EE 325: Probability and Random Processes

Homework for Modules 4 and 5

Instructions

- There will be viva on these on Wednesday, 28 October.
- Submit all the solutions.

Questions

1. From the Chebyshev inequality, find an upper bound on the probability that a random variable deviates from the mean by more than two standard deviations.

2. Let X be a Gaussian random variable with mean μ and variance σ^2. For $\mu = 3$ and $\sigma = 1.2$, find a bound on $\mathsf{Pr}(\mathsf{X} > 5)$ using Markov, Chebyshev, and Chernoff inequalities. Compare with the true value obtained from the normal distribution table.

3. Let X be a Binomial(N, p) random variable. For large N and small p, you can consider a Poisson approximation for the pmf of X. Using the Central Limit Theorem, we can also write a Gaussian approximation. Of course, you can also compute it exactly, possibly via a small computer program. For the following values of x, N, and p compare the three values—exact value, Poisson approximation, and the Gaussian approximation. (i) $N = 20, p = 0.1, x = 5$ (ii) $N = 100, p = 0.05, x = 8$ (iii) $N = 1000, p = 0.01, x = 13$. What issues do you see with the accuracy in the exact computations?

4. There n balls numbered $1, 2, \ldots, n$ and n boxes numbered $1, 2, \ldots, n$. The balls are permuted randomly with each box containing exactly one ball. i is a fixed point if ball i is in box i. Let X be the number of fixed points. Find an upper bound on $\mathsf{Pr}(\mathsf{X} > k)$ for using Markov and Chebyshev inequalities. Evaluate for $k = 2, 10$ and $n = 10, 100, 500$.

5. Zomato is giving out coupons for every order placed that works as follows. Each coupon will contain one word from their tagline “Never have a bad meal.” Every time you order food from them, they send you an e-coupon chosen randomly from the five types of coupons (one for each word in the tagline). You get a bonus if you can make a slogan out of the coupons that you obtain from Zomato. The rule is only one bonus per customer and coupons are not transferable. A customer will participate if theb probability of having to order more than K times is sufficiently small. Use the Chebyshev inequality to obtain an upper bound on the probability that a customer

has to place more than K orders and evaluate it for $K = 15, 20$. Assume that there is no reneging, i.e., all those customers that register for the scheme will continue to buy till they 'collect the slogan.'

Let us now take Zomato's view. It knows that N customers have decided to participate in the scheme. Using Chebyshev inequality, find an upper bound on the probability that the total number of orders placed by all the customers is less than NK_1. Evaluate for $N = 100, 500$ and $K_1 = 8, 11$.

Bonus: Repeat for the case of N being Poisson with known mean μ? This is more closer to the "real" situation.Evaluate for $\mu = 100, 500$?

6. Let X_n be a geometric random variable with parameter $\frac{\lambda}{n}$. Show that $\mathsf{X}_n \xrightarrow{d} \mathsf{X}$ where X is an exponential random variable with parameter λ.

7. Consider the following sequence of independent random variables.

$$\mathsf{X}_n = \begin{cases} n^3 & \text{with probability } \frac{1}{n^2} \\ 0 & \text{with probability } 1 - \frac{1}{n^2} \end{cases}$$

Show that $\mathsf{X}_n \xrightarrow{p} 0$ but not $\mathsf{X}_n \xrightarrow{d} 0$.

8. Consider the following sequence of independent random variables.

$$\mathsf{X}_n = \begin{cases} 1 & \text{with probability } \frac{1}{n} \\ 0 & \text{with probability } 1 - \frac{1}{n} \end{cases}$$

Show that $\mathsf{X}_n \xrightarrow{p} 0$ and $\mathsf{X}_n \xrightarrow{m.s.} 0$ but not $\mathsf{X}_n \xrightarrow{a.s} 0$. For the latter you can use the Borel-Cantelli Lemma suitably.

9. Consider the three coins problem discussed in class. There are three biased coins, named A, B, and C, with unknown biases designated, respectively, p_A, p_B, and p_C. You are allowed N tosses and you have to maximise the total number of heads in the N tosses. If you you knew the p, then the best thing to do would be to toss the coin with the highest p, say p^*, and obtain an expected reward of Np^* heads. Since you do not know the p, let us explore two algorithms to use in maximising the expected reward. Evaluate for the following choices of the ps. (i) $p_A = 0.2, p_B = 0.4, p_C = 0.7$ (ii) $p_A = 0.45, p_B = 0.5, p_C = 0.58$. We will use $N = 20, 100, 1000, 5000$.

   (a) **Algorithm A:** Fix $N_1 < N$ and toss each coin $N_1/3$ times. Let n_A, n_B, and n_C be the number of heads obtained for coins A, B, and C. For the remaining $N - N_1$ toss the one with the highest n. The issue here is what is the best N_1. If N_1 is small then, intuitively, the wrong coin may be chosen with higher probability (this will be made more precise later in the course). If N_1 is large then there is not enough time to use the more reliable information collected in the first N_1 times. Let $R(N_1)$ be the expected number of heads in the N tosses for a choice of N_1. Of course, this also depends on the p and N.

Peform the following computation experiment. For each of the eight cases, simulate the above algorithm for every $N_1 < N$ 1000 times and find the sample average for $R(N_1)$ and determine the best N_1. List these values. Also find the number of times the correct coin was chosen at the end of the N_1 tosses for the best N_1.

(b) **Algorithm B:** We will use Hoeffding's inequality as follows. After k tosses let $n_A(k)$, $n_B(k)$, and $n_C(k)$ be the number of times coins A, B, and C were used and let $k_A(k)$, $k_B(k)$, and $k_C(k)$ be the number of times the corresponding coins tossed heads; $n_A(k) + n_B(k) + n_C(k) = k$.

Consider coin A. Although we do not know p_A, we can use $n_A(k)$ and $k_A(k)$ in Hoeffding's inequality to determine at any time k, we can obtain an upper bound on p_A with a reasonable amount of confidence. Denote this upper bound by $UCB_A(k)$. Specifically, use Hoeffding's inequality to calculate $UCB_A(k) = \frac{k_A(k)}{n_A(k)} + X_A$ such that $\mathsf{Pr}(p_A \leq UCB_A | n_A, k_A) > \alpha$. Similarly, calculate $UCB_B(k)$ and $UCB_C(k)$. For the $(k+1)$-th toss choose the coin with the highest $UCB(k)$. If there is a tie, break it randomly. This is an elementary learning algorithm where you adaptively learn to use the best coin. This algorithm also has many nice properties that a more full fledged course will explore in detail.

Implement this algorithm. For each of the eight cases, run a simulation program that executes the algorithm 1000 times and find the sample averages for the number of heads in N tosses and the number of times the correct coin was used. Also plot the sample average of $k_A(k)/k$, $k_B(k)/k$ and $k_C(k)/$ as a function of k for $N = 5000$.

Submit the results for $\alpha = 0.1, 0.05$ and $\alpha = 0.01$ and the values p and N as in the previous algorithm.

10. A diagnostic laboratory has received 200 tubes for testing for Covid19. Before beginning testing, they would like to estimate how many of the 200 tubes are positive.

The lab technician takes 10 test tubes randomly from the 200 and tests them individually. Assume that the test is perfect, which means that a tube always tests positive if it is positive, and negative otherwise. Let $POS \in \{0, 1, 2, \ldots, 200\}$ be the random variable that describes the total number of positive tubes in the batch. Let $n \in \{0, 1, 2, \ldots, 10\}$ be the random variable that describes the number of positive tubes detected from the 10 tubes.

(a) How will the technician estimate POS from measured value of n? Write a program to compute the expected values $\mathsf{E}(POS \mid n = k)$ for $k = 0, 1, 2, 3, 4$.

(b) The technician finds that 2 of the 10 tubes are positive. She wants to know how reliable is the expected value as a measure of the number of positives. Write a program to compute the probability $\mathsf{Pr}(POS > \mathsf{E}(POS) + 1 \mid k = 2)$ that the number of positives is greater than the expected value by more than 1.

(c) Repeat the previous two parts if the total number of tubes were 400 instead of 200 and the total number of tubes tested is 20 instead of 10.

(d) Discuss how the confidence of inference changes from the first case to the second case. Why has it changed? Is it because of the increase in total number of tubes, or the increase in number of tubes sampled, or something else?

Make any reasonable assumption that you think you may need.