Remember about learning dif eqs

u: Diff Functions from $[t_0, T]$ to $\mathbb{R}^n$:
$t \to u(t)$

p: Parameters $p \in \mathbb{R}^q$

F: Zeros of F $\equiv$ Solve the dif eq with params p

$$F(u, p) = u'(t) - f(u, p, t)$$

To specify the ode we need ~~F~~ f and boundary condition $u(t_0) = 0$

$\mathcal{L}$: Loss

$$\mathcal{L}(u, p) = \text{scalar}$$

Remember p determines $\mu(t)$ determines $\mathcal{L}(u(t), p)$ and we wish to optimize over p.

$$\begin{tikzcd}
p \arrow[r, "F_p"] \arrow[rrr, bend right, "\mathcal{L}_p"'] & F \arrow[r, "-F_u^{-1}"] & u \arrow[r, "\mathcal{L}_u"] & \mathcal{L}
\end{tikzcd}$$

(in some problems $\mathcal{L}_p = 0$)

Linear operators

F_p : takes a small change to $p \in \mathbb{R}^q$
and sees what happens to F
(a continuous function ~~from~~ $[t_0, T]$ to $\mathbb{R}^n$)

F_u : takes a small change ~~s~~ to u
and sees what happens to F
/ at time t it is an $n \times n$ matrix

Following Chris we take

$$\mathcal{L}(u, p) = \int_{t_0}^{T} g(u, p)\, dt = \int_{t_0}^{T} g(u(t), p)\, dt$$

$$\mathcal{L}_p = \int_{t_0}^{T} g_p(u(t), p)\, dt \qquad g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$$

$\mathcal{L}_u$: measures how loss function reacts to small changes to u at time t $= g_u(u(t), p)$

Inner product $\langle u, v \rangle = \int_{t_0}^{T} u(t)^T v(t)\, dt$

We want to derive

$$\frac{d\mathcal{L}}{dp} = \underbrace{\mathcal{L}_p}_{\int_{t_0}^{T} g_p(u,p)\,dt} - \mathcal{L}_u(F_u^{-1} F_p) = \mathcal{L}_p - \underbrace{(\mathcal{L}_u F_u^{-1})}_{\lambda} F_p$$

$$\nabla_p \mathcal{L} = \mathcal{L}_p^T - F_p^T \underbrace{F_u^{-T} \mathcal{L}_u^T}_{\lambda(t)}$$

(3)

Easy one

$F_p^T \lambda = \cancel{\langle F_p, \lambda \rangle}$ $\langle F_p, \lambda \rangle = \int \lambda^T(t) f_p \, dt$

What are F_u, F_u^T, F_u^{-T}? $F_u^{-T} L_u$

$F_u[du] = (du)' - f_u \, du$

$F_u : \quad v(t) \to \underbrace{v'(t)}_{\text{Function of } \mathbb{R}^n} - \cancel{v(t)^T f_u(t)}$

$- \underbrace{f_u(t)}_{\substack{n \times n \\ \text{matrix} \\ \text{at time} \\ t}} \underbrace{v(t)}_{\substack{n \\ \text{vector} \\ \text{at} \\ \text{time} \\ t}}$

$F_u = \frac{d}{dt} - \underbrace{f_u}_{n \times n \text{ at time } t}$

$F_u^T = -\frac{d}{dt} - f_u^T$

F_u^T is therefore $-\frac{d}{dt} - f_u^T$

Solve $F_u^T \lambda = L_u^T$

Means $-\frac{d\lambda}{dt} - f_u^T(t) \lambda(t) = g_u^T(t)$

~~Analysis of $\left(\frac{d}{dt}\right)^T = -\frac{d}{dt}$ requires $\lambda(T) = 0$~~