HCM city University of Technology

Exercises and Problems in Linear Algebra

Phan Thi Khanh Van

0.1 Complex numbers

0.1.1 Summary

$z = a + bi$ (**rectangular form**)$= r(\cos\varphi + i\sin\varphi)$ (**polar form**)$= r.e^{i\varphi}$ (**exponential form**)
$r = \|z\|$ - **modulus** ofs z; $\varphi = arg(z)$ - **argument** of z.
$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, $z_2 = r_1(\cos\varphi_1 + i\sin\varphi_1)$

- Distance between z_1 an z_2 : $d(z_1, z_2) = \|z_1 - z_2\|$
- $z_1.z_2 = r_1r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2))$
- $z^n = r^n(\cos n\varphi + i\sin n\varphi)$
- $\sqrt[n]{z} = \sqrt[n]{r}(\cos\frac{\varphi+k2\pi}{n} + i\sin\frac{\varphi+k2\pi}{n}), k = \overline{0,(n-1)}$

Fundamental theorem of algebra: Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.
Corollary: $P(z)$ is a polynomial with real coefficients. If z_0 is a root of $P(z)$, then $\bar{z_0}$ is also a root of $P(z)$.

0.1.2 Exercises

1. Evaluate: $z = \dfrac{(1-\sqrt{3}i)^{100}}{(1-i)^{120}}$
2. Evaluate $\sqrt[5]{\sqrt{3} - i}$
3. Solve the equation $z^4 + z^3 + 6z^2 - 14z + 20 = 0$, given that $z = 1 + i$ is 1 root of the equation.
4. Find the modulus of $z = \dfrac{(-\sqrt{3}+i)^{108}}{(1-i)^120}$
5. Find n such that $z = (\dfrac{1+i}{1+\sqrt{3}i})^n$ is
   - an imaginary number.
   - real number.
6. Find the set of all z satisfying
   a) $|z-1| = |z+1+i|$
   b) $|z+2-i| = 5$
   c) $|z+1+i| + |z-2-3i| = 4$
   d) $|z-1-i| - |z+2-i| = 3$
   e) $z = e^{a+2i}, a \in \mathbb{R}$
   f) $z = e^{2+ai}, a \in \mathbb{R}$
   g) that z is the solution of $\begin{cases} |z-1| = 3 \\ |z+i| = |1-i-z| \end{cases}$

0.2 Matrix, system of linear equations, determinant

0.2.1 Matrix

Elementary row (column) operations of a matrix

1. (Interchange) Interchange two rows (columns): $r_i \leftrightarrow r_j$
2. (Scaling) Multiply all entries in a row (column) by a **nonzero** constant: $r_i \to \alpha.r_i, \alpha \neq 0$
3. (Replacement) Replace one row (column) by the sum of itself and a multiple of another row (column) $r_i \to r_i + \alpha.r_j, \forall\alpha$

Row equivalent: Two matrices A and B are called **row equivalent**: $A \sim B$ (A is row equivalent to B) if B can be obtained from A after a finite number of elementary row operations.

Rank of matrices: Given that $A \in R_{m\times n}$. $A \xrightarrow{\text{elementary operations}}$ echelon form$E$. $r(A) =$ number of nonzero rows of E.

- $r(A) = r(A^T)$
- $r(A) \leq min\{m, n\}$

Inverse of a matrix: $A.A^{-1} = A^{-1}.A = I_n$

- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$

Method to find A^{-1}:$[A|I_n] \xrightarrow{\textit{Row elementary operations}} [I_n|A^{-1}]$

0.2.2 Systems of linear equations

Gaussian elimination method: $A \to$ echelon form A_1, we have the equivalent system $A_1.X = b_1$.
Back substitution.
Croneker -Capelli theorem: Consider the linear system $A.X = b, A \in R_{m\times n}$. If:

- $r(A) < r(A|b)$: there is no solution.
- $r(A) = r(A|b) = n$: the solution is unique.
- $r(A) = r(A|b) = r < n$: there is an infinite number of solutions

Homogeneous system of linear equations $b = 0$

- $A.X = 0$ always has at least trivial solution $X = (0, 0...0)^T$: $r(A) = r(A|0)$
- $r(A) = n$: The trivial solution $X = (0, 0...0)$ is the unique solution
- $r(A) < n$: There is an infinite number of solutions

Square system

- The system has unique solution $\Leftrightarrow r(A) = n \Leftrightarrow det(A) \neq 0$
- The system has infinitely many solutions $\Leftrightarrow det(A) = 0$ và $rank(A|b) = n$
- The system has no solution $\Leftrightarrow det(A) = 0$ và $rank(A|b) = n + 1$

Homogeneous square system $b = 0$

- The system has unique solution $\Leftrightarrow r(A) = n \Leftrightarrow det(A) \neq 0$,
- The system has infinitely many solutions $\Leftrightarrow r(A) < n \Leftrightarrow det(A) = 0$

0.2.3 Determinant

A_{ij} **cofactor**: Deleting the $i-$th row and the $j-$th column of A we obtain B.

$$\boxed{A_{ij} = (-1)^{i+j}|B|}$$

Using the $i - th$ **row (** $j-$ **column) to expand the determinant**

$$\boxed{det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in} = a_{1j}A_{1j} + a_{2j}A_{2j} + ... + a_{nj}A_{nj}}$$

Effects of Elementary Row Operations on Determinants

1. $A \xrightarrow{r_i \to \alpha r_i} B \Rightarrow |B| = \alpha|A|$
2. $A \xrightarrow{r_i \to r_i + \alpha r_j} B \Rightarrow |B| = |A|$
3. $A \xrightarrow{r_i \leftrightarrow r_j} B \Rightarrow |B| = -|A|$

The following statements are **equivalent**

- A is invertible
- $r(A) = n$
- $A \sim I_n$
- $det(A) \neq 0$

0.2.4 Exercises

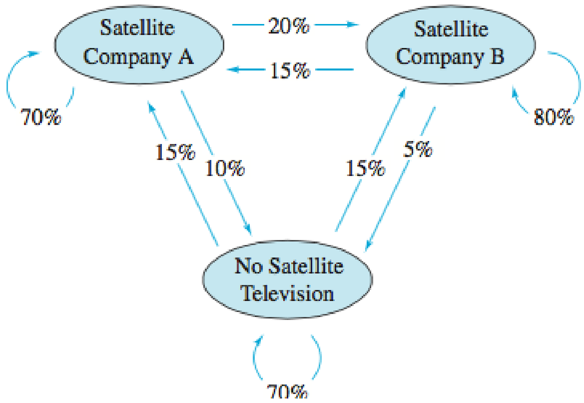
1. Given $A = \begin{bmatrix} 1 & m & 2 \\ 3 & 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 1 & 4 \\ 2 & 1 \end{bmatrix}$. Find $A + 2B^T, 3AB, 2B^TA^T$
2. Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}$. Find $f(A)$ given that $f(x) = 2x^2 + 6x - 2$.
3. A store sells commodities c_1, c_2, c_3 in two its branches B_1, B_2. The quantities of commodities sold in B_1, B_2 in a week are given in Table 1, the individual prices of commodities are given in Table 2, the costs to the store are given in Table 3. Find the store's profit for a week

4. (Markov chain) A machine can be either working or broken down on any given day. If it is working, it will break down in the next day with probability 5%, and will continue working with probability 95%. If it breaks down on a given day, it will be repaired and be working in the next day with probability 80%, and will continue to be broken down with probability 20%. Given that the probability that the machine works today is 80% (then, the probability that the machine breaks down is 20%). Find the probability that the machine will work the day after tomorrow? after 1 week.

5. (Leslie model) A population of rabbits raised in a research laboratory has the characteristics listed below.
(a) Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3 years.
(b) During the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year. The laboratory population now consists of 24 rabbits in the first age class, 24 in the second, and 20 in the third. How many rabbits will be in each age class in 1 year, 2 years?

6. (Markov chain) In a city with 1000 householders there are 3 supermarkets A, B and C. At this month, there are 200, 500 and 300 householders that go to the supermarkets A, B and C, respectively. After each month, there are 10% of customers of A change to B, 10% of those change to C; 7% of customers of B change to A, 3% of those change to C; 8.3% of customers of C change to A, 6.7% of those change to B. Find the numbers of customers of each supermarket after 1 month, 2 months.

7. (Markov chain) Two competing companies offer satellite television service to a city with 100 000 households. The transition matrix is $\begin{bmatrix} 0.7 & 0.15 & 0.15 \\ 0.2 & 0.8 & 0.15 \\ 0.1 & 0.05 & 0.7 \end{bmatrix}$ (see the figure below for the changes in satellite subscriptions each year). Company A now has 10 000 subscribers and Company B has 15 000 subscribers. How many subscribers will each company have after 1 year? 2 years?



8. Given $A = \begin{bmatrix} 1 & -3 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$. Find $X : AX + XB = 2A + 3B^T$

9. Find m such that $A = \begin{bmatrix} 1 & -2 \\ 2 & m \end{bmatrix}$ is invertible. With m found, find A^{-1}.

10. Given that $A = \begin{bmatrix} 1 & -3 & -2 \\ 3 & 6 & 2 \\ 4 & 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 3 & 1 \\ 3 & 1 & 5 \end{bmatrix}$.
    Find $X : (A + B^T)X = 3X + 2A - 4B$

11. Given that $A = \begin{bmatrix} 1 & -3 & -2 \\ 3 & 6 & 2 \\ 4 & 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 3 & 1 \\ 3 & 1 & 5 \end{bmatrix}$.
    Find $X : XA + B^T = 3X + 2A$

12. Find the rank of the following matrices:

    a) $A = \begin{bmatrix} 1 & -3 & -2 \\ 3 & 6 & 2 \\ 4 & 2 & -1 \end{bmatrix}$

    b) $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & m & 1 \end{bmatrix}$

    c) $A = \begin{bmatrix} 1 & -3 & -2 & 1 & 1 \\ 2 & 1 & 3 & 6 & 2 \\ 4 & 2 & -1 & 3 & 1 \\ 4 & -5 & -1 & 8 & 4 \end{bmatrix}$

    d) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & m & -2 \\ 3 & 5 & 4 \end{bmatrix}$

    e) $A = \begin{bmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{bmatrix}$

    f) $A = \begin{bmatrix} 1 & -2 & -1 & 2 \\ 2 & -4 & m & 2 \\ 1 & 10 & -6 & m \end{bmatrix}$

13. Application in cryptography To encode a message, we convert the text message into a stream of numerals by associating each letter with its position in the alphabet: A is 1, B is 2, $\dots$, Z is 26, and assigning the number 27 to a space between two words. After that, we break the enumerated message above into a sequence of 3×1 column vectors and place them into a matrix A. Multiplying this matrix A by the Encoder K (the key), we obtain the encrypted message $B = KA$. Decode the message, where $K = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 67 & 67 & 57 \\ 82 & 94 & 60 \\ 52 & 55 & 46 \end{bmatrix}$.

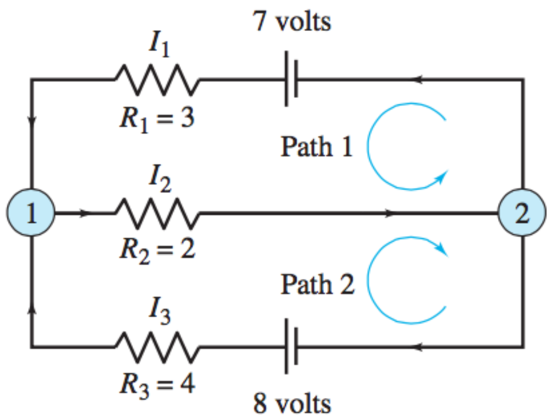
14. Solve the system $\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 3 \\ x_1 - x_2 + x_3 + 4x_4 = -1 \\ 4x_1 - x_2 + x_3 + 9x_4 = 1 \\ x_1 + 2x_2 - 2x_3 - 3x_4 = 4 \end{cases}$

15. Solve the system $\begin{cases} 2x_1 - x_2 - x_3 + x_4 = 0 \\ x_1 + 2x_2 - x_3 - 2x_5 = 0 \\ 7x_1 - x_2 - 4x_3 + x_4 + x_5 = 0 \end{cases}$

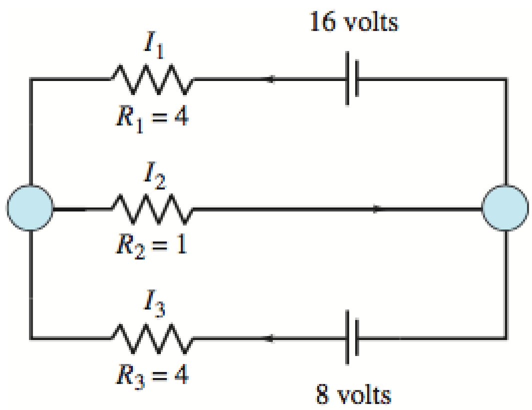
16. Find m such that $\begin{cases} mx_1 - x_2 + 2x_3 = 3 \\ x_1 - x_2 - x_3 = -2 \\ 3x_1 - x_2 - 4x_3 = 1 \end{cases}$ has unique solution

17. Find m such that $\begin{cases} x_1 + x_2 - 2x_3 = 0 \\ mx_1 + x_2 + x_3 = 0 \\ 2x_1 + 3x_2 + mx_3 = 0 \end{cases}$ has nontrivial solution

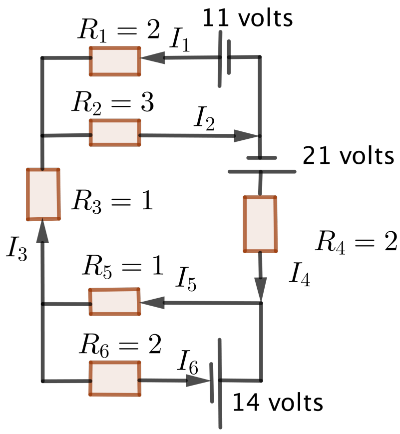
18. Find the currents I_1, I_2, I_3



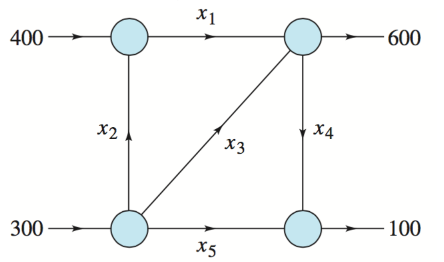
19. Find the currents I_1, I_2, I_3



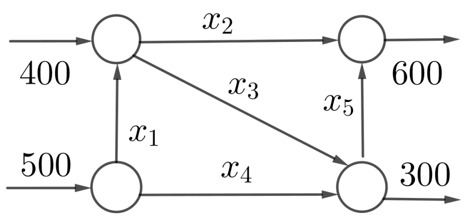
20. Find the currents I_1, I_2, I_3, I_4, I_5 and I_6 in the following electrical network:



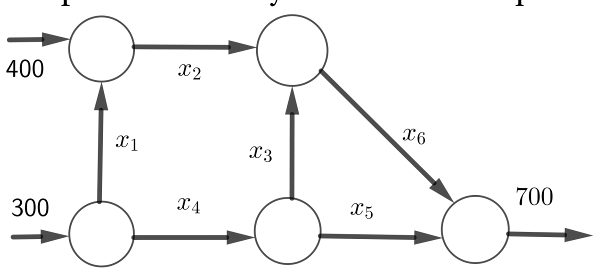
21. The figure shows the flow of traffic of a network. The diagram indicates the average number of vehicles per hour entering and leaving the intersections and the arrows indicate the direction of traffic flows. All streets are one-way. Setup and solve a system of linear equations to find the possible flows x_i



22. Setup and solve a system of linear equations to find the possible flows x_i



23. Setup and solve a system of linear equations to find the possible flows x_i



24. (Input-output Leontief model) Consider a very simple economy that runs on 3 different types of output: raw materials, services, and manufacturing. Raw materials include the output of many different industries, agriculture and mining to name two. Services include retailing, advertising, transportation, etc. The

raw materials industry needs some of the output from the other two industries to do its job. For example, it needs trucking to get its goods to market, and it uses some manufactured goods (machines.) The raw materials industry even needs some of its own output to produce its own output – iron ore to make the steel to build the rails that carry ore from the mines, for example. Each industry requires some amount of output from each of the three to do its job. All of these requirements can be summarized in the following table:

Industry	Raw materials	Services	Manufacturing
Raw materials	0.02	0.04	0.04
Services	0.05	0.03	0.01
Manufacturing	0.2	0.01	0.1

The numbers in the table tell how much output from each industry a given industry requires in order to produce one dollar of its own output. For example, to provide $1\$$ worth of service, the service sector requires $0.05\$$ worth of raw materials, $0.03\$$ worth of services, and $0.01\$$ worth of manufactured goods. The **demand matrix** D tells how much (in billions of dollars) of each type of output is demanded by consumers and others outside the economy

$D = \begin{bmatrix} 400 \\ 200 \\ 600 \end{bmatrix}, A = \begin{bmatrix} 0.02 & 0.04 & 0.04 \\ 0.05 & 0.03 & 0.01 \\ 0.2 & 0.01 & 0.1 \end{bmatrix}$. Let X denote the **production matrix**. It represents the amounts (in billions of dollars of value) produced by each of the three industries. Then, we have $X = AX + D$. Find X.

25. (Market equilibrium) Find the equilibrium price, given the supply function: $Q_s = -2 + 3p$, the demand function: $Q_d = 4 - 5p$.

26. (Multi-market equilibrium) Find the equilibrium prices in the market of 2 type of commodities, given the supply and demand functions for:
Commodity 1: $Q_{s_1} = -2 + 3p_1; Q_{d_1} = 10 - 2p_1 + p_2$.
Commodity 2: $Q_{s_2} = -1 + 2p_2; Q_{d_2} = 15 + p_1 - p_2$.

27. Find $det(A)$

a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & m & -2 \\ 3 & 5 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 9 \\ 1 & 3 & 5 & 6 \\ 1 & 3 & 5 & 7 \end{bmatrix}$

c) A^5 where $A = \begin{bmatrix} 1 & 1 & 1 & m \\ 2 & -2 & -2 & 4 \\ m & 5 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

28. Find the degree of:$P(x) = \begin{vmatrix} 1 & 1 & 1 & x \\ -2 & -2 & -2 & x^2 \\ 3 & 5 & 4 & x^3 \\ 1 & 2 & 3 & x^4 \end{vmatrix}$

29. Find m such that A is invertible: $A = \begin{bmatrix} 1 & 2 & m \\ 4 & m & 3 \\ 3 & -5 & 4 \end{bmatrix}$

30. Given $A, B \in M_3, |A| = 3, |B| = 4$. Evaluate:

    a) $|2AB|$

    b) $|3(AB)^{-1}|$

    c) $|3A^{-1}B^{2019}|$

    d) $|P_{3A}B^{-1}|$

0.3 Vector spaces

0.3.1 Summary

1. **Linear independence**
   $M = \{e_1, e_2 ... e_n\}$ is **linearly independent** (LI) $\Leftrightarrow \forall(\alpha_1, \alpha_2, ..., \alpha_n) : \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0 \Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = 0.$

   $$\boxed{\text{LI} \Leftrightarrow rank(M) = n}$$

2. **Linear dependence**
   $M = \{e_1, e_2 ... e_n\}$ is **linearly dependent** (LD) $\Leftrightarrow \exists(\alpha_1, \alpha_2, ..., \alpha_n) \neq (0, 0..0) : \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0$

   $$\boxed{\text{LD} \Leftrightarrow rank(M) < n}$$

3. **Rank of a vector set**
   Rank of a vector set M = number of vectors of any maximal independent subset of M.
   $rank(M) = rank(col(M)) = rank(row(M))$
   Theorem:

   - x is a linear combination of $M \Leftrightarrow r(M) = r(M, x)$
   - x isn't a linear combination of $M \Leftrightarrow r(M, x) = r(M) + 1$

4. **Spanning set**
   $M = \{e_1, e_2 ... e_n\}$ - spanning set of $V \Leftrightarrow \forall u \in V, \exists(\alpha_1, \alpha_2, ..., \alpha_n) : u = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$

5. **Basis**

   $$\boxed{\text{Basis= Spanning set + LI}}$$

   Dimension $dim(V)$ = number of vectors in a basis.
   Theorem:

   - M - spanning set $\Leftrightarrow rank(M) = dim(V)$
   - M - basis $\Leftrightarrow rank(M) = dim(V) =$ number of vectors of M.

6. **Coordinates**

   E is a basis of V. $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{bmatrix} = [x]_E$ if $x = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n$.

   In $\mathbb{R}^n$:

   - $[x]_E = E^{-1}[x]$, hay $[x] = E[x]_E$
   - Let E and F be 2 bases: $[x] = E[x]_E = F[x]_F \Rightarrow [x]_F = F^{-1}E.[x]_E$
     $T_{E \to F} = F^{-1}E$ - change of basis matrix from E to F.

7. **Subspace**
   The span of a vector set
   $U = span\{e_1, e_2 ... e_m\} = \{x \in V | x = \alpha_1 e_1 + .. + \alpha_m e_m\}$
   ($E = \{e_1, e_2 ... e_n\}$ is a spanning set of U)
   Nullspace of a homogeneous system
   $null(A) = \{x \in \mathbb{R}_n | A.x = 0\}$

   $$\boxed{dim(null(A)) = n - r(A)}$$

   Basis of $null(A)$ is a set of $n - r(A)$ independent solutions of the system.

0.3.2 Exercises

LI, LD

Are the following sets LI or LD:

1. In $\mathbb{R}_3$ given $M = \{(1, 2-1), (3, 1, 4), (-2, 0, 1), (1, 2, 3)\}$
2. In $P_2(x)$ given $M = \{2x^2 + 2x + 3, x - 4, x + 1\}$
3. In $M_2(R)$ given $M = \{\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 7 \\ -7 & 10 \end{bmatrix}\}$
4. Find m such that $2x^2 + x + m, x^2 - 2x - m, x + 1$ is LI.
5. Find m such that $(-2, 1, 3), (2, m, 1)(1, 4, 0), (-2, 2, 1)$ is LI.

In a vector space V given a LI set $\{x, y, z\}$. Are the following set LI or LD:

1. $M = \{x + 2y, x - 3y, x + y\}$
2. $N = \{x + y + z, x - y - z, 2y - z\}$
3. $P = \{x + y + 3z, 4x - y + 2z, 2x - 3y - 4z$
4. $R = \{x + y + 2z, x + y + z, 2x, 4y - z\}$

Spanning set, basis

1. Is $M = \{(1, 2-1), (3, 1, 4), (-2, 0, 1), (1, 2, 3)\}$ a spanning set of $\mathbb{R}_3$?

2. Is $M = \{x^2 - 2x + 3, x - 4, 2x + 1\}$ a basis of $P_2(\mathbb{R})$?

3. In V given a spanning set $\{x, y, z\}$. Is $M = \{x+2y, x-3y, x+y, x+3y\}$ a spanning set?

4. In V given a basis $\{x, y, z\}$. Is $M = \{x+2y, x-3y, x+y+z, x+3y+z\}$ a spanning set?

5. Let $\{x, y, z\}$ be a basis of V. Is $M = \{x + 2y, x + y - z, x + 3y + z\}$ a basis of V?

6. Let$\{x, y, z\}$ be a spanning set of V. Is $M = \{x + 2y - z, x + 2y - z, x + 3y + z\}$ a spanning set of V?

7. Let $\{x, y, z\}$ be a spanning set of V. Is $M = \{x + 2y - z, 2x + 2y - 3z, x - 2z\}$ a spanning set of V?

8. Let $\{x, y, z\}$ be a spanning set of V. Is $M = \{x, 2y, 3z\}$ linearly independent?

Coordinates

1. Let $E = \{(1, 2-1), (3, 1, 4), (-2, 0, 1)\}$ be a basis of $\mathbb{R}_3$.
   - Given $x = (3, 4, 5)$. Find $[x]_E$
   - Given $x = (a, b, c)$. Find $[x]_E$

2. Let $E = \{x^2 + 2x + 1, 2x^2 - x - 2, x + 3\}$, $F = \{x^2 - x + 2, x^2 - 1, 2x + 3\}$ be 2 bases of $P_2[\mathbb{R}]$. Given that $[x]_E = (3, 1, -4)^T$, find $[x]_F$

Subspaces

1. In $\mathbb{R}_3$ given a subspace $U = span\{(2, 2, -3), (0, 1, 2), (4, 5, -4), (2, 1, -5)\}$. Find one basis and the dimension of U.

2. In $\mathbb{R}_3$ given $V = \{x \in \mathbb{R}_3 | 3x_1 + 5x_2 - 1x_3 = 0\}$. Find one basis and the dimension of V.

3. In $\mathbb{R}_3$ given $V = \{x \in \mathbb{R}_3 | 2x_1 + 3x_2 - 4x_3 = 0, x_1 - x_2 + 5x_3 = 0\}$. Find one basis and the dimension of V.

4. In $\mathbb{R}_4$ given $U =< (1, 2, -1, 4), (3, 1, 2, 4), (1, -3, 4, -4), (4, 3, 1, 8), (1, 2, 3, 4) >$. Find one basis and the dimension of U.

5. In $\mathbb{R}_4$ given $V = \{x \in \mathbb{R}_4 | x_1 + 2x_2 - x_3 + x_4 = 0, 2x_1 - x_2 + 3x_3 + 2x_4 = 0, 4x_1 + 3x_2 + x_3 + 4x_4 = 0, x_1 - 3x_2 + 4x_3 + x_4 = 0$. Find one basis and the dimension of V.

6. In $\mathbb{R}_3$ given $U =< (2, 3, -4), (1, 3, 2), (5, 12, 2) >$
   - Is $x = (3, 4, 5)$ in U?
   - Find m such that $x = (1, 3, m)$ is in U

7. In $\mathbb{R}_3$ given $U = \{x \in \mathbb{R}_3 | x_1 + x_2 - x_3 = 0\}$
   - Find one basis E and the dimension of U.
   - Find m such that $x = (1, 3, m)$ is in U. Find $[x]_E$

0.4 Inner product spaces

0.4.1 Summary

Inner product spaces = Vector space + inner product.
Inner product: $(x, y) = \|x\|.\|y\|.cos(\widehat{x, y})$, **norm**:$\|x\| = \sqrt{(x, x)}$, **distance**: $d(x, y) = \|x - y\|$, angle between 2 vectors: $\widehat{x, y} = \arccos \dfrac{(x, y)}{\|x\|.\|y\|}$
Orthogonal:

- $u \perp v \Leftrightarrow (u, v) = 0$
- $u \perp M, (M = \{e_1, ...e_n\}) \Leftrightarrow u \perp e_i, \forall i$
- $M \perp N, (M = \{e_1, ...e_n\}, N = \{f_1, ...f_m\}) \Leftrightarrow e_i \perp f_j \forall i, j$
- $u \perp V \Leftrightarrow u \perp$ spanning set of V
- $U \perp V \Leftrightarrow$ spanning sets of them are orthogonal.

Orthogonal complement :
Let W be a subspace of V, $W^\perp := \{x \in V | x \perp W\}$ - is called the **orthogonal complement** of W in V.
Projection of a vector onto a subspace:
Let W be a subspace of V. Any vector $u \in V$ can be represented uniquely as: $u = x + y, x \in W, y \in W^\perp$. Then, x - orthogonal projection vector of u onto $W : x = pr_W(u)$, y - rejection.
Gram - Schmidt process: Construct an orthogonal set from $M = \{e_1, ...e_n\}$

- $f_1 = e_1$
- $f_2 = e_2 - \frac{(e_2, f_1)}{(f_1, f_1)} f_1$ (We can multiply f_2 by α so that we don't have the fractional elements in f_2.)
- $f_3 = e_3 - \frac{(e_3, f_1)}{(f_1, f_1)} f_1 - \frac{(e_3, f_2)}{(f_2, f_2)} f_2$
- ...
- $f_n = e_n - \frac{(e_n, f_1)}{(f_1, f_1)} f_1 - ... - \frac{(e_n, f_n)}{(f_n, f_n)} f_n$

Orthonormalization: From $M = \{e_1, ...e_n\} \to$ orthogonal set $\to$ orthogonal set: $g_i = \frac{f_i}{\|f_i\|}$.

0.4.2 Exercises

1. Let $(x, y) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2 + 7x_3y_3$ be an inner product of $\mathbb{R}_3$. Find:

   a) $\|(3, -2, 1)\|$

   b) Find a unit vector u that is parallel to $(3, -2, 1)$

   c) $(-1, \widehat{1, 2), (2}, 3, 4)$, $d((-1, 1, 2), (2, 3, 4))$

   d) $u = (4, 1, 2), v = (1, 3, 4)$. Find the norm of $3u - 2v$.

2. In an inner product space V given 2 vectors x and y satisfying $\|x\| = 2, \|y\| = 3, \widehat{x, y} = \frac{\pi}{6}$. Find $d(2x + 3y, x - 4y)$

3. In $\mathbb{R}_4$ with the dot product, given $U =< (1, 2, -1, 4), (3, -1, 4, -2) >$, $V = (1, 2, m, n) >$. Find m, n such that $V \perp U$

4. In $\mathbb{R}_3$ with the dot product, given $U =< (1, 2, 3), (-3, 4, 2), (1, 12, 14) >$. $u = (4, 5, 7)$

   a) Find one basis and the dimension of $U^\perp$

   b) Find one orthonormal basis of U.

   c) Find the orthogonal projection of u onto U.

5. In $\mathbb{R}_4$ with the inner product $(x, y) = 2x_1y_1 + 3x_2y_2 + x_3y_3 + 4x_4y_4$, given $U =< (1, 3, 2, 1), (2, -1, 1, 0) >$ và $z = (3, 2, 11, 16)$. Find the orthogonal projection of z onto U.

6. In $\mathbb{R}_3$ with the inner product $(x, y) = 3x_1y_1 + 2x_2y_2 + 4x_3y_3$, given $F = \{x \in \mathbb{R}_3 | 2x_1 - 3x_2 + x_3 = 0\}$ and $u = (2; 1; 1)$

   a) Find the orthogonal projection of u onto F.

   b) Find the distance from u to F.

7. In $\mathbb{R}_3$ with the inner product $(x, y) = 4x_1y_1 + 3x_2y_2 - x_2y_3 - x_3y_2 + 3x_3y_3$, given $U =< (2, 1 - 3), (1, 1, 3) >$ and $z = (2, 3, 4)$.

   a) Find m such that $(1, 2, m)$ is in U

   b) Find one basis and the dimension of $U^\perp$.

   c) Find the orthogonal projection of z onto $U^\perp$.

8. In $\mathbb{R}_4$ with the dot product, given $U =< (2, 1, 3, -1), (3, 2, 1, -2) >$. Find the orthogonal projection of $U^\perp$

9. In $\mathbb{R}_4$ with the dot product, given $U =< (1, 1, 4, -1), (3, 1, 1, -2) >$.

   a) Find one basis and the dimension of $U^\perp$

   b) Find the orthogonal projection of $z = (11, -7, 7, 7)$ onto $U^\perp$

   c) Find the orthogonal projection of $w = (3, 2, 11, 16)$ onto U.

10. In $\mathbb{R}_4$ with the dot product, given $U = \{x \in \mathbb{R}_4 | x_1 + x_2 - 2x_3 + x_4 = 0, 2x_1 + x_2 - 3x_3 + x_4 = 0, 5x_1 + 4x_2 - 9x_3 + 4x_4 = 0\}$.

    a) Find one basis and the dimension of $U^\perp$

b) Find one orthonormal basis of U.

11. In $\mathbb{R}_4$ with the inner product $(x,y) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2 + x_3y_3 + 2x_4y_4$, given $U =< (1,2,1,2), (-3,-1,2,-2), (1,-3,-4,-2) >$.

    a) Find one basis and the dimension of $U^\perp$

    b) Find the orthogonal projection of $z = (4,2,3,-1)$ onto U.

0.5 Linear transformation

0.5.1 Summary

- **Linear transformation**: $f : U \to V : f(ax+by) = af(x) + bf(y)$

- **Transformation matrix**: Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. E is a basis of $\mathbb{R}^m$, F is a basis of $\mathbb{R}^n$. A_{EF} - transformation matrix of f in the bases E, F:

$$\boxed{A_{EF} = F^{-1}f(E), \qquad [f(x)]_F = A_{EF}[x]_E}$$

- **Transformation matrix in the standard bases**: Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. A - Transformation matrix of f in the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$:

$$\boxed{[f(x)] = A[x]}$$

  Let $E = \{e_1, e_2, ..., e_m\}$ be a basis of $\mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Given $f(e_1), f(e_2), ...f(e_m)$.

$$\boxed{A = f(E).E^{-1}}$$

- **Transformation matrix** of $f : \mathbb{R}^n \to \mathbb{R}^n$ **in** E:

$$\boxed{A_E = A_{EE} = E^{-1}f(E)}$$

- **Kernel of the linear transformation**: $Kerf = \{x \in U | f(x) = 0\}$

- **Image of the linear transformation**: $Imf = \{y \in V | \exists x \in U : y = f(x)\}$.

$$\boxed{U = span\{e_1, e_2...e_m\} \Rightarrow Imf = span\{f(e_1), f(e_2)...f(e_m)\}}$$

  ($f(E)$ is a spanning set of Imf)

0.5.2 Exercises

1. Given a linear transformation $f : \mathbb{R}_3 \to \mathbb{R}_2, f(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, x_1 + x_2 + x_3)$.

   a) Find $f(3, 2, 4)$
   b) Find one basis, the dimension of $Kerf$.
   c) Find one basis, the dimension of Imf.

2. Let $f : \mathbb{R}_3 \to \mathbb{R}_3 : f(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, x_1 + x_2 + x_3, x_1 - 2x_3)$ be a linear transformation

   a) Find A_{EF} in 2 bases $E = \{(1, 1, 1), (1, 1, 2), (1, 0, 0)\}$, $F = \{(1, 1, 1), (1, 0, 1), (0, 0, 1)\}$.
   b) Fin A_E in the basis E

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying $f(1; 1; 2) = (-1; 2; 1), f(1; 1; 5) = (2; 2; 3), f(3; 2; 8) = (1; 4; 4)$

   a) Find $f(3; 4; 5)$
   b) Find $f(x_1; x_2; x_3)$
   c) Find one basis, the dimension of $Kerf$.
   d) Find one basis, the dimension of Imf.

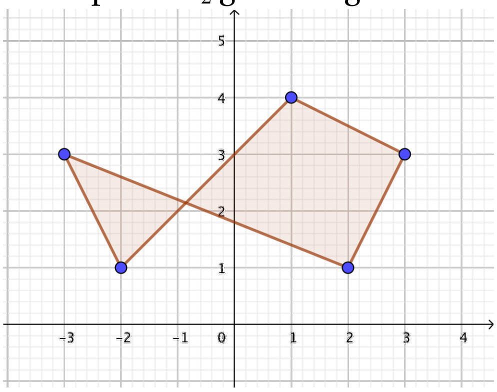
4. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying $f(u_1) = f(2; 1; 4) = (1; 2; -1), f(u_2) = f(1; 1; 5) = (2; 1; 3), f(u_3) = f(3; 2; 8) = (4; 1; 2)$.
   Find A_E of f in the basis $E = \{(1; 2; 1), (2; 1; 1), (1; 1; 1)\}$

5. Let the transformation matrix of $f : \mathbb{R}_3 \to \mathbb{R}_3$ in the basis $E = \{(1, 1, 0), (0, 1, 2), (0, 3, 1)\}$ be $A_E = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 3 & 3 \end{bmatrix}$. Find one basis, the dimension of the kernel and image of f.

6. Let $f : \mathbb{R}_3 \to \mathbb{R}_3$ be a linear transformation with the matrix in the basis $E = \{(1, 1, 1), (1, 1, 2), (1, 2, 1)\}$: $A_E = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 3 & 0 \\ 3 & 1 & 1 \end{bmatrix}$.

   a) Find $f(3, -4, 5)$
   b) Find $f(x_1, x_2, x_3)$
   c) Find the transformation matrix A_F of f in the basis $F = \{(1, 1, 1), (1, 1, 2), (-1, 5, 3)\}$.

7. In the plane $\mathbb{R}_2$ given a figure with the vertices $A(3, 3), B(1, 4), C(-3, 3), D(-2, 1), E(2, 1)$



How is this figure mapped if we use

a) Rotation transformation f_1 about the origin 0 counterclockwise by $\frac{\pi}{3}^o$. Find the dimension, one basis for Imf_1, $Kerf_1$.

b) Reflection transformation f_2 about the line $y = 2x$. Find the dimension, one basis for Imf_2, $Kerf_2$.

c) the rotation transformation f_1 followed by the reflection transformation f_2. Find the dimension, one basis for the kernel and image of this composite transformation.

8. In $\mathbb{R}^3$ find the image $f(x; y; z)$, where f is the rotation transformation about the z axis $\frac{\pi}{4}$ clockwise from the positive direction of z axis. Find $Imf, Kerf$.

9. In $\mathbb{R}^3$ find the image $f(x; y; z)$, where f is the projection transformation onto the plane $x + 2y - 3z = 0$. Find $Imf, Kerf$.

0.6 Eigenvalues, eigenvectors, diagonalization

0.6.1 Summary

- λ is one **eigenvalue** of a square matrix $A \Leftrightarrow \exists x \neq 0 : Ax = \lambda x$. The column vector x is an **eigenvector** of A

$$\boxed{\lambda \text{ is an eigenvalue of} A \Leftrightarrow |A - \lambda I| = 0}$$

- $P(\lambda) = |A - \lambda I|$- **characteristic polynomial**.
If $A \in M_2$: $P(\lambda) = \lambda^2 - trace(A)\lambda + det(A)$
If $A \in M_3$: $P(\lambda) = -\lambda^3 + trace(A)\lambda^2 - (A_{11} + A_{22} + A_{33})\lambda + det(A)$
Solve $P(\lambda) = 0$ we obtain the eigenvalues $\lambda_1, \lambda_2 ...$ where the **algebraic multiplicity** of λ_i is the multiplicity of the root in the equation.

- The set of all eigenvectors x associated with λ_i: $Ax = \lambda_i x$ - solution of $(A - \lambda_u I)x = 0$ is called the **eigenspace** E_{λ_i} associated with λ_i. $dim(E_{\lambda_i})$ is called the **geometric multiplicity** of λ_i.

- A is called **diagonalizable** if $A = PDP^{-1}$, where D-diagonal, P - invertible.

$$\boxed{A \text{ is invertible} \Leftrightarrow \forall \lambda_i : \text{algebraic multiplicity = geometric multiplicity}}$$

If A is diagonalizable, then $D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ and $P = [e_1, e_2 ..., e_n]$: the corresponding eigenvectors.

- **Properties**
Let λ be one eigenvalue of A with the corresponding eigenvector$x$.

  1. A^n one eigenvalue λ^n with the corresponding eigenvector x.
  2. If $B^m = A$, m is odd, then B has one eigenvalue $\sqrt[m]{\lambda}$ with the corresponding eigenvector x.
  3. If A is invertible then A^{-1} has one eigenvalue $\frac{1}{\lambda}$ with the corresponding eigenvector x.

If A is diagonalizable $A = PDP^{-1}$ then:

1. $A^m = PD^mP^{-1}$, where $D^m = diag(\lambda_1^m, \lambda_2^m, ...\lambda_n^m)$
2. $B^m = A$, (m is odd) then $B = PD_1P^{-1}$, where $D_1 = diag(\sqrt[m]{\lambda_1}, \sqrt[m]{\lambda_2}, ..., \sqrt[m]{\lambda_n})$

0.6.2 Exercises

1. Given $A = \begin{bmatrix} 2 & 2 & m \\ 1 & 3 & 2m \\ 2 & 4 & 1 \end{bmatrix}$

   a) Find m such that $x = (2, -1, -1)$ is an eigenvector of A.

   b) Find m such that $\lambda = 1$ is an eigenvalue of A.

   c) With m found in b), diagonalize A.

2. Diagonalize $A = \begin{bmatrix} 10 & 12 & 6 \\ -2 & 0 & 2 \\ 2 & 4 & -2 \end{bmatrix}$

3. Diagonalize $A = \begin{bmatrix} 1 & -7 & 5 \\ 0 & -7 & 6 \\ 2 & -13 & 10 \end{bmatrix}$. Find B: $B^3 = A$

4. Find B such that $B^5 = A = \begin{bmatrix} -61 & 31 & 31 \\ -93 & 63 & 31 \\ -93 & 31 & 63 \end{bmatrix}$.

5. Given $A = \begin{bmatrix} 5 & 6 & 3 \\ -1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$. Find A^{2016}

6. Given $A = \begin{bmatrix} 3 & 2 & -3 \\ 3 & 4 & -3 \\ 4 & 4 & -3 \end{bmatrix}$. Find A^{6102}

7. (Markov chain) A machine can be either working or broken down on any given day. If it is working, it will break down in the next day with probability 5%, and will continue working with probability 95%. If it breaks down on a given day, it will be repaired and be working in the next day with probability 80%, and will continue to be broken down with probability 20%. Given that the probability that the machine works today is 80% (then, the probability that the machine breaks down is 20%). Find the probability that the machine will work after 1 years.

8. (Leslie model) A population of rabbits raised in a research laboratory has the characteristics listed below.
   (a) Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3 years.
   (b) During the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year. The laboratory population now consists of 24 rabbits in the first age class, 24 in the second, and 20 in the third. Find the stable age distribution vector.

9. (Markov chain) In a city with 1000 householders there are 3 supermarkets A, B and C. At this month, there are 200, 500 and 300 householders that go to the supermarkets A, B and C, respectively. After each month, there are 10% of customers of A change to B, 10% of those change to C; 7% of customers of B change to A, 3% of those change to C; 8.3% of customers of C change to A, 6.7% of those change to B.

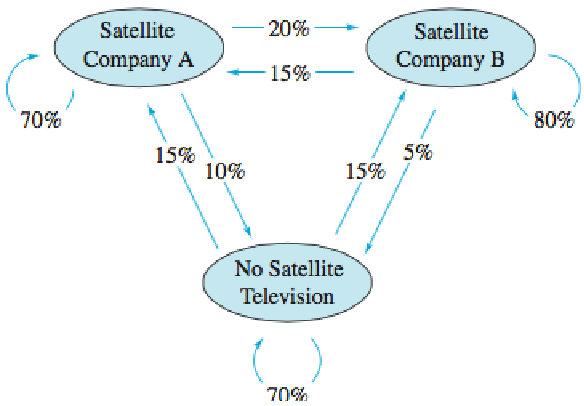
   a) Find the stable distribution vector (the numbers of supermarkets' customers won't change years after years)

   b) Use the model and the method of diagonalization to predict the numbers of customers of each supermarket after 3 years.

10. (Markov chain) Two competing companies offer satellite television service to a city with 100 000 households. The transition matrix is $\begin{bmatrix} 0.7 & 0.15 & 0.15 \\ 0.2 & 0.8 & 0.15 \\ 0.1 & 0.05 & 0.7 \end{bmatrix}$ (see the figure below for the changes in satellite subscriptions each year). Company A now has 10 000 subscribers and Company B has 15 000 subscribers.

    a) Find the stable distribution vector (the numbers of subscribers won't change years after years)

    b) Use the model and the method of diagonalization to predict the numbers of subscribers that each company will have after 50 years.



11. Orthogonally diagonalize the following matrices

    a) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

    b) $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

0.7 Quadratic forms

0.7.1 Summary

Quadratic forms:

$$\boxed{f(X) = a_{11}x_1^2 + a_{22}x_2^2 + ... + a_{nn}x_n^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + ... + 2a_{ij}x_ix_j = X^T.A.X}$$

where $A = (a_{ij})$ is symmetric: $a_{ij} = a_{ji}$, $X = (x_1, x_2, ..., x_n)^T$.
Our goal: Transform $f(X)$ into the **canonical** form for classification:

$$\boxed{f(Y) = a'_{11}y_1^2 + a'_{22}y_2^2 + ... + a'_{nn}y_n^2}$$

Orthogonal change of variables:
- Orthogonal matrix

$$\boxed{P^T = P^{-1}}$$

- Orthogonally diagonalization

$$\boxed{A = PDP^{-1} = PDP^T}$$

where P - orthogonal matrix.
With the quadratic form $f(X)$, using $Y = P^{-1}X = P^TX$ (or $X = PY$) we have:
$f = X^TAX = X^T.PDP^T.X = (P^TX)^T.D.(P^TX) = Y^TDY$ - canonical form.

To orthogonally diagonalization:

- Find the eigenvalues $\lambda_1, \lambda_2 ...$, the corresponding eigenspaces $E_{\lambda_1}, E_{\lambda_2} ...$
- Find the orthogonal basis for each (Gram-Schmidt): $f_1, f_2, ..., f_n$.
- Normalize: $e_i = \frac{f_i}{\|f_i\|}$.
- The columns of P are the orthonormal eigenvectors:

$$\boxed{P = [e_1, e_2, ..., e_n], \quad D = diag(\lambda_1, \lambda_2 ... \lambda_n).}$$

0.7.2 Exercises

Make an orthogonal change of variables to transform the following quadratic form into the canonical form (the form without cross products)

1. $f = x_1^2 + x_2^2 + x_3^2 + 14x_1x_2 + 14x_1x_3 + 14x_2x_3$
2. $f = x_1^2 + 3x_2^2 - 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 8x_2x_3$
3. $f = 3x_1^2 + 3x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$
4. $f = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$
5. $f = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$