```
(*The Jacobian matrix at the pair bonding equilibrium*)
P = 0
```

Out[1]= 0

```
M = 1
```

Out[2]= 1

```
mat = {{(F * (k * M - (P - 1) ^ 2) - (P - 1) ^ 2 * (k + gamma + 2 * P) + k * M * X) / (P - 1) ^ 2,
    (-k * (1 - P - F - X)) / (1 - P), ((-k * (1 - P - M)) / (1 - P)) - P, ((-k * (1 - P - M)) / (1 - P))},
   {-M / 2, (1 / 2) * (-F - P), (1 - M) / 2, 0}, {-(F / 2) - ((k * M) / (1 - P)) + ((k * M * (1 - P - F - X)) / (1 - P) ^ 2),
    k * (1 - P - F - X) / (1 - P), -1 - F / 2 - gamma - delta - ((k * M) / (1 - P)) + (1 / 2) * (-F - P),
    (-k * M) / (1 - P)}, {-X / 2, 0, -X / 2, -gamma + (1 / 2) * (-F - P)}}
```

Out[3]= $\left\{\{-\text{gamma} + F(-1+k) - k + k\,X, -k(1-F-X), 0, 0\}, \left\{-\frac{1}{2}, -\frac{F}{2}, 0, 0\right\}, \left\{-\frac{F}{2} - k + k(1-F-X), k(1-F-X), -1-\text{delta}-F-\text{gamma}-k, -k\right\}, \left\{-\frac{X}{2}, 0, -\frac{X}{2}, -\frac{F}{2} - \text{gamma}\right\}\right\}$

```
(*Compute the eigenvalues of the Jacobian,
in terms of the populations*){l1, l2, l3, l4} = Eigenvalues[mat]
```

Out[4]= $\left\{\frac{1}{4}\times\left(-2-2\,\text{delta}-3F-4\,\text{gamma}-2k-\sqrt{4+8\,\text{delta}+4\,\text{delta}^2+4F+4\,\text{delta}\,F+F^2+8k+8\,\text{delta}\,k+4F\,k+4k^2+8k\,X}\right),\right.$
$\frac{1}{4}\times\left(-2-2\,\text{delta}-3F-4\,\text{gamma}-2k+\sqrt{4+8\,\text{delta}+4\,\text{delta}^2+4F+4\,\text{delta}\,F+F^2+8k+8\,\text{delta}\,k+4F\,k+4k^2+8k\,X}\right),$
$\frac{1}{4}\times\left(-3F-2\,\text{gamma}-2k+2F\,k+2k\,X-\sqrt{(3F+2\,\text{gamma}+2k-2F\,k-2k\,X)^2-4\times(2F^2+2F\,\text{gamma}-2k+4F\,k-2F^2\,k+2k\,X-2F\,k\,X)}\right),$
$\left.\frac{1}{4}\times\left(-3F-2\,\text{gamma}-2k+2F\,k+2k\,X+\sqrt{(3F+2\,\text{gamma}+2k-2F\,k-2k\,X)^2-4\times(2F^2+2F\,\text{gamma}-2k+4F\,k-2F^2\,k+2k\,X-2F\,k\,X)}\right)\right\}$

```
(*Sub the equilibria in to see what the 3d plot looks like*)
F = -1 - delta - 2 * gamma - k + (1 + 2 * delta + delta ^ 2 + 4 * k + 2 * delta * k + k ^ 2) ^ (1 / 2)
```

Out[5]= $-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4k+2\,\text{delta}\,k+k^2}$

```
X = (2 * gamma) / (2 * gamma + F)
```

Out[6]= $\frac{2\,\text{gamma}}{-1-\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4k+2\,\text{delta}\,k+k^2}}$

```
{l1, l2, l3, l4}
```

Out[7]=

$$\left\{\frac{1}{4}\times\left(-2-2\,\mathrm{delta}-4\,\mathrm{gamma}-2\,k-\right.\right.$$

$$3\times\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)-$$

$$\sqrt{\left(\left(4+8\,\mathrm{delta}+4\,\mathrm{delta}^2+8\,k+8\,\mathrm{delta}\,k+4\,k^2+\right.\right.}$$

$$\frac{16\,\mathrm{gamma}\,k}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}+$$

$$4\times\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$4\,\mathrm{delta}\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$4\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$\left.\left.\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)^2\right)\right),$$

$$\frac{1}{4}\times\left(-2-2\,\mathrm{delta}-4\,\mathrm{gamma}-2\,k-3\times\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\right.\right.$$

$$\left.\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+\sqrt{\left(\left(4+8\,\mathrm{delta}+4\,\mathrm{delta}^2+8\,k+\right.\right.}$$

$$8\,\mathrm{delta}\,k+4\,k^2+\frac{16\,\mathrm{gamma}\,k}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}+$$

$$4\times\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$4\,\mathrm{delta}\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$4\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$\left.\left.\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)^2\right)\right),$$

$$\frac{1}{4}\times\left(-2\,\mathrm{gamma}-2\,k+\frac{4\,\mathrm{gamma}\,k}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}-\right.$$

$$3\times\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$2\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)-$$

$$\sqrt{\left(\left(\left(2\,\mathrm{gamma}+2\,k-\frac{4\,\mathrm{gamma}\,k}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}+\right.\right.\right.}$$

$$3\times\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)-$$

$$2\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)\Big)^{2}-$$

$$4\times\Bigg(-2\,k+\frac{4\,\mathrm{gamma}\,k}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}+$$

$$2\,\mathrm{gamma}\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$4\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)-$$

$$\frac{4\,\mathrm{gamma}\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}+$$

$$2\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)^{2}-$$

$$2\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)^{2}\Bigg)\Bigg)\Bigg),$$

$$\frac{1}{4}\times\Bigg(-2\,\mathrm{gamma}-2\,k+\frac{4\,\mathrm{gamma}\,k}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}-$$

$$3\times\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$2\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$\surd\Bigg(\Bigg(\Bigg(2\,\mathrm{gamma}+2\,k-\frac{4\,\mathrm{gamma}\,k}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}+$$

$$3\times\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)-$$

$$2\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)\Bigg)^{2}-$$

$$4\times\Bigg(-2\,k+\frac{4\,\mathrm{gamma}\,k}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}+$$

$$2\,\mathrm{gamma}\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)+$$

$$4\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)-$$

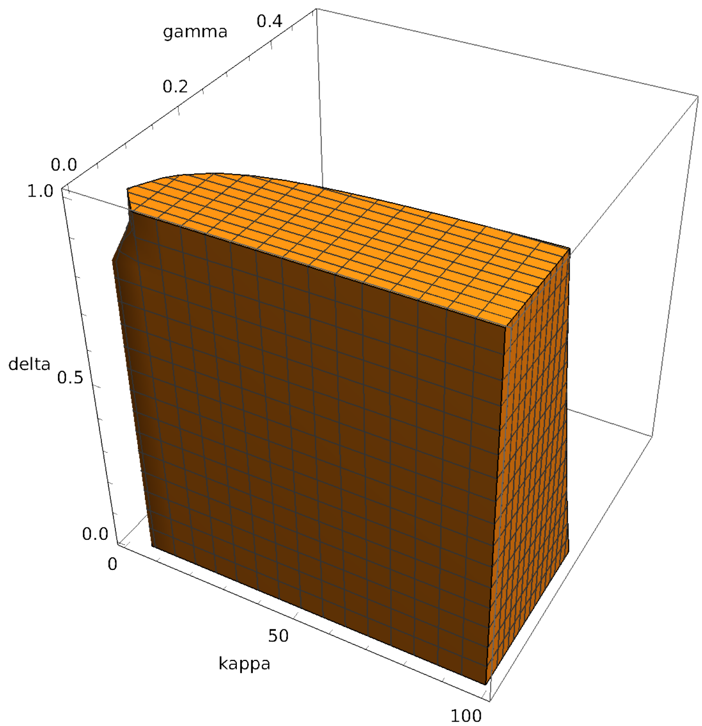
$$\frac{4\,\mathrm{gamma}\,k\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)}{-1-\mathrm{delta}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}}+$$

$$2\left(-1-\mathrm{delta}-2\,\mathrm{gamma}-k+\sqrt{1+2\,\mathrm{delta}+\mathrm{delta}^2+4\,k+2\,\mathrm{delta}\,k+k^2}\right)^{2}-$$

$$2\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)^2\Bigg)\Bigg)\Bigg)\Bigg\}$$

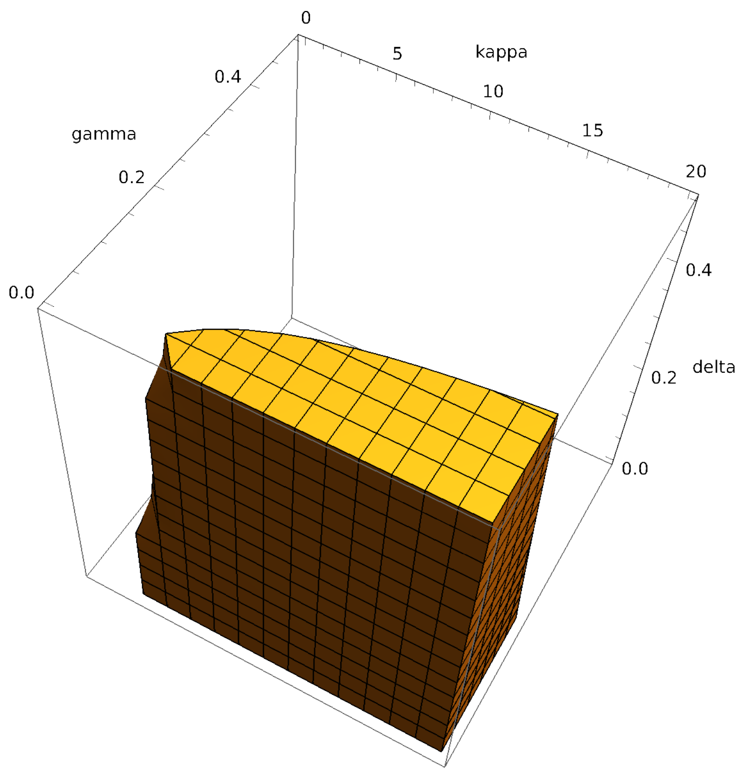
```
(*A little exploring- the 3d plot in different orientations*)RegionPlot3D[
 k > 0 && gamma > 0 && delta > 0 && l1 < 0 && l2 < 0 && Re[l3] < 0 && Re[l4] < 0 ,
 {k, 0, 100}, {gamma , 0, (1/2)}, {delta , 0.01 , 1}, AxesLabel → {kappa , gamma , delta}]
```

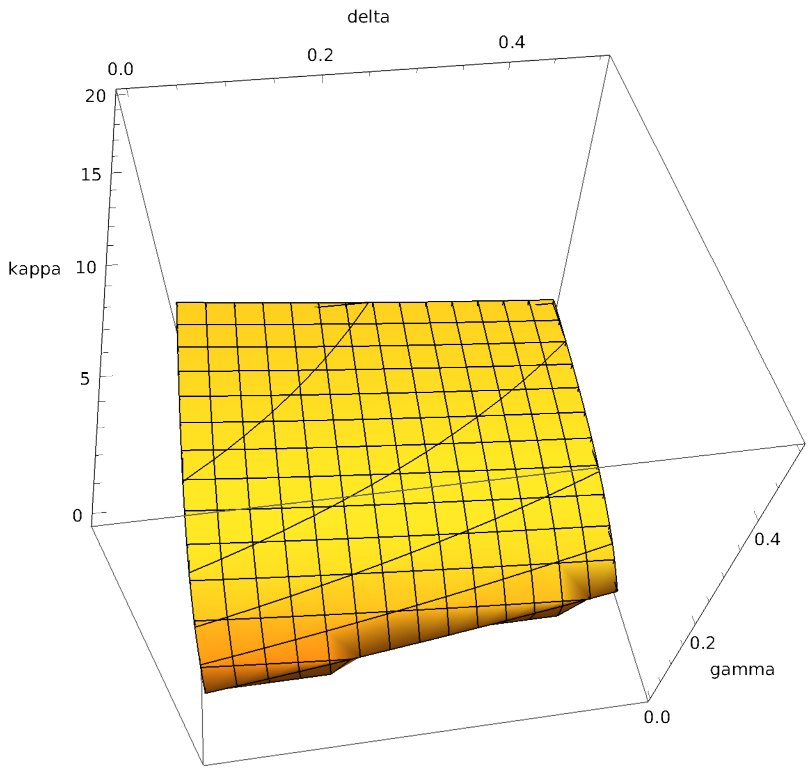
Out[8]=



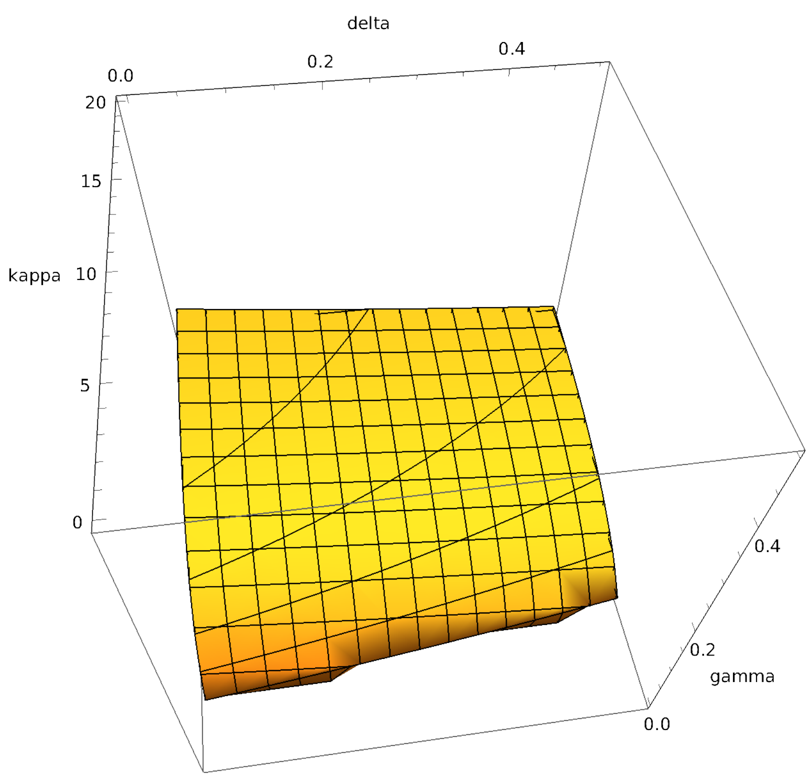
```
RegionPlot3D[k > 0 && gamma > 0 && delta > 0 && l1 < 0 && l2 < 0 && Re[l3] < 0 && Re[l4] < 0 &&
  0 ≤ F ≤ 1 && 0 ≤ X ≤ 1, {k, 0, 20}, {gamma, 0, 1/2}, {delta, 10^-10, (1/2)}, Mesh → All,
 MeshStyle → Automatic, AxesLabel → {kappa, gamma, delta}, MaxRecursion → 10]
```

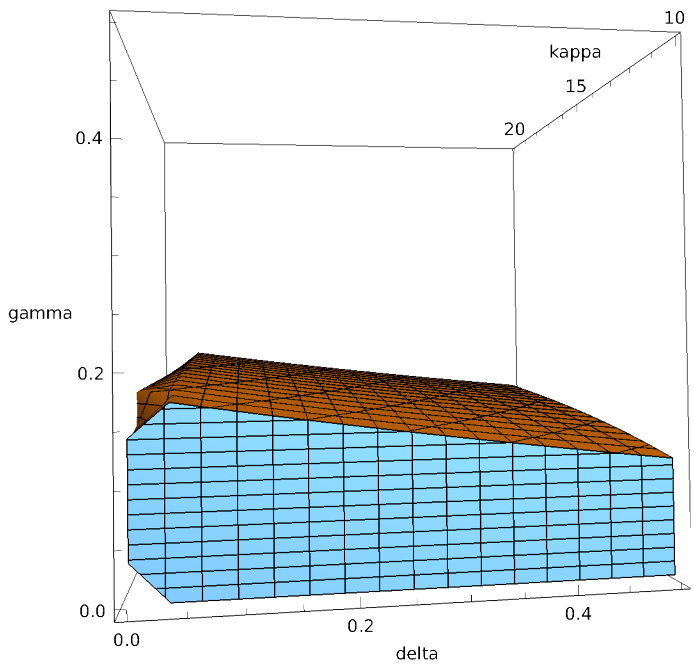
Out[9]=




delta
0.0
0.2
0.4
20
15
kappa
10
5
0
0.4
0.2
gamma
0.0

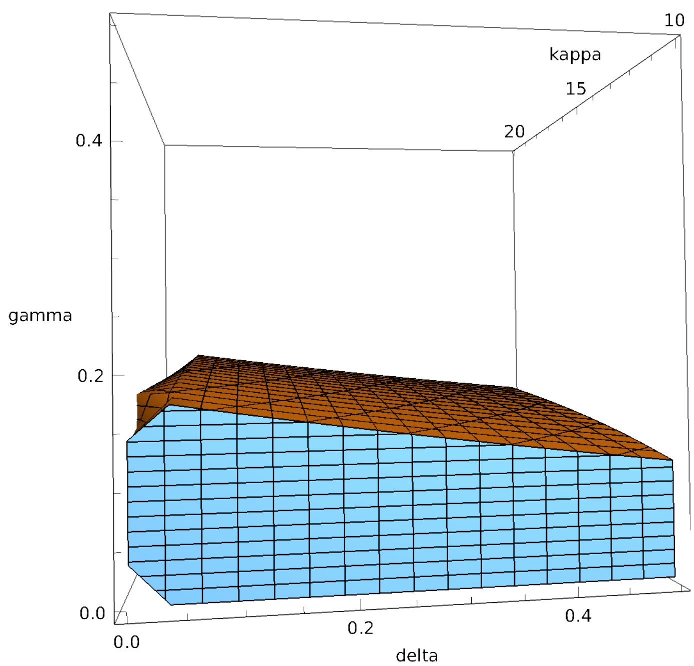
Out[10]=


delta
0.0
0.2
0.4
20
15
kappa
10
5
0
0.4
0.2
gamma
0.0

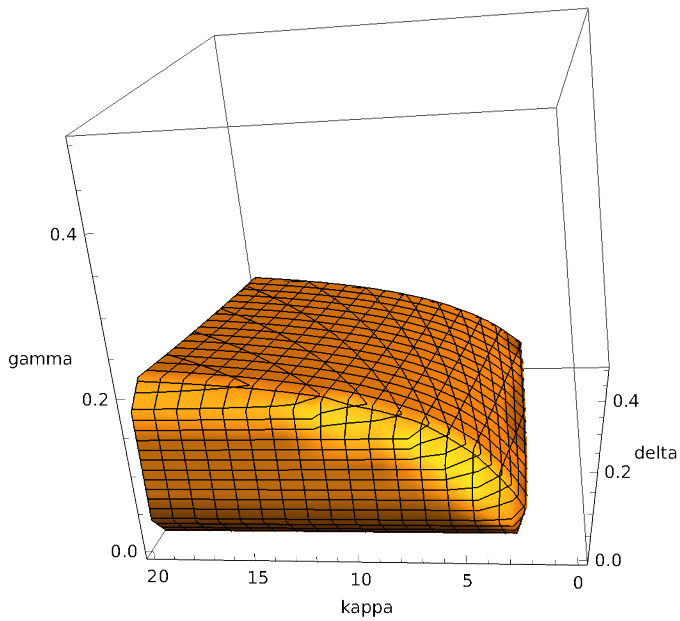


gamma
kappa
delta
0.0
0.2
0.4
10
15
20

Out[11]=

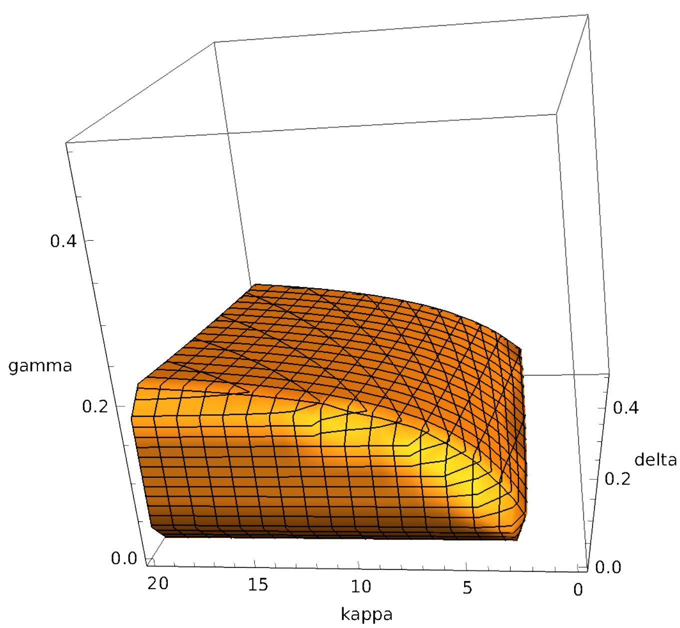


gamma
kappa
delta
0.0
0.2
0.4
10
15
20

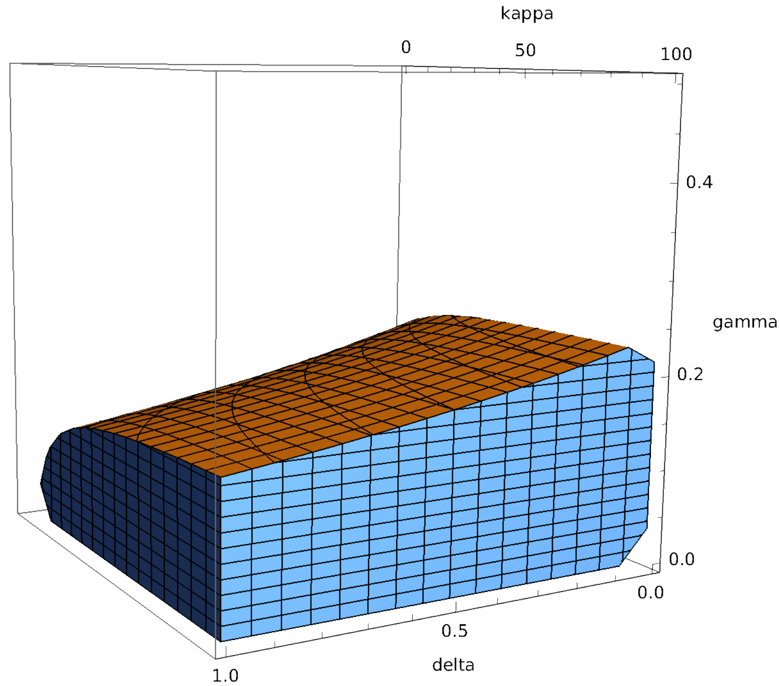


0.4
gamma
0.2
0.0
20
15
10
5
0
kappa
0.4
delta
0.2
0.0

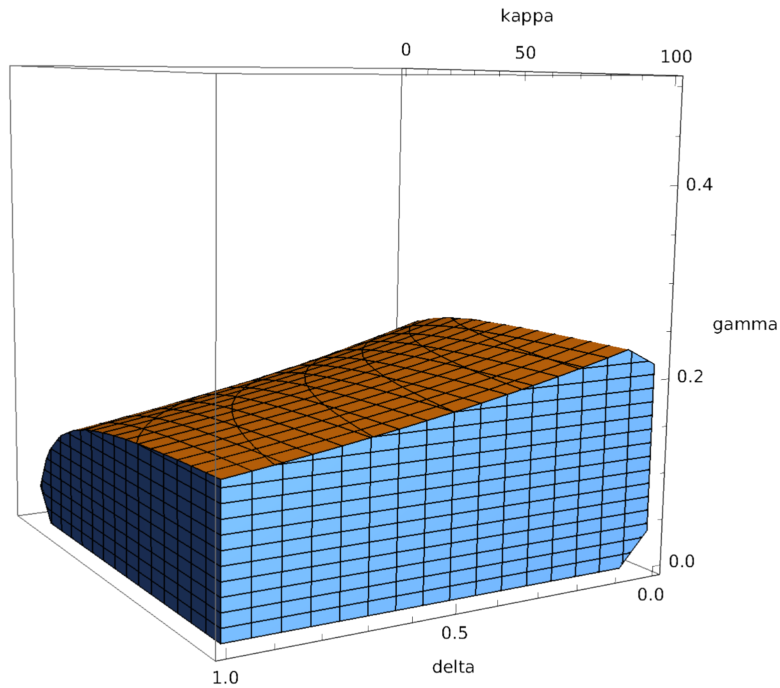
Out[12]=



0.4
gamma
0.2
0.0
20
15
10
5
0
kappa
0.4
delta
0.2
0.0



Out[13]=



In[14]:=

In[15]:=

F

Out[16]= $-1 - \text{delta} - 2\,\text{gamma} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}$

```
(*We realise that s=
 0 gives us the boundary after some mathematical reasoning given in the
   paper. This is some code to confirm our intutition that we generate
   all the points on the surface with only the inequality s > 0 *)
RegionPlot3D[2 F^2 + 2 F gamma - 2 k + 4 F k - 2 F^2 k + 2 k X - 2 F k X  > 0 &&
  k > 0 && gamma > 0 && delta > 0 && 0 ≤ F ≤ 1 && 0 ≤ X ≤ 1, {k, 0, 100},
 {gamma, 0, (1/2)}, {delta, 0.01, 1}, AxesLabel → {kappa, gamma, delta}]
```

Power : Infinite expression $\frac{1}{0.}$ encountered .

Infinity : Indeterminate expression 0. ComplexInfinity encountered .

Power : Infinite expression $\frac{1}{0.}$ encountered .

Infinity : Indeterminate expression 0. ComplexInfinity encountered .

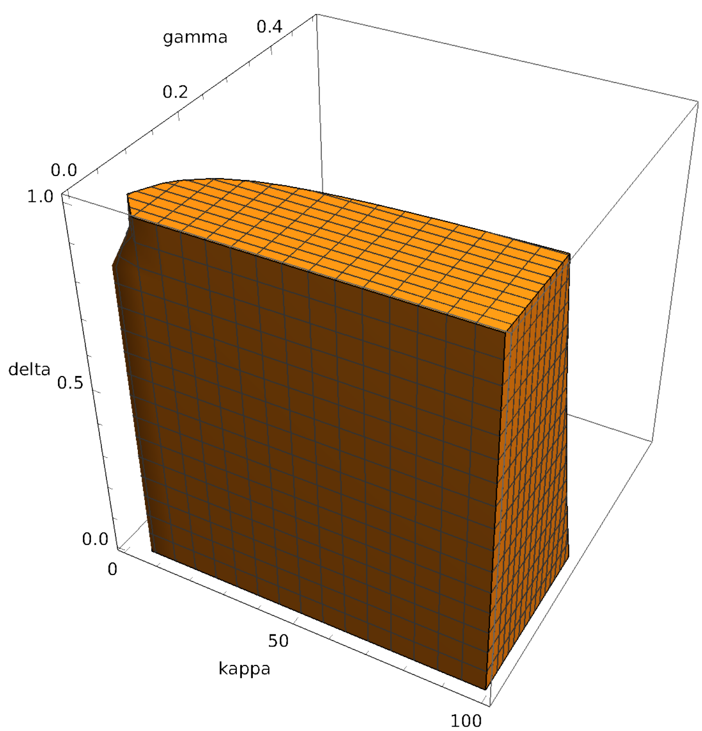
Power : Infinite expression $\frac{1}{0.}$ encountered .

General : Further output of Power ::infy will be suppressed during this calculation .

Infinity : Indeterminate expression 0. ComplexInfinity encountered .

General : Further output of Infinity ::indet will be suppressed during this calculation .

Out[17]=



```
RegionPlot3D[2 F^2 + 2 F gamma - 2 k + 4 F k - 2 F^2 k + 2 k X - 2 F k X > 0 && k > 0 &&
  gamma > 0 && delta > 0 && 0 ≤ F ≤ 1 && 0 ≤ X ≤ 1, {k, 0, 100}, {gamma, 0, 1/2},
 {delta, 0.01, 1}, PlotStyle → Automatic, AxesLabel → {kappa, gamma, delta}]
```

Power : Infinite expression $\frac{1}{0.}$ encountered .

Infinity : Indeterminate expression 0. ComplexInfinity encountered .

Power : Infinite expression $\frac{1}{0.}$ encountered .

Infinity : Indeterminate expression 0. ComplexInfinity encountered .

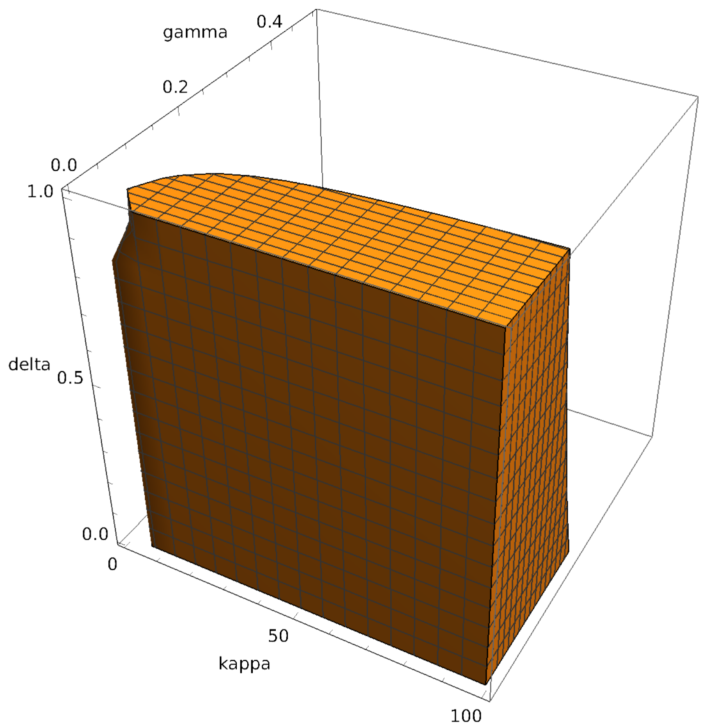
Power : Infinite expression $\frac{1}{0.}$ encountered .

General : Further output of Power ::infy will be suppressed during this calculation .

Infinity : Indeterminate expression 0. ComplexInfinity encountered .

General : Further output of Infinity ::indet will be suppressed during this calculation .

Out[18]=



In[19]:=

```
u[k, delta, gamma] = 2 F^2 + 2 F gamma - 2 k + 4 F k - 2 F^2 k + 2 k X - 2 F k X
```

Out[20]=

$$-2\,k+\frac{4\,\text{gamma}\,k}{-1-\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}}+$$
$$2\,\text{gamma}\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)+$$
$$4\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)-$$
$$\frac{4\,\text{gamma}\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)}{-1-\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}}+$$
$$2\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)^2-$$
$$2\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)^2$$

```
(* Some experiments with an alternate approach
 based on the implicit function theorem. It doesn't work
 well here. We will have to solve the equation explicitly*)
u[k_, delta_, gamma_] := 2 F^2 + 2 F gamma - 2 k + 4 F k - 2 F^2 k + 2 k X - 2 F k X
```

$$-\text{D}\Big[-2\,k+(4\,\text{gamma}\,k)\Big/\left(-1-\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)+$$
$$2\,\text{gamma}\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)+$$
$$4\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)-$$
$$\left(4\,\text{gamma}\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)\right)\Big/$$
$$\left(-1-\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)+$$
$$2\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)^2-$$
$$2\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)^2,\ \text{delta}\Big]\Big/$$
$$\text{D}\Big[-2\,k+\frac{4\,\text{gamma}\,k}{-1-\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}}+$$
$$2\,\text{gamma}\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)+$$
$$4\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)-$$
$$\left(4\,\text{gamma}\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)\right)\Big/$$
$$\left(-1-\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)+$$
$$2\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)^2-$$
$$2\,k\left(-1-\text{delta}-2\,\text{gamma}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)^2,\ \text{gamma}\Big]$$

Out[22]=

$$
\begin{aligned}
&\Bigg(-2\,\text{gamma}\left(-1+\frac{2+2\,\text{delta}+2\,\text{k}}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}\right)-\\
&4\,\text{k}\left(-1+\frac{2+2\,\text{delta}+2\,\text{k}}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}\right)+\\
&\frac{4\,\text{gamma}\,\text{k}\left(-1+\frac{2+2\,\text{delta}+2\,\text{k}}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}\right)}{\left(-1-\text{delta}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}\right)^2}+\\
&\frac{4\,\text{gamma}\,\text{k}\left(-1+\frac{2+2\,\text{delta}+2\,\text{k}}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}\right)}{-1-\text{delta}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}-\\
&4\times\left(-1+\frac{2+2\,\text{delta}+2\,\text{k}}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}\right)\times\\
&\quad\left(-1-\text{delta}-2\,\text{gamma}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}\right)+\\
&4\,\text{k}\left(-1+\frac{2+2\,\text{delta}+2\,\text{k}}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}\right)\times\\
&\quad\left(-1-\text{delta}-2\,\text{gamma}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}\right)-\\
&\Bigg(4\,\text{gamma}\,\text{k}\left(-1+\frac{2+2\,\text{delta}+2\,\text{k}}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}\right)\times\\
&\quad\left(-1-\text{delta}-2\,\text{gamma}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}\right)\Bigg)\Big/\\
&\left(-1-\text{delta}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}\right)^2\Bigg)\Big/\\
&\Bigg(-4\,\text{gamma}-8\,\text{k}+\frac{4\,\text{k}}{-1-\text{delta}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}+\\
&\frac{8\,\text{gamma}\,\text{k}}{-1-\text{delta}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}-\\
&6\times\left(-1-\text{delta}-2\,\text{gamma}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}\right)+\\
&8\,\text{k}\left(-1-\text{delta}-2\,\text{gamma}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}\right)-\\
&\frac{4\,\text{k}\left(-1-\text{delta}-2\,\text{gamma}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}\right)}{-1-\text{delta}-\text{k}+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,\text{k}+2\,\text{delta}\,\text{k}+\text{k}^2}}\Bigg)
\end{aligned}
$$

```
Reduce[
 -D[-2 k + (4 gamma k) / (-1 - delta - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2]) + 2 gamma
      (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2]) +
     4 k (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2]) -
     (4 gamma k (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2])) /
      (-1 - delta - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2]) +
     2 (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2])^2 -
     2 k (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2])^2, delta] /
   D[-2 k + (4 gamma k) / (-1 - delta - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2]) +
    2 gamma (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2]) +
    4 k (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2]) -
    (4 gamma k (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2])) /
     (-1 - delta - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2]) +
    2 (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2])^2 -
    2 k (-1 - delta - 2 gamma - k + Sqrt[1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2])^2, gamma] <
  0 && k > 0 && gamma > 0 && delta > 0, delta]
```

Out[23]=

··· 1 ···

large output | show less | show more | show all | set size limit...

$$2\,F^2 + 2\,F\,\text{gamma} - 2\,k + 4\,F\,k - 2\,F^2\,k + 2\,k\,X - 2\,F\,k\,X$$

Out[24]=

$$-2\,k + \frac{4\,\text{gamma}\,k}{-1 - \text{delta} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}} +$$
$$2\,\text{gamma}\left(-1 - \text{delta} - 2\,\text{gamma} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}\right) +$$
$$4\,k\left(-1 - \text{delta} - 2\,\text{gamma} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}\right) -$$
$$\frac{4\,\text{gamma}\,k\left(-1 - \text{delta} - 2\,\text{gamma} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}\right)}{-1 - \text{delta} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}} +$$
$$2\left(-1 - \text{delta} - 2\,\text{gamma} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}\right)^2 -$$
$$2\,k\left(-1 - \text{delta} - 2\,\text{gamma} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}\right)^2$$

```
(*Finally , we hit on the right approach . Solving the equality directly ,
and observing that the other case implies that a biologial constraint
 is being violated . F_m cannot be zero at a multiple mating or
 coexistence equilibrium point. It isn't what we're looking for. *)
Reduce[2 F^2 + 2 F gamma - 2 k + 4 F k - 2 F^2 k + 2 k X - 2 F k X == 0, gamma]
```

Out[25]=

$$\left(1 + \text{delta} + k - \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2} \neq 0\ \&\&\right.$$
$$\left.\text{gamma} == \frac{1}{2} \times \left(-1 - \text{delta} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}\right)\right) \,||$$
$$\left(2 + \text{delta} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2} \neq 0\ \&\&\right.$$
$$1 + \text{delta} + k - \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2} \neq 0\ \&\&$$
$$\text{gamma} == \left(-3 - 3\,\text{delta} + 3\,k + 2\,\text{delta}\,k + 2\,k^2 + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2} -\right.$$
$$\left. 2\,k\,\sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}\right) \Big/$$
$$\left.\left(2 \times \left(2 + \text{delta} - k + \sqrt{1 + 2\,\text{delta} + \text{delta}^2 + 4\,k + 2\,\text{delta}\,k + k^2}\right)\right)\right)$$

```
D[(-3 - 3 delta + 3 k + 2 delta k + 2 k^2 + √(1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2) -
    2 k √(1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2)) /
  (2 × (2 + delta - k + √(1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2))), delta]
```

Out[26]=

$$\frac{-3 + 2\,k + \frac{2+2\,\text{delta}+2\,k}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}} - \frac{k\,(2+2\,\text{delta}+2\,k)}{\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}}}{2 \times \left(2+\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)} - \left(\left(1+\frac{2+2\,\text{delta}+2\,k}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}}\right) \times \left(-3-3\,\text{delta}+3\,k+2\,\text{delta}\,k+2\,k^2+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2} - 2\,k\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)\right) \Big/ \left(2\left(2+\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)^2\right)$$

```
(*Now we prove the results about the
 partial derivatives with respect to kappa and delta *)
```

$$\text{Reduce}\left[\frac{-3 + 2\,k + \frac{2+2\,\text{delta}+2\,k}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}} - \frac{k\,(2+2\,\text{delta}+2\,k)}{\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}}}{2 \times \left(2+\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)} - \left(\left(1+\frac{2+2\,\text{delta}+2\,k}{2\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}}\right) \times \left(-3-3\,\text{delta}+3\,k+2\,\text{delta}\,k+2\,k^2+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2} - 2\,k\,\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)\right) \Big/ \left(2\left(2+\text{delta}-k+\sqrt{1+2\,\text{delta}+\text{delta}^2+4\,k+2\,\text{delta}\,k+k^2}\right)^2\right) < 0\ \&\&\ k > 0\ \&\&\ \text{delta} > 0,\ \text{delta}\right]$$

Out[27]= k > 0 && delta > 0

```
(*The outputs show when the statements are true-
  which constitutes a proof that the partial derivative is positive when kappa >
 0 and delta > 0. These computations would have been very
  difficult without computer algebra to help! *)
Reduce[D[(-3 - 3 delta + 3 k + 2 delta k + 2 k^2 + Sqrt[(1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2)] -
      2 k Sqrt[(1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2)]) /
    (2 × (2 + delta - k + Sqrt[(1 + 2 delta + delta^2 + 4 k + 2 delta k + k^2)])),
   k] > 0 && k > 0 && delta > 0, k]
```

Out[28]= $\text{delta} > 0 \,\&\&\, k > 0$

In[29]:=

```
Reduce[3 F + 2 * gamma + 2 * k * (1 - F - X) ≥ 0 &&
  k > 0 && gamma > 0 && delta > 0 && 0 ≤ F ≤ 1 && 0 ≤ X ≤ 1, k]
```

Out[30]= $0 < \text{gamma} < \frac{1}{2} \,\&\&\, \text{delta} > 0 \,\&\&\, k \geq \frac{-2\,\text{gamma} - 2\,\text{delta gamma} - 2\,\text{gamma}^2}{-1 + 2\,\text{gamma}}$

```
Reduce[0 ≤ F ≤ 1 && 0 ≤ X ≤ 1 && k > 0 && gamma > 0 && delta > 0, k]
```

Out[31]= $0 < \text{gamma} < \frac{1}{2} \,\&\&\, \text{delta} > 0 \,\&\&\, k \geq \frac{-2\,\text{gamma} - 2\,\text{delta gamma} - 2\,\text{gamma}^2}{-1 + 2\,\text{gamma}}$

In[32]:=

In[33]:=