# Assessment of Repairable-System Reliability Using Proportional IntensityModels: A Review

Shwu-Tzy Jiang, Thomas L. Landers, and Teri Reed Rhoads*, Member, IEEE*

***Abstract*—This paper provides an overview of methods, and surveys the literature on engineering applications of proportional intensity (PI) models with explanatory variables (covariates), for repairable systems reliability assessment. The semi-parametric PI method relaxes the assumption of an underlying distribution, and is potentially useful in engineering practice, where the underlying information for a failure process is usually not available. PI semi-parametric models initially proposed for clinical studies in medical applications include PWP (Prentice, Williams, and Peterson), AG (Andersen, and Gill), and WLW (Wei, Lin, and Weissfeld). Abundant funding received in medical research has advanced PI models to become well developed, and widely referenced in the biostatistics field. This paper reviews both the available methods for repairable-system reliability assessment, and the published engineering application case studies. An engineering application example that applies PI model to a maintainability process used in US Army M1A2 Arams Main Battle Tank is presented.**

***Index Terms*—Dormancy, overhaul, proportional intensity models, reliability assessment, repairable system, right-censoring.**

ACRONYMS[1]

| | |
|---|---|
| AG | Andersen and Gill model |
| BPP | Branching Poisson Process |
| C.I. | Confidence interval |
| CM | Corrective maintenance |
| DCO | Downtime due to corrective work |
| DPM | Downtime due to preventive maintenance |
| HPP | Homogeneous Poisson process |
| IMTBF | Instantaneous mean time between failures |
| LHD | Load-haul-dump |
| LWA | Lee, Wei, and Amato model |
| MTTF | Mean time to failure |
| NHPP | Non-homogeneous Poisson Process |
| PH | Proportional hazards model |
| PI | Proportional intensity |
| PM | Preventive maintenance |
| PWP | Prentice, Williams, and Peterson model |
| PWP-GT | Prentice, Williams, and Peterson-gap time model |
| PWP-TT | Prentice, Williams, and Peterson-total time model |
| RGF | Rapid gravity filter |
| WLW | Wei, Lin, and Weissfeld model |

NOTATION

| | |
|---|---|
| $a, b, c$ | Constants in a second-order polynomial model |
| $C_{ki}$ | Censoring time for the $i^{th}$ subject of the $k^{th}$ type of failures |
| $\mathbf{G}\ldots$ | Information matrix (2 by 2) in the Newton-Raphson algorithm |
| $L(.)$ | Likelihood function |
| $N$ | Successive failure count; observed number of failures |
| $N(t)$ | Random variable for the number of failures in $(0, t]$; a counting process |
| $n$ | An integer counting successive failure times; a stratification indicator subscript |
| $T_i$ | Cumulative time of failure event $i$ |
| $T_1, T_2$ | Random variables in bivariate exponential distribution |
| $t_{ij}$ | Observed failure time corresponding to sample unit $i$ and failure count $j$ |
| $t_n$ | Failure time variable corresponding to failure count $n$ |
| $\mathbf{U}.$ | Score vector (1 by 2) in the Newton-Raphson algorithm |
| $U$ | Sample size (number of units) |
| $\widetilde{X}$ | Observation time |
| $Y_i^{(n)}$ | An at-risk indicator in the AG model; duration of downtime |
| $\mathbf{Z}$ | Vector of covariates |
| $\boldsymbol{\beta}_n$ | $(k \times 1)$ vector of stratum-specific regression coefficients $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)$ |
| $\delta$ | Shape parameter of a power-law NHPP |
| $\lambda(t; \mathbf{z})$ | Proportional intensity function |
| $\lambda_0(t)$ | Proportional baseline intensity function |
| $\lambda_{0n}(t)$ | Event-specific baseline intensity function |
| $\lambda$ | A reference state of the system recovery |
| $\mu_0$ | Baseline value of scale parameter of a log-linear NHPP |

Manuscript received March 17, 2005; revised May 7, 2005; July 3, 2005; and January 9, 2006. Associate Editor: J.-C. Lu.

S.-T. Jiang is with Global Concepts, Inc., Little Rock, AR 72211 USA (e-mail: sjiang@gclogistics.com).

T. L. Landers and T. R. Rhoads are with the School of Industrial Engineering, University of Oklahoma, Norman, OK 73019 USA (e-mail: landers@ou.edu; teri.rhoads@ou.edu).

Digital Object Identifier 10.1109/TR.2006.874938

[1]The singular and plural of an acronym are always spelled the same.

| | |
|---|---|
| $\mu_1$ | Alternate value of a log-linear NHPP |
| $\theta$ | Shape parameter of a log-linear NHPP; correlation among recurring events |
| $\upsilon_0$ | Baseline value of scale parameter of a power-law NHPP |
| $\upsilon_1$ | Alternate value of a power-law NHPP |
| $\wedge$ | Denotes an estimator |
| $'$ | Denotes the transpose of a vector |
| $\Delta$ | Indicator of a failure or censored time |

## I. Introduction

A reliability system can be classified as non-repairable or repairable. Non-repairable systems produce single-event failure data, and repairable systems produce recurrent-event failure data. Consumer electronics provides good examples of non-repairable systems, and replacements. Aircraft, automobiles, and process machine tools are good examples of repairable systems. These systems undergo during their lifetimes multiple unscheduled failure and repair cycles and/or scheduled preventive maintenance or overhaul cycles. Cox [1] proposed a proportional hazard (PH) rate function regression model for a non-repairable system. The PH function for a time-to-failure random variable is composed of two parts: baseline hazard function, and exponential link function with covariates (regression variables). Different levels of a covariate variable affect the PH function exponentially. In a non-repairable system, a unit is replaced when a failure occurs, thus renewing the hazard rate function each time. However, the repairable system is not replaced when a failure occurs, leading to a series of recurrent failure events over the lifetime. If repair is successful, the intensity function is improved to a state between as-bad-as-old (minimal repair), and as-good-as-new (replacement).

Maintenance actions can be classified in two major categories: corrective, and preventive. Corrective maintenance (CM) restores a failed system to operating condition, whereas preventive maintenance (PM) reduces the risk of system failure. Let $\lambda$ represent the degree by which the system reliability has been recovered to a reference state. The value of $\lambda$ varies from 0 to 1 depending on the repair types, such as minimal repair ($\lambda = 0$, as-bad-as-old), imperfect repair ($0 < \lambda < 1$), and replacement ($\lambda = 1$, as-good-as-new) (Usher *et al.* [2], Kijima [3], Kijima *et al.* [4]). Pham & Wang [5] provide a thorough survey of literature on imperfect maintenance. They surveyed several imperfect maintenance models for various policies: age-dependent PM policy, periodic PM policy, failure limit policy, sequential PM policy, and repair limit policy. They also developed mathematical maintenance models to estimate reliability measures, and to determine the optimal policies for systems with imperfect maintenance. The model framework consists of PM, CM, treatment method, optimality criterion, modeling tool, and planning horizon. The imperfect maintenance models include: (p, q) rule, (p(t), q(t)) rule, improvement factor, virtual age, shock, ($\alpha$, $\beta$) rule, and multiple (p, q) rule.

The Cox [1] PH model deals with a single event, whereas many researchers have expanded the PH model capability to handle recurrent events. The failure event process for a repairable system is frequently modeled in the literature as a Non-homogeneous Poisson Process (NHPP) with either power-law or log-linear intensity functions. However, the true underlying process may not be known, in which case a distribution-free or semi-parametric model may prove useful. Prentice, Williams, and Peterson (PWP) [6] proposed a semi-parametric approach to model recurrent event data using two methods: PWP-GT (gap time), and PWP-TT (total time). Several researchers subsequently proposed alternate modeling methods, and applied them to clinical practices in medical studies, including Andersen & Gill (AG) [7], and Wei, Lin, & Weissfeld (WLW) [8]. These Cox-based proportional intensity (PI) regression models have been widely applied in medical studies (biostatistics field), such as recurrent infections in a patient. Kelly & Lim [9] defined the major components (i.e. risk sets, and risk intervals) that form a PI model, and classified the PI models into family types based on risk sets, and risk intervals. Lipschutz & Snapinn [10] suggested guidelines in choosing the appropriate models. Wei & Glidden [11] gave an overview of statistical methods for multiple failure time data in clinical trials, and urged caution required in using semi-parametric PI models when there are dependence structures among recurrent events. Some researchers proposed a few methods in dealing with this situation to enhance the model accuracy, such as Lin & Wei [12].

Compared to the extensive literature on applications of the Cox-based PI regression models in the biostatistics field, there has been a small but growing body of literature reported for engineering applications. Qureshi *et al.* [13], and Landers *et al.* [14] examined the PWP-GT model, and characterized the more favorable engineering application ranges based on sample sizes, and shape parameter values. Jiang *et al.* [15]–[17] investigated the robustness of these PI models for a set of special engineering applications, including right-censoring severity, life cycles with substantial overhaul (dormancy) intervals, and multiple failure types (major and minor). The robustness study simulates recurrent failure data, and provides a thorough examination across a relevant range of sample size, shape parameter, and scale parameter values. Performance metrics (i.e., MSE, MAD, and BIAS) were used to evaluate how well the estimators could predict the theoretical value.

## II. Proportional Intensity Models

Subsections A-B review parametric Lawless, and semi-parametric methods for reliability assessment using PI models, including PWP, AG, and WLW models. Two examples using the parametric Lawless method are demonstrated, where the underlying intensity function is set to a power-law or log-linear form. Subsection C introduces the major components for the semi-parametric PI model structure: risk set, and risk intervals. Literature for selecting risk sets, and risk intervals used in a proportional intensity model is reviewed. Subsection D compares

PI models with the competing risk model, and parametric stochastic process. Subsection E addresses a misspecification phenomenon that may exist in PWP-TT, AG, and WLW models.

### A. Parametric Lawless Method

In discussing the PI models, two methods are reviewed: parametric, and semi-parametric. For the case of a known NHPP, the more accurate method is the parametric (Lawless) [18]. When the underlying distribution is unknown, the semi-parametric PI model is preferred. Soroudi [19], Landers & Soroudi [20], Qureshi [21], and Qureshi *et al.* [13] summarized the case of a single constant covariate in implementing the parametric Lawless method, where the underlying baseline hazard function follows the power-law form. Likewise, Vithala [22] and Landers *et al.* [14] investigated a log-linear baseline hazard function. The relevant formulas are summarized in the following.

*1) Power-Law Intensity Function:* When the baseline intensity function is specified as power-law, the parametric PI function can be expressed as

$$\lambda(t;\widetilde{\mathbf{z}}) = \delta \times t^{\delta-1}\exp(\widetilde{\mathbf{z}\boldsymbol{\beta}}),$$

where $\delta$ is the shape parameter, $z$ is a vector of covariate variables, $\beta$ is a vector of regression coefficients, and $t$ denotes inter-arrival time. We define $z_0 = 1$, and $v = \exp(\beta_0)$ to incorporate the baseline intensity in the exponential link function $\exp(\tilde{\mathbf{z}}\tilde{\boldsymbol{\beta}})$. The log likelihood function for the parametric model is

$$\begin{aligned} L(\delta,\widetilde{\boldsymbol{\beta}}) =& N \times U \times \log\delta + (\delta-1)\sum_{i=1}^{U}\sum_{j=1}^{N}\log t_{ij} \\ &+ \sum_{i=1}^{U} N \times \widetilde{\mathbf{z}}\widetilde{\boldsymbol{\beta}} - \sum_{i=1}^{U} t_i^{\delta} \times N \times \exp(\widetilde{\mathbf{z}}\widetilde{\boldsymbol{\beta}}), \end{aligned}$$

where $N$ is the observed number of failures, $U$ is the number of sample units, and $\delta$ is the shape parameter. $v$ is the scale parameter of the power-law form. $v_0$, $v_1$ are scale parameters in two class levels: baseline, and alternate. The maximum likelihood estimate of $\delta$ is given by Lawless [23] as

$$\hat{\delta} = \frac{-(N \times U)}{\sum_{i=1}^{U}\sum_{j=1}^{N}\log\left(\frac{t_{ij}}{t_{iN}}\right)},$$

where $t_{iN} = \sum_{j=1}^{N} t_{ij}$, and $N$ represents the observed number of failures.

The special case of a single constant covariate adapted by Qureshi [21] from Lawless [23] yields the score vector $U_0$, $U_1$, and information matrix with elements $G_{00}$, $G_{01}$, $G_{10}$, $G_{11}$ as follows.

$$\begin{aligned} U_0 &= \sum_{i=i}^{U} N - \sum_{i=1}^{U} t_{iN}^{\delta}\exp(\beta_0 + z_1\beta_1) \\ U_1 &= \sum_{i=1}^{U} N \times z_1 - \sum_{i=1}^{U} t_{iN}^{\delta} \times z_1 \times \exp(\beta_0 + z_1\beta_1) \\ G_{00} &= -\sum_{i=1}^{U} t_{iN}^{\delta}\exp(\beta_0 + z_1\beta_1) \\ G_{01} = G_{10} &= -\sum_{i=1}^{U} t_{iN}^{\delta} \times z_1 \times \exp(\beta_0 + z_1\beta_1) \\ G_{11} &= -\sum t_{iN}^{\delta} \times z_1^2 \times \exp(\beta_0 + z_1\beta_1). \end{aligned}$$

The formula to derive $IM\hat{T}BF$ (instant mean time between failures) is in terms of $\hat{\delta}$, and $\hat{v}_0$, $\hat{v}_1$, where $\hat{\delta}$ is recursively derived by the Newton-Raphson method, and ($\hat{\delta}$, $\hat{v}_0$, $\hat{v}_1$) denotes three estimators for shape parameter, baseline scale parameter, and alternate scale parameter. The formula to calculate $IM\hat{T}BF$ is (Lawless [23])

$$\hat{v}_0 = \exp(\hat{\beta}_0), \hat{v}_1 = \exp(\hat{\beta}_0 + \hat{\beta}_1).$$

For the power-law NHPP,

$$IM\hat{T}BF(t_n) = \left(\hat{v} \times \hat{\delta} \times t_n^{\hat{\delta}-1}\right)^{-1}.$$

*2) Log-Linear Intensity Function:* Likewise, given the baseline intensity function specified as log-linear, the parametric proportional intensity function is (Cox and Lewis [24])

$$\lambda(t;\widetilde{\mathbf{z}}) = \exp(\mu + \theta t)\exp(\widetilde{\mathbf{z}\boldsymbol{\beta}}).$$

Let $z_0 = 1$; then $\exp(\mu) = \exp(\beta_0)$, yielding exponential link function $\exp(\tilde{\mathbf{z}}\tilde{\boldsymbol{\beta}}) = \exp(z_0\beta_0 + z_1\beta_1 + \ldots z_k\beta_k)$. $(\theta, \mu)$ is the shape parameter, and the scale parameter. The proportional intensity function becomes $\lambda(t;\tilde{\mathbf{z}}) = \exp(\theta \times t)\exp(\tilde{\mathbf{z}}\tilde{\boldsymbol{\beta}})$. The log likelihood function for the Lawless parametric model is

$$L(\theta,\boldsymbol{\beta}) = \theta\sum_{i=1}^{m}\sum_{j=1}^{n_i} t_{ij} + \sum_{i=1}^{m} n_i \times \mathbf{z_i} \times \boldsymbol{\beta} - \frac{1}{\theta}\sum_{i=1}^{m}(e^{\theta \times T_i} - 1)e^{\widetilde{\mathbf{z}}\widetilde{\boldsymbol{\beta}}},$$

where $t_{ij}$ is the observed failure time corresponding to sample unit $i$, and failure count $j$. $\mu_0$, $\mu_1$ are scale parameters in two class levels: baseline, and alternate. The Maximum likelihood estimator of $\theta$ can be obtained by setting $\partial \log L/\theta = 0$, $\partial \log L/\beta_0 = 0$ (Lawless [23]) to yield

$$\hat{\theta} = \frac{N - e^{\beta_0} \times \sum_{i=1}^{U} T_i}{\sum_{i=1}^{U}\sum_{j=i}^{N}(T_i - t_{ij})},$$

where

$$T_i = \sum_{j=1}^{N} t_{ij},$$

and

$U$ represents the number of sample units,

$N$ is the total failure count corresponding to unit $i$.

In the special case of single constant covariate, the score vector $(U_0, U_1)'$, and information matrix with elements $G_{00}$, $G_{01}$, $G_{10}$, $G_{11}$, is as follows (Vithala [22], adapted from Lawless [23]):

$$\begin{aligned}
U_0 &= \sum_{i=1}^{U} N_i - \frac{1}{\theta}\sum_{i=1}^{U}(e^{\theta T_i} - 1)\exp(\beta_0 + z_{i1}\beta_1) \\
U_1 &= \sum_{i=1}^{U} N_i z_{i1} - \frac{1}{\theta}\sum_{i=1}^{U}(e^{\theta T_i} - 1) z_{i1}\exp(\beta_0 + z_{i1}\beta_1) \\
G_{00} &= -\frac{1}{\theta}\sum_{i=1}^{U}(e^{\theta T_i} - 1)\exp(\beta_0 + z_{i1}\beta_1) \\
G_{01} = G_{10} &= -\frac{1}{\theta}\sum_{i=1}^{U}(e^{\theta T_i} - 1) z_{i1}\exp(\beta_0 + z_{i1}\beta_1) \\
G_{11} &= -\frac{1}{\theta}\sum_{i=1}^{U}(e^{\theta T_i} - 1) z_{i1}^2\exp(\beta_0 + z_{i1}\beta_1).
\end{aligned}$$

Utilizing the score vector and information matrix, $\hat{\beta}_0$, $\hat{\beta}_1$ are calculated, for the scale parameter estimates $\hat{\mu}_0$, $\hat{\mu}_1$ in the two strata defined by the single covariate $\hat{\mu}_0 = \hat{\beta}_0$, $\hat{\mu}_1 = \hat{\beta}_0 + \hat{\beta}_1$.

The formula to derive $IM\hat{T}BF$ is in terms of $\hat{\theta}$, and $\hat{\mu}_0$, $\hat{\mu}_1$, where $\hat{\theta}$ is recursively derived by the Newton-Raphson method. The formula to calculate $IM\hat{T}BF$ is given by Lawless [23]:

$$IM\hat{T}BF(t_n) = e^{-(\hat{\mu}_i + \hat{\theta} \times t_n)}, \quad i = 0, 1.$$

### B. Semi-Parametric Methods

*1) Prentice-Williams-Peterson:* Prentice, Williams, and Peterson (PWP) [6] proposed a method that generalizes the proportional hazards (PH) model. PWP relaxes the assumption that the failure process follows a parametric form (e.g., NHPP power-law process), and extends the case of a single event to the case of a stochastic process with recurrent events. Because the PWP does not specify the baseline intensity function, it only estimates the covariate treatment effects. The PWP-gap time (PWP-GT), and PWP-total time (PWP-TT) models, respectively, are

$$\begin{aligned}
&\lambda\{t\,|N(t), Z(t)\} = \lambda_{0s}(t - t_{n(t)})\exp\{z(t)\beta_s\}, \text{ and} \\
&\lambda\{t\,|N(t), Z(t)\} = \lambda_{0s}(t)\exp\{z(t)\beta_s\}.
\end{aligned}$$

The gap time measures elapsed time between any two consecutive events, whereas total time measures the time from the beginning of observation. PWP concluded that the gap time model usually tends to provide a more precise regression estimator at each stage of the failure count, compared with the total time model. Several researchers utilized the PWP model, and extended the PWP to similar models based on different assumptions. The AG model by Andersen & Gill, and the WLW model by Wei *et al.*, are widely cited in the literature.

*2) Andersen-Gill:* Andersen & Gill [7] developed the AG method as an extension of the Cox PH model to accommodate recurring events in a counting process. The AG method explains general covariate effects (common baseline intensity function in the concept of risk set), because each event count re-starts the failure process, and thus does not feature event-stratifying effects. The risk interval of an AG model follows a counting process, where recurrences $(N_i^{(n)}, Y_i^{(n)}, Z_i^{(n)})$ are i.i.d. replicates of $(N, Y, Z)$, and the probability of the occurrence of two events at a given time is zero. Thus, the risk set of the $(n-1)^{st}$ event is identical to the risk set of the $(n)^{th}$ event. The AG model is defined as

$$\lambda_i^{(n)}(t) = Y_i^{(n)}(\mathrm{t})\lambda_0(\mathrm{t})\exp\left\{\boldsymbol{\beta} \times \mathbf{z}_i^{(n)}(t)\right\},$$

where $Y_i^{(n)}$ is an at-risk indicator, and $Y_i^{(n)} = 1$ unless the subject is withdrawn from the study.

AG studied psychiatric admissions. Two states in a Markov process are defined as admission, and discharge, corresponding to two forces of transition $\alpha_i(t)$, $\mu_i(t)$. The number of visits to psychiatric hospitals, $N_i(t)$, is a counting process with intensity function $\lambda_i(t) = \alpha_i(t)Y_i(t)$. Parity (number of children), and age are covariates in this study. Three covariates of parity status (parity0, parity2, parity $\geq 3$, with $\lambda_0(t)$ representing parity1), and two covariates of age range (age $\leq 18$, and age $\geq 34$) are employed, and defined as follows.

$$\begin{aligned}
&Z_{i1} = \begin{cases} 1 & parity0 \\ 0 & otherwise \end{cases}, \; Z_{i2} = \begin{cases} 1 & parity2 \\ 0 & otherwise \end{cases}, \\
&Z_{i3} = \begin{cases} 1 & parity \geq 3 \\ 0 & otherwise \end{cases}, \; Z_{i4} = \begin{cases} 1 & age \leq 18 \\ 0 & otherwise \end{cases}, \\
&Z_{i5} = \begin{cases} 1 & age > 34 \\ 0 & otherwise \end{cases},
\end{aligned}$$

where $i$ represents subject $i$.

A Markov process model is considered to analyse psychiatric admissions. A time-dependent covariate $Z_{i6}$ is introduced to form a semi-Markov process model, where the covariate is defined as

$$Z_{i6} = \begin{cases} 1 & re\text{-}admitted \\ 0 & otherwise. \end{cases}$$

*3) Wei-Lin-Weissfeld:* WLW [8] in a bladder cancer study examined treatment effects by using the PWP, and WLW models. The WLW example included four recurrence times ($T_1 \sim T_4$) corresponding to four marginal proportional hazards models. Rather than fitting each $T_i$, one model at a time, WLW fits four marginal models simultaneously. This example has two response variables {failure time, censoring status}, three covariates {treatment, tumor number, tumor size}, and four event recurrences {first tumor, second tumor, third tumor, fourth tumor.} For the $k^{th}$ failure type, and the $i^{th}$ failure event count, the hazard function $\lambda_{ki}(t)$ in WLW is assumed to take the form:

$$\lambda_{ki}(t) = \lambda_{k0}(t) \exp\left\{\boldsymbol{\beta}_k' \times \mathbf{z}_{ki}(t)\right\}, \quad t \geq 0,$$

where $\lambda_{k0}(t)$ is an unspecified baseline hazard function, and $\boldsymbol{\beta}_k' = (\beta_{1k}, \ldots, \beta_{pk})'$ is a vector of failure-specific regression parameters. $\mathbf{z}_{ki}(t)$ denotes a $p \times 1$ vector of covariates for the $i^{th}$ subject at time $t$ with respect to the $k^{th}$ type of failure, expressed as $\mathbf{z}_{ki}(t) = (z_{1ki}(t), z_{2ki}(t), \ldots z_{pki}(t))'$.

### *C. Risk Set & Risk Intervals*

The risk set consists of the subjects at risk for a specified event (e.g., failure). There are three types of risk sets: conditional (e.g., PWP), counting process (e.g., AG), or marginal (e.g., WLW). The AG method also provides an index of a general covariate effect, which is expressed by the common baseline intensity (unrestricted risk set). However, a subject in the PWP method has event-specific baseline intensity (restricted risk set) in that the proportional intensity of event $k$ only considers the subjects that have experienced $(k-1)$ events. Based on the common or event-specific baseline intensities, the risk set is labeled as either unrestricted or restricted. As a marginal method, the WLW method assumes a subject is at risk regardless of event count until the observation for the subject terminates by censoring. Kelly & Lim defined three possible risk sets {1) unrestricted, 2) restricted, 3) semi-restricted} in deciding which sample units are at risk of contributing to event k. Risk interval can be defined by three formulations: 1) gap time, 2) total time, and 3) counting process. Risk interval determines whether a model is marginal in the total time or conditional in the gap time. The risk interval of any event in total time is not influenced by any previous events, but measures time from beginning of observation. However, the risk interval of the gap time begins from the end of last event (Kelly & Lim [9]). Counting processes use the total time scale, and share the same elapsed time, as does the gap time model. However, the risk interval starts from the previous event instead of the entry time. Kelly & Lim illustrated three risk interval formulations shown in Fig. 1, a hypothetical example of three subjects with recurrent events. Observations start at the same time. Subject A experiences three events, at times 2, 5, and 13; subject B experiences three events at 7, 11, and 17; and subject C experiences one single event at 14. Taking subject A for example, the first event occurs at 2. With Gap time, subject A is at risk of the first event during (0,2); and of the second, and third events during (0,3), and (0,8). Total time is the time from the beginning of the observation, (0,2), (0,5), and (0,13). The counting process uses the same time scale as total time, but subject A will not be at risk for the second event until the end of the first event. With counting process formulation, subject A is at risk for the first, second, and third events during (0,2), (2,5), and (5,13).

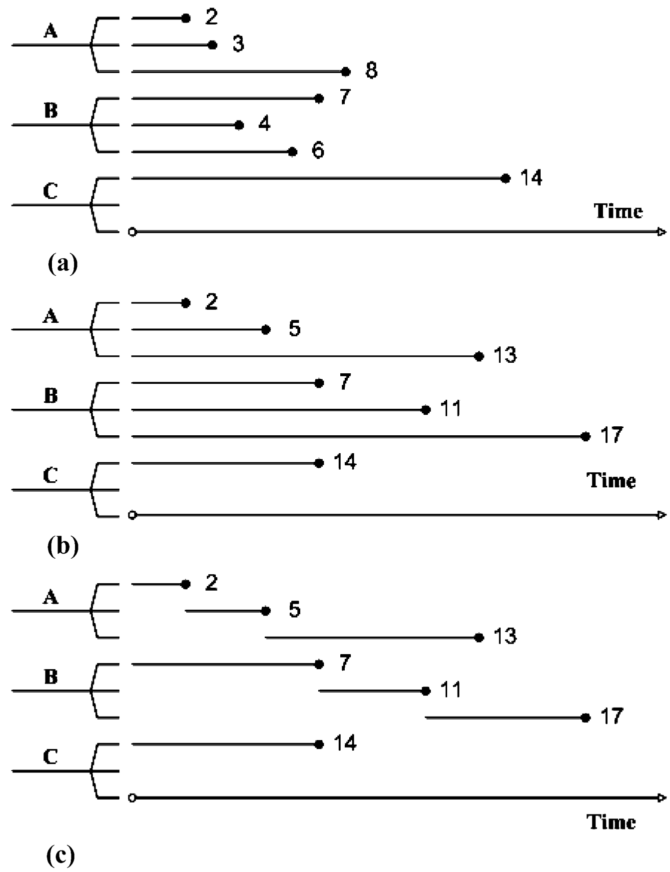


Fig. 1. Risk interval formulations (Kelly and Lim [9]). (a) Gap time; (b) total time; (c) counting process.

TABLE I
TAXONOMY OF RISK INTERVAL AND RISK SET FOR EACH MODEL (KELLY & LIM [9])

| Model | | Risk set/ baseline intensity | | |
|---|---|---|---|---|
| | | Unrestricted/ common | Semi-restricted/ event specific | Restricted/ Event specific |
| Risk interval | Gap time | Qureshi [21] | | PWP-GT [6] Vithala [22] |
| | Total time | LWA [25] | WLW [8] | |
| | Counting process | AG [7] | | PWP-TT [6] |

Table I summarizes the proportional intensity modeling methods, categorized by the Kelly & Lim [9] taxonomy on the basis of risk set versus risk interval. The LWA model, a special case of the WLW model, has a common baseline intensity function (Lin [25]). The model Qureshi [21] employed in a robustness study of the PWP model can be classified as $(\text{risk interval}, \text{risk set}) = (\text{gap time}, \text{common})$. Likewise, Vithala [22] may be classified as $(\text{risk interval}, \text{risk set}) =$

$(\text{gap time}, \text{event}-\text{specific})$. The PWP-total time (termed as PWP-CP in Kelly & Lim) is specified as a counting process instead of a total time model, due to the conditionality. PWP-CP is a stratified AG model. A marginal approach, such as the WLW method, employs the total time concept because subjects are at risk from the start of the observation.

Bowman [26] surveyed & evaluated the AG, PWP, and WLW methods applied to needlestick incidents in veterinary practice. Bowman conducted a simulation based on a bivariate exponential distribution to generate bivariate recurrent events, in order to control the correlation $(\theta)$ between recurring events. The bivariate exponential distribution $(T_1, T_2)$ was used to generate the consecutive recurring event times $T(n) = T_1(n) + T_2(n)$, where $n$ is the event count. The univariate event time $T(n)$ is composed of $T_1(n)$ & $T_2(n)$ with given correlation $\theta$. This simulation approach enables controlled correlation of recurring events. Bowman evaluated the performance of four methods (PWP-TT, PWP-GT, AG, and the WLW models), concluded that the PWP-GT model is superior, and then used the PWP-GT to analyse the needlestick injury data.

Lin [25] also evaluated the WLW, PWP, and AG methods of Cox regression analysis in multivariate failure time data using a marginal approach. Lin let $T_{ik}$ be the time when the $k^{th}$ type of failure occurs on the $i^{th}$ unit. Lin also let $C_{ik}$ be the corresponding censoring time, $X_{ik} = \min(T_{ik}, C_{ik})$ with the resulting $\Delta_{ik} = I(T_{ik} \leq C_{ik})$. The covariate vector for the $i^{th}$ unit with respect to the $k^{th}$ type of failure is $\widetilde{Z}_{ik} = (Z_{1ik}, \ldots, Z_{pik})'$. The marginal approach can be expressed in the two forms below addressed by WLW, or LWA, respectively, as

$$\lambda_k(t; Z_{ik}) = \lambda_{0k}(t) \exp\{\beta' Z_{ik}(t)\},$$
$$\lambda_k(t; Z_{ik}) = \lambda_0(t) \exp\{\beta' Z_{ik}(t)\}.$$

The WLW permits event-specific baseline intensities, whereas the LWA assumes a common baseline intensity function across all strata defined by the failure type. The partial likelihood functions for $\beta$ under WLW, and LWA, corresponding to the score vector, and information matrix, can be obtained from Lin [25].

Wei & Glidden [11] reviewed the Cox-based methods designed to model recurrent data, and summarized the strengths & weaknesses for each method. They applied these statistical methods for multiple failure time data to evaluate laser photocoagulation treatment for proliferative diabetic retinopathy. In a commentary on the Wei & Glidden paper, Lipschutz & Snapinn [10] stressed the two concepts of "event times," and "risk sets" as crucial to choosing the appropriate model. Event elapsed times are related to the total time, gap time, and counting process. The PWP-TT & WLW are modeled by total time, while only PWP-GT is modeled by gap time. The risk interval of the AG model belongs to the counting process class. Intuitively, total (global) times within a subject are highly correlated, with similar indication on the first recurrence & subsequent events. The total time model may indicate a large treatment effect throughout the entire study, although the gap time model has indicated little treatment effect beyond a certain recurrence. The counting process concept of the AG method implies no recurrence is affected by previous events, and does not contribute to future events.

Lipschutz & Snapinn suggested guidelines as follows in choosing the appropriate models:

- Use total time, common baseline hazard (unrestricted risk set) when the general effect is of interest.
- Use gap time, event-specific baseline hazards (restricted risk set) when the primary concern is how the treatment will affect the recurring events beyond the first occurrence.

### D. Comparison of PI Models with Other Reliability Models

PI models include the parametric Lawless method, and semi-parametric methods. The advantage of PI models comes from the incorporation of covariates that empowers the model to select critical factors for better reliability/maintainability, and conduct system performance assessment. Landers *et al.* [14] applied PI models to a real world example of a maintainability (restoration) process (log-linear with two explanatory covariates) for the US Army M1A2 Abrams Main Battle Tank. Two explanatory covariates are the environment in which a maintenance action is performed, and the maintenance echelon at which maintenance crews perform repairs. This example serves to demonstrate how using covariates is implemented to improve the field maintenance.

Competing risk (CR) model, another commonly used reliability model, is compared with the PI models. The distribution of the failure time random variable $T$ of the competing risk model is subject to $k$ competing risks, which can be expressed as $T = \min\{X_1, X_2, \ldots X_k\}$. The CR model requires two assumptions: 1) each failure mode meets i.i.d. conditions, and 2) the underlying distribution for each failure mode is known. Semi-parametric PI models do not require the underlying information. Additionally, the competing risk model does not account for any covariate effects as does the PI model.

Parametric stochastic processes are also used to model a recurring failure process. The assumption for using a parametric stochastic process is to obtain the underlying distribution. Two illustrations of performing the parametric stochastic process are in Section II-A: the parametric Lawless method, where two underlying distributions follow the power-law intensity function, and the log-linear intensity function. In engineering practice, the information is often not available, in which case a semi-parametric method will be more useful.

### E. Misspecification Phenomenon

Some Cox-based proportional hazards models (including the PWP-TT, AG, and WLW) are sensitive to misspecification due to a dependence structure that exists among recurrent events. The misspecification problem causes parameter estimators to become overestimated or underestimated. Kelly & Lim [9] addressed three ways to deal with misspecification problems: conditional, marginal, and random effects. The conditional method introduces time-variant covariates intended to capture the dependence structure. The marginal method utilizes a robust variance named a "sandwich estimator," added to the variance of the estimator. The approach of random effects (frailty method) includes a random covariate into the model aimed to induce the dependence structure among the failure events. Kelly & Lim [9]

TABLE II
ENGINEERING APPLICATIONS ARTICLES ON PROPORTIONAL INTENSITY MODELING

| Model Type | Paper | Dataset | | | | Covariate | | Industry | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | complete | right-censored | left-censored | failure and repair times | with | w/o | weapon system | city water treatment | pipeline | earth-moving equipment | photocopier | N/A |
| PWP | Qureshi et. al. [13] | √ | | | | √ | | | | | | | √ |
| | Landers, Soroudi [20] | √ | | | | √ | | | | | | | √ |
| | Landers et. al. [14] | √ | | | | √ | | √ | | | | | |
| | Jiang et. al. [15] | | √ | | | √ | | | | | | | √ |
| | Jiang et. al. [16] | | | | √ | √ | | | | | | | √ |
| | Kumar, Westberg [33] | √ | | | | √ | | | | | √ | | |
| AG and WLW | Jiang et. al. [15] | | √ | | | √ | | | | | | | √ |
| | Jiang et. al. [16] | | | | √ | √ | | | | | | | √ |
| | | | | | | | | | | √ | | | |
| Modified PWP | Ansell, Phillips [29] | | √ | | | √ | | | | | | | |
| | Ansell et. al. [31-32] | | | √ | | √ | | | √ | | | | |
| Phillips | [30] | √ | | | | | √ | | | | | √ | |

applied the conditional, and marginal approaches to childhood infectious disease cases, and concluded that applying the marginal method (robust variance) is not effective to resolve misspecification problems if any dependence exists.

Jiang *et al.* [27] investigated the misspecification problem, and addressed three potential misspecification factors: 1) neglect of random effects, 2) omitted covariates, and 3) measurement error. They commented that the special case of zero measurement error does not affect the point estimator. However, the variance adjustment is needed, which can be attained through a sandwich formula. If both errors exist in the model, a double-sandwich formula is derived to adjust the variance. A naïve estimator requires the adjustment to reach a consistent estimator. As for the measurement error associated with covariates, Jiang *et al.* illustrated an example in a study of skin cancers. Treatment assignment (placebo), and baseline (plasma) status are chosen as two covariates. Other researchers have worked on the robust variance model, such as Lin & Wei [12], and Therneau & Hamilton [28], to name a few.

## III. ENGINEERING APPLICATIONS

There is a growing body of literature for semi-parametric methods in engineering applications reported from varied industries. Table II provides a taxonomy of engineering applications articles grouped by model type. The following properties are used for comparison of each paper: data set, covariate, and industry.

Landers & Soroudi [20], and Qureshi *et al.* [13] conducted robustness studies of the PWP model over cases and parameter ranges typical of engineering applications. Simulation data utilized to test the model were generated from a Non-homogeneous Poisson Process (NHPP) having a power-law proportional intensity function with one covariate. $\lambda(t) = \delta \times t^{\delta-1}\exp(\mathbf{z}\boldsymbol{\beta})$, where $\delta$ is the shape parameter, $\mathbf{z}$ is the covariate variable, and $\boldsymbol{\beta}$ is the regression coefficient. The intensity function $\lambda(t)$ is strictly increasing for $\delta > 1.0$, constant for $\delta = 1.0$, and decreasing for $\delta < 1.0$. The more favorable application ranges were recommended: increasing rate of occurrence of failures (shape parameter of power-law form between 1.0 & 3.0), and a sample size of 60 or more. This model is then used to predict the successive times of occurrence of failure events. Landers *et al.* [14] examined the robustness of the PWP method for the case of an underlying failure process following the NHPP with log-linear proportional intensity function $\lambda(t) = \exp(\theta \times t)\exp(\mathbf{z}\boldsymbol{\beta})$, where $\theta$ is the shape parameter, $\mathbf{z}$ is the covariate variable, and $\boldsymbol{\beta}$ is the regression coefficient. The intensity function $\lambda(t)$ is strictly increasing for $\theta > 0$, constant for $\theta = 0$, and decreasing for $\theta < 0$. The more favorable application ranges were reported: a moderately increasing rate of occurrence of failures (shape parameter of log-linear form between 0 & 3), and a sample size of 60 or greater. A maintainability application for the US Army M1A2 Abrams Main Battle Tank demonstrated use of the PWP-GT model to identify explanatory variables, and predict successive failure times. The data set includes 2930 maintenance repair times on 18 M1A2 units that are the maintaining fielded battle tank for the US Army. These data stem from field experience in the actual operating environment, and with actual maintenance personnel performing repairs. The data system is currently used by the US Army to develop maintenance metrics and training devices. The environment (shop or field) surrounding a maintenance action is a potential covariate for the maintenance time. The maintenance echelon is another potential covariate for a maintainability model. The US Army's three lower maintenance echelons are: 1) crew, 2) organizational, and 3) direct support. The crew has limited knowledge of how to perform maintenance on the equipment, but does perform basic maintenance procedures. The organizational echelon is the lowest personnel level with primary responsibility, and specialized training to perform maintenance, both in the maintenance facility, and in the field. The direct support echelon performs maintenance primarily in the maintenance shop, but also performs some field repairs. Due to limited data, a two-echelon (shop and field) covariate was proposed for this example. The intensity function for maintenance time results in a log-linear intensity function with parameters $\beta_0 = -4.6309$, and $\beta_1 = -0.6094$:

$$\lambda(t) = \exp(\theta \times t)\exp(-4.6309 - 0.6094 \times \mathbf{z}).$$

If maintenance action is taken in the shop, $\mathbf{z} = 0$, otherwise $\mathbf{z} = 1$. Thus, the repair rate is 80% better for in-shop than for in-field repair actions.

Jiang *et al.* [15] extended the robustness characterization research to an important case of the right-censorship, and examined PWP, AG, and WLW models. Simulated data from a NHPP with power-law intensity function were used to test the model robustness. For the smaller sample size of 60, the PWP model proves to perform well when 80% of the units have some censoring. For the large sample size of 180, the PWP model performs well for severe censoring when 100% of the units have some censoring. Jiang *et al.* [16] considered repair times due to major overhaul in a recurrent failure process, examined the overhaul durations that could affect the model performance if neglected, and concluded that the PWP model performs well for relative overhaul durations less than 1/2 of the immediately preceding interval between failures. The AG model performs consistently well regardless of the repair time for a small sample size of 20 (10 per level of the covariate).

Ansell & Phillips [29] examined the failure data arising from a set of pipeline failures. Stress & temperature covariates proved statistically significant, but the interaction term proved insignificant. Phillips [30] proposed two methods to estimate the intensity function: parametric (power-law form), and nonparametric (kernel smoothing method). Ninety-two failures collected from a photocopier machine were utilized to estimate the expected ROCOF during the time period (0, 1135) days. The model indicated a decrease in the expected intensity function from more than 0.1 to 0.08 at 1135 days. Phillips compared these two methods, and concluded that the parametric method provides tight confidence bounds for ROCOF estimates, while the non-parametric estimation method depicts local activities. For instance, the photocopier has an increase near 600 days; and decrease near 400, and 800 days according to the non-parametric method.

Ansell *et al.* [31] applied a modified PWP model for a case study in municipal water treatment. Unplanned maintenance actions & repairs were analysed for a rapid gravity filter (RGF). The maintenance data were gathered from several sites spread over a large geographical area with different geologies, including clay plains and granite hills. A rapid gravity filter (RGF) cleans water by gravity filtration, and it consists of pipe work & valves covered by fine gravel, and a biological media. In the model, four covariate variables were selected: the maximum flow of water through the process, the average flow, whether the process was fit for purpose, and whether it was performing adequately. Ansell *et al.* concluded that all the variables have a significant effect, and provided an estimate of the underlying failure rate curve. Ansell *et al.* [32] performed an assessment to evaluate the maintenance impact. They utilized Crow's, and Cox & Lewis' methods to identify significant covariates, and estimate the underlying failure rate for RGF in the water treatment industry. The assessment verified refurbishment has a significant impact on the rate of occurrence of failures.

Kumar & Westberg [33] studied a class of earth-moving equipment, known as load-haul-dump (LHD) machines. LHD machines receive electric power through a cable mounted on the LHD machine. Frequent wrapping & unwrapping on the reel causes cable failures due to thermal or mechanical processes. The average time to first failure time (MTBF) is much higher than that for subsequent failures. Kumar & Westberg concluded that Cable-type (at first-failure-number), and new-welded-joint were statistically significant for the LHD machine failure rate. Time-dependence effect of a covariate was recognized by plotting the cumulative regression functions. The time point of 138 hours for the cable type covariate $(t < 138)$ indicated a change of the slope from positive to negative, which implies the cable type is a time-dependent covariate at the critical point of 138 hours.

## IV. Conclusion

The class of semi-parametric proportional intensity models provides a sound approach to obtain reliability information (e.g., failure rate, critical factor analysis, maintenance scheduling, etc) for a repairable system, when the underlying distribution is not known. The class of semi-parametric PI models has a wide range of engineering applications in terms of various industries, censoring, covariates, sample sizes, single or multiple failure types, and repair times. This paper reviewed several PI methods for repairable-system reliability assessment, both parametric Lawless, and semi-parametric (PWP, AG, and WLW). Several case studies were reported for PI model engineering applications, and we gave guidance in selecting appropriate PI models.

## Acknowledgment

The authors would like to thank the Associate Editor J.-C. Lu and the Managing Editor J. Rupe for the efforts they made during the review process and two anonymous referees for their valuable comments that have improved this paper.

## References

[1] D. R. Cox, "Regression models and lifetables (with discussion)," *Journal of the Royal Statistical Society. Series B*, vol. 34, pp. 187–220, 1972.
[2] J. S. Usher, A. H. Kamal, and W. H. Syed, "Cost optimal preventive maintenance and replacement scheduling," *IIE Transactions*, vol. 30, pp. 1121–1128, 1998.
[3] M. Kijima, "Some results for repairable systems with general repair," *Journal of Applied Probability*, vol. 26, pp. 89–102, 1989.
[4] M. Kijima, H. Morimura, and Y. Suzuki, "Periodical replacement problem without assuming minimal repair," *European Journal of Operation Research*, vol. 37, pp. 194–203, 1988.
[5] H. Pham and H. Wang, "Imperfect maintenance," *European Journal of Operation Research*, vol. 94, pp. 425–438, 1996.
[6] R. L. Prentice, B. J. Williams, and A. V. Peterson, "On the regression analysis of multivariate failure time data," *Biometrika*, vol. 68, no. 2, pp. 373–379, 1981.
[7] P. K. Andersen and R. D. Gill, "Cox's regression model for counting processes: a large sample study," *The Annals of Statistics*, vol. 10, no. 4, pp. 1100–1120, 1982.
[8] L. J. Wei, D. Y. Lin, and L. Weissfeld, "Regression analysis of multivariate incomplete failure time data by modeling marginal distributions," *Journal of the American Statistical Association*, vol. 84, no. 408, pp. 1065–1073, Dec. 1989.

[9] P. J. Kelly and L. L.-Y. Lim, "Survival analysis for recurrent data: An application to childhood infectious diseases," *Statistics in Medicine*, vol. 19, pp. 13–33, 2000.
[10] K. H. Lipschutz and S. M. Snapinn, "Discussion of paper by Wei and Glidden," *Statistics in Medicine*, vol. 16, pp. 841–851, 1997.
[11] L. J. Wei and D. V. Glidden, "An overview of statistical methods for multiple failure time data in clinical trials," *Statistics in Medicine*, vol. 16, pp. 833–839, 1997.
[12] D. Y. L. Dy and L. J. Wei, "The robust inference for the Cox proportional hazards model," *Journal of the American Statistical Association: Theory and Methods*, vol. 84, no. 408, pp. 1074–1078, Dec. 1989.
[13] W. M. Qureshi, T. L. Landers, and E. E. Gbur, "Robustness evaluation of a semi-parametric proportional intensity model," *Reliability Engineering and System Safety*, vol. 44, pp. 103–109, 1994.
[14] T. L. Landers, S. T. Jiang, and J. R. Peek, "Semi-parametric PWP model robustness for log-linear increasing rates of occurrence of failures," *Reliability Engineering and System Safety*, vol. 73, pp. 145–153, 2001.
[15] S. Jiang, T. L. Landers, and T. R. Rhoads, "Semi-parametric proportional intensity models robustness for right-censored recurrent failure data," *Reliability Engineering and System Safety*, vol. 90, no. 1, pp. 91–98, 2005.
[16] ——, "Proportional intensity models robustness with overhaul intervals," *Quality and Reliability Engineering International*, in press.
[17] S. Jiang, "Assessment of Semi-Parametric Proportional Intensity Models Applied to Recurrent Failure Data With Multiple Failure Types for Repairable-System Reliability," Ph.D. Dissertation, The University of Oklahoma, , 2004.
[18] J. F. Lawless*, Statistical Models and Methods for Lifetime Data*. New York: John Wiley & Sons, Inc., 1982.
[19] H. E. Soroudi, "Reliability Estimation for Repairable Systems With Covariates," MS thesis, The University of Arkansas, , May 1990.
[20] T. L. Landers and H. E. Soroudi, "Robustness of a semi-parametric proportional intensity model," *IEEE Transactions on Reliability*, vol. 40, no. 2, pp. 161–164, Jun. 1991.
[21] W. M. Qureshi, "Assessment of a Proportional Intensity Model for Repairable-System Reliability," Ph.D. Dissertation, The University of Arkansas, , Dec. 1991.
[22] S. Vithala, "Robustness of Semi-Parametric Proportional Intensity Model for the Case of a Log-Linear Nonhomogeneous Poisson Process," MS thesis, The University of Arkansas, , Dec. 1994.
[23] J. F. Lawless, "Regression methods for Poisson process data," *Journal of the American Statistical Association*, vol. 82, no. 399, pp. 808–815, 1987.
[24] D. R. Cox and P. A. W. Lewis*, The Statistical Analysis of Series of Events*. London: Methuen, 1966.
[25] D. Y. Lin, "Cox's regression analysis of multivariate failure time data: the marginal approach," *Statistics in Medicine*, vol. 13, pp. 2233–2247, 1994.
[26] M. E. Bowman, "An Evaluation of Statistical Models for the Analysis of Recurrent Events Data: With Application to Needlestick Injuries Among a Cohort of Female Veterinarians," Doctoral Dissertation, Ohio State University, , 1996.
[27] W. Jiang, B. W. Turnbull, and L. C. Clark, "Semiparametric regression models for repeated events with random effects and measurement error," *Journal of the American Statistical Association: Theory and Methods*, vol. 94, no. 445, pp. 111–124, Mar. 1999.
[28] T. M. Therneau and S. A. Hamilton, "rhDNase as an example of recurrent event analysis," *Statistics in Medicine*, vol. 16, pp. 2029–2247, 1997.
[29] J. I. Ansell and M. J. Phillips, "Practical aspects of modeling of repairable systems data using proportional hazards models," *Reliability Engineering and System Safety*, vol. 58, no. 2, pp. 165–171, 1997.
[30] M. J. Phillips, "Estimation of the expected ROCOF of a repairable system with bootstrap confidence region," *Quality and Reliability Engineering International*, vol. 17, pp. 159–162, 2001.
[31] J. Ansell, T. Archibald, J. Dagpunar, L. Thomas, P. Abell, and D. Duncalf, "Assessing the maintenance in a process using a semi-parametric approach," *Quality and Reliability Engineering International*, vol. 17, pp. 163–167, 2001.
[32] ——, "Analysing maintenance data to gain insight into systems performance," *Journal of Operational Research Society*, vol. 54, pp. 343–349, 2003.
[33] D. Kumar and U. Westberg, "Proportional hazards modeling of time-dependent covariates using linear regression: A case study," *IEEE Transactions on Reliability*, vol. 45, no. 3, pp. 386–392, 1996.

**Shwu-Tzy Jiang** received a Ph.D. in Industrial Engineering from the University of Oklahoma. Dr. Jiang is currently a consultant with Global Concepts, Inc., and is primarily participating in several projects in the energy industry. Her main research interests include reliability engineering, and maintainability technology in a repairable system. She was a finalist for the Best Student Paper Award, Quality, Statistics, and Reliability Section, INFORMS, 2004. She is a reviewer for IIE Transactions, and Journal of Manufacturing Science and Engineering. She is a member of IIE, and INFORMS..

**Thomas L. Landers** joined the faculty at the University of Oklahoma in 1998 as Morris R. Pitman professor and director of the School of Industrial Engineering. He is currently dean and AT&T chair in the college of engineering at the University of Oklahoma, and has served as director for Oklahoma Center for Aircraft and Systems/Support Infrastructure, Oklahoma Transportation Center, and Center for Engineering Logistics and Distribution. His professional experience includes positions as consultant with Ernst and Young, development engineer with Texas Instruments, plant manager in the alternative energy field, and as reliability and maintainability engineer for the F-16 Systems Program Office at Wright-Patterson AFB. Dr. Landers has over 80 publications to his credit.

**Teri Reed Rhoads** is associate dean for engineering education in the College of Engineering and assistant professor in the School of Industrial Engineering at the University of Oklahoma. Dr. Rhoads teaches engineering statistics and quality engineering courses. Her research interests are diverse in that she studies effective means of learning in the engineering classroom, including the incorporation of web-based learning in statistics. She is interested in not only the cognitive domain of learning, but also the affective domain and how to assess each. In addition, she researches recruitment and retention issues.