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Exercise 1:

a) Prove that $K(x,y) = (\langle x,y\rangle + c)^m$ is a kernel

where $c \geq 0$, m is a positive number , $x,y \in \mathbb{R}^d$

(*) Apply Binomial Theorem:

$$(\langle x,y\rangle + c)^m = \sum_{j=0}^{m} \binom{m}{j} c^{m-j} \cdot (\langle x,y\rangle)^j = \sum_{j=0}^{m} a_j (\langle x,y\rangle)^j$$

, where $a_j = \binom{m}{j} c^{m-j} \geq 0$, j is an integer $0 \leq j \leq m$ (i)

(*) From the conic sum of kernels :

$$\sum_{j=0}^{m} a_j \cdot K_j \quad \text{is also a kernel} , \forall a_j \geq 0 \qquad (ii)$$

(*) $(\langle x,y\rangle)^j$ is a kernel , whose feature space $\check{v}$ could be

explained as $x_1^{u_1} x_2^{u_2} x_3^{u_3} x_4^{u_4} \ldots x_j^{u_t}$, where $\sum_{v=1}^{t} u_v = j$ (iii)

(i) (ii) (iii) $\Rightarrow K(x,y) = (\langle x,y\rangle + c)^m$ is a kernel.

b) For $x, y \in \mathbb{R}^2$ (meaning we have $\{x_1, x_2, y_1, y_2\}$, and $m = 3, c = 0$

$K(x,y) = (\langle x,y\rangle)^3 = (x_1 y_1 + x_2 y_2)^3 = x_1 x_1 x_1 y_1 y_1 y_1 + x_1 x_2 x_1 y_1 y_2 y_1$

$+ x_2 x_1 x_1 y_2 y_1 y_1 + x_2 x_2 x_1 y_2 y_2 y_1 + x_1 x_1 x_2 y_1 y_1 y_2 + x_1 x_2 x_2 y_1 y_2 y_2$

$+ x_2 x_1 x_2 y_2 y_1 y_2 + x_2 x_2 x_2 y_2 y_2 y_2$

So the feature space is

$$\phi(x) = \begin{bmatrix} x_1 x_2 x_1 \\ x_1 x_2 x_1 \\ x_2 x_1 x_1 \\ x_2 x_2 x_1 \\ x_1 x_1 x_2 \\ x_1 x_2 x_2 \\ x_2 x_1 x_2 \\ x_2 x_2 x_2 \end{bmatrix}$$

$$\phi(y) = \begin{bmatrix} y_1 y_1 y_1 \\ y_1 y_2 y_1 \\ y_2 y_1 y_1 \\ y_2 y_2 y_1 \\ y_1 y_1 y_2 \\ y_1 y_2 y_2 \\ y_2 y_1 y_2 \\ y_2 y_2 y_2 \end{bmatrix}$$