A NEW SOLVER FOR LINEAR DELAY DIFFERENTIAL-ALGEBRAIC EQUATIONS*

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Abstract. We discuss the new solver COLDDAE for the numerical solution of linear differential-algebraic equations (DDAEs) of retarded and neutral type. The implementation is mainly based on the results introduced in [12]. In the single delay case, the solver can handle noncausal systems, i.e., systems, where the solution at the current time does not only depend on the system at the past and current time points, but also future time points.

Key words. Delay differential-algebraic equation, differential-algebraic equation, delay differential equations, method of steps, derivative array, classification of DDAEs.

AMS subject classifications. 34A09, 34A12, 65L05, 65H10.

1. Introduction. We discuss the new solver COLDDAE for the numerical solution of linear delay differential-algebraic equations (DDAEs) with variable coefficients of the following form

$$E(t)\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{k} B_i(t)x(t - \tau_i(t)) + f(t), \quad t \in (t_0, t_f], \qquad (1.1a)$$

where $E, A, B_i \in C([t_0, t_f], \mathbb{R}^{m \times n})$, $f \in C([t_0, t_f], \mathbb{R}^m)$ and the delay functions $\tau_i \in C([t_0, t_f], \mathbb{R})$ satisfy $\tau_i(t) > 0$ for all $t \in [t_0, t_f]$. For later use in the index reduction procedure, we further assume that all these functions are sufficiently smooth.

Setting $\underline{\tau} := \min\{\tau_i(t) | \ t \in [t_0, t_f], \ i = 1, \ldots, k\}$ and $\bar{\tau} := \max\{\tau_i(t) | \ t \in [t_0, t_f], \ i = 1, \ldots, k\}$, we assume that $\underline{\tau} > 0$, which is often referred in the literature [2, 5] as the *non vanishing delay* case. We further assume that $\bar{\tau} < \infty$. Typically, to obtain a particular solution of the DDAE (1.1a), one needs to provide a history function (or initial function) $\phi \in C([t_0 - \bar{\tau}, t_0], \mathbb{R}^n)$ and prescribe

$$x(t) = \phi(t), \quad t \in [t_0, t_f). \qquad (1.1b)$$

We assume that (1.1) has a unique solution $x \in C((t_0, t_f], \mathbb{R}^n)$. For systems with constant delays the theoretical analysis for problems of the form (1.1) has been discussed in [12–14]. These results, however, can be directly generalized to the case of time dependent delays under some minor assumptions on the delay functions τ_i. Therefore, the solver we discuss in this paper is intended for time dependent delays. The most important concepts presented in this work are the causality, the system type, the shift index and the strangeness index of the DDAE (1.1a).

Under the assumption that the continuous, piecewise differentiable solution of the problem (1.1) exists and is unique, the implementation of the new solver is based on the regularization procedure introduced in [12], which first determines the shift index and the strangeness index and then transforms the DDAE (1.1a) into a regular, strangeness-free formulation with the same solution set. Using this regularization

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procedure, we can also compute a consistent initial value, if $\phi(t_0)$ is not consistent, i.e., does not satisfy all (hidden) constraints, and apply any appropriate integration scheme on the resulting regular, strangeness-free DDAE. We implemented the three stage Radau IIA collocation method, see [15].

2. A brief survey of the basic results. The numerical solution of DDAEs, until now, has only been considered for square systems, see e.g. [1, 2, 8, 11, 22, 25–28]. For such systems, the solution is usually computed by the classical (Bellman) *method of steps* [4–6], which will be recalled below.

Since $\underline{\tau} > 0$, we have $[t_0, t_f] \subset \bigcup_{j=1,\dots,\ell+1} [t_0 + (j-1)\underline{\tau}, t_0 + j\underline{\tau}]$ with $\ell := \lfloor \frac{t_f - t_0}{\underline{\tau}} \rfloor$. For all $t \in [t_0, t_0 + \underline{\tau}]$, we have $t - \tau_i(t) \leqslant t_0 + \underline{\tau} - \underline{\tau} = t_0$, and hence $x(t - \tau_i(t)) = \phi(t - \tau_i(t))$. The DDAE (1.1a) restricted on the interval $[t_0, t_0 + \underline{\tau}]$ then becomes

$$E(t)\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{k} B_i(t)\phi(t - \tau_i(t)) + f(t). \tag{2.1}$$

This system is a DAE in the variable $x_1 := x|_{[t_0, t_0 + \underline{\tau}]}$. The initial vector of the corresponding IVP for the DAE (2.1) is $x(t_0) = \phi(0)$, if consistent. Supposing that this IVP has a unique solution x_1, we can proceed in the same way to compute the function $x_2 := x|_{[t_0 + \underline{\tau}, t_0 + 2\underline{\tau}]}$, since $t - \tau_i(t) \leqslant t_0 + 2\underline{\tau} - \underline{\tau} = t_0 + \underline{\tau}$, for all $t \in [t_0 + \underline{\tau}, t_0 + 2\underline{\tau}]$. Therefore, the solution x of (1.1) will be computed step by step on consecutive intervals $[t_0 + (j-1)\underline{\tau}, t_0 + j\underline{\tau}]$, $1 \leqslant j \leqslant \ell$, by consecutively solving DAEs of the form

$$E(t)\dot{x}_j(t) = A(t)x_j(t) + g_j(t) \quad \text{for all } t \in [t_0 + (j-1)\underline{\tau}, t_0 + j\underline{\tau}]. \tag{2.2}$$

Clearly, we see that the method of steps successfully handles the problem (1.1) if and only if for every j, the corresponding IVP for the DAE (2.2) has a unique solution. This requirement means that the solution x at the current point $t' \in (t_0, t_f]$ depends only on the system (1.1a) at current and past time points $t \leqslant t'$, but not at future time points $t > t'$. We call this property *causality*, and a DDAE that satisfies this property *causal*. Restricted to the class of causal systems, different integration strategies based on the method of steps have been successfully implemented for linear DDAEs of the form (1.1a) and also for several classes of nonlinear DDAEs, see e.g. [1, 2, 11, 16, 25]. In contrast to causal DDAEs, this approach is not feasible for noncausal systems, consider for example the equation

$$0 \cdot \dot{x}(t) = 0 \cdot x(t) - x(t - \tau) + f(t), \quad \text{for all } t \in (0, \infty). \tag{2.3}$$

The method of steps applied to the DDAE (2.3) results in a sequence of underdetermined DAEs of the form

$$0 = g_i(t), \quad \text{for all } t \in [t_0 + (j-1)\underline{\tau}, t_0 + j\underline{\tau}].$$

Nevertheless, (2.3) still has a unique solution. The reason for this failure is that the method of steps takes into account only the equation at the current time, which is not enough, due to the noncausality. Therefore, a regularization procedure for DDAEs, so that the method of steps can be used for the resulting systems, is necessary. Note that for noncausal DDAEs of the form (1.1a), the solvability analysis has only been discussed for the single delay case, i.e., $k = 1$. Even for multiple constant delays, i.e., $\tau_i(t) \equiv \tau_i$, the solvability analysis for noncausal DDAEs is not entirely understood, [12, 13]. The regularization procedure proposed in the package COLDDAE for causal and noncausal systems will be considered in the Subsections 2.1 and 2.2, respectively.

2.1. Regularization procedure for causal DDAEs with multiple delays.
Inherited from the theory of DAEs, we see that even for causal DDAEs, the numerical integration requires a reformulation of the original system in such a way that one can avoid the loss in order of convergence or the drift-off effect, see e.g. [7, 19]. Here we use the regularization procedure associated with the *strangeness index* concept, see [19], which generalizes the well-known *differentiation index* [7] for general under- and over-determined DAEs. Briefly speaking, the strangeness index μ of the DAE

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \tag{2.4}$$

is the minimum number of differentiations such that from the *derivative array* (or *differentiation-inflated* system)

$$E(t)\dot{x}(t) - A(t)x(t) = f(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E(t)\dot{x}(t) - A(t)x(t)\right) = f^{(1)}(t),$$
$$\dots$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mu}\left(E(t)\dot{x}(t) - A(t)x(t)\right) = f^{(\mu)}(t),$$

one can extract the so-called *strangeness-free formulation*

$$\begin{bmatrix}\hat{E}_1(t)\\ 0\\ 0\end{bmatrix}\dot{x}(t) = \begin{bmatrix}\hat{A}_1(t)\\ \hat{A}_2(t)\\ 0\end{bmatrix}x(t) + \begin{bmatrix}\hat{f}_1(t)\\ \hat{f}_2(t)\\ \hat{f}_3(t)\end{bmatrix}, \tag{2.5}$$

which has the same solution set as the DAE (2.4), where the matrix-valued function $\begin{bmatrix}\hat{E}_1^T & \hat{A}_2^T\end{bmatrix}^T$ has pointwise full row rank. We also call μ the strangeness index of the pair (E, A). For the numerical determination of the strangeness index and the strangeness-free formulation (2.5), we refer the readers to [20, 21].

Now we apply this regularization procedure to the DDAE (1.1a), which is assumed to be causal, and obtain the *strangeness-free DDAE*

$$\begin{bmatrix}\hat{E}_1(t)\\ 0\\ 0\end{bmatrix}\dot{x}(t) = \begin{bmatrix}\hat{A}_1(t)\\ \hat{A}_2(t)\\ 0\end{bmatrix}x(t) + \sum_{i=1}^{k}\begin{bmatrix}\hat{B}_{i,0,1}(t)\\ \hat{B}_{i,0,2}(t)\\ 0\end{bmatrix}x(t-\tau_i(t))$$
$$+\sum_{i=1}^{k}\sum_{j=1}^{\mu}\begin{bmatrix}0\\ \hat{B}_{i,j,2}(t)\\ 0\end{bmatrix}x^{(j)}(t-\tau_i(t)) + \begin{bmatrix}\hat{f}_1(t)\\ \hat{f}_2(t)\\ \hat{f}_3(t)\end{bmatrix}, \quad \begin{matrix}d\\ a\\ v\end{matrix} \tag{2.6}$$

The sizes of the block row equations are d, a, v. Due to the causality of the DDAE (1.1a), it follows that $\begin{bmatrix}\hat{E}_1^T & \hat{A}_2^T\end{bmatrix}$ is pointwise nonsingular. For the DAE (2.4), the numerical solution $x(t)$ is obtained by integrating the strangeness-free formulation (2.5), which is more advantageous, see [17–19]. For the DDAE (1.1a), integrating the strangeness-free DDAE (2.6) is not always possible. The reason is that if at least one of the matrix functions $\hat{B}_{i,j,2}$ is not identically zero, then the underlying DDE is of advanced type, which is not suitable for numerical integration, [3]. In fact, until now there has been no solver for general advanced DDEs. For the numerical solution, solvers based on the method of steps are only suitable for retarded and

neutral DDAEs, see [1, 11, 12, 16]. In these cases, the strangeness-free DDAE (2.6) takes the form

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \\ 0 \end{bmatrix} x(t) + \sum_{i=1}^{k} \begin{bmatrix} \hat{B}_{i,1}(t) \\ \hat{B}_{i,2}(t) \\ 0 \end{bmatrix} x(t - \tau_i(t)) + \begin{bmatrix} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \hat{f}_3(t) \end{bmatrix}, \quad \begin{matrix} d \\ a. \\ v \end{matrix} \qquad (2.7)$$

The integration strategy we use for causal, retarded/neutral DDAEs of the form (1.1a) is: first, determine the strangeness-free formulation (2.7), and second, apply numerical methods to (2.7) to compute $x(t)$.

Without loss of generality, we set $k = 1$, $B(t) := B_1(t)$ and $\tau(t) := \tau_1(t)$ for notational convenience. For the determination of the strangeness-free DDAE (2.7), we use the derivative array of the DDAE as follows.

$$E(t)\dot{x}(t) - A(t)x(t) = B(t)x(t - \tau(t)) + f(t),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E(t)\dot{x}(t) - A(t)x(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(B(t)x(t - \tau(t)) + f(t)\right),$$

$$\cdots$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mu}\left(E(t)\dot{x}(t) - A(t)x(t)\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mu}\left(B(t)x(t - \tau(t)) + f(t)\right),$$

which can be rewritten as

$$M(t)z(t) = P(t)z(t - \tau(t)) + g(t), \qquad (2.8)$$

where

$$M(t) := \begin{bmatrix} -A(t) & E(t) & & & & \\ -\dot{A}(t) & \dot{E}(t) - A(t) & E(t) & & & \\ -\ddot{A}(t) & \ddot{E}(t) - 2\dot{A}(t) & 2\dot{E}(t) - A(t) & E(t) & & \\ \vdots & & & \ddots & \ddots & \\ -A^{(\mu)}(t) & E^{(\mu)}(t) - \mu A^{(\mu-1)}(t) & \cdots & \cdots & \mu\dot{E}(t) - A(t) & E(t) \end{bmatrix},$$

$$P(t) := \begin{bmatrix} B(t) & & & 0 \\ \dot{B}(t) & B(t)(1 - \dot{\tau}(t)) & & 0 \\ \ddot{B}(t) & 2\dot{B}(t)(1 - \dot{\tau}(t)) - B(t)\ddot{\tau}(t) & B(t)(1 - \dot{\tau}(t))^2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ * & * & * & 0 \end{bmatrix}$$

$$z(t) := \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \vdots \\ x^{(\mu+1)}(t) \end{bmatrix}, \quad g(t) := \begin{bmatrix} f(t) \\ \dot{f}(t) \\ \vdots \\ f^{(\mu+1)}(t) \end{bmatrix}.$$

Note that the j-th block row of P is obtained by computing the coefficients of $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j-1}(B(t)x(t - \tau(t)))$, $j = 1, \ldots, \mu$. These coefficients can be computed by using Faà di Bruno's formula [9], which describes the m-th derivative of the composite function $g(f(x))$ in the variable x as follows

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m}[g(f(x))]$$

$$= \sum \frac{m!}{b_1! \ldots b_m!} g^{(k)}(f(x)) \left(\frac{f^{(1)}(x)}{1!}\right)^{b_1} \left(\frac{f^{(2)}(x)}{2!}\right)^{b_2} \cdots \left(\frac{f^{(k)}(x)}{k!}\right)^{b_k}, \qquad (2.9)$$

where the sum is over all possible combinations of nonnegative integers $b_1, \ldots, b_k$ such that $b_1 + 2b_2 + \cdots + mb_m = m$ and $k = b_1 + b_2 + \cdots + b_m$. Note that, in our solver we assume that $\mu \leqslant 3$, therefore the entries of P are hard coded.

In the following, for notational convenience, we will use Matlab notation, [23] and we also omit the time variable t in the coefficients of (2.8).

The set of algebraic constraints in the strangeness-free DDAE (2.7) is selected by finding the full row rank matrix Z_2 such that

$$Z_2^T M(:, (n+1):end) = 0. \tag{2.10}$$

Scaling the system (2.8) with Z_2^T from the left, we obtain the equation

$$Z_2^T M(:, 1:n)x(t) = Z_2^T P z(t - \tau(t)) + Z_2^T g. \tag{2.11}$$

Furthermore, the DDAE (1.1a) is not of advanced type if and only if in (2.11) the derivatives of $x(t - \tau(t))$ do not occur. This means that

$$Z_2^T P(:, (kn+1):end) = 0. \tag{2.12}$$

Consider the following spaces and matrices

$$
\begin{array}{lcc}
T_2 & \text{basis of} & \ker(Z_2^T E), \\
Z_1 & \text{basis of} & \operatorname{range}(E T_2), \\
Y_2 & \text{basis of} & \operatorname{range}(Z_2^T M(:, 1:n)).
\end{array} \tag{2.13}
$$

The set of differential equations in the strangeness-free DDAE (2.7) is then given by

$$Z_1^T E(t)\dot{x}(t) = Z_1^T A(t)x(t) + Z_1^T B(t)x(t - \tau(t)) + Z_1^T f(t).$$

In summary, we obtain the strangeness-free DDAE

$$
\begin{aligned}
Z_1^T E(t)\dot{x}(t) &= Z_1^T A(t)x(t) + Z_1^T B(t)x(t - \tau(t)) + Z_1^T f(t), \\
Y_2^T Z_2^T M(:, 1:n)x(t) &= Y_2^T Z_2^T P(:, 1:kn)x(t - \tau(t)) + Y_2^T Z_2^T g,
\end{aligned} \tag{2.14}
$$

where $\begin{bmatrix} Z_1^T E(t) \\ Y_2^T Z_2^T M(:, 1:n) \end{bmatrix}$ is nonsingular.

2.2. Regularization procedure for noncausal DDAEs with single delay. In order to handle noncausal DDAEs with single constant delays, in [12], the authors proposed the concept of the *shift index*. This concept can be directly generalized to the case of time varying delays as follows.

DEFINITION 2.1. *Consider the problem* (1.1). *For each $t \in \mathbb{I}$, consider the sequence $\{t_{(j)} | j \geq 0\}$, starting with $t_{(0)} = t$, determined via the equation*

$$t_{(j+1)} - \tau(t_{(j+1)}) = t_{(j)}, \quad \text{for all } j \geq 0. \tag{2.15}$$

Then, the minimum number $\kappa = \kappa(t)$ such that the so-called shift-inflated *system*

$$
\begin{aligned}
E(t_{(0)})\dot{x}(t_{(0)}) &= A(t_{(0)})x(t_{(0)}) + B(t_{(0)})x(t_{(0)} - \tau(t_{(0)})), \\
E(t_{(1)})\dot{x}(t_{(1)}) &= A(t_{(1)})x(t_{(1)}) + B(t_{(1)})x(t_{(1)} - \tau(t_{(1)})), \\
&\vdots \\
E(t_{(\kappa)})\dot{x}(t_{(\kappa)}) &= A(t_{(\kappa)})x(t_{(\kappa)}) + B(t_{(\kappa)})x(t_{(\kappa)} - \tau(t_{(\kappa)})),
\end{aligned} \tag{2.16}
$$

uniquely determines $x(t_{(0)})$, *is called the* shift index *of the DDAE* (1.1a) *with respect to* t.

To guarantee the existence and uniqueness of the sequence $\{t_{(j)}|j \geq 0\}$ in Definition 2.1, we assume that for every $s \in (t_0, t_f)$ the equation

$$t - \tau(t) = s, \qquad (2.17)$$

has a unique solution on the time interval (s, t_f).

REMARK 2.2. *If the DDAE* (1.1a) *is causal then the shift index* κ *is* 0 *for all* $t \in [t_0, t_f]$.

Using (2.15), we can rewrite the shift-inflated system (2.16) as

$$
\begin{bmatrix} E(t_{(0)}) & & & \\ & E(t_{(1)}) & & \\ & & \ddots & \\ & & & E(t_{(\kappa)}) \end{bmatrix} \begin{bmatrix} \dot{x}(t_{(0)}) \\ \dot{x}(t_{(1)}) \\ \vdots \\ \dot{x}(t_{(\kappa)}) \end{bmatrix}
$$
$$
= \begin{bmatrix} A(t_{(0)}) & & & \\ B(t_{(1)}) & A(t_{(1)}) & & \\ & \ddots & \ddots & \\ & & B(t_{(\kappa)}) & A(t_{(\kappa)}) \end{bmatrix} \begin{bmatrix} x(t_{(0)}) \\ x(t_{(1)}) \\ \vdots \\ x(t_{(\kappa)}) \end{bmatrix} + \begin{bmatrix} B(t_{(0)})x(t_{(0)} - \tau(t_{(0)})) + f(t_{(0)}) \\ f(t_{(1)}) \\ \vdots \\ f(t_{(\kappa)}) \end{bmatrix}.
$$
$$ (2.18) $$

As shown in [12], assuming that the DDAE (1.1a) is not of advanced type and the strangeness index μ is well-defined for the DAE (2.18), we can extract from the DAE (2.18) a strangeness-free DDAE

$$
\begin{bmatrix} \hat{E}_1(t_{(0)}) \\ 0 \end{bmatrix} \dot{x}(t_{(0)}) = \begin{bmatrix} \hat{A}_1(t_{(0)}) \\ \hat{A}_2(t_{(0)}) \end{bmatrix} x(t_{(0)}) + \begin{bmatrix} \hat{B}_1(t_{(0)}) \\ \hat{B}_2(t_{(0)}) \end{bmatrix} x(t_{(0)} - \tau(t_{(0)})) + \begin{bmatrix} \hat{f}_1(t_{(0)}) \\ \hat{f}_2(t_{(0)}) \end{bmatrix}, \quad \begin{matrix} d \\ a \end{matrix} \quad (2.19)
$$

where $\begin{bmatrix} \hat{E}_1(t_{(0)}) \\ \hat{A}_2(t_{(0)}) \end{bmatrix}$ is nonsingular.

Analogous to the case of causal DDAEs, we build the derivative arrays (2.8) for system (2.18), and extract from it the strangeness-free DDAE (2.19). To keep the brevity of this paper, we will not repeat the process. However, it should be noted, as has been observed in [12], that the main differences between regularizing causal DDAEs and regularizing noncausal DDAEs are:

i) The size of the derivative arrays (2.8) for noncausal DDAEs is bigger than for causal DDAEs.

ii) For noncausal DDAEs the set of differential equations in the strangeness-free formulation (2.7) cannot be selected from the original DDAE (1.1a), but must be selected from the derivative array (2.8).

3. Algorithms in COLDDAE. First, we observe that the resulting system of the regularization procedure for DDAEs is the following regular, strangeness-free DDAE

$$
\begin{bmatrix} \hat{E}_1(t) \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \end{bmatrix} x(t) + \sum_{i=1}^{k} \begin{bmatrix} \hat{B}_{i,1}(t) \\ \hat{B}_{i,2}(t) \end{bmatrix} x(t - \tau_i(t)) + \begin{bmatrix} \hat{\gamma}_1(t) \\ \hat{\gamma}_2(t) \end{bmatrix}, \qquad (3.1)
$$

where $k = 1$ for noncausal DDAEs. We now apply numerical methods to (3.1) to determine $x(t)$. Again, we set $k = 1$ also for the causal case without loss of generality

and for notational convenience.

Adopted from the solver RADAR5 [11], we use the Radau scheme for the numerical integration, which is given by nodes

$$0 < \delta_1 < \cdots < \delta_s = 1, \quad s \in \mathbb{N}. \tag{3.2}$$

If we know all the discontinuity points of $\phi, \dot{\phi}, \ldots, \phi^{(s)}$ then we also know all the discontinuity points of $x, \dot{x}, \ldots, x^{(s)}$ on $[t_0, t_f]$, and hence we can include them into the mesh. However, for simplicity, the solver considers only the case that there is at most one discontinuity point at t_0. We denote a mesh by $\pi : t_0 < t_1 < \cdots < t_N = t_f$. The collocation points therefore are

$$t_{ij} = t_i + h_i \delta_j, \qquad j = 1, \ldots, s, \tag{3.3}$$

where h_i is the step size used at the i-th step. For the numerical approximation of the solution, we seek for the piecewise polynomial X_π of degree s, i.e., $X_{\pi,i} := X_\pi|_{[t_i, t_{i+1}]}$ are polynomials of degree s, which are determined by the following set of equations

$$\begin{bmatrix} \hat{E}_1(t_{ij}) \\ 0 \end{bmatrix} \dot{X}_\pi(t_{ij}) = \begin{bmatrix} \hat{A}_1(t_{ij}) \\ \hat{A}_2(t_{ij}) \end{bmatrix} X_\pi(t_{ij}) + \begin{bmatrix} \hat{B}_1(t_{ij}) \\ \hat{B}_2(t_{ij}) \end{bmatrix} X_\pi(t_{ij} - \tau(t_{ij})) + \begin{bmatrix} \hat{\gamma}_1(t_{ij}) \\ \hat{\gamma}_2(t_{ij}) \end{bmatrix}, \tag{3.4}$$

for all $i = 1, \ldots, N$, $j = 1, \ldots, s$.

Due to the presence of $\begin{bmatrix} \hat{B}_1(t_{ij}) \\ \hat{B}_2(t_{ij}) \end{bmatrix} X_\pi(t_{ij} - \tau(t_{ij}))$ in (3.4), we still have to define the history function $X_\pi(t_{ij} - \tau(t_{ij}))$ which is an approximation to $x(t_{ij} - \tau(t_{ij}))$. For this we choose

$$X_\pi(t_{ij} - \tau(t_{ij}))$$
$$= \begin{cases} \phi(t_{ij} - \tau(t_{ij})) & \text{if } t_{ij} - \tau(t_{ij}) < t_0, \\ X_{\pi,K}(t_{ij} - \tau(t_{ij})) & \text{for } 1 \leqslant K \leqslant N - 1 \text{ with } t_K < t_{ij} - \tau(t_{ij}) \leqslant t_{K+1}. \end{cases}$$

The continuous output polynomial $X_{\pi,K}$ at the K-th step is given by Lagrange interpolation of order s, i.e.,

$$X_{\pi,K}(t_K + \theta h_K) = \sum_{j=0}^{s} \mathcal{L}_j(\theta) X_{\pi,K}(t_K + \delta_j h_K), \tag{3.5}$$

where $\mathcal{L}_j(\theta)$ is the Lagrange polynomial of degree s satisfying $\mathcal{L}_j(\delta_K) = \delta_{Kj}$ with δ_{Kj} being the Kronecker delta symbol.

REMARK 3.1. *As noticed in [10, 11], one can optionally replace the continuous output polynomial $X_{\pi,K}$ in (3.5) by another dense output polynomial given by*

$$X_{\pi,K}(t_K + \theta h_K) = \sum_{j=1}^{s} \mathcal{L}_j(\theta) X_{\pi,K}(t_K + \delta_j h_K).$$

The use of only s interpolation nodes δ_j, $j = 1, \ldots, s$ instead of $s + 1$ nodes δ_j, $j = 0, \ldots, s$ is beneficial in the presence of a jump in the solution at the point t_K, i.e., $X_{\pi,K}(t_K) \neq X_{\pi,K-1}(t_K)$.

The existence and uniqueness, and the convergence results for the numerical approximation X_π are stated in the following theorem.

THEOREM 3.2. *Consider the problem (1.1) and assume that it is uniquely solvable and of either retarded or neutral type. For $N \in \mathbb{N}$ and $s \geq 1$, define the mesh π and the collocation points t_{ij}, $j = 1, \ldots, s$ as in (3.3). Then the following assertions hold.*

i) *For sufficiently small mesh widths $h_0, \ldots, h_{N-1}$ there exists one and only one continuous piecewise polynomial X_π that solves the DAE sequence (3.4) and it is consistent at all the mesh point t_i.*

ii) *The convergence order of the collocation method with schemes δ_j as in (3.2) is s, i.e.,*

$$\|X_e(t) - X_\pi(t)\|_\infty = \sup_{t \in \mathbb{I}} \|X_e(t) - X_\pi(t)\| = O(h^s),$$

where X_e is the exact solution $x \in C^{s+1}(\mathbb{I}, \mathbb{C}^n)$ to the problem (1.1).

Proof. For the proof see Theorem 4, [16] or Theorem 4.2, [11]. □

4. Using COLDDAE. The solver COLDDAE, implemented in Matlab, can handle both causal and noncausal DDAEs of retarded and neutral type, but as already stated above, we restrict the noncausal systems to have only one delay. If the system is of advanced type, an appropriate error message is thrown. We have implemented step size control and so-called long steps, i.e., the step size may become bigger than the delay. Unless the user provides the exact derivative array, i.e. M, P and g from equation (2.8), the strangeness index of the system (2.18) can be at most three due to hard coding in the computation of P.

In the following we will describe the parameters inside the solver.

4.1. Input parameters.
- `E` The matrix function $E : [t_0, t_f] \to \mathbb{R}^{m,n}$.
- `A` The matrix function $A : [t_0, t_f] \to \mathbb{R}^{m,n}$.
- `B` The matrix function $B : [t_0, t_f] \to \mathbb{R}^{m,kn}$, where k is the number of delays.
- `tau` The delay functions $t \mapsto [\tau_1(t), \ldots, \tau_k(t)]$.
- `phi` The history function ϕ, i.e., $x(t) = \phi(t)$ for $t < t_0$.
- `tspan` The solution interval $[t_0, t_f]$, `tspan(1)` $= t_0$, `tspan(2)` $= t_f$.
- `options` A `struct` containing the optional parameters.

4.2. Optional input parameters. Optional parameters can be passed by the input parameter `options` by the command

$$\text{options}.\textit{field\_name} = \textit{field\_value}.$$

The following fields are available in this solver:
- `MaxIter` Upper bound for the total number of time steps (excluding rejected time steps), default: `10000`.
- `MaxReject` Upper bound for the number of rejections per time step, default: `100`.
- `MaxCorrect` Upper bound for the number of correction steps when using long steps (step size bigger than the lag), default: `10`.
- `InitStep` The initial step size of the Runge-Kutta method, default: $\frac{t_f - t_0}{100}$.
- `MinStep` A lower bound for the step size, default: `0`.
- `MaxStep` An upper bound for the step size, default: `inf`.
- `AbsTol` Absolute tolerance, default: `1e-5`.
- `RelTol` Relative tolerance, default: `1e-5`.
- `StrIdx` Lower bound for the strangeness index, default: `0`.
- `MaxStrIdx` Upper bound for the strangeness index, default: `3`.
- `Shift` Lower bound for the strangeness index, default: `0`.
- `MaxShift` Upper bound for the strangeness index, default: `3`.
- `InitVal` Initial value, not necessarily consistent, default: $\phi(t_0)$.

- **IsConst** A boolean, **true** if E, A, B, τ are constant, the regularized system is then computed only once, default: **false**.
- **DArray** Struct with three fields containing M, P (or $[P_1, \ldots, P_k]$ for $k > 1$) and g from equation (2.8). default: not set.

4.3. Output parameters.
- **t** A discretization of **tspan** with variable step size.
- **x** The numerical solution at **t**.
- **info** A struct with information, e.g. the strangeness index, shift index, etc.

5. Numerical experiments. We use COLDDAE with its default values to solve four different non-advanced DDAEs. Example 5.1, 5.2 are taken from the DDE test set [24].

EXAMPLE 5.1. *We consider the following DDE with constant delay.*

$$\begin{aligned}
\dot{x}_1(t) &= x_3(t), \\
\dot{x}_2(t) &= x_4(t), \\
\dot{x}_3(t) &= -2mx_2(t) + (1 + m^2)(-1)^m x_1(t - \pi), \\
\dot{x}_4(t) &= -2mx_1(t) + (1 + m^2)(-1)^m x_2(t - \pi),
\end{aligned}$$

(5.1a)

for $t > 0$ and $x(t) = \phi(t)$ for $t \leqslant 0$ with

$$\begin{aligned}
\phi_1(t) &= \sin(t)\cos(mt), \\
\phi_2(t) &= \cos(t)\sin(mt), \\
\phi_3(t) &= \cos(t)\cos(mt) - m\sin(t)\sin(mt), \\
\phi_4(t) &= m\cos(t)\cos(mt) - \sin(t)\sin(mt).
\end{aligned}$$

(5.1b)

Analytical solution: $x(t) = \phi(t)$ *for $t > 0$.*
The relative error of the numerical solution of (5.1) with $m = 2$ is going up fast, since additional errors are introduced by the delayed terms, see Figure 5.1.

EXAMPLE 5.2. *We consider the following DDAE*

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t - 1), \quad \text{for all } t > 0,$$

$$\phi(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad\qquad\qquad \text{for all } t \leqslant 0.$$

(5.2)

Note that the DDAE (5.2) is reformulated from the neutral DDE $\dot{x}(t) = x(t) + \dot{x}(t-1)$ by introducing a new variable to represent $x(t-1)$.

Analytical solution: *The problem (5.2) possesses the unique solution $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ given by*

$$x_1(t) = \begin{cases} e^t & \text{for } 0 < t \leqslant 1, \\ (t-1)e^{t-1} + e^t & \text{for } 1 < t \leqslant 2, \\ \frac{1}{2}(t^2 - 2t)e^{t-2} + (t-1)e^{t-1} + e^t & \text{for } 2 < t \leqslant 3, \\ \frac{1}{6}(t^3 - 3t^2 - 3t + 9)e^{t-3} + \frac{1}{2}(t^2 - 2t)e^{t-2} + (t-1)e^{t-1} + e^t & \text{for } 3 < t \leqslant 4, \end{cases}$$

and $x_2(t) = x_1(t - 1)$. The relative error of the numerical solution of (5.1) is presented in Figure 5.1 and goes up, too, due to the discretization and additional errors introduced by approximating the delayed terms.
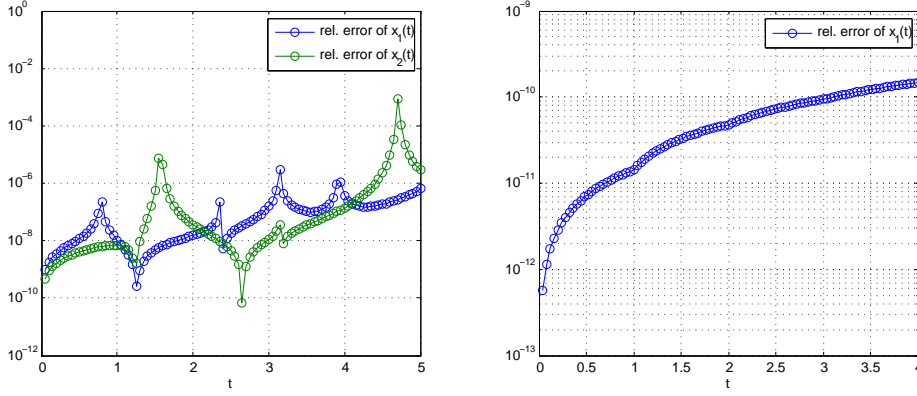
FIG. 5.1. *Relative error of the solution of* (5.1) *(left) and* (5.2) *(right).*

EXAMPLE 5.3. *We consider the following DDAE with constant coefficients and multiple time-varying delays. This DDAE is causal and it has strangeness index two.*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} x_2(t-1) + x_3(\frac{t}{2}-1) \\ 0 \\ 0 \end{bmatrix} + f(t), \quad \text{for all } t > 0,$$

$$\phi(t) = \begin{bmatrix} e^t \\ 1 \\ \sin(t) \end{bmatrix}, \qquad\qquad\qquad \text{for all } t \leqslant 0.$$

(5.3)

The function $f(t)$ is chosen such that the analytical solution is $x(t) = \phi(t)$. The relative error of the numerical solution of (5.3) *is presented in Figure 5.2. Note, that the regularized form of* (5.3) *consists of algebraic equations (with delayed terms) only, therefore the relative error does not go up as fast as in the other examples.*

EXAMPLE 5.4. *We consider the following linear DDAE with constant coefficients and time-varying delay. This DDAE is noncausal and it has strangeness index one and shift index one.*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1\left(t-1+\frac{\sin(t)}{2}\right) \\ x_2\left(t-1+\frac{\sin(t)}{2}\right) \end{bmatrix} + f(t), \quad \text{for all } t > 0,$$

$$\phi(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}, \qquad\qquad\qquad \text{for all } t \leqslant 0.$$

(5.4)

After regularization the system becomes a pure ODE and an algebraic equation. The function $f(t)$ is chosen such that the IVP (5.4) *has the unique solution $x(t) = \phi(t)$. The relative error of the numerical solution of the IVP* (5.4) *is presented in Figure 5.2.*
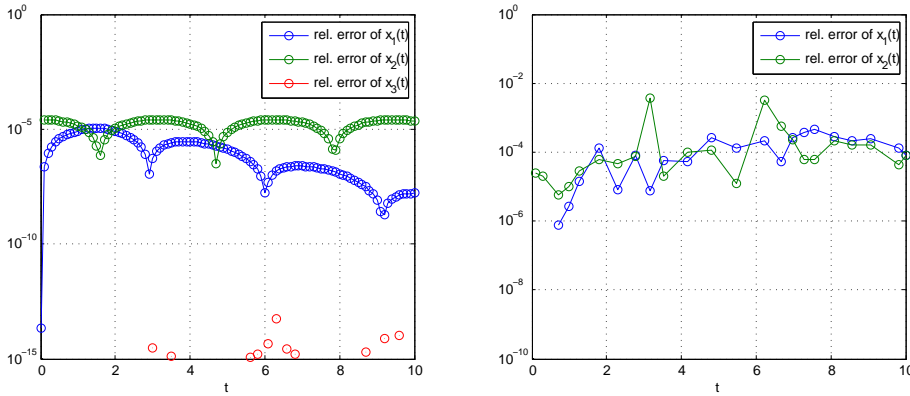
Fig. 5.2. *Relative error of the solution of* (5.3) *(left) and* (5.4) *(right).*

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