{P(x)}{Q(x)} \\ \text{冪函數},\sqrt{x} \\ \text{三角函數},\sin x \end{cases}$$

1.8.2 函數的間斷點

$$f\left(x\right),x\in\mathring{U}\left(x_0\right)\begin{cases} (1)\ x=x_0\text{處沒定義} \\ (2)\ x=x_0\text{處有定義，但}\lim\limits_{x\to x_0}f\left(x\right)\text{不存在} \\ (3)\ x=x_0\text{處有定義，且}\lim\limits_{x\to x_0}f\left(x\right)\text{存在, 但}\lim\limits_{x\to x_0}f\left(x\right)\neq f\left(x_0\right) \end{cases}$$

左右極限存在，第一類間斷點 (可去，跳躍)

不是第一類間斷點，第二類間斷點 (無窮，震蕩)

1.9 連續函數的運算與初等函數的連續性

1.9.1 連續函數的和, 差, 積, 商的連續性

函數$f\left(x\right),g\left(x\right)$在$x_0$連續 $\Rightarrow f\pm g,f\cdot g,\frac{f}{g}$在點$x_0$連續

1.9.2 反函數與復合函數的連續性

$f\left(x\right)$在D_f單增, $f^-\left(x\right)$在R_f單增

$$\left.\begin{array}{l} y=f\left(\mu\right),\mu=g\left(x\right) \\ y=f\left[g\left(x\right)\right],\mathring{U}\left(x_0\right)\in D_{f\circ g} \\ \lim\limits_{x\to x_0}g\left(x\right)=\mu_0,f\left(\mu\right)\text{在點}\mu=\mu_0\text{處連續} \end{array}\right\}\Rightarrow\lim_{x\to x_0}f\left[g\left(x\right)\right]=\lim_{\mu\to\mu_0}f\left(\mu\right)=f\left(\mu_0\right)$$

$$\left.\begin{array}{l} y=f\left(\mu\right),\mu=g\left(x\right) \\ y=f\left[g\left(x\right)\right],\mathring{U}\left(x_0\right)\in D_{f\circ g} \\ \mu=g\left(x\right)\text{在點}x=x_0\text{處連續, 且}g\left(x_0\right)=\mu_0 \\ f\left(\mu\right)\text{在點}\mu=\mu_0\text{處連續} \end{array}\right\}\Rightarrow f\left[g\left(x\right)\right]\text{在點}x=x_0\text{處連續}$$

1.9.3 初等函數的連續性

基本初等函數定義域內連續
初等函數定義區間 (定義域內的區間) 連續

1.10 閉區間上連續函數的性質

1.10.1 最值定理

$f(x)$ 在區間 $[a,b]$ 連續, $\exists M>0, \forall x \in [a,b], |f(x)| \leqslant M$

1.10.2 零點定理與介值定理

$f(x_0)=0, x_0$稱為函數$f(x)$的零點
$f(x)$ 在區間 $[a,b]$ 連續, $f(a)\cdot f(b)>0, \exists$至少一點$\xi \in (a,b)$, 使得$f(\xi)=0$
$f(x)$ 在區間 $[a,b]$ 連續, $f(x)=A, f(b)=B, \forall C \in (A,B)$, $\exists$至少一個$\xi \in (a,b)$, 使得$f(\xi)=C$
$f(x)$ 在區間 $[a,b]$ 連續, $R_f=[m,M], m=f(x)_{min}, M=f(x)_{max}$

1.10.3 一致连续性

$x \in R_f, \forall \varepsilon, \exists \delta$, 使得$\forall x_1, x_2 \in R_f$, 当$|x_1-x_2|<\delta$时, $|f(x_1)-f(x_2)|<\delta$
如果函数在 $f(x)$ 在闭区间 $[a,b]$ 连续, 那么他在该区间一定连续

2 導數與微分

2.1 導數概念

2.1.1 引例

直線運動的速度

$s=f\left(t\right)$

$\frac{s-s_0}{t-t_0}=\frac{f(t)-f(t_0)}{t-t_0}$ 平均速度

$v=\lim\limits_{t\rightarrow t_0}\dfrac{f\left(t\right)-f\left(t_0\right)}{t-t_0}$ 瞬時速度

切線問題

$\tan\varphi=\frac{y-y_0}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}$ 割線斜率

$k=\lim\limits_{x\rightarrow x_0}\dfrac{f\left(x\right)-f\left(x_0\right)}{x-x_0}$ 切線斜率

2.1.2 導數的定義

函數在一點處的導數與導函數

$\lim\limits_{x\rightarrow x_0}\dfrac{f\left(x\right)-f\left(x_0\right)}{x-x_0}$

$\lim\limits_{\Delta x\rightarrow 0}\dfrac{\Delta y}{\Delta x},\lim\limits_{\Delta x\rightarrow 0}\dfrac{f\left(x_0+\Delta x\right)-f\left(x_0\right)}{\Delta x}$

$$\begin{aligned}f^{'}\left(x_0\right) &=\lim_{\Delta x\rightarrow 0}\tfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\\ &=y^{'}|_{x=x_0}\\ &=\tfrac{d\,y}{d\,x}|_{x=x_0}\\ &=\tfrac{d\,f(x)}{d\,x}|_{x=x_0}\end{aligned}$$

$\Delta x\rightarrow 0$時, $\frac{\Delta y}{\Delta x}\rightarrow\infty\Rightarrow$ 不可導

$y\prime=\lim_{\Delta x\rightarrow 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},f^{'}\left(x\right),\frac{d\,y}{d\,x},\frac{d\,f(x)}{d\,x}$ 導函數

$f^{'}\left(x_0\right)=f^{'}\left(x\right)|_{x=x_0}$

求導舉例

2.1.3 導數的幾何意義

$f^{'}\left(x_0\right)=\infty,x=x_0$

$y-y_0=f^{'}\left(x_0\right)\left(x-x_0\right)$ 切線

$y-y_0=-\frac{1}{f^{'}(x_0)}\left(x-x_0\right)$ 法線

2.1.4 可導與連續性關係

$$\lim_{\Delta x\rightarrow 0}\frac{\Delta y}{\Delta x}=f^{'}\left(x\right)\Rightarrow\frac{\Delta y}{\Delta x}=f^{'}\left(x\right)+\alpha\Rightarrow\Delta y=f^{'}\left(x\right)\Delta x+\alpha\Delta x\Rightarrow \text{可導必連續}$$

2.2 函數求導法則

2.2.1 函數和差積商的求導法則

$$u^{'}\left(x\right),v^{'}\left(x\right)\Rightarrow\begin{array}{l}\left(u\pm v\right)=u^{'}\pm v^{'}\\ \left(uv\right)^{'}=u^{'}v+uv^{'},\left(cu\right)^{'}=cu^{'},\left(uvw\right)^{'}=u^{'}vw+uv^{'}w+uvw^{'}\\ \left(\frac{u}{v}\right)^{'}=\frac{u^{'}v-uv^{'}}{v^2}\end{array}$$

2.2.2 反函數求導法則

$x=f(y),f^{'}(y)\neq 0\rightarrow y=f^{-1}(x)$ 可導, $[f^{-1}(x)]^{'}=\frac{1}{f^{'}(y)},\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$

2.2.3 復合函數求導法則

$u=g(x)$ x點可導, $y=f(u)$ u點可導 $\rightarrow y=f[g(x)]$ x點可導, $\frac{dy}{dx}=f^{'}(u)\cdot g^{'}(x),\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

2.2.4 基本求導法則與導數公式

基本初等函數函數求導公式
和差商積求導法則
反函數求導法則
復合函數求導法則

2.3 高階導數

$\frac{d}{dx}\left(\frac{dy}{dx}\right),\frac{d^2y}{dx^2},y^{''}(x)$
$y^{'''},y^{(4)},\ldots,y^{(n)}\rightarrow\frac{d^3y}{dx^3},\frac{d^4y}{dx^4},\ldots,\frac{d^ny}{dx^n}$
$(u+v)^n=\sum\limits_{k=0}^{n}C_n^k u^{(n-k)}v^{(k)}$

2.4 隱函數, 參數方程確定的函數的導數　相關變化率

隱函數導數
$y=\sin x$　顯函數
$x+y^3-1=0$　隱函數,　$y=\sqrt[3]{1-x}$　隱函數顯化
參數方程確定的函數的導數
$$\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}\rightarrow\frac{dy}{dx}=\frac{\psi^{'}(t)}{\varphi^{'}(t)}$$
相關變化率
$\frac{dy}{dt},\frac{dx}{dt}$存在關係

2.5 函數的微分

微分的定義
$\Delta A=2x_0\Delta x+(\Delta x)^2$
$x\in U(x_0),\Delta y=A\Delta x+o(\Delta x)\rightarrow y$在$x_0$點可微, $A\Delta x$是y在x_0點相應于Δx的微分記為dy
y在任意點x的微分, 函數的微分, 記作dy
Δx自變量的微分, 記作dx
微分的幾何意義
初等函數的微分公式, 與微分運算法則

1. 基本初等
2. 和差商積
3. 復合函數

微分在近似計算中的應用

函數的近似運算

$\Delta y \approx dy = f^{'}\left(x_0\right)\Delta x$

誤差估計

A精確值$,a$測量值$,\left|A-a\right|$絕對誤差$,\frac{\left|A-a\right|}{\left|a\right|}$相對誤差$,\left|A-a\right| \leqslant \delta_A$絕對誤差極限$,\frac{\delta_A}{\left|a\right|}$相對誤差極限

3 微分中值定理與導數的應用

3.1 微分中值定理

費馬引裡 $x \in U(x_0), f'(x_0), \forall x \in U(x_0), f(x) \leqslant f(x_0)\, or f(x) \geqslant f(x_0) \to f'(x_0) = 0$駐點, 穩定點, 臨界點

儸爾定理 $[a,b]$ 連續, (a,b) 可導, $f(a) = f(b) \to \exists$至少一個$\xi \in (a,b)$, 使得$f'(\xi) = 0$

拉格朗日中值定理 $[a,b]$ 連續, (a,b) 可導 $\to \exists$至少一個$\xi \in (a,b)$, 使得$f(b) - f(a) = f'(\xi)(b-a)$

證明構造輔助函數$\varphi(x)$

微分中值定理

$x \in I$連, 導, $f'(x) = 0 \to x \in I, f(x) \equiv C$ (常數)

柯西中值定理 $[a,b]$ 連續, (a,b) 可導, $\forall x \in (a,b), F'(x) \neq 0 \to \exists$至少一點$\xi \in (a,b)$ 使得 $\frac{f(b)-f(a)}{F(b)-F(a)} = \frac{f'(\xi)}{F'(\xi)}$

證明構造輔助函數$\varphi(x)$

3.2 諾必達法則

$$\left.\begin{array}{l} when\, x \to a, f(x)\ and\, F(x) \to 0 \\ when\, x \in U(a), \exists f'(x)\ and\, F'(x), F'(x) \neq 0 \\ \exists \lim\limits_{x\to a} \dfrac{f'(x)}{F'(x)}, or\, quip\, \infty \end{array}\right\} \Rightarrow \lim_{x\to a}\frac{f(x)}{F(x)} = \lim_{x\to a}\frac{f'(x)}{F'(x)}$$

$$\left.\begin{array}{l} when\, x \to \infty, f(x)\ and\, F(x) \to 0 \\ when\, |x| > N, \exists f'(x)\ and\, F'(x), F'(x) \neq 0 \\ \exists \lim\limits_{x\to\infty} \dfrac{f'(x)}{F'(x)}, or\, quip\, \infty \end{array}\right\} \Rightarrow \lim_{x\to\infty}\frac{f(x)}{F(x)} = \lim_{x\to\infty}\frac{f'(x)}{F'(x)}$$

3.3 泰勒公式

$$\begin{array}{l} \text{if } x = x_0\, \exists f^{(n)}(x_0) \\ \text{then } \exists U(x_0)\ \forall x \in U(x_0) \end{array} \text{使得} \begin{array}{rl} f(x) & = P_n(x) + R_n(x) \\ & = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x) \\ & \text{其中} R_n(x) = o((x-x_0)^n)\ (\text{佩亚诺余项}, x_0 = x \text{麦克劳林}) \end{array}$$

$$\begin{array}{l} \text{if } x = x_0\, \exists f^{(n+1)}(x_0) \\ \text{then } \exists U(x_0)\ \forall x \in U(x_0) \end{array} \text{使得} \begin{array}{rl} f(x) & = P_n(x) + R_n(x) \\ & = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x) \\ & \text{其中} R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \xi \in (x_0, x)\ (\text{拉格朗日余项}, x_0 = x \text{麦克劳林}) \end{array}$$

3.4 函数单调性与曲线凹凸性

3.4.1 函数单调性的判定法

$$[a,b] \text{ 连续}, (a,b) \text{ 可导} \begin{array}{l} \text{if } x \in (a,b), f'(x) \geqslant 0, \text{有限点取等号 then } y = f(x) \text{ 在 } [a,b] \text{ 单增} \\ \text{if } x \in (a,b), f'(x) \leqslant 0, \text{有限点取等号 then } y = f(x) \text{ 在 } [a,b] \text{ 单减} \end{array}$$

驻点与无定义点

3.4.2 曲线的凹凸性与拐点

$$f(x) \text{ 在} x \in I \text{上连续}, \begin{array}{ll} if & \forall x_1, x_2 \in I, \text{恒有} f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2} \text{ then } f(x) \text{ 在} I \text{内图像凹} \\ if & \forall x_1, x_2 \in I, \text{恒有} f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2} \text{ then } f(x) \text{ 在} I \text{内图像凸} \end{array}$$

$$f(x) \text{ 在 } [a,b] \text{ 连续, 在 } (a,b) \text{ 一阶导, 二阶导} \begin{array}{llll} if & x \in (a,b) & f''(x) > 0, then & f(x) \text{ 在 } [a,b] \text{ 内图像凹} \\ if & x \in (a,b) & f''(x) < 0, then & f(x) \text{ 在 } [a,b] \text{ 内图像凸} \end{array}$$

$f^{''}(x)=0$点 and $f^{''}(x)$不存在点, 点两侧异号为拐点

3.5 函数的极值与最大值最小值

函数的极值及其求法

$f(x) \quad x\in \mathring{U}(x_0), if \quad \forall x\in \mathring{U}(x_0) \quad f(x)<f(x_0), then \quad f(x_0)$是函数$f(x)$的一个极大值
$f(x) \quad x\in \mathring{U}(x_0), if \quad \forall x\in \mathring{U}(x_0) \quad f(x)>f(x_0), then \quad f(x_0)$是函数$f(x)$的一个极小值

$if \quad f(x)$在x_0处可导, 取得极值, $then \quad f^{'}(x)=0$

$$\begin{array}{c} f(x)\text{在}x_0\text{处连续} \\ \text{在}\mathring{U}(x_0,\delta)\text{内可导} \end{array} \begin{array}{l} if \quad x\in(x_0-\delta,x_0), f^{'}(x)>0 \quad x\in(x_0,x_0+\delta), f^{'}(x)<0, then \quad f(x)\text{在}x_0\text{处取得极大值} \\ if \quad x\in(x_0-\delta,x_0), f^{'}(x)<0 \quad x\in(x_0,x_0+\delta), f^{'}(x)>0, then \quad f(x)\text{在}x_0\text{处取得极小值} \\ if \quad x\in\mathring{U}(x_0,\delta), f^{'}(x)\text{的符号保持不变}, then \quad f(x)\text{在}x_0\text{处没有极值} \end{array}$$

$f^{'}(x)=0$点 and $f^{'}(x)$不存在点, 点两侧异号为极值点

$$f(x)\text{在}x_0\text{处具有二阶导数}, f^{'}(x_0)=0, f^{''}(x_0)\neq 0 \begin{array}{l} if \quad f^{''}(x)<0, then \quad f(x)\text{在}x_0\text{处取得极大值} \\ if \quad f^{''}(x)<0, then \quad f(x)\text{在}x_0\text{处取得极大值} \end{array}$$

最大值最小值问题

$$\text{驻点值}a_{0,1,2,\ldots,n}, \quad \text{不可导点值}b_{0,1,2,\ldots,n}, \quad \text{端点值}f(a), f(b) \rightarrow \begin{array}{l} MAX=max\{a,b,f(a),f(b)\} \\ MIN=min\{a,b,f(a),f(b)\} \end{array}$$

3.6 函数图像的绘制

$f^{'}(x)=0, f^{''}(x)=0$点, 水平，铅锤

3.7 曲率

弧微分

$\frac{\Delta s}{\Delta x}=\sqrt{\left(\frac{\widehat{MM'}}{|MM'|}\right)\cdot\left[1+\left(\frac{\Delta y}{\Delta x}\right)^2\right]}$

$$\begin{aligned} \tfrac{ds}{dx} &= \lim_{\Delta x\to x}\frac{\Delta s}{\Delta x} \\ &= \sqrt{1\cdot\left[1+y'^2\right]} \end{aligned}$$

曲率及计算公式

$\overline{K}=\left|\frac{\Delta\alpha}{\Delta s}\right|$

$$\begin{aligned} K &= \lim_{\Delta s\to 0}\left|\frac{\Delta\alpha}{\Delta s}\right| \\ &= \frac{\left|\frac{d\alpha}{dx}\right|}{\frac{ds}{dx}} = \frac{\left|y^{''}\right|}{1+y'^2}/\sqrt{1+y'^2} = \begin{array}{ll} \text{直线} & 0 \\ \text{圆} & \frac{1}{a} \\ \text{参数方程} & \frac{\left|g^{''}f^{'}-g^{'}f^{''}\right|}{\left(g'^2+f'^2\right)^{\frac{3}{2}}} \end{array} \end{aligned}$$

曲率圆与曲率半径

$\rho=\frac{1}{K}$

曲率中心的计算公式　　渐屈线与渐伸线

$$\text{曲率中心}D(\alpha,\beta)\begin{cases} \alpha &= x-\frac{y^{'}\left(1+y'^2\right)}{y^{''}} \\ \beta &= y+\frac{1+y'^2}{y^{''}} \end{cases}$$

3.8 方程近似解

二分法

$[a,b]$ 连续, $f(a)\cdot f(b)<0$, $\exists$一个实根$\xi\in(a,b)$, $\xi_n=\frac{a+b}{2}$

切线法

$f(a)\cdot f(b)<0, x\in[a,b]\, f^{'}(x)\, f^{''}(x)$ 保持定号 $\rightarrow\exists$实根$\xi\in[a,b]$

$x_0=\mathrm{a\ or\ b}, f(x_0)\cdot f^{''}(x)>0, x_n$为过$x_{n-1}$的切线与$x$轴交点, $x_{n+1}=x_n-\frac{f(x_n)}{f^{'}(x_n)}$

割线法

初始值$x_0, x_1, x_{n+1}=x_n-\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}\cdot f(x_n)$

4 不定积分

4.1 不定积分的概念与性质

原函数与不定积分的概念

$if \quad \forall x\in I, F^{'}\left(x\right)=f\left(x\right), dF\left(x\right)=f\left(x\right)dx, then \quad F\left(x\right)$ 为 $f\left(x\right)$ 在区间 I 原函数

$if \quad x\in I, f\left(x\right)$ 连续, $then \quad \exists$ 可导函数 $F\left(x\right), \forall x\in I, F^{'}\left(x\right)=f\left(x\right)$ 连续函数一定有原函数

$$\int f\left(x\right)dx=F\left(x\right)+C\text{不定积分} \quad \begin{cases} \int & \text{积分符号} \\ f\left(x\right) & \text{被积函数} \\ f\left(x\right)dx & \text{被积表达式} \\ x & \text{积分变量} \end{cases}$$

基本积分表

...

不定积分的性质

$\int\left[f\left(x\right)+g\left(x\right)\right]dx=\int f\left(x\right)dx+\int g\left(x\right)dx$

$\int kf\left(x\right)dx=k\int f\left(x\right)dx$

4.2 积分换元法

第一类换元法

$if \quad F^{'}\left(\mu\right)=f\left(\mu\right), \mu=\varphi\left(x\right)$ 可导, $then \quad \int f\left[\varphi\left(x\right)\right]\varphi^{'}\left(x\right)dx=\left[\int f\left(\mu\right)d\mu\right]_{u=\varphi(x)}$

第二类换元法

$if \quad x=\varphi\left(t\right)$ 单调可导, $\varphi\left(t\right)\neq 0, f\left[\varphi\left(t\right)\right]\varphi^{'}\left(t\right)$ 具有原函数, $then \quad \int f\left(x\right)dx=\left[\int f\left(\varphi\left(t\right)\right)\varphi^{'}\left(t\right)dt\right]_{t=\varphi^{-1}(x)}$

4.3 分部积分法

$\int uv^{'}dx=uv-\int u^{'}vdx\to\int udv=uv-\int uvdu$

4.4 有理函数的积分

有理函数的积分

两多项式的商 $\frac{P(x)}{Q(x)}$ 称为有理函数, 有理分式

$P\left(x\right)$ 次数小于 $Q\left(x\right)$ 真分式, 否则假分式

假分式 = 多项式 + 真分式

可化为有理函数的积分举例

$\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\left(\frac{x}{2}\right)}$

$\cos x=\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

4.5 积分表的使用

...

5 定积分

5.1 定积分的概念与性质

5.1.1 定积分的概念与性质

曲边梯形的面积

$$\begin{aligned} A &\approx \sum_{i=1}^{n} f(\xi_i)\Delta x_i \\ &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i \left(\lambda = max\left\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\right\}, \Delta x_n = x_n - x_{n-1}\right) \end{aligned}$$

变速直线运动的路程

$$\begin{aligned} A &\approx \sum_{i=1}^{n} v(\tau_i)\Delta t_i \\ &= \lim_{\lambda\to 0}\sum_{i=1}^{n} v(\tau_i)\Delta t_i \left(\lambda = max\left\{\Delta t_1, \Delta t_2, \cdots, \Delta t_n\right\}, \Delta t_n = t_n - t_{n-1}\right) \end{aligned}$$

5.1.2 定积分的定义

$$\int_a^b f\{x\}\,dx = I = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i \begin{cases} f[x]\text{被积函数} \\ f[x]\,dx\text{被积表达式} \\ x\text{积分变量} \\ a\text{积分下限} \\ b\text{积分上限} \\ [a,b]\text{积分区间} \end{cases}$$

和式 (积分和)$\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$极限存在时, 极限$I$只与$f[x], [a,b]$有关

积分和存在, 函数$f(x)$可积

$[a,b]$连, $[a,b]$可积

$[a,b]$有界, 有限个间断点, $[a,b]$可积

5.1.3 定积分的近似计算

5.1.4 定积分的性质

$b=a, \int_a^b f(x)\,dx = 0; \quad a>b, \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

a,b常数, $\int_a^b [af(x)+bg(x)]\,dx = a\int_a^b f(x)\,dx + b\int_a^b g(x)\,dx$

$a<c<b, \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$积分区间可加 $\int_a^b 1dx = \int_a^b dx = b-a$

$if \quad x\in[a,b], f(x)\geqslant 0, then \quad \int_a^b f(x)\,dx \geqslant 0$

$if \quad x\in[a,b], f(x)\leqslant g(x), then \quad \int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$

$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx$

$m<f(x)<M, m(b-a) < \int_a^b f(x)\,dx < M(b-a)$

$if \quad f(x)$在$[a,b]$连续, $then \quad \exists$至少一个点$\xi\in[a,b], \int_a^b f(x)\,dx = f(\xi)(b-a)$

5.2 微积分基本公式

5.2.1 变速直线运动中位置函数与速度函数

$F'(x) = f(x), \int_a^b f(x)\,dx = F(b) - F(a)$

5.2.2 积分上限的函数及其导数

$if \quad f(x) \quad [a,b] \quad ,then \quad \Phi(x)=\int_a^x f(t)\,dt$在 $[a,b]$ 可导,$\Phi'(x)=\frac{d}{dx}\cdot\int_a^x f(t)\,dt=f(x)$
$if \quad f(x) \quad [a,b] \quad ,then \quad \Phi(x)=\int_a^x f(t)\,dt$是$f(x)$ 在 $[a,b]$ 上的原函数

5.2.3 牛顿-莱布尼兹

$if \quad x\in[a,b], F'(x)=f(x), \int_a^b f(x)\,dx=F(b)-F(a)$ 微积分基本公式

5.3 定积分的换元法和分部积分法

5.3.1 定积分的换元法

$f(x)$ 在 $[a,b]$ 连续,if $\begin{array}{l}\varphi(\alpha)=a,\varphi(\beta)=b,\\ \varphi(x)\text{ 在 }[a,b]\text{ 具有连续导数},\\ R_\varphi=[a,b]\end{array}$ $then \quad \int_a^b f(x)\,dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,dt$

$\int_a^b f[\varphi(x)]\varphi'(x)\,dx=\int_\alpha^\beta f(t)\,dt$

5.3.2 定积分的分布积分法

$\int_a^b uv'dx=[a,b]_a^b-\int_a^b u'vdx \qquad \int_a^b udv=[a,b]_a^b-\int_a^b vdu$

5.4 反常积分

5.4.1 无穷限的反常积分

$$\begin{array}{l}\text{无穷限的反常积分}\\ \text{(极限存在则收敛)}\\ F'(x)=f(x)\end{array}\begin{cases}\int_a^{+\infty} f(x)\,dx=\lim\limits_{t\to\infty}\displaystyle\int_a^t f(x)\,dx=\lim\limits_{t\to+\infty}F(t)-F(a)=I_1\\ \int_{-\infty}^{b} f(x)\,dx=\lim\limits_{t\to-\infty}\displaystyle\int_t^b f(x)\,dx=F(b)-\lim\limits_{t\to-\infty}F(t)=I_2\\ \int_{-\infty}^{+\infty} f(x)\,dx=\lim\limits_{t\to-\infty}\displaystyle\int_{-\infty}^0 f(x)\,dx+\lim\limits_{t\to+\infty}\int_0^{+\infty} f(x)\,dx=\lim\limits_{t\to+\infty}F(t)-\lim\limits_{t\to-\infty}F(t)=I_2+I_2\end{cases}$$

5.4.2 无界函数的反常积分

if $\begin{array}{l}f(x)\text{ 在 a 的任一领域无界}\\ x\in U(a,\delta),\forall\delta>0,M>0,f(x)>M\end{array}$ $,then \quad a$称为$f(x)$ 瑕点,$f(x)$ 积分称为瑕积分

$$\begin{array}{l}\text{无界函数的反常积分}\\ \text{(极限存在则收敛)}\\ F'(x)=f(x)\end{array}\begin{cases}\displaystyle\int_a^b f(x)\,dx=\lim_{t\to b^-}\int_a^t f(x)\,dx=\lim_{t\to b^-}F(t)-F(a) \quad x\in[a,b)\\ \displaystyle\int_a^b f(x)\,dx=\lim_{t\to a^+}\int_t^b f(x)\,dx=F(b)-\lim_{t\to a^+}F(t) \quad x\in(a,b]\\ \displaystyle\int_a^b f(x)\,dx=\lim_{t\to a^+}\int_t^c f(x)\,dx+\lim_{t\to b^-}\int_c^t f(x)\,dx=\lim_{t\to b^-}F(t)-\lim_{t\to a^+}F(t) \quad x\in(a,b)\end{cases}$$

5.5 反常积分的审敛法 Γ 函数

5.5.1 无穷限反常积分的审敛法

$f(x)$ 在$[a,+\infty)$连续,$f(x)\geqslant 0, if \quad F(x)=\int_a^x f(t)\,dt$在$[a,+\infty)$有上界,$then \quad \int_a^{+\infty} f(x)\,dx$收敛

$f(x), g(x)$ 在$[a,+\infty)$连续,$0\leqslant f(x)\leqslant g(x)$ $\begin{array}{l}if \quad \int_a^{+\infty} g(x)\,dx\text{收敛}, then \quad \int_a^{+\infty} f(x)\,dx\text{收敛}\\ if \quad \int_a^{+\infty} f(x)\,dx\text{发散}, then \quad \int_a^{+\infty} g(x)\,dx\text{发散}\end{array}$

$$f\left(x\right) 在 [a,+\infty)\left(a>0\right) 连续, f\left(x\right)\geqslant 0\ \begin{cases} if\quad \exists M>0,p>1,f\left(x\right)\leqslant \frac{M}{x^p}\left(a\leqslant x<+\infty\right),then\quad \int_a^{+\infty} f\left(x\right)dx 收敛\\ if\quad \exists N>0,p\leqslant 1,f\left(x\right)\geqslant \frac{N}{x^p}\left(a\leqslant x<+\infty\right),then\quad \int_a^{+\infty} f\left(x\right)dx 发散\end{cases}$$

$$f\left(x\right) 在 [a,+\infty) 连续, f\left(x\right)\geqslant 0\ \begin{cases} if\quad \exists p>1,\lim\limits_{x\to+\infty} x^p f\left(x\right)=c<+\infty,then\quad \displaystyle\int_a^{+\infty} f\left(x\right)dx 收敛\\ if\quad \exists p\leqslant 1,\lim\limits_{x\to+\infty} x^p f\left(x\right)=d>0\left(or+\infty\right),then\quad \displaystyle\int_a^{+\infty} f\left(x\right)dx 发散\end{cases}$$

$$f\left(x\right) 在 [a,+\infty) 连续, if\quad \int_a^{+\infty}\left|f\left(x\right)\right|dx 收敛, then\quad \int_a^{+\infty} f\left(x\right)dx 收敛,绝对收敛$$

5.5.2 无界函数反常积分的审敛法

$$f\left(x\right) 在 (a,b] 连续, f\left(x\right)\geqslant 0,x=a 为函数瑕点 \begin{cases} if\quad \exists M>0,p<1,f\left(x\right)\leqslant \frac{M}{(x-a)^p}\left(a\leqslant x<+\infty\right),then\quad \int_a^b f\left(x\right)dx 收敛\\ if\quad \exists N>0,p\geqslant 1,f\left(x\right)\geqslant \frac{N}{(x-a)^p}\left(a\leqslant x<+\infty\right),then\quad \int_a^b f\left(x\right)dx 发散\end{cases}$$

$$f\left(x\right) 在 (a,b] 连续, f\left(x\right)\geqslant 0,x=a 为函数瑕点 \begin{cases} if\quad \exists 0<p<1,\lim\limits_{x\to a^+}\left(x-a\right)^p f\left(x\right)=c<+\infty,then\quad \displaystyle\int_a^b f\left(x\right)dx 收敛\\ if\quad \exists p\geqslant 1,\lim\limits_{x\to a^+}\left(x-a\right)^p f\left(x\right)=d>0\left(or+\infty\right),then\quad \displaystyle\int_a^b f\left(x\right)dx 发散\end{cases}$$

5.5.3 Γ

$$\begin{aligned}&\Gamma\left(s\right)=\int_0^{\infty} e^{-x}x^{s-1}dx\\ &\Gamma\left(s+1\right)=s\Gamma\left(s\right)\left(分部积分法\right)\\ &\Gamma\left(1\right)=1,\Gamma\left(n+1\right)=n!\\ &\lim_{s\to0^+}\Gamma\left(s\right)=\lim_{s\to0^+}\frac{\Gamma\left(1+s\right)}{s}=+\infty\\ &\Gamma\left(s\right)\Gamma\left(1-s\right)=\frac{\pi}{\sin\pi s}\left(0<s<1,余元公式\right),\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}\\ &\Gamma\left(\tfrac{1}{2}\right)=\int_0^{\infty} e^{-x}x^{-\frac{1}{2}}dx \xlongequal{x=u^2} \int_0^{\infty} e^{-u^2}\left(u^2\right)^{-\frac{1}{2}}d\left(u^2\right)=2\int_0^{\infty} e^{-u^2}du\end{aligned}$$

6 定积分的应用

6.1 定积分的元素法

$$
\begin{aligned}
&A \approx \sum f(x)\,dx \\
&= \lim \sum f(x)\,dx = \int_a^b f(x)\,dx
\end{aligned}
$$

6.2 定积分在几何学上的应用

6.2.1 平面图形

直角坐标

矩形面积$A = xy$

$A = \int_a^b f(x)\,dx = \int_c^d f(y)\,dy$

极坐标

扇形面积$A = \frac{1}{2}R^2\theta$

$A = \int_a^b \frac{1}{2}\left[\rho(\theta)\right]^2 d\theta$

6.2.2 体积

旋转体体积

$V = \int_a^b \pi\left[f(x)\right]^2 dx$

平行截面积已知的立体体积

$V = \int_a^b A(x)\,dx = \int_c^d A(y)\,dy$

6.2.3 平面曲线的弧长

$if \quad \exists \sum_{i=1}^{n} \left|M_{i-1}^{i}\right|$极限存在$, then$　极限称为$\overset{\frown}{AB}$的弧长

光滑曲线弧是可求长的

$$
\begin{aligned}
ds &= \sqrt{(dx)^2 + (dy)^2} \\
&= \sqrt{(\varphi'(t)\,dt)^2 + (\psi'(t)\,dt)^2} = \sqrt{\varphi'(t)^2 + \psi'(t)^2}dt \\
&= \sqrt{[(\rho' \cos\theta - \rho\sin\theta)\,dt]^2 + [(\rho' \sin\theta + \rho\cos\theta)\,dt]^2} = \sqrt{(\rho)^2 + (\rho')^2}dt
\end{aligned}
$$

6.3 定积分在物理学上的应用

6.3.1 变力沿直线做功

$$
\begin{aligned}
W &= F \cdot s \\
&= \tfrac{kq^2}{r^2}dr
\end{aligned}
$$

6.3.2 水压力

$$
\begin{aligned}
p = \rho g h, P &= p \cdot A \\
&= \rho g x \cdot 2\sqrt{R^2 - x^2}dx
\end{aligned}
$$

6.3.3 引力

$$
\begin{aligned}
F &= G\tfrac{m_1 m_2}{r^2} \\
&= G\tfrac{m_1 \mu dy}{a^2 + y^2}
\end{aligned}
$$

7 微分方程

7.1 基本概念

$\begin{cases} \frac{dy}{dx} = 2x \\ x = 1, y = 2 \end{cases} \Rightarrow y = 2x + 1$
函數，函數導數，自變量關係的方程**微分方程**
最高階導數的階數**微分方程的階**
$F\left(x, y^{'}, \ldots, y^{(n)}\right) = 0$ **一般形式**
函數**微分方程的解**
函數含常數，常數個數同階數**微分方程的通解**
確定常數的通解**微分方程的通解**

7.2 可分離變量的微分方程

$\frac{dy}{dx} = 2x \Rightarrow dy = 2xdx \Rightarrow$
$\frac{dy}{dx} = 2xy^2 \Rightarrow \frac{dy}{y^2} = 2xdx \Rightarrow$
$g\left(y\right) dy = f\left(x\right) dx \Rightarrow y = \varphi\left(x\right)$
$G\left(y\right) = F\left(x\right) + C \Rightarrow y = \varPhi\left(x\right)$

7.3 齊次方程

$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$
$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$

7.4 一階線性微分方程

$\frac{dy}{dx} + P\left(x\right) y = Q\left(x\right)$ 關於函數，函數導數是一次方程**一階線性微分方程**
$Q\left(x\right) = 0$ 時**齊次方程**
$y = Ce^{-\int P(x)dx}$ **齊次通解**, 設 $C = \mu\left(x\right)$
$\mu\left(x\right) = \int Q\left(x\right) e^{\int P(x)\,dx}\,dx + C_2$
$y = \left(\int Q\left(x\right) e^{\int P(x)\,dx}\,dx + C_2\right) e^{-\int P(x)dx}$ **非齊次通解**
$\frac{dy}{dx} + P\left(x\right) y = Q\left(x\right) y^n$ **伯努利方程**
$\frac{dy}{dx}y^{-n} + P\left(x\right) yy^{-n} = Q\left(x\right) \xrightarrow{z = y^{1-n}, z^{'} = \left(1-n\right) y^{-n}\frac{dy}{dx}} \frac{z^{'}}{(1-n)} + P\left(x\right) z = Q\left(x\right)$ **一階線性微分方程**

7.5 可降階的高階微分方程

$y^{(n)} = f\left(x\right) \quad \Rightarrow \quad y^{(n-1)} = \int f\left(x\right)\,dx$
$y^{''} = f\left(x, y^{'}\right) \quad \xrightarrow{y^{'} = p, y^{''} = p^{'}} \quad p^{'} = f\left(x, p\right)$
$y^{''} = f\left(y, y^{'}\right) \quad \xrightarrow{y^{'} = p, y^{''} = p\frac{dp}{dy}} \quad p\frac{dp}{dy} = f\left(y, p\right) \quad \xrightarrow{p = y^{,} = \varphi\left(y, c_1\right)}$

7.6 高階性微分方程

二阶微分方程

$$\frac{d^2y}{dx^2} + P\left(x\right)\frac{dy}{dx} + Q\left(x\right) y = f\left(x\right)$$

解的结构
$y = C_1y_1\left(x\right) + C_2y_2\left(x\right)$ 是解

$y=C_1y_1(x)+C_2y_2(x)$ 无关特解是通解

$y=C_1y_1(x)+C_2y_2(x)+\cdots+C_ny_n(x)$ 无关特解是通解

$y=Y(x)+y^*(x)$ 齐次通，非齐特，非齐通

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=f_1(x)+f_2(x)$$

$y=y_1^*(x)+y_2^*(x)$ 特解，特解，特解

常数变易法

$Y(x)=C_1y_1(x)+C_2y_2(x)$

$Y(x)=v_1(x)y_1(x)+v_2(x)y_2(x)$

$$\begin{cases} y_1v_1^{'}+y_2v_2^{'}=0 \\ y_1^{'}v_1^{'}+y_2^{'}v_2^{'}=f \end{cases} \Rightarrow \begin{vmatrix} y_1 & y_2 & 0 \\ y_1^{'} & y_2^{'} & f \end{vmatrix} \xrightarrow{W=y_1y_2^{'}-y_1^{'}y_2} \begin{cases} v_1^{'}=-\frac{y_2f}{W} \\ v_2^{'}=\frac{y_1f}{W} \end{cases} \Rightarrow \begin{cases} v_1=-\int\frac{y_2f}{W}dx+c_1 \\ v_2=\int\frac{y_1f}{W}dx+c_2 \end{cases}$$

$$\begin{cases} v_1y_1=\left(-\int\frac{y_2f}{W}dx+c_1\right)y_1 \\ v_2y_2=\left(\int\frac{y_1f}{W}dx+c_2\right)y_2 \end{cases} \Rightarrow Y(x)=c_1y_1+c_2y_2-y_1\int\frac{y_2f}{W}dx+y_2\int\frac{y_1f}{W}dx$$

7.7 常系数齐次线性微分方程

$y^{''}+P(x)y^{'}+Q(x)y=0\Rightarrow y^{''}+py^{'}+qy=0$

$y=e^{rx}$

$(e^{rx})^{'}=re^{rx}$

$(r^2+pr+q)e^{rx}=0$

$$\begin{cases} p^2-4q>0, r_{1,2}=\frac{-p\pm\sqrt{p^2-4q}}{2} \\ p^2-4q=0, r_1=r_2=\frac{-p}{2}, y_2=e^{r_1x}\mu(x), \mu^{''}=0 \\ p^2-4q<0, r_{1,2}=\alpha\pm\beta i, e^{(i\theta)}=\cos(\theta)+i\sin(\theta), \overline{y_1}=\frac{1}{2}(y_1+y_2), \overline{y_1}=\frac{1}{2i}(y_1-y_2) \end{cases}$$

$$\begin{cases} p^2-4q>0, y=C_1y_1+C_2y_2=C_1e^{r_1x}+C_2e^{r_2x} \\ p^2-4q=0, y=C_1y_1+C_2\mu y_1=(C_1x+C_2)e^{r_1x} \\ p^2-4q<0, y=C_1\overline{y_1}+C_2\overline{y_2}=e^{\alpha x}(C_1\cos(\beta x)+C_2\sin(\beta x)) \end{cases}$$

n 阶常系数齐次微分方程

$y^n+p_1y^{n-1}+p_2y^{n-2}+\cdots+p_{n-1}y^{'}+p_ny=0$

$$\begin{cases} D, \frac{d}{dx} \\ Dy, \frac{dy}{dx} \\ D^ny, \frac{d^ny}{dx^n} \end{cases}$$

$(D^n+p_1D^{n-1}+\cdots+p_{n-1}D+p_n)y=0\Rightarrow L(D)y=0$, 微分算子 D 的 n 次多项式

$De^{rx}=re^{rx},\ldots,D^ne^{rx}=r^ne^{rx}\Rightarrow(r^n+p_1r^{n-1}+\cdots+p_{n-1}r+p_n)e^{rx}=L(r)e^{rx}=0\Rightarrow L(r)=0$

$$\begin{cases} \text{单实根} & Ce^{rx} \\ \text{k 重实根} & \left(C_1+C_2x+\cdots+C_kx^{k-1}\right)e^{rx} \\ \text{单复根} & e^{\alpha x}(C_1\cos(\beta x)+D_1\sin(\beta x)) \\ \text{k 重复根} & e^{\alpha x}\left(\left(C_1+C_2x+\cdots+C_kx^{k-1}\right)\cos(\beta x)+\left(D_1+D_2x+\cdots+D_kx^{k-1}\right)\sin(\beta x)\right) \end{cases}$$

7.8 常系数非齐次线性微分方程

$y^{''}+py^{'}+qy=f(x)$

$p_m(x)=a_0x^m+a_1x^{m-1}+\cdots+a_m$

$$\begin{cases} e^{\theta i}=\cos(\theta)+i\sin(\theta) \\ e^{-\theta i}=\cos(\theta)-i\sin(\theta) \end{cases} \Rightarrow \begin{cases} \cos(\theta)=\frac{1}{2}\left(e^{\theta i}+e^{-\theta i}\right) \\ \sin(\theta)=\frac{1}{2i}\left(e^{\theta i}-e^{-\theta i}\right) \end{cases} \Rightarrow \begin{cases} \cos(\omega x)P=\frac{P}{2}(e^{\omega xi}+e^{-\omega xi}) \\ \sin(\omega x)Q=\frac{Q}{2i}(e^{\omega xi}-e^{-\omega xi}) \end{cases}$$

函数 (多项式) 共轭，倒数共轭；　两对共轭，乘积共轭；　$e^{\alpha+\theta i}$与$e^{\alpha-\theta i}$ 共轭

$f(x)=e^{\lambda x}P_m(x)$	$y^*=R_?(x)e^{\lambda x}$ $R^{''}x+(2\lambda+p)R^{'}x+(\lambda^2+p\lambda+q)R(x)=p_m(x)$ $y^*=x^kP_m(x)e^{\lambda x}$
$f(x)=e^{\lambda x}\left[P_l(x)\cos(\omega x)+Q_n(x)\sin(\omega x)\right]$	$f(x)=e^{\lambda x}\left[\left(\frac{P}{2}+\frac{Q}{2i}\right)e^{\omega xi}+\left(\frac{P}{2}-\frac{Q}{2i}\right)e^{-\omega xi}\right]$ $f(x)=\left(\frac{P}{2}+\frac{Q}{2i}\right)e^{\lambda x+\omega xi}+\left(\frac{P}{2}-\frac{Q}{2i}\right)e^{\lambda x-\omega xi}$ $f(x)=\left(\frac{P}{2}+\frac{Q}{2i}\right)e^{(\lambda+\omega i)x}+\left(\frac{P}{2}-\frac{Q}{2i}\right)e^{(\lambda-\omega i)x}$ $f(x)=P_1e^{(\lambda+\omega i)x}+Q_2e^{(\lambda-\omega i)x}\left(P_1,Q_2\text{共轭}\right)$ $y_1^*=x^kR_me^{(\lambda+\omega i)x}\left(m=max\left\{P_l,Q_n\right\}\right)$ $y_2^*=x^k\overline{R_m}e^{(\lambda-\omega i)x}$ $y^*=y_1^*+y_2^*=x^ke^{\lambda x}\left(R_me^{\omega xi}+\overline{R_m}e^{-\omega xi}\right)$ $y^*=x^ke^{\lambda x}\left[R_m(\cos(\omega x)+i\sin(\omega x))+\overline{R_m}(\cos(\omega x)-i\sin(\omega x))\right]$ $y^*=x^ke^{\lambda x}\left(R_m^{(1)}\cos(\omega x)+R_m^{(2)}\sin(\omega x)\right)$

7.9 欧拉方程

$x^ny^{(n)}+p^1x^{n-1}y^{(n)}+\cdots+p^{n-1}xy^{'}+p^ny=f(x)$

$x=e^t,t=\ln x$

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{1}{x}\frac{dy}{dt}\ D\text{表示}\frac{d}{dt}\begin{cases}xy^{'}=Dy\\x^2y^{''}=(D^2-D)y\\x^3y^{'''}=(D^3-3D^2+2D)y\\\cdots\\x^ky^{(k)}=(D^n+c_1D^{n-1}+\cdots+C_{n-1}D)y\end{cases}$$

7.10 常系数线性微分方程组

$$\begin{cases}\frac{d^2x}{dt^2}+\frac{dy}{dt}-x=e^t,\\\frac{d^2y}{dt^2}+\frac{dx}{dt}+y=0,\end{cases}$$

$$\text{记}\frac{d}{dt}\text{为}D\Rightarrow\begin{cases}(D^2-1)x+Dy=e^t\\Dx+(D^2+1)y=0\end{cases}$$

8 向量代数与空间解析几何

8.1 向量及其线性运算

8.1.1 向量的概念

大小, 方向, **向量**,$\overrightarrow{AB}$
与起点无关, **自由向量**
向量的大小, **向量的模**,$\left|\overrightarrow{AB}\right|$**, 单位向量, 零向量**

$\overrightarrow{OA}=a,\overrightarrow{OB}=b,\angle AOB<\pi$向量夹角, $\left(\widehat{a,b}\right)\begin{aligned}&=0 \text{ or } \pi, a\text{与}b\text{平行, 同起点共线}\\&=\frac{\pi}{2}, a\text{与}b\text{垂直}\end{aligned}$

8.1.2 向量的运算

向量的加减法
$c=a+b$三角形法则
$a+b=b+a$交换
$(a+b)+c=a+(b+c)$ 分配
$a-b=a+(-b)$ 负向量
$|a\pm b|<|a|+|b|$
向量与数的乘法
$|\lambda a|=|\lambda|\ |a|$
$\lambda(\mu a)=\mu(\lambda a)=(\lambda\mu)a$
$|\lambda(\mu a)|=|\mu(\lambda a)|=|(\lambda\mu)a|=|\lambda\mu|\ |a|$
$(\lambda+\mu)a=\lambda a+\mu a,\lambda(a+b)=\lambda a+\lambda b$
$a\neq 0, a//b\Leftrightarrow\exists$唯一实数$\lambda, b=\lambda a$
空间直角坐标系
$[O;i,j,k]$ 右手规则, 卦限
$\overrightarrow{OM}=r=xi+yj+zk$坐标式分解, 分向量,$r=(x,y,z)$, $M(x,y,z)$ 坐标,r为M关于O向径
利用坐标向量的线性运算

$$a=(a_x,a_y,a_z),b=(b_x,b_y,b_z)\begin{cases}a+b&=(a_x+b_x,a_y+b_y,a_z+b_z)\\a-b&=(a_x-b_x,a_y-b_y,a_z-b_z)\\\lambda a&=(\lambda a_x,\lambda a_y,\lambda a_z)\end{cases}$$

向量的模, 方向角, 投影

向量的模与两点间的距离

$r=(x,y,z),|r|=\sqrt{x^2+y^2+z^2}$
$A=(x_1,y_1,z_1),B=(x_2,y_2,z_2),|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

方向角与方向余弦

r与三条坐标轴的夹角α,β,γ称为r的方向角
$(\cos\alpha,\cos\beta,\cos\gamma)=\frac{1}{|r|}(x,y,z)$, $\cos *$方向余弦

向量在轴上的投影

$$\begin{cases}(a)_u=|a|\cos\varphi\\(a+b)_u=(a)_u+(b)_u=|a|\cos\varphi+|b|\cos\varphi\\(\lambda a)_u=\lambda(a)_u\end{cases}$$

8.2 数量积　向量积　混合积

8.2.1 两向量的数量积

$$a\cdot b=|a|\ |b|\ \cos\theta\text{数量積}$$
$$\begin{cases}a\cdot a=|a|^2\\ a,b\neq 0,\ a\cdot b=0\Leftrightarrow a\perp b\\ a\cdot b=b\cdot a\text{交換}\\ (a+b)\cdot c=a\cdot c+b\cdot c\text{分配}\\ (\lambda a)\cdot b=\lambda\left(a\cdot b\right)\text{結合}\end{cases}$$
$$\begin{matrix}a=(a_x,a_y,a_z)\\ b=(b_x,b_y,b_z)\end{matrix}\Rightarrow\begin{matrix}a\cdot b=a_xb_x+a_yb_y+a_zb_z\\ \cos\theta=\frac{a\cdot b}{|a|\ |b|}=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{{a_x}^2+{a_y}^2+{a_z}^2}+\sqrt{{b_x}^2+{b_y}^2+{b_z}^2}}\end{matrix}$$

8.2.2 两向量的向量积

$$|c|=|a|\ |b|\sin\theta$$
$$c=a\times b\text{向量積}$$
$$\begin{cases}a\times a=0\\ a,b\neq 0,a\times b=0\Leftrightarrow a//b\\ a\times b=-b\times a\\ (a+b)\times c=a\times c+b\times c\\ (\lambda a)\times b=a\times(\lambda b)=\lambda\left(a\times b\right)\end{cases}$$
$$\begin{matrix}a=(a_x,a_y,a_z)\\ b=(b_x,b_y,b_z)\end{matrix}\Rightarrow\begin{aligned}a\times b&=(a_yb_z-a_zb_y,a_zb_x-a_xb_z,a_xb_y-a_yb_x)\\ &=\begin{vmatrix}i&j&k\\ a_x&a_y&a_z\\ b_x&b_y&b_z\end{vmatrix}\end{aligned}$$

8.2.3 向量的混合积

$$(a\times b)\cdot c=[abc],\text{向量的混合积}$$
$$|[abc]|=|a|\ |b|\ \sin\left(\widehat{a,b}\right)|c|\cos\left(\widehat{a\times b,c}\right),\ \text{a,b,c 右手系模大于0}$$
$$\begin{matrix}a=(a_x,a_y,a_z)\\ b=(b_x,b_y,b_z)\\ c=(c_x,c_y,c_z)\end{matrix}\Rightarrow\begin{aligned}[abc]&=(a\times b)\cdot c\\ &=\begin{vmatrix}a_x&a_y&a_z\\ b_x&b_y&b_z\\ c_x&c_y&c_z\end{vmatrix}\end{aligned}$$

8.3 平面及其方程

8.3.1 曲面方程与空间曲线方程的概念

$F\left(x,y,z\right)=0$, 曲面 S 上点满足方程, 不在曲面 S 上点不满足方程 $\Rightarrow$ 曲面 S 的方程

$$\begin{cases}F\left(x,y,z\right)=0\\ G\left(x,y,z\right)=0\end{cases}\Rightarrow\text{曲线 C 的方程}$$

8.3.2 平面的点法式方程

向量垂直平面, 平面的法线向量

$$\begin{matrix}n=(A,B,C)&\text{法向量}\\ \overrightarrow{M_0M}=(x-x_0,y-y_0,z-z_0)&\text{平面上点}\end{matrix}\Rightarrow A\left(x-x_0\right)+B\left(y-y_0\right)+C\left(z-z_0\right)=0\text{平面的点法式方程}$$

8.3.3 平面的一般方程

$Ax + By + Cz + D = 0$平面的一般方程

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$平面的截距式方程

8.3.4 平面的夹角

平面法向量的夹角, 平面的夹角 $\left(0 \leqslant \theta \leqslant \frac{\pi}{2}\right)$

$\Pi_1, n_1 = (A_1, B_1, C_1), \Pi_2, n_2 = (A_2, B_2, C_2), \cos\theta = \left|\left(\widehat{n_1, n_2}\right)\right| = \dfrac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$平面夹角

$$\begin{aligned} d &= \left|\overrightarrow{P_1P_0}\right| |\cos\theta| \\ &= \tfrac{\left|\overrightarrow{P_1P_0} \times n\right|}{|n|} \\ &= \tfrac{|A(x_0 - x) + B(y_0 - y) + C(z_0 - z)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \tfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \end{aligned}$$

8.4 空间直线及其方程

8.4.1 空间直线的一般方程

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

8.4.2 空间直线的对称式方程与参数方程

向量平行直线, 直线的方向向量

$$\begin{aligned} \overrightarrow{M_oM} &= (x - x_0, y - y_0, z - z_0) \\ S &= (m, n, p) \end{aligned} \Rightarrow \begin{array}{l} \dfrac{x - x_0}{m} = \dfrac{y - y_0}{n} = \dfrac{z - z_0}{p} = t, \text{对称式, 点向式方程} \\ \begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases} \text{参数方程} \end{array}$$

8.4.3 两直线夹角

$L_1, s_1 = (A_1, B_1, C_1), L_2, s_2 = (A_2, B_2, C_2), \cos\theta = \left|\left(\widehat{s_1, s_2}\right)\right| = \dfrac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$直线夹角

8.4.4 直线与平面夹角

直线和投影直线的夹角, 直线与平面夹角

$$\begin{array}{l} \varphi = \left|\frac{\pi}{2} - (\overrightarrow{s, n})\right| \\ s = (m, n, p) \\ n = (A, B, C) \end{array} \Rightarrow \begin{array}{l} \sin\varphi = \tfrac{|Am + Bn + Cp|}{\sqrt{A^2 + B^2 + C^2}\sqrt{m^2 + n^2 + p^2}} \\ \tfrac{A}{m} = \tfrac{B}{n} = \tfrac{C}{p} \text{垂直} \end{array}$$

8.4.5 例

$$\left.\begin{array}{ll} A_1x + B_1y + C_1z + D_1 = 0 & I \\ A_2x + B_2y + C_2z + D_2 = 0 & II \end{array}\right\} L \Rightarrow A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0 \text{过 L 平面束, 不含 } II$$

8.5 曲面及其方程

8.5.1 曲面研究的基本问题

轨迹到方程, 方程到形状

$x^2+y^2+z^2=R^2$球面

$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$球面

$Ax^2+Ay^2+Az^2+Dx+Ey+Fz+G=0$一般球面

8.5.2 旋转曲面

平面上曲线绕直线旋转一周, 母线, 轴, 旋转曲面

$f(y,z)=0\Rightarrow f\left(\pm\sqrt{x^2+y^2},z\right)=0$圆台

相交直线 L 绕 R 旋转一周, 交点顶点, 夹角 $\left(0<\alpha<\frac{\pi}{2}\right)$ 半顶角

$z^2=a^2\left(x^2+y^2\right)$ 圆锥

$$\frac{x^2}{a^2}-\frac{z^2}{b^2}=1\text{双曲线}\Rightarrow\left.\begin{array}{l}\frac{\left(x^2+y^2\right)^2}{a^2}-\frac{z^2}{b^2}=1\text{单叶双曲面}\\\frac{x^2}{a^2}-\frac{\left(z^2+y^2\right)^2}{b^2}=1\text{双叶双曲面}\end{array}\right\}\text{两种二次}$$

8.5.3 柱面

$x^2+y^2=R^2$圆柱面, 准线 (定曲线), 母线

$$\left.\begin{array}{l}y^2=ax\text{抛物柱面}\\\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\text{椭圆柱面}\\\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\text{双曲柱面}\end{array}\right\}\text{三种二次}$$

8.5.4 二次曲面九

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$椭圆锥面

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$椭圆抛物面

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$双曲抛物面

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$椭球面

8.6 空间曲线及其方程

8.6.1 空间曲线一般方程

$$\left\{\begin{array}{l}F(x,y,z)=0\\G(x,y,z)=0\end{array}\right.$$

8.6.2 空间曲线参数方程

$$\left\{\begin{array}{l}x=x(t)\\y=y(t)\\z=z(t)\end{array}\right.\Rightarrow\left\{\begin{array}{l}x=a\cos\theta\\y=a\cos\theta\\z=b\theta\end{array}\right.\text{螺旋线},2\pi b\text{螺距}$$

空间曲线参数方程

$$\Gamma\left\{\begin{array}{l}x=\varphi(t)\\y=\psi(t)\\z=\omega(t)\end{array}\right.\Rightarrow\left\{\begin{array}{l}x=\sqrt{[\varphi(t)]^2+[\varphi(t)]^2}\cos\theta\\y=\sqrt{[\varphi(t)]^2+[\varphi(t)]^2}\sin\theta z=\omega(t),\end{array}\right.\quad\text{曲面}$$

$$\Gamma\begin{cases} x = a\sin\varphi \\ y = 0 \\ z = a\cos\varphi \end{cases} \Rightarrow \begin{cases} x = \sqrt{[a\sin\varphi]^2 + [0]^2}\cos\theta = a\sin\varphi\cos\theta \\ y = \sqrt{[a\sin\varphi]^2 + [0]^2}\sin\theta = a\sin\varphi\sin\theta \\ z = a\cos\varphi, \end{cases} \text{球面}$$

8.6.3 空间曲线在坐标面投影

$H(x, y) = 0$ 投影柱面

$$\begin{cases} H(x, y) = 0 \\ z = 0 \end{cases} \text{投影曲线}$$

9 多元函数微分及其应用

9.1 多元函数的基本概念

9.1.1 平面點集 *n 維空間

平面點集

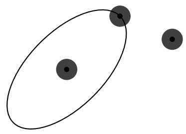
R^2 坐標平面

$C=\{(x,y)\,|x^2+y^2<r^2\}$ 平面點集

$U(P_0,\delta)=\{P|PP_0<\delta\}$ 鄰域

$\mathring{U}(P_0,\delta)=\{P|0<PP_0<\delta\}$ 去心鄰域

$$\begin{array}{ll} & \exists U(P),U(P)\subset E,P\text{為}E\text{內點}\\ & \exists U(P),U(P)\cap E=\varnothing,P\text{為}E\text{外點}\\ P\in R^2, & \left.\begin{array}{l}P=P_1\cap P_2\\ \exists U(P_1),U(P_1)\subset E\\ \exists U(P_2),U(P_2)\cap E=\varnothing\end{array}\right\},P\text{為}E\text{邊界點}\\ E\subset R^2 & \end{array}$$



E邊界點的全體, 為E邊界點,∂E

$\forall\delta>0,\mathring{U}(P,\delta)$ 內總有E中的點,P為E聚點 (內點和邊界)

開集, 內點; 閉集,$\partial E\subset E$; 連通集, 任意點連線仍在集合;

區域, 連通開集；閉區域，區域和邊界；

有界集，$\forall E\subset R^2, if\quad \exists r>0,E\subset U(O,r), then$ 有界集；無界集，不是有界集

n 維空間

$R^n=\{(x_1,x_2,\cdots,x_n)\,|x_i\in R,i=1,2,\cdots,n\}=x$集合

$$\left.\begin{array}{l}x=(x_1,x_2,\cdots,x_n)\in R^n\\ y=(y_1,y_2,\cdots,y_n)\in R^n\\ a=(a_1,a_2,\cdots,a_n)\in R^n\\ \lambda\in R\end{array}\right\}\Rightarrow\begin{array}{l}\left.\begin{array}{ll}x+y&=(x_1+y_1,x_2+y_2,\cdots,x_n+y_n)\\ \lambda x&=(\lambda x_1,\lambda x_2,\cdots,\lambda x_n)\end{array}\right\}R^n\text{中線性運算}\\ \rho(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}\\ ||x||=\rho(x,0)=\sqrt{{x_1}^2+{x_2}^2+\cdots+{x_n}^2}\\ ||x-a||\to 0\text{，變元趨于固定元，記作}x\to a\\ \text{某正數}\delta>0,\text{點集}U(a,\delta)=\{x|x\in R^n,\rho(x,a)<\delta\},a\text{的}\delta\text{鄰域}\end{array}$$

定義線性預算的集合，n 維空間

9.1.2 多元函數的概念

$$\begin{array}{l}D\subset R^2,\text{映射}f:D\to R,\text{稱為二元函數},\\ \text{記為}z=f(x,y),(x,y)\in D\end{array}\Rightarrow\begin{array}{l}D\text{定義域}\\ x,y\text{自變量}\\ z\text{因變量}\\ \text{函數值}f(x,y)\text{ 全體},f(D)=\{z|z=f(x,y),(x,y)\in D\}\text{ 值域}\end{array}$$

$D\subset R^n$, 映射$f:D\to R$, 稱為n元函數 ($n\geqslant 2$多元函數),

記為$z=f(x_1,x_2,\cdots,x_n)=f(\mathbf{x}),\mathbf{x}(x_1,x_2,\cdots,x_n)\in D$

多元函數$\mu=f(\mathbf{x})$, 有意義的變元$\mathbf{x}$的點集，自然定義域

空間點集 $\{(x,y,z)\,|z=f(x,y),(x,y)\in D\}$, 二元函數$z=f(x,y)$ 的圖形

9.1.3 多元函數的極限

$$\begin{array}{llll}\begin{array}{l}f(x,y)\text{ 定義域}D,\\ P_0(x_0,y_0)\text{ 是}D\text{聚點}\end{array} & if\ \begin{array}{l}\exists A,\forall\varepsilon>0,\exists\delta>0,\\ when\text{ 點}P(x_0,y_0)\in D\cap\mathring{U}(P_0,\delta),\\ |f(P)-A|=|f(x,y)-A|<\varepsilon\end{array} & then & \begin{array}{l}A\text{稱為函數}f(x,y)\text{ 在 }(x,y)\to(x_0,y_0)\text{ 極限},\\ \text{記為}\lim\limits_{(x,y)\to(x_0,y_0)}f(x,y)=A\end{array}\end{array}$$

任意方向趨近

9.1.4 多元函數的連續性

$f(P)=f(x,y)$ 定義域$D, P_0(x_0,y_0)$ 為D聚點$, P_0\in D,$ $\begin{array}{l} if \quad \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0), \\ then \quad f(x,y) \text{ 在} P_0(x_0,y_0) \text{ 連續} \end{array}$

$f(x,y)$ 定義域D, D內每一點都是聚點$, f(x,y)$ 在D內每一點連續$, f(x,y)$ 在D內連續

$f(x,y)$ 定義域$D, P_0(x_0,y_0)$ 為D聚點$, if \quad f(x,y)$ 在$P_0(x_0,y_0)$ 不連續$, then \quad P_0(x_0,y_0)$ 為$f(x,y)$ 間斷點

常數，不同自變量的一元基本初等函數，有限次四則運算和復合運算，**多元初等函數**

一切多元初等函數在**定義區域**（定義域內的區域或閉區域）連續 $\to \lim\limits_{P\to P_0} f(P)=f(P_0)$

	有界閉區域 D，多元連續函數，D 上有界，取得最大值，最小值	(**最值**)
性質	有界閉區域 D，多元連續函數，能取得介於最大值和最小值間的任何值	(**介值**)
	有界閉區域 D，多元連續函數，D 上一致連續	(**一致連續**)

9.2 偏導數

9.2.1 偏導數的定義及其計算法

$\begin{array}{l} f(x,y), P_0(x_0,y_0), (x,y)\in U(P_0,\delta), \\ when \quad y=y_0, x=x_0+\Delta x, \\ \text{函數增量} f(x_0+\Delta x,y_0)-f(x_0,y_0) \end{array}$ $\begin{array}{l} if \quad \exists \lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=A, \\ then \quad A\text{為} f(x,y) \text{ 在 } (x_0,y_0) \text{ 對}x\text{的偏導數}, \\ \text{記為} \left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}, f_x(x_0,y_0) \end{array}$

$$\begin{cases} \left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} = \lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x} \\ \left.\dfrac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}} = \lim\limits_{\Delta y\to 0}\dfrac{f(x_0,y_0+\Delta x)-f(x_0,y_0)}{\Delta y} \end{cases}$$

$f_x(x,y), f_y(x,y)$ 偏導函數, 偏導數

偏導數記號為整體，不能看成微分的商

(一元可導連) 各偏導存在，不一定連續

9.2.2 高階偏導數

$$z=\begin{array}{l} f(x,y) \text{ 的偏导数} \frac{\partial z}{\partial x}=f_x(x,y), \frac{\partial z}{\partial y}=f_y(x,y), \\ \text{都是}x,y\text{的函数}, \end{array} \Rightarrow \begin{array}{l} \text{若偏导数的偏导仍存在，则称为} f(x,y) \text{ 的二阶偏导数} \\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y) \\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f_{yx}(x,y) \text{ (混合偏导数)} \\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f_{xy}(x,y) \text{ (混合偏导数)} \\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y) \end{array}$$

$if \quad f(x,y)$ 的二阶偏导数$, \frac{\partial^2 z}{\partial y\partial x}, \frac{\partial^2 z}{\partial x\partial y}$ 在 D 连续$, then \quad (x,y)\in D \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$

$$\left.\begin{array}{l} z=\ln\sqrt{x^2+y^2}, \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \\ u=\frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \end{array}\right\} \text{拉普拉斯方程}$$

9.3 全微分

9.3.1 全微分定義

$$\begin{cases} f(x+\Delta x,y)-f(x,y)\approx f_x(x,y)\Delta x \\ f(x,y+\Delta y)-f(x,y)\approx f_y(x,y)\Delta y \end{cases} \text{偏增量，偏微分}$$

$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y) \quad$ 全增量

$$z=f\left(x,y\right),\text{在}\left(x,y\right)\text{的某邻域有定义,}\quad \begin{array}{ll} if & \Delta z=f\left(x+\Delta x,y+\Delta y\right)-f\left(x,y\right)=A\Delta x+B\Delta y+o\left(\rho\right), \\ then & \rho=\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2},A,B\text{不依赖}\Delta x,\Delta y \\ & f\left(x,y\right)\text{在}\left(x,y\right)\text{可微分,} \\ & A\Delta x+B\Delta y,\text{称为函数}f\left(x,y\right)\text{全微分,} \\ & \text{记作}dz=A\Delta x+B\Delta y \end{array}$$

多元函数在区域 D 内个点处都可微分，函数在 D 内可微分

多元函数在**点 P** 可微分，函数在该点连续

$$if\quad f\left(x,y\right)\text{在点}\left(x,y\right)\text{可微分,}\quad \begin{array}{l} then\quad \text{函数}f\text{x,y在点}\left(x,y\right)\text{偏导数}\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\text{必存在} \\ \text{函数}f\left(x,y\right)\text{在点}\left(x,y\right)\text{全微分}dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \end{array}$$

$z=f\left(x,y\right)$的偏導數$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$在點$\left(x,y\right)$連續,函數在該點可微分

微分疊加原裡,$u=\left(x,y,z\right),du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

9.3.2 全微分在近似計算中的應用

$g=\frac{4\pi^2 l}{T^2},\Delta g\leqslant 4\pi^2\left(\frac{1}{T^2}\delta_l+\frac{2l}{T^3}\delta_T\right)$

9.4 多元復合函數的求導法則

$$if\ \begin{array}{l} u=\varphi\left(t\right),v=\psi\left(t\right),\text{在}t\text{可導,} \\ z=f\left(u,v\right)\text{在}\left(u,v\right)\text{具有連續偏導} \end{array},then\quad \text{復合函數}z=f\left(\varphi\left(t\right),\psi\left(t\right)\right)\text{在}t\text{點可導, 且}\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}\text{全導數}$$

$$if\quad u=\varphi\left(t\right),v=\psi\left(t\right),w=\omega\left(t\right),\text{在}t\text{可導,}z=f\left(\varphi\left(t\right),then\quad \psi\left(t\right)+\omega\left(t\right)\right),\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}$$

$$if\ \begin{array}{l} u=\varphi\left(x,y\right),v=\psi\left(x,y\right), \\ \text{在}\left(x,y\right)\text{具有對}x,y\text{的偏導數,} \\ z=f\left(u,v\right)\text{在}\left(u,v\right)\text{具有連續偏導} \end{array},then\quad \begin{array}{l} \text{復合函數}z=f\left(\varphi\left(x,y\right),\psi\left(x,y\right)\right)\text{在}(x,y)\text{點兩偏導都存在, 且} \\ \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \end{array}$$

$$if\quad \begin{array}{l} u=\varphi\left(x,y\right),v=\psi\left(x,y\right),w=\omega\left(x,y\right), \\ \text{在}\left(x,y\right)\text{點兩偏導都存在} \\ z=f\left(\varphi\left(x,y\right),\psi\left(x,y\right)+\omega\left(x,y\right)\right), \end{array},then\quad \begin{array}{l} \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y} \end{array}$$

$$if\quad \begin{array}{ll} u & =\varphi\left(x,y\right)\text{在}\left(x,y\right)\text{點兩偏導都存在,} \\ v & =\psi\left(y\right)\text{在}y\text{點可導,} \\ z & =f\left(u,v\right)\text{在}\left(u,v\right)\text{具有連續偏導} \end{array}\quad then\quad \begin{array}{l} z=f\left(\varphi\left(x,y\right),\psi\left(y\right)\right)\text{在}\left(x,y\right)\text{點兩偏導都存在, 且} \\ \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \end{array}$$

$$ifz=f\left(u,x,y\right)=f\left[\varphi\left(x,y\right),x,y\right]\text{有}x,y\text{的偏導數,}then\quad \begin{array}{l} \frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x} \\ \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y} \end{array}$$

$$f\left(u,v\right),\quad \begin{array}{l} f_1^{'}=f_u\left(u,v\right) \\ f_2^{'}=f_v\left(u,v\right) \\ f_{11}^{'}=f_{uu}\left(u,v\right) \\ f_{12}^{'}=f_{uv}\left(u,v\right) \\ f_{21}^{'}=f_{vu}\left(u,v\right) \\ f_{22}^{'}=f_{vv}\left(u,v\right) \end{array}$$

$$z=f\left(u,v\right)=f\left(\varphi\left(x,y\right),\psi\left(x,y\right)\right),\quad \begin{array}{ll} dz & =\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv \\ & =\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \\ & =\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)dy \end{array}$$

9.5 隱函數求導公式

9.5.1 一個方程

$$
if\quad \begin{array}{l} P_0\left(x_0,y_0\right),\left(x,y\right)\in U\left(P_0,\delta\right)\text{時} \\ F\left(x,y\right)\text{具有連續偏導}, \\ F\left(x_0,y_0\right)=0,F_y\left(x_0,y_0\right)\neq 0 \end{array} ,then\quad \begin{array}{l} F\left(x,y\right)=0,\text{在}\left(x,y\right)\in U\left(P_0,\delta\right)\text{時}, \\ \text{能確定唯一連續，連續導數的函數}y=f\left(x\right), \\ y_0=f\left(x_0\right),\frac{dy}{dx}=-\frac{F(x)}{F(y)} \end{array}
$$

$$
if\quad \begin{array}{l} P_0\left(x_0,y_0,z_0\right),\left(x,y,z\right)\in U\left(P_0,\delta\right)\text{時} \\ F\left(x,y,z\right)\text{具有連續偏導}, \\ F\left(x_0,y_0,z_0\right)=0,F_z\left(x_0,y_0,z_0\right)\neq 0 \end{array} ,then\quad \begin{array}{l} F\left(x,y,z\right)=0,\text{在}\left(x,y,z\right)\in U\left(P_0,\delta\right)\text{時}, \\ \text{能確定唯一連續，連續導數的函數}z=f\left(x,y\right), \\ z_0=f\left(x_0,y_0\right),\frac{\partial z}{\partial x}=-\frac{F(x)}{F(z)},\frac{\partial z}{\partial y}=-\frac{F(y)}{F(z)} \end{array}
$$

9.5.2 方程組的情形

$$
if\quad \begin{array}{l} P_0\left(x_0,y_0,u_0,v_0\right),\left(x,y,u,v\right)\in U\left(P_0,\delta\right)\text{時} \\ F\left(x,y,u,v\right),G\left(x,y,u,v\right)\text{具有連續偏導}, \\ F\left(x_0,y_0,u_0,v_0\right)=0,G\left(x_0,y_0,u_0,v_0\right)=0, \\ J=\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}\neq 0\left(\begin{array}{l}\text{雅可比式}\\ \text{偏导数组成的函数行列式}\end{array}\right) \end{array} \quad then\quad \begin{array}{l} F\left(x,y,u,v\right)=0,G\left(x,y,u,v\right)=0, \\ \text{在}\left(x,y,u,v\right)\in U\left(P_0,\delta\right)\text{時}, \\ \text{能確定唯一組連續，連續偏導數的函數}, \\ u=f\left(x,y\right),v=f\left(x,y\right) \\ u_0=f\left(x_0,y_0\right),v_0=f\left(x_0,y_0\right) \\ \frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)} \\ \frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)} \\ \frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)} \\ \frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)} \end{array}
$$

$$
\left\{\begin{array}{l} F_x+F_u\frac{\partial u}{\partial x}+F_v\frac{\partial v}{\partial x}=0 \\ G_x+G_u\frac{\partial u}{\partial x}+G_v\frac{\partial v}{\partial x}=0 \end{array}\right. \Rightarrow \begin{array}{l} \frac{\partial u}{\partial x}=\dfrac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}=-\dfrac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)} \\ \frac{\partial v}{\partial x}=\dfrac{\begin{vmatrix} F_u & -F_x \\ G_u & -G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}=-\dfrac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)} \end{array} \quad \text{证明或推导}
$$

$$
\begin{array}{l} dudv,x=\varphi\left(u,v\right),y=\psi\left(u,v\right), \\ dx=\frac{\partial\varphi}{\partial u}du+\frac{\partial\varphi}{\partial v}dv \\ dy=\frac{\partial\psi}{\partial u}du+\frac{\partial\psi}{\partial v}dv \end{array} \rightarrow \begin{array}{l} \left|\left(dx,dy\right)|_{du=0}\times\left(dx,dy\right)|_{dv=0}\right| \\ =\left|\left(\frac{\partial\varphi}{\partial u}du,\frac{\partial\psi}{\partial u}du\right)\times\left(\frac{\partial\varphi}{\partial v}dv,\frac{\partial\psi}{\partial v}dv\right)\right| \\ =\left|\frac{\partial\varphi}{\partial u}\cdot\frac{\partial\psi}{\partial v}-\frac{\partial\psi}{\partial u}\cdot\frac{\partial\varphi}{\partial v}\right|dudv \\ =\begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{vmatrix}dudv \\ =\frac{\partial(\varphi,\psi)}{\partial(u,v)}dudv \end{array} \quad \text{二重积分换元和雅可比}
$$

9.6 多元函数微分学的几何应用

9.6.1 一元向量值函数及其导数

$$
\Gamma\left\{\begin{array}{l} x=\varphi\left(t\right), \\ y=\psi\left(t\right),\quad t\in\left[\alpha,\beta\right] \\ z=\omega\left(t\right) \end{array}\right.
$$

記$r=xi+yj+zk,f\left(t\right)=\varphi\left(t\right)i+\psi\left(t\right)j+\omega\left(t\right)k,$

$r=f\left(t\right),t\in\left[\alpha,\beta\right]$

數集$D \subset R$,映射$f : D \to R^n$為一元向量值函數，記為$r = f(t), t \in D$

$t \in U(t_0)$,向量值函数$f(t), if \quad \exists r_0, \forall \varepsilon > 0, \exists \delta, when \quad 0 < |t - t_0| < \delta, |f(t) - r_0| < \varepsilon, then \quad \lim_{t \to t_0} f(t) = r_0$

$t \in u(t_0)$,向量值函数$f(t), if \quad \lim_{t \to t_0} f(t) = f(t_0), f(t)$在$t_0$连续

$t \in D,$,向量值函数$f(t), if \quad D_1 \subset D, D_1$内每点连续,$D_1$上连续

$t \in U(t_0)$,向量值函数$f(t), if \quad \exists \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}, then$ 极限向量为$r = f(t)$在t_0导数或导向量,记为$f'(t_0), \frac{dr}{dt}|_{t=t_0}$

$f'(t_0) = f_1'(t_0) i + f_2'(t_0) j + f_3'(t_0) k$

向量值函数$u(t), v(t)$,数量值函数$\varphi(t) \Rightarrow$

$$\begin{array}{l} \frac{d}{dt} C = 0 \\ \frac{d}{dt}[cu(t)] = cu'(t) \\ \frac{d}{dt}[u(t) \pm v(t)] = u'(t) + v'(t) \\ \frac{d}{dt}[\varphi(t) v(t)] = \varphi'(t) v(t) + \varphi(t) v'(t) \\ \frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t) \\ \frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t) \\ \frac{d}{dt} u[\varphi(t)] = \varphi'(t) u'[\varphi(t)] \end{array}$$

$f'(t_0) = \lim_{t \to t_0} \frac{\Delta r}{\Delta t}$与曲线相切

9.6.2 空间曲线的切线与法平面

$$\Gamma \begin{cases} x = \varphi(t), \\ y = \psi(t), \\ z = \omega(t), \end{cases} \quad t \in [\alpha, \beta] \Rightarrow \begin{array}{l} \frac{x - x_0}{\varphi'(t)} = \frac{y - y_0}{\psi'(t)} = \frac{z - z_0}{\omega'(t)} \\ \varphi'(t)(x - x_0) + \psi'(t)(y - y_0) + \omega'(t)(z - z_0) = 0 \end{array}$$

$$\Gamma \begin{cases} y = \psi(x), \\ z = \omega(x), \end{cases} \Rightarrow \begin{cases} x = x, \\ y = \psi(x), \\ z = \omega(x), \end{cases} \Rightarrow \begin{array}{l} \frac{x - x_0}{1} = \frac{y - y_0}{\psi'(t)} = \frac{z - z_0}{\omega'(t)} \\ (x - x_0) + \psi'(t)(y - y_0) + \omega'(t)(z - z_0) = 0 \end{array}$$

$$\begin{array}{l} \Gamma \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \\ \frac{\partial(F,G)}{\partial(y,z)}|_{(x_0, y_0, z_0)} \neq 0 \end{array} \Rightarrow \begin{cases} y = \psi(x), \\ z = \omega(x), \end{cases} \Rightarrow \begin{array}{l} \begin{cases} F(x, \psi(x), \omega(x)) = 0 \\ G(x, \psi(x), \omega(x)) = 0 \end{cases} \\ \begin{cases} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx} = 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} + \frac{\partial G}{\partial z}\frac{dz}{dx} = 0 \end{cases} \\ \begin{cases} \frac{dy}{dx} = \psi'(x) = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,z)} = -\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} \\ \frac{dz}{dx} = \omega'(x) = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,x)} = -\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} \end{cases} \end{array}$$

9.6.3 曲面的切平面与法线

$$\begin{array}{l} \sum F(x,y,z)=0 \\ \text{任意曲线}\Gamma\left\{\begin{array}{l} x=\varphi(t), \\ y=\psi(t), \quad (\alpha<t<\beta), \\ z=\omega(t), \end{array}\right. \\ t=t_0\text{对应}M(x_0,y_0,z_0) \\ \text{曲线}\Gamma\text{在曲面}\sum\text{上，过}M\text{点} \\ F[\varphi(t),\psi(t),\omega(t)]=0, \\ F_x|_M\varphi'(t_0)+F_y|_M\psi'(t_0)+F_z|_M\omega'(t_0) \\ =(F_x|_M,F_z|_M,F_z|_M)\cdot(\varphi'(t_0),\psi'(t_0),\omega'(t_0)) \\ =n\cdot s=0, \\ \text{任一过}M\text{曲线切线垂直过}M\text{向量}n,\text{切线共面, 称为曲面切平面} \end{array} \Rightarrow \begin{array}{l} F_x|_M(x-x_0)+F_z|_M(y-y_0)+F_z|_M(z-z_0)=0 \\ \frac{(x-x_0)}{F_x|_M}=\frac{(y-y_0)}{F_y|_M}=\frac{(z-z_0)}{F_z|_M} \end{array}$$

$$z=f(x,y)\Rightarrow F(x,y,z)=f(x,y)-z=0\Rightarrow \begin{array}{l} F_x|_M=f_x(x_0,y_0),F_y|_M=f_y(x_0,y_0),F_z|_M=-1 \\ \text{记}K=\sqrt{f_x^2+f_y^2+1},\cos\gamma>0, \\ \text{方向余弦}(\cos\alpha,\cos\beta,\cos\gamma)=\left(\frac{-f_x}{K},\frac{-f_y}{K},\frac{1}{K}\right) \\ f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0 \\ \frac{(x-x_0)}{f_x(x_0,y_0)}=\frac{(y-y_0)}{f_y(x_0,y_0)}=\frac{(z-z_0)}{-1} \end{array}$$

9.7 方向导数与梯度

9.7.1 方向导数

$$\begin{array}{l} \text{单位向量}e_l=(\cos\alpha,\cos\beta), \\ \text{过}(x_0,y_0)\text{射线}l\left\{\begin{array}{l} x=x_0+t\cos\alpha \\ y=y_0+t\cos\beta \end{array}\right.(t\geqslant 0), \\ z=f(x,y) \end{array} \Rightarrow \begin{array}{l} \frac{\partial f}{\partial l}|_{(x_0,y_0)}=\lim\limits_{t\to 0^+}\dfrac{f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)}{t} \\ \text{方向导数} \end{array}$$

$f(x,y)$ 在 $P_0(x_0,y_0)$ 可微分，该点任意方向，方向导数存在,$\frac{\partial f}{\partial l}|_{(x_0,y_0)}=f_x(x_0,y_0)\cos\alpha+f_y(x_0,y_0)\cos\beta$

$f(x,y,z),\frac{\partial f}{\partial l}|_{(x_0,y_0,z_0)}=f_x(x_0,y_0,z_0)\cos\alpha+f_y(x_0,y_0,z_0)\cos\beta+f_z(x_0,y_0,z_0)\cos\gamma$

9.7.2 梯度

$f(x,y),D$内连续偏导数,$\forall P_0\in D,\mathbf{grad}f(x_0,y_0)=\Delta f(x_0,y_0)=f_x|_{P_0}i+f_y|_{p_0}j$,梯度,$\Delta=\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j$向量微分算子

$$\begin{array}{l} f(x,y)\text{在}(x_0,y_0)\text{可微分}, \\ \text{单位向量}e_l=(\cos\alpha,\cos\beta) \end{array}, \quad \begin{aligned} \frac{\partial f}{\partial l}|_{(x_0,y_0)} &= f_x(x_0,y_0)\cos\alpha+f_y(x_0,y_0)\cos\beta \\ &= \mathbf{grad}f(x_0,y_0)\cdot e_l \\ &= |\mathbf{grad}f(x_0,y_0)|\cos\theta,\theta=\left(\mathbf{grad}\widehat{f(x_0,y_0)},e_l\right) \end{aligned}$$

$$\begin{array}{l} \text{曲面}\sum z=f(x,y),\text{等值线}L\,f(x,y)=0,P_0(x_0,y_0), \\ \text{记}k=\sqrt{f_x^2(x_0,t_0)+f_y^2(x_0,y_0)}, \end{array} \Rightarrow \begin{array}{l} \begin{aligned} \text{单位法向量}n &= \tfrac{1}{k}(f_x|_p,f_y|_p) \\ &= \tfrac{\Delta f(x_0,y_0)}{|\Delta f(x_0,y_0)|} \end{aligned} \\ \Delta f(x_0,y_0)=|\Delta f(x_0,y_0)|\,n=\frac{\partial f}{\partial n}n \\ \text{梯度}=\text{梯度的模 梯度的方向} \end{array}$$

$$f(x,y,z),G\text{内连续偏导数},P_0(x_0,y_0,z_0), \begin{array}{l} \mathbf{grad}f=\Delta f=f_x|_{P_0}i+f_y|_{P_0}j+f_z|_{P_0}k, \\ \text{等值面}f(x,y,z)=c \\ \text{等值面法线方向}n \end{array}$$

空间区域$G\quad \forall M\in G\to$ 数量$f\left(M\right),G$内数量场

空间区域$G\quad \forall M\in G\to$ 向量$f\left(M\right),G$内向量场

$if\quad F\left(M\right)$ 是$f\left(M\right)$ 的梯度 $then\quad f\left(M\right)$ 势函数, $F\left(M\right)$ 势场

$\frac{m}{r}$引力势, $\mathbf{grad}\frac{m}{r}$引力场

9.8 多元函数的极值及其求法

9.8.1 多元函数的极值及最大值最小值

$$D,z=f\left(x,y\right),P_0\left(x_0,y_0\right)\in D,\ \begin{array}{l} if\quad \exists U\left(P_0\right)\subset D,\forall\left(x,y\right)\neq P_0,f\left(x,y\right)<f\left(x_0,y_0\right),then\quad \text{极大值}f\left(x_0,y_0\right)\\ if\quad \exists U\left(P_0\right)\subset D,\forall\left(x,y\right)\neq P_0,f\left(x,y\right)>f\left(x_0,y_0\right),then\quad \text{极小值}f\left(x_0,y_0\right)\\ \text{极大值，极小值统称极值}\end{array}$$

$if\quad z=f\left(x,y\right)$ 在 $\left(x_0,y_0\right)$ 具有偏导数, 且在 $\left(x_0,y_0\right)$ 点具有极值, $then\quad f_x\left(x_0,y_0\right)=0,f_y\left(x_0,y_0\right)=0$

$if\quad f_x\left(x_0,y_0\right)=0,f_y\left(x_0,y_0\right)=0,then$ 驻点 $\left(x_0,y_0\right)$

$$if\quad z=f\left(x,y,z\right)\text{ 在 }\left(x_0,y_0,z_0\right)\text{ 具有偏导数, 且在 }\left(x_0,y_0,z_0\right)\text{ 点具有极值}, then\quad \begin{array}{l} f_x\left(x_0,y_0,z_0\right)=0,\\ f_y\left(x_0,y_0,z_0\right)=0,\\ f_z\left(x_0,y_0,z_0\right)=0\end{array}$$

$$\begin{array}{l} z=f\left(x,y\right)\text{ 在 }\left(x_0,y_0\right)\text{ 的某邻域内连续，}\\ \text{且有一阶及二阶偏导数，又}f_x\left(x_0,y_0\right)=0,f_y\left(x_0,y_0\right)=0\\ \text{令}f_{xx}\left(x_0,y_0\right)=A,f_{xy}\left(x_0,y_0\right)=B,f_{yy}\left(x_0,y_0\right)=C\end{array}\ if\quad \begin{array}{l} AC-B^2>0,\text{有极值},A<0\text{极大值},A>0\text{极小值}\\ AC-B^2<0,\text{无极值}\\ AC-B^2=0,\text{可能有极值}\end{array}$$

驻点，偏导不存在点

9.8.2 条件极值　拉格朗日数乘法

无条件极值，条件极值；条件极值到无条件极值

$$\begin{array}{l} \left(x_o,y_0\right)\text{ 的某邻域内}z=f\left(x,y\right),\varphi\left(x,y\right),\\ \text{有连续一阶导数 }\varphi\left(x,y\right)=0\left(\text{条件}\right)\to y=\varphi\left(x\right),\\ z=f\left(x,\varphi\left(x\right)\right)\left(\text{函数}\right)\end{array}\ \Rightarrow\ \begin{array}{l} \frac{dz}{dx}|_{x=x_0}=0\\ f_x\left(x_0,y_0\right)+f_y\left(x_0,y_0\right)\frac{dy}{dx}|_{x=x_0}=0\\ f_x\left(x_0,y_0\right)-f_y\left(x_0,y_0\right)\frac{\varphi_x\left(x_0,y_0\right)}{\varphi_y\left(x_0,y_0\right)}=0\\ \text{记}\frac{f_y\left(x_0,y_0\right)}{\varphi_y\left(x_0,y_0\right)}=-\lambda\\ f_y\left(x_0,y_0\right)-f_y\left(x_0,y_0\right)\frac{\lambda}{\lambda}=0\\ \begin{cases} f_x\left(x_0,y_0\right)+\lambda\varphi_x\left(x_0,y_0\right)=0\\ f_y\left(x_0,y_0\right)+\lambda\varphi_y\left(x_0,y_0\right)=0\\ \varphi\left(x_0,y_0\right)=0\end{cases}\end{array}$$

$$L\left(x,y\right)=f\left(x,y\right)+\lambda\varphi\left(x,y\right)\Rightarrow\begin{cases} L_x\left(x_0,y_0\right)=0\\ L_y\left(x_0,y_0\right)=0\\ \varphi\left(x_0,y_0\right)=0\end{cases}\quad \begin{array}{l}\text{拉格朗日函数}L,\text{拉格朗日乘子}\lambda\\ \text{函数}f\left(x,y\right),\text{条件}\varphi\left(x,y\right)=0\end{array}$$

$$L\left(x,y,z,t\right)=f\left(x,y,z,t\right)+\lambda\varphi\left(x,y,z,t\right)+\mu\psi\left(x,y,z,t\right)\Rightarrow\begin{cases} L_x\left(x_0,y_0,z_0,t_0\right)=0\\ L_y\left(x_0,y_0,z_0,t_0\right)=0\\ L_z\left(x_0,y_0,z_0,t_0\right)=0\\ L_t\left(x_0,y_0,z_0,t_0\right)=0\\ \varphi\left(x_0,y_0,z_0,t_0\right)=0\\ \psi\left(x_0,y_0,z_0,t_0\right)=0\end{cases}\quad \begin{array}{l}\text{拉格朗日函数}L,\\ \text{拉格朗日乘子}\lambda,\mu\\ \text{函数}f\left(x,y,z,t\right),\\ \text{条件}\varphi\left(x,y,z,t\right)=0,\\ \psi\left(x,y,z,t\right)=0\end{array}$$

9.8.3 二元函数的泰勒公式

$$\begin{array}{l} z=f\left(x,y\right)\text{ 在点}P_0\left(x_0,y_0\right)\text{ 某邻域内连续，}\\ \text{有 n+1 阶连续偏导数,}\\ \left(x_0+h,y_0+k\right)\in U\left(P_0,\delta\right)\end{array}\ \Rightarrow\ \begin{array}{l} f\left(x_0+h,y_0+k\right)=f\left(x_0,y_0\right)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f\left(x_0,y_0\right)\\ \frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2f\left(x_0,y_0\right)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^mf\left(x_0,y_0\right)\\ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1}f\left(x_0+\theta h,y_0+\theta k\right)\quad\left(0<\theta<1\right)\end{array}$$

证明：
$$\begin{aligned}
&\Phi(t)=f(x_0+h,y_0+k)\ (0\leqslant t\leqslant 1)\\
&\Phi^{(n)}(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0+ht,y_0+kt)\\
&\Phi(t)=\Phi(0)+\Phi'(0)+\tfrac{1}{2}\Phi''(0)+\cdots+\tfrac{1}{n!}\Phi^{(n)}(0)+\tfrac{1}{(n+1)!}\Phi^{(n+1)}(\theta t)\ (0<\theta<1)\,(0\quad)\\
&\Phi(1)=\Phi(0)+\Phi'(0)+\tfrac{1}{2}\Phi''(0)+\cdots+\tfrac{1}{n!}\Phi^{(n)}(0)+\tfrac{1}{(n+1)!}\Phi^{(n+1)}(\theta)\ (0<\theta<1)\,(\quad)
\end{aligned}$$

$n=0$时$f(x_0+h,y_0+k)-f(x_0,y_0)=hf_x(x_0+\theta h,y_0+\theta k)+kf_y(x_0+\theta h,y_0+\theta k)$

$f(x,y),(x,y)\in D,f_x(x,y)\equiv 0,f_y(x,y)\equiv 0,f(x,y)=c$

9.8.4 极值充分证明

$$\begin{aligned}
\Delta f\ &=f(x,y)-f(x_0,y_0)\\
&=\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]f(x_0,y_0)+\frac{1}{2}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^2 f(x_0+\theta h,y_0+\theta k)\\
&f_x(x_0,y_0)=0,f_y(x_0,y_0)=0\\
&=\frac{1}{2}\left[h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right]^2 f(x_0+\theta h,y_0+\theta k)\\
&=\frac{1}{2}\left(h^2f_{xx}+2hkf_{xy}+k^2f_{yy}\right)\\
&=\frac{1}{2f_{xx}}\left(h^2f_{xx}^2+2hkf_{xy}f_{xx}+k^2f_{xy}^2-k^2f_{xy}^2+k^2f_{yy}f_{xx}\right)\\
&=\frac{1}{2f_{xx}}\left[\left(hf_{xx}+kf_{xy}\right)^2+k^2\left(f_{xx}f_{yy}-f_{xy}^2\right)\right]\\
&A=f_{xx},C=f_{yy},B=f_{xy}
\end{aligned}$$

$$\begin{array}{ll}
\begin{array}{l}AC-B^2>0\\ \text{有}\end{array} & \left\{\begin{array}{l}A>0,f(x,y)>f(x_0,y_0),\text{极小值}\\ A<0,f(x,y)<f(x_0,y_0),\text{极大值}\end{array}\right.\\
\begin{array}{l}AC-B^2<0\\ \text{无}\end{array} & \left\{\begin{array}{l}A=C=0,k=h,\Delta f=f_{x,y}\\ A=C=0,k=-h,\Delta f=-f_{x,y}\\ A\neq C=0,k=0,\Delta f=\frac{1}{2}h^2f_{xx}\\ A\neq C=0,h=-f_{xy}s,k=f_{xx}s,\quad \begin{aligned}\Delta f\ &=\tfrac{1}{2}\left\{f_{xy}^2f_{xx}-2f_{xy}^2f_{xx}+f_{xx}^2f_{yy}\right\}\\ &=\tfrac{1}{2}f_{xx}\left\{f_{xy}^2-2f_{xy}^2+f_{xx}f_{yy}\right\}\\ &=\tfrac{1}{2}f_{xx}\left\{-f_{xy}^2\right\}\end{aligned}\end{array}\right.\\
\begin{array}{l}AC-B^2=0\\ \text{有可能}\end{array} & \left\{\begin{array}{l}f(x,y)=x^2+y^4\\ g(x,y)=x^2+y^3\end{array}\right.
\end{array}$$

9.8.5 最小二乘法

y_i数据 $\Rightarrow f(t)=at+b$线性经验公式

$M=\sum\left[y_i-f(t_i)\right]^2=\sum\left[y_i-(at_i+b)\right]^2$ 偏差平方和,

M 最小为条件选择常数 a,b 的方法, 最小二乘法

$\sqrt{M}$均方误差

$$\text{非线性到线性}M_{min}(a,b)\Rightarrow\left\{\begin{array}{l}M_a(a,b)=0\\ M_b(a,b)=0\\ \frac{\partial M}{\partial a}=-2t_i\sum\left[y_i-(at_i+b)\right]=0\\ \frac{\partial M}{\partial b}=-2\sum\left[y_i-(at_i+b)\right]=0\\ \sum t_iy_i-a\sum {t_i}^2-b\sum y_i=0\\ \sum y_i-a\sum t_iy_i-bn=0\end{array}\right.\Rightarrow\left\{\begin{array}{l}a=\dfrac{\begin{vmatrix}\sum t_iy_i & \sum y_i\\ \sum y_i & n\end{vmatrix}}{\begin{vmatrix}\sum {t_i}^2 & \sum y_i\\ \sum t_iy_i & n\end{vmatrix}}\\ b=\dfrac{\begin{vmatrix}\sum {t_i}^2 & \sum t_iy_i\\ \sum t_iy_i & \sum y_i\end{vmatrix}}{\begin{vmatrix}\sum {t_i}^2 & \sum y_i\\ \sum t_iy_i & n\end{vmatrix}}\end{array}\right.$$

10 重积分

10.1 二重积分的概念与性质

10.1.1 二重积分的概念

曲顶柱体的体积

$V = \lim_{\lambda \to 0} \sum f(\xi_i, \eta_i) \Delta\sigma_i$

平面薄片的质量

$m = \lim_{\lambda \to 0} \sum \mu(\xi_i, \eta_i) \Delta\sigma_i$

$$\begin{array}{l} \text{闭区域}D\text{有界函数} \\ \text{分割}D, \Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_n \end{array} \Rightarrow \iint_D f(x,y)\,d\sigma = \lim_{\lambda \to 0} f(\xi_i, \eta_i)\,\Delta\sigma\text{二重积分}$$

$\Delta\sigma_i = \Delta x_j \cdot y_k, d\sigma \to dxdy, \iint_D f(x,y)\,dxdy$

10.1.2 二重积分的性质

$$\begin{array}{l} g(x,y), g \\ f(x,y), f \end{array} \Rightarrow \begin{array}{l} \iint_D [\alpha f + \beta g]\,d\sigma = \alpha \iint_D f d\sigma + \beta \iint_D g d\sigma \\ \iint_D f d\sigma = \iint_{D_1} f d\sigma + \iint_{D_2} f d\sigma \\ \iint_D 1 \cdot d\sigma = \iint_D \cdot d\sigma = \sigma \\ \iint_D f d\sigma \leqslant \iint_D g d\sigma, f \leqslant g \\ m\sigma \leqslant \iint_D f d\sigma \leqslant M\sigma, m \leqslant f \leqslant M \\ \iint_D f d\sigma = f(\xi, \eta)\,\sigma \quad (\xi, \eta) \in D \end{array}$$

10.2 二重积分的计算法

10.2.1 直角坐标计算二重积分

$$x\text{型}\begin{cases} D : \varphi_1(x) \leqslant y \leqslant \varphi_1(x), a \leqslant x \leqslant b \\ A(x_0) = \int_{\varphi_1(x_0)}^{\varphi_2(x_0)} f(x_0, y)\,dy \\ V = \int_a^b A(x)\,dx = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right] dx = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy = \iint_D f(x,y)\,d\sigma \end{cases}$$

$$y\text{型}\begin{cases} D : \psi_1(y) \leqslant x \leqslant \psi_1(y), a \leqslant y \leqslant b \\ A(y_0) = \int_{\psi_1(y_0)}^{\psi_2(y_0)} f(x, y_0)\,dx \\ V = \int_a^b A(y)\,dy = \int_a^b \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right] dy = \int_a^b dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx = \iint_D f(x,y)\,d\sigma \end{cases}$$

非 x 非 y 型，分割积分面

10.2.2 极坐标计算二重积分

$S_{\text{扇}} = \frac{r^2\theta}{2}$

$\Delta\sigma = \frac{1}{2}\left[(r + \Delta r)^2 - r^2\right]\Delta\theta = \frac{1}{2}\left[\Delta r^2 + 2r\Delta r\right]\Delta\theta = \frac{1}{2}[\Delta r + 2r]\,\Delta r\Delta\theta = \overline{r}\Delta r\Delta\theta$

$\lim_{\lambda \to 0} \sum f(\xi_i, \eta_i)\,\Delta\sigma_i = \lim_{\lambda \to 0} \sum f\left(\overline{r}\cos\overline{\theta}, \overline{r}\sin\overline{\theta}\right)\overline{r}\Delta r\Delta\theta$

$\iint_D f(x,y)\,d\sigma = \iint_D f(x,y)\,dxdy = \iint_D f(r\cos\theta, r\sin\theta)\,rdrd\theta$

$\iint_D f(r\cos\theta, r\sin\theta)\,rdrd\theta = \int_a^b \left[\int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos\theta, r\sin\theta)\,rdr\right] d\theta$

$\sigma = \iint_D rdrd\theta = \frac{1}{2}\int_a^b \left[\varphi_2(\theta)^2 - \varphi_1(\theta)^2\right] d\theta = \frac{1}{2}\int_a^b \varphi_2(\theta)^2 d\theta$

10.2.3 二重积分换元法

$f\left(x,y\right),D$内连续,
变换$T:x=x\left(u,v\right),y=y\left(u,v\right)$
uOv面$D^{'}\to xOy$面D,一对一
$x\left(u,v\right),y\left(u,v\right)$在$D^{'}$一阶连续偏导,
$D^{'}$上雅可比式,$J\left(u,v\right)=\frac{\partial(x,y)}{\partial(u,v)}\neq 0$,

$$\Rightarrow \iint_D f\left(x,y\right)dxdy=\iint_{D'} f\left(x\left(u,v\right),y\left(u,v\right)\right)\left|J\left(u,v\right)\right|dudv$$

$c^2=a^2+b^2-2ab\cos c$

$$\begin{cases} M_1:x_1=x\left(u,v\right),y_1=y\left(u,v\right) \\ M_2:x_2=x\left(u+h,v\right)=x_1+x_u\left(u,v\right)h+o\left(h\right) \\ y_2=y\left(u+h,v\right)=y_1+y_u\left(u,v\right)h+o\left(h\right) \\ M_4:x_3=x\left(u,v+h\right)=x_1+x_v\left(u,v\right)h+o\left(h\right) \\ y_3=y\left(u,v+h\right)=y_1+y_v\left(u,v\right)h+o\left(h\right) \end{cases}$$

$$\begin{aligned} s_{\text{三角形}} &=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix} \\ &=\frac{1}{2}\begin{vmatrix} x_u\left(u,v\right)h & y_u\left(u,v\right)h \\ x_v\left(u,v\right)h & y_v\left(u,v\right)h \end{vmatrix}=\frac{1}{2}h^2\begin{vmatrix} x_u\left(u,v\right) & y_u\left(u,v\right) \\ x_v\left(u,v\right) & y_v\left(u,v\right) \end{vmatrix}=\frac{1}{2}h^2\begin{vmatrix} x_u\left(u,v\right) & x_v\left(u,v\right) \\ y_u\left(u,v\right) & y_v\left(u,v\right) \end{vmatrix}=\frac{1}{2}h^2\frac{\partial(x,y)}{\partial(u,v)} \end{aligned}$$

10.3 三重积分

10.3.1 三重积分的概念

$f\left(x,y,z\right)$,有界闭区域Ω,
分割成$\Delta v_1,\Delta v_2,\cdots,\Delta v_n$,

$$\Rightarrow \iiint_D f\left(x,y,z\right)dv=\iiint_D f\left(x,y,z\right)dxdydz=\lim_{\lambda\to 0}\sum f\left(\xi_i,\eta_i,\zeta_i\right)\Delta v_i$$

10.3.2 三重积分的计算

利用直角坐标系计算

$F\left(x_0,y_0\right)=\int_{z_1(x_0,y_0)}^{z_2(x_0,y_0)} f\left(x_0,y_0,z\right)dz$

$$\iiint_\Omega f\left(x,y,z\right)dv \begin{aligned} &=\iint_{D_{xy}} F\left(x,y\right)d\sigma=\iint_{D_{xy}}\left[\int_{z_1(x,y)}^{z_2(x,y)} f\left(x,y,z\right)dz\right]d\sigma=\int_a^b dx\int_{y_1(x)}^{y_2(x)} dy\int_{z_1(x,y)}^{z_2(x,y)} f\left(x,y,z\right)dz \\ &=\int_{c_1}^{c_2} dz\iint_{D_z} f\left(x,y,z\right)dxdy \end{aligned}$$

利用柱坐标系计算

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases} \Rightarrow \iiint_\Omega f\left(x,y,z\right)dxdydz=\iiint_\Omega f\left(r\cos\theta,r\sin\theta,z\right)rdrd\theta dz$$

$dv=rdrd\theta dz$

利用球面坐标计算

$$\begin{cases} x=OP\cos\theta=r\sin\varphi\cos\theta \\ y=OP\sin\theta=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases} \quad \iiint_\Omega f\left(x,y,z\right)dxdydz=\iiint_\Omega f\left(r\sin\varphi\cos\theta,r\sin\varphi\sin\theta,r\cos\varphi\right)r^2\sin\varphi drd\varphi d\theta$$

φ,Z 轴;θ,X 轴
$dv=r^2\sin\varphi drd\varphi d\theta$

10.4 重积分的应用

曲面的面积

曲面$S:z=f\left(x,y\right)$

$\left|\left(f_x,f_y,-1\right)\cdot\left(0,0,1\right)\right|=1=\sqrt{{f_x}^2+{f_y}^2+-1^2}\cdot1\cdot\cos\theta,$ $A=\iint_D\sqrt{{f_x}^2+{f_y}^2+1}d\sigma$

$\cos\theta=\frac{1}{\sqrt{{f_x}^2+{f_y}^2+(-1)^2}}$

$\frac{d\sigma}{dA}=\cos\theta,\frac{\sum\sigma_k}{\sum A_k}=\cos\theta,dA=\frac{1}{\cos\theta}d\sigma$

* 利用曲面的参数方程求曲面的面积

$$\begin{cases}x=x\left(u,v\right)\\y=y\left(u,v\right)\quad\left(u,v\right)\in D\\z=z\left(u,v\right)\end{cases}\Rightarrow\begin{array}{l}A=\iint_D\sqrt{EG-F^2}dudv\\E=x_u^2+y_u^2+z_u^2\\F=x_ux_v+y_uy_v+z_uz_v\\G=x_v^2+y_v^2+z_v^2\end{array}$$

x,y,z在D内连续一阶

$\frac{\partial(x,y)}{\partial(u,v)},\frac{\partial(y,z)}{\partial(u,v)},\frac{\partial(z,x)}{\partial(u,v)}$不全为零

质心

$$\begin{array}{l}\overline{x}=\frac{M_y}{M}=\frac{\sum m_ix_i}{\sum m_i},\\\overline{y}=\frac{M_x}{M}=\frac{\sum m_iy_i}{\sum m_i},\\M_y,M_x\text{净距},M\text{质点系总质量}\end{array}\Rightarrow\begin{array}{l}dM_y=x\mu\left(x,y\right)d\sigma,M_y=\iint_Dx\mu\left(x,y\right)d\sigma\\dM_x=y\mu\left(x,y\right)d\sigma,M_x=\iint_Dy\mu\left(x,y\right)d\sigma\\dM=\mu\left(x,y\right)d\sigma,M=\iint_D\mu\left(x,y\right)d\sigma\\\overline{x}=\frac{M_y}{M}=\frac{\iint_Dx\mu(x,y)d\sigma}{\iint_D\mu(x,y)d\sigma}\xlongequal[A=\iint_Dd\sigma]{\mu(x,y)=c}\frac{1}{A}\iint_Dx\mu\left(x,y\right)d\sigma\\\overline{y}=\frac{M_x}{M}=\frac{\iint_Dy\mu(x,y)d\sigma}{\iint_D\mu(x,y)d\sigma}\xlongequal[A=\iint_Dd\sigma]{\mu(x,y)=c}\frac{1}{A}\iint_Dy\mu\left(x,y\right)d\sigma\end{array}$$

$$M=\iiint_\Omega\rho\left(x,y,z\right)dv,\begin{cases}\overline{x}=\frac{1}{M}\iiint_\Omega x\rho\left(x,y,z\right)dv,\\\overline{y}=\frac{1}{M}\iiint_\Omega y\rho\left(x,y,z\right)dv,\\\overline{z}=\frac{1}{M}\iiint_\Omega z\rho\left(x,y,z\right)dv,\end{cases}$$

转动惯量

$$\begin{array}{l}I_x=\sum y_i^2m_i,\\I_y=\sum x_i^2m_i\end{array}\Rightarrow\begin{array}{l}dI_x=y^2\mu\left(x,y\right)d\sigma\\dI_y=x^2\mu\left(x,y\right)d\sigma\\I_x=\iint_Dy^2\mu\left(x,y\right)d\sigma\\I_y=\iint_Dx^2\mu\left(x,y\right)d\sigma\end{array}$$

$I_x=\iiint_\Omega\left(y^2+z^2\right)\rho\left(x,y,z\right)dv$

$I_y=\iiint_\Omega\left(x^2+z^2\right)\rho\left(x,y,z\right)dv$

$I_z=\iiint_\Omega\left(y^2+x^2\right)\rho\left(x,y,z\right)dv$

引力

$F=G\frac{mM}{r^2}$

$\left(\cos\alpha,\cos\beta,\cos\gamma\right)=\left(\frac{x-x_0}{r},\frac{y-y_0}{r},\frac{z-z_0}{r}\right)$

$F=\left(\left|F\right|\cos\alpha,\left|F\right|\cos\beta,\left|F\right|\cos\gamma\right)\xlongequal{\left|dF\right|=G\frac{1\cdot\rho(x,y,z)dv}{r^2}}\left(\iiint_\Omega\frac{G\rho(x,y,z)(x-x_0)}{r^2\cdot r}dv,,\right)$

10.5 含参变量的积分

$f\left(x,y\right),R=\left[a,b\right]\times\left[c,d\right]=\left\{\left[x,y\right]\left|x\in\left[a,b\right],y\in\left[c,d\right]\right.\right\}$

$\varphi\left(x\right)=\int_c^df\left(x,y\right)dy\quad\left(a\leqslant x\leqslant b\right)$ 含参数变量积分

$f\left(x,y\right)$ 在R连续,$\varphi\left(x\right)$ 在 $\left[a,b\right]$ 连续

$f\left(x,y\right)$ 在R连续, $\int_a^bdx\int_c^df\left(x,y\right)dy=\int_c^ddy\int_a^bf\left(x,y\right)dx$

$f\left(x,y\right),f_x\left(x,y\right)$ 在R连续,$\varphi\left(x\right)$ 在 $\left[a,b\right]$ 可微,$\varphi'\left(x\right)=\frac{d}{dx}\int_c^df\left(x,y\right)dy=\int_c^df_x\left(x,y\right)dy$

$\Phi\left(x\right)=\int_{\alpha(x)}^{\beta(x)}f\left(x,y\right)dy\quad\left(a\leqslant x\leqslant b\right)$ 含参数变量积分

$f\left(x,y\right)$ 在R连续,$\alpha\left(x\right),\beta\left(x\right)$ 在 $\left[a,b\right]$ 连续,$c\leqslant\alpha\left(x\right),\beta\left(x\right)\leqslant d,\Phi\left(x\right)$ 在 $\left[a,b\right]$ 连续,

$f\left(x,y\right),f_x\left(x,y\right)$ 在R连续,$\alpha\left(x\right),\beta\left(x\right)$ 在 $\left[a,b\right]$ 可微,$c\leqslant\alpha\left(x\right),\beta\left(x\right)\leqslant d$ $\Phi\left(x\right)$ 在 $\left[a,b\right]$ 可微,

$$\Phi'\left(x\right)=\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)}f\left(x,y\right)dy=\int_{\alpha(x)}^{\beta(x)}f_x\left(x,y\right)dy+f\left[x,\beta\left(x\right)\right]\beta'\left(x\right)-f\left[x,\alpha\left(x\right)\right]\alpha'\left(x\right)$$

11 曲线积分与曲面积分

11.1 对弧长的曲线积分

11.1.1 对弧长的曲线积分的概念与性质

曲柄构件的质量

$$\begin{aligned} m &\approx \sum \mu\left(\xi_i,\eta_i\right)\Delta s_i \\ &= \lim_{\lambda\to 0}\sum \mu\left(\xi_i,\eta_i\right)\Delta s_i \end{aligned}$$

$$\begin{array}{l} f\left(x,y\right)\text{在直线}L\text{上有界}, \\ L\text{插入点}M_1,M_2,\cdots,M_{n-1}\text{分割弧}\Delta s_i, \end{array} \Rightarrow \begin{array}{l} \int_L f\left(x,y\right)ds = \lim_{\lambda\to 0}\sum f\left(\xi_i,\eta_i\right)\Delta s_i, \\ \text{函数对弧长积分，第一类曲线积分} \end{array}$$

$\int_\Gamma f\left(x,y,z\right)ds = \lim_{\lambda\to 0}\sum f\left(\xi_i,\eta_i,\zeta_i\right)\Delta s_i$

封闭曲线L, $\int_L f\left(x,y\right)ds = \oint_L f\left(x,y\right)ds$

分段光滑 $\int_{L_1+L_2} f\left(x,y\right)ds = \int_{L_1} f\left(x,y\right)ds + \int_{L_2} f\left(x,y\right)ds$

常数α,β, $\int_L \left[\alpha f\left(x,y\right)+\beta g\left(x,y\right)\right]ds = \alpha\int_L f\left(x,y\right)ds + \beta\int_L g\left(x,y\right)ds$

曲线L上$f\left(x,y\right)\leqslant g\left(x,y\right)$, $\int_L f\left(x,y\right)ds \leqslant \int_L g\left(x,y\right)ds$

$$\begin{array}{l} -\left|c\right|\leqslant a\leqslant\left|c\right|, \\ \left|a\right|\leqslant\left|c\right| \end{array} \Rightarrow -\int\left|c\right|ds \leqslant \begin{array}{l} \int a\,ds \leqslant \int\left|c\right|ds \\ \left|\int a\,ds\right| \leqslant \int\left|c\right|ds \end{array} \xrightarrow{a=c=f(x,y)} \left|\int f\left(x,y\right)ds\right| \leqslant \int\left|f\left(x,y\right)\right|ds$$

11.1.2 对弧长的曲线积分的计算法

$$\begin{array}{l} f\left(x,y\right)\text{在曲线}L\text{连续} \\ L:\begin{cases} x=\varphi\left(t\right) \\ y=\psi\left(t\right) \end{cases} t\in\left[\alpha,\beta\right] \\ \varphi\left(t\right),\psi\left(t\right),\text{在}\left[\alpha,\beta\right]\text{连续一阶导} \end{array} \Rightarrow \begin{aligned} \int_L f\left(x,y\right)ds &= \lim_{\lambda\to 0}\sum f\left(\varphi\left(t\right),\psi\left(t\right)\right)\sqrt{\varphi'^2\left(t\right)+\psi'^2\left(t\right)}\Delta t_i \\ &= \int_\alpha^\beta f\left(\varphi\left(t\right),\psi\left(t\right)\right)\sqrt{\varphi'^2\left(t\right)+\psi'^2\left(t\right)}dt \end{aligned}$$

$\varphi'^2\left(t\right)+\psi'^2\left(t\right)\neq 0$

$$y=\psi\left(x\right)\to\begin{cases} x=\varphi\left(x\right)=x \\ y=\psi\left(x\right) \end{cases}\to\sqrt{\varphi'^2\left(x\right)+\psi'^2\left(x\right)}=\sqrt{1+\psi'^2\left(x\right)}$$

$$x=\varphi\left(y\right)\to\begin{cases} x=\varphi\left(y\right) \\ y=\psi\left(y\right)=y \end{cases}\to\sqrt{\varphi'^2\left(y\right)+\psi'^2\left(y\right)}=\sqrt{1+\varphi'^2\left(y\right)}$$

$$\begin{cases} x=\varphi\left(t\right) \\ y=\psi\left(t\right) \\ z=\omega\left(t\right) \end{cases} t\in\left[\alpha,\beta\right]\to\int_\Gamma f\left(x,y,z\right)ds=\int_\Gamma f\left(\varphi\left(t\right),\psi\left(t\right),\omega\left(t\right)\right)\sqrt{\varphi'^2\left(t\right)+\psi'^2\left(t\right)+\omega'^2\left(t\right)}dt$$

11.2 对坐标的曲线积分

11.2.1 对坐标的曲线积分的概念与性质

变力沿曲线所作的功

$$\begin{aligned} W &= \sum\Delta W \approx \sum\left[P\left(\xi_i,\eta_i\right)\Delta x_i+Q\left(\xi_i,\eta_i\right)\Delta y_i\right] \\ &= \lim_{\lambda\to 0}\sum\left[P\left(\xi_i,\eta_i\right)\Delta x_i+Q\left(\xi_i,\eta_i\right)\Delta y_i\right] \\ &\mathbf{F}\left(x,y\right)=\left(P\left(x,y,z\right),Q\left(x,y,z\right)\right) \\ &\mathbf{A}\left(x,y,z\right)=\left(P\left(x,y,z\right),Q\left(x,y,z\right),R\left(x,y,z\right)\right) \\ &d\mathbf{r_2}=\left(dx,dy\right) \\ &d\mathbf{r_3}=\left(dx,dy,dz\right) \end{aligned} \to \begin{array}{l} \int_L P\left(x,y\right)dx=\lim_{\lambda\to 0}\sum P\left(\xi_i,\eta_i\right)\Delta x_i \\ \int_L P\left(x,y\right)dy=\lim_{\lambda\to 0}\sum Q\left(\xi_i,\eta_i\right)\Delta y_i \\ \text{第二类曲线积分} \\ W=\int_L\mathbf{F}\left(x,y\right)\cdot d\mathbf{r_2} \\ W_3=\int_\Gamma\mathbf{A}\left(x,y,z\right)\cdot d\mathbf{r_3} \end{array}$$

$\int_L \alpha\mathbf{F}_1+\beta\mathbf{F}_2\cdot d\mathbf{r}_2=\alpha\int_L\mathbf{F}_1\cdot d\mathbf{r}_2+\beta\int_L\mathbf{F}_2\cdot d\mathbf{r}_2$

$\int_L\mathbf{F}\cdot d\mathbf{r}_2=\int_{L_1}\mathbf{F}\cdot d\mathbf{r}_2+\int_{L_2}\mathbf{F}\cdot d\mathbf{r}_2$

$\int_L\mathbf{F}\cdot d\mathbf{r}=-\int_{L^-}\mathbf{F}\cdot d\mathbf{r}$

11.2.2 对坐标的曲线积分的计算法

$$\begin{array}{l} P(x,y), Q(x,y)\text{ 在}L\text{连续} \\ L:\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \qquad \rightarrow \int_L P(x,y)\,dx+Q(x,y)\,dy=\int_\alpha^\beta \left[P(\varphi(t),\psi(t))\,\varphi'+Q(\varphi(t),\psi(t))\,\psi'\right]dt \\ \varphi(t),\psi(t)\text{ 在 }[\alpha,\beta]\text{ 连续一阶导} \\ \varphi'^2(t)+\psi'^2(t)\neq 0 \\ x=\varphi(y)\rightarrow L:\begin{cases} x=\varphi(y) \\ y=y \end{cases} \rightarrow \int_L P(x,y)\,dx+Q(x,y)\,dy=\int_\alpha^\beta\left[P(\varphi(y),y)\,\varphi'(y)+Q(\varphi(y),y)\right]dy \\ y=\psi(x)\rightarrow L:\begin{cases} x=x \\ y=\psi(x) \end{cases} \rightarrow \int_L P(x,y)\,dx+Q(x,y)\,dy=\int_\alpha^\beta\left[P(x,\psi(x))+Q(x,\psi(x))\,\psi'(x)\right]dx \end{array}$$

$$\Gamma:\begin{cases} x=\varphi(t) \\ y=\psi(t) \\ z=\omega(t) \end{cases} \rightarrow \int_\Gamma (P,Q,R)\cdot(dx,dy,dz)=\int_\Gamma P\varphi'+Q\psi'+R\omega' dt$$

11.2.3 两类曲线积分的关系

$$\begin{array}{l} L:\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \\ \text{记}k=\sqrt{\varphi'^2(t)+\psi'^2(t)} \\ \text{切向量}n=(\cos\alpha,\cos\beta)=\left(\frac{\varphi'}{k},\frac{\psi'}{k}\right) \end{array} \rightarrow \begin{array}{l} \begin{aligned} \int_L[P\cos\alpha+Q\cos\beta]\,ds &=\int_L\left[P\frac{\varphi'}{k}+Q\frac{\psi'}{k}\right]kdt \\ &=\int_L\left[P\varphi'+Q\psi'\right]dt \end{aligned} \\ \int_L(P,Q)\cdot(\cos\alpha,\cos\beta)\,ds=\int(P,Q)\cdot(dx,dy)=\int(P,Q)\,dr \\ \int_L \mathbf{F}\cdot nds=\int_L\mathbf{F}_n ds=\int_L\mathbf{F}\cdot dr \\ \int_L(P,Q,R)\cdot(\cos\alpha,\cos\beta,\cos\gamma)\,ds=\int(P,Q,R)\cdot(dx,dy,dz)=\int(P,Q,R)\,dr \\ \int_\Gamma\mathbf{A}\cdot nds=\int_\Gamma\mathbf{A}_n ds=\int_\Gamma\mathbf{A}\cdot dr \\ dr=nds\text{有向曲线元} \end{array}$$

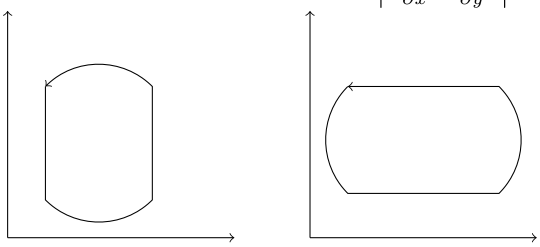
11.3 格林公式及其应用

11.3.1 格林公式

D 内任意封闭曲线所围部分都属于 D, **单连通**，非单连通**复连通**

边界曲线正向，左边是 D

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=-\iint_D\begin{vmatrix} P & Q \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{vmatrix}=\oint_L Pdx+Qdy$$



$$\iint_D\frac{\partial P}{\partial y}dxdy=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\frac{\partial Q}{\partial y}dy\right]dx=\int_a^b P(x,\varphi_2(x))-P(x,\varphi_1(x))\,dx$$

$$\iint_D\frac{\partial Q}{\partial x}dxdy=\int_c^d\left[\int_{\psi_1(y)}^{\psi_2(y)}\frac{\partial Q}{\partial x}dx\right]dy=\int_c^d Q(\psi_2(y),y)-Q(\psi_1(y),y)\,dy$$

$\oint Pdx = \int_{L_1} Pdx + \int_{L_2} Pdx + \int_{L_3} Pdx + \int_{L_4} Pdx$
$= \int_{L_1} P\left(x, \varphi_2\left(x\right)\right) dx + \int_{L_2} P\left(x, \varphi_1\left(x\right)\right) dx$
$= \int_b^a P\left(x, \varphi_2\left(x\right)\right) dx + \int_a^b P\left(x, \varphi_1\left(x\right)\right) dx$
$= \int_a^b P\left(x, \varphi_1\left(x\right)\right) - P\left(x, \varphi_2\left(x\right)\right) dx$
$= -\left(\int_a^b P\left(x, \varphi_2\left(x\right)\right) - P\left(x, \varphi_1\left(x\right)\right) dx\right)$
$L_1 : y = \varphi_2\left(x\right), x \in [b, a]$
$L_2 : y = \varphi_1\left(x\right), x \in [a, b]$
$L_3 : x = a, y \in [d, c]$
$L_4 : x = b, y \in [a, b]$
$\oint Qdy = \int_{L1} Qdy + \int_{L2} Qdy + \int_{L3} Qdy + \int_{L4} Qdy$
$= \int_{L_3} Q\left(\psi_1, y\right) dy + \int_{L_4} Q\left(\psi_2, y\right) dy$
$= \int_d^c Q\left(\psi_1, y\right) dy + \int_c^d Q\left(\psi_2, y\right) dy$
$= \int_c^d Q\left(\psi_2, y\right) - Q\left(\psi_1, y\right) dy$
$L_1 : y = d, x \in [b, a]$
$L_2 : y = c, x \in [a, b]$
$L_3 : x = \psi_1\left(y\right)$
$L_4 : x = \psi_2\left(y\right)$
非标准分割

11.3.2 平面上曲线积分与路径无关的条件

区域 G 内任意点 A,B, 任意 A 到 B 曲线,
$if \quad \int_{L_1} Pdx + Qdy = \int_{L_2} Pdx + Qdy, then$ 曲线积分 $\int_L Pdx + Qdy$在 G 内与路径无关, $\oint_{L_1 - L_2} Pdx + Qdy = 0$
if 单连通区域G, P, Q在 G 内连续一阶偏导数, $then$ 曲线积分 $\int_L Pdx + Qdy$在 G 内与**路径无关** $\Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
奇点

11.3.3 二元函数的全微分求积

单连通区域 $G, P\left(x, y\right), Q\left(x, y\right)$ 在 G 内具有一阶连续偏，
$Fdx + Qdy$ 在 G 内为某一函数 $\mu\left(x, y\right)$ 的全微分 $\Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
曲线积分 $\int_L Pdx + Qdy$ 在 G 内与路径无关 $\Leftrightarrow \exists \mu\left(x, y\right), du = Pdx + Qdy$

全微分方程

微分方程 $P\left(x, y\right) dx + Q\left(x, y\right) dy = 0$ 左端是某函数 $\mu(x, y)$ 全微分, **全微分方程**

11.3.4 曲线积分的基本定理

曲线积分 $\int_L \mathbf{F} \cdot d\mathbf{r}$ 在区域 G 内与积分路径无关，向量场 $\mathbf{F}$ 为保守场

$F\left(x, y\right) = \left(P\left(x, y\right) + Q\left(x, y\right)\right)$ 是区域 G 内向量场,$P\left(x, y\right), Q\left(x, y\right)$ 在 G 内连续，且存在数量函数使得 $F = \nabla f$, 则曲线积分 $\int_L \mathbf{F} \cdot d\mathbf{r}$ 在 G 内与路径无关，且 $\int_L \mathbf{F} \cdot d\mathbf{r} = f\left(B\right) - f\left(A\right)$ **曲线积分基本公式**

11.4 对面积的曲面积分

11.4.1 对面积的曲面积分的概念与性质

光滑曲面 Σ, 函数 $f\left(x, y, z\right)$ 在 Σ 有界，任意分割 n 小块 ΔS_i, 无关面分法，点取法，极限 $\lim_{\lambda \to 0} \sum f\left(\xi_i, \eta_i, \zeta_i\right) \Delta S_i = \iint_\Sigma f\left(x, y, z\right) dS$ **第一类曲面积分**

11.4.2 对面积的曲面积分的计算法

曲面 $\Sigma: z=z(x,y), n=(-z_x,-z_y,1)=\left(\frac{-z_x}{\sqrt{{z_x}^2+{z_y}^2+1}},\frac{-z_y}{\sqrt{{z_x}^2+{z_y}^2+1}},\frac{1}{\sqrt{{z_x}^2+{z_y}^2+1}}\right)$

$|n\cdot(1,0,0)|=\frac{z_x}{\sqrt{{z_x}^2+{z_y}^2+1}}=\cos\alpha \quad \Delta S_i\cos\alpha=(\Delta\sigma_i)_{yz} \quad \Delta S_i=\frac{1}{\cos\alpha}(\Delta\sigma_i)_{yz}$

$|n\cdot(0,1,0)|=\frac{z_y}{\sqrt{{z_x}^2+{z_y}^2+1}}=\cos\beta \quad \Delta S_i\cos\beta=(\Delta\sigma_i)_{xz} \quad \Delta S_i=\frac{1}{\cos\beta}(\Delta\sigma_i)_{xz}$

$|n\cdot(0,0,1)|=\frac{1}{\sqrt{{z_x}^2+{z_y}^2+1}}=\cos\gamma \quad \Delta S_i\cos\gamma=(\Delta\sigma_i)_{xy} \quad \Delta S_i=\frac{1}{\cos\gamma}(\Delta\sigma_i)_{xy}$

$\lim_{\lambda\to0}\sum f(\xi_i,\eta_i,\zeta_i)\Delta S_i=\lim_{\lambda\to0}\sum f(\xi_i,\eta_i,\zeta_i)\frac{1}{\cos\gamma}(\Delta\sigma_i)_{xy}$

$\int_\Sigma f(x,y,z)\,dS=\iint_{D_{xy}}f[x,y,(x,y)]\sqrt{{z_x}^2+{z_y}^2+1}dxdy$

11.5 对坐标的曲面积分

11.5.1 对坐标的曲面积分的概念与性质

ΔS 单位法向量 n ,$n\cdot(0,0,1)=\cos\gamma\Rightarrow\Delta S$在$xOy$面投影$(\Delta S)_{xy}=\begin{cases}(\Delta\sigma)_{xy}, & \cos\gamma>0\\ -(\Delta\sigma)_{xy}, & \cos\gamma<0\\ 0, & \cos\gamma=0\end{cases}$

流向曲面一侧的流量

$Av\cdot n$

光滑有向曲面 Σ, 函数 $R(x,y,z)$ 在 Σ 有界，任意分割 Σ 为 ΔS_i, $\begin{array}{l}\Delta S_i\text{在}xOy\text{面投影为}(\Delta S_i)_{xy}\\ \Delta S_i\text{在}xOz\text{面投影为}(\Delta S_i)_{xz}\\ \Delta S_i\text{在}yOz\text{面投影为}(\Delta S_i)_{yz}\end{array}$

第二类曲面积分,函数在有向面上对坐标轴 x,y(x,z/y,z) 的积分 $\begin{cases}\iint_\Sigma R(x,y,z)\,dxdy=\lim_{\lambda\to0}R(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{xy}\\ \iint_\Sigma Q(x,y,z)\,dxdz=\lim_{\lambda\to0}Q(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{xz}\\ \iint_\Sigma P(x,y,z)\,dydz=\lim_{\lambda\to0}P(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{yz}\end{cases}$

$\iint_\Sigma(P,Q,R)\cdot(dydz,dxdz,dxdy)=\iint_{\Sigma_1}(P,Q,R)\cdot(dydz,dxdz,dxdy)+\iint_{\Sigma_2}(P,Q,R)\cdot(dydz,dxdz,dxdy)$

$\iint_\Sigma(P,Q,R)\cdot(dydz,dxdz,dxdy)=-\iint_{\Sigma^-}(P,Q,R)\cdot(dydz,dxdz,dxdy)$

$\iint_{\Sigma^-}Pdydz=-\iint_\Sigma Pdydz$

$\iint_{\Sigma^-}Pdxdz=-\iint_\Sigma Pdxdz$

$\iint_{\Sigma^-}Pdxdy=-\iint_\Sigma Pdxdy$

11.5.2 对坐标的曲面积分的计算法

$\Sigma: z=z(x,y),(\Delta S_i)_{xy}=\pm(\Delta\sigma)_{xy},\iint_\Sigma R(x,y,z)\,dxdy=\pm\iint_{D_{xy}}R(x,y,z(x,y))\,dxdy$

11.5.3 两类曲面积分之间的关系

$\Sigma: z=z(x,y)$,Σ 法向量 $\mathbf{n}=\left(\frac{-z_x}{|n|},\frac{-z_y}{|n|},\frac{1}{|n|}\right)=(\cos\alpha,\cos\beta,\cos\gamma)$

$\iint_\Sigma(P,Q,R)\cdot(dydz,dxdz,dxdy)=\iint_\Sigma(P,Q,R)\cdot(\cos\alpha,\cos\beta,\cos\gamma)\,dS$

$\iint_\Sigma\mathbf{A}\cdot d\mathbf{S}=\iint_\Sigma\mathbf{A}\cdot\mathbf{n}dS=\iint_\Sigma\mathbf{A}_n dS$

$d\mathbf{S}=\mathbf{n}dS$ 有向曲面元

11.6 高斯公式　通量与散度

11.6.1 高斯公式

空间闭区域 Ω 由分片光滑闭曲面 (有向，曲面外侧)Σ 组成，$P(x,y,z),Q(x,y,z),R(x,y,z)$ 在 Ω 上有一阶连续偏导数，Σ 方向余弦 $n=(\cos\alpha,\cos\beta,\cos\gamma)$ 则

$$\iint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dv=\oiint_{\Sigma}(P,Q,R)\cdot(dydz,dxdz,dxdy)=\oiint_{\Sigma}(P,Q,R)\cdot(\cos\alpha,\cos\beta,\cos\gamma)\,dS$$

$$\begin{aligned}\iiint_{\Omega}\tfrac{\partial R}{\partial z}dv&=\iint_{D_{xy}}\left\{\int_{z_1(x,y)}^{z_2(x,y)}\tfrac{\partial R}{\partial z}dz\right\}dxdy\\&=\iint_{D_{xy}}\left\{R(x,y,z_2)-R(x,y,z_1)\right\}dxdy\\\oiint_{\Sigma}R(x,y,z)\,dxdy&=\iint_{\Sigma_1}Rdxdy+\iint_{\Sigma_2}Rdxdy+\iint_{\Sigma_3}Rdxdy\\&=\iint_{\Sigma_1}R(x,y,z)\,dxdy+\iint_{\Sigma_2}R(x,y,z)\,dxdy+0\\&=-\iint_{D_{xy}}R(x,y,z_1)\,dxdy+\iint_{D_{xy}}R(x,y,z_2)\,dxdy\end{aligned}\Rightarrow\begin{aligned}\iint_{\Omega}\tfrac{\partial P}{\partial x}dv&=\oiint_{\Sigma}Pdydz\\\iint_{\Omega}\tfrac{\partial Q}{\partial y}dv&=\oiint_{\Sigma}Qdxdz\\\iint_{\Omega}\tfrac{\partial R}{\partial z}dv&=\oiint_{\Sigma}Rdxdy\end{aligned}$$

不规则体积分割

函数 $u(x,y,z),v(x,y,z)$ 在闭区域 Ω 上具有一二阶连续偏导数，

则 $\iint_{\Omega}u\Delta vdxdydz=\oiint_{\Sigma}u\dfrac{\partial v}{\partial n}dS-\iiint_{\Omega}\left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y}+\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y}\right)$ **格林第一公式**,Σ 为闭区域 Ω 边界曲面,$\frac{\partial v}{\partial n}$ 为函数 $v(x,y,z)$ 沿 Σ 外法线方向的方向导数,$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ 称为**拉普拉斯算子**

$$\begin{aligned}\oiint_{\Sigma}u\frac{\partial v}{\partial n}dS&=\oiint_{\Sigma}u\left(\frac{\partial v}{\partial x}\cos\alpha+\frac{\partial v}{\partial y}\cos\beta+\frac{\partial v}{\partial z}\cos\gamma\right)dS\\&=\oiint_{\Sigma}\left[\left(u\frac{\partial v}{\partial x}\right)\cos\alpha+\left(u\frac{\partial v}{\partial y}\right)\cos\beta+\left(u\frac{\partial v}{\partial z}\right)\cos\gamma\right]dS\\&=\iiint_{\Omega}\left[\tfrac{\partial}{\partial x}\left(u\tfrac{\partial v}{\partial x}\right)+\tfrac{\partial}{\partial y}\left(u\tfrac{\partial v}{\partial y}\right)+\tfrac{\partial}{\partial z}\left(u\tfrac{\partial v}{\partial z}\right)\right]dxdydz\\&=\iiint_{\Omega}\left[u\tfrac{\partial^2 v}{\partial x^2}+u\tfrac{\partial^2 v}{\partial y^2}+u\tfrac{\partial^2 v}{\partial z^2}+\tfrac{\partial u}{\partial x}\tfrac{\partial v}{\partial x}+\tfrac{\partial u}{\partial y}\tfrac{\partial v}{\partial y}+\tfrac{\partial u}{\partial z}\tfrac{\partial v}{\partial z}\right]dxdydz\\&=\iiint_{\Omega}\left[u\Delta v+\tfrac{\partial u}{\partial x}\tfrac{\partial v}{\partial x}+\tfrac{\partial u}{\partial y}\tfrac{\partial v}{\partial y}+\tfrac{\partial u}{\partial z}\tfrac{\partial v}{\partial z}\right]dxdydz\\&=\iiint_{\Omega}u\Delta vdxdydz+\iiint_{\Omega}\tfrac{\partial u}{\partial x}\tfrac{\partial v}{\partial x}+\tfrac{\partial u}{\partial y}\tfrac{\partial v}{\partial y}+\tfrac{\partial u}{\partial z}\tfrac{\partial v}{\partial z}dxdydz\end{aligned}$$

11.6.2 沿任意闭曲面的曲面积分为零的条件

空间区域 G,G 内任意闭曲面所围区域完全属于 G, **空间二维单连通**

空间区域 G,G 内任意闭曲线总可以张成 (做曲面) 完全属于 G, **空间一维单连通**

二维单连通区域 G，若 P,Q,R, 在 G 内具有一阶连续偏导数，则曲面积分 $\iint_{\Sigma}Pdydz+Qdxdz+Rdxdy$ 在 G 内与选取曲面选取无关，只与边界曲线有关 $\Leftrightarrow\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$

11.6.3 通量与散度

向量场$\mathbf{A}(x,y,z)=(P,Q,R)$

有向曲面Σ,法向量$\mathbf{n}$

通量 $\iint_{\Sigma}\mathbf{A}\cdot\mathbf{n}dS$

速度场$v(x,y,z)=(P,Q,R)$

$$\begin{aligned}\iiint_{\Omega}\left(\tfrac{\partial P}{\partial x}+\tfrac{\partial Q}{\partial y}+\tfrac{\partial P}{\partial y}\right)dv&=\oiint_{\Sigma}v_ndS\\\tfrac{1}{V}\iiint_{\Omega}\left(\tfrac{\partial P}{\partial x}+\tfrac{\partial Q}{\partial y}+\tfrac{\partial P}{\partial y}\right)dv&=\tfrac{1}{V}\oiint_{\Sigma}v_ndS\\\left(\tfrac{\partial P}{\partial x}+\tfrac{\partial Q}{\partial y}+\tfrac{\partial P}{\partial y}\right)|_{(\xi,\eta,\zeta)}&=\tfrac{1}{V}\oiint_{\Sigma}v_ndS\quad(\xi,\eta,\zeta)\in\Omega\\\tfrac{\partial P}{\partial x}+\tfrac{\partial Q}{\partial y}+\tfrac{\partial P}{\partial y}&=\lim_{\Omega\to M}\tfrac{1}{V}\oiint_{\Sigma}v_ndS=\mathrm{div}v(M)=\mathbf{\Delta}\cdot\mathbf{A}\end{aligned}$$

速度场v在点M的通量密度, 源头强度**散度**

div$\mathbf{V}$ 处处为零，**无源场**

散度体积分，通量面积分

11.7 斯托克斯公式 * 环流量与旋度

11.7.1 斯托克斯公式

分段光滑空间有向闭曲线 Γ,Γ 张成分片光滑有向曲面 Σ，符合右手定则若 P,Q,R 在曲面 Σ 上具有一阶连续偏导数，则 $\iint_\Sigma\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dxdz+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\oint_\Gamma Pdx+Qdy+Rdz$

曲面$\Sigma: z=z\left(x,y\right)$,法向量$n=(\cos\alpha,\cos\beta,\cos\gamma)=\left(-\frac{z_x}{|n|},-\frac{z_y}{|n|}\ \frac{1}{|n|}\right)$

$$\begin{aligned}\iint_\Sigma\frac{\partial P}{\partial z}dzdx-\frac{\partial P}{\partial y}dydx &=\iint_\Sigma\left(\frac{\partial P}{\partial z}\cos\beta-\frac{\partial P}{\partial y}\cos\gamma\right)dS\\ &\xlongequal{\cos\beta=-f_y\cos\gamma}-\iint_\Sigma\left(\frac{\partial P}{\partial z}f_y+\frac{\partial P}{\partial y}\right)\cos\gamma dS\\ &=-\iint_{D_{xy}}\left(\frac{\partial P}{\partial z}f_y+\frac{\partial P}{\partial y}\right)dxdy\\ &=-\iint_{D_{xy}}\frac{\partial P}{\partial y}dxdy\\ &=\oint_L Pdx\end{aligned}$$

$$\iint_\Sigma\frac{\partial Q}{\partial x}dxdy-\frac{\partial Q}{\partial z}dzdy=\oint_\Gamma Qdy$$

$$\iint_\Sigma\frac{\partial R}{\partial y}dydz-\frac{\partial R}{\partial x}dxdz=\oint_\Gamma Rdz$$

$$\iint_\Sigma\begin{vmatrix}dydz & dxdz & dxdy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}\begin{aligned}&=\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}dS=\oint_\Gamma Pdx+Qdy+Rdz\\ &=\begin{vmatrix}\cos\alpha & \cos\beta & 1\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0\\ P & Q & 0\end{vmatrix}dS=\oint_\Gamma Pdx+Qdy\end{aligned}$$

11.7.2 空间曲线积分与路径无关的条件

一维单连通区域 G，若函数 P,Q,R 在 G 内具有一阶连续偏导数，

则曲线积分 $\int_\Gamma Pdx+Qdy+Rdz$ 在 G 内与路径无关 $\Leftrightarrow\begin{cases}\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\\ \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}\\ \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}\end{cases}$

一维单连通区域 G，若函数 P,Q,R 在 G 内具有一阶连续偏导数，

则存在 $u\left(x,y,z\right),du=Pdx+Qdy+Rdz\Leftrightarrow\begin{cases}\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\\ \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}\\ \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}\end{cases}$

$$\begin{aligned}u\left(x,y,z\right)&=\int_{(x_0,y_0,z_0)}^{(x,y,z)}Pdx+Qdy+Rdz\\ &=\int_{x_0}^{x}Pdx+\int_{y_0}^{y}Qdy+\int_{z_0}^{z}Rdz\end{aligned}$$

11.7.3 环流量与旋度

向量场 $\mathbf{A}=(P,Q,R)$, 分段光滑有向闭曲线 Γ, 单位切向量 τ , 则 $\oint_\Gamma\mathbf{A}\cdot\tau ds$ 称为 A 沿 τ **环流量**

$\mathbf{rot}\ \mathbf{A}=\Delta\times\mathbf{A}=\begin{vmatrix}i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$ **旋度**

旋度处处为零，**无旋场**

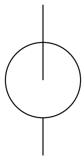
无旋，无源，**调和场**

Σ 单位法向量 $n=(\cos\alpha,\cos\beta,\cos\gamma)$

$\mathbf{rot}\ \mathbf{A}\cdot n=\Delta\times\mathbf{A}\cdot n=\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$

$\iint_\Sigma\mathbf{rot}\ \mathbf{A}\cdot\mathbf{n}dS=\oint_\Gamma\mathbf{A}\cdot\tau ds$

旋度通量，环流量



$$r=\overrightarrow{OM},\omega=(0,0,w)\,,v=\omega\times r,v=\begin{vmatrix} i & j & k \\ 0 & 0 & w \\ x & y & z \end{vmatrix}=(-wy,wx,0)\,,\mathbf{rot}\ \mathbf{v}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix}=(0,0,2w)=2\omega$$

12 无穷级数

12.1 常数项级数的概念和性质

12.1.1 数项级数的概念

数列 $\mu_1, \mu_2, \mu_3, \cdots + \mu_n, \ldots$，表达式 $\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n + \ldots$ 称为 (常数项) 无穷级数, 简称 **(常数项) 级数**，记为 $\sum_{i=1}^{\infty} \mu_i = \mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n + \ldots$, 第 n 项 μ_n 叫做级数的一般项

$S_n = \sum_{i=1}^{n} \mu_i = \mu_1, \mu_2, \mu_3, \cdots + \mu_n$ 级数部分和

新数列 $\{S_n\}$ $\quad S_1, S_2, \ldots, S_n, \ldots$

无穷级数 $\sum_{i=1}^{\infty} \mu_i$，无穷级数的部分和数列 $\{S_n\}$， $\begin{matrix} \lim_{n\to\infty} S_n = s \Rightarrow \sum_{i=1}^{\infty} \mu_i \text{收敛}, s = \mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n + \ldots \\ \lim_{n\to\infty} S_n = \infty(\text{不存在}) \Rightarrow \sum_{i=1}^{\infty} \mu_i \text{发散} \end{matrix}$

$r_n = s - S_n = S_{n+1} + S_{n+2} + \ldots$ 余项

$$\sum_{i=1}^{\infty} \mu_i = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_i$$

$\sum_{i=1}^{\infty} aq^i = a + aq + aq^2 + \cdots + aq^i + \ldots$ 等比级数, $\begin{matrix} S_n = a + aq + \cdots + aq^{n-1} = \frac{a - aq^n}{1-q} \text{部分和数列} \\ |q| \geqslant 1, \text{发散}, |q| \leqslant 1, \text{收敛} \end{matrix}$

$$S_n = S_1 + (S_2 - S_1) + \cdots + (S_n - S_{n-1}) + \ldots = S_1 + \sum_{i=2}^{\infty} = \sum_{i=1}^{\infty} \mu_i$$

12.1.2 收敛级数的基本性质

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s, 则级数 $\sum_{n=1}^{\infty} k\mu_n$ 收敛于 s, 和为 ks, 数乘同敛散

级数 $\sum_{n=1}^{\infty} \mu_n$ 与 $\sum_{n=1}^{\infty} v_n$ 分别收敛于 s 与 σ, 级数 $\sum_{n=1}^{\infty} (\mu_n \pm v_n)$ 也收敛, 和为 $s \pm \sigma$, 收敛和收敛

级数中去掉，加上，改变有限项，不会改变级数的收敛性

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s，任意项加括号组成的级数仍收敛，和为 s, 括号发散源发散

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s(必要条件), 一般项趋近 0，即 $\lim_{n\to\infty} \mu_n = 0$

调和级数 $\sum_{n=1}^{\infty} \frac{1}{n}$ 发散

12.1.3 柯西审敛原理

级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛 $\Leftrightarrow \forall \varepsilon > 0, \exists N > 0, when \quad n > N, \forall p \in Z^+, |\mu_{n+1} + \mu_{n+2} + \cdots + \mu_{n+p}| < \varepsilon$

12.2 常数项级数审敛法

12.2.1 正项级数及其审敛法

各项是正数或零的级数**正项级数**

正项级数 $\sum_{n=1}^{\infty} \mu_n$ 收敛于 s $\Leftrightarrow$ 部分和数列 $\{S_n\}$ 有界

正项级数 $\sum_{n=1}^{\infty} \mu_n, \sum_{n=1}^{\infty} v_n, \mu_n \leqslant v_n, \sum_{n=1}^{\infty} v_n$ 收敛，$\sum_{n=1}^{\infty} \mu_n$ 收敛；$\sum_{n=1}^{\infty} \mu_n$ 发散,$\sum_{n=1}^{\infty} v_n$ 发散

正项级数 $\sum\limits_{n=1}^{\infty}\mu_n,\sum\limits_{n=1}^{\infty}v_n$, 存在正整数 N, $\quad \mu_n \leqslant kv_n\,(n>N,k>0),\sum\limits_{n=1}^{\infty}v_n$ 收敛，$\sum\limits_{n=1}^{\infty}\mu_n$ 收敛；$\sum\limits_{n=1}^{\infty}\mu_n$ 发散,$\sum\limits_{n=1}^{\infty}v_n$ 发散

$$\text{正项级数} \sum_{n=1}^{\infty}\mu_n,\begin{array}{l} if \quad \lim_{n\to\infty}\frac{\mu_n}{v_n}=l,\{0\leqslant l<\infty\},\sum_{n=1}^{\infty}v_n\text{收敛, 则}\sum_{n=1}^{\infty}\mu_n\text{收敛} \\ if \quad \lim_{n\to\infty}\frac{\mu_n}{v_n}=l,\{l>0,+\infty\},\sum_{n=1}^{\infty}v_n\text{发散, 则, }\sum_{n=1}^{\infty}\mu_n\text{发散}\end{array}$$

$$\text{正项级数} \sum_{n=1}^{\infty}\mu_n,\lim_{n\to\infty}\frac{\mu_{n+1}}{\mu_n}=\rho,\begin{array}{l} if \quad \rho>1\ (=\infty)\text{ 发散 (不可能收敛)} \\ if \quad \rho<1\text{收敛} \\ if \quad \rho=1\text{可能收敛}\end{array}$$

$when \quad \rho<1,\forall\rho+\varepsilon=r<1,\exists m, when \quad n\geqslant m,\frac{\mu_{n+1}}{\mu_n}=\rho<\rho+\varepsilon=r<1,\mu_{n+1}<r\mu_n,\mu_{n+k}<r^k\mu_n$

级数 $\sum_{n=1}^{\infty}r^k\mu_n$ 收敛，级数 $\sum_{n=1}^{\infty}\mu_{n+k}$ 收敛

$\sum\limits_{n=1}^{\infty}\mu_n=\sum\limits_{n=1}^{\infty}\mu_{n+k}+\sum\limits_{n=1}^{k}\mu_n$ 则级数 $\sum_{n=1}^{\infty}\mu_n,\sum_{n=1}^{\infty}\mu_{n+k}$ 同敛散，级数 $\sum_{n=1}^{\infty}\mu_n$ 收敛

$$\text{正项级数} \sum_{n=1}^{\infty}\mu_n,\lim_{n\to\infty}\sqrt[n]{\mu_n}=\rho,\begin{array}{l} if \quad \rho>1\ (=\infty)\text{ 发散 (不可能收敛)} \\ if \quad \rho<1\text{收敛} \\ if \quad \rho=1\text{可能收敛}\end{array}$$

$when \quad \rho<1,\mu_n<r^n\,(r<1)$, 级数$\sum\limits_{n=1}^{\infty}r^n$ 收敛, 则 $\sum\limits_{n=1}^{\infty}\mu_n$ 收敛

$$\text{正项级数} \sum_{n=1}^{\infty}\mu_n,\begin{array}{l} if \quad \lim_{n\to\infty}n^p\mu_n=\frac{\mu_n}{\frac{1}{n^p}}=l,\{0\leqslant l<\infty,p>1\},\sum_{n=1}^{\infty}\frac{1}{n^p}\text{收敛, 则级数}\sum_{n=1}^{\infty}\mu_n\text{收敛} \\ if \quad \lim_{n\to\infty}n\mu_n=\frac{\mu_n}{\frac{1}{n}}=l,\{l>0,\infty\},\sum_{n=1}^{\infty}\frac{1}{n}\text{发散, 则级数}\sum_{n=1}^{\infty}\text{发散}\mu_n\end{array}$$

12.2.2 交错级数审敛法

$$\text{各项正负交错，可以写成} \begin{array}{l}\mu_1-\mu_2+\mu_3-\mu_4+\dots, \\ -\mu_1+\mu_2-\mu_3+\mu_4-\dots,\end{array} \quad \text{其中 } \mu_n>0$$

$$\text{交错级数} \sum_{n=1}^{\infty}\mu_n \text{ 满足条件：}\begin{array}{l}(1)\quad \mu_n\geqslant\mu_{n+1}\,(n=1,2,3,\dots) \\ (2)\quad \lim_{n\to\infty}\mu_n=0\end{array}, \text{那么级数收敛,其和 } s\leqslant\mu_1, \text{余项绝对值 } |r_n|\leqslant\mu_{n+1}$$

12.2.3 绝对收敛与条件收敛

$$\text{级数} \sum_{n=1}^{\infty}\mu_n,\begin{array}{l}\text{正项级数}\sum_{n=1}^{\infty}|\mu_n|\text{ 收敛, 则级数}\sum_{n=1}^{\infty}\mu_n\text{绝对收敛} \\ \text{正项级数}\sum_{n=1}^{\infty}|\mu_n|\text{ 发散,, 级数}\sum_{n=1}^{\infty}\mu_n\text{收敛, 则级数}\sum_{n=1}^{\infty}\mu_n\text{条件收敛}\end{array}$$

级数 $\sum_{n=1}^{\infty}\mu_n$绝对收敛, 则级数 $\sum_{n=1}^{\infty}\mu_n$必定收敛 $\quad v_n=\frac{1}{2}\left(\mu_n+|\mu_n|\right)$

12.2.4 绝对收敛级数的性质

绝对收敛级数经改变项位置后构成的级数也收敛，且与原级数有相同的和 $S_n{}^*\leqslant S_m\leqslant s,\mu_n=2v_n-|\mu_n|$

级数 $\sum\mu_n,\sum v_n$ 绝对收敛，和分别为 s,σ, 所有项的可能乘积 $\mu_i v_i$

$$\begin{array}{cccc}\mu_1v_1 & \mu_1v_2 & \mu_1v_3 & \cdots \\ \mu_2v_1 & \mu_2v_2 & \mu_2v_3 & \cdots \\ \mu_3v_1 & \mu_3v_2 & \mu_3v_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots\end{array} \qquad\qquad \begin{array}{cc|c|c}\mu_1v_1 & \mu_1v_2 & \mu_1v_3 & \cdots \\ \mu_2v_1 & \mu_2v_2 & \mu_2v_3 & \cdots \\ \mu_3v_1 & \mu_3v_2 & \mu_3v_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots\end{array}$$

柯西乘积 (级数) $\mu_1v_1+(\mu_1v_2+\mu_2v_1)+\cdots+(\mu_1v_n+\mu_2v_{n-1}+\cdots+\mu_nv_1)+\dots$ 绝对收敛，和为 $s\sigma$

$\mu_1v_1+(\mu_1v_2+\mu_2v_2+\mu_2v_1)+(\mu_1v_3+\mu_2v_3+\mu_3v_3+\mu_3v_2+\mu_3v_1)+\cdots=(\mu_1+\mu_2+\cdots+\mu_n)\cdot(v_1+v_2+\cdots+v_n)$

12.3 幂级数

12.3.1 函数项级数的概念

区间 I 上函数列 $\mu_1(x), \mu_2(x), \mu_3(x), \ldots, \mu_n(x), \ldots,$

表达式 $\mu_1(x) + \mu_2(x) + \mu_3(x) + \cdots + \mu_n(x) + \ldots,$ 区间 I 上的 (函数项) 无穷级数，(函数项) 级数

对于每个确定值 $x_0 \in I, \mu_1(x_0) + \mu_2(x_0) + \mu_3(x_0) + \cdots + \mu_n(x_0) + \ldots$ 常数项级数，收敛点 x_0

$S(x) = \mu_1(x) + \mu_2(x) + \mu_3(x) + \cdots + \mu_n(x) + \ldots,$ 和函数

部分和 $S_n(x), \lim_{n\to\infty} S_n(x) = S(x)$, 余项 $r_n(x) = S(x) - S_n(x), \lim_{n\to\infty} r_n(x) = 0$

12.3.2 幂级数及其收敛性

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \text{ 幂级数}$$

$$\text{幂级数} \sum_{n=0}^{\infty} a_n x^n \quad \begin{matrix} x = x_0, \text{幂级数收敛}, if \quad |x| < |x_0|, \text{幂级数绝对收敛} \\ x = x_0, \text{幂级数发散}, if \quad |x| > |x_0|, \text{幂级数发散} \end{matrix}$$

$$\lim_{n\to\infty} a_n x_0^n = 0 \Rightarrow |a_n x_0^n| \leqslant M \Rightarrow |a_n x^n| = |a_n x_0^n| \cdot \left|\frac{x}{x_0}\right|^n$$

$$\text{幂级数} \textstyle\sum_{n=0}^{\infty} a_n x^n \text{不是一点收敛，也不是整个数轴收敛}, \exists R > 0, \begin{matrix} |x| < R, \text{幂级数绝对收敛} \\ |x| > R, \text{幂级数发散} \\ |x| = \pm R, \text{幂级数可能收敛} \end{matrix}$$

$$\text{幂级数} \textstyle\sum_{n=0}^{\infty} a_n x^n, \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho, \text{收敛半径} R = \begin{cases} \frac{1}{\rho}, & \rho \neq 0 \\ +\infty, & \rho = 0 \\ 0, & \rho = +\infty \end{cases}$$

$$\frac{|a_{n+1} x^{n+1}|}{|a_n x^n|} = \frac{|a_{n+1}|}{|a_n|} |x| = \rho |x| \overset{?}{<} 1$$

12.3.3 幂级数的运算

$$\text{幂级数} \begin{matrix} a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \\ b_0 + a_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots \end{matrix}$$

加减 $(a_0 \pm b_0) + (a_1 \pm b_1) x + (a_2 \pm b_2) x^2 + \cdots + (a_n \pm b_n) x^n + \cdots$ 区间取小

乘 $(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots) \cdot (b_0 + a_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots)$

$a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) x^n + \cdots$ 区间取小

幂级数的柯西乘积

除 $\frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots}{b_0 + a_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots} = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$ 区间取小

$$\begin{matrix} a_0 = c_0 b_0 \\ a_1 = c_0 b_1 + c_1 b_0 \\ a_2 = c_0 b_2 + c_1 b_1 + c_2 b_0 \\ \cdots \end{matrix} \quad \text{解得 } c_0, c_1, c_2, \cdots, c_n, \cdots$$

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $S_n(x)$ 在收敛域 I 上连续

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $S_n(x)$ 在收敛域 I 上可积，所得收敛同半径

$$\int_0^x S(t)\,dt = \int_0^x \left[\sum_{n=0}^{\infty} a_n t^n\right] dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \, (x \in I)$$

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $S_n(x)$ 在收敛域 I 上可导，所得收敛同半径

$$S'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=0}^{\infty} n a_n x^{n-1} \, (x \in I)$$

幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $S_n(x)$ 在收敛域 I 上具有任意阶导数

12.4 函数展开成幂级数

$f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n, x\in U(x_0)$ 泰勒级数（极限），泰勒展开式
函数 $f(x)$ 在 $U(x_0)$ 内具有各阶导数，$f(x)$ 能展开成泰勒级数 $\Leftrightarrow$ 余项满足 $\sum_{n=0}^{\infty}R_n(x)=0$
$f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(0)(x-0)^n, x\in U(0)$ 麦克劳林级数（极限），麦克劳林展开式
$\sin^{(n)}x=\sin(x+n\pi)$

$$\begin{cases} e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n, (-\infty<x<\infty) \\ \sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}, (-\infty<x<\infty) \\ \frac{1}{x+1}=\sum_{n=0}^{\infty}(-1)^n x^n, (-1<x<1) \end{cases}$$

已知展开，运算得未知
$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\cdots+\frac{m(m-1)\cdots(m-n+1)}{n!}x^n+\ldots, (-1<x<1)$ 二项展开式
$(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\frac{1}{2}\cdot-\frac{1}{2}}{2!}x^2+\cdots+\frac{\frac{1}{2}\cdot-\frac{1}{2}\cdots\frac{3-2n}{2}}{n!}x^n+\cdots=\frac{\prod(3-2n)}{(2n)!!}x^n, (-1<x<1)$ 二项展开式
$(1+x)^{-\frac{1}{2}}=1-\frac{1}{2}x+\frac{-\frac{1}{2}\cdot-\frac{3}{2}}{2!}x^2+\cdots+\frac{-\frac{1}{2}\cdot-\frac{1}{2}\cdots\frac{1-2n}{2}}{n!}x^n+\cdots=\frac{\prod(1-2n)}{(2n)!!}x^n, (-1<x<1)$ 二项展开式

证明：

$$\begin{aligned}
&F(x)=(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\cdots+\frac{m(m-1)\cdots(m-n+1)}{n!}x^n+\ldots \\
&F'(x)=m\left[1+\frac{m-1}{1}x+\cdots+\frac{(m-1)\cdots(m-n+1)}{(n-1)!}x^{n-1}\ldots\right] \\
&xF'(x)=m\left[x+\frac{m-1}{1}x^2+\cdots+\frac{(m-1)\cdots(m-n+1)}{(n-1)!}x^n\ldots\right] \\
&(1+x)F'(x)=m\left[1+mx+\frac{*(m-n+1)}{(n-1)!}x^{n-1}+\frac{*}{(n-2)!}x^{n-1}+\frac{*1(m-n)}{(n)!}x^n+\frac{*1}{(n-1)!}x^n+\ldots\right] \\
&(1+x)F'(x)=m\left[1+mx+\frac{m*}{(n-1)!}x^{n-1}+\frac{m*1}{n!}x^n+\ldots\right]=mF(x) \\
&\varphi(x)=\frac{F(x)}{(1+x)^m}, \varphi(0)=F(0)=1 \\
&\varphi'(x)=\frac{F'(x)(1+x)^m-m(1+x)^{m-1}F(x)}{(1+x)^{2m}}=\frac{(1+x)^{m-1}\left[(1+x)F'(x)-mF(x)\right]}{(1+x)^{2m}}=0 \\
&\varphi(x)=1, F(x)=(1+x)^m
\end{aligned}$$

12.5 函数的幂级数展开应用

12.5.1 近似计算

12.5.2 微分方程的幂级数解法

$\frac{dy}{dx}=f(x,y), y=\sum_{n=0}^{\infty}a_nx^n$
$y''+P(x)y'+Q(x)y=0, P, Q$定义域内可展开为幂级数，方程解形如$y=\sum_{n=0}^{\infty}a_nx^n$

12.5.3 欧拉公式

$u_1+u_2+\cdots+u_n+\ldots$ (1) 實部組成的級數收斂于 u
$v_1+v_2+\cdots+v_n+\ldots$ (2) 虛部組成的級數收斂于 v
$(u_1+v_1i)+(u_2+v_2i)+\cdots+(u_n+v_ni)+\ldots$ (3) 復數項級數收斂於 $u+vi$
$\sqrt{u_1^2+v_1^2}+\sqrt{u_2^2+v_2^2}+\cdots+\sqrt{u_n^2+v_n^2}+\cdots$ (4) (3) 各項的模組成的級數收斂,$(3),(2),(1)$ 絕對收斂
$e^z=1+z+\frac{1}{2!}z^2+\cdots+\frac{1}{n!}z^n+\ldots, z=x+yi$ 復變量指數函數

$$\begin{aligned}
e^{yi}&=1+(yi)+\frac{1}{2!}(yi)^2+\cdots+\frac{1}{n!}(yi)^n+\ldots \\
&=\left(1+\frac{1}{2!}(yi)^2+\cdots+\frac{1}{(2n)!}(yi)^{2n}\right)+\left((yi)+\frac{1}{3!}(yi)^3+\cdots+\frac{1}{(2n-1)!}(yi)^{2n-1}\right)+\ldots \\
&=\left(1-\frac{1}{2!}y^2+\cdots+(-1)^n\frac{1}{(2n)!}y^{2n}\right)+i\left(y+\frac{1}{3!}(yi)^2+\cdots+\frac{1}{(2n-1)!}y^{2n-1}i^{2n-2}\right)+\ldots \\
&=\left(1-\frac{1}{2!}y^2+\cdots+(-1)^n\frac{1}{(2n)!}y^{2n}\right)+i\left(y+\frac{1}{3!}(yi)^2+\cdots+(-1)^{n-1}\frac{1}{(2n-1)!}y^{2n-1}\right)+\ldots \\
&=\left(1-\frac{1}{2!}y^2+\cdots+(-1)^n\frac{1}{(2n)!}y^{2n}\right)+i\left(y+\frac{1}{3!}(yi)^2+\cdots+(-1)^{n-1}\frac{1}{(2n-1)!}y^{2n-1}+(-1)^n\frac{1}{(2n+1)!}y^{2n+1}\right)+\ldots \\
&=\cos y+i\sin y \quad \text{欧拉公式}
\end{aligned}$$

$z=x+yi=\rho\cos\theta+i\rho\sin\theta=\rho(\cos\theta+i\sin\theta)=\rho e^{\theta i}$
$e^{x+yi}=e^x(\cos y+i\sin y)$

$$\begin{cases} \cos y = \frac{e^{yi}+e^{-yi}}{2} \\ \sin y = \frac{e^{yi}-e^{-yi}}{2i} \end{cases}$$ 欧拉公式

12.6 函数项级数的一致收敛性及一致收敛函数的基本性质

12.6.1 函数项级数的一致收敛性

$x+(-x+x^2)+(-x^2+x^3)+\cdots+(-x^{n-1}+x^n)+\ldots \quad s(x)=\lim_{n\to\infty}s_n(x)=\begin{cases}0,0\leqslant x<1\\1,x=1\end{cases}$

函数项级数 $\sum_{n=0}^{\infty}u_n(x)$ 在区间 I 收敛于 $s(x)\Leftrightarrow\forall x_0\in I$, 数项级数 $\sum_{n=0}^{\infty}u_n(x_0)$ 收敛于 $s(x_0)$

$\forall\varepsilon,x_0\in I,\exists N(\varepsilon,x_0),when\quad n>N,|r_n(x_0)|=\left|\sum_{i=n+1}^{\infty}u_i(x_0)\right|<\varepsilon$ 函数项级数收敛

$\forall\varepsilon,x_0\in I,\exists N(\varepsilon),when\quad n>N,|r_n(x_0)|=\left|\sum_{i=n+1}^{\infty}u_i(x_0)\right|<\varepsilon$ 函数项级数一致收敛

函数项级数 $\sum_{n=0}^{\infty}u_n(x)$ 在区间 I 满足 $\begin{array}{l}|u_n(x)|\leqslant a_n\quad(n=0,1,2,\cdots)\\ \text{正项级数}\sum_{n=0}^{\infty}a_n\text{收敛}\end{array}$ 函数项级数在区间 I 一致收敛

12.6.2 一致收敛级数的基本性质

在区间 I，级数 $\sum_{n=0}^{\infty}u_n(x)$ 一致收敛于 $s(x)$, 各项 $u_n(x)$ 连续，$s(x)$ 在区间 I 连续

在区间 I，级数 $\sum_{n=0}^{\infty}u_n(x)$ 一致收敛于 $s(x)$, 各项 $u_n(x)$ 连续，级数 $\sum_{n=0}^{\infty}u_n(x)$ 可逐项积分

$\int_{x_0}^{x}s(x)dx=\int_{x_0}^{x}u_1(x)dx+\int_{x_0}^{x}u_2(x)dx+\cdots+\int_{x_0}^{x}u_n(x)dx+\ldots(x_0<x\in I)$ 右端级数在区间 I 一致收敛

在区间 I，级数 $\sum_{n=0}^{\infty}u_n(x)$ 收敛于 $s(x)$, 各项 $u_n(x)$ 具有连续导数 $u_n'(x)$, 级数 $\sum_{n=0}^{\infty}u_n'(x)$ 一致收敛，则级数 $\sum_{n=0}^{\infty}u_n(x)$ 一致收敛，且可逐项求导

$s'(x)=u_1'(x)+u_2'(x)+\cdots+u_n'(x)+\ldots$

幂级数 $\sum_{n=0}^{\infty}a_nx^n$ 的收敛半径 R>0, 在闭区间 $I\in(-R,R)$，幂级数一致收敛

幂级数 $\sum_{n=0}^{\infty}a_nx^n$ 的收敛半径 R>0, 在闭区间 $I\in(-R,R)$ 其和函数可导，级数可逐项求导

$s'(x)=\left(\sum_{n=0}^{\infty}a_nx^n\right)'=\sum_{n=0}^{\infty}a_nnx^{n-1}$ 右端级数与原级数同收敛半径

12.7 傅里叶级数

三角函数组成的函数项级数 (三角级数)

12.7.1 三角级数　三角函数的正交性

$y=A\sin(\omega t+\varphi)\quad\begin{array}{l}A\text{振幅}\\ \omega\text{角频率}\\ \varphi\text{初相}\end{array}$

$$\begin{aligned}f(t)&=A_0+\sum_{n=1}^{\infty}A_n\sin(n\omega t+\varphi_n)\\&=A_0+\sum_{n=1}^{\infty}A_n\cdot[\sin(n\omega t)\cos(\varphi_n)+\cos(n\omega t)\sin(\varphi_n)]\\&=A_0+\sum_{n=1}^{\infty}[A_n\cdot\cos(\varphi_n)\sin(n\omega t)+A_n\cdot\sin(\varphi_n)\cos(n\omega t)]\\&=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\sin(n\omega t)+b_n\cos(n\omega t))\\&\xlongequal{\omega=\frac{\pi}{l}}\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\sin\left(n\frac{\pi}{l}t\right)+b_n\cos\left(n\frac{\pi}{l}t\right)\right],(T=2l)\\&\xlongequal{\omega t=x}\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\sin(nx)+b_n\cos(nx)],(T=2\pi)\end{aligned}$$

谐波分析
A_0直流分量
$A_1\sin(\omega t+\varphi_1)$ 一次谐波 (基波)

三角函数系 $1,\sin x,\cos x,\sin 2x,\cos 2x,\ldots,\sin nx,\cos nx,\ldots$ 任意两函数乘积在区间 $[-\pi,\pi]$ 上积分为零

$\int_{-\pi}^{\pi}\cos nxdx=0\quad(n=1,2,3,\cdots)$

$\int_{-\pi}^{\pi}\sin nxdx=0\quad(n=1,2,3,\cdots)$

$\int_{-\pi}^{\pi}\sin kx\cos nxdx=0\quad(k,n=1,2,3,\cdots)$

$\int_{-\pi}^{\pi}\cos kx\cos nxdx=0\quad(k,n=1,2,3,\cdots,k\neq n)$

$\int_{-\pi}^{\pi}\sin kx\cos nxdx=0\quad(k,n=1,2,3,\cdots,k\neq n)$

$\int_{-\pi}^{\pi}1^2dx=\frac{\pi}{2},\int_{-\pi}^{\pi}\sin^2(nx)^2dx=\pi,\int_{-\pi}^{\pi}\cos^2(nx)^2dx=\pi$

12.7.2 函数展开成傅里叶级数

$$
\begin{aligned}
&f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]\\
&\int_{-\pi}^{\pi}f(x)\,dx=\int_{-\pi}^{\pi}\frac{a_0}{2}dx+\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]dx\\
&\quad=\int_{-\pi}^{\pi}\frac{a_0}{2}dx+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(nx)\,dx+b_n\int_{-\pi}^{\pi}\sin(nx)\,dx\right]\\
&\quad=\int_{-\pi}^{\pi}\frac{a_0}{2}dx\\
&\quad=\pi a_0\\
&f(x)\sin(kx)=\frac{a_0}{2}\sin(kx)+\sin(kx)\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]\\
&\int_{-\pi}^{\pi}f(x)\sin(kx)\,dx=\int_{-\pi}^{\pi}\frac{a_0}{2}\sin(kx)\,dx+\int_{-\pi}^{\pi}\sin(kx)\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]dx\\
&\quad=\int_{-\pi}^{\pi}\frac{a_0}{2}\sin(kx)\,dx+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\sin(kx)\cos(nx)\,dx+b_n\int_{-\pi}^{\pi}\sin(kx)\sin(nx)\,dx\right]\\
&\quad=b_k\int_{-\pi}^{\pi}\sin(kx)\sin(kx)\,dx\\
&\quad=\pi b_k\\
&f(x)\cos(kx)=\frac{a_0}{2}\cos(kx)+\cos(kx)\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]\\
&\int_{-\pi}^{\pi}f(x)\cos(kx)\,dx=\int_{-\pi}^{\pi}\frac{a_0}{2}\cos(kx)\,dx+\int_{-\pi}^{\pi}\cos(kx)\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]dx\\
&\quad=\int_{-\pi}^{\pi}\frac{a_0}{2}\cos(kx)\,dx+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(kx)\cos(nx)\,dx+b_n\int_{-\pi}^{\pi}\cos(kx)\sin(nx)\,dx\right]\\
&\quad=a_k\int_{-\pi}^{\pi}\cos(kx)\cos(kx)\,dx\\
&\quad=\pi a_k
\end{aligned}
$$

$$
\text{傅里叶系数}\quad
\begin{aligned}
a_0&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx\\
a_k&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(kx)\,dx\\
b_k&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(kx)\,dx
\end{aligned}
$$

$f(x)$ 是周期为2π的周期函数，
满足 (1) 一个周期内连续，或只有有限个第一类间断点
(2) 一个周期内至多只有有限个极值点

且 $f(x)$ 的傅里叶级数收敛，
(1) x是连续点，级数收敛于$f(x)$
(2) x是间断点，级数收敛于$\frac{1}{2}\left[f(x^-)+f(x^+)\right]$

$C=\left\{x|f(x)=\frac{1}{2}\left[f(x^-)+f(x^+)\right]\right\},f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right],x\in C$

扩广函数定义域的过程，周期延拓

12.7.3 正弦级数和余弦级数

$f(x)$,是奇函数 $\quad a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(kx)\,dx=0,\quad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(kx)\,dx\quad f(x)=\sum_{n=1}^{\infty}b_n\sin(nx)$,正弦级数

$f(x)$,是偶函数 $\quad a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(kx)\,dx,\quad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(kx)\,dx=0\quad f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)$,余弦级数

以扩广函数为奇函数，扩展定义域的过程，奇延拓 $f(x)\neq 0$,则规定$f(x)=0$

以扩广函数为偶函数，扩展定义域的过程，偶延拓

$|x|=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)x\,(-\infty<x<+\infty)$

$x=0\Rightarrow 0=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\Rightarrow\frac{\pi^2}{8}=\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$

$$
\begin{aligned}
&\delta=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots+\frac{1}{n^2}+\dots && \delta=\delta_1+\delta_2\\
&\delta_1=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots+\frac{1}{(2n-1)^2}+\dots=\frac{\pi^2}{8} && \delta_2=\frac{1}{4}\delta=\frac{1}{4}\frac{\pi^2}{6}=\frac{\pi^2}{24}\\
&\delta_2=\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\dots+\frac{1}{(2n)^2}+\dots && \frac{3}{4}\delta=\delta_1,\delta=\frac{4}{3}\delta_1,\delta=\frac{4}{3}\frac{\pi^2}{8}=\frac{\pi^2}{6}\\
&\delta_3=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots+(-1)^{n-1}\frac{1}{n^2}+\dots && \delta_3=\delta_1-\delta_2=\frac{\pi^2}{8}-\frac{\pi^2}{24}=\frac{\pi^2}{12}
\end{aligned}
$$

12.7.4 一般周期函数的傅里叶级数

周期为 $2l$ 的周期函数 $f(x)$ 满足收敛条件，傅里叶级数展开式为

$$\begin{aligned} f\left(x\right) &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} \left[a_n \cos\left(\tfrac{n\pi x}{l}\right) + b_n \sin\left(\tfrac{n\pi x}{l}\right)\right] \\ a_0 &= \tfrac{1}{l}\int_{-l}^{l} f\left(x\right) dx \\ a_k &= \tfrac{1}{l}\int_{-l}^{l} f\left(x\right)\cos\left(\tfrac{n\pi x}{l}\right) dx \\ b_k &= \tfrac{1}{l}\int_{-l}^{l} f\left(x\right)\sin\left(\tfrac{n\pi x}{l}\right) dx \end{aligned}$$

$$f\left(x\right)\text{是奇函数}\ \begin{aligned} a_k &= \tfrac{1}{l}\int_{-l}^{l} f\left(x\right)\cos\left(\tfrac{n\pi x}{l}\right) dx = 0 \\ b_k &= \tfrac{1}{l}\int_{-l}^{l} f\left(x\right)\sin\left(\tfrac{n\pi x}{l}\right) dx \end{aligned}\ f\left(x\right) = \textstyle\sum_{n=1}^{\infty} b_n \sin\ \tfrac{n\pi x}{l}$$

$$f\left(x\right)\text{是偶函数}\ \begin{aligned} a_k &= \tfrac{1}{l}\int_{-l}^{l} f\left(x\right)\cos\left(\tfrac{n\pi x}{l}\right) dx \\ b_k &= \tfrac{1}{l}\int_{-l}^{l} f\left(x\right)\sin\left(\tfrac{n\pi x}{l}\right) dx = 0 \end{aligned}\ f\left(x\right) = \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} a_n \cos\ \tfrac{n\pi x}{l}$$

证明 $z = \frac{\pi x}{l}$

12.7.5 傅里叶级数的复数形式

$$\begin{aligned} f\left(x\right) &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} \left[a_n \cos\left(\tfrac{n\pi x}{l}\right) + b_n \sin\left(\tfrac{n\pi x}{l}\right)\right] \\ &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} \left[a_n \cos\alpha + b_n \sin\alpha\right] \\ &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} \left[a_n \tfrac{e^{i\alpha}+e^{-i\alpha}}{2} + b_n \tfrac{e^{i\alpha}-e^{-i\alpha}}{2i}\right] \\ &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} \left[\tfrac{a_n+b_n i}{2} e^{i\alpha} + \tfrac{a_n-b_n i}{2} e^{-i\alpha}\right] \\ &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} \left[\tfrac{a_n-b_n i}{2} e^{-i\frac{n\pi x}{l}} + \tfrac{a_n+b_n i}{2} e^{i\frac{n\pi x}{l}}\right] \\ &= c_0 + \textstyle\sum_{n=1}^{\infty} \left[c_n e^{-i\frac{n\pi x}{l}} + c_{-n} e^{i\frac{n\pi x}{l}}\right] \\ &= c_k \textstyle\sum_{k=-\infty}^{\infty} e^{-i\frac{k\pi x}{l}} \qquad (k = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots) \\ c_0 &= \tfrac{a_0}{2} = \tfrac{1}{2l}\int_{-l}^{l} f\left(x\right) dx \\ c_n &= \tfrac{a_n-b_n i}{2} = \tfrac{1}{2}\left[\tfrac{1}{l}\cos\left(\tfrac{n\pi x}{l}\right) dx - \tfrac{1}{li}\int_{-l}^{l} f\left(x\right)\sin\left(\tfrac{n\pi x}{l}\right) dx\right] \\ &= \tfrac{1}{2l}\int_{-l}^{l} f\left(x\right)\left[\cos\left(\tfrac{n\pi x}{l}\right) - \sin\left(\tfrac{n\pi x}{l}\right)\right] dx \\ &= \tfrac{1}{2l}\int_{-l}^{l} f\left(x\right) e^{-\frac{n\pi x}{l}} dx \quad (n = 1, 2, 3, \cdots) \\ c_{-n} &= \tfrac{a_n+b_n i}{2} = \tfrac{1}{2l}\int_{-l}^{l} f\left(x\right) e^{\frac{n\pi x}{l}} dx \quad (n = 1, 2, 3, \cdots) \\ c_k &= \tfrac{1}{2l}\int_{-l}^{l} f\left(x\right) e^{-\frac{k\pi x}{l}} dx \quad (k = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots) \end{aligned}$$