Homework 9

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Due date: Thursday, April 18

1. (2 points) **Exercise 6.66** If X is a random variable having a normal distribution, what are the probabilities of getting a value

a. within *one* standard deviation of the mean;

```
> pnorm(1) - pnorm(-1)
[1] 0.6826895
```

b. within *two* standard deviation of the mean;

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> pnorm(2) - pnorm(-2)
[1] 0.9544997
```

c. within *three* standard deviation of the mean;

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> pnorm(3) - pnorm(-3)
[1] 0.9973002
```

2. (2 points) Let X be an exponential random variable. Show that $\int_0^\infty S(x)\,dx = E(X)$, where $S(x) = P(X > x) = 1 - F(X)$ is the survival function of X.

$$
\begin{aligned}
f(x) &= \frac{1}{\beta}e^{-\frac{x}{\beta}} \quad \text{for} \quad x > 0 \\
F(x) &= \int f(x)\;dx = \int \frac{1}{\beta}e^{-\frac{x}{\beta}}\;dx = -e^{-\frac{x}{\beta}} + C \\
&- e^{-\frac{0}{\beta}} = -1 \quad \text{so} \quad C = 1, \quad \text{therefore} \\
F(x) &= 1 - e^{-\frac{x}{\beta}} \\
S(x) &= 1 - F(x) = 1 - (1 - e^{-\frac{x}{\beta}}) = e^{-\frac{x}{\beta}}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty S(x)\;dx &= \int_0^\infty e^{-\frac{x}{\beta}}\;dx = \left[-\beta\, e^{-\frac{x}{\beta}}\right]_0^\infty \\
&= (0 - (-\beta)\,) = \beta = E(X)
\end{aligned}
$$

3. (2 points) Suppose that a random variable X has a probability function given by

$$f(x) = kx^3 - 2kx^2 + kx, 0 < x < 1,$$

for some constant k. Find the value of k that makes $f(x)$ a density function.

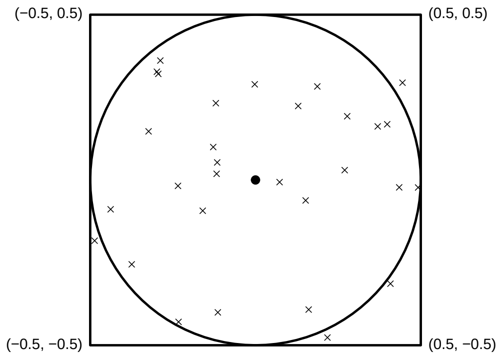
$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\, dx = 1 &= \int_0^1 \left(kx^3 - 2kx^2 + kx\right) dx \\
1 &= k\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1 \\
\frac{1}{k} &= \frac{1}{4} - \frac{2}{3} + \frac{1}{2} \\
\frac{1}{k} &= \frac{1}{12} \\
k &= 12
\end{aligned}$$

4. (2 points) Let Z be a standard normal random variable, show that

$$E(|Z|) = \sqrt{2/\pi} \approx 0.798.$$

$$\begin{aligned}
f(x) &= \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
P(Z = z) &= Norm(\mu = 0,\ \sigma = 1) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}} \\
E(|Z|) &= \int_{-\infty}^{\infty} |z|\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\, dz \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} -z\, e^{-\frac{z^2}{2}}\, dz \quad + \quad \int_0^{\infty} z\, e^{-\frac{z^2}{2}}\, dz\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_0^{-\infty} z\, e^{-\frac{z^2}{2}}\, dz \quad + \quad \int_0^{\infty} z\, e^{-\frac{z^2}{2}}\, dz\right] \\
&\text{Due to symmetry the two integrals are equivalent, so} \\
&= \frac{1}{\sqrt{2\pi}} \cdot 2\int_0^{\infty} z\, e^{-\frac{z^2}{2}}\, dz \\
&\text{Let} \quad u = -\frac{z^2}{2} \quad \text{and} \quad du = -z\, dz \\
&= \frac{1}{\sqrt{2\pi}} \cdot 2\left(-\int_0^{-\infty} e^u\, du\right) \\
&= -\sqrt{\frac{2}{\pi}}\, \left[e^u\right]_0^{-\infty} = -\sqrt{\frac{2}{\pi}}\, (0 - 1) \\
&= \sqrt{\frac{2}{\pi}} \approx 0.798
\end{aligned}$$

5. (2 points) **The estimation of π** Suppose that the random vector (X, Y) is uniformly distributed in the unit square centered at the origin below. The probability that this random point in the square is contained within the inscribed circle of diameter 1 is the area of the inscribed circle. Hence, if we generate a large number of random points in the square, the proportion of points that fall within the circle will be approximately $\pi/4$. Use this property to approximate π with 100,000 random vector (X, Y).




```r
> set.seed(20190418)
> x <- runif(1e5, -0.5, 0.5)
> y <- runif(1e5, -0.5, 0.5)
> d <- sqrt(x^2 + y^2)
> 4 * length(d[d < 0.5]) / length(d)
[1] 3.14396
```