1/ Introduction

In this lab report, we investigate the response of a motorcycle and rider system as it travels on a road with two bumps. The bumps are triangular shape, each with a height A and horizontal distance L, and the bumps are separated by a distance L_{wb}. The tires of the motorcycle have mass and stiffness, so they can be modeled as a mass-spring system. There are two suspension systems for the motorcycle, each being modeled as a spring and a damper in parallel. In this lab, we model the motorcycle and rider as a single rigid body with a mass m_{cr} and a moment of inertia J_{cr}.
We want to determine how the two-bump road input affects the response of the motorcycle system. Plots will be made showing the time-varying response of the front and rear suspension deflections, the heave velocity (vertical velocity of the rider and motorcycle), and the pitch angular velocity (angular velocity of the rider and motorcycle). To do this, we will define the state equations, inputs, outputs, and the initial equilibrium conditions.

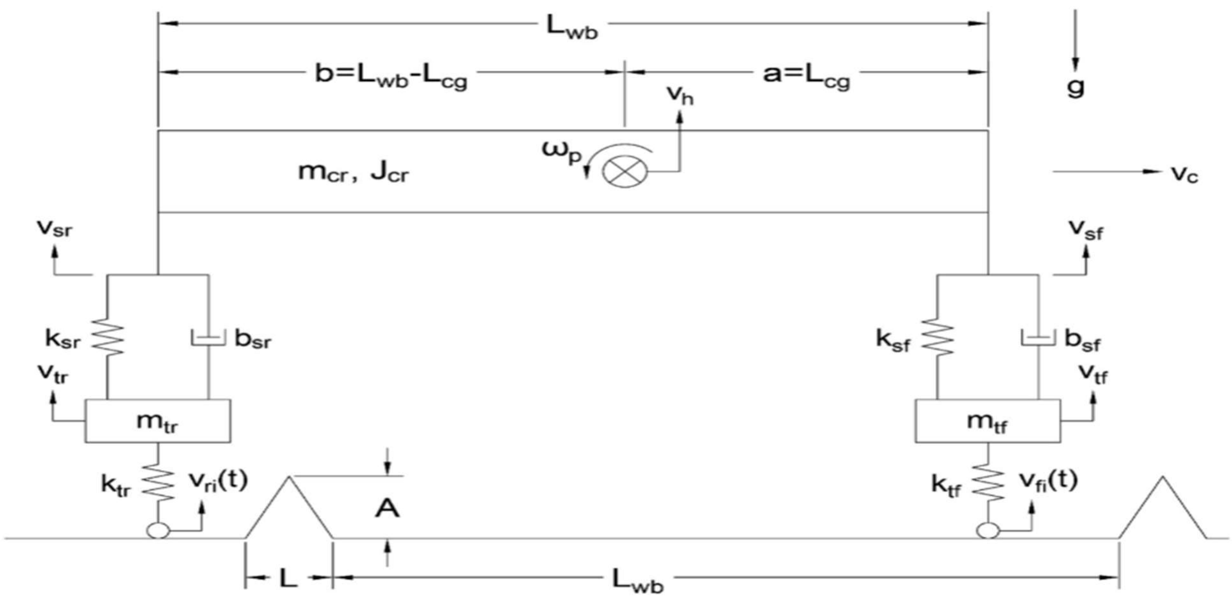


Fig 1: Motocross motorcycle pitch-heave model equivalent bond graph (right)

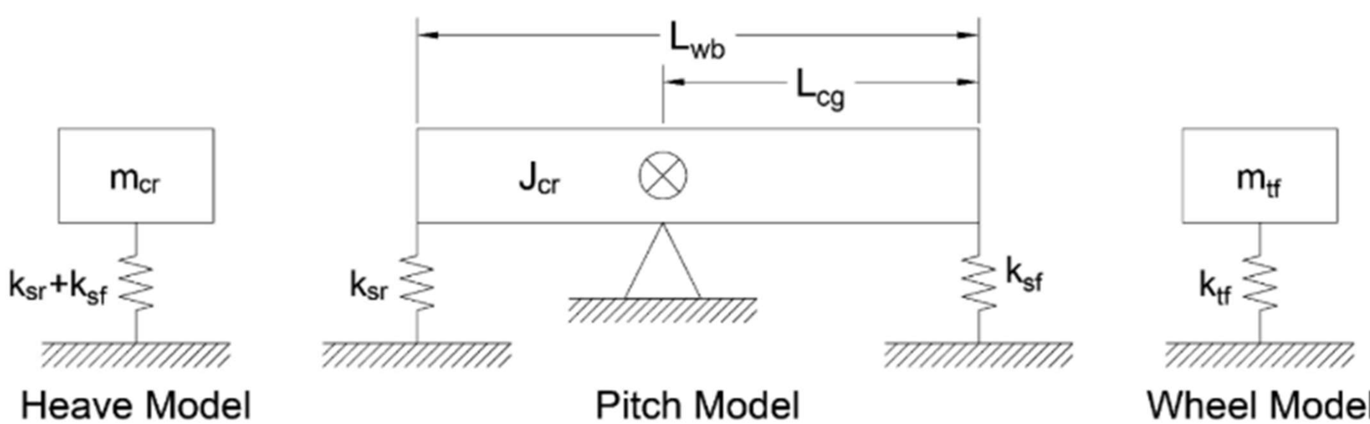


Fig 2: Three independent systems for natural frequencies calculation purpose

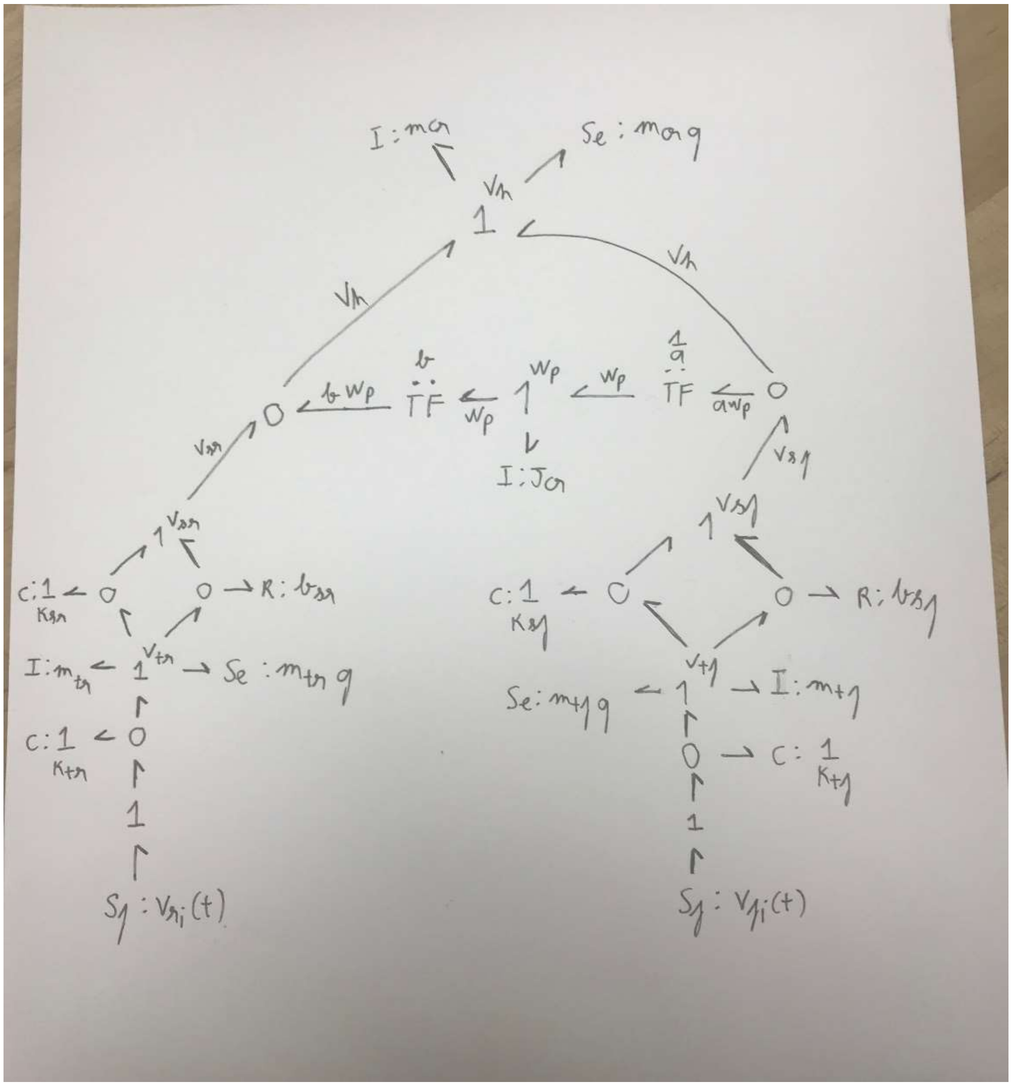


Fig 3: Bond graph of the system

There are 8 equations of state for the system, which are listed below. The equations are solved explicitly for the time derivatives of momentums and displacements.

$$\dot{p}_J = a\left[q_{sf}k_{sf} + b_{sf}\left(\frac{p_{tf}}{m_{tf}} - \frac{p_{cr}}{m_{cr}} - a\frac{p_J}{J_{cr}}\right)\right] - b\left[q_{sr}k_{sr} + b_{sr}\left(\frac{p_{tr}}{m_{tr}} - \frac{p_{cr}}{m_{cr}} + b\frac{p_J}{J_{cr}}\right)\right]$$

$$\dot{p}_{cr} = -m_{cr}g + q_{sf}k_{sf} + b_{sf}\left(\frac{p_{tf}}{m_{tf}} - \frac{p_{cr}}{m_{cr}} - a\frac{p_J}{J_{cr}}\right) + q_{sr}k_{sr} + b_{sr}\left(\frac{p_{tr}}{m_{tr}} - \frac{p_{cr}}{m_{cr}} + b\frac{p_J}{J_{cr}}\right)$$

$$\dot{q}_{sf} = \frac{p_{tf}}{m_{tf}} - \frac{p_{cr}}{m_{cr}} - a\frac{p_J}{J_{cr}}$$

$$\dot{q}_{sr} = \frac{p_{tr}}{m_{tr}} - \frac{p_{cr}}{m_{cr}} + b\frac{p_J}{J_{cr}}$$

$$\dot{p}_{tf} = q_{tf}k_{tf} - m_{tf}g - q_{sf}k_{sf} - b_{sf}\left(\frac{p_{tf}}{m_{tf}} - \frac{p_{cr}}{m_{cr}} - a\frac{p_J}{J_{cr}}\right)$$

$$\dot{p}_{tr} = q_{tr}k_{tr} - m_{tr}g - q_{sr}k_{sr} - b_{sr}\left(\frac{p_{tr}}{m_{tr}} - \frac{p_{cr}}{m_{cr}} + b\frac{p_J}{J_{cr}}\right)$$

$$\dot{q}_{tf} = v_{fi}(t) - \frac{p_{tf}}{m_{tf}}$$

$$\dot{q}_{tr} = v_{ri}(t) - \frac{p_{tr}}{m_{tr}}$$

Figure 1: System equations of state

The variables in the above equations are listed below along with a description of what they represent. There are 8 state variables in total.

p_J	Pitch angular momentum
p_{cr}	Vertical momentum of motorcycle and rider
q_{sf}	Front suspension spring displacement
q_{sr}	Rear suspension spring displacement
p_{tf}	The momentum of front tire mass
p_{tr}	The momentum of rear tire mass
q_{tf}	Front tire deflection
q_{tr}	Rear tire deflection

Table 1: State variables

2/ Calculations

Constant Parameters

There are also 20 constant parameters defined in the system. They are listed below:

v_c = 10 m/s	forward velocity of motorcycle
L_{cg} = 0.9 m	center of gravity distance in standard configuration
L_{cg} = 0.7 m	center of gravity distance in forward configuration
m_{cr} = 300 kg	combined mass of rider and cycle
r_{gy} = 0.5 m	body radius of gyration
k_{sf} = 3000 N/m	front suspension stiffness
k_{sr} = 3500 N/m	rear suspension stiffness
b_{sf} = 400 N*s/m	front damping coefficient
b_{sr} = 500 N*s/m	rear damping coefficient
m_{tf} = 15 kg	front tire un-sprung mass
m_{tr} = 20 kg	rear tire un-sprung mass
k_{tf} = 30000 N/m	front tire stiffness
k_{tr} = 40000 N/m	rear tire stiffness
L_{wb} = 1.6 m	distance between wheel bases
A	bump height
L = 0.5 m	bump distance
g = 9.81 m/s^2	acceleration due to gravity
δ_{max} = 0.1 m	maximum deflection of suspension
a	a = L_{cg}
b	b = $L_{wb} - L_{cg}$

Table 2: Constant Parameters

Initial Conditions

The initial conditions of each state at equilibrium must also be calculated. We can use the fact that at equilibrium, all state derivatives (derivatives of momentum and displacement) are equal to zero. The velocities of the system are also zero at equilibrium, which means all initial momentums are zero as well. Using this information, we can solve the system of equations and find the equilibrium values of the displacements. The equilibrium values are boxed below.

$$0 = a(q_{sf} * k_{sf}) - b(q_{sr} * k_{sr})$$

$$a * q_{sf} * k_{sf} = b * q_{sr} * k_{sr}$$

$$q_{sf} = \frac{b * q_{sr} * k_{sr}}{a * k_{sf}} \quad (1)$$

$$0 = -m_{cr}g + q_{sf} * k_{sf} + q_{sr} * k_{sr}$$
$$q_{sf} * k_{sf} + q_{sr} * k_{sr} = m_{cr}g \quad (2)$$

$$0 = q_{tf} * k_{tf} - m_{tf}g - q_{sf} * k_{sf}$$
$$q_{tf} * k_{tf} - q_{sf} * k_{sf} = m_{tf}g \quad (3)$$

$$0 = q_{tr} * k_{tr} - m_{tr}g - q_{sr} * k_{sr}$$
$$q_{tr} * k_{tr} - q_{sr} * k_{sr} = m_{tr}g \quad (4)$$

We can solve for q_{sr} by plugging in q_{sf} from equation (1) into equation (2):

$$\frac{b * q_{sr} * k_{sr}}{a * k_{sf}} * k_{sf} + q_{sr} * k_{sr} = m_{cr}g$$
$$q_{sr} * k_{sr}\left(\frac{b + a}{a}\right) = m_{cr}g$$
$$\boxed{q_{sr} = \frac{m_{cr}ga}{k_{sr}(a + b)}}$$

Now, going back to equation (1), we can solve for q_{sf}:

$$q_{sf} = \frac{b * q_{sr} * k_{sr}}{a * k_{sf}} = \frac{b * \frac{m_{cr}ga}{k_{sr}(a + b)} * k_{sr}}{a * k_{sf}}$$
$$\boxed{q_{sf} = \frac{m_{cr}gb}{k_{sf}(a + b)}}$$

Next, we can plug in q_{sf} into equation (3) to find q_{tf}:

$$q_{tf} * k_{tf} - \frac{m_{cr}gb}{k_{sf}(a + b)} * k_{sf} = m_{tf}g$$
$$q_{tf} * k_{tf} = m_{tf}g + \frac{m_{cr}gb}{a + b}$$
$$\boxed{q_{tf} = \frac{g}{k_{tf}}\left(m_{tf} + \frac{m_{cr}b}{a + b}\right)}$$

Finally, our last step is to find q_{tr} by plugging in q_{sr} into equation (4):

$$q_{tr} * k_{tr} - \frac{m_{cr}ga}{k_{sr}(a + b)} * k_{sr} = m_{tr}g$$
$$q_{tr} * k_{tr} = m_{tr}g + \frac{m_{cr}ga}{a + b}$$
$$\boxed{q_{tr} = \frac{g}{k_{tr}}\left(m_{tr} + \frac{m_{cr}a}{a + b}\right)}$$

Input Equations

To determine the input, an initial start time $t_i = 0.5$ s was chosen. Then four important points of time were determined to find the input velocity for the front and rear tire as shown below:

- The times when the front tire reaches the apex and end of the first bump.
- The times when the front tire reaches the start, apex, and end of the second bump.
- The times when the rear tire reaches the start, apex, and end of the first bump.
- The times when the rear tire reaches the start, apex, and end of the second bump.

Other variables to consider when determining the input velocities are: the time to go through one bump $T_b = L/v_c$ the time the tire takes to go from the first bump to the second bump $del_T = L_{wb}/v_c$ and the slope of the bump $A/(0.5L)$. Then the input velocity amplitude when going up and down the bump is $v_{fi} = v_{ri} = \pm$ slope*v_c depending on the point of time. The gravitational forces on the front tire, rear tire and on cycle and rider are determined and shown below:

$$F_{cr} = m_{cr} * g = 2943 \text{ (N)}$$
$$F_{tf} = m_{tf} * g = 147.15 \text{ (N)}$$
$$F_{tr} = m_{tr} * g = 196.2 \text{ (N)}$$

Maximum Bump Height

Since the system is linear, we determine the maximum bump height by choosing a random value of $A_{initial}$ and find the corresponding maximum suspension deflection $\delta_{intial,max}$. Then the maximum bump height is determined as $A_{max} = \frac{A_{initial} * \delta_{max}}{\delta_{intial,max}}$. The maximum deflection is required to be $\delta_{max} = 0.1$ (m). After doing these calculations for both the standard configuration and the forward configuration, the max bump height for each configuration is found as shown below.

$$A_{max,std} = 0.16\ (m) \quad standard\ configuration$$
$$A_{max,for} = 0.162\ (m) \quad forward\ configuration$$

Other parameters

The center of gravity position changes depending on the configuration. This position is determined based on the variable a and b where $a = L_{cg}$ and $b = L_{wb} - L_{cg}$. In the standard configuration, $a = L_{cg} = 0.9$ (m) and $b = L_{wb} - L_{cg} = 0.7$ (m). In the forward configuration, $a = L_{cg} = 0.7$ (m) and $b = L_{wb} - L_{cg} = 0.9$ (m).

Time Parameter

In order to determine the time parameter, we find the pitch natural frequency, the heave natural frequency and the tire natural frequency. From those three natural frequencies, we calculate the corresponding period.

$$\boldsymbol{w_{n,tire}} = \sqrt{\frac{k_{tf}}{m_{tf}}} \quad and \quad \boldsymbol{T_{tire}} = \frac{\boldsymbol{2\pi}}{\boldsymbol{w_{n,tire}}} \quad (\boldsymbol{wheel\ model})$$

$$\boldsymbol{w_{n,heave}} = \sqrt{\frac{k_{sf} + k_{sr}}{m_{cr}}} \quad and \quad \boldsymbol{T_{heave}} = \frac{\boldsymbol{2\pi}}{\boldsymbol{w_{n,heave}}} \quad (\boldsymbol{heave\ model})$$

$$\boldsymbol{w_{n,pitch}} = \sqrt{\frac{k_{sf}a^2 + k_{sr}b^2}{j_{cr}}} \quad and \quad \boldsymbol{T_{pitch}} = \frac{\boldsymbol{2\pi}}{\boldsymbol{w_{n,pitch}}} \quad (\boldsymbol{pitch\ model})$$

	Standard Configuration			Forward Configuration		
	Heave model	Pitch model	Wheel model	Heave model	Pitch model	Wheel model
$\omega_n(\frac{rad}{s})$	4.655	7.434	44.721	4.655	7.576	44.721
T (s)	1.350	0.845	0.140	1.350	0.829	0.140

Table 3: Natural frequencies and period of different model

The largest natural frequency, which also results in shortest period is used to determine the sample time. Therefore, the tire natural frequency is used to find the sample time. The maximum step size is at most about one-tenth of the shortest vibration period.

$$\boldsymbol{dt = min([T_{heave}, T_{pitch}, T_{tire}])/10}$$

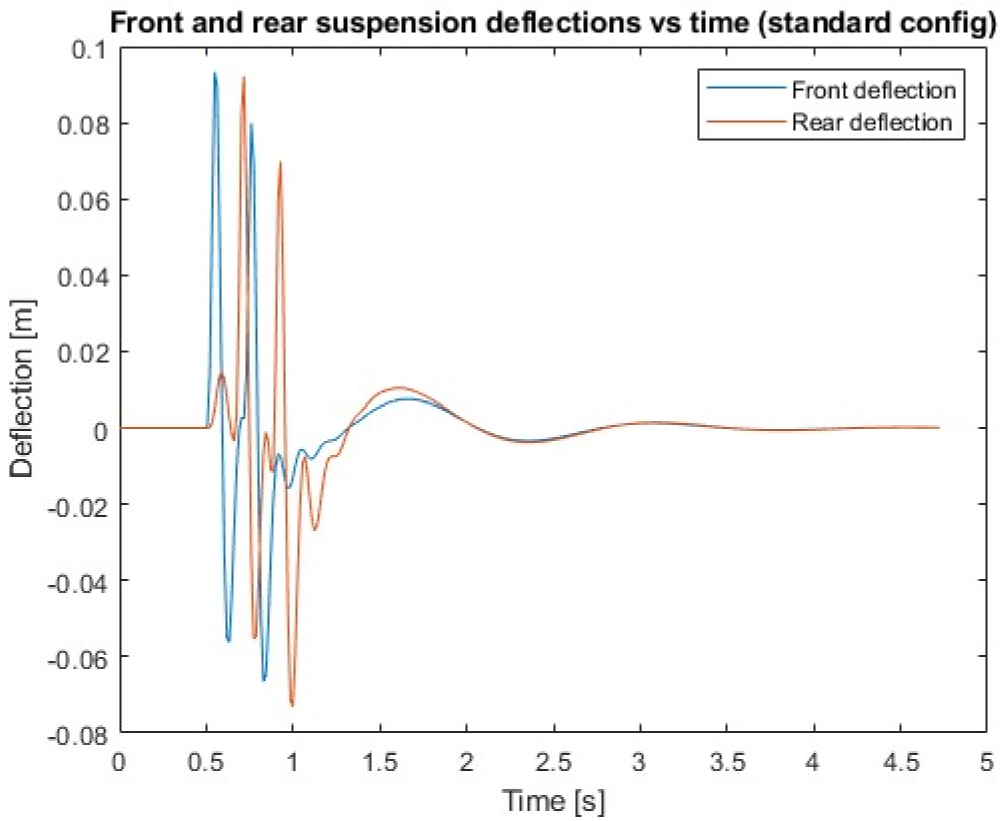
The smallest natural frequency, which also results in longest period is used to determine the simulation length. Therefore, the heave natural frequency is used to find the simulation length. The finish time t_{final} is about three of the heave period after the rear tire reaches the end of the second bump.

$$\boldsymbol{t_{final} = max([T_{heave}, T_{pitch}, T_{tire}]) * 3.5}$$

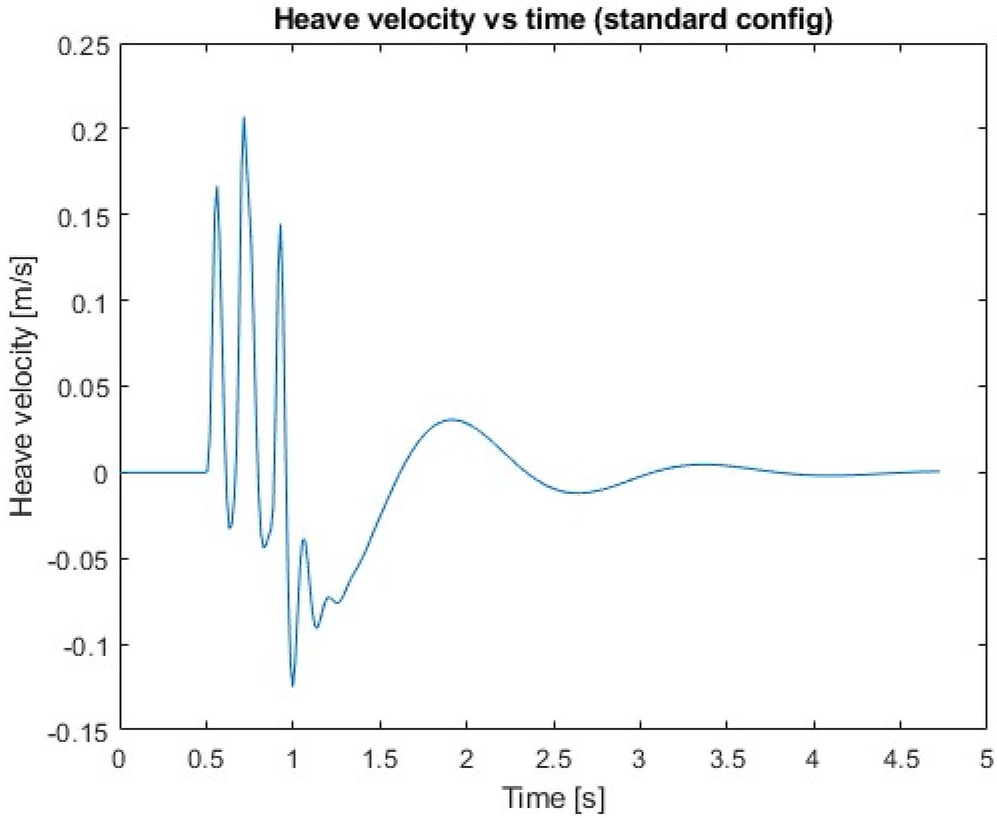
Lastly, the number of step and the simulation time span are shown below.

$$\boldsymbol{N_{steps}} = \boldsymbol{round}([\boldsymbol{T_{heave}}/\boldsymbol{dt}]) + \mathbf{1}$$
$$\boldsymbol{ts} = \boldsymbol{linspace}([\mathbf{0}, \boldsymbol{t_{final}}, \boldsymbol{N_{steps}}])$$

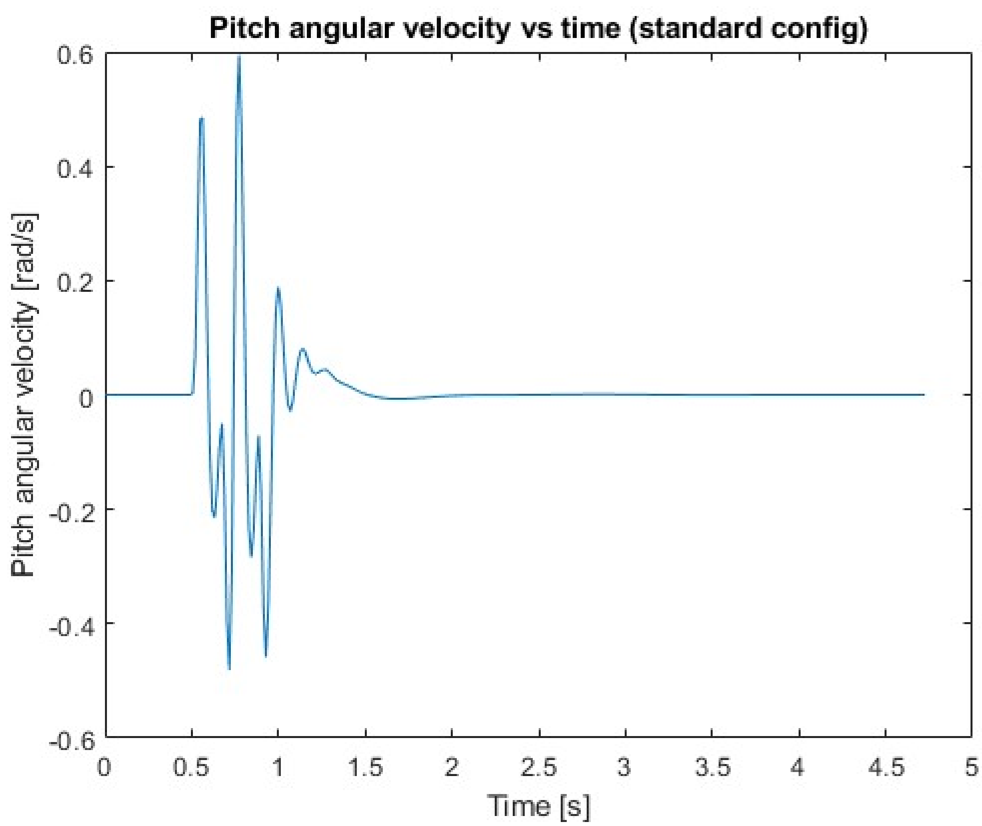
3/ Simulations



4a/ Front and rear suspension deflection vs time

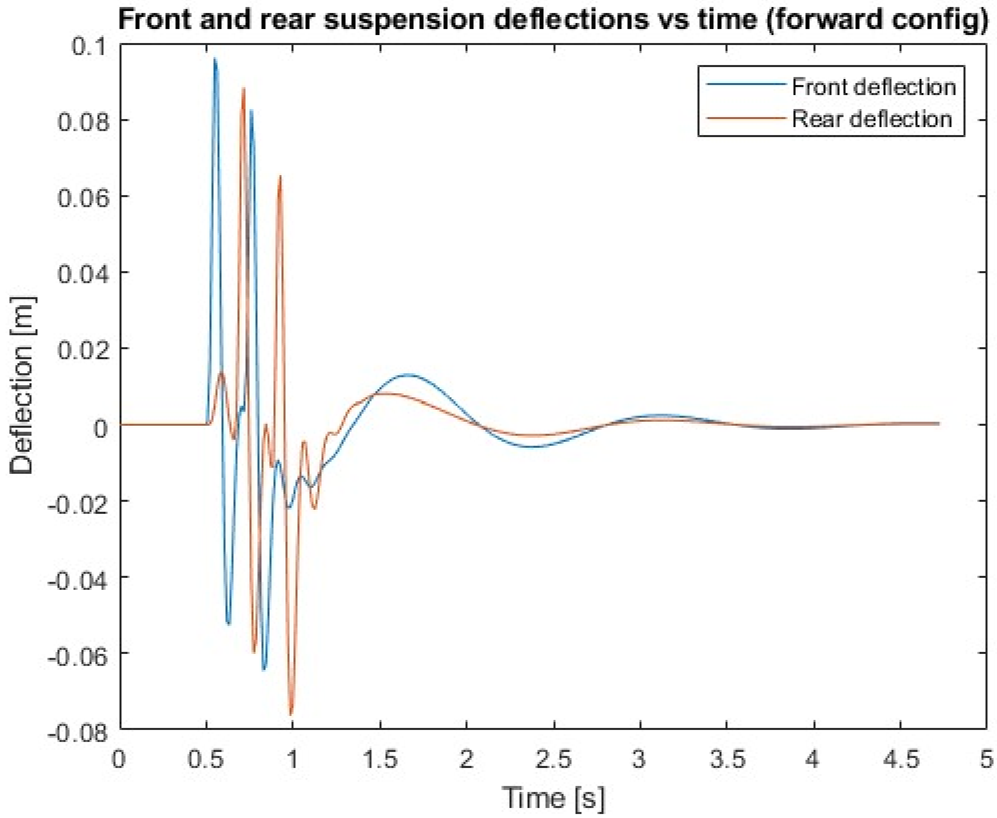


4b/ Heave velocity vs time

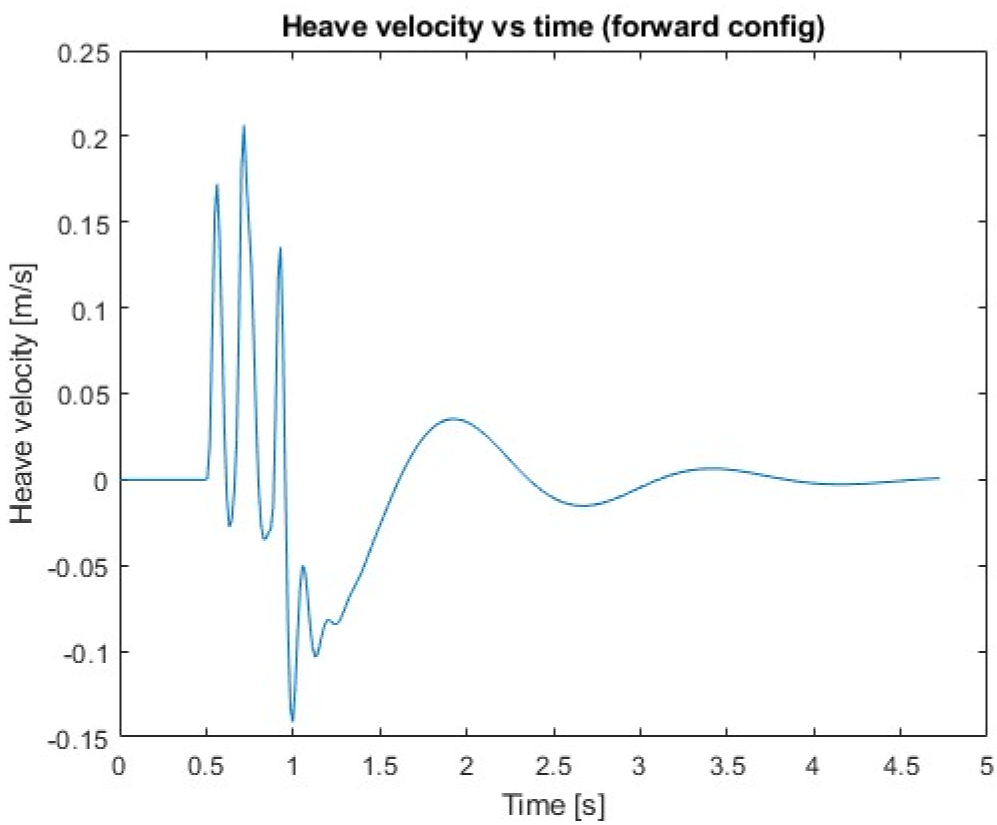


4c/ Angular pitch velocity vs time

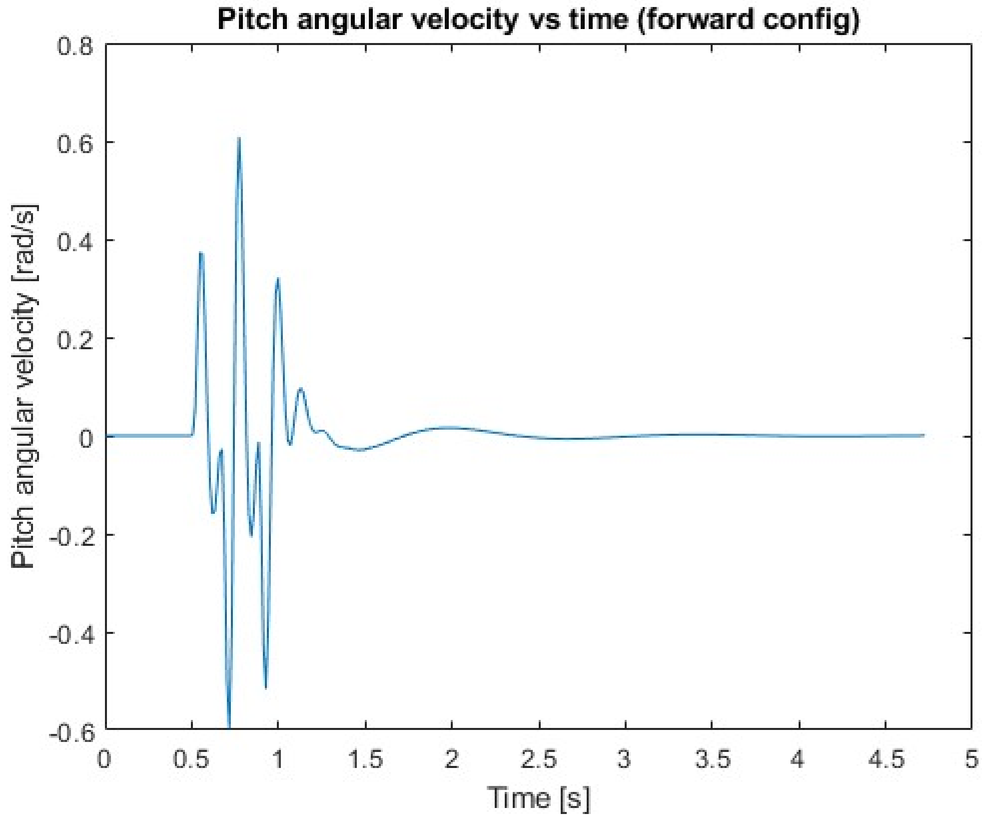
Fig 4a-c: Suspension system under standard configuration



5a/ Front and rear suspension deflection vs time



5b/ Heave velocity vs time



5c/ Angular pitch velocity vs time

Fig 5a-c: Suspension system under forward configuration

Analysis:

1/ There are three natural frequencies in the system: the pitch natural frequency, the heave natural frequency and the tire natural frequency associated with this system. The values of these natural frequencies in standard and forward configuration are listed in table 3. The smallest natural frequency, which also results in longest period is used to determine the simulation length. Therefore, the heave natural frequency is used to find the simulation length. The finish time t_{final} is about three of the heave periods after the rear tire reaches the end of the second bump. The largest natural frequency, which also results in shortest period is used to determine the sample time. Therefore, the tire natural frequency is used to find the sample time. The maximum step size is at most about one-tenth of the shortest vibration period.

2/ Based on the power flow on the bond graph in figure 2, the deflections of the suspension are positive in compression. For instance, in the case of the front suspension, v_{tf} and v_{sf} are in the same direction and v_{tf} is larger than v_{sf}. Therefore, the power flow to the suspension system is positive since the relative velocity is $v_{relative} = v_{tf} - v_{sf}$. When this positive power flow is integrated it would result in positive deflection in compression for the suspension system.

3/ The plots of the suspension deflections for the two center of gravity configurations are very similar in term of behaviors as shown in figure 4a and 5a. They both have four spikes before the oscillation is damped out. However, there are still some differences in the maximum displacements and shape of response between those two plots. In the forward configuration, the front tire has higher spike amplitude in comparison with the front tire in the standard configuration. In the forward configuration, the rear tire has lower spike amplitude in comparison to the rear tire in the standard configuration. In addition, when comparing the plots of heave velocity and angular pitch velocity, the standard configuration has lower heave velocity and angular pitch velocity than the forward configuration. Overall, when the center of gravity is in forward configurations, the vehicle experiences more weight in the front than in the rear. Therefore, the front suspension will experience more deflection than the rear suspension. This is what would happen in real life when the rider leans toward the front on a motorcycle.