Homework 6

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Due date: Tuesday, March 26

1. (2 points) Assume that X_1, X_2, and X_3 are random variables, with

$$\begin{array}{rrr} E(X_1) = 2 & E(X_2) = -1 & E(X_3) = 4 \\ Var(X_1) = 4 & Var(X_2) = 6 & Var(X_3) = 8 \\ Cov(X_1, X_2) = 1 & Cov(X_1, X_3) = -1 & Cov(X_2, X_3) = 0 \end{array}$$

Find $Var(3X_1 + 4X_2 - 6X_3)$.

Applying Theorem 14 we have the following relation:

$$Var(aX + bY + cZ) =$$

$$a^2Var(X) \ + \ b^2Var(Y) \ + \ c^2Var(Z) \ + \ 2ab \cdot Cov(X,Y) \ + \ 2ac \cdot Cov(X,Z) \ + \ 2bc \cdot Cov(Y,Z)$$

By substituting in the coefficients and random variables, we have

$$Var(3X_1 + 4X_2 - 6X_3) =$$

$$Var(3X_1)+Var(4X_2)+Var(-6X_3)+2\cdot3\cdot4Cov(X_1,X_2)+2\cdot3\cdot-6Cov(X_1,X_3)+2\cdot4\cdot-6Cov(X_2,X_3)$$

$$= 9Var(X_1) + 16Var(X_2) + 36Var(X_3) + 24Cov(X_1,X_2) - 36Cov(X_1,X_3) - 48Cov(X_2,X_3)$$

$$= 9 \cdot 4 + 16 \cdot 6 + 36 \cdot 8 + 24 \cdot 1 - 36 \cdot -1 - 48 \cdot 0$$

$$= \mathbf{480}$$

2. (2 points) Let X be a standard uniform random variable, e.g., $X \sim \text{unif}(0,1)$. Find $\text{Cov}(X, e^X)$.

$$
\begin{aligned}
Cov(X,Y) &= E(XY) - E(X) \cdot E(Y) \\
Cov(X,e^X) &= E(X \cdot e^X) - E(X) \cdot E(e^X) \\
&= \int_0^1 x \cdot e^x \; dx \; - \int_0^1 x \; dx \; \int_0^1 e^x \; dx \\
&= [(x-1)e^x]_0^1 \; - \left[\frac{1}{2}x^2\right]_0^1 \; [e^x]_0^1 \\
&= (1-0) - (0.5-0)(e-1) \\
&= \mathbf{1 - 0.5(e-1)} \approx \mathbf{0.141}
\end{aligned}
$$

```
> set.seed(20190326)
> X <- runif(1e6)
> cov(X, exp(X))
[1] 0.1407736
```

3. (2 points) Let $X \sim \text{unif}(\alpha, \beta)$. Identify α and β so that $\text{E}(X) = \text{Var}(X) = 1$.

We can use the formula for the variance of the uniform distribution and set it equal to 1 to obtain an equation.

$$V(X) = \frac{(\beta - \alpha)^2}{12} = 1$$
$$12 = (\beta - \alpha)^2$$
$$\beta - \alpha = \sqrt{12}$$

We can also use the formula for the expectation of the uniform distribution and set it equal to 1 to find another equation.

$$E(X) = \frac{\alpha + \beta}{2} = 1$$
$$\alpha + \beta = 2$$

These two equations are a system of equations, with two unknown variables, that can be solved.

$$\begin{cases} \beta - \alpha & = & \sqrt{12} \\ \alpha + \beta & = & 2 \end{cases}$$

Solving the second equation for β gives

$$\beta = 2 - \alpha$$

which can be substituted into the first equation to give

$$2 - \alpha - \alpha = \sqrt{12}$$
$$2 - 2\alpha = \sqrt{12}$$
$$\alpha = \frac{2 - \sqrt{12}}{2}$$
$$\alpha = \mathbf{1 - \sqrt{3} \approx -0.732}$$

This value of α can be substituted back into the second equation to solve for β.

$$\beta = 2 - 1 + \sqrt{3}$$
$$\beta = \mathbf{1 + \sqrt{3} \approx 2.732}$$

4. (2 points) We know

$$\int_0^1 4\sqrt{1-x^2}\ dx = \pi.$$

Use 1,000,000 uniform random numbers to approximate the above integral (the empirical mean should be close to π).

```
> set.seed(20190326)
> f <- function(x) 4*sqrt(1-x^2)
> x <- runif(1e6)
> mean(f(x))
[1] 3.14226
```

5. (2 points) We know

$$\int_0^1 e^{-x}\ dx = 1 - e^{-1}.$$

Based on the above relationship, generate 1,000,000 uniform random numbers to approximate the number $e \approx 2.718281$.

In order to approximate the number e we must manipulate the above equation so that one side is e.

$$e^{-1} = 1 - \int_0^1 e^{-x}\ dx$$

$$e = \frac{1}{1 - \int_0^1 e^{-x}\ dx}$$

```
> set.seed(20190326)
> f <- function(x) exp(-x)
> x <- runif(1e6)
> 1/(1-mean(f(x)))
[1] 2.718629
```