Homework 10

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Due date: Thursday, April 25

1. (2 points) **Exercise 6.80** To illustrate the law of large numbers, use the normal approximation to the binomial distribution to determine the probabilities that the proportion of heads will be anywhere from 0.49 to 0.51 when a balanced coin is flipped

a. 10 times;

```
> n = 10; theta = 0.5
> mu <- n*theta
> s <- sqrt(n*theta*(1-theta))
> pnorm(0.51*n, mu, s) - pnorm(0.49*n, mu, s)
[1] 0.05042903
```

b. 100 times.

```
> n = 100; theta = 0.5
> mu <- n*theta
> s <- sqrt(n*theta*(1-theta))
> pnorm(0.51*n, mu, s) - pnorm(0.49*n, mu, s)
[1] 0.1585194
```

2. (2 points) Consider #5 from the last homework assignment (Homework 9) where we inscribed a circle in a unit square and use random vectors to approximate the irrational number, π. Suppose we only dropped 100 random points, what is the *exact* probability that at least 80 of these points will fall inside of the circle?

```
> 1-pbinom(79, 100, pi/4)
[1] 0.4162444
```

3. (2 points) Use a normal approximation (with continuity correction) to approximate the exact probability in #2.

```
> n = 100; theta = pi/4
> mu <- n*theta
> s <- sqrt(n*theta*(1-theta))
> 1-pnorm(79.5, mu, s)
[1] 0.4075392
```

4. (2 points) Let x be the number of random points to fall inside of the circle among the $n = 100$ random points considered in #2. Define $\hat{\theta} = x/n$ as the observed proportion that we used to approximate θ, the true proportion of points that fall within the circle with $n \to \infty$. Use a normal approximation (central limit theorem) to find the probability that the $P(|\hat{\theta} - \theta| < 0.01)$

$$\hat{\theta} = \frac{x}{n}, \quad \theta = \frac{\pi}{4}, \quad n = 100$$

$$\begin{aligned} P(|\hat{\theta} - \theta| < 0.01) &= P\left(\left|\hat{\theta} - \frac{\pi}{4}\right| < 0.01\right) \\ &= P\left(-0.01 < \hat{\theta} - \frac{\pi}{4} < 0.01\right) \\ &= P\left(\frac{\pi}{4} - 0.01 < \hat{\theta} < \frac{\pi}{4} + 0.01\right) \end{aligned}$$

```
> n = 100; theta = pi/4
> mu <- n*theta
> s <- sqrt(n*theta*(1-theta))
>
> pnorm(n*(theta + 0.01), mu, s/10) - pnorm(n*(theta - 0.01), mu, s/10)
[1] 0.9851403
```

5. (2 points) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables. Suppose $E(X_i) = \mu$ and $V(X_i) = \sigma^2$ for all $i = 1, 2, \ldots, n$, and μ and σ are constants. What is $Cov(X_i, \bar{X})$?

$$\begin{aligned} Cov(X_i, \bar{X}) &= Cov\left(X_i, \ \frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n} Cov\left(X_i, \ \sum_{i=1}^{n} X_i\right) \\ &= \frac{V(X_i)}{n} \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Bonus question (2 points)

Suppose X follows a beta distribution with parameters $\alpha = 2$ and $\beta = 3$.

$$X \sim Beta(\alpha = 2,\ \beta = 3)$$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1}\,(x-1)^{\beta-1} \quad \text{for} \quad 0 < x < 1 \tag{Eq. 1}$$

$$E(X) = \frac{\alpha}{\alpha+\beta} \tag{Eq. 2}$$

$$P(X = x) = \frac{\Gamma(5)}{\Gamma(2)\,\Gamma(3)}\, x\,(x-1)^2 = \frac{4!}{1!\,2!}\, x\,(x-1)^2 = 12x(1-x)^2 \tag{Eq. 3}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx \tag{Eq. 4}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \tag{Eq. 5}$$

a. Find $E(X^9)$.

Using Eq. 4 for expectation and Eq. 3 for $X \sim Beta(\alpha = 2,\ \beta = 3)$

$$\begin{aligned}
E(X^9) &= \int_0^1 x^9 \cdot 12x(1-x)^2\, dx \\
&= 12 \int_0^1 x \cdot x^9 (1-x)^2\, dx \\
&= 12 \int_0^1 x \cdot x^{10-1} (1-x)^{3-1}\, dx \\
&= 12\, \frac{\Gamma(10)\,\Gamma(3)}{\Gamma(13)} \int_0^1 x \cdot \frac{\Gamma(13)}{\Gamma(10)\,\Gamma(3)}\, x^{10-1}(1-x)^{3-1}\, dx
\end{aligned}$$

Using Eq. 5 for expectation and Eq. 1 for the Beta distribution

$$= 12\, \frac{9!\,2!}{12!}\, E(Y) \quad \text{for} \quad Y \sim Beta(\alpha = 10,\ \beta = 3)$$

Using Eq. 2 for expectation of a Beta random variable

$$\begin{aligned}
&= 12\, \frac{2}{12 \cdot 11 \cdot 10} \cdot \frac{10}{10+3} \\
&= \frac{2}{11 \cdot 13} \\
&= \frac{2}{143} \approx 0.0140
\end{aligned}$$

b. Find $E[(1-x)^9]$.

Using Eq. 4 for expectation and Eq. 3 for $X \sim Beta(\alpha = 2,\ \beta = 3)$

$$
\begin{aligned}
E[(1-X)^9] &= \int_0^1 (1-x)^9 \cdot 12x(1-x)^2 \, dx \\
&= 12 \int_0^1 x \cdot x^0 (1-x)^{11} \, dx \\
&= 12 \, \frac{\Gamma(1)\,\Gamma(12)}{\Gamma(13)} \int_0^1 x \cdot \frac{\Gamma(13)}{\Gamma(1)\,\Gamma(12)} \, x^{1-1}(1-x)^{12-1} \, dx
\end{aligned}
$$

Using Eq. 5 for expectation and Eq. 1 for the Beta distribution

$$
= 12 \, \frac{0!\ 11!}{12!} \, E(Y) \quad \text{for} \quad Y \sim Beta(\alpha = 1,\ \beta = 12)
$$

Using Eq. 2 for expectation of a Beta random variable

$$
\begin{aligned}
&= 12 \, \frac{1}{12} \cdot \frac{1}{1+12} \\
&= \frac{1}{13} \approx 0.0769
\end{aligned}
$$