# LINEAR ALGEBRA EXERCISES

## WEEK 1: COMPLEX NUMBERS

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# Contents

# CONTENTS

# CONTENTS

# Contents

# Exercise 1

Find $z \in \mathbb{C}$ such that $Re(z(1+i)) + z\bar{z} = 0$.

**Solution**: Let $z = a + bi,\ a, b \in \mathbb{R}$, then

$$Re(z(1+i)) = Re\left[(a+bi)(1+i)\right] = a - b,$$

and $z\bar{z} = a^2 + b^2$. It is equivalent to find $a, b \in \mathbb{R}$ such that

$$\begin{aligned} & a - b + a^2 + b^2 = 0 \\ \Leftrightarrow \quad & \left(a + \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 = \frac{1}{2} \end{aligned}$$

Hence, the solutions $(a, b)$ are the points of the circle with centered in $\left(\frac{-1}{2}, \frac{1}{2}\right)$ and radius $\sqrt{2}/2$.

## Exercise 2

Find $z \in \mathbb{C}$ such that
$Re(z^2) + iIm(\bar{z}(1+2i)) = -3$

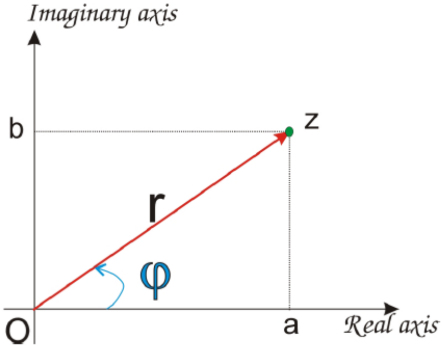
**Summary:**

- The algebraic form: $z = a + bi$, where $a$ is the real part and $b$ is the imaginary part.
- $\bar{z} = a - bi$ is the conjugate of $z$.
- $z\bar{z} = a^2 + b^2$.
- $z_1 = a + bi$, $z_2 = c + di$ then $z_1 \pm z_2 = (a \pm c) + (b \pm d)i$
- $i^2 = -1$

**EXAMPLE**

*Write in the "trigonometric" form* $r(\cos(\alpha) + i\sin(\alpha))$ *the following complex number* $-\sqrt{3} + i$

**Trigonometric form:** $z = r\left(\cos\varphi + i\sin\varphi\right)$,




$r = \sqrt{a^2+b^2}$: modulus of $z$;

$\varphi$: argument of $z$, $\cos(\varphi) = \frac{a}{r}$, $\sin(\varphi) = \frac{b}{r}$, $0 \le \varphi < 2\pi,\ or\ -\pi \le \varphi < \pi$

**Solution**: $a=-\sqrt{3}$, $b=1$, then $r=\sqrt{a^2+b^2}=2$ and
$$\begin{cases}\cos(\alpha)=\frac{-\sqrt{3}}{2}\\ \sin(\alpha)=\frac{1}{2}\end{cases}$$
Hence $\alpha=\frac{5\pi}{6}$ and $-\sqrt{3}+i=2\left(\cos\frac{5\pi}{6}+\sin\frac{5\pi}{6}\right)$.

## Exercise 3

Let $z = 9\left(\cos\left(\frac{\pi}{7}\right) + i\sin\left(\frac{\pi}{7}\right)\right)$. Find the minimum positive integer $n$ such that $z^n$ is a real number.

**Power form:**

If $z = a + bi = r(\cos\varphi + i\sin\varphi)$ is any complex number, then $z$ can be expressed in the exponential form $z = re^{i\varphi}$.

THEOREM

*If $z \in \mathbb{C}$ and $n$ is a positive integer then*

$$z^n = r^n e^{in\varphi} = r^n \big(\cos(n\varphi) + i\sin(n\varphi)\big)$$

## Exercise 4

Let $z = (2-2i)(3+3\sqrt{3}i)$. Compute $z^{100}$.

## Exercise 5

Find all points on the complex plane which represent the complex numbers $w = (\sqrt{3} - 1)z + 1 - 2i$, given by $z = 2e^{i\phi}$, $\forall \phi \in \mathbb{R}$.

## Exercise 6

Find the square roots of the following complex numbers and illustrate the roots on the complex plane

(A) $z = -i$

(B) $z = -1 - i$

(C) $z = \frac{1-2i}{1+i} + \frac{2-i}{1-i}$

**Theorem**

*Let* $z = a + bi = r(\cos\varphi + i\sin\varphi)$ *and let* $n$ *be a positive integer. Then* $z$ *has the* $n$ *distinct nth roots*

$$\sqrt[n]{z} = z_k = \sqrt[n]{r}\left(\cos\left(\frac{\varphi + k2\pi}{n}\right) + i\sin\left(\frac{\varphi + k2\pi}{n}\right)\right)$$

*where* $k = 0, \ldots, n-1$

**Solution**:

$$z = -i = \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)$$

Then the $\sqrt{z}$ has 2 values

$$\sqrt{z} = z_1 = \cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} = e^{-i\frac{\pi}{4}}$$

and

$$\sqrt{z} = z_2 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$
$$= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = e^{i\frac{3\pi}{4}}$$

**Summary:**

- The trigonometric form: $z = r\left(\cos(\varphi) + i\sin(\varphi)\right)$
- The exponential form: $z = re^{i\varphi}$
- The power of $z$: $z^n = r^n\left(\cos(n\varphi) + i\sin(n\varphi)\right)$, $n \in N$
- The nth of $z$:

$$\sqrt[n]{z} = z_k = \sqrt[n]{r}\left(\cos\left(\frac{\varphi + k2\pi}{n}\right) + i\sin\left(\frac{\varphi + k2\pi}{n}\right)\right)$$

where $k = 0, \ldots, n-1$.

## Exercise 7

Find all solutions of the following equation
$z^4 - 2z^3 + 6z^2 - 2z + 5 = 0$ given that $z = i$ is a root.

# FUNDAMENTAL THEOREM OF ALGEBRA

**THEOREM**

*The equation* $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ *where* $n \in \mathbb{N}*$, $a_n \neq 0$, $a_k \in \mathbb{C}$, $k = 0, \ldots, n$ *has exactly* $n$ *roots(real, complex and multiple roots).*

**THEOREM**

*if $x = x_0$ is a root of the equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where $n \in \mathbb{N}*$, $a_n \neq 0$, $a_k \in \mathbb{R}$, $k = 0, \ldots, n$. Then $x = \bar{x}_0$ is also a root of this equation.*

Solution: Note that equation (a) has all real coefficients, hence if $z = i$ is a root, $z = -i$ is also another root. We factor (a) as

$$(z-i)(z+i)(Az^2+Bz+C) = z^4 - 2z^3 + 6z^2 - 2z + 5$$

Then we have $\begin{cases} A = 1 \\ B = -2 \\ C = 5 \end{cases}$

It means that we have to find roots for $z^2 - 2z + 5 = 0$. It leads to

$$\begin{aligned} & z^2 - 2z + 5 = && 0 \\ \Leftrightarrow\quad & \left(z^2 - 2z + 1\right) - 1 + 5 = && 0 \\ \Leftrightarrow\quad & (z-1)^2 = && 4i^2 \\ \Leftrightarrow\quad & z = 1 + 2i \text{ or } z = 1 - 2i \end{aligned}$$

## Exercise 8

Find $a \in \mathbb{R}$ such that $z = i$ is a root for the polynomial $P(z) = z^3 - z^2 + z + 1 + a$. Furthermore, for such value of $a$ find all solutions of the equation $P(z) = 0$.