Lecturer: **Approved by:**

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UNIVERSITY OF TECHNOLOGY - VNUHCM	**FINAL EXAM**	**Semester/ Academic year**	**2**	**2019 - 2020**
FACULTY OF AS		**Date**	**31/07/2020**	
	Course title	**Linear Algebra**		
	Course ID	**MT1007**		
	Duration	100 mins	Question sheet code	

Notes: - This is a closed book exam. Only your calculator is allowed. Total available score: 10.
- You MUST fill in your full name and student ID on this question sheet. There are 5 questions in 2 pages.
- You MUST write your answers in the question sheet.

Student ID: .... Student Fullname: ....

Question 1 (**1.5 points**) (L.O.1.4): Consider an economy that runs on 3 different types of output: raw materials, services, and manufacturing. Each industry requires some amount of output from each of the three to do its job. All of these requirements can be summarized in the following table:

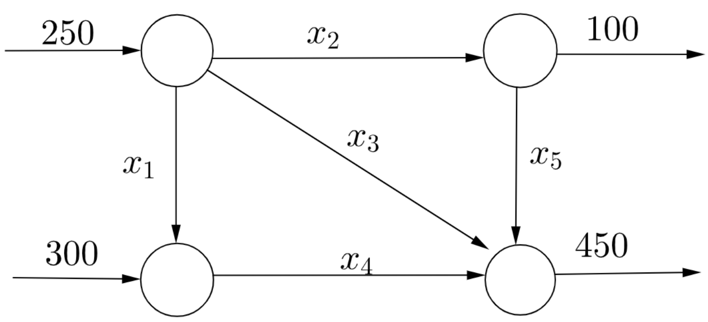
Industry	Raw materials	Services	Manufacturing
Raw materials	0.03	0.02	0.04
Services	0.03	0.03	0.01
Manufacturing	0.25	0.05	0.15

The demand matrix D tells how much (in billions of dollars) of each type of output is demanded by consumers and others outside the economy $D = \begin{bmatrix} 300 \\ 500 \\ 450 \end{bmatrix}$, $A = \begin{bmatrix} 0.03 & 0.02 & 0.04 \\ 0.03 & 0.03 & 0.01 \\ 0.25 & 0.05 & 0.15 \end{bmatrix}$. Let X denote the production matrix. It represents the amounts (in billions of dollars of value) produced by each of the three industries. Then, we have $X = AX + D$. Find X.

Question 2. (**2 points**) (L.O.1.3): The figure shows the flow of traffic of a network. The diagram indicates the average number of vehicles per hour entering and leaving the intersections and the arrows indicate the direction of traffic flows. All streets are one-way.

a. (1.5 points) Setup and solve a system of linear equations to find the possible flows x_1, x_2, x_3, x_4, and x_5.

b. (0.5 point) Find the flows when $x_5 = 50, x_3 = 50$.



Question 3. (**2.5 points**) (L.O.1.2): Let $F = \{x \in \mathbb{R}^3 | x_1 + 3x_2 - x_3 = 0\}$ be a subspace of $\mathbb{R}^3$ with the inner product $(x, y) = x_1y_1 + 2x_2y_1 + 2x_1y_2 + 5x_2y_2 + 3x_3y_3$.

a. (0.5 point) Find the dimension and one basis for F.

b. (1 point) Find the dimension and one basis of the orthogonal complement of F.

c. (1 point) Find the orthogonal projection of $u = (2, 1, -3)$ onto F.

Question 4. (**2.5 points**) (L.O.1.3): Let
$f : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection transformation about the line $d : 2x + 3y = 0$,
$g : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation transformation about the origin by an angle α counterclockwise.
Let ABC be a triangle with 3 vertices: $A(1, 1), B(1, 2), C(3, 1)$. Applying g on ABC, we obtain $A_1B_1C_1$: $g(ABC) = A_1B_1C_1$ and applying f on $A_1B_1C_1$, we obtain $A_2B_2C_2$: $f(A_1B_1C_1) = A_2B_2C_2$. Given $A_1(\frac{-1-\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2})$.

a. (1 point) Find the angle α and the coordinates of B_1, C_1.

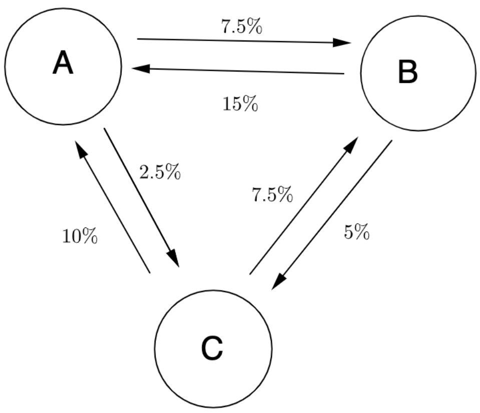
b. (1 point) Find the coordinates of A_2, B_2, C_2.

c. (0.5 point) Find the dimension and one basis for $Kerf$.

Question 5. (**2 points**) (L.O.1.4): In a city with 100.000 householders, there are 3 supermarkets A, B and C. In this month there are $30,000$ householders going to A, $40,000$ going to B and $30,000$ going to C. The state diagram shows us the state transitions after 1 month.

a. (0.5 point) How many householders that go to A, B and C after 1 month?

b. (1.5 points) Using the method of diagonalizing the transition matrix to find the numbers of householders going to A, B and C after 5 years. Round each answer to the nearest integer.



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Answers:

Question 1. $(I-A)X=D$

$$\Leftrightarrow X=(I-A)^{-1}D \text{ (1 point)} = \begin{bmatrix} 347.6095 \\ 533.0498 \\ 663.0057 \end{bmatrix} \text{ (0.5 point)}$$

Question 2. $\begin{cases} x_1+x_2+x_3=200 \\ -x_1+x_4=400 \\ x_2+x_4-x_5-x_6=0 \\ x_3+x_5=300 \end{cases}$ (0.5 point)

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 250 \\ 0 & 1 & 0 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 1 & 450 \\ -1 & 0 & 0 & 1 & 0 & 300 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 250 \\ 0 & 1 & 0 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 1 & 450 \end{array}\right] \text{ (0.5 point)}$$

$\Rightarrow$ General solution: $\begin{cases} x_1=\alpha-300 \\ x_2=\beta+100 \\ x_3=450-\alpha-\beta \\ x_4=\alpha \\ x_5=\beta, \end{cases}$. (0.5 point)

b. $\begin{cases} x_1=50 \\ x_2=150 \\ x_3=50 \\ x_4=350 \\ x_5=50, \end{cases}$. (0.5 point)

Question 3. a. One basis of $F: \{f_1=(1,0,1),(-3,1,0)\}$, $\mathbf{dim}(F)=2$. (0.5 point)

b. The matrix of the inner product: $A=\begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$$x \in F^{\perp} \Leftrightarrow \begin{cases} (x,f_1)=0 \\ (x,f_2)=0 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0. \text{ (0.5 point)}$$

$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$. One basis of $F^{\perp}: \{f_3=(3,-3,1)\}$, $dimF^{\perp}=1$. (0.5 point)

c. $u=\alpha f_1+\beta f_2+\gamma f_3 \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -13/7 \\ -17/7 \\ -8/7 \end{bmatrix}$.(0.5 point)

The projection: $pr_F(u)=\alpha f_1+\beta f_2=(-\frac{38}{7},-\frac{17}{7},-\frac{13}{7})$.(0.5 point)

Question 4. a. $\alpha=\frac{2\pi}{3}$ (0.5 point). $A_g=\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$$[OB_1\ OC_1]=A_g[OB\ OC]=\begin{bmatrix} -\frac{1}{2}-\sqrt{3} & -\frac{3}{2}-\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2}-1 & \frac{3\sqrt{3}}{2}-\frac{1}{2} \end{bmatrix} \text{ (0.5 point).}$$

b. $f(2,3) = (-2,-3), f(3,-2) = (3,-2)$.

$A_f = fB.B^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{-12}{13} & \frac{-5}{13} \end{bmatrix}$. (0.5 point)

$[OA_2\ OB_2\ OC_2] = A_f[OA_1\ OB_1\ OC_1] = \begin{bmatrix} -0.8633 & -0.7348 & -2.8467 \\ 1.1202 & 2.1119 & 1.3771 \end{bmatrix}$ (0.5 point)

c. $u \in Kerf \Leftrightarrow A_f u = 0$

$\Leftrightarrow \left[\begin{array}{cc|c} \frac{5}{13} & \frac{-12}{13} & 0 \\ \frac{-12}{13} & \frac{-5}{13} & 0 \end{array}\right] \Leftrightarrow u = 0.$

$Kerf = \{0\}, dim(Kerf) = 0$. (0.5 point)

Question 5. Let x, y and z be the numbers of householders that go to A, B and C. At the present: $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 30000 \\ 40000 \\ 30000 \end{bmatrix}$.

The transition matrix: $T = \begin{bmatrix} 0.9 & 0.15 & 0.1 \\ 0.075 & 0.8 & 0.075 \\ 0.025 & 0.05 & 0.825 \end{bmatrix}$

a. After 1 month $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = T \begin{bmatrix} 30000 \\ 40000 \\ 30000 \end{bmatrix} = \begin{bmatrix} 36000 \\ 36500 \\ 27500 \end{bmatrix}$ (0.5 point)

b. $T = PDP^{-1}$.

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.725 \end{bmatrix}, P = \begin{bmatrix} 25 & -1 & 2 \\ 12 & 0 & -3 \\ 7 & 1 & 1 \end{bmatrix}$. (1 point)

After 5 years: $\begin{bmatrix} x_{60} \\ y_{60} \\ z_{60} \end{bmatrix} = T^{60} \begin{bmatrix} 30000 \\ 40000 \\ 30000 \end{bmatrix} = PD^{50}P^{-1} \begin{bmatrix} 30000 \\ 40000 \\ 30000 \end{bmatrix} = \begin{bmatrix} 56818 \\ 27273 \\ 15909 \end{bmatrix}$ (0.5 point)