Lecturer: **Approved by:**

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UNIVERSITY OF TECHNOLOGY VNUHCM FACULTY OF AS	**FINAL EXAM**	**Semester/ Academic year**		**202**	**2020 - 2021**
		Date	**24 May 2021**		
	Course title	**Linear Algebra**			
	Course ID	**MT1007**			
	Duration	90 mins	Question sheet code	1254	

Intructions to students: - There are 5 pages in the exam
-This is a closed book exam. Only your calculator is allowed. Total available score: 10.
-At the beginning of the working time, you MUST fill in your full name and student ID on this question sheet.

Student's full name: . . . .

Student ID: . . . .

Invigilator 1:. . . . .

Invigilator 2:. . . . .

Part I. Multiple choice (6 points, 60 minutes)

Question 01. Find all values of m so that $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 3 & 7 & m \end{pmatrix}$ is invertible.

(A) $m \neq -5$ (B) None of them (C) $m \neq 5$ (D) $m \neq 3$

Question 02. In $\mathbb{R}_3$, let the inner product be given $(x, y) = x_1y_1 + x_2y_2 + 6x_3y_3 + 2x_1y_2 + 2x_2y_1$ and $x = (1;2;3), y = (2;-1;4)$. Calculate the distance between x and y.

(A) $d(x,y) = \sqrt{5}$ (B) $d(x,y) = \sqrt{3}$ (C) $d(x,y) = 1$ (D) $d(x,y) = 2$

Question 03. In $\mathbb{R}_2$, let the inner product be given $(x, y) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2$. Find m so that vector $u = (1;1)$ is perpendicular to vector $v = (2;m)$.

(A) $m = 2$ (B) $m = 1$ (C) $m = -2$ (D) $m = -\frac{6}{7}$

Question 04. In $\mathbb{R}_3$, let the standard inner product be given. Which set of vectors is orthogonal system?

(A) $F = \Big\{(1;2;1), (-1;0;1), (1;1;1)\Big\}$ (B) $F = \Big\{(1;2;1), (-1;0;1), (1;-1;1)\Big\}$

(C) $F = \Big\{(1;2;1), (1;0;1), (1;1;1)\Big\}$ (D) $F = \Big\{(1;2;1), (1;0;1), (1;-1;1)\Big\}$

Question 05. In $\mathbb{R}_2$, let the inner product be given $(x, y) = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$. Calculate the length of vector $v = (-1;1)$.

(A) $\sqrt{2}$ (B) 3 (C) 2 (D) $\sqrt{3}$

Question 06. In $\mathbb{R}_3$, the standard inner product and the subspace $F = spans\{(-5;2;-1), (2;-2;-2), (1;2;5)\}$ are given. Find the dimension of $F^{\perp}$.

(A) 1 (B) 2 (C) 0 (D) 3

Question 07. In $\mathbb{R}_3$, the standard inner product and the subspace $F = \Big\{(x_1;x_2;x_3) | x_1 - x_2 + 2mx_3 = 0\Big\}$ are given. Find m so that $\dim(F^{\perp}) = 1$.

(A) $\forall m$ (B) $m = 1$ (C) $\nexists m$ (D) $m \neq 1$

Question 08. Let f be the linear transformation $f : \mathbb{R}_2 \to \mathbb{R}_2$ so that $f(1;2) = (-2;1)$, $f(1;1) = (3;2)$. Calculate $f(4;-2)$.

(A) $(34;4)$ (B) $(42;14)$ (C) None of them (D) $(22;-12)$

Question 09. Let f be the linear transformation $f : \mathbb{R}_3 \to \mathbb{R}_3$ so that $f(1;1;1) = (2;3;3)$, $f(1;2;-2) = (1;1;2)$, $f(3;3;1) = (2;1;1)$. If $f(-1;3;-2) = (a;b;c)$, then find $a+b+c$.

(A) 12 (B) 52 (C) 86 (D) None of them

Question 10. Let $f : \mathbb{R}_2 \to \mathbb{R}_2$ be linear transformation and $f(x_1, x_2) = (2x_1 + 3x_2; 4x_2)$. Find matrix of f relative to standard basis $E = \left\{(1;0), (0;1)\right\}$.

(A) $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ (B) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (C) None of them (D) $A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

Question 11. Let $f : \mathbb{R}_2 \to \mathbb{R}_2$ be linear transformation and the matrix of f relative to basis $E = \left\{(7;5), (3;2)\right\}$ is $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$. Calculate $f(1;3)$.

(A) $(-13;1)$ (B) None of them (C) $(11;15)$ (D) $(218;-507)$

Question 12. The linear transformation f is the orthogonal projection onto the plane $(P) : 2x + y - z = 0$ in the coordinate system $Oxyz$. Which vector belongs to $Kerf$?

(A) $(1;1;1)$ (B) $(1;0;2)$ (C) $(0;1;1)$ (D) $(4;2;-2)$

Question 13. Let f be the linear transformation $f : \mathbb{R}_3 \to \mathbb{R}_3$ so that $\forall x \in \mathbb{R}_3, f(x) = mx$. Find m such that $\dim(Kerf) = 3$.

(A) $m = 1$ (B) $m = 0$ (C) $\forall m$ (D) $\nexists m$

Question 14. Let f be the linear transformation $f : \mathbb{R}_3 \to \mathbb{R}_2$ so that $f(x_1;x_2;x_3) = (2x_1 + x_2 - x_3; x_1 + x_2 - 2x_3)$. Which vector belongs to $Kerf$?

(A) $(1;-1;1)$ (B) $(2;-3;1)$ (C) $(0;1;1)$ (D) $(-1;3;1)$

Question 15. Let f be the linear transformation $f : \mathbb{R}_2 \to \mathbb{R}_2$ so that $f(1;1) = (1;-1)$, $f(1;2) = (-2;2)$. Find m so that vector $v = (-3;m)$ belongs to Imf?

(A) $\forall m$ (B) $m = 1$ (C) $m = -3$ (D) $m = 3$

Question 16. Let f be the linear transformation $f : \mathbb{R}_2 \to \mathbb{R}_2$ so that $Kerf = \left\langle (1;2) \right\rangle$, and $f(1;1) = (3;6)$. Find m so that vector $v = (-1;m)$ belongs to Imf?

(A) $\nexists m$ (B) $\forall m$ (C) $m = 2$ (D) $m = -2$

Question 17. Let the linear transformation f be the reflection over the line $2x - y = 0$ on the Oxy plane. A is the matrix of linear transformation f relative to basis $E = \left\{(1;0), (0;1)\right\}$. Which vector is an eigenvector of A?

(A) $(2;3)^T$ (B) $(2;-1)^T$ (C) $(0;1)^T$ (D) $(1;3)^T$

Question 18. Let $A = \begin{pmatrix} 6 & 2 \\ -2 & 1 \end{pmatrix}$. Which statement is FALSE?

(A) A is invertible (B) A has different real eigenvalues
(C) The sum of all eigenvalues of A is 7 (D) A is not diagonalizable

Question 19. Let $X = (1;2;-1)^T$ be the eigenvector of matrix A corresponding to the eigenvalue $\lambda_0 = -1$. Calculate AX.

(A) $(2;1;-1)^T$ (B) $(-1;-2;1)^T$ (C) $(0;0;0)^T$ (D) $(1;2;-1)^T$

Question 20. Let $A = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$. Calculate the sum of all eigenvalues of A^{-1}.

(A) $-\frac{7}{10}$ (B) $\frac{7}{10}$ (C) 7 (D) $\frac{1}{7}$

Question 21. Find m such that $(1; m)$ is an eigenvector of matrix $A = \begin{pmatrix} 7 & -3 \\ 10 & -4 \end{pmatrix}$.

(A) None of them (B) $m = 1$ (C) $m = 2$ (D) $m = 3$

Question 22. Find the eigenvalue of matrix $A = \begin{pmatrix} 5 & -1 \\ -3 & 3 \end{pmatrix}$ corresponding to the eigenvector $X = (-1; 1)^T$.

(A) 1 (B) None of them (C) 6 (D) 2

Question 23. Let $A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$. Find all eigenvalues of A^{10}.

(A) $\left\{2^{10}; 4^{10}\right\}$ (B) $\left\{4^{10}; 9^{10}\right\}$ (C) None of them (D) $\left\{1; 5^{10}\right\}$

Question 24. Let $A = \begin{pmatrix} 3 & -2 \\ -3 & 8 \end{pmatrix}$. Which vector is an eigenvector of A.

(A) $(2; 1)^T$ (B) $(5; 3)^T$ (C) $(3; 2)^T$ (D) None of them

Question 25. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find algebraic multiplicity (AM) and geometric multiplicity (GM) of eigenvalue $\lambda = 1$.

(A) $AM = 2; GM = 2$ (B) None of them (C) $AM = 3; GM = 2$ (D) $AM = 3; GM = 3$

Part II. Essay (4 points, 30 minutes)

Question 01. Consider a simple economic system consisting of three industries: electricity, water, and coal. Production, or output, of one unit of electricity requires 0.5 unit of itself, 0.25 unit of water, and 0.25 unit of coal. Production of one unit of water requires 0.1 unit of electricity, 0.6 unit of itself, and 0 unit of coal. Production of one unit of coal requires 0.2 unit of electricity, 0.15 unit of water, and 0.5 unit of itself. Find the input-output matrix D for this system and the output matrix X when the external demands are $E = \begin{pmatrix} 20000 \\ 30000 \\ 25000 \end{pmatrix}$.

Question 02. Suppose that two competing television channels, channel 1 and channel 2, each have 50% of the viewer market at some initial point in time. Assume that over each one-year period channel 1 captures 10% of channel 2's share, and channel 2 captures 20% of channel 1's share. What is each channel's market share after one year?

SOLUTION.

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ANSWER

Part I. Multiple choice (6 points, 60 minutes)

01. A **02.** D **03.** D **04.** B **05.** B **06.** A **07.** A **08.** B **09.** C **10.** A **11.** B **12.** D **13.** B **14.** D **15.** D **16.** D **17.** C **18.** D **19.** B **20.** B **21.** D **22.** C **23.** B **24.** A **25.** C

Part II. Essay (4 points, 30 minutes)

Question 01. **Solution.** The column entries show the amounts each industry requires from the others, and from itself, to produce one unit of ouput.

$$\begin{array}{cl} \overbrace{\begin{array}{ccc}\text{E} & \text{W} & \text{C}\end{array}}^{\text{User (Output)}} & \\ \begin{bmatrix} 0.5 & 0.1 & 0.2 \\ 0.25 & 0.6 & 0.15 \\ 0.25 & 0 & 0.5 \end{bmatrix} & \left.\begin{array}{c}\text{E}\\ \text{W}\\ \text{C}\end{array}\right\} \text{Supplier (Input)} \end{array}$$

The row entries show the amounts each industry supplies to the others, and to itself, for that industry to produce one unit of output.

Letting I be the identity matrix, write the equation $X = DX + E$ as $IX - DX = E$, which means that

$$(I - D)X = E \Rightarrow X = (I - D)^{-1}E = \begin{pmatrix} 46750 \\ 50950 \\ 37800 \end{pmatrix}$$

To produce the given external demands, the outputs of the three industries must be approximately 46750 units for industry electricity, 50950 units for industry water, and 37800 units for industry coal.

Question 02. Solution. Let us begin by introducing the time-dependent variables.
$x_1(t) =$ fraction of the market held by channel 1 at time t
$x_2(t) =$ fraction of the market held by channel 1 at time t
If we take $t = 0$ to be the starting point at which the two channels had 50% of the market, then the state of the system at that time is $x_1(0) = 0.5$; $x_2(0) = 0.5$. Now let us try to find the state of the system at time $t = 1$ (one year later). Over the one-year period, channel 1 retains 80% of its initial 50%, and it gains 10% of channel 2's initial 50%. Thus,

$$x_1(1) = 0.8 \times 0.5 + 0.1 \times 0.5 = 0.45$$

Similarly, channel 2 gains 20% of channel 1's initial 50%, and retains 90% of its initial 50%. Thus,

$$x_2(1) = 0.2 \times 0.5 + 0.9 \times 0.5 = 0.55.$$

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FINAL EXAM AND THE ANSWER

Part I. Multiple choice (6 points, 60 minutes)

Question 01. Find all values of m so that $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 3 & 7 & m \end{pmatrix}$ is invertible.

(A) $m \neq -5$ (B) None of them (C) $m \neq 5$ (D) $m \neq 3$

Solution. The correct answer (A). Matrix A is invertible if and only if $\det(A) \neq 0$

$$\Leftrightarrow -1 \times (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} + (-1) \times (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} + m \times (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2m + 10 \neq 0 \Leftrightarrow m \neq -5.$$

□

Question 02. In $\mathbb{R}_3$, let the inner product be given $(x, y) = x_1y_1 + x_2y_2 + 6x_3y_3 + 2x_1y_2 + 2x_2y_1$ and $x = (1;2;3), y = (2;-1;4)$. Calculate the distance between x and y.

(A) $d(x,y) = \sqrt{5}$ (B) $d(x,y) = \sqrt{3}$ (C) $d(x,y) = 1$ (D) $d(x,y) = 2$

Solution. The correct answer (D). $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \Rightarrow d(x,y) = \sqrt{(x-y)A(x-y)^T} = \sqrt{4} = 2.$ □

Question 03. In $\mathbb{R}_2$, let the inner product be given $(x, y) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2$. Find m so that vector $u = (1;1)$ is perpendicular to vector $v = (2;m)$.

(A) $m = 2$ (B) $m = 1$ (C) $m = -2$ (D) $m = -\frac{6}{7}$

Solution. The correct answer (D). $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \Rightarrow (u,v) = uAv^T = 7m + 6$. So $u \perp v \Leftrightarrow (u,v) = 0 \Rightarrow m = -\frac{6}{7}$

□

Question 04. In $\mathbb{R}_3$, let the standard inner product be given. Which set of vectors is orthogonal system?

(A) $F = \{(1;2;1), (-1;0;1), (1;1;1)\}$ (B) $F = \{(1;2;1), (-1;0;1), (1;-1;1)\}$

(C) $F = \{(1;2;1), (1;0;1), (1;1;1)\}$ (D) $F = \{(1;2;1), (1;0;1), (1;-1;1)\}$

Solution. The correct answer (B). We have

$\langle (1;2;1), (-1;0;1) \rangle = 1 \times (-1) + 2 \times 0 + 1 \times 1 = 0$

$\langle (1;2;1), (1;-1;1) \rangle = 1 \times 1 + 2 \times (-1) + 1 \times 1 = 0$

$\langle (-1;0;1), (1;-1;1) \rangle = (-1) \times 1 + 0 \times (-1) + 1 \times 1 = 0$

□

Question 05. In $\mathbb{R}_2$, let the inner product be given $(x, y) = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$. Calculate the length of vector $v = (-1;1)$.

(A) $\sqrt{2}$ (B) 3 (C) 2 (D) $\sqrt{3}$

Solution. The correct answer (B). $A = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \Rightarrow ||v|| = \sqrt{(v,v)} = \sqrt{vAv^T} = \sqrt{9} = 3.$ □

Question 06. In $\mathbb{R}_3$, the standard inner product and the subspace $F = spans\{(-5;2;-1), (2;-2;-2), (1;2;5)\}$ are given. Find the dimension of $F^{\perp}$.

(A) 1 (B) 2 (C) 0 (D) 3

Solution. The correct answer (A). Let $x = (x_1, x_2, x_3) \in F^{\perp} \Rightarrow x \perp F$. Then we receive linear system

$$\begin{cases} -5x_1 + 2x_2 - x_3 = 0 \\ 2x_1 - 2x_2 - 2x_3 = 0 \\ x_1 + 2x_2 + 5x_3 = 0 \end{cases} \Rightarrow rank(A) = 2 \Rightarrow \dim(F^{\perp}) = 3 - 2 = 1.$$

□

Question 07. In $\mathbb{R}_3$, the standard inner product and the subspace $F = \{(x_1;x_2;x_3) | x_1 - x_2 + 2mx_3 = 0\}$ are given. Find m so that $\dim(F^{\perp}) = 1$.

(A) $\forall m$ (B) $m = 1$ (C) $\nexists m$ (D) $m \neq 1$

Solution. The correct answer (A). We have $\dim(F) + \dim(F^{\perp}) = 3$ and $\dim(F) = 2 \Rightarrow \dim(F^{\perp}) = 3 - 2 = 1, \forall m$.

□

Question 08. Let f be the linear transformation $f : \mathbb{R}_2 \to \mathbb{R}_2$ so that $f(1;2) = (-2;1)$, $f(1;1) = (3;2)$. Calculate $f(4;-2)$.

(A) $(34;4)$ (B) $(42;14)$ (C) None of them (D) $(22;-12)$

Solution. The correct answer (B). We have $(4;2) = -6(1;2) + 10(1;1) \Rightarrow f(4;-2) = -6f(1;2) + 10f(1;1) = -6(-2;1) + 10(3;2) = (42;14)$. □

Question 09. Let f be the linear transformation $f : \mathbb{R}_3 \to \mathbb{R}_3$ so that $f(1;1;1) = (2;3;3)$, $f(1;2;-2) = (1;1;2)$, $f(3;3;1) = (2;1;1)$. If $f(-1;3;-2) = (a;b;c)$, then find $a + b + c$.

(A) 12 (B) 52 (C) 86 (D) None of them

Solution. The correct answer (C). We have $(-1;3;-2) = \frac{23}{2}(1;1;1) + 4(1;2;-2) - \frac{11}{2}(3;3;1) \Rightarrow f(-1;3;-2) = \frac{23}{2}f(1;1;1) + 4f(1;2;-2) - \frac{11}{2}f(3;3;1) = \frac{23}{2}(2;3;3) + 4(1;1;2) - \frac{11}{2}(2;1;1) = (16;33;37) = (a;b;c) \Rightarrow a + b + c = 86.$ □

Question 10. Let $f : \mathbb{R}_2 \to \mathbb{R}_2$ be linear transformation and $f(x_1, x_2) = (2x_1 + 3x_2; 4x_2)$. Find matrix of f relative to standard basis $E = \left\{(1;0), (0;1)\right\}$.

(A) $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ (B) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (C) None of them (D) $A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

Solution. The correct answer (A). We have $f(1;0) = (2;0)$ and $f(0;1) = (3;4)$. Therefore, $[f(1;0)]_E = (2;0)^T$ and $[f(0;1)]_E = (3;4)^T$. Thus $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ □

Question 11. Let $f : \mathbb{R}_2 \to \mathbb{R}_2$ be linear transformation and the matrix of f relative to basis $E = \left\{(7;5), (3;2)\right\}$ is $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$. Calculate $f(1;3)$.

(A) $(-13;1)$ (B) None of them (C) $(11;15)$ (D) $(218;-507)$

Solution. The correct answer (B). We have $[f(1;3)]_E = A[(1;3)]_E$ and $[(1;3)]_E = (7;-16)^T$. Therefore, $[f(1;3)]_E = (23;-50)^T$ and $f(1;3) = 23(7;5) - 50(3,2) = (11;15)$. Thus $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ □

Question 12. The linear transformation f is the orthogonal projection onto the plane $(P) : 2x + y - z = 0$ in the coordinate system $Oxyz$. Which vector belongs to $Kerf$?

(A) $(1;1;1)$ (B) $(1;0;2)$ (C) $(0;1;1)$ (D) $(4;2;-2)$

Solution. The correct answer (D). $Kerf = \left\{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = 0\right\} \Rightarrow (x,y,z) = \alpha(2;1;-1)$, where $\alpha \in \mathbb{R} \Rightarrow (4;2;-2) = 2(2;1;-1) \in Kerf$. □

Question 13. Let f be the linear transformation $f : \mathbb{R}_3 \to \mathbb{R}_3$ so that $\forall x \in \mathbb{R}_3, f(x) = mx$. Find m such that $\dim(Kerf) = 3$.

(A) $m = 1$ (B) $m = 0$ (C) $\forall m$ (D) $\nexists m$

Solution. The correct answer (B). $Kerf = \left\{x \in \mathbb{R}^3 : f(x) = mx = 0\right\}$.
Since $\dim(Kerf) = 3 \Rightarrow f(x) = mx = 0, \forall x \Rightarrow m = 0$ □

Question 14. Let f be the linear transformation $f : \mathbb{R}_3 \to \mathbb{R}_2$ so that $f(x_1; x_2; x_3) = (2x_1 + x_2 - x_3; x_1 + x_2 - 2x_3)$. Which vector belongs to $Kerf$?

(A) $(1;-1;1)$ (B) $(2;-3;1)$ (C) $(0;1;1)$ (D) $(-1;3;1)$

Solution. The correct answer (D). $Kerf = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : f(x_1,x_2,x_3) = 0\right\} \Rightarrow (x_1,x_2,x_3) = (-1;3;1) \in Kerf$.
□

Question 15. Let f be the linear transformation $f : \mathbb{R}_2 \to \mathbb{R}_2$ so that $f(1;1) = (1;-1)$, $f(1;2) = (-2;2)$. Find m so that vector $v = (-3;m)$ belongs to Imf?

(A) $\forall m$ (B) $m = 1$ (C) $m = -3$ (D) $m = 3$

Solution. The correct answer (D). $Imf = \left\{(y_1,y_2) \in \mathbb{R}^2 | \exists (x_1,x_2) \in \mathbb{R}^2 : f(x_1,x_2) = (y_1,y_2)\right\}$. Since $(x_1,x_2) \in \mathbb{R}^2 \Rightarrow (x_1,x_2) = \alpha(1;1) + \beta(1;2) \Rightarrow f(x_1,x_2) = \alpha f(1;1) + \beta f(1;2) = \alpha(1;-1) + \beta(-2;2)$, so $f(x_1,x_2) = v = (-3;m) \Rightarrow \begin{cases} \alpha - 2\beta = -3 \\ -\alpha + 2\beta = m \end{cases} \Rightarrow m = 3.$ □

Question 16. Let f be the linear transformation $f : \mathbb{R}_2 \to \mathbb{R}_2$ so that $Kerf = \left\langle(1;2)\right\rangle$, and $f(1;1) = (3;6)$. Find m so that vector $v = (-1;m)$ belongs to Imf?

(A) $\nexists m$ (B) $\forall m$ (C) $m = 2$ (D) $m = -2$

Solution. The correct answer D. Since $Kerf = \left\langle (1;2) \right\rangle \Rightarrow f(1;2) = (0;0)$
$Imf = \left\{ (y_1, y_2) \in \mathbb{R}^2 | \exists (x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = (y_1, y_2) \right\}$. Since $(x_1, x_2) \in \mathbb{R}^2 \Rightarrow (x_1, x_2) = \alpha(1;2) + \beta(1;1)$ $\Rightarrow f(x_1, x_2) = \alpha f(1;2) + \beta f(1;1) = \alpha(0;0) + \beta(3;6)$, so $f(x_1, x_2) = v = (-1; m) \Rightarrow \begin{cases} 0\alpha + 3\beta = -1 \\ 0\alpha + 6\beta = m \end{cases} \Rightarrow m = 6\beta = 6 \times \left(-\frac{1}{3}\right) = -2$. □

Question 17. Let the linear transformation f be the reflection over the line $2x - y = 0$ on the Oxy plane. A is the matrix of linear transformation f relative to basis $E = \left\{ (1;0), (0;1) \right\}$. Which vector is an eigenvector of A?

A $(2;3)^T$ B $(2;-1)^T$ C $(0;1)^T$ D $(1;3)^T$

Solution. The correct answer C. We have $f(1;2) = (1;2)$ and $f(2;-1) = (-2;1)$.
Matrix $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \Rightarrow A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/5 \end{pmatrix} = 3/5 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector. □

Question 18. Let $A = \begin{pmatrix} 6 & 2 \\ -2 & 1 \end{pmatrix}$. Which statement is FALSE?

A A is invertible
B A has different real eigenvalues
C The sum of all eigenvalues of A is 7
D A is not diagonalizable

Solution. The correct answer D. A has 2 eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. □

Question 19. Let $X = (1;2;-1)^T$ be the eigenvector of matrix A corresponding to the eigenvalue $\lambda_0 = -1$. Calculate AX.

A $(2;1;-1)^T$ B $(-1;-2;1)^T$ C $(0;0;0)^T$ D $(1;2;-1)^T$

Solution. The correct answer B. $AX = \lambda_0 X = -1X = (-1;-2;1)^T$ □

Question 20. Let $A = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$. Calculate the sum of all eigenvalues of A^{-1}.

A $-\frac{7}{10}$ B $\frac{7}{10}$ C 7 D $\frac{1}{7}$

Solution. The correct answer B. A has 2 eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$. So $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$ where X_1, X_2 are nonzero vectors. It implies that $A^{-1}X_1 = \frac{1}{\lambda_1} X_1$ and $A^{-1}X_2 = \frac{1}{\lambda_2} X_2$. Therefore, $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ are 2 eigenvalues of $A^{-1} \Rightarrow$ the sum of all eigenvalues of A^{-1} is $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$ □

Question 21. Find m such that $(1; m)$ is an eigenvector of matrix $A = \begin{pmatrix} 7 & -3 \\ 10 & -4 \end{pmatrix}$.

A None of them B $m = 1$ C $m = 2$ D $m = 3$

Solution. The correct answer D. We have $\begin{pmatrix} 7 & -3 \\ 10 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \Rightarrow \begin{cases} 7 \times 1 - 3 \times m = \lambda \\ 10 \times 1 - 4 \times m = \lambda \end{cases} \Rightarrow$
$\begin{cases} \lambda + 3 \times m = 7 \\ \lambda + 4 \times m = 10 \end{cases} \Rightarrow \begin{cases} \lambda = -2 \\ m = 3 \end{cases}$ □

Question 22. Find the eigenvalue of matrix $A = \begin{pmatrix} 5 & -1 \\ -3 & 3 \end{pmatrix}$ corresponding to the eigenvector $X = (-1;1)^T$.

A 1 B None of them C 6 D 2

Solution. The correct answer (C). We have $AX = \lambda X \Rightarrow \begin{pmatrix} 5 & -1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} 5 \times (-1) - 1 \times 1 = -\lambda \\ -3 \times (-1) + 3 \times 1 = \lambda \end{cases} \Rightarrow \lambda = 6$$

□

Question 23. Let $A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$. Find all eigenvalues of A^{10}.

(A) $\{2^{10}; 4^{10}\}$ (B) $\{4^{10}; 9^{10}\}$ (C) None of them (D) $\{1; 5^{10}\}$

Solution. The correct answer (B). A has 2 eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 9$. So $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$ where X_1, X_2 are nonzero vectors. It implies that $A^{10}X_1 = \lambda_1^{10} X_1$ and $A^{10}X_2 = \lambda_2^{10} X_2$. Therefore, 4^{10} and 9^{10} are 2 eigenvalues of A^{10}. □

Question 24. Let $A = \begin{pmatrix} 3 & -2 \\ -3 & 8 \end{pmatrix}$. Which vector is an eigenvector of A.

(A) $(2;1)^T$ (B) $(5;3)^T$ (C) $(3;2)^T$ (D) None of them

Solution. The correct answer (A). A has 2 eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 9$. When $\lambda_1 = 2$, the corresponding eigenvectors are $X_1 = \alpha(2;1)^T$, where $\alpha \neq 0$. When $\lambda_1 = 9$, the corresponding eigenvectors are $X_2 = \beta(1;-3)^T$, where $\beta \neq 0$. □

Question 25. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find algebraic multiplicity (AM) and geometric multiplicity (GM) of eigenvalue $\lambda = 1$.

(A) $AM = 2; GM = 2$ (B) None of them (C) $AM = 3; GM = 2$ (D) $AM = 3; GM = 3$

Solution. The correct answer (C). $\det(A - \lambda I) = (1 - \lambda)^3 = 0 \Rightarrow AM = 3$.
When $\lambda = 1$, $rank(A - 1I) = 1 \Rightarrow GM = 3 - 1 = 2$. □

Part II. Essay (4 points, 30 minutes)

Question 01. Consider a simple economic system consisting of three industries: electricity, water, and coal. Production, or output, of one unit of electricity requires 0.5 unit of itself, 0.25 unit of water, and 0.25 unit of coal. Production of one unit of water requires 0.1 unit of electricity, 0.6 unit of itself, and 0 unit of coal. Production of one unit of coal requires 0.2 unit of electricity, 0.15 unit of water, and 0.5 unit of itself. Find the input-output matrix D for this system and the output matrix X when the external demands are $E = \begin{pmatrix} 20000 \\ 30000 \\ 25000 \end{pmatrix}$.

Solution. The column entries show the amounts each industry requires from the others, and from itself, to produce one unit of ouput.

$$\begin{array}{c} \overbrace{\begin{array}{ccc} \text{E} & \text{W} & \text{C} \end{array}}^{\text{User (Output)}} \\ \begin{bmatrix} 0.5 & 0.1 & 0.2 \\ 0.25 & 0.6 & 0.15 \\ 0.25 & 0 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ \left.\begin{array}{c} \text{E} \\ \text{W} \\ \text{C} \end{array}\right\} \text{Supplier (Input)} \end{array}$$

The row entries show the amounts each industry supplies to the others, and to itself, for that industry to produce one unit of output.

Letting I be the identity matrix, write the equation $X = DX + E$ as $IX - DX = E$, which means that

$$(I - D)X = E \Rightarrow X = (I - D)^{-1}E = \begin{pmatrix} 46750 \\ 50950 \\ 37800 \end{pmatrix}$$

To produce the given external demands, the outputs of the three industries must be approximately 46750 units for industry electricity, 50950 units for industry water, and 37800 units for industry coal.

Question 02. Suppose that two competing television channels, channel 1 and channel 2, each have 50% of the viewer market at some initial point in time. Assume that over each one-year period channel 1 captures 10% of channel 2's share, and channel 2 captures 20% of channel 1's share. What is each channel's market share after one year?

Solution. Let us begin by introducing the time-dependent variables.
$x_1(t) =$ fraction of the market held by channel 1 at time t
$x_2(t) =$ fraction of the market held by channel 1 at time t
If we take $t = 0$ to be the starting point at which the two channels had 50% of the market, then the state of the system at that time is $x_1(0) = 0.5$; $x_2(0) = 0.5$. Now let us try to find the state of the system at time $t = 1$ (one year later). Over the one-year period, channel 1 retains 80% of its initial 50%, and it gains 10% of channel 2's initial 50%. Thus,

$$x_1(1) = 0.8 \times 0.5 + 0.1 \times 0.5 = 0.45$$

Similarly, channel 2 gains 20% of channel 1's initial 50%, and retains 90% of its initial 50%. Thus,

$$x_2(1) = 0.2 \times 0.5 + 0.9 \times 0.5 = 0.55.$$