Homework 8

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Due date: Thursday, April 18

1. (2 points) Suppose X follows a Gamma distribution with parameters α and β. Identify these parameters if $E(X) = Var(X) = 1$.

Theorem 6.2 implies that the expectation, $E(X) = \alpha\beta$ and the variance, $V(X) = \alpha\beta^2$. Setting these two equations equal to 1 allows us to solve for the parameters α and β.

$$1 = E(X) = \alpha\beta$$
$$1 = V(X) = \alpha\beta^2$$

$$\alpha\beta = \alpha\beta^2$$
$$\text{Since } \alpha > 0 \quad \text{and} \quad \beta > 0,$$
$$\alpha = 1, \ \beta = 1$$

2. (2 points) Find the median for the random variable in #1.

Since $\alpha = 1$ the Gamma distribution simplifies into an Exponential distribution.

$$f(x) = \frac{1}{\beta}e^{-\frac{x}{\beta}} \quad \text{for} \quad x > 0$$
$$F(x) = \int f(x)\,dx = \int \frac{1}{\beta}e^{-\frac{x}{\beta}}\,dx = -e^{-\frac{x}{\beta}} + C$$
$$-e^{-\frac{0}{\beta}} = -1 \quad \text{so} \quad C = 1, \quad \text{therefore}$$
$$F(x) = 1 - e^{-\frac{x}{\beta}}$$

$$\text{Using } \beta = 1:$$
$$F(x) = 1 - e^{-x}$$
$$\frac{1}{2} = 1 - e^{-x}$$
$$\frac{1}{2} = e^{-x}$$
$$x = -\log\left(\frac{1}{2}\right) \approx 0.693$$

```
> qexp(p = 0.5)
[1] 0.6931472
> qgamma(p = 0.5, shape = 1, rate = 1)
[1] 0.6931472
```

3. (2 points) Suppose the lifetime of a light bulb is model with an exponential distribution with a mean of 100 days. Find the probability that the light bulb's life exceeds 100 days.

$$
\begin{aligned}
&\mu = E(X) = \beta = 100 \\
&f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}} = \frac{1}{100} e^{-\frac{x}{100}} \\
&F(x) = 1 - e^{-\frac{x}{\beta}} = 1 - e^{-\frac{x}{100}} \\
\\
&P(X > 100) = 1 - F(100) \\
&P(X > 100) = 1 - \left(1 - e^{-\frac{100}{100}}\right) \\
&P(X > 100) = e^{-1} \approx 0.368
\end{aligned}
$$

```
> 1 - pexp(q = 100, rate = 1/100)
[1] 0.3678794
```

4. (2 points) Based on the information in #3, find the probability that the light bulb's life exceeds 400 days given it exceeds 100 days.

$$
\begin{aligned}
P(X > 400 | X > 100) &= \frac{P(X > 100 \ \cap \ X > 400)}{P(X > 100)} \\
&= \frac{P(X > 400)}{P(X > 100)} \\
&\text{Using} \quad F(x) = 1 - e^{-\frac{x}{\beta}} \\
&= \frac{1 - \left(1 - e^{-\frac{400}{100}}\right)}{1 - \left(1 - e^{-\frac{100}{100}}\right)} = \frac{e^{-4}}{e^{-1}} \\
&= e^{-3} \approx 0.0498
\end{aligned}
$$

```
> 1 - pexp(q = 400 - 100, rate = 1/100)
[1] 0.04978707
```

5. (2 points) Use random numbers to approximate the integral

$$\int_0^\infty \log(x+1)e^{-x}\,dx,$$

where $\log(\cdot)$ denotes natural log.

$$\begin{aligned}
X &\sim exp(\lambda = 1) \\
P(X = x) = f(x) &= e^{-x} \quad \text{for} \quad x > 0 \\
g(x) &= \log(x+1) \quad \text{for} \quad x > 0 \\
E[g(X)] &= \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx \\
E[g(X)] &= \int_0^\infty \log(x+1)e^{-x} \\
&= E[\log(X+1)]
\end{aligned}$$

```
> set.seed(20190418)
> x <- rexp(1e6, 1)
> mean(log(x + 1))
[1] 0.5955806
```

```
> integrate(f = function(x) log(x + 1)*exp(-x), 0, Inf)
0.5963474 with absolute error < 8.6e-07
```