Loop calculations with FeynCalc 9

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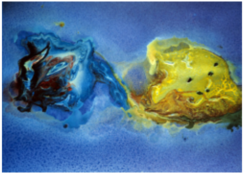


Physik-Department T30f

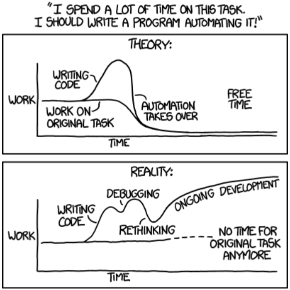
Outline

What

- Automation of QFT $\approx$ automatic symbolic or numeric evaluation of Feynman diagrams
- This talks: only symbolics

Why

- Feynman diagrams $\rightarrow$ theoretical predictions for experimental observables: cross-sections, decay rates etc.
- Rising experimental precision $\rightarrow$ theoretical errors must be reduced.
- Leading order (LO) in perturbation theory is mostly not enough, need to go at least to NLO or even higher.
- Often too many Feynman diagrams to do everything by pen and paper :(
- Also calculations that are doable by pen and paper need to be checked!



[xkcd.com/1319]

How

- CAS or CAS-like environment: Mathematica, Reduce, FORM, Sympy, GiNaC, RedBerry etc.
- Specific codes running on top of it.

Automation tools classified by their usage

- Single purpose tools: FeynArts[1], Tracer [2], FIRE[3], LoopTools[4], ...
- Multi purpose tools (semi-automatic): HEPMath[5], FeynCalc, Package X[6], ...
- Multi purpose tools (fully-automatic): CalcHEP[7], GRACE[8], FormCalc[1] ...

[1] [Hahn, 2001], [2] [Jamin & Lautenbacher, 1993], [3] [Smirnov, 2008], [4] [Hahn & Perez-Victoria, 1999], [5] [Wiebusch, 2014], [6] [Patel, 2015], [7] [Belyaev et al., 2012], [8] [Ishikawa et al., 1993]

Some single purpose tools for making coffee

Semi-automatic coffee maker

Fully-automatic coffee maker

- FeynCalc is a Mathematica package for symbolic semi-automatic evaluation of Feynman diagrams and algebraic expressions in QFT.

[Mertig et al., 1991]
[Shtabovenko et al., 2016]

Feyn
Calc

Features

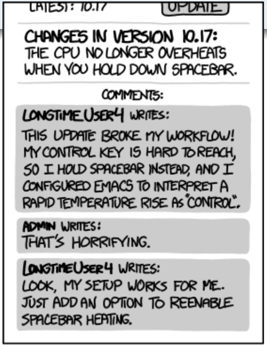
- Suitable for evaluating both single expressions and full Feynman diagrams.
- The calculation can be organized in many different ways (flexibility)
- Extensive typesetting for better readability
- Tools for frequently occurring tasks like Lorentz index contraction, $SU(N)$ algebra, Dirac matrix manipulation and traces, etc.
- Passarino-Veltman reduction of one-loop amplitudes to standard scalar integrals
- Basic support for manipulating multi-loop integrals
- General tools for non-commutative algebra

FeynCalc 9

- FeynCalc 9 was officially released this year (see arXiv:1601.01167)
- The source code is now hosted on GitHub: `github.com/FeynCalc`
- Works with Mathematica versions 8, 9 and 10.
- Test driven development: over 3000 unit and integration tests to prevent new bugs and regressions.

What's new

- Improved 1-loop tensor reduction.
- New partial fractioning algorithm.
- Basic multi-loop tensor reduction.
- Simpler interfacing with other tools (e.g. for IBP-reduction)



EVERY CHANGE BREAKS SOMEONE'S WORKFLOW.
[xkcd.com/1172]

What minimal capabilities do we need to compute 1-loop integrals in dimensional regularization using software for symbolic manipulations?

$$\mu^{D-4} \int \frac{d^D q}{(2\pi)^D} \frac{\gamma^\mu (\not{q} + m) \gamma_\mu}{(q^2 - m^2)(q-p)^2} = \quad ?$$

- Simplification of the Dirac algebra (✓FeynCalc)
- Tensor integral reduction
    - Standard technique: Passarino-Veltman (✓FeynCalc)
- Further simplification of scalar integrals
    - Partial fractioning (✓FeynCalc)
    - Optional: IBP reduction (requires external tools)
- Analytic/numeric evaluation of the master integrals (requires external tools)

Dirac algebra

FeynCalc implements the conventional generalization of the Dirac algebra to D dimensions

['t Hooft & Veltman, 1972]

$$\{\bar{\gamma}^{\mu}, \bar{\gamma}^{\nu}\} = 2\bar{g}^{\mu\nu}, \quad \bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = 4 \to \{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}, \quad g^{\mu\nu}g_{\mu\nu} = D$$

- DiracSimplify simplifies chains of Dirac matrices
- DiracTrace computes traces of Dirac matrices

In[1]:= FCI[GAD[μ].(m + GSD[q]).GAD[μ] FAD[{q, m}, −p + q]]

Out[1]= $\dfrac{\gamma^{\mu}.(m+\gamma\cdot q).\gamma^{\mu}}{(q^2-m^2).(q-p)^2}$

In[2]:= %//DiracSimplify

Out[2]= $-\dfrac{D\gamma\cdot q}{(q^2-m^2).(q-p)^2} + \dfrac{Dm}{(q^2-m^2).(q-p)^2} + \dfrac{2\gamma\cdot q}{(q^2-m^2).(q-p)^2}$

In[3]:= GAD[μ, ν, ρ, σ, τ, κ, μ, ν, ρ, σ, τ, κ]

Out[3]= $\gamma^{\mu}.\gamma^{\nu}.\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\tau}.\gamma^{\kappa}.\gamma^{\mu}.\gamma^{\nu}.\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\tau}.\gamma^{\kappa}$
In[4]:= DiracTrace[%, DiracTraceEvaluate −> **True**] // Factor2
Out[4]= $4(2-D)^2 D(-D^3+26D^2-152D+128)$

Tensor reduction

The default behavior of TID is to reduce each tensor integral into Passarino-Veltman scalar functions (A_0, B_0, C_0, D_0)

In[1]:= TID[GAD[μ].(m + GSD[q]).GAD[μ] FAD[{q, m}, −p + q], q];
ToPaVe[%, q]

Out[1]= $$\frac{i\pi^2(D-2)\mathrm{A}_0\left(m^2\right)\gamma\cdot p}{2p^2} - \frac{i\pi^2\mathrm{B}_0\left(p^2,0,m^2\right)\left(Dm^2\gamma\cdot p-2Dmp^2+Dp^2\gamma\cdot p-2m^2\gamma\cdot p-2p^2\gamma\cdot p\right)}{2p^2}$$

If the function detects zero Gram determinants, it automatically switches to Passarino-Veltman coefficient functions (e.g. B_1, B_{00}, C_{222} etc.)

In[3]:= ScalarProduct[p, p] = 0;
TID[FCI[GAD[μ].(m + GSD[q]).GAD[μ] FAD[{q, m}, −p + q]], q];
ToPaVe[%, q] // Collect2[#, B0] &

Out[3]:= $$-\frac{1}{2}i\pi^2\mathrm{B}_0\left(0,0,m^2\right)(-2Dm+D\gamma\cdot p-2\gamma\cdot p)-\frac{1}{4}i\pi^2(D-2)\gamma\cdot p$$

Partial fractioning

- Scalar loop integrals can be often simplified even further by using partial fractioning.
- Well known identities (implemented in `SPC` and APART2) are

$$q \cdot p = \frac{1}{2}[(q+p)^2 + m_2^2 - (q^2 + m_1^2) - p^2 - m_2^2 + m_1^2],$$
$$\frac{1}{(q^2 - m_1^2)(q^2 - m_2^2)} = \frac{1}{m_1^2 - m_2^2}\Big(\frac{1}{q^2 - m_1^2} - \frac{1}{q^2 - m_2^2}\Big).$$

- But: Many decompositions, e.g.

$$\int d^D q \, \frac{1}{q^2 (q-p)^2 (q+p)^2} = \frac{1}{p^2} \int d^D q \left(\frac{1}{q^2 (q-p)^2} - \frac{1}{(q-p)^2 (q+p)^2} \right),$$

require more sophisticated algorithms.
- New in FEYNCALC 9: `ApartFF` introduces partial fractioning algorithm from [Feng, 2012]
- Compared to the reference MATHEMATICA implementation (`https://github.com/F-Feng/APart`), it is fully integrated into FEYNCALC

In[1]:= ApartFF[FAD[{q}, {q − p}, {q + p}], {q}]

Out[2]:= $\frac{1}{p^2 q^2.(q-p)^2} - \frac{1}{p^2 q^2.(q-2p)^2}$

Tools for IBP-reduction

- Reduction of scalar loop integrals using integration-by-parts (IBP) identities [Chetyrkin & Tkachov, 1981] is a standard technique in modern loop calculations.
- Many publicly available IBP-packages on the market: FIRE [Smirnov & Smirnov, 2013], LiteRED [Lee, 2012], Reduze [Studerus, 2009], AIR [Anastasiou & Lazopoulos, 2004], ...
- Expected input: loop integrals with propagators that form a basis.
- What about integrals with an incomplete or overdetermined basis?
  - `FCLoopBasisIncompleteQ` detects integrals that require a basis completion
  - `FCLoopBasisFindCompletion` gives a list of propagators (with zero exponents) required to complete the basis
  - `FCLoopBasisOverdeterminedQ` checks if the propagators are linearly dependent. Such integrals can be decomposed further using `ApartFF`.

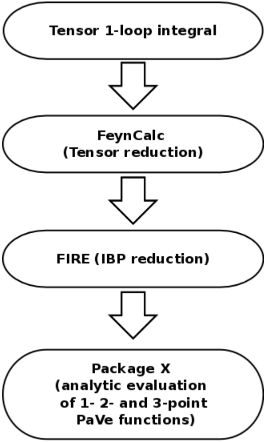
In[1]:= FCI[FAD[{q1, m, 2}, {q1 + q3, m}, {q2 − q3}, q2]]

Out[1]:= $\frac{1}{(\mathrm{q1}^2 - m^2).(\mathrm{q1}^2 - m^2).((\mathrm{q1}+\mathrm{q3})^2 - m^2).(\mathrm{q2}-\mathrm{q3})^2.\mathrm{q2}^2}$

In[2]:= FCLoopBasisIncompleteQ[%, {q1, q2, q3}]
Out[2]:= **True**

In[3]:= FCLoopBasisFindCompletion[%%, {q1, q2, q3}][[2]]
Out[3]:= $\left\{-(\mathrm{q1}\cdot\mathrm{q3}) + \mathrm{q2}\cdot\mathrm{q3} + 2\mathrm{q3}^2, \mathrm{q1}\cdot\mathrm{q2}\right\}$

- Real life computations: combination of different tools is often advantageous
- Useful setup for evaluation of 1-loop integrals: FeynCalc+FIRE+Package X
- Need to convert between conventions used in each package!
- Work in progress (VS): Interface for seamless integration between these (and possibly) other packages and FeynCalc.
- Private version available, public release in near future.



- Call FIRE from FeynCalc:

In[1]:= FCI[FAD[q1, q2, {q1 − p, 0, 2}, {q1 − q2, 0, 2}]]

Out[1]:= $\frac{1}{\mathrm{q1}^2.\mathrm{q2}^2.(\mathrm{q1}-p)^2.(\mathrm{q1}-p)^2.(\mathrm{q1}-\mathrm{q2})^2.(\mathrm{q1}-\mathrm{q2})^2}$

In[2]:= FIREBurn[%, {q1, q2}, {p}]

Out[2]:= $\frac{3(D-5)(D-3)(3D-10)(3D-8)}{(D-6)(D-4)p^6\mathrm{q2}^2.(\mathrm{q1}-\mathrm{q2})^2.(\mathrm{q1}-p)^2}$

- `FIREBurn` actually generates the code to run FIRE standalone

```
FIREPath="/home/vs/.Mathematica/Applications/FIRE5/";
Get["/home/vs/.Mathematica/Applications/FIRE5/FIRE5.m"];
Internal ={q1, q2};
External={p};
Propagators={q1^2, (−p + q1)^2, (q1 − q2)^2, q2^2, p∗q2};
Replacements={};
PrepareIBP [];
Prepare[ AutoDetectRestrictions −> True];
SaveStart ["/home/vs/.Mathematica/Applications/FeynCalc/Database/FIREStartFile"];
```

- Easy and convenient evaluation of Passarino-Veltman functions with PACKAGE-X directly from FEYNCALC:

In[1]:= FAD[{q, m0}]

$$\textbf{Out}[1]:= \frac{1}{q^2 - \mathrm{m0}^2}$$

In[2]:= PaXEvaluate[%, q, PaXImplicitPrefactor −> 1/(2 **Pi**)^**D**]

$$\textbf{Out}[2]:= \frac{i\mathrm{m0}^2}{16\pi^2\varepsilon} - \frac{i\mathrm{m0}^2\left(-\log(\frac{\mu^2}{\mathrm{m0}^2}) + \gamma - 1 - \log(4\pi)\right)}{16\pi^2}$$

In[3]:= **I Pi** PaVe[1, 2, {m^2, 0, m^2}, {0, m^2, m^2}]

$$\textbf{Out}[3]:= i\pi \mathrm{C}_{12}\left(m^2, 0, m^2, 0, m^2, m^2\right)$$

In[4]:= PaXEvaluate[%, q, PaXImplicitPrefactor −> 1/(2 **Pi**)^**D**]

$$\textbf{Out}[4]:= -\frac{i}{192\pi^3 m^2}$$

Evaluating Feynman diagrams

- To simplest way to generate Feynman diagrams for FeynCalc is to use FeynArts[9].
- Both tools are not 100% compatible, some auxiliary steps are needed
  - FeynArts needs to be patched to avoid variable conflicts with FeynCalc.
  - The output of FeynArts needs to be converted into FeynCalc notation using `FCFAConvert`.
- FeynCalc 9 comes with many examples for evaluating QED/QCD diagrams

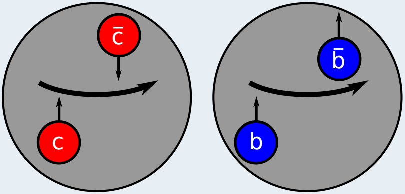
Working with non-relativistic theories

- FeynCalc was originally designed to be used in calculations, where the Lorentz covariance is manifest.
- Description of physical systems at low-energies is often done using Effective Field Theories (EFTs) that are non-relativistic.
- Can we extend FeynCalc to handle these cases as well?

[9] FeynArts is a Mathematica package for generation and visualization of Feynman diagrams and amplitudes. The original FeynArts was created by J. Küblbeck, M. Böhm and A. Denner in 1990. Since 1998 it is developed further by Thomas Hahn.

Heavy Quarkonia

Heavy quarkonium is a bound state of a heavy quark with its antiparticle



It contains not only two quarks but also virtual quark and gluons.
Quark and anti-quark interact with each other by exchanging an arbitrary number of gluons.

History

1974: Discovery of J/ψ and ψ' mesons which are $c\bar{c}$-bound states.
1977: Discovery of $\Upsilon(1S)$, a $b\bar{b}$-bound state.

Challenges

The evolution of a $Q\bar{Q}$-pair to a bound state is a non-perturbative process.
Need to factorize perturbative and non-perturbative parts of the production or decay.

Understaing QCD

The more we know about quarkonia, the better we can understand the strong force and confinement.

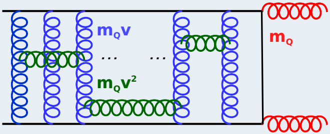
Relevant scales

$m \gg mv \gg mv^2$, $m \gg \Lambda_{\mathrm{QCD}}$ with

- $|\vec{p}_{\mathrm{rel}}| \sim mv$
- $E_{\mathrm{bind}} = m_H - 2m \sim mv^2$
- $\Lambda_{\mathrm{QCD}} \approx 0.2\,\mathrm{MeV}$
- $m_c \approx 1.3\,\mathrm{GeV}$, $m_b \approx 4.2\,\mathrm{GeV}$
- $v_c \approx 0.55$, $v_b \approx 0.32$

Full QCD

Different scales are entangled, their contributions are difficult to separate



Effective Field Theory

Motivation: Let us deal only with scales we are interested in.
General Idea: To study production and annihilation we are interested in scales below m_Q.
Approach: Start with full QCD and integrate out the scale m_Q. Make sure that the resulting theory reproduces full QCD below that scale. This theory is called *Non-Relativistic QCD (NRQCD)*. [Bodwin et al., 1995]
Advantage: In NRQCD one has to deal only with two dynamical scales which are $m_Q v$ and $m_Q v^2$. The effects of the m_Q scale on physics below $m_Q v$ are encoded in the short distance coefficients of the NRQCD operators.

Features of NRQCD

- Not a model, rigorous derivation from the full QCD.
- $\mathcal{L}_{\text{NRQCD}}$ is an expansion in v.
- Infinite number of operators with increasing mass dimension.
- Operators are compatible with QCD symmetries.
- Contributions to a process at the given accuracy estimated by velocity scaling rules.

NRQCD Lagrangian

$$\mathcal{L}_{\text{NRQCD}} = \sum_n \frac{c_n(\alpha_s(m), \mu)}{m^n} O_n(\mu)$$

- $c_n(\alpha_s(m), \mu)$ - Wilson coeffs
- $O_n(\mu)$ - NRQCD operators

Predictions

- Predictive power of the theory relies on the knowledge of non-perturbative NRQCD matrix elements $\langle O_n(\mu) \rangle$.
- Matrix elements must be extracted from experiment but do not depend on the process.
- Since the matrix elements are universal, values extracted from one experiment can be used to make predictions for a completely different experiment.

NRQCD factorization

Factorized decay rates for $\chi_{c0,2} \to \gamma\gamma$ in NRQCD are given by (at leading order in v) [Bodwin et al., 1995]

$$\Gamma(\chi_{c0} \to \gamma\gamma) = \frac{2\mathrm{Im} f_{em}(^3P_0)}{3m^4} \langle \chi_{c0} | \chi^\dagger (-\tfrac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \boldsymbol{\sigma}) \psi | 0 \rangle \langle 0 | \psi^\dagger (-\tfrac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \boldsymbol{\sigma}) \chi | \chi_{c0} \rangle$$

$$\Gamma(\chi_{c2} \to \gamma\gamma) = \frac{2\mathrm{Im} f_{em}(^3P_2)}{m^4} \langle \chi_{c2} | \chi^\dagger (-\tfrac{i}{2} \overleftrightarrow{\mathbf{D}}^{(i} \boldsymbol{\sigma}^{j)}) \psi | 0 \rangle \langle 0 | \psi^\dagger (-\tfrac{i}{2} \overleftrightarrow{\mathbf{D}}^{(i} \boldsymbol{\sigma}^{j)}) \chi | \chi_{c2} \rangle .$$

- Pauli spinor field ψ (χ) annihilates (creates) a heavy quark (antiquark)
- $i\mathbf{D} = i\nabla + g\mathbf{A}$

Matching

- To determine the decay short distance coefficients, use the matching condition

$$A(Q\bar{Q} \to Q\bar{Q})|_{\text{pert. QCD}} = \sum_n \frac{f_n}{M^{d_n - 4}} \langle Q\bar{Q} | \mathcal{O}_n^{Q\bar{Q}} | Q\bar{Q} \rangle |_{\text{pert. NRQCD}}$$

- In general, we have $|\chi_{c0,2}\rangle = |Q\bar{Q}\rangle + \mathcal{O}(v) |Q\bar{Q}g\rangle + \ldots$
- Higher order operators start contributing through $|Q\bar{Q}g\rangle$ or even higher
- Read-off the matching coefficients by comparing the expressions on both sides
- Implies that we need to expand the QCD amplitude up to given order in $1/m$ and rewrite it in terms of Pauli spinors and 3-vectors.

Why not projectors?

- Covariant projector technique [Bodwin & Petrelli, 2002]: Project out contributions of different spin and color from the amplitude.
- Doesn't break covariant notation.
- Often employed in matching calculations that use FeynCalc
- However:
  - Need separate calculations for every angular momentum
  - Generalization of projectors to Fock states beyond $|Q\bar{Q}\rangle$ is not straight-forward
- Non-covariant matching in the spirit of [Bodwin et al., 1995] doesn't exhibit these problems
- Price to pay: Need to break covariant notation, thus more difficult to automatize.

Making non-covariant matching look "more covariant"

- Idea: Extend FeynCalc to automatize non-covariant matching in a simpler way.
- In the amplitude there are no free Cartesian indices.
- Eliminate Cartesian indices alltogether by introducing special Lorentz tensors.
- This approach is inspired by the threshold expansion method [Braaten & Chen, 1996]
- Introduce E-tensor to write scalar products of 3-vectors

$$E^{\mu\nu} = \begin{cases} 0 \text{ for } \mu = 0 \text{ or } \nu = 0, \\ \delta^{ij} \text{ for } \mu \neq 0 \text{ and } \nu \neq 0. \end{cases}$$

- Then,

$$x^i y^i = x^i y^j \delta^{ij} = E^{\mu\nu} x_\mu y_\nu \equiv E(x,y),$$
$$x^\mu y_\mu = x^0 y^0 + E(x,y).$$

- Introduce C-tensor to write vector products of 3-vectors

$$C_{\mu\nu\rho} = \begin{cases} 0 \text{ for } \mu = 0 \text{ or } \nu = 0 \text{ or } \rho = 0, \\ \varepsilon_{ijk} \text{ for } \mu \neq 0 \text{ and } \nu \neq 0 \text{ and } \rho \neq 0, \end{cases}$$

- So,

$$\varepsilon^{ijk} x^i y^j z^k = -\varepsilon_{ijk} x^i y^j z^k = -C_{\mu\nu\rho} x^\mu y^\nu z^\rho \equiv -C(x,y,z),$$
$$\varepsilon^{\sigma\mu\nu\rho} a_\sigma x_\mu y_\nu z_\rho = a^0 C(x,y,z) - x^0 C(a,y,z) + y^0 C(a,x,z) - z^0 C(a,x,y).$$

Why is it useful?

- Technically, we are not introducing anything new, just rewriting things in a different way.
- By completely eliminating Cartesian indices our task to automatize such computations simplifies a lot!
- Extending FeynCalc to work with tensors that carry Cartesian and mixed indices requires a lot of work.
- Adding a few Lorentz tensors with special properties is, however, quite easy.

So far

- Work in progress: Provide an add-on for FeynCalc for automatic QCD/NRQCD tree level (relativistic corrections) matching calculations.
- Successfully tested this approach to reproduce all color singlet matching coefficients of $\chi_{c_J} \to \gamma\gamma$ decays up to $\mathcal{O}(1/m^6)$.
  [Brambilla et al., 2006]
- Finishing reproducing color octet matching coefficients for the same process up to $\mathcal{O}(1/m^6)$ as well.
- Useful technique to obtain new matching coefficients for phenomenologically relevant processes.

Summary

- FeynCalc is a very flexible and versatile tool for semi-automatic calculations in QFT.
- FeynCalc 9 introduces several interesting features that facilitate 1-loop and multi-loop calculations.
- Several FeynCalc add-ons that provide direct interfaces to other useful HEP tools or facilitate calculations in non-relativistic theories are under active development.

Issues with γ^5

- There is no unique way to handle $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in DR
- In D-dimensions, the relations

$$\{\gamma^5, \gamma^\mu\} = 0$$

and

$$\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \neq 0$$

cannot be simultaneously satisfied.
- In other words, there is a conflict between the anticommutativity of γ^5 and the cyclicity property of Dirac traces that involve and odd number of γ^5

[Chanowitz et al., 1979]
[Jegerlehner, 2001]

Issues with γ^5

- We can stick to the anticommuting γ^5 in D-dimensions. This is fine, as long as we have only traces with an even number of γ^5.

  $\Rightarrow$ Naive dimensional regularization (NDR)

- To compute traces with an odd number of γ^5 unambiguously, we need an additional prescription

  $\Rightarrow$ Kreimer's prescription

  [Kreimer, 1990]

  $\Rightarrow$ Larin-Gorishny-Akyeampong-Delburgo prescription

  [Larin, 1993]

- Or we can accept that γ^5 is a purely 4-dimensional object and therefore doesn't anitcommute with D-dimensional Dirac matrices

  ['t Hooft & Veltman, 1972]
  [Breitenlohner & Maison, 1977]

  $\Rightarrow$ Breitenlohner-Maison- t'Hooft Veltman scheme (BMHV)

By default, FeynCalc works with an anticommuting γ^5

In[1]:= GAD[μ,ν,ρ].GA[5].GAD[σ,τ,κ].GA[5]
Out[1]= $\gamma^{\mu}.\gamma^{\nu}.\gamma^{\rho}.\bar{\gamma}^{5}.\gamma^{\sigma}.\gamma^{\tau}.\gamma^{\kappa}.\bar{\gamma}^{5}$
In[2]:= %//DiracSimplify
Out[2]= $-\gamma^{\mu}.\gamma^{\nu}.\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\tau}.\gamma^{\kappa}$

Trying to compute a chiral trace in the naive scheme produces an error message:

In[1]:= DiracTrace[GAD[μ,ν,ρ,σ,τ,κ].GA[5]]
Out[1]= tr $\left(\gamma^{\mu}.\gamma^{\nu}.\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\tau}.\gamma^{\kappa}.\bar{\gamma}^{5}\right)$
In[2]:= % /. DiracTrace $->$ **Tr**

DiracTrace::ndranomaly :
You are using naive dimensional regularization (NDR), such that in D dimensions gamma^5 anticommutes with all other Dirac matrices. In this scheme (without additional prescriptions) it is not possible to compute traces with an odd number of gamma^5 unambiguously. The trace

of DiracGamma[LorentzIndex[μ, D], D].
DiracGamma[LorentzIndex[ν, D], D].
DiracGamma[LorentzIndex[ρ, D], D].
DiracGamma[LorentzIndex[σ, D], D].
DiracGamma[LorentzIndex[τ, D], D].
DiracGamma[LorentzIndex[κ, D], D].DiracGamma[5]

is illegal in NDR. Evaluation aborted!

D-dimensional traces with anticommuting γ^5 can be evaluated using Larin-Gorishny-Akyeampong-Delburgo prescription

In[1]:= \$Larin = **True**;
In[2]:= \$West = **False**;
In[3]:= \$BreitMaison = **False**;
In[4]:= DiracTrace[GAD[μ, ν, ρ, σ, τ, κ].GA[5]]
Out[4]:= $\mathrm{tr}\left(\gamma^{\mu}.\gamma^{\nu}.\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\tau}.\gamma^{\kappa}.\bar{\gamma}^{5}\right)$
In[5]:= % /. DiracTrace -> **Tr**
Out[5]:= $4\left(ig^{\mu\nu}\epsilon^{\kappa\rho\sigma\tau} - ig^{\mu\rho}\epsilon^{\kappa\nu\sigma\tau} + ig^{\mu\sigma}\epsilon^{\kappa\nu\rho\tau} - ig^{\mu\tau}\epsilon^{\kappa\nu\rho\sigma} + ig^{\nu\rho}\epsilon^{\kappa\mu\sigma\tau} - ig^{\nu\sigma}\epsilon^{\kappa\mu\rho\tau} + ig^{\nu\tau}\epsilon^{\kappa\mu\rho\sigma} + ig^{\rho\sigma}\epsilon^{\kappa\mu\nu\tau} - ig^{\rho\tau}\epsilon^{\kappa\mu\nu\sigma} + ig^{\sigma\tau}\epsilon^{\kappa\mu\nu\rho}\right)$

Or in the BMHV scheme

In[6]:= \$Larin = **False**;
In[7]:= \$West = **True**;
In[8]:= \$BreitMaison = **False**;
In[9]:= DiracTrace[GAD[μ, ν, ρ, σ, τ, κ].GA[5]]
Out[9]:= $\mathrm{tr}\left(\gamma^{\mu}.\gamma^{\nu}.\gamma^{\rho}.\gamma^{\sigma}.\gamma^{\tau}.\gamma^{\kappa}.\bar{\gamma}^{5}\right)$
In[10]:= % /. DiracTrace -> **Tr**
Out[10]:= $4\left(-ig^{\kappa\mu}\epsilon^{\nu\rho\sigma\tau} + ig^{\kappa\nu}\epsilon^{\mu\rho\sigma\tau} - ig^{\kappa\rho}\epsilon^{\mu\nu\sigma\tau} + ig^{\kappa\sigma}\epsilon^{\mu\nu\rho\tau} - ig^{\kappa\tau}\epsilon^{\mu\nu\rho\sigma} + ig^{\mu\nu}\epsilon^{\kappa\rho\sigma\tau} - ig^{\mu\rho}\epsilon^{\kappa\nu\sigma\tau} + ig^{\mu\sigma}\epsilon^{\kappa\nu\rho\tau} - ig^{\mu\tau}\epsilon^{\kappa\nu\rho\sigma} + ig^{\nu\rho}\epsilon^{\kappa\mu\sigma\tau} - ig^{\nu\sigma}\epsilon^{\kappa\mu\rho\tau} + ig^{\nu\tau}\epsilon^{\kappa\mu\rho\sigma} + ig^{\rho\sigma}\epsilon^{\kappa\mu\nu\tau} - ig^{\rho\tau}\epsilon^{\kappa\mu\nu\sigma} + ig^{\sigma\tau}\epsilon^{\kappa\mu\nu\rho}\right)$

Issues with γ^5 in NDR

Assuming that both

$$\{\gamma^5, \gamma^\mu\} = 0,$$
$$\mathrm{Tr}\{\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\} \neq 0$$

hold in D-dimensions leads to a contradiction. The reason is the assumed cyclicity of the Dirac trace

$$\begin{aligned}
D\,\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) &= \mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\tau \gamma^\tau) \\
&= -2 g^{\tau\mu}\,\mathrm{Tr}(\gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\tau) + 2 g^{\tau\nu}\,\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\rho \gamma^\sigma \gamma_\tau) \\
&- 2 g^{\tau\rho}\,\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma \gamma_\tau) + 2 g^{\tau\sigma}\,\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\tau) \\
&- D\,\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) \\
&= -2\,\mathrm{Tr}(\gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu) + 2\,\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\rho \gamma^\sigma \gamma^\nu) \\
&- 2\,\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho) + 2\,\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) \\
&- D\,\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma)
\end{aligned}$$

Issues with γ^5 in NDR

Using that $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$ we have

$$D\,\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = (8-D)\,\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$$

or

$$(4-D)\,\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 0.$$

- This implies that $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$ is zero for all $D \neq 4$.
- But if we demand the trace to be meromorphic in D, then the above trace should be zero also for $D = 4$,
- Hence, we cannot recover the 4-dimensional Dirac algebra at $D = 4$.

Non-naive scheme for γ^5

In the Breitenlohner-Maison-'t Hooft-Veltman scheme we are dealing with matrices in D, 4 and D-4 dimensions. Many identities of the BMHV algebra can be proven by by decomposing Dirac matrices into two pieces

$$\begin{aligned} \dim(\gamma^\mu) &= d, & \gamma^\mu &= \bar{\gamma}^\mu + \hat{\gamma}^\mu, \\ \dim(\bar{\gamma}^\mu) &= 4, & g^{\mu\nu} &= \bar{g}^{\mu\nu} + \hat{g}^{\mu\nu}, \\ \dim(\hat{\gamma}^\mu) &= d-4. & p^\mu &= \bar{p}^\mu + \hat{p}^\mu. \end{aligned}$$

(Anti)commuatators between γ's in different dimensions

$$\begin{aligned} &\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \\ &\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = \{\gamma^\mu, \bar{\gamma}^\nu\} = 2\bar{g}^{\mu\nu}, \\ &\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = \{\gamma^\mu, \hat{\gamma}^\nu\} = 2\hat{g}^{\mu\nu}, \\ &\{\bar{\gamma}^\mu, \hat{\gamma}^\nu\} = 0 \\ &\{\bar{\gamma}^\mu, \gamma^5\} = [\hat{\gamma}^\mu, \gamma^5] = 0, \\ &\{\gamma^\mu, \gamma^5\} = \{\hat{\gamma}^\mu, \gamma^5\} = 2\hat{\gamma}^\mu \gamma^5 = 2\gamma^5 \hat{\gamma}^\mu. \end{aligned}$$

Non-naive scheme for γ^5

Contractions of Dirac matrices and vectors with the metric

$$g^{\mu\nu}\gamma_\nu = \gamma^\mu, \qquad g^{\mu\nu}p_\nu = p^\mu,$$
$$\bar{g}^{\mu\nu}\bar{\gamma}_\nu = g^{\mu\nu}\bar{\gamma}_\nu = \bar{g}^{\mu\nu}\gamma_\nu = \bar{\gamma}^\mu, \qquad \bar{g}^{\mu\nu}\bar{p}_\nu = g^{\mu\nu}\bar{p}_\nu = \bar{g}^{\mu\nu}p_\nu = \bar{p}^\mu,$$
$$\hat{g}^{\mu\nu}\hat{\gamma}_\nu = g^{\mu\nu}\hat{\gamma}_\nu = \hat{g}^{\mu\nu}\gamma_\nu = \hat{\gamma}^\mu, \qquad \hat{g}^{\mu\nu}\hat{p}_\nu = g^{\mu\nu}\hat{p}_\nu = \hat{g}^{\mu\nu}p_\nu = \hat{p}^\mu,$$
$$\bar{g}^{\mu\nu}\hat{\gamma}_\nu = \hat{g}^{\mu\nu}\bar{\gamma}_\nu = 0, \qquad \bar{g}^{\mu\nu}\hat{p}_\nu = \hat{g}^{\mu\nu}\bar{p}_\nu = 0.$$

Contractions of the metric with itself

$$g^{\mu\nu}g_{\nu\rho} = g^\mu_\rho \qquad g^{\mu\nu}g_{\mu\nu} = d,$$
$$\bar{g}^{\mu\nu}\bar{g}_{\nu\rho} = g^{\mu\nu}\bar{g}_{\nu\rho} = \bar{g}^{\mu\nu}g_{\nu\rho} = \bar{g}^\mu_\rho \qquad \bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = g^{\mu\nu}\bar{g}_{\mu\nu} = \bar{g}^{\mu\nu}g_{\mu\nu} = 4,$$
$$\hat{g}^{\mu\nu}\hat{g}_{\nu\rho} = g^{\mu\nu}\hat{g}_{\nu\rho} = \hat{g}^{\mu\nu}g_{\nu\rho} = \hat{g}^\mu_\rho \qquad \hat{g}^{\mu\nu}\hat{g}_{\mu\nu} = g^{\mu\nu}\hat{g}_{\mu\nu} = \hat{g}^{\mu\nu}g_{\mu\nu} = d-4,$$
$$\bar{g}^{\mu\nu}\hat{g}_{\nu\rho} = \hat{g}^{\mu\nu}\bar{g}_{\nu\rho} = 0, \qquad \bar{g}^{\mu\nu}\hat{g}_{\mu\nu} = \hat{g}^{\mu\nu}\bar{g}_{\mu\nu} = 0.$$

Contractions of Dirac matrices and vectors with themselves

$$\gamma^\mu\gamma_\mu = D, \qquad p^\mu p_\mu = p^2,$$
$$\bar{\gamma}^\mu\bar{\gamma}_\mu = \gamma^\mu\bar{\gamma}_\mu = \bar{\gamma}^\mu\gamma_\mu = 4, \qquad \bar{p}^\mu\bar{p}_\mu = \bar{p}^\mu p_\mu = p^\mu\bar{p}_\mu = \bar{p}^2,$$
$$\hat{\gamma}^\mu\hat{\gamma}_\mu = \gamma^\mu\hat{\gamma}_\mu = \hat{\gamma}^\mu\gamma_\mu = D-4, \qquad \hat{p}^\mu\hat{p}_\mu = \hat{p}^\mu p_\mu = p^\mu\hat{p}_\mu = \hat{p}^2,$$
$$\bar{\gamma}^\mu\hat{\gamma}_\mu = \hat{\gamma}^\mu\bar{\gamma}_\mu = 0, \qquad \bar{p}^\mu\hat{p}_\mu = \hat{p}^\mu\bar{p}_\mu = 0.$$

Larin-Gorishny-Akyeampong-Delburgo prescription allows one to use anticommuting γ^5 in D-dimensions but compute the chiral traces, such, that the result is expected to be equivalent with the BMHV scheme, if we have only one axial-vector current. The prescription is essentially

- Anticommute γ^5 to the right inside the trace
- Replace $\gamma^\mu \gamma^5$ with $-\frac{i}{6} \varepsilon^{\mu\alpha\beta\sigma} \gamma^\alpha \gamma^\beta \gamma^\sigma$
- Treat $\varepsilon^{\mu\alpha\beta\sigma}$ as if it were D-dimensional, i.e. $\varepsilon^{\mu\alpha\beta\sigma} \varepsilon_{\mu\alpha\beta\sigma} = -D(D^3 - 6D^2 + 11D - 6)$ instead of -24.

Schouten's identity

In an n-dimensional space, a totally antisymmetric tensor with $n+1$ indices vanishes. For example, $e^{ijkl} = 0$ if i,j,k and l are Cartesian indices that run from 1 to 3, because no matter how you choose the values of the indices, you will always have at least two indices with the same value.

- 4D space

$$\varepsilon^{\mu\nu\rho\sigma} p^{\tau} + \varepsilon^{\nu\rho\sigma\tau} p^{\mu} + \varepsilon^{\rho\sigma\tau\mu} p^{\nu} + \varepsilon^{\sigma\tau\mu\nu} p^{\rho} + \varepsilon^{\tau\mu\nu\rho} p^{\sigma} = 0$$

$$\varepsilon^{\mu\nu\rho\sigma} g^{\tau\kappa} + \varepsilon^{\nu\rho\sigma\tau} g^{\mu\kappa} + \varepsilon^{\rho\sigma\tau\mu} g^{\nu\kappa} + \varepsilon^{\sigma\tau\mu\nu} g^{\rho\kappa} + \varepsilon^{\tau\mu\nu\rho} g^{\sigma\kappa} = 0$$

- 3D space

$$\varepsilon^{ijk} p^{l} - \varepsilon^{jkl} p^{i} + \varepsilon^{klj} p^{j} - \varepsilon^{lij} p^{k} = 0$$

$$\varepsilon^{ijk} g^{lm} - \varepsilon^{jkl} g^{im} + \varepsilon^{kli} g^{jm} - \varepsilon^{lij} g^{km} = 0$$

Definitions of the PaVe scalar integrals (LoopTools convention)

$$A_0(m_0) = \mu^{4-D}(4\pi)^{\frac{4-D}{2}} \int \frac{d^D q}{i\pi^{\frac{D}{2}}} \frac{1}{q^2 - m_0^2}$$

$$B_0(p_1, m_0, m_1) = \mu^{4-D}(4\pi)^{\frac{4-D}{2}} \int \frac{d^D q}{i\pi^{\frac{D}{2}}} \frac{1}{(q^2 - m_0^2)((q+p_1)^2 - m_1^2)}$$

$$C_0(p_1, p_2, m_0, m_1, m_2)$$

$$= \mu^{4-D}(4\pi)^{\frac{4-D}{2}} \int \frac{d^D q}{i\pi^{\frac{D}{2}}} \frac{1}{(q^2 - m_0^2)((q+p_1)^2 - m_1^2)((q+p_1+p_2)^2 - m_2^2)}$$

$$D_0(p_1, p_2, p3, m_0, m_1, m_2, m_3)$$

$$= \mu^{4-D}(4\pi)^{\frac{4-D}{2}} \int \frac{d^D q}{i\pi^{\frac{D}{2}}} \frac{1}{(q^2 - m_0^2)((q+p_1)^2 - m_1^2)((q+p_1+p_2)^2 - m_2^2)}$$

$$\times \frac{1}{((q+p_1+p_2+p_3)^2 - m_3^2)}$$

Passarino-Veltman scalar functions are normally related to the text book integrals by a factor of $(16\pi^2)/i$, e.g.

$$\mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 - m_0^2} = \frac{i}{16\pi^2} A_0(m_0)$$

To see this observe that

$$\frac{1}{(2\pi)^D} = \frac{1}{16\pi^2} \frac{1}{2^{D-4}\pi^{D-2}} = \frac{1}{16\pi^2} \frac{4^{\frac{4-D}{2}}}{\pi^{D-2}} = \frac{1}{16\pi^2} \frac{(4\pi)^{\frac{4-D}{2}}}{\pi^{\frac{D}{2}}}$$

However, in FeynCalc the PaVe functions are normalized as $\mu^{4-D} \frac{1}{i\pi^2} \int d^D q (\ldots)$. Hence, we have

$$A_{0,FC}(m_0) = (2\pi)^{D-4} A_0(m_0) = \frac{(2\pi)^D}{i\pi^2} \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 - m_0^2}$$

On the other hand, if the prefactor $\frac{1}{(2\pi)^D}$ is implicit (i.e. it is understood but not written down explicitly) in the calculation, then it is enough to perform the replacement

$$A_{0,FC}(m_0) \to \frac{1}{i\pi^2} \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 - m_0^2}$$

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