Homework 5

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Due date: Tuesday, March 12

1. (2 points) The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{4} & 0 < x < 2, \\ \frac{1}{2} & 2 \le x < 3, \\ 0 & \text{elsewhere.} \end{cases}$$

Check if $f(x)$ is a valid density function by verifying if $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

We can take the definite integral from $-\infty$ to ∞ by splitting the integral at each of the bounds for the piecewise-defined function.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} \frac{x}{4}\,dx + \int_{2}^{3} \frac{1}{2}\,dx + \int_{3}^{\infty} 0\,dx$$

$$= 0 + \left[\frac{x^2}{8}\right]_0^2 + \left[\frac{x}{2}\right]_2^3 + 0$$

$$= \left(\frac{1}{2} - 0\right) + \left(\frac{3}{2} - \frac{2}{2}\right) = \frac{1}{2} + \frac{1}{2}$$

$$\int_{-\infty}^{\infty} \mathbf{f(x)\,dx = 1}$$

2. (2 points) Construct the distribution function, $F(x)$, for the probability density function in problem #1 and use it to find $P(X < 2.5) = F(2.5)$.

Here we can take the indefinite integral for each section of $f(x)$ to find $F(x)$, similar to the previous problem. However, the constant of integration, C, must be chosen so that $F(x)$ is continuous, begins at 0, and increases to 1 as x increases.

$$F(x) = \begin{cases} 0 & -\infty < x \leq 0, \\ \frac{x^2}{8} & 0 < x < 2, \\ \frac{x}{2} - \frac{1}{2} & 2 \leq x < 3, \\ 1 & 3 \leq x < \infty \end{cases}$$

$$P(X < 2.5) = F(2.5) = \frac{2.5}{2} - \frac{1}{2}$$

$$= \frac{5}{4} - \frac{2}{4} = \frac{3}{4} = 0.75$$

$$\mathbf{P(X < 2.5) = 0.75}$$

3. (2 points) Find the expected value, $E(X)$, for the random variable in problem #1.

We can use the expectation formula for each section of the piecewise-defined function to find the total expectation.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{-\infty}^{0} x \cdot 0\, dx + \int_{0}^{2} x \cdot \frac{x}{4}\, dx + \int_{2}^{3} x \cdot \frac{1}{2}\, dx + \int_{3}^{\infty} x \cdot 0\, dx$$

$$= \int_{0}^{2} \frac{x^2}{4}\, dx + \int_{2}^{3} \frac{x}{2}\, dx$$

$$= \left[\frac{x^3}{12}\right]_0^2 + \left[\frac{x^2}{4}\right]_2^3$$

$$= \left(\frac{8}{12} - 0\right) + \left(\frac{9}{4} - \frac{4}{4}\right) = \frac{8}{12} + \frac{15}{12} = \frac{23}{12}$$

$$\mathbf{E(X)} = \frac{\mathbf{23}}{\mathbf{12}} \approx \mathbf{1.92}$$

4. (2 points) Given the following cumulative distribution function for X:

$$e^{-e^{-x}}, \text{ for } -\infty < x < \infty.$$

What is the corresponding probability density function, $f(x)$?

The derivative of $F(x)$ is equal to $f(x)$.

$$f(x) = \frac{d}{dx} F(x)$$

$$= \frac{d}{dx} e^{-e^{-x}} = e^{-e^{-x}} \cdot e^{-x}$$

$$\mathbf{f(x) = e^{-x-e^{-x}}}$$

5. (2 points) Consider an experiment that consists of repeating independent tosses of a fair coin until a head is obtained. Let X be the random variable representing the number of tosses in the experiement. Use `R` to find an empirical average of X with 10,000 replications.

```
> set.seed(20190312)
> trials = c(1:10000) ## vector for 1, 2, 3, ..., 10,000 trials
>
> for(val in trials) ## for each trial, do the following...
+ {
+   X = 1 ## X is number of tosses. Initialized to 1 to
+         ## account for final toss resulting in "heads."
+   while(sample(1:0, 1, TRUE) == 0) ## while sample returns 0 (tails)...
+   {
+     X = X + 1 ## ...add 1 to tosses.
+   }
+   trials[val] = X ## record tosses for the trial
+ }
>
> head(trials, 10)
 [1] 1 1 2 5 7 2 1 3 1 1
> mean(trials)
[1] 2.007
```