DETERMINANTS

ELECTRONIC VERSION OF LECTURE

HoChiMinh City University of Technology
Faculty of Applied Science, Department of Applied Mathematics

HCMC — 2020.

OUTLINE

Outline

OUTLINE

OUTLINE

OUTLINE

DEFINITION 1.1

If $A = (a_{ij})$ is a square matrix, then the determinant of A is a number. We denote it by $det(A)$ or $|A|$.

DEFINITION 1.1

If $A = (a_{ij})$ is a square matrix, then the determinant of A is a number. We denote it by $det(A)$ or $|A|$.

So

$$det : M_n(K) \to K$$

$$A \to det A.$$

Definition 1.2

If $A = (a_{ij})_{n\times n}$ is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix of order $(n-1)$ that remains after the $i-$th row and $j-$column are deleted from A.

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{i1} & \cdots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \cdots & a_{in} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n)(j-1)} & a_{nj} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}_{n\times n}$$

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{i1} & \cdots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \cdots & a_{in} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n)(j-1)} & a_{nj} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}_{n\times n}$$

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{i1} & \cdots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \cdots & a_{in} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n)(j-1)} & a_{nj} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}_{n\times n}$$

$$M_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$M_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}_{(n-1)\times(n-1)}$$

Definition 1.3

If $A = (a_{ij})_{n\times n}$ is a square matrix, then the number $C_{ij} = (-1)^{i+j} M_{ij}$ is called the cofactor of entry a_{ij}.

Definition 1.4

If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the determinant of A, and the sums themselves are called cofactor expansion of A. That is,

$$det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

$$det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

Cofactor expansion along the first row

$$det(A) = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

$$= \sum_{j=1}^{n} a_{1j} C_{1j} = \sum_{j=1}^{n} a_{1j}.(-1)^{1+j} M_{1j}.$$

1. $n = 1, A = (a_{11}) \Rightarrow |A| = a_{11}.$

1. $n = 1, A = (a_{11}) \Rightarrow |A| = a_{11}.$
2. $n = 2, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\Rightarrow |A| = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} = a_{11}a_{22} - a_{12}a_{21}.$$

1. $n = 1, A = (a_{11}) \Rightarrow |A| = a_{11}.$
2. $n = 2, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

   $$\Rightarrow |A| = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} = a_{11}a_{22} - a_{12}a_{21}.$$
3. $n = 3, A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

   $$\Rightarrow |A| = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13}$$
   $$= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
   $$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

EXAMPLE 1.1

Find the determinant $\det A$ *of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 5 \end{pmatrix}$

Example 1.1

Find the determinant $\det A$ *of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 5 \end{pmatrix}$

Solution. Cofactor expansion along the first row:

$|A| = 1.C_{11} + 2.C_{12} + 3.C_{13}.$

$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = 2 \times 5 - 1 \times 1 = 9,$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 1 \\ 3 & 5 \end{vmatrix} = -(4 \times 5 - 1 \times 3) = -17,$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = 4 \times 1 - 2 \times 3 = -2.$

Therefore, $|A| = 1 \times 9 + 2 \times (-17) + 3 \times (-2) = -31.$

Smart Choice of Row or Column

Smart Choice of Row or Column

We can find determinant using cofactor expansion along any row.

$$\det A = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Determinant also can be found using cofactor expansion along any column.

$$\det A = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Determinant also can be found using cofactor expansion along any column.

$$detA = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

It will be easiest to use cofactor expansion along the row or column which has the most zeros.

Example 1.2

Evaluate $\det A$ *where* $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix}$

EXAMPLE 1.2

Evaluate $\det A$ *where* $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix}$

Solution. Cofactor expansion along the second row:

$$|A| = 0.C_{21} + 2.C_{22} + 0.C_{23}$$

$$= 2.(-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = 2(1 \times 5 - 3 \times 3) = -8.$$

Example 1.3

$$Evaluate\ det A\ where\ A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

EXAMPLE 1.3

Evaluate $\det A$ *where* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix}$

Solution. Cofactor expansion along the third column

$$|A| = 3.C_{13} + 0.C_{23} + 0.C_{33}$$
$$= 3.(-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 3(2\times 1 - 1\times 3) = -3.$$

How an elementary row (or column) operation on a square matrix affects the value of its determinant

How an elementary row (or column) operation on a square matrix affects the value of its determinant

1. If $A \xrightarrow{r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)} B$ then $detB = -detA$.

How an elementary row (or column) operation on a square matrix affects the value of its determinant

1. If $A \xrightarrow{r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)} B$ then $detB = -detA$.
2. If $A \xrightarrow{r_i \to \lambda r_i (c_i \to \lambda c_i)} B$ then $detB = \lambda detA$ where $\lambda \neq 0$.

How an elementary row (or column) operation on a square matrix affects the value of its determinant

1. If $A \xrightarrow{r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)} B$ then $detB = -detA$.
2. If $A \xrightarrow{r_i \to \lambda r_i (c_i \to \lambda c_i)} B$ then $detB = \lambda detA$ where $\lambda \neq 0$.
3. If $A \xrightarrow{r_i \to r_i + \lambda . r_j (c_i \to c_i + \lambda c_j)} B$ then
$$detB = detA, \forall \lambda \in K$$

Corollary 1.1

1. *If A is a square matrix with 2 equal rows or 2 equal columns then $det(A) = 0$.*

COROLLARY 1.1

1. *If A is a square matrix with 2 equal rows or 2 equal columns then $det(A) = 0$.* $A \xrightarrow{r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)} A$ *where i, j are 2 equal rows or 2 equal columns* $det A = -det A \Rightarrow det A = 0$.

Corollary 1.1

1. *If A is a square matrix with 2 equal rows or 2 equal columns then $det(A) = 0$.* $A \xrightarrow{r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)} A$ *where i, j are 2 equal rows or 2 equal columns* $det A = -det A \Rightarrow det A = 0$.
2. *If A is a square matrix with 2 proportional rows or 2 proportional columns then $det(A) = 0$.*

Corollary 1.1

1. *If A is a square matrix with 2 equal rows or 2 equal columns then $det(A) = 0$.* $A \xrightarrow{r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)} A$ *where i, j are 2 equal rows or 2 equal columns* $detA = -detA \Rightarrow detA = 0$.
2. *If A is a square matrix with 2 proportional rows or 2 proportional columns then $det(A) = 0$. Since* $A \xrightarrow{r_i \to \lambda r_i (c_i \to \lambda c_i)} B$ *where $\lambda \neq 0$ is the ratio of 2 rows or 2 columns,* $detB = \lambda detA$, *where* $detB = 0 \Rightarrow detA = 0$.

EXAMPLE 1.4

Use Row Reduction to evaluate the determinant

$$\begin{vmatrix} 2 & 3 & -4 \\ 3 & -5 & 2 \\ 5 & 4 & 3 \end{vmatrix}$$

EXAMPLE 1.4

Use Row Reduction to evaluate the determinant

$$\begin{vmatrix} 2 & 3 & -4 \\ 3 & -5 & 2 \\ 5 & 4 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & -4 \\ 3 & -5 & 2 \\ 5 & 4 & 3 \end{vmatrix} \overline{\overline{\underset{r_2 \to r_2 - r_1}{\qquad}}}$$

Example 1.4

Use Row Reduction to evaluate the determinant

$$\begin{vmatrix} 2 & 3 & -4 \\ 3 & -5 & 2 \\ 5 & 4 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & -4 \\ 3 & -5 & 2 \\ 5 & 4 & 3 \end{vmatrix} \xlongequal[r_2 \to r_2 - r_1]{} \begin{vmatrix} 2 & 3 & -4 \\ 1 & -8 & 6 \\ 5 & 4 & 3 \end{vmatrix} \xlongequal[r_3 \to r_3 - 5r_2]{r_1 \to r_1 - 2r_2}$$

Example 1.4

Use Row Reduction to evaluate the determinant

$$\begin{vmatrix} 2 & 3 & -4 \\ 3 & -5 & 2 \\ 5 & 4 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & -4 \\ 3 & -5 & 2 \\ 5 & 4 & 3 \end{vmatrix} \underset{r_2 \to r_2 - r_1}{=\!=\!=\!=} \begin{vmatrix} 2 & 3 & -4 \\ 1 & -8 & 6 \\ 5 & 4 & 3 \end{vmatrix} \overset{r_1 \to r_1 - 2r_2}{\underset{r_3 \to r_3 - 5r_2}{=\!=\!=\!=}}$$

$$\begin{vmatrix} 0 & 19 & -16 \\ 1 & -8 & 6 \\ 0 & 44 & -27 \end{vmatrix} \underset{}{\text{Cofactor expansion along the first column}}$$

$$= 1.(-1)^{2+1}.\begin{vmatrix} 19 & -16 \\ 44 & -27 \end{vmatrix} = -191.$$

DETERMINANT OF A MATRIX PRODUCT

THEOREM 1.1

If A, B are square matrices of the same size, then

$$\boxed{det(AB) = det(A).det(B)} \tag{1}$$

EXAMPLE 1.5

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 6 \\ 2 & 8 & 9 \end{pmatrix}, B = \begin{pmatrix} 7 & 8 & 9 \\ 4 & -3 & 6 \\ -1 & 2 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 12 & 8 & 30 \\ 14 & 50 & 42 \\ 37 & 10 & 93 \end{pmatrix}$$

Verify that

$$det(A).det(B) = (-6).(-246) = det(AB) = 1476$$

COROLLARY 1.2

If A, B are square matrices of the same size

1. $det(A^k) = (det A)^k$.

COROLLARY 1.2

If A, B are square matrices of the same size

1. $det(A^k) = (det A)^k$. *Indeed,* $det(A^k) = det(\underbrace{A.A\ldots A}_{k\ times}) = \underbrace{det A.det A\ldots det A}_{k\ times} = (det A)^k$.

COROLLARY 1.2

If A, B are square matrices of the same size

1. $det(A^k) = (det A)^k$. *Indeed,* $det(A^k) = det(\underbrace{A.A\dots A}_{k\ times}) = \underbrace{det A.det A\dots det A}_{k\ times} = (det A)^k$.
2. $det(\alpha AB) = \alpha^n.det A.det B$.

Corollary 1.2

If A, B are square matrices of the same size

1. $det(A^k) = (det A)^k$. *Indeed,* $det(A^k) = det(\underbrace{A.A\dots A}_{k\ times}) = \underbrace{det A.det A\dots det A}_{k\ times} = (det A)^k$.
2. $det(\alpha AB) = \alpha^n.det A.det B$.
   Indeed,
   $det(\alpha AB) = det(\alpha A).det B = \underbrace{\alpha.\alpha\dots\alpha}_{n\ times} det A.det B$

Example 1.6

Evaluate $det(X)$ *if* X *satisfies*

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 5 & 2 \end{pmatrix}$$

Example 1.6

Evaluate $det(X)$ if X satisfies

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 5 & 2 \end{pmatrix}$$

We have $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} .det(X) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 5 & 2 \end{vmatrix}$

$\Rightarrow$

EXAMPLE 1.6

Evaluate $det(X)$ *if* X *satisfies*

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 5 & 2 \end{pmatrix}$$

We have $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} .det(X) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 5 & 2 \end{vmatrix}$

$$\Rightarrow 1.det(X) = 3 \Rightarrow det(X) = 3.$$

EXAMPLE 1.7

If $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}$, *then evaluate* $det(A^{2011})$.

Example 1.7

If $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}$, *then evaluate* $det(A^{2011})$.

We have

$$det(A^{2011}) = (det A)^{2011} = (-1)^{2011} = -1.$$

EXAMPLE 1.8

If $A = \begin{pmatrix} 3 & -2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 5 \\ 1 & -2 & 7 \end{pmatrix}$, *then evaluate* $det(2AB)$.

EXAMPLE 1.8

If $A = \begin{pmatrix} 3 & -2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 5 \\ 1 & -2 & 7 \end{pmatrix}$, *then evaluate* $det(2AB)$.

We have

$$det(2AB) = 2^3.detA.detB = 8 \times 3 \times 2 = 48.$$

Definition 2.1

Let A be an $m \times n$ matrix. A minor of A of order k is a determinant of a $k \times k$ submatrix of A.

DEFINITION 2.1

Let A be an $m \times n$ matrix. A minor of A of order k is a determinant of a $k \times k$ submatrix of A.

EXAMPLE 2.1

Find the minors of order 3 of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

SOLUTION. We obtain the determinants of order 3 by keeping all the rows and deleting one column from A. So there are 4 different minors of order 3. We compute one of them to illustrate:

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \times (-4) + 2 \times 0 = -4.$$

SOLUTION. We obtain the determinants of order 3 by keeping all the rows and deleting one column from A. So there are 4 different minors of order 3. We compute one of them to illustrate:

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \times (-4) + 2 \times 0 = -4.$$

The minors of order 3 are called the maximal minors of A, since there are no 4×4 submatrices of A.

Proposition 2.1

Let A be an $m \times n$ matrix. The rank of A is maximal order of a non-zero minor of A.

Computing the rank

Start with the minors of maximal order k. If there is one that is non-zero then $rank(A) = k$. If all maximal minors are zero, then $rank(A) < k$, and we continue with the minors of order $k - 1$ and so on, until we find a minor that is non-zero. If all minors of order 1 (i.e. all entries in A) are zero, then $rank(A) = 0$.

Example 2.2

Find the rank of the matrix $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$

EXAMPLE 2.2

Find the rank of the matrix $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$

The maximal minors have order 3, and we found that the one obtained by deleting the last column is $-4 \neq 0$. Hence $rank(A) = 3$.

EXAMPLE 2.3

Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$

Example 2.3

Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$

The maximal minors have order 3, so we compute the 4 minors of order 3. The first one is

$$\begin{vmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{vmatrix} = 7 \times (-1) + (-1) \times (-7) = 0.$$

The other 3 are also zero. Since all minors of order 3 are zero, $rank(A) < 3$. We continue to look at the minors of order 2. The first one is $\begin{vmatrix} 1 & 2 \\ 9 & 5 \end{vmatrix} = 1 \times 5 - 2 \times 9 = -13 \neq 0.$

The other 3 are also zero. Since all minors of order 3 are zero, $rank(A) < 3$. We continue to look at the minors of order 2. The first one is $\begin{vmatrix} 1 & 2 \\ 9 & 5 \end{vmatrix} = 1 \times 5 - 2 \times 9 = -13 \neq 0$. It is not necessary to compute any more minors, and we conclude that $rank(A) = 2$.

SOME PROPERTIES OF INVERSE OF AN MATRIX

1. A square matrix A is invertible if and only if $detA \neq 0$. Since $A.A^{-1} = I \Rightarrow detA.det(A^{-1}) = detI = 1 \Rightarrow detA \neq 0.$

Some properties of Inverse of an matrix

1. A square matrix A is invertible if and only if $det A \neq 0$. Since $A.A^{-1} = I \Rightarrow det A.det(A^{-1}) = det I = 1 \Rightarrow det A \neq 0$.
2. $det(A^{-1}) = \dfrac{1}{det A}$.

Some properties of Inverse of an matrix

1. A square matrix A is invertible if and only if $detA \neq 0$. Since $A.A^{-1} = I \Rightarrow detA.det(A^{-1}) = detI = 1 \Rightarrow detA \neq 0$.
2. $det(A^{-1}) = \dfrac{1}{detA}$. Since $A.A^{-1} = I \Rightarrow detA.det(A^{-1}) = 1$.

Some properties of Inverse of an matrix

1. A square matrix A is invertible if and only if $det A \neq 0$. Since $A.A^{-1} = I \Rightarrow det A.det(A^{-1}) = det I = 1 \Rightarrow det A \neq 0$.
2. $det(A^{-1}) = \dfrac{1}{det A}$. Since $A.A^{-1} = I \Rightarrow det A.det(A^{-1}) = 1$.
3. If A, B are invertible, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Some properties of Inverse of an matrix

1. A square matrix A is invertible if and only if $detA \neq 0$. Since $A.A^{-1} = I \Rightarrow detA.det(A^{-1}) = detI = 1 \Rightarrow detA \neq 0$.
2. $det(A^{-1}) = \dfrac{1}{detA}$. Since $A.A^{-1} = I \Rightarrow detA.det(A^{-1}) = 1$.
3. If A, B are invertible, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$. Since $(B^{-1}A^{-1}).(AB) = B^{-1}(A^{-1}.A)B = B^{-1}B = I$

Some properties of Inverse of an matrix

1. If A is invertible and $\alpha \neq 0$, then $(\alpha A)^{-1} = \dfrac{1}{\alpha} A^{-1}$.

Some properties of Inverse of an matrix

1. If A is invertible and $\alpha \neq 0$, then $(\alpha A)^{-1} = \dfrac{1}{\alpha} A^{-1}$.
   Since $\left(\dfrac{1}{\alpha} A^{-1}\right).(\alpha A) = I$.

Some properties of Inverse of an matrix

1. If A is invertible and $\alpha \neq 0$, then $(\alpha A)^{-1} = \dfrac{1}{\alpha} A^{-1}$.
   Since $\left(\dfrac{1}{\alpha} A^{-1}\right).(\alpha A) = I$.
2. If A is invertible, then A^{-1}, A^T are also invertible and $(A^{-1})^{-1} = A, \quad (A^T)^{-1} = (A^{-1})^T$.

Some properties of Inverse of an matrix

1. If A is invertible and $\alpha \neq 0$, then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$.
   Since $\left(\frac{1}{\alpha}A^{-1}\right).(\alpha A) = I$.
2. If A is invertible, then A^{-1}, A^T are also invertible and $(A^{-1})^{-1} = A, \quad (A^T)^{-1} = (A^{-1})^T$. Indeed, $A^{-1}.A = I$, $(A^{-1})^T.A^T = (A.A^{-1})^T = I^T = I$.

1. If A is an $n \times n$ square matrix and $det(A) \neq 0$; B is an $n \times p$ matrix, then $AX = B$ has an unique solution $X = A^{-1}B$.

1. If A is an $n \times n$ square matrix and $det(A) \neq 0$; B is an $n \times p$ matrix, then $AX = B$ has an unique solution $X = A^{-1}B$.
2. If A is an $n \times n$ square matrix and $det(A) \neq 0$; B is an $p \times n$ matrix, then $XA = B$ has an unique solution $X = BA^{-1}$.

1. If A is an $n \times n$ square matrix and $det(A) \neq 0$; B is an $n \times p$ matrix, then $AX = B$ has an unique solution $X = A^{-1}B$.
2. If A is an $n \times n$ square matrix and $det(A) \neq 0$; B is an $p \times n$ matrix, then $XA = B$ has an unique solution $X = BA^{-1}$.
3. If A is an $n \times n$ square matrix and $det(A) \neq 0$; B is an $m \times m$ square matrix and $det(B) \neq 0$; C is an $n \times m$ matrix, then $AXB = C$ has an unique solution $X = A^{-1}CB^{-1}$.

Example 3.1

Find matrix X which satisfies

$$\begin{pmatrix} 0 & -8 & 3 \\ 1 & -5 & 9 \\ 2 & 3 & 8 \end{pmatrix} X = \begin{pmatrix} -25 & 23 & -30 \\ -36 & -2 & -26 \\ -16 & -26 & 7 \end{pmatrix}$$

Example 3.1

Find matrix X which satisfies

$$\begin{pmatrix} 0 & -8 & 3 \\ 1 & -5 & 9 \\ 2 & 3 & 8 \end{pmatrix} X = \begin{pmatrix} -25 & 23 & -30 \\ -36 & -2 & -26 \\ -16 & -26 & 7 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & -8 & 3 \\ 1 & -5 & 9 \\ 2 & 3 & 8 \end{pmatrix}^{-1} . \begin{pmatrix} -25 & 23 & -30 \\ -36 & -2 & -26 \\ -16 & -26 & 7 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 5 & 7 \\ 2 & -4 & 3 \\ -3 & -3 & -2 \end{pmatrix}$$

Example 3.1

Find matrix X which satisfies

$$\begin{pmatrix} 0 & -8 & 3 \\ 1 & -5 & 9 \\ 2 & 3 & 8 \end{pmatrix} X = \begin{pmatrix} -25 & 23 & -30 \\ -36 & -2 & -26 \\ -16 & -26 & 7 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & -8 & 3 \\ 1 & -5 & 9 \\ 2 & 3 & 8 \end{pmatrix}^{-1} . \begin{pmatrix} -25 & 23 & -30 \\ -36 & -2 & -26 \\ -16 & -26 & 7 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 5 & 7 \\ 2 & -4 & 3 \\ -3 & -3 & -2 \end{pmatrix}$$

EXAMPLE 3.2

Find matrix X which satisfies $X.\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}$

Example 3.2

Find matrix X which satisfies $X.\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}$

Solution.

$$X = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix} . \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}^{-1} =$$

EXAMPLE 3.2

Find matrix X which satisfies $X.\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}$

Solution.

$$X = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix} . \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$$

EXAMPLE 3.2

Find matrix X which satisfies $X.\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}$

Solution.

$$X = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix} . \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$$

Example 3.3

Find matrix X which satisfies

$$\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} .X. \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix}$$

Example 3.3

Find matrix X which satisfies
$$\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} .X. \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix}$$

Solution.

$$X = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}^{-1} . \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix} . \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Example 3.3

Find matrix X which satisfies

$$\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}.X.\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix}$$

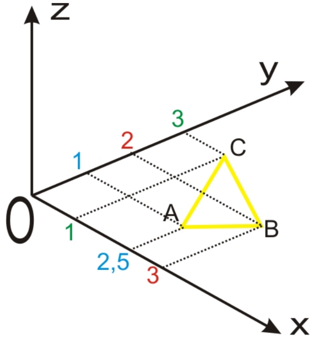
Solution.

$$X = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}^{-1} . \begin{pmatrix} 14 & 16 \\ 9 & 10 \end{pmatrix} . \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

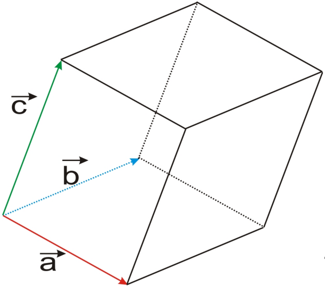
EVALUATING THE AREA OF THE TRIANGLE

$\Longrightarrow$



$$S = \frac{1}{2} abs|[\overrightarrow{AB}, \overrightarrow{AC}]| = \frac{1}{2} abs \begin{vmatrix} 2,5 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \frac{5}{4}$$

Evaluating the volume of the parallelepiped



$\overrightarrow{a} = (a_1, a_2, a_3)$;
$\overrightarrow{b} = (b_1, b_2, b_3)$; $\overrightarrow{c} = (c_1, c_2, c_3)$

$$\Rightarrow V = abs([\overrightarrow{a} \times \overrightarrow{b}], \overrightarrow{c}) = abs \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

MatLab

- **Evaluating determinant of matrix A:** $det(A)$
- **Finding inverse of Matrix A:**
  $A\hat{}(-1)$ or $inv(A)$

THANK YOU FOR YOUR ATTENTION