5-40 Let $\mathbf{x}$ denote the event "*the number of failures that precede the n^{th} success*" so that $\mathbf{x}+n$ represents the total number of trials needed to generate n successes. In that case, the event $\{\mathbf{x}=k\}$ occurs if and only if the last trial results in a success and among the previous $(\mathbf{x}+n-1)$ trials there are $n-1$ successes (or $\mathbf{x}$ failures). This gives an alternate formulation for the *Pascal* (or *negative binomial*) distribution as follows: (see Table 5-2)

$$P\{\mathbf{x}=k\} = \binom{n+k-1}{k} p^n q^k = \binom{-n}{k} p^n (-q)^k \qquad k = 0, 1, 2, \ldots$$

find $\Gamma(z)$ and show that $\eta_x = nq/p$, $\sigma_x^2 = nq/p^2$.

5-40)

$$\Gamma(z) = \mathcal{F}_x(z) = \sum z^k \cdot p_x(k)$$

$$= \sum_k p^n \cdot (-q)^k \cdot \binom{-n}{k} \cdot z^k$$

$$= p^n \sum_{k=0}^{\infty} \binom{-n}{k} \cdot (-qz)^k$$

The summation term looks like $(1-qz)^x$, with $x=-n$. Since we've defined $\binom{-n}{k}$, we can write it as $(1-qz)^{-n}$.

$$\therefore \Gamma(z) = \boxed{p^n \cdot (1-qz)^{-n}}$$

Now $E(X) = \mathcal{F}_x'(1) = \left(p^n \cdot -n \cdot (1-qz)^{-n-1} \cdot -q\right)\Big|_{z=1}$

$$\therefore E(X) = \frac{np^n q}{(1-q)^{n+1}} = \boxed{\frac{nq}{p}}$$

Also, $E(X(X-1)) = E(X^2) - E(X) = \mathcal{F}_x''(1)$

$$\therefore E(X^2) - E(X) = \frac{n \cdot n+1 \cdot p^n \cdot q^2}{(1-q)^{n+2}} = \frac{n \cdot n+1 \cdot q^2}{p^2}$$

$$\therefore Var(X) = E(X^2) - E(X) + E(X) - E(X)^2$$

$$= \frac{n \cdot n+1 \cdot q^2}{p^2} + \frac{n \cdot q \cdot p}{p^2} - \frac{n^2 q^2}{p^2}$$

$$= \frac{nq(q+p)}{p^2} = \boxed{\frac{nq}{p^2}}$$

5-41 Let $\mathbf{x}$ be a negative binomial random variable with parameters r and p. Show that as $p \to 1$ and $r \to \infty$ such that $r(1-p) \to \lambda$, a constant, then

$$P(\mathbf{x} = n + r) \to e^{-\lambda}\frac{\lambda^n}{n!} \qquad n = 0, 1, 2, \ldots$$

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We know that,

$$P(X=k) = \binom{k-1}{r-1} p^r q^{k-r}, \quad k \geq r$$

so, Let $k = n + r$

$$\Rightarrow P(X = n+r) = \binom{n+r-1}{r-1} p^r q^{n}, \quad n \geq 0$$

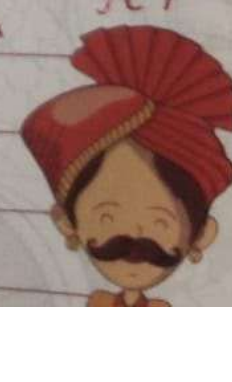
$$= \frac{(n+r-1)!}{n!\,(r-1)!} p^r (1-p)^{n}$$

$$= \frac{1}{n!}\left(\frac{(n+r-1)(n+r-2)\cdots(r) \times r^n}{r^n \times \cancel{(r-1)!}}\right) p^r (1-p)^{n}$$

$$= \frac{1}{n!}\left(\left(1+\frac{n-1}{r}\right)\left(1+\frac{n-2}{r}\right)\cdots(1)\right) \times \underbrace{(r(1-p))}_{\lambda}{}^{n}\, p^r$$

$$= \frac{\lambda^n}{n!}\left\{\prod_{k=1}^{n}\left(1+\frac{n-k}{r}\right)\right\}\left(1-\frac{\lambda}{r}\right)^r$$

where $\lambda = r(1-p)$



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Thus,

$$\lim_{r \to \infty} P(x = n + r) = \frac{\lambda^n}{n!} \left\{ \lim_{r \to \infty} \cancel{\prod_{k=1}^{n} \left(1 + \frac{n-k}{r} \right)} \xrightarrow{} 1 \right\}$$

$$\times \lim_{r \to \infty} \left(\cancel{1} - \frac{\lambda}{r} \right) \xrightarrow{} e^{-\frac{\lambda}{r}}$$

$$\rightarrow \frac{\lambda^n}{n!} e^{-\lambda} \sim P(\lambda)$$

5-50 A biased coin is tossed and the first outcome is noted. The tossing is continued until the outcome is the complement of the first outcome, thus completing the first run. Let **x** denote the length of the first run. Find the p.m.f of **x**, and show that

$$E\{\mathbf{x}\} = \frac{p}{q} + \frac{q}{p}$$

5-50) Let $p(H) = p$.

x is the length of the first run.

i.e. ① toss a coin and see outcome

② start run and keep tossing till complement is observed.

$$\therefore f(x) = p(H)\cdot p(\underbrace{HH\cdots HT}_{x}) + p(T)\cdot p(\underbrace{T\cdot T\cdots TH}_{x})$$

$$\therefore \boxed{f(x) = p^x\cdot q + q^x\cdot p}$$

$$E(x) = \sum_{x=1}^{\infty} x(p^x q + q^x p) = q\sum_1^{\infty} x p^x + p\sum_1^{\infty} x q^x$$

We know that $\sum_1^{\infty} x p^x = \frac{p}{(1-p)^2} = \frac{p}{q^2}$

$$\therefore E(x) = q\cdot\frac{p}{q^2} + p\cdot\frac{q}{p^2}\left[\because \sum_1^{\infty} x q^x = \frac{q}{p^2}\right]$$

$$= \boxed{\frac{p}{q} + \frac{q}{p}}$$

Let $s = \sum_1^{\infty} x r^x = r + 2r^2 + 3r^3 \cdots$

$\therefore rs = \quad r^2 + 2r^3 \cdots$

$\therefore s(1-r) = r + r^2 + r^3 \cdots$

$\therefore s(1-r) = r(1+r+r^2\cdots)$

$\therefore s(1-r) = \frac{r}{1-r}$

$\therefore s = \sum_1^{\infty} x r^x = \frac{r}{(1-r)^2}$

6-25 Let x be the lifetime of a certain electric bulb, and y that of its replacement after the failure of the first bulb. Suppose x and y are independent with common exponential density function with parameter λ. Find the probability that the combined lifetime exceeds 2λ. What is the probability that the replacement outlasts the original component by λ?

(6-25)

As X, Y are independent, we can say that,

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{1}{\lambda^2} e^{-(x+y)/\lambda}$$

(i) Let $Z = X + Y$

$$\Rightarrow f_Z(z) = \frac{z}{\lambda^2} e^{-z/\lambda}$$ [from book]

Now, we need $P(Z > 2\lambda)$

$$\Rightarrow P(Z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{z}{\lambda^2} e^{-z/\lambda} dz$$

$$= \int_{2}^{\infty} x e^{-x} dx \quad ; \left[\begin{array}{c} x = z/\lambda \\ dx = dz/\lambda \end{array} \right]$$

$$= \left[-x e^{-x}\right]_2^{\infty} + \int_2^{\infty} e^{-x} dx$$

$$= \cancel{0} + 2e^{-2} + e^{-2}$$

$$= 3e^{-2}$$

$$\approx \underline{0.406}$$

Let; $\boxed{W = Y - X}$ (for 2nd condition)

$$P(Y-X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} f_W(w)\, dw$$

For $w = y - x$; $F_W(w)$ is given by:

$$F_W(w) = \int_{x=-w}^{\infty} \int_{y=0}^{w+x} f_{yx}(y, x)\, dy\, dx$$

$$\Rightarrow f_W(w) = \begin{cases} \displaystyle\int_0^{\infty} f_{yx}(w+x, x)\, dx & w \geq 0 \\ \displaystyle\int_{-w}^{\infty} f_{yx}(w+x, x)\, dx & w < 0 \end{cases}$$

for $w > 0$;

$$f_W(w) = \int_0^{\infty} \frac{1}{\lambda^2} e^{-\frac{(w+2x)}{\lambda}}\, dx$$

$$= \frac{1}{\lambda^2} e^{-w/\lambda} \int_0^{\infty} e^{-2x/\lambda}\, dx$$

$$= \frac{1}{2\lambda} e^{-w/\lambda}, \quad w > 0$$

$$\therefore P(Y-X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} \frac{1}{2\lambda} e^{-w/\lambda}\, dW$$

$$= \frac{1}{2\cancel{\lambda}} \times -\cancel{\lambda} \left[e^{-w/\lambda} \right]_{\lambda}^{\infty}$$

$$= \frac{1}{2e}$$

6-46 Let $\mathbf{x}$ and $\mathbf{y}$ be independent Poisson random variables with parameters λ_1 and λ_2, respectively. Show that the conditional density function of $\mathbf{x}$ given $\mathbf{x}+\mathbf{y}$ is binomial.

As X, Y are Poisson random variables,

$$f_X(x) = e^{-\lambda_1}\frac{\lambda_1^x}{x!}, \quad f_Y(y) = e^{-\lambda_2}\frac{\lambda_2^y}{y!}$$

Let $Z = X+Y$

$$f_Z(z) = \sum_{x+y=z} f_X(x)\, f_Y(y)$$

$$f_Z(z) = \sum_{x=0}^{z} e^{-\lambda_1}\frac{\lambda_1^x}{x!} \cdot e^{-\lambda_2}\frac{\lambda_2^{z-x}}{(z-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!}\sum_{x=0}^{z} {}^{z}C_x \cdot \lambda_1^x\, \lambda_2^{z-x}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!}(\lambda_1+\lambda_2)^z$$

Hence $f_Z(z)$ is also a Poisson random variable with parameter $\lambda_1+\lambda_2$.

As $Z = X+Y$, $z = x+y$

$$\therefore f_{XZ}(x,z) = f_{XY}(x, z-x)$$

$$= f_X(x)\, f_Y(z-x) \quad (\because \text{independent})$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}\lambda_1^x \lambda_2^{z-x}}{x!\,(z-x)!}$$

Now

$$f_X(x|z) = \frac{f_{XZ}(x,z)}{f_Z(z)}$$

$$= \frac{\dfrac{e^{-(\lambda_1+\lambda_2)}\lambda_1^x\lambda_2^{z-x}}{x!\,(z-x)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^z}{z!}}$$

$$\therefore \boxed{f_X(x|z) = {}^{z}C_x\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{z-x}}$$

Hence conditional density fn. of X given $X+Y$ is binomial

(a) The probability that a family has n children is p_n for $n = 0, 1, \ldots$. You meet a random person on the street. Let X be the number of siblings that this person has. Let Y be the number of elder siblings that this person has. Obtain the marginal and the joint expectations of X and Y. Also obtain the expectations of X and Y. Make any reasonable assumptions that you may need.

Expected number of children in a family = $\sum_{i=0}^{\infty} i p_i$

Suppose there ~~is~~ are total n families.

Expected number of children = $n \sum_{i=0}^{\infty} i p_i$

Expected number of families with $(x+1)$ children

$= n P_{(x+1)}$

Number of children with x siblings = $(x+1) \cdot n \cdot P_{(x+1)}$

$\therefore$ Probability of meeting a random ~~guy~~ person with x siblings =

$$\boxed{p(X=x) = \frac{(x+1) P_{(x+1)}}{\sum_{i=0}^{\infty} i p_i}}$$

Probability that the person has x siblings and y elder siblings =

$$\left.\begin{array}{ll} 0 & \text{if } y > x \\ p(X=x) \times \frac{1}{x+1} & \text{if } y \leq x \end{array}\right\} \quad x, y \geq 0$$

Joint PMF:

$\cancel{x=z+r}$ $\cancel{z=y+r}$

$\cancel{x=z+r}$ $\cancel{z=y+r}$

$$P(X=x, Y=y) = \begin{cases} \dfrac{p(x+1)}{\sum_{i=0}^{\infty} i p_i} & x \geq y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(Y=y) = \sum_{x=0}^{\infty} P(X=x, Y=y)$$

$$= \frac{\sum_{x=y}^{\infty} p(x+1)}{\sum_{i=0}^{\infty} i p_i}$$

$$F_X(x) = P(X \leq x) = \sum_{i=0}^{x} p(X=i) = \boxed{\frac{\sum_{i=0}^{x} p(i+1) \cdot (i+1)}{\sum_{i=0}^{\infty} i p_i}}$$

$$F_Y(y) = P(Y \leq y) = \sum_{i=0}^{y} P(Y=i) = \boxed{\frac{\sum_{i=0}^{y} \left(\sum_{x=i}^{\infty} p(x+1) \right)}{\sum_{i=0}^{\infty} i p_i}}$$

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

$$= \sum_{m=0}^{x} \sum_{n=0}^{y} P(X=x)(Y=y)$$

$$\boxed{F_{XY}(x,y) = \frac{\sum\limits_{m=0}^{x} \sum\limits_{n=0}^{\min(m,y)} p(m+1)}{\sum\limits_{i=0}^{\infty} i p_i}}$$

$$E(X) = \sum_{x=0}^{\infty} x P(X=x)$$

$$= \boxed{\frac{\sum\limits_{x=1}^{\infty} x(x+1)\, p(x+1)}{\sum\limits_{i=0}^{\infty} i p_i}}$$

$$E(Y) = \sum_{y=0}^{\infty} y P(Y=y) = \frac{\sum\limits_{y=0}^{\infty} \left(y \sum\limits_{x=y}^{\infty} p(x+1) \right)}{\sum\limits_{i=0}^{\infty} i p_i}$$

$$= \boxed{\frac{\sum\limits_{y=0}^{\infty} \frac{y(y+1)}{2} \cdot p(y+1)}{\sum\limits_{i=0}^{\infty} i p_i}} = \boxed{\frac{E(X)}{2}}$$

(b) A graph, or a network, is represented by the set of vertices V and the set of edges E. An edge is a 2-tuple (unordered) (v_1, v_2). You can visualise an edge as a connection between the two vertices. A graph is connected if there is a path, a sequence of edges, between every pair of nodes. Construct a random graph as follows. Choose points uniformly and independently in $(0, 1)$ for the vertices. Thus $\mathsf{X_i}$ is a uniform random variable in $(0, 1)$. Nodes i and j are connected if $|\mathsf{X_i} - \mathsf{X_j}| \leq \mathsf{r}$.

For example, a for a two node network let $\mathsf{X_1}$ and $\mathsf{X_2}$ be the 'random' locations of nodes 1 and 2 chosen as above. They are connected if $|\mathsf{X_1} - \mathsf{X_2}| < \mathsf{r}$ for a give r. What is the probability that this 2-node graph is connected. Repeat this for a three node graph.

Let $|X_i - X_j| = d$ $\qquad d \in [0, r]$

∴ Assume $X_i < X_j$:

$X_i \in [0, 1-d]$ $\qquad X_j = X_i + d$

∴ Probability of choosing $X_i, X_j = (1-d)$ $\qquad$ for $X_i < X_j$, $|X_i - X_j| = d$

∴ Total probability for $|X_i - X_j| = d$:

$$\text{Prob} = 2(1-d)$$

$$\text{Prob}(|X_i - X_j| \leq r) = \int_0^r 2(1-s)\,ds = \boxed{2r - r^2}$$

Probability of 2 pts being connected.

~~For~~ There are 8 cases let x = conn. b/w 1,2
y = conn. b/w 2,3
z = conn b/w 1,3

$x, y, z = 0 \Rightarrow$ No connection
$x, y, z = 1 \Rightarrow$ connected directly

Following 8 possibilities for x, y, z:

x	y	z	case #
0	0	0	0
0	0	1	1
0	1	0	2
0	1	1	3
1	0	0	4
1	0	1	5
1	1	0	6
1	1	1	7

→ Let C_i denote Case # i $i \in [0, 7]$
→ Each case is mutually exclusive

We want ~~the~~ the 3 points to be indirectly connected → at least 2 to be connected.

$$P(x=1) = 2r - r^2 = P(C_4) + P(C_5) + P(C_6) + P(C_7)$$
$$P(y=1) = 2r - r^2 = P(C_2) + P(C_3) + P(C_6) + P(C_7)$$
$$P(z=1) = 2r - r^2 = P(C_1) + P(C_3) + P(C_5) + P(C_7)$$

8.00 Now, $P(G)$ is probability that node 1, 2, 3 are directly connected.

10.00 WLOG, consider smallest, middle, largest node.

11.00 No. of ways 1, 2, 3 can take these places = $3!$

12.00 Now for all three to be connected,

1.00 largest - smallest $\leq r$

2.00 Suppose we want to find prob. of largest - smallest = k

3.00 ~~the~~ let the value assumed by smallest be s.

4.00 . s has to lie in $[0, 1-k]$, middle value

5.00 has to lie in $[s, s+k]$ and largest value

6.00 is $s+k$.

7.00 • Hence prob. (largest - smallest = k) = $3!\,(1-k)(k)$

8.00 $\therefore$ Prob (largest - smallest $\leq r$) = $\int_0^r 3!\,(1-k)k\,dk$

$$P(G) = 3r^2 - 2r^3$$

Now, $P(C_0)$ is probability of no node being directly connected.

WLOG consider smallest, middle, largest node.

No. of ways ~~to~~ 1, 2, 3 can take thes places = 3!

for all 3 to be disconnected,

middle - smallest $> r$
largest - middle $> r$

let middle node assume value m

$m \in (r, 1-r)$ clearly, $\boxed{r < 1/2}$

smallest value lies in $[0, m-r)$

largest value lies in $(m+r, 1]$

~~Probab~~ $P(C_0) = 3! \int_r^{1-r} (m-r)(1-m-r)\, dm$

$$= 6 \int_r^{1-r} \left(-m^2 + m - r(1-r)\right) dm$$

$$P(C_0) = -\left(8r^3 - 12r^2 + 6r - 1\right) \quad (r < 1/2)$$

$$P(C_0) = 0 \quad r \geq 1/2$$

As largest - smallest > 1

Now, for at least 2 nodes to be connected, cases of interest : C_3, C_5, C_6, C_7

$$P(C_3)+P(C_5)+P(C_6)+P(C_7) = P(x=1)+P(y=1)+P(z=1) + P(C_0) - P(C_7) - 1$$

$$= 3(2r-r^2) - (3r^2-2r^3) - 1 + P(C_0)$$

for $r<1/2$:

$$Prob = 6r - 3r^2 - 3r^2 + 2r^3 - 1 - 8r^3 + 12r^2 - 6r + 1$$

$$= -6r^3 + 6r^2$$

for $r \geq 1/2$

$$Prob = 6r - 3r^2 - 3r^2 + 2r^3 - 1 + 0$$

$$= 2r^3 - 6r^2 + 6r - 1$$

$\therefore$ Probability of graph with 3 nodes being connected is:

$$\boxed{\begin{array}{ll} 6r^2 - 6r^3 & \text{for } 0 \leq r < 1/2 \\ 2r^3 - 6r^2 + 6r - 1 & \text{for } 1 \geq r \geq 1/2 \end{array}}$$

(c) Consider two random variables X and Y. Define VAR(X|Y). Show that VAR(X) = E(VAR(X|Y)) + VAR(E(X|Y)).

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4) (c)

$$VAR(X|Y) = E\left[(X - E[X|Y])^2 \,|\, Y\right]$$

$$= E\left[X^2 - E(X|Y)^2 \,|\, Y\right]$$

$$\Rightarrow E(VAR(X|Y)) = E(E(X^2|Y)) - E(E(X|Y)^2|Y)$$

$$= E(X^2) - E(E(X|Y)^2)$$

$$= E(X^2) - E(X^{\cancel{2}})^2 + E(E(X|Y))^2 - E(E(X|Y)^2)$$

$$\left(\text{As } E(X^{\cancel{2}})^2 = E(E(X|Y))^2\right)$$

$$= Var(X) - Var(E(X|Y))$$

$$\therefore Var(X) = Var(E(X|Y)) + E(Var(X|Y))$$

Hence proved