Mathematical Appendix for

Competitive Strategy for Open Source Software

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Appendix A: Proofs

Denote the uniform *pdf* and *cdf* of $\theta \sim U[0, M]$ by $g(\theta) = \frac{1}{M}$ and $G(\theta) = \frac{\theta}{M}$.

Lemma 1. *In a vertically differentiated duopoly with quality levels q_1 and q_2, with $q_1 > q_2$, the optimal prices are set at: $p_1(q_1, q_2) = \frac{2Mq_1(q_1-q_2)}{4q_1-q_2}$ and $p_2(q_1, q_2) = \frac{Mq_2(q_1-q_2)}{4q_1-q_2}$. The revenues of the firms are $R_1(q_1, q_2) = \frac{4Mq_1^2(q_1-q_2)}{(4q_1-q_2)^2}$ and $R_2(q_1, q_2) = \frac{Mq_1(q_1-q_2)q_2}{(4q_1-q_2)^2}$. The consumer surplus is $CS(q_1, q_2) = \frac{Mq_1^2(4q_1+5q_2)}{2(4q_1-q_2)^2}$.*

Proof of Lemma 1. The product qualities, prices, and the choices of consumers determine the revenues as follows: $R_1 = p_1(M - \theta_{12})$ and $R_2 = p_2\theta_{20}$. Firms focus only on revenue to set prices, as quality choices are determined (sunk) in the prior stages. The FOCs with respect to price are

$$\left.\frac{\partial R_1}{\partial p_1}\right|_{p_1,p_2} = M - \frac{p_1}{q_1 - q_2} - \frac{p_1 - p_2}{q_1 - q_2} = 0, \left.\frac{\partial R_2}{\partial p_2}\right|_{p_1,p_2} = \frac{p_1 - p_2}{q_1 - q_2} + p_2\left(-\frac{1}{q_2} - \frac{1}{q_1 - q_2}\right) - \frac{p_2}{q_2} = 0$$

Solving these FOCs simultaneously, we obtain

$$p_1(q_1, q_2) = 2Mq_1\left(\frac{q_1 - q_2}{4q_1 - q_2}\right) \text{ and } p_2(q_1, q_2) = \frac{Mq_2(q_1 - q_2)}{4q_1 - q_2}$$

Substituting these prices in the revenue functions, we obtain the expressions in the proposition. The consumer surplus with these quality levels is:

$$CS(q_1, q_2) = \int_{\theta_{20}}^{\theta_{12}} (\theta q_2 - p_2)\, g(\theta)d\theta + \int_{\theta_{12}}^{M} (\theta q_1 - p_1)\, g(\theta)d\theta = \frac{M}{2}\frac{q_1^2\,(4q_1 + 5q_2)}{(4q_1 - q_2)^2}$$

Lemma 2. ***[Quality Decomposition]*** *The optimal level of features and usability to contribute to a quality target q_j for firm j when the first stage produces F_0 features due to developers' signaling actions is as follows:*

$$f_j = \sqrt{\frac{c_s}{w\eta}}q_j^2 - \frac{F_0}{\eta}, \quad s = \sqrt{\frac{w}{c_s\eta}}q_j^2, \quad \text{and } C(q_j) = 2\sqrt{\frac{wc_s}{\eta}}q_j^2 - w\frac{F_0}{\eta} \tag{1}$$

where f_j and s_j are the optimal levels of features and usability, and $C(q)$ is the minimum cost of obtaining quality q.

Proof of Lemma 2. The firm's problem is $(f, s) = \text{argmin}_{f,s}\, wf + c_s s$ subject to the constraint $((F_0 + \eta f)s)^{\frac{1}{4}} = q$ can be transformed to $C_f(f) = wf + c_s\frac{q^4}{F_0+\eta f}$ and its solution is $\arg\min_{f\geq 0} wf + c_s\frac{q^4}{F_0+\eta f}$. The interior solution yields $f(q) = q^2\sqrt{\frac{c_s}{\eta w}} - \frac{F_0}{\eta}$ and $s(q) = q^2\sqrt{\frac{w}{\eta c_s}}$. The necessary condition for a corner solution $f = 0$ to hold is that: $\left.\frac{\partial C_f}{\partial f}\right|_{f=0} > 0 \implies \left. w - \eta c_s\frac{q^4}{F_0+\eta f}\right|_{f=0} > 0$

$\implies wF_0^2 > \eta c_s q^4$; that is, the wage must be above a certain level.[1] Thus, if wages are not excessive, we will always have an interior solution given by the proposition, and we focus on that situation. The overall cost of providing quality q with an interior solution is $C(q) = wf(q) + c_s s(q)$, so $C(q) = 2\sqrt{\frac{wc_s}{\eta}}q^2 - w\frac{F_0}{\eta}$. $\square$

Proof of Proposition 1. We consider the two markets separately because the different strategic interactions imply that similar proof methods will not work in both cases.

Private Features Market. The quality decomposition result from Lemma 2 implies we can reduce the duopoly market competition in the private features market as represented by the following profits (note that F_0 is exogenously fixed for Stage 2):

$$\Pi_1(q_1, q_2) = \frac{4Mq_1^2(q_1 - q_2)}{(4q_1 - q_2)^2} - \left(2\sqrt{\frac{wc_s}{\eta}}q_1^2 - w\frac{F_0}{\eta}\right) \qquad \Pi_2(q_1, q_2) = \frac{Mq_1(q_1 - q_2)q_2}{(4q_1 - q_2)^2} - \left(2\sqrt{\frac{wc_s}{\eta}}q_2^2 - w\frac{F_0}{\eta}\right)$$

The quality best responses for each firm is given by the solution to these FOCs:

$$\frac{\partial \Pi_1}{\partial q_1} = \frac{2Mq_1\left(8q_1^2 - 6q_2q_1 + 4q_2^2\right)}{(4q_1 - q_2)^3} - 2cq_1 = 0 \text{ and } \frac{\partial \Pi_2}{\partial q_2} = \frac{Mq_1^2\left(4q_1 - 7q_2\right)}{(4q_1 - q_2)^3} - 2cq_2 = 0$$

where $c = 2\sqrt{\frac{wc_s}{\eta}}$. Solving these FOCs focusing on positive real solutions, we obtain $q_1 = \frac{M\phi_1}{2}\sqrt{\frac{\eta}{wc_s}}$ and $q_2 = \frac{M\phi_2}{2}\sqrt{\frac{\eta}{wc_s}}$, where ϕ_1 and ϕ_2 are constants.[2] Using these and the optimal quality decomposition results from Lemma 2, we obtain the stated results. The constants for the shared features market are:

$$\pi_j^Q = \frac{\phi_j}{2}, \pi_j^F = \pi_j^S = \frac{\phi_j^2}{4}$$

We next show that no symmetric equilibria of the product design subgame exist. Consider any potential symmetric equilibrium characterized by the equilibrium features and usability outcomes $f_1 = f_2 = f^*$ and $s_1 = s_2 = s^*$, implying $q_1^* = q_2^*$. With equal quality levels, the firms will charge equal prices (Lemma 1), and obtain half the market. If the firms charge different prices (say $p_1 > p_2$), all consumers will prefer firm 2's product since the qualities are equal. Further, we demonstrate that both firms charge zero prices. If either firm charges $p' > 0$, its competitor can obtain the entire market by offering a price of $p'' = p' - \epsilon$, where $\epsilon > 0$ is a small deviation. Recall

[1] This condition at the equilibrium quality level for firm 2, that is $q_2 = \phi_2 M\sqrt{\frac{\eta}{c_s w}}$ and reduces to $wF_0^2 > \frac{\phi_2^4\eta^3M^4}{c_s}$. Observe that the LHS is increasing in w.

[2] The constants are the positive real solutions to the polynomials below:

$$(\phi_1) \quad -128 + 1168x - 31111x^2 + 235824x^3 = 0$$
$$(\phi_2) \quad -16 + 944x - 13057x^2 + 58956x^3 = 0$$

that sunk product development costs do not affect pricing in Stage 3. Therefore, firms must set zero prices and earn zero revenue and have positive costs in any symmetric pricing equilibrium. We prove by contradiction that a symmetric product development equilibrium cannot exist. Suppose we have a symmetric equilibrium with (f^*, s^*) being the equilibrium strategies of both firms. Consider a deviation by firm 2, setting $s_2 = s' = s^* - \delta$, where $\delta > 0$ is a small deviation. Firm 2 can obtain higher revenues by part (i), and have lower development costs, thereby increasing profits beyond the symmetric equilibrium outcome. Thus, no symmetric equilibrium exists.

Shared Features Market. For the shared features market, recall that the timing is different than the private features market. The firms in Stage 2_a^S determine the features in the shared market and the decisions concerning usability levels in which to invest as well as the decision concerning what fraction of features to copy from the competing firm are made in Stage 2_b^S, which follows Stage 2_a^S. We begin with the analysis of stage 3 and 4, which are identical in both markets, and the pricing is derived as a function of the firms' quality levels. Next, the backwards induction process begins with Stage 2_b^S, where the firms know each other's features levels, and decide simultaneously on both usability (s_1, s_2) and the degree of copying $(\delta_1, \delta_2 \in [0,1])$. We detail the profit functions for the subgame beginning at Stage 2_b^S to be:

$$\Pi_1^{(2_b^S)} = -s_1 c_s + \frac{4\sqrt{F_1} M \sqrt{s_1}\left(\sqrt[4]{F_1}\sqrt[4]{s_1} - \sqrt[4]{F_2}\sqrt[4]{s_2}\right)}{\left(\sqrt[4]{F_2}\sqrt[4]{s_2} - 4\sqrt[4]{F_1}\sqrt[4]{s_1}\right)^2} \quad \Pi_2^{(2_b^S)} = -s_2 c_s + \frac{\sqrt[4]{F_1}\sqrt[4]{F_2} M \sqrt[4]{s_1}\sqrt[4]{s_2}\left(\sqrt[4]{F_1}\sqrt[4]{s_1} - \sqrt[4]{F_2}\sqrt[4]{s_2}\right)}{\left(\sqrt[4]{F_2}\sqrt[4]{s_2} - 4\sqrt[4]{F_1}\sqrt[4]{s_1}\right)^2}$$

Observe that at Stage 2_b^S, the features levels contributed by both firms f_1 and f_2 are fixed and exogenous, but the level of features in the firm's products denoted by F_j are not fixed since they depend on the copying strategies, i.e. $F_j = (F_0 + \eta f_j + \eta\delta_j f_{-j})$. We take the derivative w.r.t. δ_1 to find $\frac{\partial \Pi_1^{(2_b^S)}}{\partial \delta_1} =$

$$\frac{f_2 M\sqrt{s_1}\eta\left(4\sqrt{s_1}\sqrt{\eta(f_2\delta_1+f_1)+F_0} - 3\sqrt[4]{s_1}\sqrt[4]{s_2}\sqrt[4]{\eta(f_1\delta_2+f_2)+F_0}\sqrt[4]{\eta(f_2\delta_1+f_1)+F_0} + 2\sqrt{s_2}\sqrt{\eta(f_1\delta_2+f_2)+F_0}\right)}{\sqrt{\eta(f_2\delta_1+f_1)+F_0}\left(4\sqrt[4]{s_1}\sqrt[4]{\eta(f_2\delta_1+f_1)+F_0} - \sqrt[4]{s_2}\sqrt[4]{\eta(f_1\delta_2+f_2)+F_0}\right)^3}$$

The denominator is clearly positive, and we demonstrate that the numerator is also positive. The numerator can be rewritten as: $f_2 M\sqrt{s_1}\eta\left(4q_1^2 - 3q_1q_2 + 2q_2^2\right) = f_2 M\sqrt{s_1}\eta\left[(2q_1 - q_2)^2 + q_1q_2\right]$, which is always positive since $q_1, q_2 > 0$. Note this inequality is true no matter what the values of δ_1 and δ_2 are. We therefore find that $\frac{\partial \Pi_1^{(2_b^S)}}{\partial \delta_1} > 0$ at all values of the quality levels, implying there is no interior solution where $\delta_1 < 1$. Therefore, the corner solution $\delta_1 = 1$ is the best response of firm 1 no matter what firm 2 does.

Consider the FOC for firm 2 with respect to δ_2:

$$\frac{\partial \Pi_2^{(2_b^S)}}{\partial \delta_2} = \frac{f_1 M \sqrt{s_1}\sqrt[4]{s_2}\eta\sqrt{F_0+\eta(f_1+f_2\delta_1)}\left(4\sqrt[4]{s_1}\sqrt[4]{F_0+\eta(f_1+f_2\delta_1)}-7\sqrt[4]{s_2}\sqrt[4]{F_0+\eta(f_1\delta_2+f_2)}\right)}{4(F_0+\eta(f_1\delta_2+f_2))^{3/4}\left(4\sqrt[4]{s_1}\sqrt[4]{F_0+\eta(f_1+f_2\delta_1)}-\sqrt[4]{s_2}\sqrt[4]{F_0+\eta(f_1\delta_2+f_2)}\right)^3}$$

The denominator is clearly positive and the sign of the numerator $\left(4\sqrt[4]{s_1}\sqrt[4]{F_0+\eta(f_1+f_2\delta_1)}-7\sqrt[4]{s_2}\sqrt[4]{F_0+\eta(f_1\delta_2+f_2)}\right)$ is to be determined. We have already proved that the dominant strategy for firm 1 is $\delta_1^*=1$.

Consider the candidate for (partial) equilibrium specified by :$(\delta_1=1,\delta_2=1,s_1^e,s_2^e)$, where s_1^e and s_2^e are given by the solution to the FOCs $\frac{\partial \Pi_1^{(2_b^S)}}{\partial s_1}=0$ and $\frac{\partial \Pi_1^{(2_b^S)}}{\partial s_2}=0$. At the candidate for equilibrium, we note that the sign of numerator of $\left.\frac{\partial \Pi_1^{(2_b^S)}}{\partial \delta_2}\right|_{\delta_1=1,\delta_2=1}$ is determined by $\left(4\sqrt[4]{s_1}\sqrt[4]{F_0+\eta(f_1+f_2\delta_1)}-7\sqrt[4]{s_2}\sqrt[4]{F_0+\eta(f_1\delta_2+f_2)}\right)>0$, which we find to be positive. At this candidate equilibrium, we need to check for deviations to $\delta_2<1$. We find that for any deviation by firm 2 to $\tilde{\delta}_2<1$, the numerator is still positive, since

$$\left(4\sqrt[4]{s_1}\sqrt[4]{F_0+\eta(f_1+f_2)}-7\sqrt[4]{s_2}\sqrt[4]{F_0+\eta(f_1\tilde{\delta}_2+f_2)}\right)>\left(4\sqrt[4]{s_1}\sqrt[4]{F_0+\eta(f_1+f_2)}-7\sqrt[4]{s_2}\sqrt[4]{F_0+\eta(f_1+f_2)}\right)>0$$

Note that at any value of $\delta_2<1$, this will still hold true since the above expression from the numerator increases as δ_2 becomes smaller. Thus, at any interior $\tilde{\delta}_2<1$, we have $\frac{\partial R_2}{\partial \delta_2}>0$, implying that only a corner solution $(\delta_1^*=1,\delta_2^*=1)$ can be an equilibrium.

Another way to prove $\delta_2=1$ is by contradiction. Consider firm 1's best response $(\bar{s}_1,\delta_1=1)$ to firm 2's strategy $(\bar{s}_2,\bar{\delta}_2)$, with $\bar{\delta}_2<1$. We can rewrite firm 1's FOC as:

$$\frac{\partial \Pi_1^{(2_b^S)}}{\partial \delta_1}=\frac{f_2 M\sqrt{s_1}\eta\left(4\sqrt{s_1}\sqrt{\eta\left(f_2\delta_1+f_1\right)+F_0}-3\sqrt[4]{s_1}q_2\sqrt[4]{\eta\left(f_2\delta_1+f_1\right)+F_0}+2q_2^2\right)}{\sqrt{\eta\left(f_2\delta_1+f_1\right)+F_0}\left(4\sqrt[4]{s_1}\sqrt[4]{\eta\left(f_2\delta_1+f_1\right)+F_0}-q_2\right)^3}$$

Observe that f_2 does appear in the FOC, but has already been determined in the previous stage. The key idea here is that s_2 and δ_2 do not appear separately from q_2 in this first-order condition. So, for a specified quality q_2, firm 1's best response is invariant to exactly how the quality is specifically achieved by usability and copying strategies (s_2,δ_2).[3] We prove now that $(\bar{s}_2,\bar{\delta}_2)$ cannot be an equilibrium strategy when $\bar{\delta}_2<1$. Firm 2 can use an altered strategy to improve profitability, that is, copying the full amount and simultaneously reducing its usability by ϵ so its overall quality level is maintained, denoted as $(\bar{s}_2-\epsilon,\delta_2=1)$, where

$$\left[\bar{s}_2\left(F_0+\eta\left(\bar{\delta}_2 f_1+f_2\right)\right)\right]^{\frac{1}{4}}=\left[\left(\bar{s}_2-\epsilon\right)\left(F_0+\eta\left(f_1+f_2\right)\right)\right]^{\frac{1}{4}},\text{ or }\epsilon=\bar{s}_2\left[1-\frac{F_0+\eta\left(\bar{\delta}_2 f_1+f_2\right)}{F_0+\eta\left(f_1+f_2\right)}\right]$$

Note that for any strategy without full copying $(\bar{s}_2,\bar{\delta}_2)$, firm 2 can deviate to this altered strategy of

[3] The same reasoning can easily be shown to be true for firm 1's FOC with respect to s_1, $\frac{\partial \Pi_1^{(2_a^S)}}{\partial s_1}=0$.

$\left(\bar{s}_2\left[\frac{F_0+\eta\left(\bar{\delta}_2 f_1+f_2\right)}{F_0+\eta(f_1+f_2)}\right],1\right)$ and maintain the same quality level for firm 2, and firm 1 will not change its best response as a result of the altered strategy. However, it reduces costs for firm 2 by $c_s\epsilon$ and is therefore more profitable. This finding holds only because firm 1's best response is only to the overall level of firm 2's quality level and its (predetermined) features. This change in firm 2's decisions will not alter firm 1's best response. Thus this reasoning implies $\delta_2 = 1$ must hold in equilibrium since $\delta_j \in [0,1]$. Observe that when $\delta_1 = \delta_2 = 1$, $F_1 = F_2 = (F_0 + \eta f_1 + \eta f_2)$.

Having proved the equilibrium copying strategy, we determine the usability levels by determining the following FOCs, substituting in $\delta_1 = 1, \delta_2 = 1$ to obtain:

$$\frac{\partial \Pi_1^{\left(2_b^S\right)}}{\partial s_1} = 0 = \frac{\sqrt[4]{F_0+\eta\left(f_1+f_2\right)}M\left(\left(2\sqrt[4]{s_2}-3\sqrt[4]{s_1}\right)\sqrt[4]{s_2}+4\sqrt{s_1}\right)}{\sqrt{s_1}\left(4\sqrt[4]{s_1}-\sqrt[4]{s_2}\right)^3} - c_s \qquad \frac{\partial \Pi_2^{\left(2_b^S\right)}}{\partial s_2} = 0 = \frac{M\sqrt[4]{F_0+\eta(f_1+f_2)}\sqrt{s_1}\left(4\sqrt[4]{s_1}-7\sqrt[4]{s_2}\right)}{4\left(4\sqrt[4]{s_1}-\sqrt[4]{s_2}\right)^3 s_2^{3/4}} - c_s$$

Solving the above equation, we obtain the usability levels as a function of the total number of features $F = F_0 + \eta f_1 + \eta f_2$ as [4] :

$$s_1(F) = \sqrt[3]{F}\sigma_1^4\left(\frac{M}{c_s}\right)^{4/3} \quad \text{and} \quad s_2(F) = \sqrt[3]{F}\sigma_2^4\left(\frac{M}{c_s}\right)^{4/3} \tag{2}$$

Next, we examine stage 2_a^S, where firms invest in features, anticipating further usability investments and copying of features to take place in Stage 2_b^S. We replace the usability levels in the profit function by the above functions $s_1(F)$ and $s_2(F)$ to obtain each firm's profit level as a function only of the features levels:

$$\Pi_1^{\left(2_a^S\right)} = -\frac{M^{4/3}\left(4\sigma_2+\sigma_1\left(\sigma_1\left(\sigma_2-4\sigma_1\right)^2-4\right)\right)\sigma_1^2\sqrt[3]{\frac{F_0+\eta(f_1+f_2)}{c_s}}}{\left(\sigma_2-4\sigma_1\right)^2} - wf_1$$
$$\Pi_2^{\left(2_a^S\right)} = M^{4/3}\sqrt[3]{\frac{F_0+\eta(f_1+f_2)}{c_s}}\left(\frac{\sigma_1\left(\sigma_1-\sigma_2\right)\sigma_2}{\left(\sigma_2-4\sigma_1\right)^2}+\sigma_2^4\right) - wf_2$$

Note that this profit function $\Pi_j^{\left(2_a^S\right)}$ denotes the profit of firm j conditional only on the features, assuming the subgame perfect equilibrium holds for the subgame beginning at Stage 2_b^S.

We take the FOCs of the profit function of firm j with respect to f_j. An interior solution would lead us to the conditions $\frac{\partial \Pi_j^{\left(2_a^S\right)}}{\partial f_j} = 0$ for each firm $j = 1, 2$; however, we find the difference between

[4] The constants are defined as the positive real roots of the polynomial equations:

$$\begin{aligned}(\sigma_1):&7051514643711391827138205384474361856x^{15} - 3361057834920770359509566464597 45x^{12}\\&+ 5537059002451953297015516800x^9 - 3337752309009754911 1296x^6\\&+ 58148577673216x^3 - 268435456 = 0\\(\sigma_2):&203695116x^5 - 28488008x^4 + 1778324x^3 - 50117x^2 + 593x - 2 = 0\end{aligned}$$

the FOCs of the two firms is

$$\frac{\partial \Pi_1^{\left(2_a^S\right)}}{\partial f_1} - \frac{\partial \Pi_2^{\left(2_a^S\right)}}{\partial f_2} = \frac{M^{4/3}\eta\left(-4\sigma_1^5 + \sigma_2\sigma_1^4 + \sigma_1^2 + \sigma_2\left(4\sigma_2^3 - 1\right)\sigma_1 - \sigma_2^5\right)}{3\sqrt[3]{\mathrm{cs}}\left(4\sigma_1 - \sigma_2\right)\left(F_0 + \eta(f_1 + f_2)\right)^{2/3}} > 0$$

This inequality implies that both FOCs $\frac{\partial \Pi_1^{\left(2_a^S\right)}}{\partial f_1} = 0$ and $\frac{\partial \Pi_2^{\left(2_a^S\right)}}{\partial f_2} = 0$ cannot be simultaneously satisfied, and we cannot have an interior solution for both firms.

Thus either $\frac{\partial \Pi_1^{\left(2_a^S\right)}}{\partial f_1} > 0$ or $\frac{\partial \Pi_2^{\left(2_a^S\right)}}{\partial f_2} < 0$ or both must hold. If $\frac{\partial \Pi_1^{\left(2_a^S\right)}}{\partial f_1} > 0$, firm 1 will find hiring more developers to be optimal and will add features. Since firm 1 can improve its profits when $\frac{\partial \Pi_1^{\left(2_a^S\right)}}{\partial f_1} > 0$, we cannot have an equilibrium. Therefore, $\frac{\partial \Pi_1^{\left(2_a^S\right)}}{\partial f_1} = 0$ in equilibrium and $f_1^* > 0$. On the other hand, $\frac{\partial \Pi_2^{\left(2_a^S\right)}}{\partial f_2} < 0$ implies firm 2 would find hiring fewer developers to be optimal, and the inequality is consistent with $f_2^* = 0$, where it would still apply. Note that this result does not depend on specific values of w or any other parameters, and thus holds generally.

Setting $f_2^* = 0$ and using the FOCs for the others, after much algebraic manipulation, we obtain,

$$f_1^* = \psi_1^F M^2\sqrt{\frac{\eta}{c_s w^3}} - \frac{F_0}{\eta}, \text{ where } \psi_1^F = \frac{\sigma_1^3\sqrt{3}\left(\sigma_1\left[4 - \sigma_1\left(4\sigma_1 - \sigma_2\right)^2\right] - 4\sigma_2\right)^{\frac{3}{2}}}{9\left(4\sigma_1 - \sigma_2\right)^3} \tag{3}$$

The other results in the proposition follow from substituting this value of f_1^* into the expressions of s_1, and so on. The constants are defined as follows:

$$\psi_1^S = \frac{\sigma_1^5\sqrt{\left(\sigma_1\left[4 - \sigma_1\left(4\sigma_1 - \sigma_2\right)^2\right] - 4\sigma_2\right)}}{\sqrt{3}\left(4\sigma_1 - \sigma_2\right)}, \psi_2^S = \frac{\sigma_1\sigma_2^4\sqrt{\left(\sigma_1\left[4 - \sigma_1\left(4\sigma_1 - \sigma_2\right)^2\right] - 4\sigma_2\right)}}{\sqrt{3}\left(4\sigma_1 - \sigma_2\right)}$$
$$\psi_1^Q = \frac{\sigma_1^2\sqrt{\left(\sigma_1\left[4 - \sigma_1\left(4\sigma_1 - \sigma_2\right)^2\right] - 4\sigma_2\right)}}{\sqrt{3}\left(4\sigma_1 - \sigma_2\right)}, \psi_2^Q = \frac{\sigma_1\sigma_2\sqrt{\left(\sigma_1\left[4 - \sigma_1\left(4\sigma_1 - \sigma_2\right)^2\right] - 4\sigma_2\right)}}{\sqrt{3}\left(4\sigma_1 - \sigma_2\right)}$$

□

Proof of Proposition 2. For a separating equilibrium, we need to characterize the conditions on contributions to open source e_L and e_H for each type of developer. The market wage is w for the high-type developer and 0 for the low-type, where w is determined in equilibrium. Consider the binding constraints IR_H and IC_L specified in equation (2). We argued previously that $e_L = 0$, because low-type developers do not receive a positive wage in a separating equilibrium. The IR_H condition then implies $w - c_H e_H \geq r$ for a high-type developer with reservation utility r, imposing the constraint $e_H \leq \frac{w-r}{c_H}$ for this developer to signal. Given a specific e_H and w, only high-type developers with low reservation option, that is, $r < w - c_H e_H$, will choose to enter by signaling. Note that these are necessary conditions.

The least-cost separation is achieved by the high-type contributing just enough to open source that it deters the low-type from masquerading: $e_H^{LCS}(w) = \frac{w}{c_L}$ and the corresponding belief by the firms to support this behavior, that is, $\mu(H|e,w) = \begin{cases} 0, e < \frac{w}{c_L} \\ 1, e > \frac{w}{c_L} \end{cases}$. To prove this LCS contribution is an equilibrium for both types, we consider their possible deviations in turn. Suppose the high-type deviates from the least-cost contribution to $e' = e_H^{LCS}(w) + \epsilon$, where $\epsilon > 0$. The best possible belief for any type at this level e' is $\mu(H|e',w) = 1$. For the high type, this deviation to e' is not profitable since $w - c_H e' < w - c_H e_H^{LCS}(w) = w\left(1 - \frac{c_H}{c_L}\right)$. Therefore, no such deviation contributing beyond $e^{LCS}(w)$ is profitable, and any deviation below $e_H^{LCS}(w)$ will result in the firm believing the developer is low-type. These factors imply $e_H^{LCS}(w)$ is an equilibrium strategy for the high-types. For the low-type developer, for any $e > e_H^{LCS}$, we have $w - c_L e < 0$, so it is unprofitable even when the firms believe the developer is high-type. Therefore, the low-type will not deviate from $e_L^{LCS} = 0$.

We apply the intuitive criterion to eliminate non-LCS equilibria: suppose another equilibrium exists where the high type contributes e'. This equilibrium requires that everyone's beliefs on the equilibrium path are

$$\mu(H|e,w) = \begin{cases} 0, e < e' \\ 1, e \geq e' \end{cases}.$$

Consider a deviation $\tilde{e}$ from the equilibrium path where $\frac{w}{c_L} < \tilde{e} < e'$. The best possible belief, $\mu(H|\tilde{e},w) = 1$, is still not sufficient to induce the low-type developers to contribute $\tilde{e}$ since $w - c_L\tilde{e} < 0$. Therefore, only the high-type developer could have deviated to $\tilde{e}$, and the intuitive criterion requires the firms to assign beliefs $\mu(H|e',w) = 1$ after observing $\tilde{e}$. This leads to an inconsistent off-equilibrium-path belief, and we can therefore eliminates this equilibrium. We can apply this criterion to filter any equilibrium with the high-type contributing $e' > e_H^{LCS}(w)$, and the only remaining equilibrium is the least-cost separating equilibrium.

In a least-cost equilibrium, the high-type developers who have low reservation utilities signal by contributing, that is, $r < \tilde{r}^{LCS} = w - c_H e_H^{LCS}(w) = w\left(1 - \frac{c_H}{c_L}\right)$ so the number of contributing developers is $\Psi(\tilde{r}^{LCS}) = \Psi\left(w\left[1 - \frac{c_H}{c_L}\right]\right)$. □

Proof of Proposition 3. For part (i), we compare the excess demand functions (i.e., demand at wage w - supply at wage w) in the shared features and private features markets ξ^S and ξ^P:

$$\begin{aligned} \xi^S(w) &= D(w) + f_1^S(w) + f_2^S(w) - f_0(w) \\ \xi^S(w) &= D(w) + \frac{\sigma_1^3\sqrt{3}\left(\sigma_1\left[4 - \sigma_1\left(4\sigma_1 - \sigma_2\right)^2\right] - 4\sigma_2\right)^{\frac{3}{2}}}{9\left(4\sigma_1 - \sigma_2\right)^3} M^2\sqrt{\frac{\eta}{c_s}}(w)^{-\frac{3}{2}} - \Psi\left(w\left[1 - \frac{c_H}{c_L}\right]\right)\left[1 + \frac{w}{c_L}\right] \end{aligned}$$

Observe that $\xi^P(w)$ can be rewritten as:

$$\begin{aligned}
\xi^P(w) &= D(w) + f_1^P(w) + f_2^P(w) - \Psi\left(w\left[1-\frac{c_H}{c_L}\right]\right) \\
&= D(w) + \underbrace{\frac{M^2\left(\phi_1^2\right)}{4}\sqrt{\frac{\eta}{c_s}}(w)^{-\frac{3}{2}} - \Psi\left(w\left[1-\frac{c_H}{c_L}\right]\right)\frac{w}{c_L}}_{f_1^P(w)} + f_2^P(w) - \Psi\left(w\left[1-\frac{c_H}{c_L}\right]\right) \\
&= \xi^S(w) + \left(f_2^P(w) - f_2^S(w)\right) + M^2\left(\frac{\phi_1^2}{4} - \frac{\sigma_1^3\sqrt{3}\left(\sigma_1\left[4-\sigma_1\left(4\sigma_1-\sigma_2\right)^2\right]-4\sigma_2\right)^{\frac{3}{2}}}{9\left(4\sigma_1-\sigma_2\right)^3}\right)\sqrt{\frac{\eta}{c_s}}(w)^{-\frac{3}{2}}
\end{aligned}$$

On the RHS, we know $f_2(w) \geq 0$ and find that the constant

$$\left(\frac{\phi_1^2}{4} - \frac{\sigma_1^3\sqrt{3}\left(\sigma_1\left[4-\sigma_1\left(4\sigma_1-\sigma_2\right)^2\right]-4\sigma_2\right)^{\frac{3}{2}}}{9\left(4\sigma_1-\sigma_2\right)^3}\right) > 0.$$

Therefore, $\xi^P(w) > \xi^S(w)$.

Next, observe that $\xi^P(w^S) > \xi^S(w^S) = \xi^P(w^P) = 0$. We know that the excess demand for developers ξ^P is a decreasing function of wage (since each of its individual components are decreasing functions of wage), we find that $w^P > w^S$. The functional form of the external demand $D(w)$ does not affect this result.

For part (ii), consider the private features market: we apply the implicit function theorem to the excess demand equation $\xi^P(w) = 0$, to examine the effect of a change in c_H on w, and find that $\frac{dw^P}{dc_H} = -\frac{\partial\xi^P}{\partial c_H}\Big/\frac{\partial\xi^P}{\partial w}\Big|_{w^P} =$

$$\left.\frac{\left(-\frac{w}{c_L}\right)\psi\left(w\left[1-\frac{c_H}{c_L}\right]\right)}{D'(w) + \left(\frac{-3}{2}w^{\frac{-5}{2}}\frac{M^2\left(\phi_1^2+\phi_2^2\right)}{4}\sqrt{\frac{\eta}{c_s}}\right) + \left(-\left[1-\frac{c_H}{c_L}\right]\psi\left(w\left[1-\frac{c_H}{c_L}\right]\right)\left(1+\frac{2w}{c_L}\right)\right) + \left(-\frac{2}{c_L}\Psi\left(w\left[1-\frac{c_H}{c_L}\right]\right)\right)}\right|_{w=w^P}$$

Observe that the numerator is always positive, whereas each of the terms in the denominator is negative. Therefore, the overall effect is positive. We find the comparative statics of wages with respect to c_s, η, and M for both the private features and shared features markets in a similar manner. $\square$

Proof of Proposition 4. The contribution to open source in the private features market is $F^P = F_0 = \Psi\left(w^P\left[1-\frac{c_H}{c_L}\right]\right)\frac{w^P}{c_L}$, whereas in the shared features market, contributions made by the firms and due to developers signaling is $F^S = F_0 + \eta f_1 + \eta f_2 = M^2\frac{\sigma_1^3\sqrt{3}\left(\sigma_1\left[4-\sigma_1(4\sigma_1-\sigma_2)^2\right]-4\sigma_2\right)^{\frac{3}{2}}}{9(4\sigma_1-\sigma_2)^3}\sqrt{\frac{\eta^3}{\left(w^S\right)^3 c_s}}$. We obtain $F^P > F^S \iff \Psi\left(w^P\left[1-\frac{c_H}{c_L}\right]\right)\frac{w^P}{c_L} > \frac{4}{\phi_1^2}\frac{\sigma_1^3\sqrt{3}\left(\sigma_1\left[4-\sigma_1(4\sigma_1-\sigma_2)^2\right]-4\sigma_2\right)^{\frac{3}{2}}}{9(4\sigma_1-\sigma_2)^3}\frac{1}{\left(w^S\right)^{\frac{3}{2}}}$, which holds when w^S or w^P are high, which in turn occurs when M is large, η is high, c_H is large or c_s is low (from Proposition 3). $\square$

Proof of Proposition 5. For part (i), we obtain the quality levels from Proposition 1 to find: $\frac{q_2^P(w^P)}{q_2^S(w^S)} = \left(\frac{\phi_2}{2} \Big/ \frac{\sigma_1\sigma_2\sqrt{\left(\sigma_1\left[4-\sigma_1(4\sigma_1-\sigma_2)^2\right]-4\sigma_2\right)}}{\sqrt{3}(4\sigma_1-\sigma_2)}\right)\sqrt{\frac{w^S}{w^P}} < \sqrt{\frac{w^S}{w^P}}$. We know the final fraction is less than 1 since $w^S < w^P$ from Proposition 3. Comparing the quality levels for the high-quality product in both the shared features and private features markets, we find $\frac{q_1^P(w^P)}{q_1^S(w^S)} = \left(\frac{\phi_1}{2} \Big/ \frac{\sigma_1^2\sqrt{\left(\sigma_1\left[4-\sigma_1(4\sigma_1-\sigma_2)^2\right]-4\sigma_2\right)}}{\sqrt{3}(4\sigma_1-\sigma_2)}\right)\sqrt{\frac{w^S}{w^P}}$. For $q_1^P > q_1^S$, we must have $w^P < w^S\left(\frac{\phi_1}{2} \Big/ \frac{\sigma_1^2\sqrt{\left(\sigma_1\left[4-\sigma_1(4\sigma_1-\sigma_2)^2\right]-4\sigma_2\right)}}{\sqrt{3}(4\sigma_1-\sigma_2)}\right)^2$. This condition can only hold for small market sizes and low signaling costs and proves part (ii). $\frac{q_1^P}{q_2^P} = \frac{\phi_1}{\phi_2} > \frac{q_1^S}{q_2^S} = \frac{\sigma_1}{\sigma_2}$ is immediate from the constants and parameters determining the quality levels. $\square$

Proof of Proposition 6. For (i), the profit of the high-quality firm under different markets is

$$\Pi_1^S = \Psi\left(w^S\left[1-\frac{c_H}{c_L}\right]\right)\frac{(w^S)^2}{\eta c_L}+\gamma_1^S M^2\sqrt{\frac{\eta}{w^S c_s}}, \quad \Pi_1^P = \Psi\left(w^P\left[1-\frac{c_H}{c_L}\right]\right)\frac{(w^P)^2}{\eta c_L}+\gamma_1^P M^2\sqrt{\frac{\eta}{w^P c_s}}$$

where

$$\gamma_1^S = \frac{\sigma_1^3\sqrt{\left(\sigma_1\left[4-\sigma_1\left(4\sigma_1-\sigma_2\right)^2\right]-4\sigma_2\right)}}{3\sqrt{3}\left(4\sigma_1-\sigma_2\right)^3}\left(\left(12\left(\sigma_1-\sigma_2\right)-3\sigma_1^2\left(4\sigma_1-\sigma_2\right)^2\right)-\left(\sigma_1\left[4-\sigma_1\left(4\sigma_1-\sigma_2\right)^2\right]-4\sigma_2\right)\right)$$

and $\gamma_1^P = \frac{\phi_1^2\left(4(2\phi_2\phi_1+\phi_1)-16\phi_1^2-\phi_2(\phi_2+4)\right)}{2(\phi_2-4\phi_1)^2}$.

When $w^S = w^P$, we find that $\Pi_1^P > \Pi_1^S$ since $\gamma_1^S < \gamma_1^P$. Note that profits are decreasing in wage, i.e. $\frac{\partial \Pi_1}{\partial w} < 0$, and we know that $w^P > w^S$ from Proposition 3. When the wages diverge significantly so that $w^P \gg w^S$, e.g. because of high M, then we find that Π_1^P can be lower than Π_1^S.

For (ii): From Lemma 1, we can rewrite the consumer surplus expression as: $CS(q_1,q_2) = q_1\left[\left(\frac{q_1}{q_2}\right)\frac{4\frac{q_1}{q_2}+5}{\left(4\frac{q_1}{q_2}-1\right)^2}\right]$. Observe that the term in square brackets is constant; that is, it only depends on the quality ratio, which is independent of the wage and other model primitives but depends on the market structure, namely shared features or private features. We know that when the wages are identical $CS^S(w) > CS^P(w)$ and the surplus increases with the quality level, which decreases with wage. Since $w^S < w^P$, the surplus inequality will also hold at the equilibrium wage. $\square$