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PRODUCTION OF HEAVY QUARKONIA FROM GLUONS

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By introducing, in a covariant way, explicit polarization vectors for the gluons, we calculate the various helicity amplitudes for the processes $g + g \to (^{2S+1}L_J) + g$, with L = S or P. In this way, we obtain simple formulae for the cross sections of these processes, which, in the framework of the perturbative QCD, are related to the hadroproduction of heavy quarkonia.

1. Introduction

The recent observation of a J/ψ signal at the CERN $p\bar{p}$ collider [1] offers several new possibilities for high energy physics. The hadroproduction of heavy quarkonia, in general, can reasonably be expected to yield information on $B-\bar{B}$ mixing [1–3] and the existence of extra b quarks [4]. Of course, these processes also provide further tests of perturbative QCD [5].

For a detailed comparison between the theory and experiment in the case of J/ψ hadroproduction, it is not sufficient to calculate direct J/ψ production only. Indeed, many of the $(c\bar{c})$ excited states are known to have large branching ratios into the J/ψ [6]. It thus becomes necessary to know, e.g., the production cross sections for the ^3P states.

Within the framework of perturbative QCD, the hadroproduction of heavy quarkonia is described by several subprocesses, such as $gg \to {}^{2S+1}L_J$, $q\bar{q} \to {}^{2S+1}L_J$, $gq \to {}^{2S+1}L_J\, q$, $g\bar{q} \to {}^{2S+1}L_J\, \bar{q}$, $q\bar{q} \to {}^{2S+1}L_J\, g$, $gg \to {}^{2S+1}L_J\, g$, etc., where the quarkonia are denoted by their spectroscopic notation. Using the standard Feynman techniques, one readily evaluates the cross sections for these processes [5], except for the case $gg \to {}^{2S+1}L_J\, g$, described by the Feynman diagrams of fig. 1. The problem

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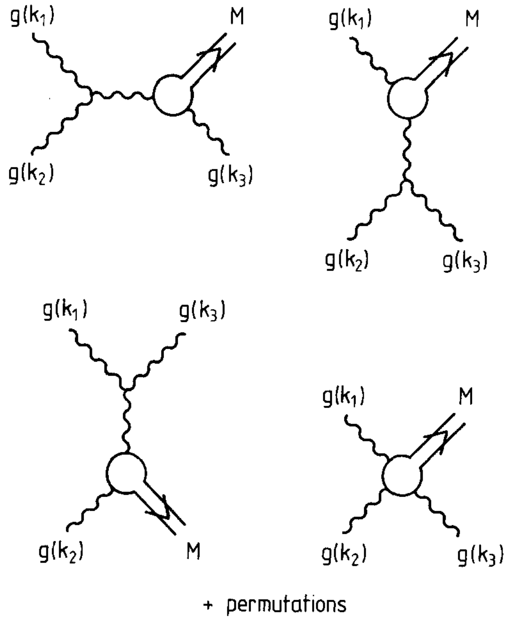

Fig. 1. Feynman diagrams for $g + g \to {}^{2S+1}L_J + g$.

in this case is the relatively large number of Feynman diagrams and the long expressions which are generated by the covariant summation over the polarization degrees of freedom. As a result, one had to resort to symbolic manipulation programs and the cross section formulae one obtained were quite cumbersome.

In a previous letter [7], we presented our cross section formulae for the process $gg \to {}^{2S+1}L_J\, g$, which were derived using the helicity formalism [8] developed earlier. It is the purpose of this paper to show how the introduction in a covariant way, of explicit polarization vectors for the three gluons leads to relatively simple expressions for the two independent helicity amplitudes for $gg \to q\bar{q}g$. With the method of Guberina, Kühn, Peccei, and Rückl [9] for projecting out the proper spin and angular momentum states, it then becomes a straightforward task to obtain the cross section formulae. As in the previous works [5], the formulae are all derived in lowest order, i.e. α_S^3, without much more complicated higher order QCD corrections. Our final formulae are, however, quite simple, and, more importantly, much easier to program.

For reasons of simplicity, we prefer to present the calculations for the process $ggg \to {}^{2S+1}L_J$, which is a more symmetric example. It is, however, well-known how crossing of a gluon can be applied to yield the desired formulae for $gg \to {}^{2S+1}L_J\, g$.

This paper is organized as follows. In sect. 2, we briefly describe the framework [9] for the calculations, and, in sect. 3, we show how to apply the helicity amplitude method [8] to the process ggg $\to {}^{2S+1}\mathrm{L}_J\,\mathrm{g}$. In the following sections, we present our results for the ${}^1\mathrm{S}_0$, ${}^3\mathrm{S}_1$, ${}^1\mathrm{P}_1$, ${}^3\mathrm{P}_0$, ${}^3\mathrm{P}_1$, and ${}^3\mathrm{P}_2$ cases successively. Finally, in sect. 10, we list our conclusions.

2. Framework

We consider first the process

$$\mathrm{g}(k_1) + \mathrm{g}(k_2) + \mathrm{g}(k_3) \to \mathrm{q}(\tfrac{1}{2}p + q) + \mathrm{q}(\tfrac{1}{2}p - q), \tag{1}$$

where k_1, k_2, and k_3 represent the four-momenta of the gluons, while $\frac{1}{2}p + q$ and $\frac{1}{2}p - q$ denote those of the quark and the anti-quark. As we are only interested in color singlet $\mathrm{q\bar{q}}$ states, we only have to consider the Feynman diagrams of fig. 1. We introduce the notation

$$s = (k_1 + k_2)^2, \qquad t = (k_2 + k_3)^2, \qquad u = (k_3 + k_1)^2. \tag{2}$$

For the first diagrams (s-channel), we have the amplitude

$$M_s = \bar{u}(\tfrac{1}{2}p + q)\mathcal{O}_s(p, q)v(\tfrac{1}{2}p - q), \tag{3}$$

where

$$\begin{aligned}\mathcal{O}_s(p, q) = {}& \frac{g_S^3 f^{abc}}{2s\sqrt{3}}\left[(k_2 - k_1)^\alpha(\varepsilon_1\varepsilon_2) + 2(k_1\varepsilon_2)\varepsilon_1^\alpha - 2(k_2\varepsilon_1)\varepsilon_2^\alpha\right] \\ & \times\left[\frac{\gamma_\alpha(\not{p} - 2\not{q} - 2\not{k}_s - 2m)\not{\varepsilon}_3}{2(pk_3) - 4(qk_3)} - \frac{\not{\varepsilon}_3(\not{p} + 2\not{q} - 2\not{k}_s + 2m)\gamma_\alpha}{2(pk_3) + 4(qk_3)}\right],\end{aligned} \tag{4}$$

with m the quark mass. The last term in eq. (4) is obtained from the previous one by interchanging the quark lines. The quantity f^{abc} is the completely antisymmetric structure constant of color SU(3), where the labels a, b, and c refer to the color states of the three gluons, with polarization vectors ε_i, $i = 1,2,3$. Obviously, the quantity g_S is the QCD coupling constant. For the t- and u-channel amplitudes, one readily writes down similar expressions.

For the diagrams, not involving the three-gluon vertex, one finds the amplitudes

$$\begin{aligned}\mathcal{O}_D(p, q) = {}& \sqrt{\tfrac{1}{3}}\,g_S^3(f^{abc} + id^{abc})\frac{[(p\varepsilon_1) + 2(q\varepsilon_1) + \not{k}_1\not{\varepsilon}_1]\not{\varepsilon}_2[(p\varepsilon_3) - 2(q\varepsilon_3) + \not{\varepsilon}_3\not{k}_3]}{[2(pk_1) + 4(qk_1)][2(pk_3) - 4(qk_3)]} \\ & + 5 \text{ permutations of } (1,2,3 \text{ and } a, b, c),\end{aligned} \tag{5}$$

where this time d^{abc} is the completely symmetric structure constant.

In what follows, we shall limit ourselves to S- and P-wave quarkonia and use the non-relativistic bound state approximation

$$M \simeq 2m\,, \tag{6}$$

where M is the quarkonium mass. This means that for the S-waves we can put $q=0$, while for P-waves we have to retain the terms linear in q in eqs. (4) and (5).

Following the method of Guberina, Kühn, Peccei, and Rückl [9], we introduce spin projection operators, which combine the quark and antiquark spins to the appropriate singlet ($S=0$) or triplet ($S=1$) states, and take into account the non-relativistic bound-state wave function of the quarks. We refer to ref. [9] for the details of the method and simply list the different amplitudes for quarkonium production:

$$A(^1\mathrm{S}_0) = \frac{R_0}{\sqrt{16\pi M}}\mathrm{Tr}[\mathcal{O}(\not{p}-M)\gamma_5]\,, \tag{7}$$

$$A(^3\mathrm{S}_1) = \frac{R_0}{\sqrt{16\pi M}}\mathrm{Tr}[\mathcal{O}(\not{p}-M)\not{\varepsilon}]\,, \tag{8}$$

$$A(^1\mathrm{P}_1) = -iR_1'\left(\frac{3}{16\pi M}\right)^{1/2}\mathrm{Tr}[\varepsilon^\varepsilon\mathcal{O}_\alpha(\not{p}-M)\gamma_5 + 2\mathcal{O}\not{\varepsilon}\not{p}\gamma_5/M]\,, \tag{9}$$

$$A(^3\mathrm{P}_0) = \frac{iR_1'}{\sqrt{16\pi M}}\mathrm{Tr}[\mathcal{O}_\alpha(\gamma^\alpha - p^\alpha/M)(\not{p}+M) + 6\mathcal{O}]\,, \tag{10}$$

$$A(^3\mathrm{P}_1) = -R_1'\left(\frac{3}{32\pi M^3}\right)^{1/2}\varepsilon^{\alpha\beta\gamma\delta}p_\gamma\varepsilon_\delta\mathrm{Tr}\left[\mathcal{O}_\alpha\gamma_\beta(\not{p}+M) - 2\mathcal{O}\gamma_\alpha\not{p}\gamma_\beta/M\right]\,, \tag{11}$$

$$A(^3\mathrm{P}_2) = iR_1'\left(\frac{3}{16\pi M}\right)^{1/2}\varepsilon^{\alpha\beta}\mathrm{Tr}\left[\mathcal{O}_\alpha\gamma_\beta(\not{p}+M)\right]. \tag{12}$$

In the formulae (7)–(12), we introduced

$$\mathcal{O} = \mathcal{O}_s(p,0) + \mathcal{O}_t(p,0) + \mathcal{O}_u(p,0) + \mathcal{O}_D(p,0)\,, \tag{13}$$

$$\mathcal{O}_\alpha = \frac{\partial}{\partial q^\alpha}[\mathcal{O}_s(p,q) + \mathcal{O}_t(p,q) + \mathcal{O}_u(p,q) + \mathcal{O}_D(p,q)]|_{q=0}\,, \tag{14}$$

and the quantity R_0, which is the S-wave wave function evaluated at the origin, and R_1', the derivative of the P-wave wave function also evaluated at the origin. The quantity R_0 is simply related to the leptonic width of the $^3\mathrm{S}_1$ state through the formula

$$R_0^2 = M^3\Gamma(^3\mathrm{S}_1 \to \mathrm{e}^+\mathrm{e}^-)/4\alpha^2 e_\mathrm{q}^2\,, \tag{15}$$

with e_q the fractional charge of the quarks. Using the quarkonium potential of ref. [10] with $\Lambda = 0.2$ GeV, one finds for the $(c\bar{c})$ system

$$R_1'^2/M_X^2 = 0.006\ (\mathrm{GeV})^3, \tag{16}$$

where M_X is the mass of a ^{3}P state.

Finally, the quantities of ε^{α} and $\varepsilon^{\alpha\beta}$ in eqs. (8), (9), (11), (12) describe the polarization states for the massive spin-1 and spin-2 quarkonia.

3. Helicity amplitude method

It is well-known that the helicity amplitude method [8] can advantageously be applied to QED [11] and QCD [12] processes, in the high energy limit, where fermion masses can be neglected. For the production of heavy quarkonia, the quark masses can, however, not be neglected, but still, the introduction of explicit polarization vectors for the three gluons greatly simplifies the algebra.

Let us denote by $M(\lambda_1, \lambda_2, \lambda_3)$ any helicity amplitude for ggg $\to {}^{2S+1}\mathrm{L}_\mathrm{J}$, where the labels λ_i, $i = 1, 2, 3$, refer to the helicities of the three gluons. We then have to consider two cases: all helicities are the same or one helicity differs. It is sufficient to consider $M(+,+,+)$ and $M(+,+,-)$ only, as all other helicity amplitudes can be obtained by permuting the gluons, or by applying a parity conjugation, which flips all helicities.

3.1. THE AMPLITUDE $M(+,+,+)$

For incoming gluons with positive helicities, we can choose the following polarization vectors:

$$\begin{aligned} \not{\varepsilon}_1 &= N\left[\not{k}_1\not{k}_2\not{k}_3(1-\gamma_5) + \not{k}_3\not{k}_2\not{k}_1(1+\gamma_5)\right], \\ \not{\varepsilon}_2 &= N\left[\not{k}_2\not{k}_3\not{k}_1(1-\gamma_5) + \not{k}_1\not{k}_3\not{k}_2(1+\gamma_5)\right], \\ \not{\varepsilon}_3 &= N\left[\not{k}_3\not{k}_1\not{k}_2(1-\gamma_5) + \not{k}_2\not{k}_1\not{k}_3(1+\gamma_5)\right], \end{aligned} \tag{17}$$

with the normalization factor

$$N = (2stu)^{-1/2}. \tag{18}$$

With the identity

$$(ab) = \tfrac{1}{4}\mathrm{Tr}[\not{a}\not{b}], \tag{19}$$

one can now express all the scalar products of four-vectors which appear in a given amplitude in terms of s, t, and u; e.g.,

$$(p\varepsilon_1) = (k_2\varepsilon_1) = Nst\,,$$

$$(p\varepsilon_2) = (k_3\varepsilon_2) = Ntu\,,$$

$$(p\varepsilon_3) = (k_1\varepsilon_3) = Nus\,,$$

$$(\varepsilon_1\varepsilon_2) = (\varepsilon_2\varepsilon_3) = (\varepsilon_3\varepsilon_1) = 1\,, \tag{20}$$

while many other scalar products, like $(k_1\varepsilon_1)$, $(k_3\varepsilon_1)$, $(k_1\varepsilon_2)$, etc., vanish. Also note that

$$p = k_1 + k_2 + k_3\,, \qquad s + t + u = M^2\,. \tag{21}$$

Substitution of the formulae (17) and (20) in the amplitudes (7)–(12) leads to many simplifications. Furthermore, the calculation of the traces in (7)–(12) has now become very simple: only the four-vectors k_1, k_2, and k_3 appear and the relations $k_1^2 = k_2^2 = k_3^2 = 0$ can be used repeatedly.

3.2. THE AMPLITUDE $M(+,+,-)$

This time, the third gluon has an opposite helicity, and we find it convenient to choose.

$$\not{\varepsilon}_1 = \left[\not{k}_1\not{k}_2\not{k}_3(1-\gamma_5) + \not{k}_3\not{k}_2\not{k}_1(1+\gamma_5)\right],$$

$$\not{\varepsilon}_2 = \left[\not{k}_2\not{k}_1\not{k}_3(1-\gamma_5) + \not{k}_3\not{k}_1\not{k}_2(1+\gamma_5)\right],$$

$$\not{\varepsilon}_3 = \left[\not{k}_3\not{k}_1\not{k}_2(1+\gamma_5) + \not{k}_2\not{k}_1\not{k}_3(1-\gamma_5)\right]. \tag{22}$$

With this choice, all scalar products among the ε's are made to vanish and

$$(p\varepsilon_1) = (k_2\varepsilon_1) = Nst\,,$$

$$(p\varepsilon_2) = (k_1\varepsilon_2) = Nsu\,,$$

$$(p\varepsilon_3) = (k_1\varepsilon_3) = Nsu\,. \tag{23}$$

Note that, because of the opposite helicities of gluons 2 and 3, it is perfectly possible to take $\varepsilon_2 = \varepsilon_3$. Again, many other scalar products vanish, and the evaluation of the expressions (7)–(12) is greatly simplified.

3.3. COMMENTS

The choices for the gluon polarizations made in eqs. (17) and (22) are not unique. As explained in ref. [8], any other choice, say ε', is however related through the equation

$$\varepsilon' = \mathrm{e}^{i\phi}\varepsilon + \beta k\,, \tag{24}$$

where k is the momentum of the gluon and ϕ a phase given by

$$\mathrm{e}^{i\phi} = -(\varepsilon'\varepsilon^*)\,. \tag{25}$$

Because of gauge invariance, the constant β is irrelevant.

If one is interested in the production of quarkonia from polarized gluons, one can readily combine the helicity amplitudes, which are listed in the following sections, but one should take into account the phase factors of eq. (25) when different polarization vectors are chosen.

Ultimately, we are interested in the process $\mathrm{gg} \to {}^{2S+1}\mathrm{L}_J\,\mathrm{g}$. For the cross section formulae, this simply means that one has to make the substitution $k_3 \to -k_3$, which changes the definitions of t and u in eqs. (2). At the level of the helicity amplitudes, one must not only replace k_3 by $-k_3$, but also, at the same time, flip the spin of gluon 3.

4. ${}^1\mathrm{S}_0$ production

Substitution of eqs. (17) and the use of eqs. (20) in the amplitude $A({}^1\mathrm{S}_0)$ of eq. (7) yields directly

$$M(+,+,+) = -\frac{4g_{\mathrm{S}}^3 R_0 f^{abc}}{\sqrt{3\pi M}} N^3 \frac{stuM^4(st+tu+us)}{(s-M^2)(t-M^2)(u-M^2)}\,, \tag{26}$$

while the same operations with eqs. (22) lead to

$$M(+,+,-) = \frac{4g_{\mathrm{S}}^3 R_0 f^{abc}}{\sqrt{3\pi M}} N^3 \frac{stus^2(st+tu+us)}{(s-M^2)(t-M^2)(u-M^2)}\,. \tag{27}$$

Obviously, $M(+,-,+)$ is obtained from eq. (27) by replacing $s \leftrightarrow u$.

For the process, $\mathrm{gg} \to {}^1\mathrm{S}_0\mathrm{g}$, one then obtains the cross section, averaged over the initial polarizations and color degrees of freedom,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\pi\alpha_{\mathrm{S}}^3 R_0^2}{2Ms^2}\left[\frac{st+tu+us}{(s-M^2)(t-M^2)(u-M^2)}\right]^2 \frac{M^8+s^4+t^4+u^4}{stu}\,. \tag{28}$$

Using the variables

$$P = st+tu+us\,,$$

$$Q = stu\,, \tag{29}$$

which are symmetrical under all permutations of s, t, and u, one can rewrite this cross section in the following form:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}=\frac{\pi\alpha_S^3R_0^2}{Ms^2}\frac{P^2(M^8-2M^4P+P^2+2M^2Q)}{Q(Q-M^2P)^2}. \tag{30}$$

5. $^3S_1$ production

For this process, only the last diagrams of fig. 1, not involving the three-gluon vertex, contribute, and with eqs. (17) and the relation $(p\varepsilon)=0$, we find readily

$$M(+,+,+)=0. \tag{31}$$

Similarly, with eqs. (22), we find

$$\begin{aligned}M(+,+,-)=&\frac{4ig_S^3R_0d^{abc}}{\sqrt{3\pi M}}\frac{MN^3s^2tu}{(s-M^2)(t-M^2)(u-M^2)}\\&\times\left[t(t+u)(k_1\varepsilon)-u(t+u)(k_2\varepsilon)-s(t-u)(k_3\varepsilon)\right.\\&\left.+2i(t+u)\varepsilon(k_1,k_2,k_3,\varepsilon)\right],\end{aligned} \tag{32}$$

where we use the notation

$$\varepsilon(a,b,c,d)=\varepsilon^{\alpha\beta\gamma\delta}a_\alpha b_\beta c_\gamma d_\delta. \tag{33}$$

Using the relation

$$\sum_{\mathrm{pol}}\varepsilon_\mu\varepsilon_\nu^*=P_{\mu\nu}=-g_{\mu\nu}+p_\mu p_\nu/M^2, \tag{34}$$

and summing over the gluon color degrees of freedom, we find

$$\sum_{\mathrm{pol}}|M(+,+,-)|^2=\frac{10240\pi^2\alpha_S^3R_0^2Ms^2}{9(t-M^2)^2(u-M^2)^2}. \tag{35}$$

Symmetrizing eq. (35) with respect to s, t, and u, and averaging over the initial state degrees of freedom, then yields the cross section:

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t}=&\frac{5\pi\alpha_S^3R_0^2M}{9s^2}\\&\times\left[\frac{s^2}{(t-M^2)^2(u-M^2)^2}+\frac{t^2}{(u-M^2)^2(s-M^2)^2}+\frac{u^2}{(s-M^2)^2(t-M^2)^2}\right].\end{aligned} \tag{36}$$

In terms of the variables P and Q, defined in eqs. (29), this reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}=\frac{10\pi\alpha_S^3R_0^2}{9s^2}\frac{M(P^2-M^2Q)}{(Q-M^2P)^2}. \tag{37}$$

6. $^1\mathrm{P}_1$ production

Like in the $^3\mathrm{S}_1$ case, the s-, t-, and u-channel diagrams do not contribute, and the amplitude $A(^1\mathrm{P}_1)$ is again proportional to d^{abc}. With our choice (17), we obtain

$$M(+,+,+)=\frac{g_S^3R_1'd^{abc}}{\sqrt{\pi M}}\bigg[a_1(\varepsilon\varepsilon_1)+a_2(\varepsilon\varepsilon_2)+a_3(\varepsilon\varepsilon_3)+\frac{2b_1}{t-M^2}(k_1\varepsilon)+\frac{2b_2}{u-M^2}(k_2\varepsilon)+\frac{2b_3}{s-M^2}(k_3\varepsilon)\bigg], \tag{38}$$

with

$$a_1=-\frac{2(2t+u)}{(s-M^2)(t-M^2)},$$

$$a_2=-\frac{2(2u+s)}{(t-M^2)(u-M^2)},$$

$$a_3=-\frac{2(2s+t)}{(u-M^2)(s-M^2)},$$

$$b_1=\frac{2N}{t-M^2}\left[\frac{-s(t^2+ut-uM^2)}{s-M^2}+\frac{t(u^2+su-sM^2)}{u-M^2}\right],$$

$$b_2=\frac{2N}{u-M^2}\left[\frac{-t(u^2+su-sM^2)}{t-M^2}+\frac{u(s^2+st-tM^2)}{s-M^2}\right],$$

$$b_3=\frac{2N}{s-M^2}\left[\frac{-u(s^2+ts-tM^2)}{u-M^2}+\frac{s(t^2+tu-uM^2)}{t-M^2}\right]. \tag{39}$$

After the summation over the polarization states of the $^1\mathrm{P}_1$ and the color states of the gluons, we obtain

$$\sum_{\mathrm{pol}}|M(+,+,+)|^2=\frac{40960\pi^2\alpha_S^2R_1'^2}{3M}\frac{M^8(-M^2P+5Q)}{(Q-M^2P)^3}. \tag{40}$$

Similar manipulations with the choice of polarizations (22) give for the $M(+,+,-)$ amplitude a formula as in eq. (38), but now,

$$\begin{aligned}
a_1 &= \frac{2s(t+u-2M^2)}{(s-M^2)(t-M^2)(u-M^2)}, \\
a_2 &= 0, \\
a_3 &= \frac{2s}{(t-M^2)(u-M^2)}, \\
b_1 &= \frac{-2Ns^2u}{t-M^2}\left[\frac{1}{s-M^2}+\frac{1}{u-M^2}\right], \\
b_2 &= \frac{2Ns^2u}{u-M^2}\left[\frac{1}{t-M^2}-\frac{1}{s-M^2}\right], \\
b_3 &= \frac{2Ns^2u}{s-M^2}\left[\frac{1}{u-M^2}+\frac{1}{t-M^2}\right],
\end{aligned} \tag{41}$$

and, consequently,

$$\begin{aligned}
&\sum_{\text{pol}}|M(+,+,-)|^2 \\
&\quad = \frac{40960\pi^2\alpha_S^2 R_1'^2}{3M}\left[\frac{M^4s^2}{(s-M^2)^2(t-M^2)^2(u-M^2)^2}\right. \\
&\qquad\left. + \frac{2stu(s^4+s^2M^4)}{(s-M^2)^3(t-M^2)^3(u-M^2)^3}\right].
\end{aligned} \tag{42}$$

With eqs. (40) and (42), we obtain the cross section for ${}^1P_1$ production:

$$\begin{aligned}
\frac{d\sigma}{dt} &= \frac{20\pi\alpha_S^3 R_1'^2}{3Ms^2}\frac{1}{[(s-M^2)(t-M^2)(u-M^2)]^2} \\
&\quad\times\left\{M^4(s^2+t^2+u^2+M^4)+\frac{2stu[s^4+t^4+u^4+M^4(s^2+t^2+u^2)+2M^8]}{(s-M^2)(t-M^2)(u-M^2)}\right\} \\
&= \frac{40\pi\alpha_S^3 R_1'^2}{3Ms^2}\frac{[-M^{10}P+M^6P^2+Q(5M^8-7M^4P+2P^2)+4M^2Q^2]}{(Q-M^2P)^3},
\end{aligned} \tag{43}$$

with the variables P and Q defined in eqs. (29).

7. $^3P_0$ production

The evaluation of the helicity amplitudes for $^3P_0$ production is straightforward, and we merely list our results

$$M(+,+,+) = -\frac{12ig_S^3 R_1' f^{abc}}{\sqrt{3\pi M^3}} \frac{NM^4 P}{(s-M^2)(t-M^2)(u-M^2)}, \tag{44}$$

$$M(+,+,-) = \frac{4ig_S^3 R_1' f^{abc}}{\sqrt{3\pi M^3}} \frac{Ns^2\left[t^2u^2 - stu(s-6M^2) - 3M^2s^2(s-M^2)\right]}{(s-M^2)(t-M^2)^2(u-M^2)^2}. \tag{45}$$

The resulting cross section reads:

$$\begin{aligned}
\frac{d\sigma}{dt} &= \frac{2\pi\alpha_S^3 R_1'^2}{M^3 s^2} \frac{1}{\left[(s-M^2)(t-M^2)(u-M^2)\right]^2} \\
&\quad\times\left\{8M^2\left[\frac{tu(t^4-t^2u^2+u^4)}{(s-M^2)^2} + \frac{us(u^4-u^2s^2+s^4)}{(t-M^2)^2} + \frac{st(s^4-s^2t^2+t^4)}{(u-M^2)^2}\right]\right. \\
&\quad + 4M^2\left[M^2(st+tu+us) - 5stu\right] + (st+tu+us)^2 \\
&\quad\times\left[\frac{9M^8}{stu} + \frac{1}{(s-M^2)(t-M^2)(u-M^2)}\right. \\
&\quad\left.\left.\times\left(8M^4(s^2+t^2+u^2) - 16M^2stu + \left(1 - 9M^2\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)\right)(s^4+t^4+u^4)\right)\right]\right\} \\
&= \frac{4\pi\alpha_S^3 R_1'^2}{M^3 s^2} \frac{1}{Q(Q-M^2P)^4}\left[9M^4P^4(M^8 - 2M^4P + P^2) - 6M^2P^3Q(2M^8 - 5M^4P + P^2)\right. \\
&\quad\left. - P^2Q^2(M^8 + 2M^4P - P^2) + 2M^2PQ^3(M^4 - P) + 6M^4Q^4\right].
\end{aligned} \tag{46}$$

8. $^3P_1$ production

For the $^3P_1$ case, a somewhat lengthy, but straightforward calculation shows that

$$M(+,+,+) = 0. \tag{47}$$

The substitution of eqs. (22) in the $A(^3P_1)$ amplitude, eq. (11), leads to

$$M(+,+,-) = -\frac{4g_S^3 R_1' f^{abc}}{\sqrt{2\pi M}} \frac{Ns}{(s-M^2)(t-M^2)^2(u-M^2)^2}$$
$$\times \{2\varepsilon(k_1,k_2,k_3,\varepsilon)(t-u)(st+tu+us-s^2)$$
$$-i(k_1\varepsilon)(s+t)[st(t-s)+tu(t-u)+us(u-s)]$$
$$-i(k_2\varepsilon)(s+u)[su(u-s)+ut(u-t)+ts(t-s)]\}, \quad (48)$$

where ε is the polarization vector of the $^3P_1$ quarkonium. Using eq. (34), we obtain

$$\sum_{\text{pol}} |M(+,+,-)|^2 = \frac{6144\pi^2\alpha_S^3 R_1'^2}{M^3} \frac{s^2}{(s-M^2)^2(t-M^2)^4(u-M^2)^4}$$
$$\times \{(s+t)(t+u)(u+s)[st(s-t)^2+tu(t-u)^2+us(u-s)^2]$$
$$+M^2(t-u)^2(st+tu+us-s^2)^2\}. \quad (49)$$

Note that in eq. (49), we also summed over all the color degrees of freedom. Symmetrizing eq. (49) in s, t and u, and multiplying with the appropriate factors, we obtain the cross section formula

$$\frac{d\sigma}{dt} = \frac{12\pi^2\alpha_S^3 R_1'^2}{M^3 s^2} \frac{1}{[(s-M^2)(t-M^2)(u-M^2)]^2}$$
$$\times \left\{ M^2 \left[\frac{t^2u^2(t^2+u^2)}{(s-M^2)^2} + \frac{u^2s^2(u^2+s^2)}{(t-M^2)^2} + \frac{s^2t^2(s^2+t^2)}{(u-M^2)^2} \right] \right.$$
$$\left. + \frac{2(s^2t^2+t^2u^2+u^2s^2)(s^2t^2+t^2u^2+u^2s^2+M^2stu)}{(s-M^2)(t-M^2)(u-M^2)} \right\}$$
$$= \frac{12\pi\alpha_S^3 R_1'^2}{M^3 s^2} \frac{P^2[M^2P^2(M^4-4P)-2Q(M^8-5M^4P-P^2)-15M^2Q^2]}{(Q-M^2P)^4}. \quad (50)$$

9. $^3P_2$ production

For the case of $^3P_2$ production, we find again that

$$M(+,+,+)=0. \tag{51}$$

The calculation of the $M(+,+,-)$ amplitude is somewhat more involved. With the choice of gluon polarization vectors (22), we have

$$\begin{aligned}
&M(+,+,-)\\
&\quad=\frac{8ig_S^2R_1'f^{abc}}{\sqrt{\pi}}\varepsilon^{\alpha\beta}N\sqrt{M}\\
&\quad\times\left\{s^2\left[\frac{k_{1\alpha}}{t(t-M^2)}-\frac{k_{2\alpha}}{u(u-M^2)}\right]\left[t(t+u)k_{1\beta}-u(t+u)k_{2\beta}-s(t-u)k_{3\beta}\right]\right.\\
&\quad+\frac{st+tu+us}{stu}\left[(s-t-u)(tk_{1\alpha}-uk_{2\alpha})(tk_{1\beta}-uk_{2\beta})\right.\\
&\qquad\left.+sk_{3\alpha}\left(t(2t-s)k_{1\beta}+u(2u-s)k_{2\beta}\right)+\frac{s^2(t^2+u^2)}{s-M^2}k_{3\alpha}k_{3\beta}\right]\\
&\quad+2i\varepsilon_{\mu\nu\rho\alpha}k_1^\mu k_2^\nu k_3^\rho\left[k_{1\beta}\left(\frac{u^2(t-s)}{t(t-M^2)}-t+u-\frac{ut}{s}-\frac{t(st+tu+us)}{su}\right)\right.\\
&\qquad\left.\left.-k_{2\beta}\left(\frac{t^2(u-s)}{u(u-M^2)}-u+t-\frac{ut}{s}-\frac{u(st+tu+us)}{st}\right)\right]\right\}.
\end{aligned} \tag{52}$$

In eq. (52), $\varepsilon^{\alpha\beta}$ is the polarization tensor of the $^3P_2$ quarkonium, which satisfies the relations

$$\varepsilon^{\alpha\beta}=\varepsilon^{\beta\alpha},\qquad p_\alpha\varepsilon^{\alpha\beta}=0,\qquad \varepsilon_\alpha^\alpha=0. \tag{53}$$

To obtain the cross section, we have to sum over the polarization states of the $^3P_2$ system using

$$\sum_{\text{pol}}\varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}^*=\tfrac{1}{2}\left(P_{\alpha\mu}P_{\beta\nu}+P_{\alpha\nu}P_{\beta\mu}\right)-\tfrac{1}{3}P_{\alpha\beta}P_{\mu\nu}, \tag{54}$$

where $P_{\mu\nu}$ is defined in eq. (34). Also summing over the color degrees of freedom, we find

$$\sum_{\text{pol}}|M(+,+,-)|^2$$
$$= \frac{4096\pi^2\alpha_S^3 R_1'^2}{M^3 Q(Q-M^2P)^4}\big[12M^8P^4(3s^2-4M^2s+M^4)$$
$$-12M^4P^5s(s-3M^2)-3M^6P^3Q(25s^2-33M^2s+8M^4)$$
$$+12M^2P^4Q(s^2-4M^2s-3M^4)+M^4P^2Q^2(8s^2+9M^2s-15M^4)$$
$$-2P^3Q^2(s^2-5M^2s-30M^4)+M^2PQ^3(29s^2-51M^2s+18M^4)$$
$$+2P^2Q^3(s-11M^2)-M^2Q^4(9s-11M^2)\big]. \tag{55}$$

It follows that, for $^3P_2$ production,

$$\frac{d\sigma}{dt}=\frac{4\pi\alpha_S^2 R_1'^2}{M^3s^2}\frac{1}{Q(Q-M^2P)^4}$$
$$\times\big[12M^4P^4(M^8-2M^4P+P^2)-3M^2P^3Q(8M^8-M^4P+4P^2)$$
$$-2P^2Q^2(7M^8-43M^4P-P^2)+M^2PQ^3(16M^4-61P)+12M^4Q^4\big]$$
$$=\frac{4\pi\alpha_S^3 R_1'^2}{M^3s^2}\frac{1}{\left[(s-M^2)(t-M^2)(u-M^2)\right]^2}$$
$$\times\Bigg\{M^2\left[\frac{t^2u^2(t^2+4tu+u^2)}{(s-M^2)^2}+\frac{u^2s^2(u^2+4us+s^2)}{(t-M^2)^2}+\frac{s^2t^2(s^2+4st+t^2)}{(u-M^2)^2}\right]$$
$$+12M^2\left[3(s^3t+t^3u+u^3s+st^3+tu^3+us^3)+4M^2stu\right]$$
$$+\frac{2(st+tu+us-M^4)(st+tu+us)^2}{(s-M^2)(t-M^2)(u-M^2)}$$
$$\times\left[st+tu+us-24M^4-6M^2(st+tu+us-M^4)\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)\right]\Bigg\}. \tag{56}$$

10. Conclusions

We have shown that the application of the helicity amplitude formalism [8] leads to simple and useful formulae for the processes $g+g\to {}^{2S+1}L_J+g$, with $L=S$ or

P. The simplicity of our formulae, especially in the ^{3}P-cases, allows one to make more refined studies because of the savings in computer time.

One can expect several applications of our formulae. An outstanding example is related to the recent study of Glover, Halzen and Martin [13] on the physics of J/ψ tags in $p\bar{p}$ collisions. Also, the fact that we obtained the separate helicity amplitudes for these processes, enables one to consider the more complicated case of quarkonium production from polarized gluons [14].

Although the helicity amplitude formalism was originally designed for zero-mass fermion processes, it is seen from these examples that the formalism can also be used advantageously in the case where massive quark pairs are produced.

Finally, we point out that for $ggg \to {}^3S_1$, ${}^3P_1$, and ${}^3P_2$ the helicity amplitude $M(+,+,+)$ vanishes. We have no simple explanation for this intriguing fact.

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