Simulation and Analysis of Learjet C-21 Response Characteristics

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1/ Abstract:

From the parameters and stability derivatives data obtained from wind tunnel testing of the Learjet C-21. An engineer is asked to write a detail report that describes the dynamic characteristics of the aircraft. This report should contain enough information so that it can be used by the control system team and the design team. The detail calculation and explanation in this report confirm that the damping ratio for both the phugoid and the short period mode of this aircraft are in between -1 and 1. This proves that the aircraft is dynamically stable.

Introduction:

Longitudinal static stability is the first and most important criterion in designing an aircraft. The second important factor in designing an aircraft is dynamic stability. This is an important factor because it give the designers insight of aircraft handling qualities and design features that make the aircraft fly well or not as well while performing specific mission tasks. In the calculation below, we are using the parameters and stability derivatives data of the aircraft to verify that our aircraft is dynamically stable. Detail calculation is shown in the discussion part.

2/ Results:

Figure 2.1 below, show the result for the three longitudinal transfer functions, obtained using Matlab and the given data from Table 2.2 for the Learjet C-21. Please note that data from Table 3.2 is recorded from wind tunnel testing and analyzing the static stability of the aircraft.

$$\frac{\mu(s)}{\delta_e(s)} = \frac{-0.5211s^2 + 59.19s + 53.55}{s^4 + 1.786s^3 + 2.563s^2 + 0.1819s + 0.1362}$$

$$\frac{\alpha(s)}{\delta_e(s)} = \frac{-0.04598s^3 - 2.919s^2 - 0.1718s - 0.2078}{s^4 + 1.786s^3 + 2.563s^2 + 0.1819s + 0.1362}$$

$$\frac{\theta(s)}{\delta_e(s)} = \frac{-2.865s^2 - 1.832s - 0.1711}{s^4 + 1.786s^3 + 2.563s^2 + 0.1819s + 0.1362}$$

Figure 2.1 Three Transfer Functions.

Derivation process:

The three longitudinal transfer functions shown in figure 2.1 are derived from solving the longitudinal linearized EOM in Laplace form using the data found in table 2.2.

The longitudinal linearized EOM equation is listed as:

$$\underbrace{\begin{bmatrix} (s - X_u - X_{T_u}) & -X_\alpha & g\cos\Theta_1 \\ -Z_u & [s(U_1 - Z_{\dot\alpha}) - Z_\alpha] & [-(Z_q + U_1)s + g\sin\Theta_1] \\ -(M_u + M_{T_u}) & -[M_{\dot\alpha}s + M_\alpha + M_{T_\alpha}] & (s^2 - M_q s) \end{bmatrix}}_{\textbf{Matrix A}} \underbrace{\begin{bmatrix} \frac{u(s)}{\delta_e(s)} \\ \frac{\alpha(s)}{\delta_e(s)} \\ \frac{\theta(s)}{\delta_e(s)} \end{bmatrix}}_{\textbf{Matrix TF}} = \underbrace{\begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \end{bmatrix}}_{\textbf{Matrix b}}$$

In order to solve for the transfer functions in Matlab, first step is to load all the parameters according to table 2.2 into matrix A and matrix b. Then set **TF = inv(A) * b** and cancel the repeated zeros and poles by inputting **TF = mineral(TF)**. After typing this procedure into Matlab, the transfer functions as shown in figure 2.1 are obtained.

Parameter (unit)	Value	Parameter (unit)	Value
Airspeed V_∞ (ft/s)	170	Dynamic pressure $\bar{q}$ (lb/ft^2)	34.3
Weight W (lb)	13,000	Wing area S (ft^2)	230
Wing span b (ft)	34.0	Wing chord $\bar{c}$ (ft)	7.0
c.g. ($\bar{c}$)	0.32	Trim AOA α_{trim} (deg)	5.0
I_{xx} (slug-ft^2)	2.79*10^4	I_{yy} (slug-ft^2)	1.88*10^4
I_{zz} (slug-ft^2)	4.11*10^4	I_{xz} (slug-ft^2)	-3.60*10^2
X_u (1/s)	-0.0589	X_α (ft/s^2)	11.3335
Z_u (1/s)	-0.3816	Z_α (ft/s^2)	-103.4862
Z_{δ_e} (ft/s^2)	-7.8162	M_α (1/s^2)	-1.9387
$M_{\dot{\alpha}}$ (1/s)	-0.3024	M_q (1/s)	-0.8164
M_{δ_e} (1/s^2)	-2.8786	M_u (1/ft.s)	-0.0002

Table 2.2: Learjet C-21 aircraft parameter and stability derivatives. If a stability parameter is not listed in this table, assume it is zero.

3/ Discussion:

Model characteristics Analysis

From the three transfer functions obtained from the result part. It is clear that all of the equations have the same denominator which is also our characteristic equation. This is an indication that the transfer functions are correct because characteristic equations determine the dynamic stability characteristics of the response and therefore the 3 transfer functions will have the same dynamic characteristic. The numerators of the transfer functions affect the magnitude of the response therefore each motion will have a different magnitude of response.

In order to identify the phugoid and short period mode, the characteristic equations are set to be equal to zero and solve for the roots. The results are 2 pair of complex roots. We can identify the characteristic equation with the higher natural frequency as the short period mode characteristic equation. The equation with the lower natural frequency is identified as the phugoid mode.

Characteristic equation

$$s^4 + 1.816s^3 + 2.589s^2 + 0.1833s + 0.136$$

Set the characteristic equation equal to zero to get the roots

$$s_{1,2} = -0.8760 \pm 1.2959\, i$$

$$s_{3,4} = -0.0172 \pm 0.2353\, i$$

The following formulas are used to calculate the phugoid and short period mode damping frequency, natural frequency and damping ratio.

Since the real part of the first set of roots is smaller we identify it as the short period mode

$$s_{1,2} = \zeta_{sp}\omega_{Nsp} \pm i\omega_{Dsp}$$

$$s_{3,4} = \zeta_{ph}\omega_{Nph} \pm i\omega_{Dph}$$

In this case

$$\zeta_{sp}\,\omega_{Nsp} = -0.8760 \text{ and } \omega_{Dsp} = 1.2959$$

$$\zeta_{ph}\,\omega_{Nph} = -0.0172 \text{ and } \omega_{Dph} = 0.2353$$

$$\omega_{Nsp} = \sqrt{\left(-\zeta\omega_{N}\right)^{2}{}_{sp} + \left(\omega_{Dsp}\right)^{2}}$$

$$\omega_{Nph} = \sqrt{\left(-\zeta\omega_{N}\right)^{2}{}_{ph} + \left(\omega_{Dph}\right)^{2}}$$

$$\zeta_{sp} = \frac{\zeta_{sp}\,\omega_{Nsp}}{\omega_{Nsp}}$$

$$\zeta_{ph} = \frac{\zeta_{ph}\,\omega_{Nph}}{\omega_{Nph}}$$

$$\tau_{sp} = \frac{1}{\zeta_{sp}\,\omega_{Nsp}}$$

$$\tau_{ph} = \frac{1}{\zeta_{ph}\,\omega_{Nph}}$$

The following result is obtained

Phugoid Mode	**Short period mode**
ω_{Nph} = 0.2359 rad/s	**ω_{Nsp} = 1.5642 rad/s**
ξ_{ph} = 0.0731	**ξ_{sp} = 0.5600**
τ_{ph} = 57.9897 s	**τ_{sp} = 1.1416 s**

Time Response

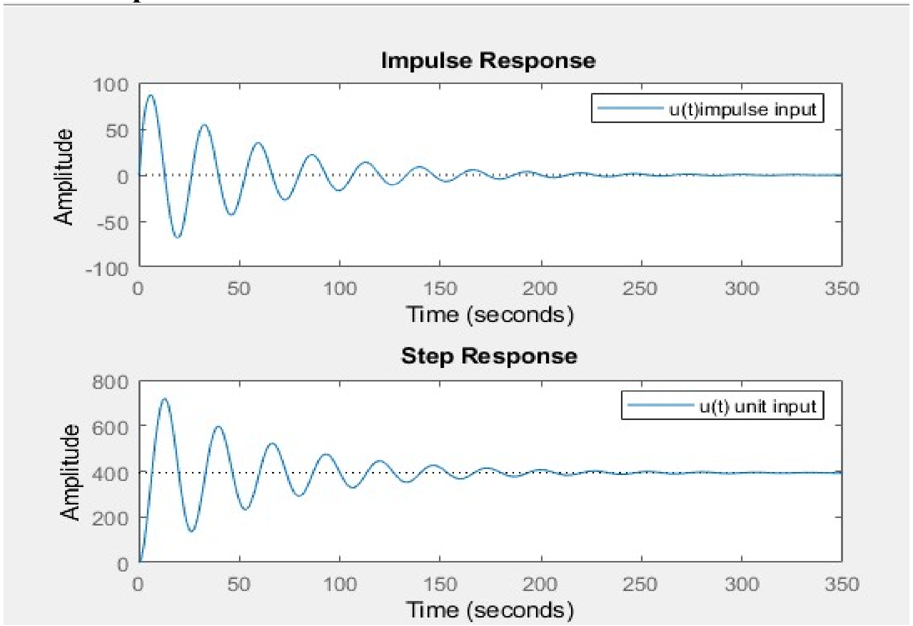

Figure 3.1: Impulse and Step response of u

For $\frac{u(s)}{\delta_e(s)}$

With impulse input there was oscillation from 0 to 250s. Then the oscillation is damp out after 250^{th} second.

With step input there was oscillation from 0 to 250s. Then the oscillation is damp out after 250^{th} second.

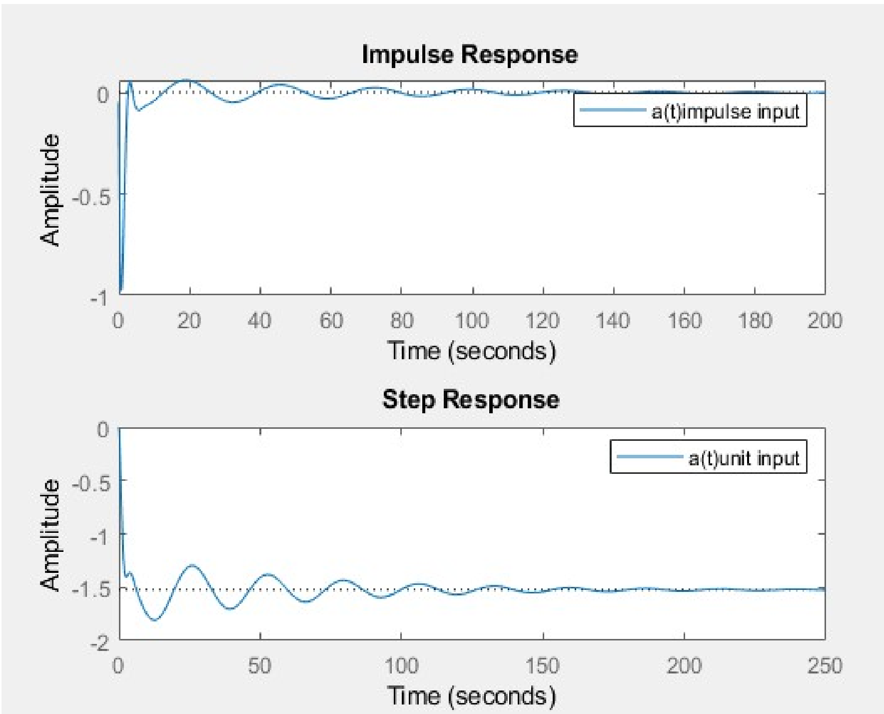

Figure 3.2: Impulse and step response of α

For $\frac{\alpha(s)}{\delta_e(s)}$

With impulse input there was oscillation from 0 to 200s. Then the oscillation is damp out after 200^{th} second.

With step input there was oscillation from 0 to 200s. Then the oscillation is damp out after 200^{th} second.

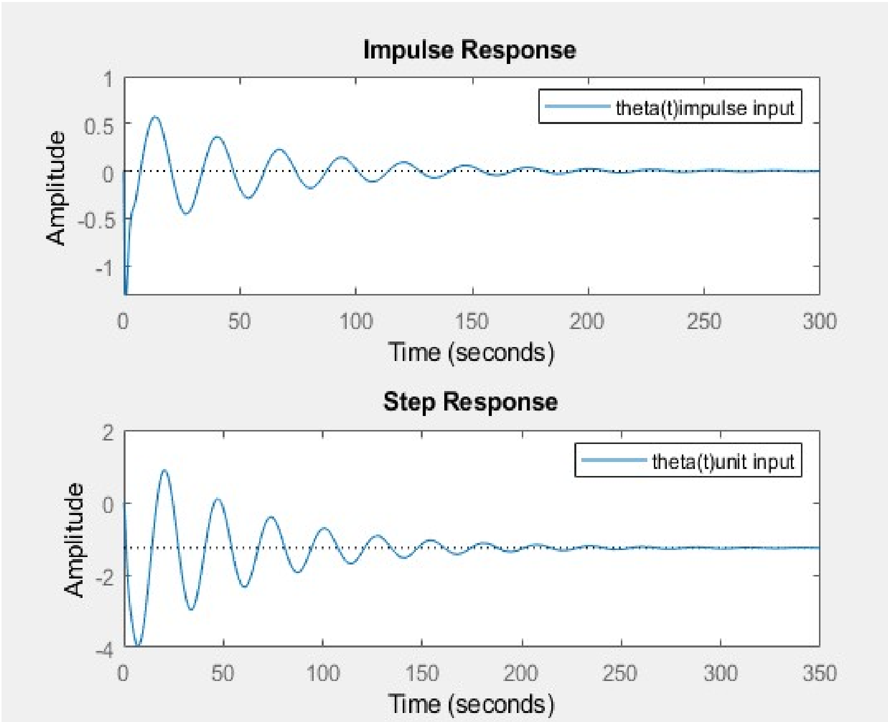

Figure 3.3: Impulse and Step response of θ

For $\frac{\theta(s)}{\delta_e(s)}$

With impulse input there was oscillation from 0 to 250s. Then the oscillation is damp out after 250th second.

With step input there was oscillation from 0 to 250s. Then the oscillation is damp out after 250th second.

Reduced Order Analysis

Using the characteristic equation of the transfer function of two-degrees-of-freedom short period approximation and data from table 3.2.

$$s^2 - (M_q + \frac{Z_\alpha}{U_1} + M_{\dot{\alpha}})s + (\frac{Z_\alpha M_q}{U_1} - M_\alpha) = 0$$

Characteristic equation

$s^2 + 1.7275s + 2.4357$

Set the characteristic equation equal to zero to get the roots

$s_1 = -0.8638 + 1.2998\,i$

$s_1 = -0.8638 - 1.2998\,i$

Short period mode (Reduced Order Analysis)	Short period mode (from Model Characteristic)	% Different
ω_{Nsp} = 1.5607 rad/s	ω_{Nsp} = 1.5642 rad/s	0.22 %
ξ_{sp} = 0.5535	ξ_{sp} = 0.5600	1.17 %

Comment: In comparison to the Model Characteristic Analysis, the Reduced Order Analysis gave accurate results within the range of 0.22% to 1.17% error. The reduced order analysis approximation is accurate. This can also be proved through the graphical results bellow. It is much faster and convenient to use reduced order analysis since its equation are simplified (assume u remains near zero and eliminate x-force equation) does not require Matlab to solve for the damping ration and natural frequency.

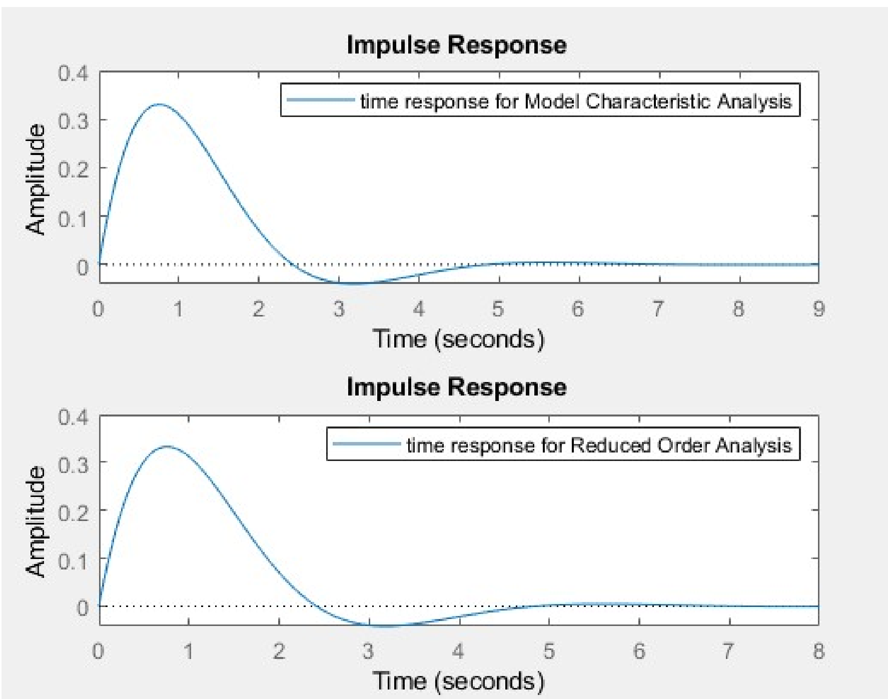

Figure 3.4: Time response for unit impulse input for short period mode found in model characteristic analysis and reduced order analysis.

Description: there was oscillation at the beginning, but the oscillation is damp from the 7th second.

Handling Qualities Investigation

	Minimum	Maximum
Level 1	0.30	2.00
Level 2	0.20	2.00
Level 3	0.15	No maximum

	Requirement
Level 1	$\zeta_{PH} > 0.04$
Level 2	$\zeta_{PH} > 0$
Level 3	$T_2 > 55\text{s}$

Since the damping ratio for the short period mode and phugoid mode are

$\xi_{sp} = 0.5600$

$\xi_{ph} = 0.0731$

Both of the equations are second order **stable** system (real part of the roots<0). The time to double T_2 does not exist.

The aircraft is flying at constant speed of v = 170 ft/s. and at trim condition. Therefore the handling qualities level for the aircraft **is level 1, category B cruising**.

Comment: In this case, since both of the damping ratio for the phugoid and the short period mode are in between -1 and 1, both modes are stable. Therefore, it can be said that this aircraft has dynamic stability. The aircraft's flying qualities clearly adequate for the mission flight phase.

4/ Conclusion:

Overall, from the data obtained from the wind tunnel testing and static stability analysis in table 2.2 and the longitudinal linearized EOM in Laplace form, engineers can derive and plot the three transfer functions for the Learjet C-21. Then from the three transfer functions obtained, the damping ratio, natural frequency and damping frequency for the short period mode and phugoid mode of the aircraft can be calculated. The calculated damping ratio and natural frequency for the short period mode from using the longitudinal linearized EOM in Laplace form was also double checked by using Reduced Order Analysis. Those two results were identical with very small percentage different (from 0.22% to 1.17%). Lastly, the damping ratio for both the phugoid and short period mode are 0.5600 and 0.0731 respectively. Since both damping ratios are in between -1 and 1, the phugoid and short period mode are both stable. The aircraft is flying at constant speed of v = 170 ft/s. and at trim condition. Therefore, the handling qualities level for the aircraft is level 1, category B cruising. Overall, it can be said that this aircraft has dynamic stability. The aircraft's flying qualities clearly adequate for the mission flight phase.

Reference: Introduction to Aircraft Flight Mechanics. Thomas R. Yechout.