EIGENVALUES AND EIGENVECTORS

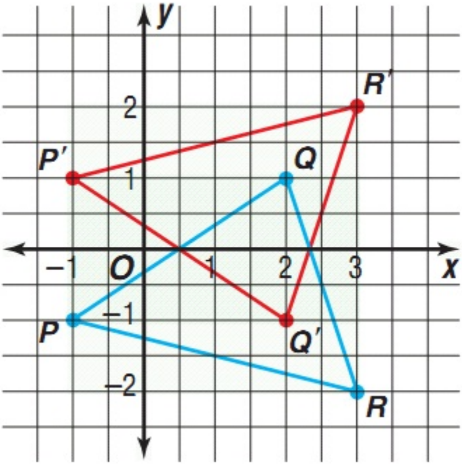
ELECTRONIC VERSION OF LECTURE

HoChiMinh City University of Technology
Faculty of Applied Science, Department of Applied Mathematics

HCMC — 2021.

OUTLINE

MODELLING MOTION



$\triangle PQR \rightarrow \triangle P'Q'R'$ is the reflection over the $x-$axis.

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the reflection matrix. Therefore, for every point in the plane (x_1, x_2), the matrix that results in a reflection over the $x-$axis and then we obtain a new point in the plane (y_1, y_2)

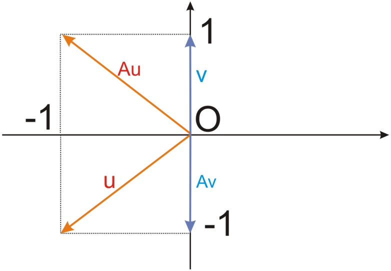
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

Question: For every point (x_1, x_2), find

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A^k . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ (k \in \mathbb{N}).$$

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $u = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We have

$A\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1.\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



EIGENVALUES AND EIGENVECTORS OF A MATRIX

DEFINITION 2.1

If A is an $n \times n$ matrix, then a nonzero vector $X \in \mathbb{R}^n, X \neq 0$ is called an ***eigenvector*** *of A if $AX = \lambda.X$ for some scalar λ. The scalar λ is called an* ***eigenvalue*** *of A and X is said to be an eigenvector corresponding to λ.*

EXAMPLE 2.1

Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The equation $AX = \lambda X$ can be rewritten as

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This homogeneous linear system has non-zero solution $X \neq 0$, thus

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 1 = 0$$

$$\Leftrightarrow \lambda_1 = -1, \lambda_2 = 1.$$

In the case where $\lambda_1 = -1$, we have

$$\begin{cases} 2x_1 + 0x_2 = 0 \\ 0x_1 + 0x_2 = 0 \end{cases} \Leftrightarrow x_1 = 0, x_2 = \alpha.$$

Therefore, the eigenvectors corresponding to $\lambda_1 = -1$ are $\alpha(0,1), \alpha \neq 0$.

In the case where $\lambda_2 = 1$. We have

$$\begin{cases} 0x_1 + 0x_2 = 0 \\ 0x_1 - 2x_2 = 0 \end{cases} \Leftrightarrow x_1 = \beta, x_2 = 0.$$

Therefore, the eigenvectors corresponding to $\lambda_2 = 1$ are $\beta(1,0), \beta \neq 0$.

Example 2.2

Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$AX = \lambda X$ can be rewritten

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This homogeneous linear system has non-zero solution $X \neq 0$, thus

$$\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)^2 + 4 = 0$$

$$\Leftrightarrow \lambda_{1,2} = 1 \pm 2i.$$

In the case where $\lambda_1 = 1 + 2i$. We have

$$\begin{cases} -2ix_1 + 2x_2 = 0 \\ -2x_1 - 2ix_2 = 0 \end{cases} \Leftrightarrow x_1 = \alpha, x_2 = \alpha i.$$

Therefore, the eigenvectors corresponding to λ_1 are $\alpha(1, i), \alpha \neq 0$.

In the case where $\lambda_2 = 1 - 2i$. We have

$$\begin{cases} 2ix_1 + 2x_2 = 0 \\ -2x_1 + 2ix_2 = 0 \end{cases} \Leftrightarrow x_1 = \beta, x_2 = -\beta i.$$

Therefore, the eigenvectors corresponding to λ_2 are $\beta(1, -i), \beta \neq 0$.

If λ is an eigenvalue of $A \Leftrightarrow \exists X \neq 0 : AX = \lambda.X$
$\Leftrightarrow AX - \lambda X = 0 \Leftrightarrow (A - \lambda I).X = 0.$
This homogeneous linear system has non-zero solution $X \neq 0$, thus $det(A - \lambda I) = 0$

DEFINITION 2.2

If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if $\chi_A(\lambda) = det(A - \lambda I) = 0$. This is called the characteristic equation of A. The polynomial $\chi_A(\lambda) = det(A - \lambda I)$ is called the characteristic polynomial.

FINDING EIGENVALUES AND EIGENVECTORS OF A SQUARE MATRIX

STEP 1. Finding the characteristic equation $\det(A - \lambda I) = 0$.

STEP 2. Solving this equation to find eigenvalues.

STEP 3. For every eigenvalue λ_i, solve the homogeneous system $(A - \lambda_i I)X = 0$ to find eigenvectors X corresponding to the eigenvalue λ_i.

THEOREM 2.1

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, *then*

$$\chi_A(\lambda) = |A - \lambda I| = -\lambda^3 + tr(A)\lambda^2 - \left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \right) \lambda + det(A)$$

where $tr(A) = a_{11} + a_{22} + a_{33}$ *is called the trace of* A.

DEFINITION 2.3

The eigenvetors corresponding to the eigenvalue λ, together with the zero vector, form the null space of the matrix $(A - \lambda I)$. This subspace is called the eigenspace corresponding to the eigenvalue λ.

Definition 2.4

If λ_0 is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the ***geometric multiplicity*** *of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the* ***algebraic multiplicity*** *of λ_0.*

Theorem 2.2

For every eigenvalue of A, the geometric multiplicity $\leqslant$ the algebraic multiplicity.

EXAMPLE 2.3

Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$

1. *Find the characteristic polynomial of* A
2. *Find eigenvalues and eigenvectors of* A

1. The characteristic polynomial of A

$$\chi_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 10\lambda^2 - 28\lambda + 24 = -(\lambda-2)^2(\lambda-6)$$

2. The characteristic equation of A

$$\chi_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow -(\lambda-2)^2(\lambda-6) = 0 \Leftrightarrow \lambda_1 = 2, \lambda_2 = 6.$$

In the case where $\lambda_1 = 2$, we have

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

$$\Rightarrow X_1 = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \alpha^2 + \beta^2 \neq 0.$$

The algebraic multiplicity of $\lambda_1 = 2$ is 2 and the geometric multiplicity of $\lambda_1 = 2$ is 2.

In the case where $\lambda_2 = 6$, we have

$$\begin{cases} -3x_1 + x_2 + x_3 = 0 \\ 2x_1 - 2x_2 + 2x_3 = 0 \\ x_1 + x_2 - 3x_3 = 0 \end{cases} \Rightarrow X_2 = \gamma \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \gamma \neq 0.$$

The algebraic multiplicity of $\lambda_2 = 6$ is 1 and the geometric multiplicity of $\lambda_2 = 6$ is 1.

DEFINITION 3.1

*If A and B are square matrices, then we say that B **is similar to** A if there is an invertible matrix S such that $B = S^{-1}AS$.*

DEFINITION 3.2

*A square matrix A is said to be **diagonalizable** if it is similar to some diagonal matrix D, that is, if there exists an invertible matrix S such that $S^{-1}AS = D$. In this case the matrix S is said to diagonalize A.*

We have $S^{-1}AS = D = dig(\lambda_1, \lambda_2, \dots, \lambda_n)$. It follows that $AS = SD$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

$$S = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \dots & \dots & \dots & \dots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{pmatrix} = \begin{pmatrix} S_{*1} & S_{*2} & \dots & S_{*n} \end{pmatrix}$$

$$AS = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} . \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \dots & \dots & \dots & \dots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{pmatrix}$$

$$= A\begin{pmatrix} S_{*1} & S_{*2} & \dots & S_{*n} \end{pmatrix} = \begin{pmatrix} AS_{*1} & AS_{*2} & \dots & AS_{*n} \end{pmatrix}$$

$$SD = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \dots & \dots & \dots & \dots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 S_{*1} & \lambda_2 S_{*2} & \dots & \lambda_n S_{*n} \end{pmatrix}$$

Therefore,

$$(AS)_{*i} = AS_{*i} = (SD)_{*i} = \lambda_i S_{*i}, (i = 1, 2, \ldots, n).$$

So, S_{*i} is the eigenvector corresponding to eigenvalue $\lambda_i (i = 1, 2, \ldots, n)$ of A.

Form the matrix S whose column vectors are the n basis eigenvectors of A.

EXAMPLE 3.1

Let $A = \begin{pmatrix} 15 & -18 & -16 \\ 9 & -12 & -8 \\ 4 & -4 & -6 \end{pmatrix}$. *Find a matrix* S *that diagonalizes* A.

Step 1. Find eigenvalues, eigenvectors of A.

$$\chi_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 15-\lambda & -18 & -16 \\ 9 & -12-\lambda & -8 \\ 4 & -4 & -6-\lambda \end{vmatrix} = 0$$

$\Leftrightarrow -(\lambda+3)(\lambda+2)(\lambda-2) = 0 \Leftrightarrow \lambda_1 = -3$ (AM=1), $\lambda_2 = -2$ (AM=1), $\lambda_3 = 2$ (AM=1).

In the case where $\lambda_1 = -3$ **(AM=1)**, we have

$$\left\{\begin{array}{r} 18x_1 - 18x_2 - 16x_3 = 0 \\ 9x_1 - 9x_2 - 8x_3 = 0 \\ 4x_1 - 4x_2 - 3x_3 = 0 \end{array}\right. \Rightarrow X_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha \neq 0.$$

In the case where $\lambda_2 = -2$ **(AM=1)**, we have

$$\begin{cases} 17x_1 - 18x_2 - 16x_3 = 0 \\ 9x_1 - 10x_2 - 8x_3 = 0 \\ 4x_1 - 4x_2 - 4x_3 = 0 \end{cases} \Rightarrow X_2 = \beta \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \beta \neq 0.$$

In the case where $\lambda_3 = 2$ **(AM=1)**, we have

$$\begin{cases} 13x_1 - 18x_2 - 16x_3 = 0 \\ 9x_1 - 14x_2 - 8x_3 = 0 \\ 4x_1 - 4x_2 - 8x_3 = 0 \end{cases} \Rightarrow X_3 = \gamma \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \gamma \neq 0.$$

Step 2. Find a matrix S that diagonalizes A.

$$S = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Then $S^{-1}AS = D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Computing Powers of a Matrix

Suppose A is diagonalizable, that is

$$\begin{aligned} S^{-1}AS = D = dig(\lambda_1, \lambda_2, \ldots, \lambda_n) \\ \Rightarrow (S^{-1}AS)^k = D^k, k \in \mathbb{N} \\ \Rightarrow S^{-1}A(S.S^{-1})AS.\ldots.S^{-1}AS = S^{-1}A^kS = D^k \\ \Rightarrow A^k = SD^kS^{-1}. \end{aligned}$$

Therefore,

$$A^k = S \begin{pmatrix} \lambda_1^k & 0 & \ldots & 0 \\ 0 & \lambda_2^k & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n^k \end{pmatrix} S^{-1}$$

EXAMPLE 3.2

Let $A = \begin{pmatrix} 0 & -8 & 6 \\ -1 & -8 & 7 \\ 1 & -14 & 11 \end{pmatrix}$. *Find* A^k, $k \in \mathbb{N}$.

$$\chi_A(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & -8 & 6 \\ -1 & -8-\lambda & 7 \\ 1 & -14 & 11-\lambda \end{vmatrix} = 0$$

$\Leftrightarrow -(\lambda-2)(\lambda+2)(\lambda-3) = 0 \Leftrightarrow \lambda_1 = -2$ (AM=1), $\lambda_2 = 2$ (AM=1), $\lambda_3 = 3$ (AM=1).

In the case where $\lambda_1 = -2$ **(AM=1)**, we have

$$\begin{cases} 2x_1 - 8x_2 + 6x_3 = 0 \\ -x_1 - 6x_2 + 7x_3 = 0 \\ x_1 - 14x_2 + 13x_3 = 0 \end{cases} \Rightarrow X_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha \neq 0.$$

In the case where $\lambda_2 = 2$ **(AM=1)**, we have

$$\begin{cases} -2x_1 - 8x_2 + 6x_3 = 0 \\ -x_1 - 10x_2 + 7x_3 = 0 \\ \quad x_1 - 14x_2 + 9x_3 = 0 \end{cases} \Rightarrow X_2 = \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta \neq 0.$$

In the case where $\lambda_3 = 3$ **(AM=1)**, we have

$$\left\{\begin{array}{r} -3x_1 - 8x_2 + 6x_3 = 0 \\ -x_1 - 11x_2 + 7x_3 = 0 \\ x_1 - 14x_2 + 8x_3 = 0 \end{array}\right. \Rightarrow X_3 = \gamma \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \gamma \neq 0.$$

A matrix S that diagonalizes A is $S = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$

$\Rightarrow S^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Therefore,

$A^k = SD^kS^{-1} =$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} (-2)^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

THEOREM 3.1

*The square matrix A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is **equal** to the algebraic multiplicity.*

EXAMPLE 3.3

Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ -2 & 0 & -1 \end{pmatrix}$. *Diagonalize A if A is diagonalizable.*

Step 1. Find eigenvalues, eigenvectors of A.

$$\chi_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ -2 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow -\lambda(\lambda-1)^2 = 0$$

$\Leftrightarrow \lambda_1 = 0$ (AM=1), $\lambda_2 = 1$ (AM=2).

In the case where $\lambda_1 = 0$ **(AM=1)**, we have

$$\begin{cases} 2x_1 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ -2x_1 - x_3 &= 0 \end{cases} \Rightarrow X_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \alpha \neq 0.$$

In the case where $\lambda_2 = 1$ **(AM=2)**, we have

$$\begin{cases} x_1 + x_3 &= 0 \\ x_1 + x_3 &= 0 \\ -2x_1 - 2x_3 &= 0 \end{cases}$$

$$\Rightarrow X_2 = \begin{pmatrix} \alpha \\ \beta \\ -\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \alpha^2 + \beta^2 \neq 0.$$

Step 2. The matrix that diagonalizes A is

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}$$

Then $S^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$

$$D = S^{-1}AS = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 3.4

Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. *Diagonalize A if A is diagonalizable.*

Step 1. Find eigenvalues, eigenvectors of A.

$$\chi_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow -(\lambda - 4)(\lambda - 2)^2 = 0$$

$\Leftrightarrow \lambda_1 = 4$ (AM=1), $\lambda_2 = 2$ (AM=2).

In the case where $\lambda_1 = 4$ **(AM=1)**, we have

$$\begin{cases} -2x_1 &= 0 \\ x_1 - 2x_3 &= 0 \end{cases} \Rightarrow X_1 = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \alpha \neq 0.$$

In the case where $\lambda_2 = 2$ **(AM=2)**, we have

$$\begin{cases} 2x_2 & = & 0 \\ x_1 & = & 0 \end{cases} \Rightarrow X_2 = \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \beta \neq 0.$$

Step 2. Since the algebraic multiplicity =2> geometric multiplicity=1 then A is not diagonalizable.

Orthogonal Diagonalization

Definition 3.3

A square matrix A is said to be symmetric if $A = A^T$ or equivalently if $A = (a_{ij})_n$ then $a_{ij} = a_{ji}, \forall i, j = 1, 2, \ldots, n$.

Theorem 3.2

*If A is a **symmetric** matrix with real entries, then the eigenvalues λ of A are all real numbers and eigenvectors from different eigenspaces are orthogonal.*

Definition 3.4

A square matrix P is said to be ***orthogonal*** *if its transpose is the same as its inverse, that is, if $P^T = P^{-1}$, or equivalently, if $PP^T = P^TP = I$.*

Theorem 3.3

If A is a symmetric matrix with real entries, then there exists the orthogonal matrix P such that $P^TAP = P^{-1}AP$ is diagonal.

Orthogonal Diagonalization

Step 1. Find the eigenvalues.
Step 2. Find a basis for each eigenspace.
Step 3. Apply the Gram-Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.
Step 4. Form the matrix P whose comlumns are the vectors constructed in Step 3. The eigenvalues on the diagonal of $D = P^T AP$ will be the same order as their corresponding eigenvectors in P.

EXAMPLE 3.5

Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Step 1. Find eigenvalues of A.

$$\chi_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow -\lambda(\lambda-3)^2 = 0$$

$\Leftrightarrow \lambda_1 = 0$, (AM=1) $\lambda_2 = 3$ (AM=2).

Step 2, 3. In the case where $\lambda_1 = 0$ (AM=1), we have

$$\begin{cases} 2x_1 - x_2 - x_3 & = & 0 \\ -x_1 + 2x_2 - x_3 & = & 0 \\ -x_1 - x_2 + 2x_3 & = & 0 \end{cases} \Rightarrow X_0 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha \neq 0.$$

Therefore, $P_{*1} = \dfrac{X_0}{||X_0||} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

In the case where $\lambda_2 = 3$ **(AM=2)**, we have

$$\begin{cases} -x_1 - x_2 - x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \end{cases}$$

$$\Rightarrow X = \begin{pmatrix} -\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \alpha^2 + \beta^2 \neq 0.$$

Applying the Gram-Shmidt process, yields the orthogonal eigenvectors $B = \{y_1, y_2\}$.

$$y_1 = X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \; y_2 = X_2 - \frac{< X_2, y_1 >}{< y_1, y_1 >} y_1 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

Normalize the orthogonal basis to obtain

$$P_{*2} = \frac{y_1}{||y_1||} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \text{ and } P_{*3} = \frac{y_2}{||y_2||} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

Step 4. The matrix that orthogonally diagonalizes A is $P = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$

Then $D = P^T AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

MATLAB

1. **Finding the characteristic polynomial of A:** $p = poly(A)$
2. **Finding the roots of characteristic equation of A:** $roots(p)$
3. **Finding eigenvalues and eigenvectors of of A:** $[V, D] = eig(A)$

THANK YOU FOR YOUR ATTENTION