Group Work Project - M3

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- **Course:** MScFE 620 Discrete-time Stochastic Processes
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[1]:
```python
# Import helpers
from binomial import binomial_tree, binomial_freq
from filtration import filtration_set
# Import stats
import numpy as np
np.set_printoptions(precision=4, suppress=True) # prettify numpy
import pandas as pd
pd.set_option('precision', 4) # prettify pandas
import matplotlib.pyplot as plt
```

1 Question 1) Build a binomial tree for stock price evolution.

Specifically, set the upper movement of the price (u) to an expression that will depend on the number on your group.

$$u = (1.10 + \frac{GroupNumber}{100})$$

,

Group 1, u = 1.11. Group 2 = 1.12. Group 3 = 1.13, …, Group 50 = 1.60

You can define d = 1/u.

1.1 a) Calculate and show the new Binomial tree for N=6.

[2]:
```python
# Variables
N = 6  # Number of layers
S0 = 1  # Initial price
group_n = 2  # Group Number 2
```

```python
tree_6 = binomial_tree(N, S0, group_n)
stock_prices_6 = tree_6["stock_prices"]
df_stock_price_6 = pd.DataFrame(stock_prices_6)
df_stock_price_6.index.name = 'Time Step'
df_stock_price_6
```

[2]:

Time Step	0	1	2	3	4	5	6
0	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.8929	1.1200	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.7972	1.0000	1.2544	0.0000	0.0000	0.0000	0.0000
3	0.7118	0.8929	1.1200	1.4049	0.0000	0.0000	0.0000
4	0.6355	0.7972	1.0000	1.2544	1.5735	0.0000	0.0000
5	0.5674	0.7118	0.8929	1.1200	1.4049	1.7623	0.0000
6	0.5066	0.6355	0.7972	1.0000	1.2544	1.5735	1.9738

[3]:

```python
### b) What are the terminal values of each path?
df_stock_price_6.iloc[-1]
```

[3]:

```
0    0.5066
1    0.6355
2    0.7972
3    1.0000
4    1.2544
5    1.5735
6    1.9738
Name: 6, dtype: float64
```

1.1.1 b) Define the appropriate filtrations for each of these values.

To define the filtration $\mathbb{F}$, we first define a stochastic process $S = \{S_t : t = 1, ..., T\}$ where:

$$S_t(\omega) = \omega_n, \omega \in \Omega, n \in \{1, ..., 2^T\}$$

Define a filtration $\mathbb{F} = \{\mathcal{F}_t : t = 1, ..., T\}$ where $\mathcal{F}_T$ is the natural filtration of S.

$$\mathcal{F}_T = \sigma(\{S_t : t \leq T\})$$

Specifically, in our binomial model with $T = 6$:

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$
- $\mathcal{F}_1 = \sigma(\{w_1, ..., w_{16}\}, \{w_{17}, ..., w_{32}\}, \{w_{33}, ..., w_{48}\}, \{w_{49}, ..., w_{64}\})$
- $\mathcal{F}_2 = \sigma(\{w_1, ..., w_8\}, \{w_9, ..., w_{16}\}, \{w_{17}, ..., w_{24}\}, \{w_{25}, ..., w_{32}\},$ $\{w_{33}, ..., w_{40}\}, \{w_{41}, ..., w_{48}\}, \{w_{49}, ..., w_{56}\}, \{w_{57}, ..., w_{64}\})$
- $\mathcal{F}_3 = \sigma(\{w_1, ..., w_4\}, \{w_5, ..., w_8\}, \{w_9, ..., w_{12}\}, \{w_{13}, ..., w_{16}\},$ $\{w_{17}, ..., w_{20}\}, \{w_{21}, ..., w_{24}\}, \{w_{25}, ..., w_{28}\}, \{w_{29}, ..., w_{32}\},$ $\{w_{33}, ..., w_{36}\}, \{w_{37}, ..., w_{40}\}, \{w_{41}, ..., w_{44}\}, \{w_{45}, ..., w_{48}\},$ $\{w_{49}, ..., w_{52}\}, \{w_{53}, ..., w_{56}\}, \{w_{57}, ..., w_{60}\}, \{w_{61}, ..., w_{64}\})$

- $\mathcal{F}_4 = \sigma(\{w_1, w_2\}, \{w_3, w_4\}, \{w_5, w_6\}, \{w_7, w_8\}, \{w_9, w_{10}\}, \{w_{11}, w_{12}\}, \{w_{13}, w_{14}\}, \{w_{15}, w_{16}\}, \{w_{17}, w_{18}\}, \{w_{19}, w_{20}\}, \{w_{21}, w_{22}\}, \{w_{23}, w_{24}\}, \{w_{25}, w_{26}\}, \{w_{27}, w_{28}\}, \{w_{29}, w_{30}\}, \{w_{31}, w_{32}\}, \{w_{33}, w_{34}\}, \{w_{35}, w_{36}\}, \{w_{37}, w_{38}\}, \{w_{39}, w_{40}\}, \{w_{41}, w_{42}\}, \{w_{43}, w_{44}\}, \{w_{45}, w_{46}\}, \{w_{47}, w_{48}\}, \{w_{49}, w_{50}\}, \{w_{51}, w_{52}\}, \{w_{53}, w_{54}\}, \{w_{55}, w_{56}\}, \{w_{57}, w_{58}\}, \{w_{59}, w_{60}\}, \{w_{61}, w_{62}\}, \{w_{63}, w_{64}\})$
- $\mathcal{F}_5 = \mathcal{F} = 2^\Omega$

[4]:
```python
### b) Define the appropriate filtrations for each of these values.
print("Sampling first 3 filtrations:\n")
S0 = 1
u = 1.12
d = 1/u
N = 3
for i in range(N):
    print(f"\tF{i} = {filtration_set(S0,u,d,N,i)}\n")
```

```
Sampling first 3 filtrations:

	F0 = [(), ([1.4049280000000004, 1.12, 1.12, 0.8928571428571428, 1.12,
0.8928571428571428, 0.8928571428571429, 0.711780247813411],)]

	F1 = [(), ([1.4049280000000004, 1.12, 1.12, 0.8928571428571428],),
([1.12, 0.8928571428571428, 0.8928571428571429, 0.711780247813411],),
([1.4049280000000004, 1.12, 1.12, 0.8928571428571428], [1.12,
0.8928571428571428, 0.8928571428571429, 0.711780247813411])]

	F2 = [(), ([1.4049280000000004, 1.12],), ([1.12, 0.8928571428571428],),
([1.12, 0.8928571428571428],), ([0.8928571428571429, 0.711780247813411],),
([1.4049280000000004, 1.12], [1.12, 0.8928571428571428]), ([1.4049280000000004,
1.12], [1.12, 0.8928571428571428]), ([1.4049280000000004, 1.12],
[0.8928571428571429, 0.711780247813411]), ([1.12, 0.8928571428571428], [1.12,
0.8928571428571428]), ([1.12, 0.8928571428571428], [0.8928571428571429,
0.711780247813411]), ([1.12, 0.8928571428571428], [0.8928571428571429,
0.711780247813411]), ([1.4049280000000004, 1.12], [1.12, 0.8928571428571428],
[1.12, 0.8928571428571428]), ([1.4049280000000004, 1.12], [1.12,
0.8928571428571428], [0.8928571428571429, 0.711780247813411]),
([1.4049280000000004, 1.12], [1.12, 0.8928571428571428], [0.8928571428571429,
0.711780247813411]), ([1.12, 0.8928571428571428], [1.12, 0.8928571428571428],
[0.8928571428571429, 0.711780247813411]), ([1.4049280000000004, 1.12], [1.12,
0.8928571428571428], [1.12, 0.8928571428571428], [0.8928571428571429,
0.711780247813411])]
```

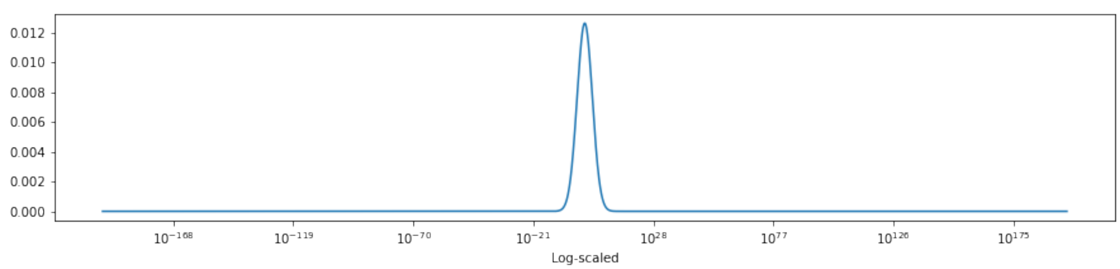
2 Question 2) Finally, recalculate the tree for N=4,000.

2.1 a) Plot the terminal prices produced by the model

[5]:
```
# Variables
S0 = 1  # Initial Price
N = 4000  # number of layers
group_n = 2  # Group Number 2
# Recreate the tree
tree_4000 = binomial_tree(N, S0, group_n)
stock_prices_4000 = tree_4000["stock_prices"]
# Mangling
df_P_4000 = pd.DataFrame(stock_prices_4000)
P_4000 = df_P_4000.iloc[-1:].to_numpy()
```

[6]:
```
# Calculate the frequency
freq_arr = binomial_freq(N, relative=True)
```

[7]:
```
# Plot
fig, ax = plt.subplots(
    figsize=(15,3)
)
ax.plot(
    P_4000[0],
    freq_arr
)
ax.set_xscale('log')
ax.set_xlabel('Log-scaled')
plt.show()
```



2.2 b) Can you identify what type of statistical distribution do these prices resemble to?

After a log-rescale of the x-axis, the post-rescale prices resemble a normal distribution. Hence, the prices themselves resemble a log-normal distribution.

2.3 c) Which statistical distribution would Returns follow, that is, using a log-function from previous prices? Indicate the appropriate Probability Density Function for it.

2.3.1 Case 1: if we defined $Return = \ln(\frac{Price_t}{Price_{t-1}})$

In this case, then *Returns* are either (roughly) $u - 1 \approx .12$ or $1 - d \approx -.12$.

Its distribution is:

$$p_{Returns}(u-1) = p_{Returns}(.12) = \mathbb{P}(Returns = .12) = p$$
$$p_{Returns}(1-d) = p_{Returns}(-.12) = \mathbb{P}(Returns = -.12) = 1 - p$$

where p is the probability of an up-movement.

2.3.2 Case 2: if we defined $Return = \ln(\frac{Price_t}{Price_0})$

then its distribution is the normal distribution,

and its PDF is, accordingly:

$$p_{Returns}(x) = \varphi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

3 Question 3)

3.1 a) How many fundamental securities are there in the market?

There are many fundamental securities in the markets:

- Cash & Cash equivalents
- Accounts Receivable / Notes Receivables
- Fixed Deposits
- Equity Shares
- Debentures/ Bonds
- Preference Shares
- Mutual Funds
- Interests in subsidiaries, associates and joint ventures
- Insurance contracts
- Rights and Obligations under leases
- Share-Based Payments

3.2 b) At any given node, how many states of the world are there in the binomial tree?

Each node represents 2 states of the world:

- The state where the price goes up
- The state where the price goes down

3.3 c) Define market completeness using parts a and b

A complete market is one where contracts can be made to gamble on all states of the world using the existing instruments

3.4 d) Suppose the underlying stock price jumped. By jumps, we mean that it moves by a factor larger than u (or smaller than d) from 1 node to the next. Would that market still be complete? Why or why not?

That market is still complete.

- If the price has increased by a factor larger than u, then it has also increased (at least) u. That state of the world is still represented in the model.
- If the price has decreased by a factor larger than d, then it has also decreased (at least) d. That state of the world is still represented in the model.

4 References

- Complete Market - Defintions.net
- Crow, Edwin L. Lognormal Distributions: Theory and Applications (2018). CRC Press.ISBN.URL:https://books.google.co.uk/books?id=glMPEAAAQBAJ
- John Van Der Hoek, Robert J. Elliott (2006).Binomial Models in Finance. Springer