ARTICLE IN PRESS

ISA Transactions xxx (xxxx) xxx

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

Research article

Interval linear quadratic regulator and its application for speed control of DC motor in the presence of uncertainties

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ARTICLE INFO

Article history:
Received 11 February 2019
Received in revised form 1 July 2021
Accepted 1 July 2021
Available online xxxx

Keywords:
LQR optimal control
Interval analysis
Chebyshev inclusion method
Monte Carlo method
DC motor

ABSTRACT

An analytical investigation of a DC motor with interval uncertainties is performed in this study and a new approach by interval analysis is suggested for optimal control of the system. The main advantage of using an interval model for uncertainties is that makes the system independent from the probability distribution models of the system; therefore, it can be analyzed by only having information about minimum and maximum bounds. Here, the interval analysis deals with linear quadratic feedback control (LQR) to simulate and optimal control of the DC motor in the realistic state. To do this, the Pontryagins principle is used to solve the interval linear quadratic regulator to obtain the essential conditions, and thus, they have been reconstructed as ordinary differential equation by applying several algebraic manipulations. Afterward, by solving the interval nonlinear system of the ODE, the confidence interval for the feedback controller is achieved. The confidence interval is to guarantee the solution which is included in it. The Chebyshev inclusion approach is applied here to find solution for the ODE system with uncertainties. A comparison of the step response of the suggested approach with the centered approach and Monte Carlo methods a statistical approach is performed. The simulation results indicated that the suggested approach retains tighter and more sensible results than the Monte Carlo method.

1. Introduction

In the last decades, designing systems with robust characteristics have been turned into a significant problems in the control theory [1]. Therefore, there are a lot of research works that have been working on robust control problems, especially the methods based on matrix Riccati equations and inequalities. These methods are often based on H_2 and H_∞ methods [2]. Another technique about the optimal control of these kinds of systems is to use feedback control theory. Linear quadratic regulators (LQRs) are one of the most popular feedback-based methods which can be employed for optimal control of a system [3]. LQR is one of the popular optimal controller design techniques for linear and time-invariant control systems. This approach is founded on the fact that a definite and considered performance index should be optimized to achieve the dynamic output feedback with optimal values for the control and state variables [4].

The main purpose of this research is to design a robust controller that has the features of an optimal controller; in other words, this paper aims to address the LQR controller within an uncertain framework. The presence of uncertainties in the system model makes its solution to get so complicated or even sometimes impossible. Generally, in the real world, there are always some uncertainties that can be made for various causes such as neglecting small phenomena, failure to consider several unidentified processes, round off error, etc. Using classical methods for solving these types of problems makes some errors in the solution which sometimes results in a wrong solution. This problem leads researchers to study and try various approaches on the systems by considering their uncertainties [5].

Recently, various approaches were presented to overcome this problem. For example, Stochastic methods, fuzzy programming approaches, and interval-based approaches have been placed among the most popular methods for problems under uncertainties [6].

Fuzzy methods can be used when the membership functions are clear and stochastic approaches require probabilistic distribution for utilizing in the system dynamics. However, sometimes there is no information about memberships and distribution. In this situation, the optimum solution is to apply the interval approach [7]. Generally, interval analysis is one of the best methods for considering system uncertainties [8,9]. Interval-based methods can be employed for the problems with uncertainties only

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https://doi.org/10.1016/j.isatra.2021.07.004

Please cite this article as: Y. Zhi, W. Weiqing, C. Jing et al., Interval linear quadratic regulator and its application for speed control of DC motor in the presence of uncertainties. ISA Transactions (2021), https://doi.org/10.1016/j.isatra.2021.07.004.

by knowing the lower and upper bounds. Interval methods have been introduced over the years [10]; but from 1996, interval arithmetic was introduced in simple propositions [11–13].

DC motors are commonly employed in different areas from industry to robotics [14–17]. While different types of control strategies have been introduced for DC motors, they have not considered any uncertainties in the system. Therefore, the next target here is to investigate a DC motor with its uncertainties in the interval structure. Assume a mathematical model of DC motor with interval uncertainties. The major purpose is to present an optimum feedback controller for the system so that stays fixed in the considered confidential interval [18]. Some methods were studied in the systems' feedback control with uncertain intervals [19–22]. But, all of them designed the final feedback control by assuming a real-valued (not interval-based) controller. This drawback leads us to design a new strategy to design a feedback controller based on interval analysis.

After introducing the Kharitonov Theorem, different methods for robust control of the systems with interval uncertainties were created which utilized the interval transfer function to model the system dynamics [23]. The application of this theory is limited; because a transfer function representation can be made difficult or even impossible to solve some classes of dynamical objects especially those who need to be approximated by state–space models. The main target of the present research is to outline a sub-optimal, robust controller based on LQR for a DC motor with interval uncertainties. The main point is that the proposed method produces an interval (not a fixed constant) for the solution which not only can be used for obtaining a correct range of solutions, but also as a pre-conditioner approach for checking the accuracy of the solution for the other robust methods. The final results are compared by the Monte Carlo technique and center-based interval technique to show the proposed system's performance. The main contributions of the paper are highlighted in the following:

– A new extended version for LQR optimal control for the systems under a type of uncertainty is introduced.
– Interval assessment is applied for considering the uncertainties for the system.
– The interval controllability of the proposed DC motor is testified before controlling it.
– The parameters of the DC motor model are achieved from the lab.
– The proposed approach introduced a pre-conditioning technique to guarantee the solution of any optimal control method.

In the following, this study is outlined as: in Section 2, a primary description of the interval arithmetic is presented. Section 2 describes the major conception of the inverse operator of the interval matrices. Section 3 describes a detailed description of the Chebyshev inclusion method. Section 4 explains the Monte Carlo method as a method for solving the systems with uncertainties to carry out a comparison with the suggested approach. In Section 5, a mathematical model of the DC motor is described under its interval uncertainties. The proposed interval quadratic regulator is then presented in Section 6. In Section 7 the simulation results have been analyzed and at the end, conclusions are included in Section 8.

2. Preliminaries on the interval analysis

Because of the application of the interval analysis in the present research, a brief review of interval arithmetic has been illustrated in the following [24,25]. An interval uncertainty can be bounded by a set of actual integers such that they include all the possible uncertainties of the system. To understand better, assume a real interval integer A; this interval integer can be illustrated as follows [26]:

$$\mathbb{IR} = \left\{ \tilde{\mathrm{A}} | \tilde{\mathrm{A}} = [\underline{\mathrm{a}}, \bar{\mathrm{a}}] \right\}, X = \left\{ x | x \in \mathbb{R} \cup \{-\infty, \infty\}, \underline{\mathrm{a}} \le \mathrm{a} \le \bar{\mathrm{a}} \right\} \quad (1)$$

where, $\tilde{\mathrm{A}}$ describes an interval integer over $\mathbb{IR}$ and $\underline{\mathrm{a}}, \bar{\mathrm{a}}$ are the minimum and the maximum bounds of $\tilde{\mathrm{A}}$, respectively.

From [27], the algebraic properties of classical interval analysis are not satisfactory for real-world problems which are because nonzero width intervals have not inversed in $\mathbb{IR}$ in the classic interval arithmetic. Kaucher [28] presented a new definition of interval arithmetic, wherein the standard interval has been extended by improper intervals and interval arithmetic. The set of generalized intervals (suitable and unsuitable) is defined by $\mathbb{D} = \mathbb{IR} \cup \overline{\mathbb{IR}} = \{[a_1, a_2] : a_1, a_2 \in \mathbb{R}\}$. The generalized interval set, $\mathbb{D}$, contains a collection of zero-free addition and multiplication operations to keep the inclusion of monotonicity. Based on this conception and the fact that $\tilde{A} = \left[\underline{a}, \bar{a}\right], \tilde{B} = [\underline{b}, \bar{b}] \in \mathbb{IR}^{m\times n}$, the midpoint $\hat{A}$ and the radius $\breve{A}$, are achieved by the following equations [29]:

$$\hat{A} = \frac{\underline{a} + \bar{a}}{2} \quad (2)$$

$$\breve{A} = \frac{\bar{a} - \underline{a}}{2} \quad (3)$$

The primary interval operations between two interval integers $\tilde{A}$ and $\tilde{B}$ are described as follows [24]:

$$\tilde{A} + \tilde{B} = [\hat{A} + \hat{B} - \eta; \breve{A} + \breve{B} + \eta] \quad (4)$$

$$\tilde{A} - \tilde{B} = [\hat{A} - \hat{B} - \eta; \breve{A} + \breve{B} + \eta] \quad (5)$$

$$\tilde{A} \times \tilde{B} = \left[\hat{A} \times \hat{B} - \eta, \hat{A} \times \hat{B} + \eta\right] \quad (6)$$

$$\frac{\tilde{A}}{\tilde{B}} = \tilde{A} \times \frac{1}{\tilde{B}}, \quad (7)$$

$$\frac{1}{\tilde{B}} = \left[\frac{1}{\hat{B}} - \gamma, \frac{1}{\hat{B}} + \gamma\right], 0 \notin [\underline{b}, \bar{b}] \quad (8)$$

where,

$$\eta = \left[min\left(\hat{A} \times \hat{B}\right) - \alpha, \beta - \max\left(\hat{A} \times \hat{B}\right)\right]$$

$$\alpha = \min\left\{\underline{ab}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\right\}$$

$$\beta = \max\left\{\underline{ab}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\right\}$$

$$\gamma = \left\{\frac{1}{\bar{b}}\left(\frac{\bar{b} - \underline{b}}{\bar{b} + \underline{b}}\right), \frac{1}{\underline{b}}\left(\frac{\bar{b} - \underline{b}}{\bar{b} + \underline{b}}\right)\right\}$$

Interval conception can be also extended to an interval matrix. An interval matrix $\tilde{A}$ can be defined as follows [30]:

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{1,1} & \cdots & \tilde{a}_{1,n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m,1} & \cdots & \tilde{a}_{m,n} \end{pmatrix} = \left(\tilde{a}_{i,j}\right)_{1\le i\le m, 1\le j\le n}$$

where,

$\tilde{a}_{i,j} \in \mathbb{IR}^{m\times n}$: $\underline{a}_{i,j} \le \tilde{a}_{i,j} \le \bar{a}_{i,j}$.

Matrix operations for interval integers are like real integers (non-interval), i.e. [31]

$$\tilde{A} + \tilde{B} = \left(\tilde{a}_{i,j} + \tilde{b}_{i,j}\right)_{1\le i\le m, 1\le j\le n} \quad (9)$$

$$\tilde{A} - \tilde{B} = \begin{cases} \left(\tilde{a}_{i,j} - \tilde{b}_{i,j}\right)_{1\le i\le m, 1\le j\le n}, if \tilde{A} \ne \tilde{B} \\ \tilde{A} - \mathrm{dual}\left(\tilde{A}\right) = \tilde{0} = 0, if \tilde{A} = \tilde{B} \end{cases} \quad (10)$$

$$\tilde{A} \times \tilde{B} = \left[\sum_{k=1}^{P} \left(\tilde{a}_{i,k} \times \tilde{b}_{k,j} \right)_{1 \leq i \leq m, 1 \leq j \leq n} \right] \tag{11}$$

where, k describes any dimension and dual $\left(\tilde{A}\right)$ = dual $\left([\underline{a}, \overline{a}]\right) = [\overline{a}, \underline{a}] \in \mathbb{D}$.

More details about interval arithmetic can be found in [11,29,31]. Also, the generalized Hukuhara difference between of two sets A, B is considered in the following:

$$A \ominus_g B = C \leftrightarrow \begin{cases} (a) & A = B + C \\ (b) & B = A + (-1)\, C \end{cases} \tag{12}$$

The standard interval arithmetic with gH-differentiability (generalized Hukuhara differentiability) has been used in [32].

The final operator is the division. Generally, in interval arithmetic, most of the determinant properties of the classical matrices and the interval matrices are equal. In classical mathematics, a matrix describes invertible if and only if its determinant is not equal to zero.

Definition 1 (*[31]*)**.** When $\left|\tilde{A}\right|$ is invertible (i.e. $0 \notin \left|\tilde{A}\right|$), a square interval matrix $\tilde{A}$ is defined for being non-singular or regular. A square interval matrix $\left|\tilde{A}\right|$ is defined to be invertible if $\left|\tilde{A}\right|$ is invertible (i.e. $0 \notin \left|\tilde{A}\right|$).

Definition 2 (*[31]*)**.** When $\left|\tilde{A}\right|$ is invertible, for any $\tilde{A} \in <\mathbb{R}^{n\times n}$, thus the equations' usual solution $\tilde{A}\tilde{X} = \tilde{I}$ and $\tilde{X}\tilde{A} = \tilde{I}$ is defined as the inverse of $\tilde{A}$ and is signified by $\tilde{A}^{-1} = \frac{adj\left(\tilde{A}\right)}{\left|\tilde{A}\right|} = \frac{\tilde{A}^{*}}{\left|\tilde{A}\right|}$ where $\tilde{A}^{*}$ is the adjoint matrix of $\tilde{A}$.

3. Chebyshev inclusion approach

3.1. Chebyshev approximation method

If the $f(x)$ is a continuous function over the closed interval $[a, b]$, For any $\varepsilon > 0$, there are polynomials that can consistently approach the $f(x)$ as nearly as appropriate by a polynomial function such that:

$$\|f(x) - p(x)\|_{\infty} < \varepsilon, \ \ x \in [a, b]. \tag{13}$$

The above equation was first introduced as the Weierstrass theorem [33,34]. It should be noted that creating a balance between the precision and the computational expense in polynomials is significant; because by incrementing the polynomials inevitably's order, the computational expense can be increased.

Consider a Chebyshev polynomial (C_i) that is specified in the range $t = [-1, 1]$ of degree i [35]:

$$C_i(t) = \cos i\theta,\ \theta = \arccos(t) \in [0, \pi] \tag{14}$$

where, i denotes an integer that is not negative.

In this study, a truncated Chebyshev series are utilized to approximate the $f(x)$ with a precision of degree k [36]:

$$[f]([t]) \approx p(t) = \frac{1}{2}f_0 + \sum_{i=1}^{k} f_i C_i([t]) = \frac{1}{2}f_0 + \sum_{i=1}^{k} f_i \ \cos(i[\theta]) \tag{15}$$

where f_i illustrates the ith constant coefficient. With applying the Mehler (Gaussian-Chebyshev) integration [37] on the formula, the Chebyshev polynomial coefficient is calculated as below:

$$f_i = \frac{2}{p} \sum_{j=1}^{p} f(\cos\theta_j) \cos i\theta_j \tag{16}$$

where, p shows the truncation order.

3.2. Chebyshev inclusion approach

Applying the interval analysis for making the truncated Chebyshev polynomial made a robust position that regulates the overestimation effectively [36]. By developing the interval assessment into the Chebyshev polynomials, t is turned into an interval parameter ($[t]$). As presented in [36], the trigonometric representation of the Chebyshev polynomial in the interval analysis makes the solution to control the overestimation efficiency better than the ordinary polynomial representation. Chebyshev inclusion function is as below [38]:

$$[f_{C_k}] = \frac{1}{2}f_0 + \sum_{i=1}^{k} f_i \cos(i[t]) \tag{17}$$

where, $[t] \subset [-1, 1]$ and $[\theta] = \arccos([t]) = [0, \pi]$. Since $[\cos](i[\theta]) = [\cos]([0, i\pi]) = [-1, 1]$ the Chebyshev inclusion polynomial is redefined as below [38]:

$$[f_{C_k}] = \frac{1}{2}f_0 + [-1, 1] \times \sum_{i=1}^{k} |f_i| \tag{18}$$

Notably the Chebyshev inclusion approach is not deterministic and it fails the truncated error and numerical error of the integration error. Nevertheless, interval analysis can cover these errors.

4. Monte Carlo method

A famous statistical approach to find solution for ODEs with uncertain parameters is the Monte Carlo (MC) method [39]. Because of the easy implementation of the MC method, they are employed in most engineering applications. In the MC method, random variable samples are taken based on the probability distribution and then the probability distribution of the response is directly evaluated as the output. The accuracy in the Monte Carlo method has a direct relationship with the sampling size; in other words, the convergence ratio is $\left(\sqrt{N}\right)^{-1}$ while considering N as the sampling size and the *weak law of large number*. In complicated engineering applications, the running time of the MC method is so slow. To solve this problem, we first converted the main optimal control into an ODE, and then, this problem is solved by MC.

5. Mathematical model of DC motor in the presence of interval uncertainties

For a long time, DC motors have become one of the most widely used prime actuators in most industrial applications. The reason is behind their inherent straightforward characteristics and stability. The speed of DC motors is proportional to the applied voltage. There are different techniques for controlling the DC motor speed, like using electronic controllers and battery trapping [40,41].

One of the important problems which are not usually considered in the mathematical modeling of DC motor is neglecting its uncertainties like unknown terms, resistant value variations due to temperature rises and falls, etc. This problem makes the researchers design a controller which is not completely cover the system uncertainties.

Here, a new approach by interval arithmetic is utilized to consider the motor in the presence of uncertainties. As before mentioned, it is important to consider that the interval differentiability here is generalized Hukuhara [42]. This method will be made the controller more robust and more practical against

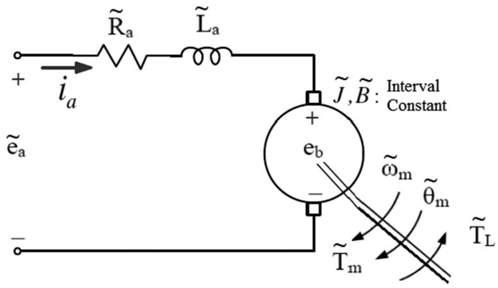

Fig. 1. Schematic of DC motor with interval parameters.

the motor condition changes. The Schematic of a DC motor with interval parameters ($\sim$) is depicted in Fig. 1.

Consider the motor torque (T_m). This term is related to the armature current (i_a) with the following formula:

$$\tilde{T}_m = \widetilde{K}_i \tilde{i}_a, \tag{19}$$

The back *emf* ($\tilde{e}_b$) is also relative to angular velocity ($\tilde{\omega}_m$) by:

$$\tilde{e}_b = \widetilde{K}_b \omega_m = \widetilde{K}_b \frac{d\theta}{dt}, \tag{20}$$

By applying *Newton's law* and *Kirchoff's law*, the system equations can be achieved as follows:

$$\tilde{L}_a \frac{d\tilde{i}_a}{dt} + \widetilde{R}_a \tilde{i}_a = \tilde{e}_a - \widetilde{K}_b \frac{d\theta}{dt}, \tag{21}$$

$$\tilde{J}_m \frac{d^2\theta}{dt^2} + \tilde{B}_m \frac{d\theta}{dt} = \widetilde{K}_i \tilde{i}_a.$$

$$\tilde{e}_b = \widetilde{K}_b \omega_m = \widetilde{K}_b \frac{d\theta}{dt}, \tag{22}$$

From the equations above, the state–space representation of the system is considered as follows:

$$\begin{bmatrix} \dot{\tilde{i}}_a \\ \dot{\tilde{\omega}}_m \\ \dot{\tilde{\theta}}_m \end{bmatrix} = \begin{bmatrix} -\dfrac{\widetilde{R}_a}{\tilde{L}_a} & -\dfrac{\widetilde{K}_b}{\tilde{L}_a} & \{0\} \\ \widetilde{K}_i/\tilde{J}_m & -\dfrac{\tilde{B}_m}{\tilde{J}_m} & \{0\} \\ \{0\} & \{1\} & \{0\} \end{bmatrix} \begin{bmatrix} \tilde{i}_a \\ \tilde{\omega}_m \\ \tilde{\theta}_m \end{bmatrix} + \begin{bmatrix} 1/\tilde{L}_a \\ \{0\} \\ \{0\} \end{bmatrix} \tilde{e}_a, \tag{23}$$

$$\tilde{\omega}_m = \begin{bmatrix} \{0\} & \{1\} & \{0\} \end{bmatrix} \begin{bmatrix} \tilde{i}_a \\ \tilde{\omega}_m \\ \tilde{\theta}_m \end{bmatrix}. \tag{24}$$

where, the terms $\{0\} = [0, 0]$ and $\{1\} = [1, 1]$ called degenerate integers [43]. An interval standard test model with interval parameters is considered and listed in Table 1.

Notably, the parameters interval is achieved based on the identification of the motor in 1 month in different conditions in the lab at Tafresh university.

6. Interval quadratic regulator

Consider a linear multivariable state–space model of the plant dynamics with interval uncertainties as follows:

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t), \tag{25}$$

where, $x(t) \in \mathbb{R}^n$ are a state vector and $u(t) \in \mathbb{R}^p$ is an inputted vector. The arrays $a_{i,j},\ b_{i,k}(i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, p)$ of matrix $\tilde{A} \in \mathbb{IR}^{n\times n}$ and matrix $\tilde{B} \in \mathbb{IR}^{n\times p}$ are interval integers bounded by a definite minimum and maximum interval; i.e. $\tilde{A} = [\underline{A}, \overline{A}]$ and $\tilde{B} = [\underline{B}, \overline{B}]$ are the interval system matrix and the input matrix where their elements lie between minimum and maximum bounds. The boundary conditions of the system are:

Table 1
DC motor parameters with interval values.

Symbol	Value with interval uncertainty
$\tilde{\boldsymbol{E}}$	[11, 13] (volt)
$\tilde{\boldsymbol{J}}_{\boldsymbol{m}}$	[0.008, 0.02] (kgm^2)
$\tilde{\boldsymbol{B}}_{\boldsymbol{m}}$	[0.00002, 0.00005] (kgm^2/s)
$\tilde{\boldsymbol{K}}_{\boldsymbol{i}}$	[0.021, 0.024] (Nm/A)
$\tilde{\boldsymbol{K}}_{\boldsymbol{b}}$	[0.020, 0.024] (V/rad/s)
$\tilde{\boldsymbol{R}}_{\boldsymbol{a}}$	[0.5, 1.5] (Ω)
$\tilde{\boldsymbol{L}}_{\boldsymbol{a}}$	[0.25, 0.75] (H)

$$x(t_0) = X_0,\ \ x(t_f) = X_f. \tag{26}$$

where, X_0 and X_f describe the primary and the ultimate system's states, respectively.

Assume the following performance index:

$$J(x(t),\ u(t),\ \Delta) = \frac{1}{2}x^T(t_f)F(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix} x^T\ (t) & u^T\ (t) \end{bmatrix} \times \begin{bmatrix} \tilde{Q}\ (t) & 0 \\ 0 & \widetilde{R}\ (t) \end{bmatrix} \begin{bmatrix} x\ (t) \\ u\ (t) \end{bmatrix} dt, \tag{27}$$

where, Δ illustrates the system uncertainties, $\tilde{Q}(t)$ is a positive semi-definite interval matrix and $\widetilde{R}(t)$ is a positive definite interval matrix and $J(t,\ x(t),\ u(t),\ \Delta) = [\underline{j}(t,\ x(t),\ u(t)),\ \bar{j}t,\ x(t),\ u(t)]$ describes the interval-valued performance index.

By considering the interval conception in the aforementioned sections and using it for expanding the interval arithmetic into the *Pontryagin principle* [44,45], the interval Hamiltonian equation of the problem is achieved as follows:

$$\tilde{H}(x(t), u(t), \lambda(t)) = \frac{1}{2}x^T(t)\tilde{Q}x(t) + \frac{1}{2}u^T(t)\widetilde{R}u(t) + \lambda(t)(\tilde{A}x(t) + \tilde{B}u(t)), \tag{28}$$

Based on the introduction conception and also using *gH*-difference, and using the optimum control on the interval Hamiltonian matrix [46],

$$\frac{\partial\tilde{H}}{\partial u} = \tilde{0} \rightarrow \widetilde{R}u(t) + \tilde{B}^T\lambda = 0, \quad \Rightarrow \underset{\sim}{\tilde{u}^*}(t) = -\widetilde{R}^{-1}\tilde{B}^T\lambda, \tag{29}$$

$$\dot{x}(t) = \frac{\partial\tilde{H}}{\partial\lambda} \rightarrow \dot{x}(t) = \tilde{A}\,x(t) + \tilde{B}\,u(t)$$
$$\dot{\lambda}(t) = -\frac{\partial\tilde{H}}{\partial x} \rightarrow \dot{\lambda}(t) = -\tilde{Q}\,x(t) - \tilde{A}^T\,\lambda(t) \tag{30}$$

That is:

$$\begin{bmatrix} \dot{x}\ (t) \\ \dot{\lambda}\ (t) \end{bmatrix} = \begin{bmatrix} \tilde{A} & -\tilde{E} \\ -\tilde{Q} & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} x\ (t) \\ \lambda\ (t) \end{bmatrix}, \quad \tilde{E} = \tilde{B}\widetilde{R}^{-1}\tilde{B}^T, \tag{31}$$

Note that if $\widetilde{R}$ is a matrix (i.e., multiple-inputted systems), the inverse matrix of the equation should be first achieved by the explained interval inverse matrix method. For closed-loop optimum control, $\lambda(t) = \tilde{P}(t)x(t)$. Therefore,

$$\tilde{u}^*(t) = -\widetilde{R}^{-1}\tilde{B}^T\tilde{P}x(t) = -\tilde{k}x(t) \tag{32}$$

and,

$$\begin{cases} \dot{x}(t) = \tilde{A}x(t) - \tilde{B}\widetilde{R}^{-1}\tilde{B}^T\tilde{P}x(t) \\ \dot{\lambda}(t) = -\tilde{Q}x - \tilde{A}^T\tilde{P}x(t) \end{cases} \tag{33}$$

By solving the equation above, the final equation will become as follows:

$$\tilde{P}\tilde{A} + \tilde{A}^T\tilde{P} + \tilde{P}\tilde{B}\widetilde{R}^{-1}\tilde{B}^T\tilde{P}x + Q + \dot{\tilde{P}} = 0 \tag{34}$$

This is an extended version of the Algebraic Riccati Equation (*IMARE*) based on interval analysis, where the interval solution $\tilde{P}$ is required to achieve the optimal interval feedback gain $\tilde{k}$ such that $\tilde{k} = \widetilde{R}^{-1}\tilde{B}^T\tilde{P}$.

Since an interval method is required to solve the Interval ODEs, by simplification the achieved IMARE, some interval ordinary differential equations (IODEs) have been extracted. Based on [40], the following theorem can be used for this purpose:

Theorem 1 (*[40]*). *Let $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q)$, $y_0 \in \mathbb{I}$ nontrivial and $f : R_0 \to \mathbb{I}$ be continuous, nontrivial (i.e., y nontrivial interval gives $f(x, y)$ a nontrivial interval). If f satisfies the Lipschitz condition $D(f(x, y), f(x, z)) \le L \cdot D(y, z)$, $\forall (x, y), (x, z) \in R_0$ then the interval problem,*

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

gives two single solutions $y^i, y^{ii} : [x_0, x_0 + r] \to \overline{B}(y_0, q)$ and the continuous iterations in

$$y_0(x) = y_0$$

$$y_{n+1}(x) \ominus_g y_0 = \int_{x_0}^{x} f(t, y_n(t))\, dt$$

Or further accurately,

$$y^i_{n+1}(x) = y_0 + \int_{x_0}^{x} f\left(t, y^i_n(t)\right) dt$$

The proof and more information can be found in [40].

Since interval methods intrinsic wrapping effect, overestimation occurs in them [47]. This shortcoming leads researchers to work on different methods for reducing the overestimation of the interval calculations [6,48]. Here, the Chebyshev inclusion approach is used due to its ability in compressing (tightening) the interval bounds [5]. From the above explanations, it is clear that the final coefficients of the feedback controller will be interval integer; that is, the final controller determines a confident bound to provide the considered sub-optimal controller under interval uncertainties. After achieving the feedback gain coefficients, the final feedback law can be achieved from the Eq. (31).

7. Simulation results

Consider the optimal control problem for the DC motor speed model with interval uncertainties which is described in the previous section. A linear differential equation with $n = 3$, $p = 1$ and interval matrices $\tilde{A}$ and $\tilde{B}$. To confirm the validity of the linear quadratic regulator controller, we should first analyze the system controllability. Controllability in the interval matrices is different from the real matrices. In the following, an interval-based method is introduced and utilized for analyzing the DC motor controllability. The graphical abstract of the suggested control procedure is depicted in Fig. 2.

7.1. Controllability test for the suggested DC motor with interval uncertainties

In the following, the proposed controller is applied to the nonlinear system that is modeled as a linear system with interval uncertainties. So, the controllability must be analyzed on the model. The system is controllable if for each primary condition of $x(0) = x_0$ and for each given vector x_f, there is a limited time like t_f and input $u(t)$ in the interval $[0, t_f]$ where the input mapping the system from the x_0 into x_f in time t_f; i.e., $x(t_f) = x_f$. In otherwise, the given equation is uncontrollable. There are different methods for analyzing the controllability of the systems with real integer values [49]. Consider the introduced third-order interval DC with the following interval parameters:

$$\tilde{A} = \begin{bmatrix} [-6, -0.6] & [-0.096, -0.027] & \{0\} \\ [1.05, 3] & [-0.0062, -0.001] & \{0\} \\ \{0\} & \{1\} & \{0\} \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} [1.33, 4] \\ \{0\} \\ \{0\} \end{bmatrix}, \quad \tilde{C} = [\{0\} \quad \{1\} \quad \{0\}].$$

The system $(\tilde{A}, \tilde{B})$ will be controllable if the controllability matrix $\widetilde{R} = [\tilde{B}\tilde{A}\tilde{B}\tilde{A}^2\tilde{B}\tilde{A}^{n-1}\tilde{B}]$ has full row rank (i.e., $\rho(\widetilde{R}) = \{n\} = [n, n]$). By extending the interval arithmetic to this assumption and calculating the system controllability matrix, we have:

$$\rho(\widetilde{R}) = \rho\left(\begin{bmatrix} [1.33, 4] & [-7.98, -0.798] & [0.371, 47.746] \\ \{0\} & [-1.40, 3.99] & [-8.4656, -2.398] \\ \{0\} & \{0\} & [1.396, 3.99] \end{bmatrix}\right) = \{3\},$$

From the above, it is clear that the studied system is controllable and hence its closed-loop poles can be placed anywhere in the s-plane.

7.2. Optimal control of the interval-valued DC motor using proposed interval LQR

The presented method has a significant difference from other methods based on interval analysis. Generally, in interval optimal control, this depends on the order relation, as also, of interval derivative concepts that can be found in [50].

In the proposed method, instead of using the interval analysis to find a signal for robust control, at last, it gives a limitation for the signal which guarantees a sub-optimal area for controlling the system. This method can be also a pre-conditioning method for validating the methods before practical implementation.

To outline the optimum control of the assumed interval LQR, the augmented system matrices should be define first. Based on the prior subsection, we have $\tilde{A}$ and $\tilde{B}$ matrices. The other required matrices are given in the following:

$$\tilde{Q}(t) = \begin{bmatrix} \tilde{\gamma}_1 & \tilde{\gamma}_3 & \tilde{\gamma}_3 \\ \tilde{\gamma}_3 & \tilde{\gamma}_1 & \tilde{\gamma}_3 \\ \tilde{\gamma}_3 & \tilde{\gamma}_3 & \tilde{\gamma}_2 \end{bmatrix}, \widetilde{R} = \{8\}$$

$$t_0 = 0, \ t_f = 10, \ F(t_f) = 0$$

where, $\tilde{\gamma}_1 = [1/2, 2]$, $\tilde{\gamma}_2 = [1, 2]$ and $\tilde{\gamma}_3 = \{1/2\}$.

By considering the given matrices and applying it to the interval Algebraic Riccati Equation, the following interval Differential

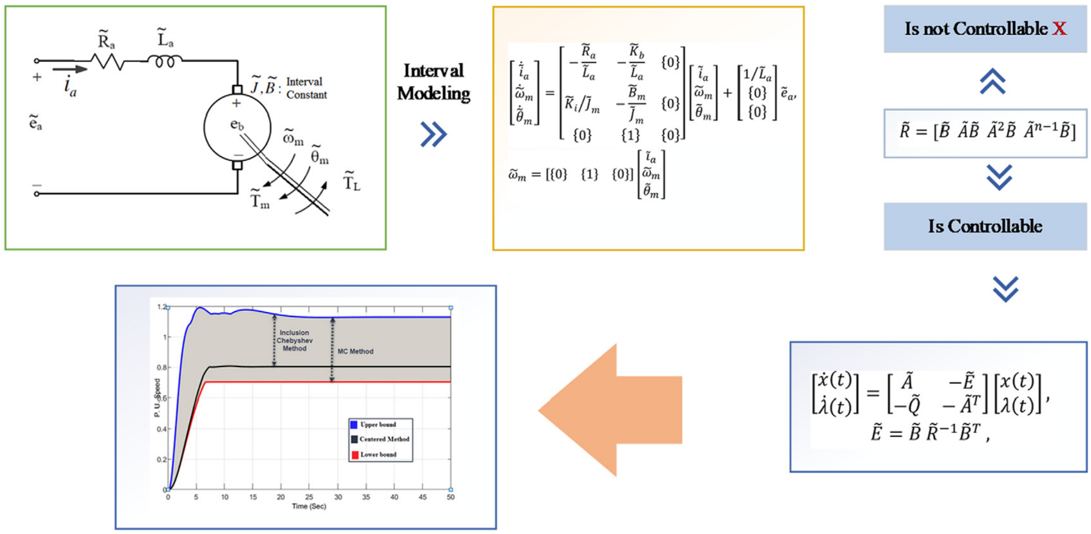

Fig. 2. The graphical abstract of the suggested control procedure.

system is achieved:

$$\dot{\tilde{P}}_1 = \tilde{\alpha}_1\tilde{P}_1 + \tilde{\alpha}_2\tilde{P}_2 + \tilde{\beta}\tilde{P}_1^2 - \tilde{\gamma}_1$$

$$\dot{\tilde{P}}_2 = \tilde{\alpha}_3\tilde{P}_2 + \tilde{\alpha}_4\tilde{P}_1 + \tilde{\alpha}_5\tilde{P}_4 - \tilde{P}_3 + \tilde{\beta}\tilde{P}_1\tilde{P}_2 - \tilde{\gamma}_3$$

$$\dot{\tilde{P}}_3 = \tilde{\alpha}_6\tilde{P}_3 + \tilde{\alpha}_7\tilde{P}_5 + \tilde{\beta}\tilde{P}_1\tilde{P}_3 + \tilde{\gamma}_3$$

$$\dot{\tilde{P}}_4 = 2\tilde{\alpha}_4\tilde{P}_2 + \tilde{\alpha}_8\tilde{P}_4 - 2\tilde{P}_5 + \tilde{\beta}\tilde{P}_2^2 - \tilde{\gamma}_1$$

$$\dot{\tilde{P}}_5 = \tilde{\alpha}_4\tilde{P}_3 + \tilde{\alpha}_8\tilde{P}_5 - \tilde{P}_6 + \tilde{\beta}\tilde{P}_2\tilde{P}_3 - \tilde{\gamma}_3$$

$$\dot{\tilde{P}}_6 = \tilde{\beta}\tilde{P}_3^2 - \tilde{\gamma}_2$$

where,

$\tilde{\alpha}_1 = [1.2, 12]$, $\tilde{\alpha}_2 = [-6, -2.1]$, $\tilde{\alpha}_3 = [0.601, 6.0062]$, $\tilde{\alpha}_4 = [0.027, 0.096]\tilde{\alpha}_5 = [-3, -1.05]$, $\tilde{\alpha}_6 = [0.6, 6]$, $\tilde{\alpha}_7 = [0.002, 0.0124]$, $\tilde{\alpha}_8 = [0.001, 0.0062]$, $\tilde{\beta} = [1/9, 1]$.

After applying the Chebyshev inclusion method on the ODE system, the optimum feedback gain ($K = [\underline{k}, \overline{k}]$) is obtained. The eigenvalues for different closed-loop interval controllers are given in Table 2.

There have been some different methods introduced for determining a robust optimal control [51,52]; but the main disadvantage of these methods is that the final controller is designed based on real-valued arithmetic about the center part of the interval.

Indeed, they neglect the confidence interval. Here, we consider all the confidence intervals for preventing the system from sudden problems. In other words, our work is a pre-condition operation to achieve the confidence interval for the next steps like optimal control, optimal tracking which can be also utilized for testifying a designed robust method; if the solution of the designed method does not include in the obtained interval, then, the method will not give proper results. For analyzing the system ability, it is compared by the MC method with 50 iterations. The comparison is shown in Fig. 3.

From Fig. 2, it can be observed that the centered method is laid in the proposed MC interval method. In the next step, the suggested approach is applied to reduce the overestimation of the interval and to find the optimal interval gain for closed-loop control of a DC motor with interval uncertainties. The step response of the DC motor is depicted in Fig. 4; here, the suggested Chebyshev inclusion approach is compared with the MC approach with 1000 iterations.

The achievements indicate that the suggested Chebyshev inclusion approach can obtain a rigorous result than the Monte Carlo simulation method. Since the proposed approach can efficiently handle the overestimation of the wrapping effect of

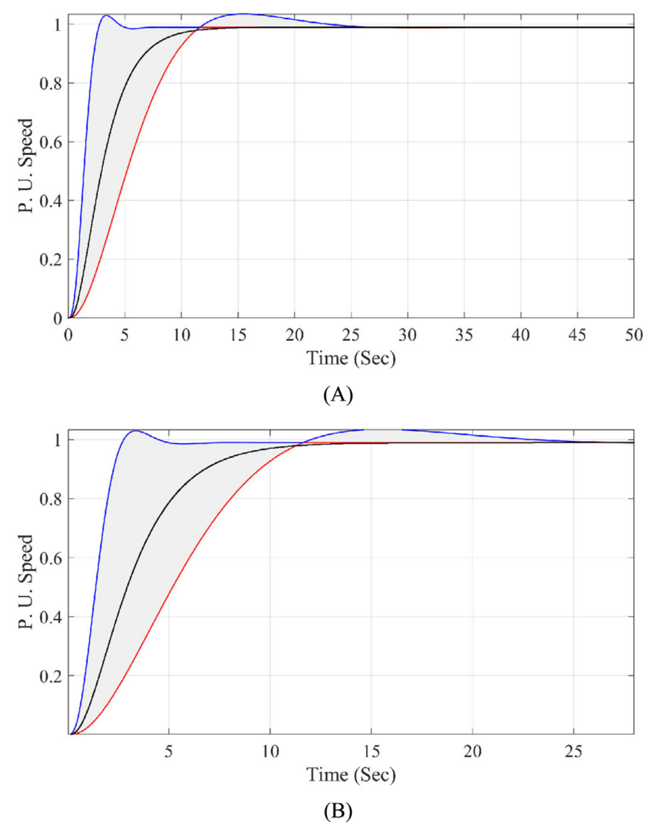

Fig. 3. Comparison of the step response for the DC motor with interval uncertainties for the suggested inclusion Chebyshev Method with MC method (50 iterations) in the (A) general range and (B) time range [0, 12]: Blue: upper bound, Red: Lower bound, and Black: Centered method.. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the interval optimal control. The suggested approach has a fair computational expense that is lower compared the extensively utilized MC approach. The calculation time for the Chebyshev inclusion-based and MC-based interval approaches is illustrated in Table 3. As observed in this table, the Chebyshev inclusion approach needs less running time rather than the MC approach.

Table 2
Eigenvalues for different closed-loop interval controllers.

Plant	Eigenvalues		
	λ_1	λ_2	λ_3
Lower bounds	−5.0518 + 0.0000i	−0.6030 + 0.5449i	−0.6030 − 0.5449i
Intermediate	−4.0342 + 0.0000i	−0.3432 + 0.3406i	−0.3432 − 0.3406i
Intermediate	−2.6723 + 0.0000i	−0.7530 + 0.6674i	−0.7530 − 0.6674i
Upper bounds	−1.4909 + 0.0000i	−0.6271 + 0.7666i	−0.6271 − 0.7666i

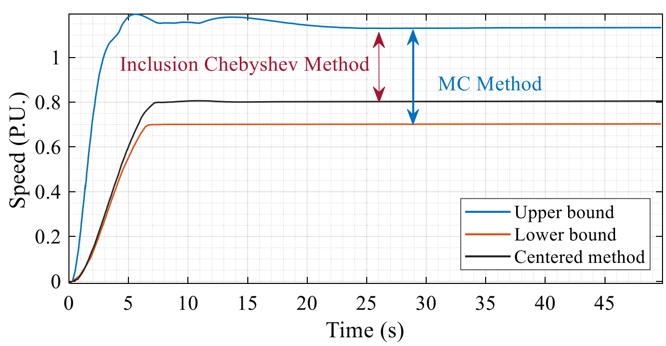

Fig. 4. Step response for DC motor with uncertainties by MC and Chebyshev inclusion methods.

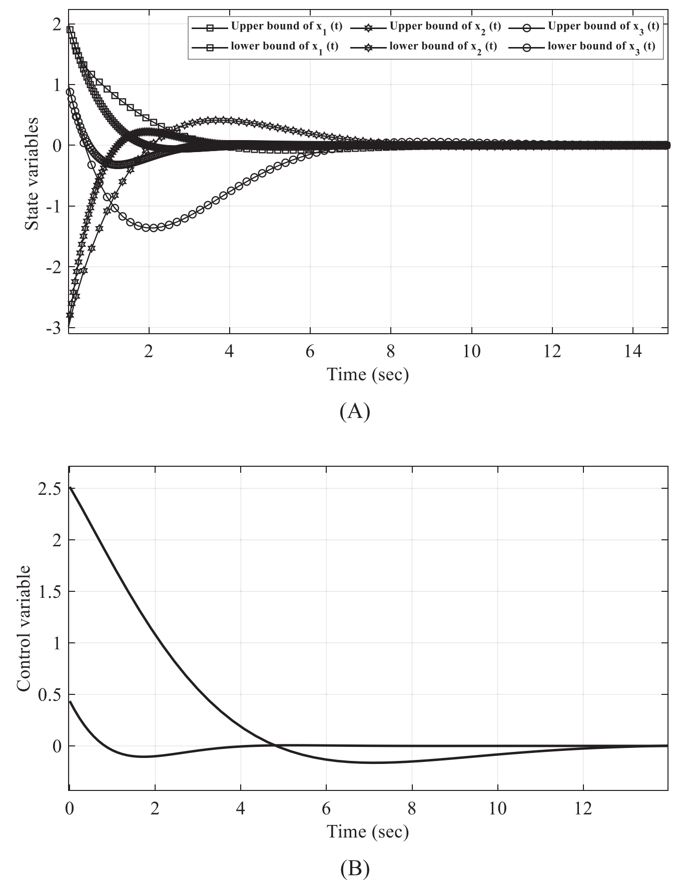

Fig. 5. Optimal State (A) and Control (B) variables for DC motor with interval uncertainties.

Table 3
Running time of the suggested method.

Method	Chebyshev inclusion approach	MC method
Running time (s.)	10.2	1000

The state and the control variables of the Chebyshev inclusion approach for the system are shown in Fig. 5.

Based on the results in simulations, it is observed that the interval-based proposed methodology can be considered as a first step method for studying rough mathematical programming optimal control problems. It allows considering errors in variables that are not dependent. An indefinite approach has the subjectivity factor. The known statistical approaches are applicable not in all cases. The approach guarantees to contain the precise achievement, it does not matter its maximum and minimum bounds occur to be overestimated. We gain probably the most reliable method here to deal with outliers when fitting unknown parameters to theoretical curves. Excellent tool, comparable with complex numbers in electrical engineering.

8. Conclusions

An interval-based optimal control method is proposed from a new point of view. The proposed method utilizes the Chebyshev inclusion method for robust control of a model of DC motor with uncertainties. The approach based on interval is applied to expand the optimum control problem with uncertain-but-bounded variables, without requiring complete information about uncertainties. The optimal control here is based on a new expanded design of the linear quadratic regulator with interval uncertainties. In the suggested approach, after applying the Pontryagin principle and achieving the final interval ordinary differential equation, the Chebyshev inclusion approach is applied to find solution for this problem. This method is utilized for controlling the overestimation led by the wrapping effect. Afterward, a comparison of the proposed Chebyshev inclusion approach with the ordinary centered approach and the Monte Carlo method was performed. During the simulation, the required run time for the Monte Carlo approach and the suggested Chebyshev inclusion approach are achieved 1000 s and 10.2 s that shows a significant prominence of the proposed method. Also, the results indicated a more compact interval for the suggested Chebyshev inclusion method which shows its prominence toward the Monte Carlo method and can better handle the overestimation of the wrapping effect of the problem. The main feature of the proposed method which can be considered as both advantages and disadvantages is its output which gives an interval instead of a definite value. So, on the one hand, it can be utilized as a pre-conditioner for validating the designed methods or a guarantee for designing the controller, and on the other hand, it does not a fixed signal for direct control of the system.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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