3) a) Given $F(x)$ is a valid distribution,

$$G(x) = aF(x) + (1-a)F(x) \qquad 0 \leq a \leq .$$

$$= \cancel{aF(x)} + F(x) - \cancel{aF(x)}$$

$$G(x) = F(x)$$

Hence, $G(x)$ is also a valid distribution.

b) $(F(x))^r = G(x)$

$\lim_{x\to\infty} F(x) = 1$; $\lim_{x\to -\infty} F(x) = 0$

$F(x)$ is non decreasing and $F(x) \in [0,1]$

∴ for any $r > 0$;

$\lim_{x\to\infty} (F(x))^r = 1$ and $\lim_{x\to-\infty} (F(x))^r = 0$

Let $H(x) = x^r$. As $H(x)$ is increasing for $x \geq 0$,

$H(F(x))$ is also increasing for $1 \geq F(x) \geq 0$

and $G(x) = H(F(x)) = (F(x))^r$

and hence it is <u>valid</u>.

For $r = 0$, $G(x) = 1$ ⇒ not a valid dist.

as $\lim_{x\to-\infty} G(x) \neq 0$

For $r < 0$; $\lim_{x\to\infty} G(x) = 1$ but $\lim_{x\to-\infty} G(x) = \infty$

⇒ not a valid distribution

c) $G(x) = 1 - (1 - F(x))^r$

(a) $r = 0$, $G(x) = 1 \Rightarrow$ Invalid

(b) $r > 0$,

$$\lim_{x \to -\infty} G(x) = \lim_{x \to -\infty} \left(1 - (1 - (F(x)))^r\right) = 0$$

$$\lim_{x \to \infty} G(x) = \lim_{x \to \infty} \left(1 - (1 - F(x))^r\right)$$

$$= 1$$

We know that

$$\forall \quad a \geq b$$

$$\longleftrightarrow \quad F(a) \geq F(b)$$

$$\longleftrightarrow \quad -F(a) \leq -F(b)$$

$$\longleftrightarrow \quad 1 - F(a) \leq 1 - F(b)$$

$$\longleftrightarrow \quad (1 - F(a))^r \leq (1 - F(b))^r$$

$$\longleftrightarrow \quad -(1 - F(a))^r \geq -(1 - F(b))^r$$

$$\longleftrightarrow \quad 1 - (1 - F(a))^r \geq 1 - (1 - F(b))^r$$

$$\longleftrightarrow \quad G(a) \geq G(b)$$

$\therefore$ $G(x)$ is valid.

d) F(x) is valid

$G(F(x)) = F(x) + (1 - F(x))(\log(1-F(x)))$

This is valid only if F(x) never attains 1 for a finite x.

$G(t) = t + (1-t)(\log(1-t))$

$G'(t) = 1 - \frac{\cancel{(1-t)}}{\cancel{(1-t)}}1 - \log(1-t)$

$= -\log(1-t)$

when $t \in [0,1)$, $G'(t) \geq 0$ and hence $G(t)$ is non-decreasing.

For $t_2 > t_1$; $t_1, t_2 \in [0,1)$:

$$G(t_1) \leq G(t_2)$$

Also, $F(t_2) \geq F(t_1)$; $F(t_1), F(t_2) \in [0,1)$

Hence $G \circ F$ is also non decreasing.

Checking the limits,

$\lim_{x \to -\infty} F(x) + (1-F(x))(\log(1-F(x)))$

$= \lim_{x \to -\infty} F(x) + \left(\lim_{x \to -\infty}(1-F(x))\right) \cdot \left(\lim_{x \to -\infty} \log(1-F(x))\right)$

$= 0 + 1 \cdot \log(1)$

$= 0 = \lim_{x \to -\infty} G(F(x))$

$$\lim_{x\to\infty} (F(x)) + \left(\lim_{x\to\infty} (1-F(x))\right)\left(\lim_{x\to\infty} \log(1-F(x))\right)$$

$$= 1 + \lim_{t\to 0}[t\cdot\log t] \qquad (t = 1-F(x))$$

$$= 1$$

We verified the above 2 limits using common limit rules.
Now we need to check for right continuity.

$$\lim_{x\to x_0^+}\left(F(x) + (1-F(x))\log(1-F(x))\right) = \lim_{x\to x_0^+} F(x) + \lim_{x\to x_0^+}(1-F(x))\log(1-F(x))$$

$$= F(x_0) + (1-F(x_0))\lim_{x\to x_0^+}(1-F(x))$$

$$= F(x_0) + (1-F(x_0))\cdot\log(1-F(x_0))$$

Hence, this is a valid Distribution for every valid $F(x)$ which doesn't attain '1' for a finite x.

4. Without loss of generality, consider a given black ball, say ball i. It is always chosen from an urn, say A, with n balls. This black ball will be in urn A if it is chosen an even number of times out of the m time steps. Let $X_i(m)$ be the indicator variable that this given black ball is in urn A after m time steps i.e. $X_i(m) = 1$ if this black ball is in urn A. Therefore, the expectation of the number of black balls in urn A,$E_B(A)$ after m time steps is simply the sum of this $X_i(m)$ being 1 for all the black balls.
   Now, for one black ball, the probability of choosing that ball, $P(B)$ is $1/n$ and total number of balls in each urn at any time is n. So the probability for that one ball to stay inside urn A is given by :

$$P_B(A) = \sum_i \binom{m}{i} \left(\frac{1}{n}\right)^i \left(\frac{n-1}{n}\right)^{m-i} \tag{1}$$

where $0 \leq i \leq m$; i is even

   Now, (1) can be simplified to the following :

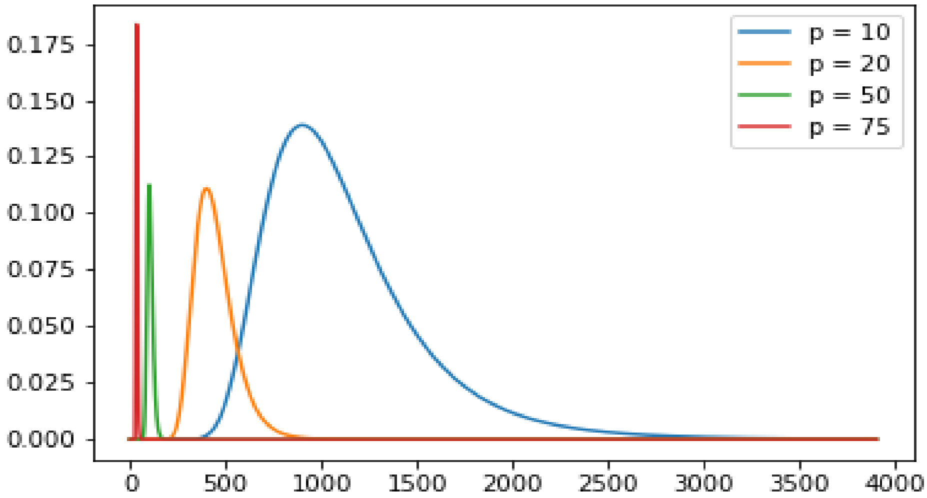
$$P_B(A) = \left(\frac{1}{2}\right) \left(1 + \left(\frac{n-2}{n}\right)^m\right) \tag{2}$$

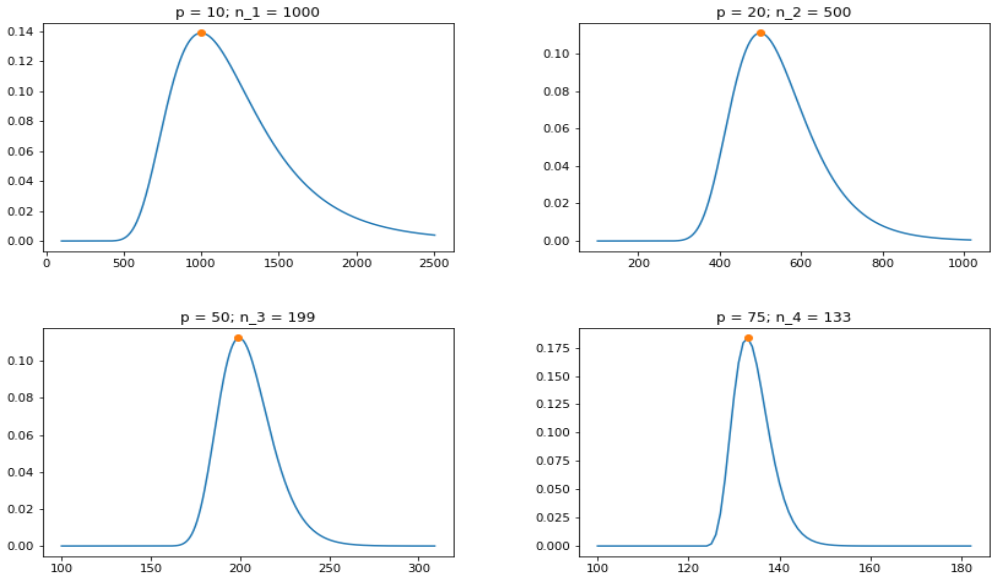
   So, (2) gives the expectation for one black ball in urn A. Since $P_B(A)$ is independent for each black ball, so $E_B(A)$ is simply given as :

$$E_B(A) = \left(\frac{n}{2}\right) \left(1 + \left(\frac{n-2}{n}\right)^m\right)$$

   Now, the above result is under the assumption that event of a ball being swapped even times is independent for each black ball. Since the total number of swaps is fixed i.e m, the event of swapping of 1 ball influences the net remaining swaps for other $n-1$ balls.Thus, pmf of number of balls in urn A after m swaps can't be simply commented upon via this logic.

5. $P_{m,p}(n)$ is the probability of p marked fish being caught, given m fish out of n were marked in the first place.
   Total favourable outcomes are given by choosing p out of m marked fish and $m-p$ out of $n-m$ unmarked fish.
   Total possible outcomes are given by choosing m out of n fish.
   Hence, $P_{m,p}(n) = \dfrac{\binom{m}{p}\binom{n-m}{m-p}}{\binom{n}{m}}$
   Here are the plots of $P_{m,p}(n)$ plotted as a function of n, for the given values of p :





Hence, the n_i obtained are 1000, 500, 199, 133.

6. After having run the simulation for the n_i calculated above, the following p values were obtained:

   - $p = 9.972$ for $n_i = 1000$
   - $p = 19.706$ for $n_i = 500$
   - $p = 50.386$ for $n_i = 199$
   - $p = 75.332$ for $n_i = 133$

These values are very close to the values of p that yielded these n_i in the first place.
The following statement can now be concluded.
Keeping m fixed, if n_0 total fish result in the expected value of p to be p_0, then the most probable value of n, given p_0 fish out of m were marked in the second catch, is indeed n_0.

7. Probability of getting exactly n 6's in the first $6n$ tosses :

$$\rho_n = \binom{6n}{n}\left(\frac{1}{6}\right)^n\left(\frac{5}{6}\right)^{5n}$$

Proving that ρ_n is monotonically decreasing by showing $\dfrac{\rho_{n+1}}{\rho_n} < 1\ \forall n \geq 1$:

$$\begin{aligned}
\frac{\rho_{n+1}}{\rho_n} &= \left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right)\frac{(6n+6)(6n+5)\cdots(6n+1)}{(n+1)(5n+5)\cdots(5n+1)} \\
&= \left(\frac{5}{6}\right)^5\frac{(6n+5)\cdots(6n+1)}{(5n+5)\cdots(5n+1)} \\
&= \left(\frac{5}{6}\right)^5\left(\frac{n}{5n+5}+1\right)\left(\frac{n}{5n+4}+1\right)\cdots\left(\frac{n}{5n+1}+1\right)
\end{aligned}$$

$\forall n \geq 1,\ \dfrac{n}{5n+i}$ is strictly increasing $\forall i \in \{1,2,3,4,5\}$

$$\therefore \frac{n}{5n+i} < \lim_{n\to\infty}\frac{n}{5n+i} = \frac{1}{5}\quad \forall i \in \{1,2,3,4,5\}$$

Hence $\left(\dfrac{n}{5n+5}+1\right)\left(\dfrac{n}{5n+4}+1\right)\cdots\left(\dfrac{n}{5n+1}+1\right)$ is strictly increasing and its value is bounded above by $\left(\dfrac{6}{5}\right)^5$

Hence $\dfrac{\rho_{n+1}}{\rho_n}$ is strictly increasing and is bounded above by $\lim_{n\to\infty}\dfrac{\rho_{n+1}}{\rho_n}$

$$\begin{aligned}
\lim_{n\to\infty}\frac{\rho_{n+1}}{\rho_n} &= \left(\frac{5}{6}\right)^5\left(\frac{6}{5}\right)^5 = 1 \\
\therefore \frac{\rho_{n+1}}{\rho_n} &< 1 \\
\therefore \rho_{n+1} &< \rho_n\ (\forall n \geq 1)
\end{aligned}$$

Hence ρ_n is monotonic function.