Notes on Mathematical Statistics

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Contents

Chapter 1

Multivariate Probability Distributions

1.1 Bivariate and Multivariate Probability Distributions

Definition 1.1: Joint Probability Function for Discrete RVs

The *joint (or bivariate) probability function* for discrete random variables Y_1 and Y_2 is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2), -\infty < y_1, y_2 < \infty.$$

Remark.

> The joint probability function p of two discrete random variables Y_1 and Y_2 satisfies:
>
> 1. $p(y_1, y_2) \geq 0$ for all y_1, y_2.
> 2. $\sum_{y_1, y_2} p(y_1, y_2) = 1$

Definition 1.2: Joint Distribution Function

The *joint (bivariate) ditribution function* for any random variables Y_1 and Y_2 is given by

$$F(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2), -\infty < y_1, y_2 < \infty.$$

Definition 1.3: Jointly Continuous RVs

Let Y_1 and Y_2 be continuous RVs with joint distribution function $F(y_1, y_2)$. If there exists a nonnegative function $f(y_1, y_2)$ such that

$$F(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(t_1, t_2) dt_2 dt_1$$

for all $-\infty < y_1, y_2 < \infty$ then Y_1 and Y_2 are said to be *jointly continuous RVs*. The function $f(y_1, y_2)$ is called the *joint probability density function.*

Remark.

> If $f(y_1, y_2)$ is a joint density function for jointly continuous random variables Y_1 and Y_2, then
>
> 1. $f(y_1, y_2) \geq 0$ for all y_1, y_2.
> 2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2)dy_1dy_2 = 1$.

Remark.

> $P((X,Y) \in R) = \iint_R f(x,y)dA$ for any "reasonable" set R in the (y_1, y_2) plane.

Exercises

Exercise 1.1

Contracts for two construction jobs are randomly assigned to one or more of three firms A, B, C. Let $Y_1 = $ #contracts assigned to firm A and $Y_2 = $ #contracts assigned to firm B.

1. Find the joint probability function for Y_1 and Y_2.
2. Find $F(1, 0)$.

Solution:

1. Let (α, β) where $\alpha, \beta \in \{A, B, C\}$ denote the firms that job 1 and job 2 are assigned to, respectively. The sample space is

$$\Omega = \{(A,A),(A,B),(A,C),(B,A),(B,B),(B,C),(C,A),(C,B),(C,C)\}.$$

Each outcome has probability 1/9. The joint probability function for Y_1 and Y_2 is given by the following table

	y_1		
y_2	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

2. We have

$$\begin{aligned} F(1,0) &= P(Y_1 \leq 1, Y_2 \leq 0) \\ &= P(Y_1 = 0, Y_2 = 0) + P(Y_1 = 1, Y_2 = 0) \\ &= p(0,0) + p(1,0) \\ &= \frac{1}{9} + \frac{2}{9} \\ &= \frac{1}{3} \end{aligned}$$

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Exercise 1.2

Three balanced coins are tossed independently. Let $Y_1 = \#\text{heads}$ and $Y_2 =$ amount of money won. The amount of money won is determined as follows: if the first head occurs on toss i, you win $\$i$ for $i = 1, 2$ or 3. If no heads appear, you lose \$1.

1. Find $p(y_1, y_2)$.
2. Find the probability that fewer than three heads will occur and you will win \$1 or less, i.e., find $F(2, 1)$.

Solution:

1. The function $p(y_1, y_2)$ is given by

	y_1			
y_2	0	1	2	3
-1	1/8	0	0	0
1	0	1/8	2/8	1/8
2	0	1/8	1/8	0
3	0	1/8	0	0

2. We have

$$
\begin{aligned}
F(2,1) &= p(0,-1) + p(1,1) + p(2,1) \\
&= \frac{1}{8} + \frac{1}{8} + \frac{2}{8} \\
&= \frac{1}{2}
\end{aligned}
$$

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Exercise 1.3

Of nine executives in a business firm, 4 are married, 3 have never married and 2 are divorced. 3 of the executives are to be randomly selected for promotion. Let Y_1 be the number of married executives and Y_2 be the number of never-married executives among the 3 selected. Find the joint probability function of Y_1 and Y_2.

Solution: The sample space, the possible values of Y_1 and Y_2 and the associated probabilities are given by

outcome	3m	2m1n	2m1d	3n	2n1m	2n1d	2d1m	2d1n	1m1n1d
(y_1, y_2)	$(3,0)$	$(2,1)$	$(2,0)$	$(0,3)$	$(1,2)$	$(0,2)$	$(1,0)$	$(0,1)$	$(1,1)$
probability	$\frac{\binom{4}{3}}{\binom{9}{3}}$	$\frac{\binom{4}{2}\binom{3}{1}}{\binom{9}{3}}$	$\frac{\binom{4}{2}\binom{2}{1}}{\binom{9}{3}}$	$\frac{\binom{3}{3}}{\binom{9}{3}}$	$\frac{\binom{4}{1}\binom{3}{2}}{\binom{9}{3}}$	$\frac{\binom{3}{2}\binom{2}{1}}{\binom{9}{3}}$	$\frac{\binom{4}{1}}{\binom{9}{3}}$	$\frac{\binom{3}{1}}{\binom{9}{3}}$	$\frac{\binom{4}{1}\binom{3}{1}\binom{2}{1}}{\binom{9}{3}}$

The joint probability function $p(y_1, y_2)$ is given by

	y_1			
y_2	0	1	2	3
0	0	1/21	1/7	1/21
1	1/28	2/7	3/14	0
2	1/14	1/7	0	0
3	1/84	0	0	0

More concisely, one can write down the formula for $p(y_1, y_2)$ as

$$p(y_1, y_2) = \frac{\binom{4}{y_1}\binom{3}{y_2}\binom{2}{3-y_1-y_2}}{\binom{9}{3}}.$$

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