MATRICES AND MATRIX OPERATIONS

ELECTRONIC VERSION OF LECTURE

HCMC — 2021.

OUTLINE

Definition 2.1

*A **matrix** A with size $m \times n$ is a rectangular array of numbers, which contains **m rows** and **n columns**.*

$$A = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

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The numbers a_{ij} are called the entries.

DEFINITION 2.2

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A matrix with only one row $\begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix}$ *is called a row vector.*

Relation between matrix and its row vectors, column vectors

If $A_{i*} = \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix}$ is the i–th row of matrix A,

$1 \leqslant i \leqslant m$,

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If $A_{i*} = \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix}$ is the i–th row of matrix A, $1 \leqslant i \leqslant m$, and $A_{*j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ is the j–th column if matrix A, $1 \leqslant j \leqslant n$ then

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$$A = \begin{pmatrix} A_{1*} \\ A_{2*} \\ \vdots \\ A_{m*} \end{pmatrix} = \begin{pmatrix} A_{*1} & A_{*2} & \dots & A_{*n} \end{pmatrix}$$

EXAMPLE 2.1

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and 3 column vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

Zero Matrices

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Example 2.2

$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *is the zero matrix of order* 2×3.

Definition 2.4

A matrix A with n rows and n columns is called a square matrix of order n

$$A = \begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ii} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{ni} & \dots & a_{nn} \end{pmatrix}.$$

DEFINITION 2.4

A matrix A with n rows and n columns is called a square matrix of order n

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The entries $a_{11}, a_{22}, \dots, a_{nn}$ are said to be on the main diagonal of A.

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$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 5 & 4 & -5 \end{pmatrix}$$ *is the square matrix of order* 3.

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$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 5 & 4 & -5 \end{pmatrix}$ *is the square matrix of order* 3. *The entries on the main diagonal of* A *are* $1, -3, -5$

Identity Matrices

Definition 2.5

A ***square matrix*** $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$, *with* $1's$ *on the main diagonal and zeros elsewhere, i.e.* $(a_{ii} = 1, i = 1, ..n; a_{ij} = 0, \forall i \neq j)$ *is called an* *identity matrix* *of order* n *and is denoted by* I *or* I_n

EXAMPLE 2.4

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *is the identity matrix of order 3.*

TRANSPOSE OF A MATRIX

Transpose of a Matrix

Definition 2.6

If $A = (a_{ij})_{m\times n}$ is any $m \times n$ matrix, then the ***transpose of*** A*, denoted by $A^T = (a_{ji})_{n\times m}$ is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is,*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

EXAMPLE 2.5

If

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

then

$$\Rightarrow A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

SOLVING A SYSTEM OF LINEAR EQUATIONS

Which system is easier to solve algebraically?

$$\begin{cases} x-2y+3z &= 9 \\ -x+3y &= -4 \\ 2x-5y+5z &= 17 \end{cases} \text{ or } \begin{cases} x-2y+3z &= 9 \\ y+3z &= 5 \\ z &= 2 \end{cases}$$

Solving a system of linear equations

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The system on the right is clearly easier to solve. This system is in **row-echelon form,** which means that it follows a stair-step pattern and has leading coefficients of 1.

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3. *Add a constant times one row to another:*
   $r_i \to r_i + \lambda . r_j, \quad \forall \lambda$

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Using Elementary Row Operations to Solve a System

$$\begin{cases} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right)$$

Add the first equation to the second equation.

Add the first row to the second row to produce a new second row.

Using Elementary Row Operations to Solve a System

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right)$$

Add -2 times the first equation to the third equation.

Add -2 times the first row to the third row to produce a new third row.

Using Elementary Row Operations to Solve a System

$$\begin{cases} x - 2y + 3z & = & 9 \\ y + 3z & = & 5 \\ -y - z & = & -1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right)$$

Add the second equation to the third equation.	Add the second row to the third row to produce a new third row.

USING ELEMENTARY ROW OPERATIONS TO SOLVE A SYSTEM

$$\left\{\begin{array}{rcl} x-2y+3z &=& 9 \\ y+3z &=& 5 \\ 2z &=& 4 \end{array}\right. \qquad \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right)$$

Multiply the third equation by $\frac{1}{2}$	Multiply the third row by $\frac{1}{2}$ to produce a new third row.

Using Elementary Row Operations to Solve a System

$$\begin{cases} x-2y+3z &= 9 \\ y+3z &= 5 \\ z &= 2 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$x=1; y=-1; z=2$

The last matrix is said to be in **row-echelon form**

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2. *If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.*

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2. *If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.*
3. *In any two successive rows that do not consist entirely of zeros, the leading* 1 *in the lower row occurs farther to the right than the leading* 1 *in the higher row.*
4. *Each column that contains a leading* 1 *has zeros everywhere else in that column.*

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4. *Each column that contains a leading* 1 *has zeros everywhere else in that column.*

Definition 3.3

A matrix that has the first 3 *properties is said to be in row-echelon form.*

Example 3.1

The following matrices are in reduced row-echelon form:
$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix};$$

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The following matrices are in reduced row-echelon form: $\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix}$; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\begin{pmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

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The following matrices are in row-echelon form:

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EXAMPLE 3.2

The following matrices are in row-echelon form:

$$\begin{pmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix};\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix};\begin{pmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

EXAMPLE 3.3

Transform the following matrix to row-echelon form and reduced row-echelon form

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column.

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first and second rows in the preceding matrix were interchanged.

Step 3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $\frac{1}{a}$ in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first row of the preceding matrix was multiplied by $\frac{1}{2}$

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

(-2) times the first row of the preceding matrix was added to the third row.

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the **entire** matrix is in row-echelon form.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \xrightarrow{r_2 \to -\frac{1}{2} r_2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \xrightarrow{r_3 \to r_3 - 5r_2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} \xrightarrow{r_3 \to 2r_3}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The procedure produces a row-echelon form is called Gaussian Elimination

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading $1's$.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_1 \to r_1 + r_3 \times (-6)]{r_2 \to r_2 + r_3 \times 7/2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 \to r_1 + r_2 \times 5} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The last matrix is in reduced row echelon form. The procedure described for reducing a matrix to reduced row echelon form is called **Gauss-Jordan elimination**.

Rank of a matrix

Definition 4.1

We denote $A \longrightarrow B$ to show that B is the matrix that results from A by performing some of the Elementary Row Operations.

Definition 4.2

If $A_{m\times n} \longrightarrow B_{m\times n}$, where B is in row-echelon form, then rank of the matrix A is the number of leading 1's of matrix B and is denoted by $r(A)$.

Properties of Rank of a matrix

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3. If $A \xrightarrow{\text{Elementary Row Operations}} B$ then
$$r(B) = r(A).$$
4. $r(A) = r(A^T)$.

EXAMPLE 4.1

For matrix $A = \begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \end{pmatrix}$, *transform* A *to row-echelon form using elementary row operations. Determine the rank of the matrix* A.

$$\begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \end{pmatrix} \xrightarrow[]{\substack{r_2 \to -r_2 \\ r_2 \leftrightarrow r_1}}$$

$$\begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \end{pmatrix} \xrightarrow[]{\substack{r_2 \to -r_2 \\ r_2 \leftrightarrow r_1}} \begin{pmatrix} 1 & 4 & -5 \\ 0 & 2 & -4 \\ 3 & 1 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \end{pmatrix} \xrightarrow[]{\substack{r_2 \to -r_2 \\ r_2 \leftrightarrow r_1}} \begin{pmatrix} 1 & 4 & -5 \\ 0 & 2 & -4 \\ 3 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow[]{\substack{r_3 \to r_3 - 3r_1 \\ r_2 \to \frac{1}{2} r_2}}$$

$$\begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \end{pmatrix} \xrightarrow[]{\substack{r_2 \to -r_2 \\ r_2 \leftrightarrow r_1}} \begin{pmatrix} 1 & 4 & -5 \\ 0 & 2 & -4 \\ 3 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow[]{\substack{r_3 \to r_3 - 3r_1 \\ r_2 \to \frac{1}{2} r_2}} \begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & -11 & 22 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \end{pmatrix} \xrightarrow[\;]{\substack{r_2 \to -r_2 \\ r_2 \leftrightarrow r_1}} \begin{pmatrix} 1 & 4 & -5 \\ 0 & 2 & -4 \\ 3 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow{\substack{r_3 \to r_3 - 3r_1 \\ r_2 \to \frac{1}{2} r_2}} \begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & -11 & 22 \end{pmatrix} \xrightarrow{r_3 \to r_3 + 11 r_2}$$

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$$\begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

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$$\xrightarrow[]{\substack{r_3 \to r_3 - 3r_1 \\ r_2 \to \frac{1}{2} r_2}} \begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & -11 & 22 \end{pmatrix} \xrightarrow{r_3 \to r_3 + 11 r_2}$$

$$\begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$. Rank of matrix A is 2.

Definition 5.1

Two matrices A and B are defined to be equal if they have the same size and their corresponding entries are equal, i.e.

$$\boxed{A = (a_{ij})_{m\times n} = B = (b_{ij})_{m\times n} \Leftrightarrow a_{ij} = b_{ij}, \forall i, j} \tag{1}$$

Example 5.1

Find real numbers x, y, z, t such that the following 2 matrices are equal

$$\begin{pmatrix} x+y & 2z+t \\ x-y & z-t \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 1 & 5 \end{pmatrix}$$

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$$\begin{pmatrix} x+y & 2z+t \\ x-y & z-t \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 1 & 5 \end{pmatrix}$$

$$\begin{cases} x+y &= 3 \\ x-y &= 1 \\ 2z+t &= 7 \\ z-t &= 5 \end{cases} \Leftrightarrow \begin{cases} x &= 2 \\ y &= 1 \\ z &= 4 \\ t &= -1 \end{cases}$$

DEFINITION 5.2

If $A = (a_{ij})_{m \times n}$ is any matrix and α is any scalar, then the product αA obtained by the multiplying each entry of the matrix A by α. The matrix αA is said to be a scalar multiple of A, i.e.

$$\boxed{\alpha A = (\alpha . a_{ij})_{m \times n}} \tag{2}$$

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3. $\alpha.\mathbf{0} = \mathbf{0}, \forall \alpha \in \mathbb{R}, \mathbf{0}$ is the zero matrix.

PROPERTIES

1. $1.A = A$
2. $0.A = \mathbf{0}$,
3. $\alpha.\mathbf{0} = \mathbf{0}, \forall \alpha \in \mathbb{R}, \mathbf{0}$ is the zero matrix.
4. $\alpha(\beta A) = (\alpha\beta)A, \quad \forall \alpha, \beta \in \mathbb{R}.$

EXAMPLE 5.2

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & -5 \end{pmatrix}$ *then*

$$3A = \begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 5 & 3 \times 4 & 3 \times (-5) \end{pmatrix} =$$

Example 5.2

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & -5 \end{pmatrix}$ *then*

$$3A = \begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 5 & 3 \times 4 & 3 \times (-5) \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 15 & 12 & -15 \end{pmatrix}$$

DEFINITION 5.3

If A and B are matrices of the same size, then the sum $A+B$ is the matrix obtained by adding the entries of B to the corresponding entries of A.

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IN ORDER TO ADD 2 MATRICES A AND B

1. A and B must have the same size $m \times n$

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IN ORDER TO ADD 2 MATRICES A AND B

1. A and B must have the same size $m \times n$
2. $A+B = (a_{ij} + b_{ij})_{m\times n}$

PROPERTIES

IF A, B, C ARE MATRICES WITH THE SAME SIZE, THEN

Properties

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5. $A + \mathbf{0} = \mathbf{0} + A = A$

EXAMPLE 5.3

$$\begin{pmatrix} 1 & 4 & 3 \\ 8 & -3 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 1 \\ 4 & -1 & 0 \end{pmatrix} =$$

EXAMPLE 5.3

$$\begin{pmatrix} 1 & 4 & 3 \\ 8 & -3 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 1 \\ 4 & -1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1+3 & 4+1 & 3+1 \\ 8+4 & -3-1 & 2+0 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 4 \\ 12 & -4 & 2 \end{pmatrix}$$

PREMIER LEAGUE 2012-2013

Team	Won	Draw	Lost
Man. Utd	28	5	5
Man. City	23	9	6
Chelsea	22	9	7
Arsenal	21	10	7
Tottenham	21	9	8

Find the number of points each team scored if a team scores 3 points for winner, 0 point for loser, and 1 point for draw match.

$$\begin{pmatrix} 28 & 5 & 5 \\ 23 & 9 & 6 \\ 22 & 9 & 7 \\ 21 & 10 & 7 \\ 21 & 9 & 8 \end{pmatrix} . \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 28 \times 3 + 5 \times 1 + 5 \times 0 \\ 23 \times 3 + 9 \times 1 + 6 \times 0 \\ 22 \times 3 + 9 \times 1 + 7 \times 0 \\ 21 \times 3 + 10 \times 1 + 7 \times 0 \\ 21 \times 3 + 9 \times 1 + 8 \times 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 89 & \text{(Man. Utd)} \\ 78 & \text{(Man. City)} \\ 75 & \text{(Chelsea)} \\ 73 & \text{(Arsenal)} \\ 72 & \text{(Tottenham)} \end{pmatrix}$$

HISTORICAL NOTE

The concept of matrix multiplication is due to the German mathematician Gotthold Eisenstein, who introduced the idea around 1844 to simplify the process of making substitutions in linear system. Eisenstein was a pupil of Gauss, who ranked him as the equal of Isaac Newton and Archimedes. However, Eisenstein, suffering from bad health his entire life, died at age 30, so his potential was never realized.

Gotthold Eisenstein
(1823–1852)

MULTIPLICATION OF TWO MATRICES

DEFINITION 5.4

If $A = (a_{ij})_{m\times n} \in M_{m\times n}(K), B = (b_{ij})_{n\times p} \in M_{n\times p}(K)$.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m\times n} \cdot \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nj} & \dots & b_{np} \end{pmatrix}_{n\times p} =$$

$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1j} & \dots & c_{1p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{i1} & c_{i2} & \dots & c_{ij} & \dots & c_{ip} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mj} & \dots & c_{mp} \end{pmatrix}_{m\times p}$$

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$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1j} & \dots & c_{1p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{i1} & c_{i2} & \dots & c_{ij} & \dots & c_{ip} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mj} & \dots & c_{mp} \end{pmatrix}_{m\times p}$$

then the product AB is the matrix

$C = A.B = (c_{ij})_{m\times p}$ *whose entries are defined by* $c_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj}$, $i = 1..m; j = 1..p$

Multiplication of two matrices

Definition 5.4

If $A = (a_{ij})_{m\times n} \in M_{m\times n}(K), B = (b_{ij})_{n\times p} \in M_{n\times p}(K).$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix}_{n\times p} =$$

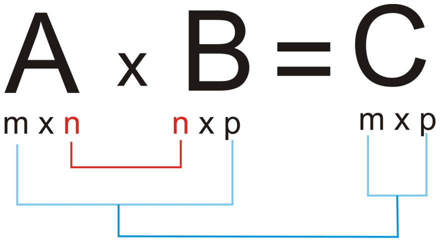
$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}_{m\times p}$$

then the product AB is the matrix

$C = A.B = (c_{ij})_{m\times p}$ *whose entries are defined by* $c_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj},\ i = 1..m; j = 1..p$

REMARK

IN ORDER TO FORM THE PRODUCT *AB*



EXAMPLE 5.4

Find product A.B where

$$A = \begin{pmatrix} 2 & -1 & 4 & 5 \end{pmatrix}_{1\times 4}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}_{4\times 1}$$

Example 5.4

Find product $A.B$ where

$$A = \begin{pmatrix} 2 & -1 & 4 & 5 \end{pmatrix}_{1\times 4}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}_{4\times 1}$$

$$A.B = \begin{pmatrix} 2 & -1 & 4 & 5 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} =$$

Example 5.4

Find product A.B where

$$A = \begin{pmatrix} 2 & -1 & 4 & 5 \end{pmatrix}_{1\times 4}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}_{4\times 1}$$

$$A.B = \begin{pmatrix} 2 & -1 & 4 & 5 \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} =$$

$$\begin{pmatrix} 2\times 1 + (-1)\times 2 + 4\times 0 + 5\times(-1) \end{pmatrix} = \begin{pmatrix} -5 \end{pmatrix}_{1\times 1}$$

Example 5.5

Find product $C = A.B$ *where*

$$A = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix}_{2\times 3}, \quad B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \\ 0 & 2 & 1 \end{pmatrix}_{3\times 3}.$$

Example 5.5

Find product $C = A.B$ *where*

$$A = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix}_{2\times 3}, \quad B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \\ 0 & 2 & 1 \end{pmatrix}_{3\times 3}.$$

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \\ 0 & 2 & 1 \end{pmatrix} =$$

$$c_{11} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 2\times 2 + 3\times 1 + 1\times 0 = 7$$

$$c_{12} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 2 \times 1 + 3 \times 3 + 1 \times 2 = 13$$

$$c_{13} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} . \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = 2 \times (-1) + 3 \times (-2) + 1 \times 1 = -7$$

$$c_{21} = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = (-1) \times 2 + 0 \times 1 + 1 \times 0 = -2$$

$$c_{22} = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = (-1) \times 1 + 0 \times 3 + 1 \times 2 = 1$$

$$c_{23} = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} . \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} =$$

$$= (-1) \times (-1) + 0 \times (-2) + 1 \times 1 = 2$$

$$c_{23} = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} . \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} =$$

$$= (-1) \times (-1) + 0 \times (-2) + 1 \times 1 = 2$$

Therefore,

$$C = A.B = \begin{pmatrix} 7 & 13 & -7 \\ -2 & 1 & 2 \end{pmatrix}.$$

PROPERTIES

1. $(A.B).C = A.(B.C) = A.B.C$

Properties

1. $(A.B).C = A.(B.C) = A.B.C$
2. $A.(B+C) = A.B + A.C.$

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PROPERTIES

1. $(A.B).C = A.(B.C) = A.B.C$
2. $A.(B+C) = A.B + A.C.$
3. $(B+C).A = B.A + C.A$
4. $\lambda(AB) = (\lambda A).B = A.(\lambda B), \quad \lambda \in \mathbb{R}.$

Definition 5.5

If A is a square matrix and

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$$

is any polynomial, then we define the matrix $p(A)$ to be

$$\boxed{p(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_m A^m} \tag{3}$$

EXAMPLE 5.6

Find $p(A)$ *for* $p(x) = x^2 - 2x - 3$ *and* $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$

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Find $p(A)$ *for* $p(x) = x^2 - 2x - 3$ *and* $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$

$$p(A) = A^2 - 2A - 3I =$$

$$= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Elementary Matrix

Definition 5.6

A matrix E is called an ***elementary matrix*** *if it can be obtained from an identity matrix I by performing a single elementary row operation.*

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Example 5.7

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \to r_1 + r_3 \times 3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

THEOREM 5.1 (ROW OPERATIONS BY MATRIX MULTIPLICATION)

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when ***this same row operation*** *is performed on A.*

Example 5.8

Consider an 3×4 *matrix* A *and elementary row operation: adding 2 times the first row of matrix* A *to the third row. Find the elementary matrix* E *such that the product* EA *is the matrix that results when this same row operation is performed on* A.

Example 5.8

Consider an 3×4 matrix A and elementary row operation: adding 2 times the first row of matrix A to the third row. Find the elementary matrix E such that the product EA is the matrix that results when this same row operation is performed on A.

$$A_{3\times 4} \xrightarrow{r_3 \to r_3 + 2r_1} B_{3\times 4} \Leftrightarrow$$

Example 5.8

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$$A_{3\times 4} \xrightarrow{r_3 \to r_3 + 2r_1} B_{3\times 4} \Leftrightarrow B_{3\times 4} = E_{3\times 3}.A_{3\times 4}$$

Example 5.8

Consider an 3×4 matrix A and elementary row operation: adding 2 times the first row of matrix A to the third row. Find the elementary matrix E such that the product EA is the matrix that results when this same row operation is performed on A.

$$A_{3\times 4} \xrightarrow{r_3 \to r_3 + 2r_1} B_{3\times 4} \Leftrightarrow B_{3\times 4} = E_{3\times 3}.A_{3\times 4}$$

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_3 \to r_3 + 2r_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = E_{3\times 3}$$

Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \xrightarrow{r_3 \to r_3 + 2r_1} B =$$

Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \xrightarrow{r_3 \to r_3 + 2r_1} B =$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} + 2a_{11} & a_{32} + 2a_{12} & a_{33} + 2a_{13} & a_{34} + 2a_{14} \end{pmatrix}$$

$$E.A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} . \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = B =$$

$$E.A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} . \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = B =$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31}+2a_{11} & a_{32}+2a_{12} & a_{33}+2a_{13} & a_{34}+2a_{14} \end{pmatrix}$$

Inverse of a Matrix

Definition 6.1

If A is a square matrix, and if a matrix B of the same size can be found such that

$$\boxed{BA = AB = I,} \tag{4}$$

then A is said to be invertible (or non-singular) and B is called an inverse of A and is denoted by A^{-1}.

INVERSE OF A MATRIX

DEFINITION 6.1

If A is a square matrix, and if a matrix B of the same size can be found such that

$$\boxed{BA = AB = I,} \tag{4}$$

then A is said to be invertible (or non-singular) and B is called an inverse of A and is denoted by A^{-1}. If no such matrix B can be found, then A is said to be singular.

REMARK

The relationship $AB = BA = I$ is not changed by interchanging A and B, so if A is invertible and B is an inverse of A, then it is also true that B is invertible, and A is an inverse of B. Thus, when

$$AB = BA = I$$

we say that A and B are inverses of one another.

Equivalent Statements

Theorem 6.1

If A is an $n \times n$ matrix, then the following statements are equivalent

1. *A is invertible*

EQUIVALENT STATEMENTS

THEOREM 6.1

If A is an $n \times n$ matrix, then the following statements are equivalent

1. *A is invertible*
2. $A \xrightarrow{\textit{Elementary Row Operations}} I_n$

Equivalent Statements

Theorem 6.1

If A is an $n \times n$ matrix, then the following statements are equivalent

1. A *is invertible*
2. $A \xrightarrow{\text{Elementary Row Operations}} I_n$
3. $r(A) = n$

Using Elementary Row Operations to find inverse of an invertible matrix

Inversion Algorithm

$$(A|I) \xrightarrow{\text{Elementary Row Operations}} (I|A^{-1})$$

USING ELEMENTARY ROW OPERATIONS TO FIND INVERSE OF AN INVERTIBLE MATRIX

INVERSION ALGORITHM

$$(A|I) \xrightarrow{\text{Elementary Row Operations}} (I|A^{-1})$$

$$E_n.E_{n-1}.\dots.E_2E_1.A = I$$

$$\Rightarrow A^{-1} = E_n.E_{n-1}.\dots.E_2E_1$$

EXAMPLE 6.1

Find the inverse A^{-1} *of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 8 \end{pmatrix}$

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Find the inverse A^{-1} *of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 8 \end{pmatrix}$

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 3 & 7 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[]{\substack{r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 3r_1}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 \end{array}\right) \xrightarrow[]{\substack{r_1 \to r_1 - 2r_2 \\ r_3 \to r_3 - r_2}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 \end{array}\right) \xrightarrow[]{\substack{r_1 \to r_1 - 2r_2 \\ r_3 \to r_3 - r_2}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 5 & -2 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow[]{\substack{r_1 \to r_1 - 7r_3 \\ r_2 \to r_2 + 2r_3}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 \end{array}\right) \xrightarrow[r_3 \to r_3 - r_2]{r_1 \to r_1 - 2r_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 5 & -2 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow[r_2 \to r_2 + 2r_3]{r_1 \to r_1 - 7r_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 12 & 5 & -7 \\ 0 & 1 & 0 & -4 & -1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) =$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 \end{array}\right) \xrightarrow[r_3 \to r_3 - r_2]{r_1 \to r_1 - 2r_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 5 & -2 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow[r_2 \to r_2 + 2r_3]{r_1 \to r_1 - 7r_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 12 & 5 & -7 \\ 0 & 1 & 0 & -4 & -1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) = (I_3|A^{-1})$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 12 & 5 & -7 \\ -4 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

LESLEI MODEL

EXAMPLE 7.1

A population of rabbits raised in a research laboratory has the characteristics listed below. (a) Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3 years. (b) During the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year. The laboratory population now consists of 24 rabbits in the first age class, 24 in the second, and 20 in the third. How many rabbits will be in each age class in 1 year, 2 years?

The current age distribution vector is

$$x = \begin{bmatrix} 24 & (0 \leq age \leq 1) \\ 24 & (1 \leq age \leq 2) \\ 20 & (2 \leq age \leq 3) \end{bmatrix}$$

and the age transition matrix is

$$A = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

After 1 year the age distribution vector will be

$$Ax = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 304 \\ 12 \\ 12 \end{bmatrix}$$

After 2 year the age distribution vector will be

$$A^2x = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 304 \\ 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 168 \\ 152 \\ 6 \end{bmatrix}$$

Markov chain

Example 8.1

In a city with 1000 householders there are 3 supermarkets A, B and C. At this month, there are 200, 500 and 300 householders that go to the supermarkets A, B and C, respectively. After each month, there are 10% of customers of A change to B, 10% of those change to C; 7% of customers of B change to A, 3% of those change to C; 8.3% of customers of C change to A, 6.7% of those change to B. Find the numbers of customers of each supermarket after 1 month, 2 months.

Theorem 8.1

Let P be the transition matrix of a Markov chain. The ij-th entry $p_{ij}^{(n)}$ of the matrix P^n gives the probability that the Markov chain, starting in state s_j, will be in state s_i after n steps.

The numbers of customers of A, B and C at this month

$$X = \begin{bmatrix} 200 \\ 500 \\ 300 \end{bmatrix}$$

The transition matrix

$$A = \begin{bmatrix} 0.8 & 0.07 & 0.83 \\ 0.1 & 0.9 & 0.067 \\ 0.1 & 0.03 & 0.85 \end{bmatrix}$$

The numbers of customers of each supermarket after 1 month

$$AX \approx \begin{bmatrix} 220 \\ 490 \\ 290 \end{bmatrix}$$

The numbers of customers of each supermarket after 2 month

$$A^2X \approx \begin{bmatrix} 234 \\ 483 \\ 283 \end{bmatrix}$$

LEONTIEF INPUT-OUTPUT MODELS

Consider an economic system that has n different industries I_1, I_2, ..., I_n, each having input needs (raw materials, utilities, etc.) and an output (finished product). In producing each unit of output, an industry may use the outputs of other industries, including itself. For example, an electric utility uses outputs from other industries, such as coal and water, and also uses its own electricity.

Let d_{ij} be the amount of output the j-th industry needs from the i-th industry to produce one unit of output per year. The matrix of these coefficients is the input-output matrix

$$D = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \vdots & \vdots & \dots & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{pmatrix}$$

EXAMPLE 9.1

Consider a simple economic system consisting of 3 industries: electricity, water, and coal. Production, or output, of one unit of electricity requires 0.5 unit of itself, 0.25 unit of water, and 0.25 unit of coal. Production of one unit of water requires 0.1 unit of electricity, 0.6 unit of itself, and 0 units of coal. Production of one unit of coal requires 0.2 unit of electricity, 0.15 unit of water, and 0.5 unit of itself. Find the input-output matrix for this system.

SOLUTION. The column entries show the amounts each industry requires from the others, and from itself, to produce one unit of output

$$D = \begin{pmatrix} 0.5 & 0.1 & 0.2 \\ 0.25 & 0.6 & 0.15 \\ 0.25 & 0 & 0.5 \end{pmatrix}$$

The row entries show the amounts each industry supplies to the others, and to itself, for that industry to produce one unit of output.

CLOSED SYSTEM

Let the total output of the i-th industry be denoted by x_i. If the economic system is closed (that is, the economic system sells its products only to industries within the system), then the total output of the i-th industry is

$$x_i = d_{i1}x_1 + d_{i2}x_2 + \ldots + d_{in}x_n$$

Open system

If the industries within the system sell products to non-producing groups (such as governments or charitable organizations) outside the system, then system is open and the total output of the ith industry is

$$x_i = d_{i1}x_1 + d_{i2}x_2 + \ldots + d_{in}x_n + e_i$$

where e_i represents the external demand for the i-th industry's product.

The collection of total outputs for an open system is

$$\begin{cases} x_1 = d_{11}x_1 + d_{12}x_2 + \ldots + d_{1n}x_n + e_1 \\ x_2 = d_{21}x_1 + d_{22}x_2 + \ldots + d_{2n}x_n + e_2 \\ \ldots \\ x_n = d_{n1}x_1 + d_{n2}x_2 + \ldots + d_{nn}x_n + e_n \end{cases}$$

The matrix form of this system is

$$X = DX + E,$$

where X is the output matrix and E is the external demand matrix.

EXAMPLE 9.2

An economic system composed of 3 industries has the input-output matrix

$$D = \begin{pmatrix} 0.1 & 0.43 & 0 \\ 0.15 & 0 & 0.37 \\ 0.23 & 0.03 & 0.02 \end{pmatrix}$$

Find the output matrix X when the external demands are

$$E = \begin{pmatrix} 20000 \\ 30000 \\ 25000 \end{pmatrix}$$

SOLUTION.

$$X = DX + E \Rightarrow (I - D)X = E \Rightarrow X = (I - D)^{-1}E.$$

$$X \approx \begin{pmatrix} 46616 \\ 51058 \\ 38014 \end{pmatrix}$$

To produce the given external demands, the outputs of the 3 industries must be approximately 46616 units for the industry I, 51058 units for industry II, and 38014 units for industry III.

Application in cryptography

We need to encrypt the series of characters **R U CRAZY**. Choose a rule of cryptography, for ex., transform the characters $A \to Z$ into the numbers $1 \to 26$, respectively, the white space is 27:

$$\begin{array}{ccccccccc} R & & U & & C & R & A & Z & Y \\ 18 & 27 & 21 & 27 & 3 & 18 & 1 & 26 & 25 \end{array}$$

Rearrange the series into the matrix with 3 rows:

$$A = \begin{bmatrix} 18 & 27 & 1 \\ 27 & 3 & 26 \\ 21 & 18 & 25 \end{bmatrix}$$

Chose a square matrix of order 3: K as the "symmetric key" and then multiply it to the left of A

$$K = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow B = KA = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 18 & 27 & 1 \\ 27 & 3 & 26 \\ 21 & 18 & 25 \end{bmatrix} = \begin{bmatrix} 39 & 45 & 26 \\ 63 & 57 & 28 \\ 21 & 18 & 25 \end{bmatrix}$$

We have the encrypted series of numbers

$$\begin{array}{cccccccccc} & R & & U & & C & R & A & Z & Y \\ \Rightarrow & 18 & 27 & 21 & 27 & 3 & 18 & 1 & 26 & 25 \\ \Rightarrow & 39 & 63 & 21 & 45 & 57 & 18 & 26 & 28 & 25 \end{array}$$

We can recover A from B by $A = K^{-1}B$.

Graph theory

Graph theory investigates the structure, properties, and algorithms associated with graphs. Graphs have a number of equivalent representations; one representation, in particular, is widely used as the primary definition, a standard which this paper will also adopt.

A graph, denoted G, is defined as an ordered pair composed of two distinct sets:

1. A set of vertices, denoted $V(G)$
2. A set of edges, denoted $E(G)$

The **order** of a graph G refers to $|V(G)|$ and the **size** of a graph G refers to $|E(G)|$. In other words, order refers to the number of vertices and size refers to the number of edges. In order to perform computations with these graphs, we utilize matrices as an incredibly valuable, alternative representation. Such representations include incidence, adjacency, distance, and Laplacian matrices.

ADJACENCY MATRICES

DEFINITION 11.1

*For a graph G of order n, the **adjacency matrix**, denoted $A(G)$, of graph G is an n by n matrix whose (i, j)-th entry is determined as follows:*

$$A_{ij} = \begin{cases} 1, & \text{if vertex } v_i \text{ is adjacent to vertex } v_j \\ 0, & \text{otherwise} \end{cases}$$

Adjacency matrices not only encapsulate the structure and relationships of a graph, but also provide for an efficient method of storage and access in a computer.

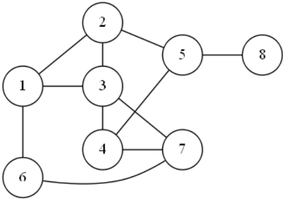

Figure 1: Graph of Order 8

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Distance and Powers of A

The distance between vertices v_i and v_j, denoted $d(i, j)$, of a graph G is defined by the path of minimum length between the two vertices.
The adjacency matrix of a graph provides a method of counting these paths by calculating the powers of the matrices.

Theorem 11.1

Let G be a graph with adjacency matrix A and k be a positive integer. Then the matrix power A^k gives the matrix where A_{ij} counts the the number of paths of length k between vertices v_i and v_j.

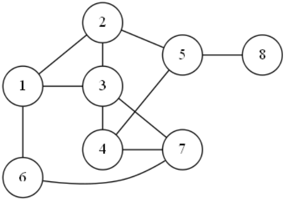

Figure 1: Graph of Order 8

$$A^4 = \begin{pmatrix} 17 & 11 & 13 & 11 & 10 & 4 & 16 & 2 \\ 11 & 18 & 13 & 17 & 4 & 9 & 11 & 6 \\ 13 & 13 & 28 & 13 & 16 & 14 & 13 & 2 \\ 11 & 17 & 13 & 18 & 4 & 9 & 11 & 6 \\ 10 & 4 & 16 & 4 & 15 & 4 & 10 & 0 \\ 4 & 9 & 14 & 9 & 4 & 10 & 4 & 2 \\ 16 & 11 & 13 & 11 & 10 & 4 & 17 & 2 \\ 2 & 6 & 2 & 6 & 0 & 2 & 2 & 3 \end{pmatrix}$$

Constructing Matrix

Example 12.1

$$A = [1\ 2\ 3\ 4; 5\ 6\ 7\ 8; 9\ 10\ 11\ 12; 13\ 14\ 15\ 16]$$

$$A = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{matrix}$$

Some special matrices

1. Zero matrix: zeros(m, n)
2. Square zero matrix of order n: zeros(n)
3. Identity matrix of order n: eye(n)

1. Rank of the matrix A: $rank(A)$
2. Transforming the matrix A to reduced row echelon form: $rref(A)$ (Reduced row echelon form)
3. Addition of Matrices: $A+B$
4. Subtraction of Matrices : $A-B$
5. Matrix multiplication: $A*B$
6. Power of matrices: $A\hat{}n$
7. Scalar Multiples: $k*A$
8. Transpose of Matrices: $A.'$

Extracting the entries of a matrix

1. Extracting the entry at the $i-$th row and the $j-$column of the matrix A: $A(i,j)$
2. Extracting the main diagonal of the square matrix A: $diag(A)$
3. Extracting the $i-$row of the matrix A: $A(i,:)$
4. Extracting the $j-$column of the matrix A: $A(:,j)$

Elementary Row Operations

1. Multiply the $i-$row through by a nonzero constant k $A(i,:) = A(i,:) * k$
2. Add a constant times one row to another: $A(i,:) = A(i,:) + A(j,:) * k$
3. Interchange 2 rows $A = A(\text{[the order of rows in A]},:)$

$$A = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{matrix}$$

If $A([1\ 3\ 2\ 4],:)$ we obtain

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 13 & 14 & 15 & 16 \end{matrix}$$

Using Elementary Row Operations to find inverse of an invertible matrix

Adjoin the identity matrix I to right side of A

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 4 & 7 \\ 3 & 7 & 8 & 12 \\ 4 & 8 & 14 & 19 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Elementary Row Operations to find inverse of an invertible matrix

Adjoin the identity matrix I to right side of A

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 4 & 7 \\ 3 & 7 & 8 & 12 \\ 4 & 8 & 14 & 19 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$>> B = [A \quad I]$

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 2 & 5 & 4 & 7 & 0 & 1 & 0 & 0 \\ 3 & 7 & 8 & 12 & 0 & 0 & 1 & 0 \\ 4 & 8 & 14 & 19 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$>> C =$ rref(B) $\Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 10 & 7 & -9 & 1 \\ 0 & 1 & 0 & 0 & -2 & -3 & 4 & -1 \\ 0 & 0 & 1 & 0 & 1 & -3 & 3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 2 & -2 & 1 \end{pmatrix}$$

$$>> [C(:,5) \quad C(:,6) \quad C(:,7) \quad C(:,8)]$$

$$\Rightarrow \begin{pmatrix} 10 & 7 & -9 & 1 \\ -2 & -3 & 4 & -1 \\ 1 & -3 & 3 & -1 \\ -2 & 2 & -2 & 1 \end{pmatrix}$$

THANK YOU FOR YOUR ATTENTION