Homework 4

Vineet Rai

Due date: Tuesday, March 5

1. (2 points) Find a constant k so that the following function is a valid probability mass function:

$$f(x) = k\left(\frac{1}{2}\right)^{|x|} \quad \text{for } x \in \mathbb{Z},$$

where $\mathbb{Z}$ denotes all integer. In this case, the random variable, X, is discrete and $X = \ldots, -2, -1, 0, 1, 2, \ldots$.

A function $f(x)$ can serve as the probability distribution for a discrete random variable X if it satisfies these two conditions: 1) $f(x) \geq 0$ for all domain values; 2) $\sum_x f(x) = 1$, where the summation extends over all domain values. The constant k must be chosen such that both conditions are met.

First we can rewrite $f(x)$ as the following 3-part, piecewise-defined discrete function

$$f(x) = \begin{cases} k\left(\frac{1}{2}\right)^{-x} & x = -1, -2, -3\ldots \\ k & x = 0 \\ k\left(\frac{1}{2}\right)^{x} & x = 1, 2, 3\ldots \end{cases}$$

At this point we can verify that $f(x)$ will always be positive so long as k is positive. Now we can solve for k by setting $\sum_x f(x) = 1$ and equating this to finding 3 individual sums for the 3-part function.

$$\sum_{x=-\infty}^{\infty} f(x) = 1 = \sum_{x=-\infty}^{-1} k\left(\frac{1}{2}\right)^{-x} + k + \sum_{x=1}^{\infty} k\left(\frac{1}{2}\right)^{x}$$

Due to symmetry about the point $x = 0, f(-1) = f(1), f(-2) = f(2), \ldots$ making the two sums equivalent. The constant k can be pulled out of the sum, simplifying to

$$1 = k + 2k\sum_{x=1}^{\infty}\left(\frac{1}{2}\right)^{x}$$

The infinite sum is similar to a geometric series of the form $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$. By rewriting it slightly, we have

$$1 = k + 2k\sum_{x=1}^{\infty}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{x-1}$$

Using $x = n$, $a = 0.5$, and $r = 0.5$ we have $\frac{a}{1-r} = \frac{0.5}{1-0.5} = 1$. So the infinite series will sum to 1.

$$1 = k + 2k(1)$$

$$\mathbf{k = \frac{1}{3}}$$

2. (2 points) Find a constant k so that the following function is a valid probability denity function:

$$f(x) = ke^{-|x|} \text{ for } x \in \mathbb{R},$$

where $\mathbb{R}$ denotes all real number. In this case, the random variable, X, is continuous and $X \in (-\infty, \infty)$.

A function $f(x)$ can serve as the probability density of a continuous random variable X if it satisfies the following two conditions: 1) $f(x) > 0$ for $-\infty < x < \infty$; 2) $\int_{-\infty}^{\infty} f(x)dx = 1$. The constant k must be chosen such that that both conditions are met.

Similarly to the last problem, we can rewrite $f(x)$ as the following 2-part, piecewise-defined continuous function

$$f(x) = \begin{cases} ke^{x} & x < 0 \\ ke^{-x} & x \geq 0 \end{cases}$$

At this point we can verify that $f(x)$ will always be positive so long as k is positive. Now we can solve for k by setting $\int_{-\infty}^{\infty} f(x)dx = 1$ and equating this to solving 2 indefinite integrals for the 2-part function.

$$\int_{-\infty}^{\infty} f(x)dx = 1 = \int_{-\infty}^{0} ke^{x}dx + \int_{0}^{\infty} ke^{-x}dx$$

These two indefinite integrals are equivalent and symmetric about $x = 0$, because in either case any x value will result in e being raised to the same negative power. The constant k can be pulled out of the integral, which simplifies to

$$1 = 2k\int_{0}^{\infty} e^{-x}dx$$

$$1 = -2ke^{-x}|_{0}^{\infty}$$

$$1 = -2k(0 - 1)$$

$$\mathbf{k = \frac{1}{2}}$$

3. (2 points) **Exercise 3.6:** Show that there are no values of c such that

$$f(x) = \frac{c}{x}$$

can serve as the values of the probability distribution of a random variable with the countably infinite range $x = 1, 2, 3, \ldots$.

The function $f(x)$ can serve as the probability distribution for a discrete random variable if it satisfies these two conditions: 1) $f(x) \geq 0$ for all domain values; 2) $\sum_x f(x) = 1$, where the summation extends over all domain values. If we choose any $c \geq 0$ then the first condition is satisfied. However, the sum of the function over its domain can never equal 1.

$$\sum_{x=1}^{\infty} f(x) = c \sum_{x=1}^{\infty} \frac{1}{x} = \infty$$

This series is identical to the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is known to diverge to ∞. This can be demonstrated by applying the integral test for convergence of series

$$\int_1^{\infty} \frac{1}{x} dx = ln(x)|_1^{\infty} = \infty$$

Since the integral of the function diverges to ∞, the series must also diverge since its sum is even greater. No constant value of c can prevent the series from diverging.

4. (2 points) Starbucks gives a free beverage to its rewards members on their birthday. A loyal customer woke up from brain trauma forgot his birthday. He decides to purchase a cup of Starbucks coffee every day, and the day he receives his coffee for free would be determined his birthday. Suppose he starts this process today and let X be the number of days from today until he receives his free coffee. The random variable X in this example is discrete can take on possible values of $0, 1, \ldots, \cancel{365}$ 364. Assuming there are 365 days in a year and his birthday is not on Feburary 29th, What is $P(X = \cancel{365}\ 364)$?

Note: as discussed in class, if the first day is X = 0 then the last day is X = 364. The 365th day from the beginning would be the same calendar date of the year as X = 0.

Every day has an equal chance of being the customer's birthday.

$$P(X = 364) = \frac{1}{365} \approx 0.00274$$

5. (2 points) Write down the probability distribution for X in #4.

We have a discrete, uniform probability distribution.

$$f(x) = \frac{1}{365} \text{ for } x = 0, 1, 2, \ldots, 362, 363, 364$$