“Mixed Convection of Nanofluid Flow in a Vertical Channel with General Boundary Conditions”

Dissertation

Submitted in partial fulfillment of the requirements for the degree of

M.Sc. Mathematics

By

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Certificate

This is to certify that the Group Dissertation titled "**Mixed Convection of Nanofluid Flow in a Vertical Channel with General Boundary Conditions** " is a bonafide record of the work carried out by Vinay M S, Reg. No.:21MPMT076011 in partial fulfillment of requirements for the award of M. Sc. Degree of M. S. Ramaiah University of Applied Sciences in the Department of Mathematics and Statistics

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Declaration

“Mixed convection of nanofluid flow in a vertical channel with general boundary conditions”

The dissertation is submitted in partial fulfilment of academic requirements for the M.Sc. Degree of Ramaiah University of Applied Sciences in the Department of Mathematics and Statistics. This dissertation is a result of my research. All sections of the text and results, which have been obtained from other sources, are fully referenced. I understand that cheating and plagiarism constitute a breach of university regulations, hence this dissertation has been passed through plagiarism check and the report has been submitted to the supervisor.

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Table of contents

Preface

Differential transform method (DTM) is a simple technique based on the Taylor series. Applying DTM to given nonlinear mathematical problem is converted into a recurrence relation or a set of recurrence relations for the Taylor coefficients. This ultimately leads to the solution of the mathematical problem in the form of an infinite power series with an appropriate region of convergence. DTM has been successfully applied in solving general boundary value problems involving ordinary and partial differential equations. The present dissertation aims to apply DTM in solving mixed convection of nanofluid flow in a vertical channel with general boundary conditions.

The dissertation is organized into four chapters. **Chapter 1** is of introductory nature and presents the objective and scope of the dissertation. A detailed literature survey concerning nanofluid and DTM is also presented in **Chapter 1**. Conservation laws of fluid mechanics, namely, conservation of mass, linear momentum and energy are discussed in **Chapter 2**. Further, **Chapter 2** provides details of approximations, boundary conditions and dimensionless parameters involved in mixed convection in nanofluid.

Chapter 4 is devoted to study of mixed convection of nanofluid flow in a vertical channel with GBC arising from a real world problem. The vertical plates that are made of tightly-packed porous medium enable in exchanging fluid and heat with the surroundings and the same are modeled through third-kind boundary conditions temperature field and on velocity Beveris-Joseph boundary condition. The single phase nanofluid model is used to model the flow through porous medium in a vertical channel. The plates are permeable they exchange heat with the external fluid. The mathematical models gives rise to nonlinear equation involving Beveries-Joseph boundary condition and third-kind boundary conditions with respect to the physical problem considered. The velocity and temperature distributions are obtained as a function of physical parameters such as mixed convection parameter, Brinkman number, porous parameter, Biot numbers, and slip-Darcy number by using mathematical software MATLAB. This chapter also discuss with special cases. The solution by DTM is compared with that obtained by MATLAB bvp4c routine and found in excellent agreement, establishing the accuracy of DTM.

Chapter 1

Introduction

1.1 Background

Flow and heat transfer problems are encountered in many areas of science and engineering, like, mechanical engineering, civil engineering, fluid mechanics and so on. The variation of fluid flow and heat transfer is the primary focus of present time. The dissertation is devoted to apply differential transform method in solving such boundary value problem involving ordinary differential equations. The fluid saturated sparsely-packed porous medium in a vertical channel exchanging heat and fluid with the surroundings that give rise to a boundary value problem will be investigated with the help of differential transform method.

1.2 Objective and Scope

Most scientific problems in fluid mechanics are inherently nonlinear. All these problems and phenomena are modelled by a set of ordinary/partial differential equations. Solutions of these differential equations are important in predicting future states of phenomena under study. Most of the differential equations are highly nonlinear and exact solutions are not always possible. For those cases where exact solutions are not possible, numerical methods often provide approximate solutions. Both numerical and analytical methods have their own advantages and disadvantages. This study sought to introduce new and improved semi-numerical-analytical technique known as differential transform method for solving linear/nonlinear differential and integral equations and especially boundary value problems. This technique aims to combine the strength of both numerical and analytical methods.

The differential method used to solve wide range of problems whose mathematical models yield equations or system involving algebraic, differential, integral, and integro-differential equations. Mixed convection of nanofluid flow in a vertical channel to various problems is science and engineering fields include solutions of linear/nonlinear initial value problems, linear/nonlinear boundary value problem, linear/nonlinear integral equations and general boundary conditions.

Thermal convection in fluid-saturated porous media is of considerable interest due to its numerous applications in different fields, such as geothermal energy utilization, oil reservoir modelling, building thermal insulation, and nuclear waste disposal, to mention a few. It is in particular, of great importance to petroleum engineers concerned with the movement of gas and to chemical engineers in connection with filtration processes.

Keeping in view of the several applications of convection in fluid-saturated porous media, the objective of the present work is to understand thermal convective instability in fluid-saturated porous media with boundary conditions of third kind with the help of differential transform method.

The following are the main objectives of the dissertation

- State and prove properties and important theorems concerning DTM
- Illustrate basic concepts of nanofluids
- Illustrate application of nanofluids
- Validate the results of convection of nanofluid flow method with any known analytical/numerical methods

1.2.1 Literature on DTM:

Ming et al. [1] have applied differential transform method of fixed and adaptive grid size to obtain approximate solution of linear and nonlinear initial value problem involving ordinary differential equation. Abdel-Halim [2] has discussed the application of differential transform method to obtain eigenvalues and normalized eigen functions of Sturm-Liouville problem. Zaid et al. [3] generalized one-dimensional differential transform method to differential equations of fractional order, based on generalized Taylor formula and Caputo fractional derivatives. Parviz et al. [4] applied two-dimensional differential transform to present an efficient procedure for solving the two-dimensional nonlinear Volterra integro-differential equations. Abdel-Halim [2] has studied the analytical solution for different systems of linear and nonlinear differential equations using differential transform method. Mohammad and

Abdollah [5] have applied differential transform method to find exact solutions of nonlinear delay integro-differential equations. Mohseni and Saeedi [6] have applied differential transform method to solve the Volterra integro-partial differential equations based on Taylor series expansion for functions of several variables. Che-Haziqah et al. [7] applied the generalized differential transformation method to solve higher order linear boundary value problem. Rostam and Botan [8] considered implementation of differential transform method to solve system of linear/nonlinear delay differential equations. Salah and Saied [9] have extended the differential transformation method to higher order nonlinear Volterra-Fredholm integro-differential equations with separable kernels. Yuzbasi and Nurbol [10] have applied the differential transform method to solve system of Volterra integral and integro-differential equations with proportional delay. Sucheta and Narhari [11] have applied differential transform method to singular initial value problem (Lane-Emden differential equation) and Volterra integral equation of second kind. Behiry [12] have introduced new approach for differential transform method, based on Adomain polynomials, to solve nonlinear initial value problem and illustrated the features of related theorems, several numerical examples with different types of nonlinearities. Uayip et al. [13] applied the differential transform method to solve linear and nonlinear two-dimensional Volterra integral equation with proportional Volterra integral equation with proportional delays. Raslan et al. [14] have used the differential transform method for finding approximate solutions of some partial differential equations with variable coefficients. Villafuerte [15] have applied the differential transform method to time-dependent random linear differential equation. Zdenek et al. [16] studied the differential transformation method by providing new theorems to develop approximate solutions of nonlinear differential and integro-differential equations with proportional delays represented by nonlinear multi-pantograph equations. Saurabh et al. [17] obtained the series solution of nonlinear differential equations using differential transform method. Tari et al. [18] have applied two-dimensional differential transform method for double integrals to solve a class of two-dimensional linear and nonlinear Volterra integral equations. Mohammed and Al-Amr [19] have applied the reduced differential transform method for solving two types of nonlinear partial differential equations, namely, generalized Drinfeld-Sokolov equation and Kaup-Kupershmidt equation. Dimatteoa and Pirrotta [20] have applied the generalized differential transform method to solve linear/nonlinear boundary value problem involving differential equation of fractional order. Matteoa and Pirrotta [21] have also applied the generalized differential transform method to obtain approximate solution of linear/nonlinear boundary value problems involving differential equations of fractional order. Tari and Shahmorad [22] have applied differential transform method for solving a class of system of two-dimensional linear and nonlinear Volterra integro-differential equations of the second kind. Ziyaee and Tari [23] have applied differential transform method to solve two-dimensional Fredholm integral equation of second kind. Recently, Chandrali [24] obtained the solution of nonlinear singular initial value problem of Emden-Fowler type using differential transform method based on Adomian polynomials. Munaza et al. [25] have implemented differential transform method for solving various forms of nonlinear Klein-Gordon type equations. Zaid [26] applied differential transform method to solve Volterra integral equations with separable kernels.

It should be noted from the above literature survey that the application of differential transform method to mixed convection in a vertical channel filled with porous medium with BJ and mixed type boundary conditions are very limited.

1.2.2 Literature on convection in porous media

Darcy's law is a phenomenological law that describes the flow of a fluid through a porous medium and it results in a constitutive equation which is empirical in nature. The law was formulated by Henry Darcy [27] based on the results of experiments on flow of water through beds of sand. Extensive details of Darcy model, Forchhiemer model (Forchheimer [28] and Brinkman model (Brinkman [29]) and comprehensive literature on the subject can be found in the books of Bejan [30], Pop and Ingham [31], Vafai [32], Vadasz [33] and Nield and Bejan [34]. Beji and Gobin [35] numerically studied thermal dispersion effects on the natural dispersion heat transfer using Darcy-Forchheimer-Brinkman model. Mishra and Sarkar [36] have provided comparative numerical studies concerning differentially heated square cavity using Darcy, Brinkman and Darcy-Brinkman-Forchheimer models and confirmed that boundary friction and quadratic drag lead to a reduction in heat transfer. Vafai and Kim [37] have presented numerical study based on the Darcy-Forchheimer-Brinkman model for the forced convection in a composite system containing fluid and porous regions. Nield [38] obtained a closed form solution of the Brinkman-Forchheimer equation for different values of Darcy number using the Brinkman-Forchheimer model and stress jump boundary condition at the porous interface. Nakayama [39] has presented a unified treatment of Darcy-Forchheimer boundary layer flows. Paul et al. [40] studied natural convection flow in a vertical channel partially filled with a porous medium using Brinkman-Forchheimer extended Darcy model and reported that the effect of viscous dissipation is significant on the heat transfer inside the channel. The thermal instability of porous layers of finite or infinite width, subject to heating from below, is a classical topic of growing interest for the heat transfer community.

In the recent decades, several new results have been achieved on this subject, for instance, by Nield [38], Rees [40] and Tyvand [41]. In particular, recent studies have shown the role played by temperature boundary conditions of the third kind, describing a wall heat transfer regime intermediate between the first kind (Dirichlet-type) temperature conditions, and the second kind (Neumann-type) temperature conditions. The onset of thermohaline convection in a layer of fluid-saturated porous medium was considered by Nield [38] with the horizontal boundaries subjected to general boundary conditions on velocity, temperature and concentration fields. Nield [38] obtained an eigenvalue equation involving thermal Rayleigh number R and an analogous solute Rayleigh S number based on Fourier series method. Based on the numerical results it was reported that the net destabilizing effect can be expressed by the sum of R and S for identical boundary conditions on temperature and concentration fields.

The studies on the effect of third-kind temperature conditions, prescribed at the sidewalls of a vertical layer, on the thermal instability in a porous medium were initiated by Weidman and Kassoy [42]. Kubitschek and Weidman [43] analyzed the thermal instability in a fluid saturated porous box where the lower horizontal boundary is heated through an external forced convection process, parameterized by a Biot number, while the upper horizontal boundary is maintained isothermal. The influence of side wall heat transfer on convection was investigated by Kubitschek and Weidman [43] by assuming general boundary condition on temperature for lower horizontal surface. Kubitschek and Weidman [43] found that, at the onset of convection, the isoflux/isothermal horizontal boundaries lead to a value 27.096 for the critical Darcy-Rayleigh number.

The Biot number is, in fact, the dimensionless parameter involved when third-kind temperature conditions are used to model the boundary heat transfer (see, for instance, Incropera et al. [44]). It is to be mentioned that Nygard and Tyvand [45], on studying the thermal instability in a porous box (Nygard and Tyvand [45]), and in a vertical porous cylinder (Nygard and Tyvand [45]), considered third-kind boundary conditions for the temperature field in the case of partially permeable boundary walls. Mojtabi and Rees [47], as well as, Rees and Mojtabi [46], obtained third-kind boundary conditions for the temperature disturbances in the stability analysis of a saturated porous layer bounded by walls with a finite and non-negligible thermal conductance. A peculiar feature in this case is that the Biot number depends on the wave number of the disturbance. Barletta and Storesletten [48] studied the onset of the thermal instability in a circular porous duct saturated by a fluid and subject to a temperature boundary condition of the third kind.

A three-dimensional study of the onset of convection in a horizontal, rectangular porous channel heated from below was considered by Barletta and Storesletten [48] highlighting the effect of the longitudinal wave number of the normal mode disturbances on the motionless rest state. In this study, the vertical impermeable sidewalls were subjected to an external convection process, modelled by temperature boundary conditions of the third kind. The role of the Biot number in determining the onset conditions of the instability was described. The assumption of an infinite channel length was finally relaxed in order to establish the behavior in the case of discrete spectrum for the longitudinal wave number of the normal modes employed in the linear stability analysis.

Effects of confinement, variable permeability, and thermal boundary conditions on the natural convection in a porous medium were considered by Ribando and Torrance [49]. The ratio of fluid viscosity μ and matrix permeability K was assumed to be an exponential function of the depth z of the porous layer. Wang [50] studied the onset of convection in a fluid-saturated rectangular box with lower horizontal wall heated by constant flux and showed that temperature deviations are completely asymmetrical about the mid-horizontal plane which is unlike the constant temperature case.

The literature survey concerning differential transform method and convection in porous medium reveal that there are no studies on the application of differential transform method in solving problem arising from convection in porous medium in a vertical channel with BJ and mixed type boundary condition. With the prime focus of application of differential transform method to boundary value problem the dissertation considers Darcy-Brinkman convection problem subject to BJ and general boundary conditions as a representation.

Chapter 2

Fundamental equations, boundary conditions, approximations and non-dimensional numbers

2.1 Introduction

The theoretical mechanics characterizes the physical behavior of a fluid by means of suitable analytical expression, so that one can specify the properties of an arbitrary element of the material and also predict the behavior of the material in bulk by means of differential equations with suitable initial and boundary conditions. The convective flow and heat transfer can be modelled mathematically by the set of partial differential equations derived from the conservation of mass, Newton's second law and the first law of thermodynamics. A study of these differential equations, their derivation together with their solution, forms the basis of the present work. With this view point, the basic equations, boundary conditions, approximations and dimensionless numbers are discussed in this chapter.

2.2 Fundamental equations of fluid dynamics

The investigation of any fluid motion involves solving a set of non-linear partial differential equations called the fundamental equations of fluid dynamics. The fundamental equations governing any flow phenomena are stated below:

2.2.1 Equation of continuity:

<u>Physical principle</u>: *Mass can neither be created nor be destroyed* (Law of conservation of mass)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0. \tag{2.1}$$

Here, ρ is the density fluid, $\vec{q} = (u, v, w)$ is the velocity of the fluid in Cartesian frame of reference and t is time. The vector differential operator ∇ is given by

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}.$$

2.2.2 Equation of linear momentum:

<u>Physical principle</u>: *The total force acting on a fluid mass enclosed in an arbitrary volume fixed in space is equal to the time rate of change of linear momentum* (Law of conservation of momentum)

$$\rho \frac{D\vec{q}}{Dt} = \rho \vec{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}, \tag{2.2}$$

where, $\boldsymbol{\tau}$ is stress tensor, p is the hydrostatic pressure and $\rho \vec{g}$ is the body force. The operator $\frac{D}{Dt}$, known as the material derivative, is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{q} \cdot \nabla$$

The momentum equation is popularly known as the Navier-Stokes equation.

2.2.3 Equation of angular momentum:

<u>Physical principle</u>: *The total moment of the forces acting on a fluid mass enclosed in an arbitrary volume fixed in space is equal to the time rate of change of angular momentum* (Law of conservation of angular momentum)

$$\rho \frac{D\vec{\omega}}{Dt} = \vec{r} \times (\rho \vec{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}). \tag{2.3}$$

Here, $\vec{\omega} = \vec{r} \times \vec{q}$ is the angular velocity and $\vec{r}$ being the position vector of a fluid element in space. Equation (2.3) can be obtained by taking cross product of momentum equation (2.2) with $\vec{r}$.

2.2.4 Equation of energy

Physical principle: *The energy added to a closed system increases the initial energy per unit mass of the fluid* (Law of conservation of energy)

$$\rho C_p \frac{DT}{Dt} = -\nabla \cdot \vec{Q} - p(\nabla \cdot \vec{q}) + \Phi. \tag{2.4}$$

Here, C_p is the specific heat at constant pressure, T is the temperature, $\vec{Q} = -k\nabla T$ is heat flux vector, k is the thermal conductivity of the fluid and $\Phi = \nabla \cdot (\boldsymbol{\tau}\vec{q}) - \vec{q}\nabla \cdot \boldsymbol{\tau}$ is the dissipation function. Equations (2.1) - (2.4) are known as thse fundamental equations of fluid dynamics.

2.3 Basic equations for Newtonian liquid

The general basic equations of fluid dynamics mentioned in **Section 2.2** take particular forms based in the fluid under consideration. The stress tensor $\boldsymbol{\tau}$ in equation (2.2) is a characteristic of the fluid chosen. The stress tensor is related to rate of strain by means of constitutive relations that are similar to stress-strain relationships for elastic solids. Newton proposed a linear constitutive relation between stress and rate of strain for viscous fluids. Fluids obeying Newton's stress-strain relationship are referred to as Newtonian fluids. Many fluids deviate from the Newton's law and are referred to as non-Newtonian fluids. In this section, the basic equations concerning Newtonian fluid are presented.

2.3.1 Newtonian law of viscosity

According to the Newton's law of viscosity, for laminar flows, the shear stress is directly proportional to the strain rate or velocity gradient,

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}, \tag{2.5}$$

where μ is the constant of proportionality known as the coefficient of dynamic viscosity of the fluid. The shear stress is maximum at the surface, the fluid is in contact with, due to no slip condition.

2.3.2 Continuity equation

For incompressible flow, the density of the fluid is assumed to remain constant. It is interesting to note that flow of a compressible fluid can be regarded as incompressible, if the Mach number for the flow is less than 0.3. Therefore, for incompressible fluids, from equation (2.1) the continuity equation reduces to the following form

$$\nabla \cdot \vec{q} = 0.$$

2.3.3 Momentum equation

The Navier-Stokes equation, describe the momentum balance in the fluid flow. The Navier-Stokes equation is a differential equation, which unlike algebraic equation do not explicitly establish a relation among the variables of interest rather they establish relation among the rate of change of these quantities. Solution of the Navier-Stokes equation gives velocity field, which is a description of the velocity of the fluid at a given point in space and time. Once the velocity field is found, other quantities of interest such as flow rate, drag force, the path of a fluid particle and so on, may be found. The constitutive relation for Newtonian fluid is given by

$$\boldsymbol{\tau} = \mu\left[\nabla\vec{q} + (\nabla\vec{q})^T - \frac{2}{3}(\nabla \cdot \vec{q})\mathbf{I}\right] + \mu_v(\nabla \cdot \vec{q})\mathbf{I}.$$

Here, the coefficient μ_v is known as second coefficient of viscosity or volume viscosity. Substituting the above expression for stress tensor in equation (2.2), the momentum equation reduces to

$$\rho\left[\frac{\partial\vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q}\right] = -\nabla p + \rho\vec{g} + \mu\nabla^2\vec{q} + \left(\mu_v - \frac{2}{3}\mu\right)(\nabla \cdot \vec{q})\mathbf{I}$$

From equation (2.6), for incompressible fluids, the divergence of velocity is zero and hence the momentum equation for the viscous incompressible fluid, further reduces to

$$\rho\left[\frac{\partial\vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q}\right] = -\nabla p + \rho\vec{g} + \mu\nabla^2\vec{q}. \tag{2.7}$$

2.3.4 Energy equation

For incompressible fluids, on using equation (2.6), the energy equation (2.4) reduces to

$$\rho C_p \left[\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T\right] = k\nabla^2 T + \Phi, \tag{2.8}$$

where Φ is the viscous dissipation defined by

$$\Phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^2 + \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right]^2 + \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]^2.$$

Due to the assumption of a homogeneous liquid, the temperature field does not alter the velocity field and hence the effects of viscous dissipation, internal heat generation/absorption on the velocity field come into play only in the presence of buoyancy as it results in a two-way coupling between velocity and temperature fields.

2.4 Boussinesq approximation

Navier-Stokes equation in its simplest form is considered in order to illustrate Boussinesq approximation. If p and ρ are now expanded about the values p_0 and ρ_0 in a reference state of hydrostatic equilibrium for which

$$\nabla p_0 = \rho_0 \vec{g}.$$

Setting $p = p_0 + p'$, $\rho = \rho_0 + \rho'$ and $\vec{q} = \vec{q}_0 + \vec{q}\,'$, and noting $\vec{q}_0 = 0$ for equilibrium quiescent state, the Navier-Stokes equation (2.7) takes the form

$$(\rho_0 + \rho')\frac{D\vec{q}\,'}{Dt} = -\nabla p' + \rho'\,\vec{g} + \mu\,\nabla^2\vec{q}\,'. \tag{2.9}$$

This equation implies that the difference of densities ρ' between standard and reference values is relevant in determining the effect of gravity. Vowing to the approximation that density variation ρ' is small compared to ρ_0, equation (2.9) is rewritten in the following form:

$$\left(1 + \frac{\rho'}{\rho_0}\right)\frac{D\vec{q}\,'}{Dt} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}g + \nu\,\nabla^2\vec{q}\,'. \tag{2.10}$$

It is to be noted that the density ratio $\frac{\rho'}{\rho_0}$ appears twice, in the inertial term and in the buoyancy term. If $\frac{\rho'}{\rho_0} \ll 1$, it produces only a small correction to the inertial term, but it is of primary importance in the buoyancy term. The approximation introduced by Boussinesq [51] consists essentially of neglecting variation of density in so far as they affect inertia, but retaining them in the buoyancy term. When viscosity and diffusion are included, variations of liquid properties are also neglected in this approximation (see, Turner [52]).

2.5 Equation of state

The equation of state for a viscous fluid can be derived by expanding the density $\rho(T)$ in Taylor's series about the reference temperature $T = T_0$ in the following form:

$$\rho(T) = \rho(T_0) + \left(\frac{\partial\rho}{\partial T}\right)_{T=T_0}(T-T_0) + \left(\frac{\partial^2\rho}{\partial T^2}\right)_{T=T_0}\frac{(T-T_0)^2}{2!} + \dots\dots\dots \quad (2.11)$$

Neglecting the terms of degree ≥ 2 in $(T - T_0)$, following linear equation is obtained

$$\rho(T) = \rho(T_0) + \left(\frac{\partial\rho}{\partial T}\right)_{T=T_0}(T-T_0). \quad (2.12)$$

The thermal expansion coefficient β is defined as

$$\beta = -\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial T}\right)_{T=T_0} \Rightarrow \left(\frac{\partial\rho}{\partial T}\right)_{T=T_0} = -\rho_0\beta. \quad (2.13)$$

Using equation (2.13) in equation (2.12) the equation of state can be obtained as

$$\rho(T) = \rho_0[1 - \beta(T - T_0)]. \quad (2.14)$$

2.6 Different model in flow through porous medium

2.6.1 Darcy's law

Darcy's law states that the rate at which the fluid flows through a permeable substance per unit area is equal to the permeability (a property only of the substance thorough which the fluid is flowing) times the pressure drop per unit length of flow, divided by the viscosity of the fluid. Henri Darcy's investigations into the hydrology of the fountains of Dijon and his experiments on steady-state unidirectional flow in an uniform porous medium revealed a

$$\frac{\partial p}{\partial x} = -\frac{\mu}{K}u. \quad (2.15)$$

Here, $\frac{\partial p}{\partial x}$ is the pressure gradient in the flow direction and μ is dynamic viscosity of the fluid. The permeability K is independent of the nature of the fluid but depends on the geometry of the medium. Equation (2.15) can be written, in vectorial form, as follows:

$$\nabla p = -\frac{\mu}{K}\vec{q}. \quad (2.16)$$

Many early authors have considered an extension of Darcy's equation in studying convection in porous media in the following form:

$$\rho\left(\frac{\partial\vec{q}}{\partial t}+(\vec{q}\cdot\nabla\vec{q})\right)=-\nabla p-\frac{\mu}{K}\vec{q}. \tag{2.17}$$

proportionality between flow rate and the applied pressure difference. Mathematically Darcy's law can be written as

$$\frac{\partial p}{\partial x}=-\frac{\mu}{K}u. \tag{2.15}$$

Here, $\frac{\partial p}{\partial x}$ is the pressure gradient in the flow direction and μ is dynamic viscosity of the fluid. The permeability K is independent of the nature of the fluid but depends on the geometry of the medium. Equation (2.15) can be written, in vectorial form, as follows:

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2.6.2 Brinkman's form of Darcy's law

An alternative to Darcy's equation is commonly known as Brinkman's equation, with inertial terms omitted, is of the form:

$$\nabla p=-\frac{\mu}{K}\vec{q}+\tilde{\mu}\,\nabla^2\vec{q}. \tag{2.18}$$

In right hand side, the first term is the usual Darcy-drag term and the second is analogous to the Laplacian term that appears in the Naviers-Stokes equation with $\tilde{\mu}$ being the coefficient of effective viscosity. This correction term accounts for flow through sparsely-packed in nanoparticle.

2.6.3 Darcy-Forchheimer law

Inertial effects become important for flows through porous media with moderate Reynolds number ($0\leq Re\leq 10$). An inertial term, therefore, can be included in Darcy's equation in the form of quadratic drag which is known as Forchheimer term. The modified law is known as Darcy-Forchheimer law and its mathematical form is given by

$$\nabla p = -\frac{\mu}{K}\vec{q} - \frac{\mu}{K_1}\vec{q}|\vec{q}|, \tag{2.19}$$

where K_1 is known as inertial permeability. One may use Darcy-Forchheimer equation for flows with high flow rate in the fractures.

2.7 Flow through nanoparticle

The flow behavior of fluids in the vicinity of nanoparticles can differ significantly from bulk fluid behavior due to various factors, such as surface effects, intermolecular interactions, and confinement effects. Understanding the flow through nanoparticle systems is of great interest in many fields, including nanotechnology, materials science, and biomedicine.

2.8.2 Brinkman model

Brinkman [29], in the year 1952 proposed a model for viscosity in a concentrated suspension. The viscosity μ_{nf} of the nanofluid is

$$\mu_{\mathrm{nf}} = \frac{\mu_f}{(1-\phi)^{2.5}}, \tag{2.20}$$

The thermal expansion coefficient of the nanofluid (proposed by Xuan and Roetzel [53] in the year 2000) can be determined by

$$(\rho\beta)_{\mathrm{nf}} = (1-\phi)(\rho\beta)_f + \phi(\rho\beta)_s, \tag{2.21}$$

where subscript f indicates a base fluid, p a particle, ϕ is the solid volume fraction and μ_f the viscosity of the base fluid.

2.8.3 Maxwell's equation

Maxwell was the first person to investigate conduction analytically through a suspension particle. Maxwell [54] in the year 1873) considered a very dilute suspension of spherical particles by ignoring interactions among particles. The naofluid thermal conductivity K_{nf} is defined as

$$\mathrm{K}_{\mathrm{nf}} = K_f + 3\phi\left(\frac{K_s - K_f}{2K_f + K_s - \phi(K_s - K_f)}\right)K_f. \tag{2.22}$$

For low particle-volume concentrations,

$$\mathrm{K}_{\mathrm{nf}} = K_f + 3\phi\left(\frac{K_s - K_f}{2K_f + K_s}\right)K_f. \tag{2.23}$$

where K_f is the thermal conductivity of the base fluid and K_s the thermal conductivity of the solid nano particles.

2.8 Boundary conditions

The boundary conditions on velocity depend on the nature of the fluid flow and geometry of the boundaries. The mathematical forms of various velocity boundary conditions are discussed below:

2.8.1 Boundary condition on velocity for convection

The fluid, in contact with the solid surface, assumes the velocity of the solid surface and the resulting condition is termed as no-slip condition. This is characteristic of all viscous fluid flows. According to "no-slip condition", the velocity components vanish at the stationary impermeable boundaries, i.e.,

$$u = v = w = 0.$$

If the wall is permeable allowing constant injection w_0 in a direction perpendicular to the wall, then the no-slip condition takes the form:

$$u = v = 0 \text{ and } w = w_0.$$

In connection to convection process the boundary conditions on velocity, are obtained from the mass balance, the no-slip condition and the stress principle of Cauchy, depending on the nature of boundary surfaces (rigid or stress-free).

The boundary surfaces that are generally considered in convection problems are:

i. Both lower and upper boundary surfaces are rigid.

ii. Lower surface is rigid and upper surface is stress-free / vice versa.

iii. Both lower and upper boundary surfaces are stress-free.

2.8.1.1 Stress-free boundaries:

The lower and upper horizontal surfaces are considered to be at $y = 0$ and $y = H$. The horizontal surfaces are assumed to be impermeable so that $v = 0$. At the stress-free boundary, the tangential component of stress must vanish, i.e.,

$$\tau_{xy} = \tau_{zy} = 0,$$

which imply

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} = 0,$$

on using equation of continuity for incompressible fluid in the component form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

one obtains the condition for impermeable stress-free boundaries as follows:

$$v = \frac{\partial^2 v}{\partial y^2} = 0.$$

2.8.1.2 Rigid boundaries:

For rigid boundaries, since $u = w = 0$, $\forall x, y$ on the boundaries, their derivatives also vanish, i.e.,

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial w}{\partial z} = 0.$$

Hence, from the continuity equation for an incompressible liquid, one obtains

$$\frac{\partial v}{\partial y} = 0.$$

Thus in the case of rigid boundaries the boundary condition on velocity are as follows:

$$v = \frac{\partial v}{\partial y} = 0.$$

2.8.1.3 Stress-jump at permeable boundaries:

The following velocity boundary conditions hold for the permeable walls:

$$\frac{\partial u}{\partial y} = \frac{\alpha}{\sqrt{K^*}} u \quad \text{at} \quad y = 0,$$

$$\frac{\partial u}{\partial y} = \frac{-\alpha}{\sqrt{K^*}} u \quad \text{at} \quad y = H.$$

Here, α is dimensionless quantity depending on the material parameters which characterizes the structure of the permeable boundary membranes and K^* is the permeability of the porous medium. In literature, such boundary conditions are known as Beaver-Joseph boundary conditions.

2.8.2 Boundary condition on temperature

The boundary condition on temperature depends on the nature of the boundaries in terms of heat conductivity.

3.3.2.1 Boundary condition of first kind

If the boundaries of the fluid layer have high heat conductivity implying large heat capacity, then their temperature would be uniform and unchanging in time. The boundary temperature in this case would be unperturbed by any flow or temperature perturbations in the fluid, thus

$$T = T_1 \quad \text{on} \quad y = 0,$$
$$T = T_2 \quad \text{on} \quad y = H,$$

where T_1 and T_2 are constants. This is known as the boundary condition of first kind or fixed surface temperature (i.e., isothermal) condition. First kind temperature boundary condition is a *Dirichlet* type boundary condition.

3.3.2.2 Boundary condition of second kind

In the case of stress-free/rigid surface, there will be heat exchange between the surface and the environment. According to Fourier's law, the heat flux Q_T through the boundary per unit time and area is

$$Q_T = -k\frac{\partial T_b}{\partial y},$$

where $\frac{\partial T_b}{\partial y}$ is the basic state temperature gradient in the liquid at the boundary. If Q_T is unperturbed by thermal or flow perturbations in the liquid, the condition on the perturbation field T is given by

$$\frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0, H.$$

These boundary conditions are known as adiabatic boundary condition or insulated boundary condition or boundary condition of second kind or fixed surface heat flux condition. This condition is a *Neumann* type boundary condition.

3.3.2.3 Boundary condition of third kind

Third kind boundary conditions (or Robin boundary conditions) are a weighed combination of *Dirichlet* and *Neumann* boundary conditions. This contrasts to mixed boundary conditions, which are boundary conditions of different types specified on different subsets of the boundary. Robin boundary conditions are also called as *impedance boundary* conditions, due to their application in electromagnetic problems. These are general type of boundary conditions on temperature, given by

$$\frac{\partial T}{\partial y} = B\,T \quad \text{at} \quad y = 0, H,$$

where B is the Biot number that depends on the external heat transfer coefficient and the conductivity of the body that is losing heat. Such boundary conditions arise when the system exchanges heat with the surrounding environment. The case $B \to \infty$ corresponds to isothermal boundary condition $T = 0$ and the case $B \to 0$ corresponds to adiabatic boundary condition $\frac{\partial T}{\partial y} = 0$. Robin boundary conditions are commonly used in solving Sturm–Liouville problems, which appear in many contexts in science and engineering and forms an integral part of the present work.

4.4 Non-dimensional parameters

Dimensional analysis of any problem provides information on qualitative behavior of the physical problem. The dimensionless parameters help us to understand the physical significance of particular phenomenon associated with the problem. The following dimensionless parameters appear in the dissertation.

Reynolds number – Re

It is the ratio of the magnitude of the inertial force to viscous force in the flow.

$$Re = \frac{UD}{\nu} = \frac{\text{inertia force}}{\text{viscous force}}$$

where U is some characteristic velocity, D is the characteristic length and ν is the kinematics viscosity.

Prandtl number – Pr

The Prandtl number is defined as the ratio of momentum diffusivity to thermal diffusivity and it is given by:

$$Pr = \frac{\nu}{\kappa},$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity and $\kappa = \frac{k}{\rho C_p}$ is the thermal diffusivity of the fluid.

Grashof number – Gr

It may effectively be called as the ratio of the buoyancy force to the viscous force.

$$Gr = \frac{g\beta b^3 \Delta T}{\nu^2}.$$

Viscosity ratio – Λ

The viscosity ratio is defined as:

$$\Lambda = \frac{Gr}{Re},$$

Brinkman number – Br

Brinkman number is the ratio of the kinetic energy dissipated in the flow to the thermal energy conducted into or away from the fluid.

$$Br = \frac{\mu U_0^2}{K\,\Delta T}$$

Darcy number – Da

The Darcy number is defined as ratio of relative permeability of the medium to cross section area and it is commonly used in heat transfer through porous medium. The Darcy number is given by:

$$Da = \frac{K}{H^2},$$

where K is the permeability of the medium, H is the characteristic length.

Biot number – Bi

The Biot number is a dimensionless group that compares the relative transport resistances, external and internal. It arises when formulating and non-dimensionalizing the boundary conditions for the typical conservation of species or energy equation for heat/mass transfer problems.

The Biot number is defined as:

$$Bi = \frac{hD}{K}$$

Temperature difference - R_T

The temperature difference is defined as

$$R_T = \frac{T_2 - T_1}{\Delta T}$$

where T_2, T_1 is the reference temperature of the external fluid and ΔT is the reference temperature difference.

Reference velocity - U_0

The reference velocity U_0 is defined as

$$U_0 = -\frac{AD^2}{48\mu}$$

Where A is the constant, D is the hydraulic diameter, μ is the dynamic viscosity.

Dimensionless parameter – S

The dimensionless parameter is a ratio defined on

$$S = \frac{Bi_1 Bi_2}{Bi_1 Bi_2 + 2Bi_1 + 2Bi_2}$$

where $Bi_{1,} Bi_2$ are the biot numbers.

Chapter 3

Differential transform method :

Introduction

3.1 Introduction

The basic ideas of differential transform were first proposed by Zhou [55] in the year 1986 for ordinary differential equations. The main advantage of differential transform method is that it can be applied directly to nonlinear differential equations without requiring linearization, discretization or perturbation. Another important advantage is that the method is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate. The differential transform method is thus free of discretization errors and yields good approximations to closed form solutions. It is a semi-numerical-analytical method which does not need the presence or introduction of small parameters. This method constructs an approximate analytical solution in the form of a polynomial (truncation of an infinite power series) and different from the traditional higher-order Taylor series method. The differential transform method is an alternative procedure for obtaining an analytic Taylor series solution of differential equations. In this chapter the basic concepts of differential transform and its application to simple mathematical problems are presented.

Definition:

An arbitrary function $u(x)$ can expanded in Taylor series about a point $x = x_0$ as

$$\begin{aligned} u(x) &= u(x_0) + \frac{u^{(1)}(x_0)}{1!}(x - x_0) + \frac{u^{(2)}(x_0)}{2!}(x - x_0)^2 + \ \ldots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} u^{(k)}(x_0)(x - x_0)^k. \end{aligned} \tag{3.1}$$

Differential transform of k^{th} order of function $u(x)$ is defined as follows:

$$U[k] = \frac{1}{k!}\left(\frac{d^k u(x)}{dx^k}\right)\Bigg|_{x=x_0}, \tag{3.2}$$

which is the coefficient of $(x - x_0)^k$ in the Taylor series expansion of $u(x)$ about $x = x_0$.

The inverse differential transform is defined as

$$u(x) = \sum_{k=0}^{\infty} U[k]\,(x - x_0)^k. \tag{3.3}$$

In equations (3.2) and (3.3), $u(x)$ represents the given function and $U[k]$ represents the transformed function, i.e., the lower case letter stands for original function and upper case letter represent the transformed function. It is evident from equations (3.2) and (3.3) that the idea of differential transform is derived from the Taylor series expansion.

3.2 Differential transform of some standard functions

$$\left.\begin{array}{l}
\text{If } u(x) = x^m, \text{ then} \\
\qquad U[k] = \delta(k-m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases} \\
\text{If } u(x) = e^{\lambda x}, \text{ then} \\
\qquad U[k] = \dfrac{\lambda^k}{k!}. \\
\text{If } u(x) = \sin(\omega x + \alpha), \text{ then} \\
\qquad U[k] = \dfrac{\omega^k}{k!}\sin\left(\dfrac{k\pi}{2} + \alpha\right). \\
\text{If } u(x) = \cos(\omega x + \alpha), \text{ then} \\
\qquad U[k] = \dfrac{\omega^k}{k!}\cos\left(\dfrac{k\pi}{2} + \alpha\right).
\end{array}\right\} \tag{3.4}$$

Further proof of properties and applications of DTM has given in Mahesha et al. [79].

3.3 Properties of differential transform

Assume that $U[k], V[k]$ and $W[k]$ are the differential transforms of the functions $u(x), v(x)$ and $w(x)$ respectively, then the following properties hold:

If $u(x) = v(x) \pm w(x)$, then $U[k] = V[k] \pm W[k]$. (3.5)

If $u(x) = \alpha\, v(x)$, then $U[k] = \alpha\, V[k]$. (3.6)

If $u(x) = v(x)\, w(x)$, then

$$U[k] = \sum_{m=0}^{k} V[m]\, W[k-m]. \tag{3.7}$$

If $u(x)$ is m^{th} derivative of some function of the form

$$u(x) = \frac{d^m w(x)}{dx^m}, \tag{3.8}$$

then $U[k] = (k+m)(k+m-1) \dots \dots \dots (k+1)\, W[k+m]$.

If

$$u(x,y) = \frac{\partial^{m+n}}{\partial x^m \partial y^n}\{w(x,y)\}, \tag{3.9}$$

then $U[h,k] = (h+m)(h+m-1) \dots \dots (h+1)$

$$\times (k+n)(k+n-1) \dots \dots (k+1) W[h+m, k+n].$$

If $u(x) = \int_{x_0}^{x} v(t)dt$, then

$$U[k] = \frac{V[k-1]}{k}, \qquad k \geq 1, \qquad U[0] = 0. \tag{3.10}$$

If $u(x) = v(x) \int_{x_0}^{x} w(t)dt$, then

$$U[k] = \sum_{m=0}^{k-1} \frac{V[m]\, W[k-m-1]}{k-m}, \qquad k \geq 1, \qquad U[0] = 0. \tag{3.11}$$

If $u(x) = \int_{x_0}^{x} v(t)w(t)dt$, then

$$U[k] = \sum_{m=0}^{k-1} \frac{V[m]\, W[k-m-1]}{k}, \qquad k \geq 1, \qquad U[0] = 0. \tag{3.12}$$

Chapter 4

Convection of nanofluid flow in a vertical channel with general boundary conditions

4.1 Introduction

The frequent occurrence of the mixed convection flow phenomenon in a wide varite of applications, such as crystal formation, food processing, electronic component cooling, heat exchangers, nuclear reactor insulation, building insulation, solar energy collectors, etc., has long drawn researcher's attention [56,57]. In 1960, Tao [56] conducted the first investigation into the mixed convection heat transfer properties of fully developed laminar flow in a vertical channel. Hamadah and Wirtz [57] investigated the problem of mixed convection in vertical channels with asymmetric heated walls. The results showed that the buoyancy effect inside the bulk flow resulted in a flow reversal near the cold wall and enhanced the transfer of heat efficiency towards the hot wall. Using the perturbation method, Barletta [48] examined the heat transfer effectiveness of mixed convection flow in a vertical channel subject to viscous dissipation effects. It was demonstrated that, in the presence of asymmetric heating, the viscous dissipation effect increased the buoyancy force acting on the bulk flow and had an impact on the efficiency of heat transfer at the hot surface. In a plane vertical channel, Zanchini [58] examined the laminar and fully developed mixed convection with viscous dissipation. He used the perturbation series method to examine the combined effects of buoyant forces and viscous dissipation.

The aerodynamic extrusion of a plastic sheet, the cooling of metallic plates in cooling baths, and thin-film solar energy collectors are just a few examples of the numerous industrial and engineering applications that call for heating and cooling. Due to their limited thermal conductivity, Traditional heat transfer fluids, such as motor oil, ethylene glycol mixes, and pure water, only have partial heat transfer capabilities. This allows many modern items of equipment achieve the high performance and necessary compactness. Metals have a high heat conductivity compared to fluids. In order to generate a heat-transfer medium that behaves like a fluid but possesses the thermal conductivity of metals, it makes sense to combine metals and fluids. Compared to bigger particles, nanoparticles cling to the air significantly longer. The addition of nanoparticles in the fluids improves the parameters of heat transmission by considerably increasing the base fluid's effective thermal conductivity and viscosity. Nanofluids can therefore be utilized for a variety of purposes, such as nuclear reactor coolant, transformers for car cooling, and electronic cooling. The term "nanofluids" appears to have been first used by Choi [59] to describe a liquid containing a dispersion of submicronic solid particles (nanoparticles). Shariat et al. [60] investigated the flow field and heat transfer characteristics of a mixed convection flow of Al_2O_3-water nanofluid in an elliptical tube. The results showed that, for a given Reynolds number and Richardson number, the Nusselt number increased as the volume percentage of nanoparticles increased. Additionally, it was shown that a high Richardson number improved heat transfer efficiency and grew the mixed convection effect. Al_2O_3- water, Cu-water, and TiO_2-water nanofluids were examined by Xu and Pop [61] in a fully developed mixed convection flow in a vertical channel. One can analyse the properties of the flow field using an analytical solution by varying the Rayleigh numbers magnitude and looking into the impact of gain reductions and buoyant growth. Chen et al. [62] analysed the heat transfer and entropy generation in fully-developed mixed convection nanofluid flow in vertical channel. Recently, Dhrubajyoti and Dass [63] studied the effect of boundary conditions on heat transfer and entropy generation during two-phase mixed convection hybrid Al2O3-Cu/water nanofluid flow in a cavity.

With boundary conditions of the third sort, Umavathi and Veershetty [64] explored the combined free and forced convection flow in a parallel plate vertical channel filled with porous matrix in the fully developed region. More recently, Narayana et al. [65] studied on the differential transform method of solving boundary eigenvalue problems: An illustration.

The differential transform method is one of the semi-analytical techniques that does not require tiny parameters. Zhou [55] was the first to introduce the DTM concept for the purpose of tackling linear and nonlinear issues in electrical circuit problems. This approach was developed for partial differential equations by Chen and Ho [66] and used for the differential equation system by Ayaz [67]. Unlike the conventional higher-order Taylor series approach, this method creates an analytical solution in the form of a polynomial. For large orders, the

Taylor series approach is computationally demanding. An alternate method for finding a Taylor series analytic solution to differential equations is the differential transform method. Many scientists have given this approach careful thought [68-74]. Engineering applications for vertical channels between two plates include heat exchangers, machinery for processing chemicals, heat dissipation for electronic components, and many more. As a result, the DTM approach is utilized in the current investigation, the heat transport was evaluated efficiency in a vertical channel containing a fully developed mixed convective flow of Al_2O_3 water nanofluids with mixed boundary conditions. The research explicitly examines how the velocity and temperature distributions within the channel are influenced by the Brinkman number, the dimensionless mixed convection parameter, and the concentration of nanoparticles. Additionally, it is noted that there are any studies on the slip velocity and temperature jump boundary conditions of a nanofluid in a vertical channel, which are introduced in this study for the first time.

4.2 Mathematical formulation

The momentum balance equation, the energy balance equation, and the boundary condition are solved in this part in a way that is appropriate for the resolution of the issue at hand. This is followed by a dimensionless rewrite of the equations.

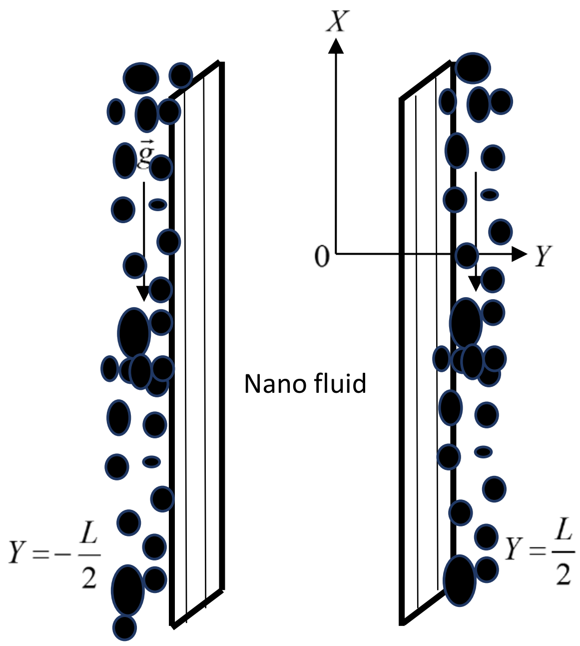

Figure 4.1: Physical configuration.

Let's begin by think about a Newtonian fluid flowing steadily and laminarly in a completely developed area of a parallel plate vertical channel. The channel's axial plane contains the x-axis, which runs in the exact opposite direction as the gravitational fluid. The walls and y-axis are perpendicular. The canal is located in the area of space $-\frac{l}{2} \leq y \leq \frac{l}{2}$. We'll make the assumption that the fluid's thermal conductivity, thermal diffusivity, dynamic viscosity, and thermal expansion coefficient are all constant. The equation of state and the Boussinesq approximation were used as usual.

$$\rho = \rho_0[1 - \beta(\theta_1 - \theta_0)] \tag{4.1}$$

will be adopted. Moreover, it will be assumed that the only nonzero compact of the velocity field U is the X component U. Thus, since $\nabla. u = 0$, one has

$$\frac{du}{dx} = 0 \tag{4.2}$$

Consider the mixed convection flow between two parallel, long, vertical, nanofluid-filled plane walls that is separated by a distance of 2L and is influenced by both a buoyancy force and an external pressure gradient. The x-axis of the chosen coordinate system is parallel to, but in the opposite direction from, the gravitational acceleration vector g. The origin of the axes is such that the positions of the channel walls are $y = -L/2$ and $y = L/2$, respectively. The y-axis is orthogonal to the channel walls. Figure 4.1 reports a sketch of the system and the coordinate axis.

Following Tao [53], we assume that the temperature of both walls is constant, with θ_0 being the upstream reference wall temperature. This temperature is positive for buoyancy-assisted flow and negative for buoyancy-opposed flow. The continuity equation reduces to $du/dx = 0$ in the case where the velocity field is supplied by $(u, 0)$, which indicates that $u = u(y)$. Additionally, $dp/dy = 0$ in the pressure gradient makes $p = p(x)$. The momentum balance and energy equations according to the Boussinesq approximation can be expressed as using the nanofluid model developed by Tiwari and Das [75].

$$0 = \left(\emptyset\rho_s\beta_s + (1-\emptyset)\rho_f\beta_f\right)g(\theta - \theta_0) - \frac{dp}{dx} + \mu_{nf}\frac{d^2u}{dy^2} \tag{4.3}$$

$$\frac{\mathrm{dp}}{\mathrm{dy}} = 0 \tag{4.4}$$

where $p = p + \rho_0 gx$. Since on account of equation (4.4), p depends only on x, equation (4.3) can be rewritten as

$$\alpha_{nf}\left[\frac{d^2\theta}{dy^2}\right] + \frac{\gamma_{nf}}{(cp)_{nf}}\left(\frac{du}{dy}\right)^2 = 0 \tag{4.5}$$

From the equation (4.5), one obtain by differentiating (4.3) with respect to x

$$\frac{d\theta}{dx} = \frac{1}{g(\emptyset\rho_s\beta_s + (1-\emptyset)\rho_f\beta_f)}\left(\frac{d^2u}{dx^2}\right) \tag{4.6}$$

with respect to y we get

$$\frac{d\theta}{dy} = \frac{\mu_{nf}}{g(\emptyset\rho_s\beta_s + (1-\emptyset)\rho_f\beta_f)}\left(\frac{d^3u}{dx^3}\right) \tag{4.7}$$

Again with respect to y we get

$$\frac{d^2\theta}{dy^2} = -\frac{\mu_{nf}}{g\left(\emptyset\rho_s\beta_s + (1-\emptyset)\rho_f\beta_f\right)}\left(\frac{d^4u}{dy^4}\right) \tag{4.8}$$

We'll assume that the channel's two walls are both extremely thin and that they exchange heat with an outside fluid by convection. In specifically, the fluid in the region around $y = -l/2$ will be assumed to have a uniform reference temperature t_1 and the external convection coefficient will be assumed to be uniform with a value of h_1. The fluid in the region between $y > l/2\ and\ y = l/2$ will be assumed to have a uniform reference temperature of $\theta_2 \geq \theta_1$, and the external conviction coefficient will be assumed to be uniform at $y = l/2$ with the value of h_2. Consequently, the temperature field's boundary conditions can be specified

$$-k_{nf}\left(\frac{d\theta}{dy}\right) = h_1\left[\theta_1 - \theta\left(x, -\frac{l}{2}\right)\right] \text{ at } y = -l/2 \tag{4.9}$$

$$-k_{nf}\left(\frac{d\theta}{dy}\right) = h_2\left[\theta\left(x, \frac{l}{2}\right) - \theta_2\right] \quad \text{at } \mathrm{y} = \ l/2 \tag{4.10}$$

On account of equation (4.7), equation (4.9) and (4.10) can be rewritten as

$$\frac{d^3u}{dy^3} = -\frac{gh_1(\emptyset\rho_s\beta_s + (1-\emptyset)\rho_f\beta_f}{\mu_{nf}k_{nf}}\left(\left[\theta_1 - \theta\left(x, -\frac{l}{2}\right)\right]\right) \text{at } \ y = -l/2 \tag{4.11}$$

$$\frac{d^3u}{dy^3} = \frac{gh_2(\emptyset\rho_s\beta_s + (1-\emptyset)\rho_f\beta_f}{\mu_{nf}k_{nf}}\left(\left[\theta\left(x, -\frac{l}{2}\right) - \theta_2\right]\right) \text{at } \quad y = l/2 \tag{4.12}$$

$$\frac{dp}{dx} = A \tag{4.13}$$

For the problem under exam, the energy balance equation in the presence of viscous dissipation can be written as (4.12)

$$\frac{d^2\theta}{dy^2} = -\frac{\gamma_{nf}}{\alpha_{nf}(cp)_{nf}}\left(\frac{du}{dy}\right)^2 = 0 \tag{4.14}$$

Equation (4.8) and (4.14) yields a differential equation for $u(y)$, namely

$$\left(\emptyset\rho_s\beta_s + (1-\emptyset)\rho_f\beta_f\right)g(\theta-\theta_0) - A + \mu_{nf}\frac{d^2u}{dy^2} = 0$$

Differentiate with respect to y

$$\frac{d^4u}{dy^4} = \frac{g\gamma_{nf}\left(\emptyset\rho_s+(1-\emptyset)\rho_f\beta_f\right)}{\alpha_{nf}(cp)_{nf}\mu_{nf}}\left(\left(\frac{du}{dy}\right)^2\right) \tag{4.15}$$

The boundary condition on

$$\frac{du}{dy} = \frac{\alpha}{\sqrt{K^*}}u \; at \; y = -\frac{l}{2} \tag{4.16}$$

$$\frac{du}{dy} = \frac{\alpha}{\sqrt{K^*}}u \; at \; y = -\frac{l}{2} \tag{4.17}$$

From equation (4.11)

$$\left(\frac{d^3u}{dy^3}\right)_{y=-\frac{l}{2}} - \left(\frac{h_1}{k_{nf}}\frac{d^2u}{dy^2}\right)_{y=-\frac{l}{2}} = \frac{-Ah_1}{\mu_{nf}k_{nf}} - \frac{gh_1}{\mu_{nf}k_{nf}}(\theta_0-\theta_1)\left[\emptyset\rho_s\beta_s + (1-\emptyset)\rho_f\beta_f\right] \tag{4.18}$$

$$\left(\frac{d^3u}{dy^3}\right)_{y=\frac{l}{2}} + \left(\frac{h_1}{k_{nf}}\frac{d^2u}{dy^2}\right)_{y=\frac{l}{2}} = \frac{Ah_1}{\mu_{nf}k_{nf}} + \frac{gh_1}{\mu_{nf}k_{nf}}(\theta_0-\theta_2)\left[\emptyset\rho_s\beta_s + (1-\emptyset)\rho_f\beta_f\right] \tag{4.19}$$

Together with equation (4.11) and (4.12), on account of equation (4.5), can be written as

$$\theta(x,y) = \frac{A}{g\left(\emptyset\rho_s\beta_s+(1-\emptyset)\rho_f\beta_f\right)} - \frac{\mu_{nf}}{g\left(\emptyset\rho_s\beta_s+(1-\emptyset)\rho_f\beta_f\right)}\left(\frac{d^2u}{dy^2}\right) + \theta_0$$

Equation (4.15) – (4.18) determine the velocity distribution. They can be written in a dimensionless form by means of the following dimensionless parameters:

$$u^* = \frac{u}{u_0}, \qquad \theta^* = \frac{\theta - \theta_0}{\Delta\theta}, \qquad y^* = \frac{y}{D}, \qquad Pr = \frac{\gamma}{\alpha}, \qquad Gr = \frac{g\beta\Delta\theta D^3}{\gamma^2}, \qquad Re = \frac{u_0 D}{\gamma},$$
$$Br = \frac{\mu {u_0}^2}{k\Delta\theta}, \qquad \Xi = \frac{Gr}{Re}, \qquad Bi_1 = \frac{h_1 D}{k}, \qquad R_s = \frac{\theta_2 - \theta_1}{\Delta\theta},$$
$$Bi_2 = \frac{h_2 D}{k}, \qquad S = \frac{Bi_1 Bi_2}{Bi_1 Bi_2 + 2Bi_1 + 2Bi_2}, \qquad \mu_{nf} = \frac{\mu_f}{(1-\emptyset)^{2.5}},$$
$$\alpha_{nf} = \frac{k_{nf}}{\left(\rho c_p\right)_{nf}}, \qquad \left(\rho c_p\right)_{nf} = (1-\emptyset)((\rho c_p)_f + \emptyset((\rho c_p)_s,$$
$$\frac{k_{nf}}{k_f} = \frac{\left(k_s + 2k_f\right) - 2\emptyset\left(k_f - k_s\right)}{\left(k_s + 2k_f\right) + \emptyset\left(k_f - k_s\right)}, \tag{4.20}$$

In equation (4.19), $D = 2l$ is the hydraulic diameter, while the reference velocity u_o and the reference temperature θ_0 are given by

$$u_0 = -\frac{AD^2}{48\mu}, \qquad \theta_0 = \frac{\theta_1 + \theta_2}{2} + S\left(\frac{1}{Bi_1} - \frac{1}{Bi_2}\right)(\theta_2 - \theta_1) \tag{4.21}$$

The reference temperature difference $\Delta\theta$ is given either by

$$\Delta\theta = \theta_2 - \theta_1 \tag{4.22}$$

If $\theta_1 < \theta_2$, or by

$$\Delta\theta = \frac{\gamma^2}{c_p D^2} \tag{4.23}$$

If $\theta_1 = \theta_2$. Therefore, As a result, similar to ref. [76], the dimensionless parameters R_θ's value can either be 0 or 1. To be more specific, R_θ is equal to 1 for asymmetric fluid temperatures $\theta_1 < \theta_2$ and to 0 for asymmetric fluid temperatures $\theta_1 = \theta_2$.

The dimensionless bulk temparature $\theta^*{}_b$ and the dimensionless mean velocity u are given by (4.11)

$$u^* = 2\int_{-\frac{1}{4}}^{\frac{1}{4}} u^* dy \tag{4.24}$$

$$\theta_b = \frac{2}{u^*}\int_{-\frac{1}{4}}^{\frac{1}{4}} u^* \theta dy \tag{4.25}$$

We skip the asterisk to keep things simple. Due to equation (4.13), for upward flow, $A < 0$, resulting in positive values for u_0, R_e and Ξ. $A > 0$ is positive for downward flow, while u_0, R_e and Ξ are negative. Equations (4.15)-(4.18) can be expressed as if the dimensionless quantities defined in equation (4.19) are used.

$$\frac{d^4u}{dy^4} = \Lambda\, Br\, C_1\, C_2 \left(\frac{du}{dy}\right)^2 \tag{4.26}$$

Considering the velocity boundary conditions

$$\left(\frac{du}{dy}\right)_{y=-\frac{1}{4}} = D_s u \tag{4.27}$$

$$\left(\frac{du}{dy}\right)_{y=\frac{1}{4}} = -D_s u \tag{4.28}$$

$$\frac{1}{Bi_1}\left(\frac{d^3u}{dy^3}\right)_{y=-\frac{1}{4}} - \left(\frac{d^2u}{dy^2}\right)_{y=-\frac{1}{4}} = (1-\emptyset)^{2.5}48 - \frac{\Lambda R_\theta S}{2}\left(1+\frac{4}{Bi_1}\right)C_1 \tag{4.29}$$

$$\frac{1}{Bi_1}\left(\frac{d^3u}{dy^3}\right)_{y=+\frac{1}{4}} + \left(\frac{d^2u}{dy^2}\right)_{y=+\frac{1}{4}} = -(1-\emptyset)^{2.5}48 + \frac{\Lambda R_\theta S}{2}\left(1+\frac{4}{Bi_2}\right)C_1 \tag{4.30}$$

where $C_1 = (1-\emptyset)^{2.5}\left[\emptyset\left(\frac{\rho_s\beta_s}{\rho_f\beta_f}\right) + (1-\emptyset)\right]$, $C_2 = \frac{[(k_s+2k_f)+\emptyset(k_f-k_s)]}{[(k_s+2k_f)-2\emptyset(k_f-k_s)]}$.

Similarly, equation (4.14) and (4.19) yields

$$\frac{d^2\theta}{dy^2} = -\frac{Br}{(1-\emptyset)^{2.5}}\left(\frac{du}{dy}\right)^2 \tag{4.31}$$

While from equation (4.5) and (4.19) one obtain

$$\theta = \frac{-1}{\Lambda\left(\left(\frac{\rho_s\beta_s}{\rho_f\beta_f}\right)\emptyset + (1-\emptyset)\right)}\left[48 + \frac{1}{(1-\emptyset)^{2.5}}\left(\frac{d^2u}{dy^2}\right)\right] \tag{4.32}$$

Table 4.1: Thermal properties of fluid and Nanoparticles:

Physical properties	Fluid phase (water)	Cu	Al_2O_3	Tio_2
$C_p\left(\frac{J}{kgK}\right)$	4179	385	765	686.2
$\rho\left(\frac{kg}{m^3}\right)$	999.7	8933	3970	4250
$k\left(\frac{W}{mK}\right)$	0.613	400	40	8.9538
$\alpha \times 10^7\left(\frac{m^2}{s}\right)$	1.47	11163.1	131.7	30.2

$\beta \times 10^5 \left(\frac{1}{K}\right)$	21	1.67	0.85	0.9

4.3. Method of solutions

Case 4.3.1 Separated effects of buoyancy forces and viscous dissipation

The case of negligible viscous dissipation can be obtained by setting $Br = 0$ in the dimensionless energy equations (4.26). As a consequence; the dimensionless temperature field θ is independent of the dimensionless velocity field u. Moreover, equations (4.26) – (4.30) can be easily solved and yield

$$\begin{aligned} u(y) = {} & \frac{1}{192(2\mathrm{Bi}_2 + \mathrm{Bi}_1(2 + \mathrm{Bi}_2))D_s(4 + D_s)}\Bigg(M_1\Big(\Big(-12(4 + D_s)(-8 \\ & + (-1 + 16y^2)D_s \\ & + \mathrm{Bi}_2(96 + (-3 + 4y)D_s(-12 + 64y^2 + (-1 + 16y^2)D_s))\Big)\Big) \\ & - 96(4 + \mathrm{Bi}_1)M_2 \\ & + D_s M_2(12 + (3 + 4y)\,\mathrm{Bi}_1)(64y^2 + (-1 + 16y^2)\,D_s - 12)\Bigg). \end{aligned} \tag{4.33}$$

where $M_1 = \mathrm{Bi}_1\left(48(1 - \phi)^{2.5} - \frac{\mathrm{Rt}\Lambda ss C_1}{2}\left(1 + \frac{4}{\mathrm{Bi}_1}\right)\right)$,

$M_2 = \mathrm{Bi}_2\left(-48(1 - \phi)^{2.5} - \frac{\mathrm{Rt}\Lambda ss C_1}{2}\left(1 + \frac{4}{\mathrm{Bi}_2}\right)\right)$.

On account of equation (4.33), equation (4.32) in the absence of nano particles can be written as

$$\theta = 2R_\theta s y. \tag{4.34}$$

Now that we have established that there are no buoyancy forces and a meaningful viscous dissipation, to $\Lambda = \frac{Gr}{Re} = 0$, let's examine this scenario. Since there is only forced convection in this case, the answer to the velocity problem is

$$
\begin{aligned}
u(y) = \frac{-1}{\left(\left(2\mathrm{Bi}_2 + \mathrm{Bi}_1(2+\mathrm{Bi}_2)\right)D_s\right)} &\Big(\Big(24(1-\phi)^{5/2}\big(\mathrm{Bi}_2(-1 \\
&+ (-0.125 + 2y^2)D_s) \\
&+ \mathrm{Bi}_1(-1 - 0.5\mathrm{Bi}_2 \\
&+ (-0.125 + 2y^2 + (-0.0625 + y^2)\mathrm{Bi}_2)D_s))\big)\Big)\Big).
\end{aligned} \tag{4.35}
$$

The equation (4.26) along with boundary conditions (4.27) – (4.30) are solved using DTM, we get

$$
U(k+4) = \frac{\Lambda Br\, C_1 C_2}{(k+1)(k+2)(k+3)(k+4)} \sum_{r=0}^{k} (k-r+1)(r+1)U(k-r+1)U(r+1) \tag{4.36}
$$

The initial conditions are $U(0) = \alpha,\ U(1) = \beta,\ U(2) = \gamma,\ U(3) = \delta$. The unknowns $\alpha,\ \beta,\ \gamma$, and δ are obtained by using boundary conditions (4.27) – (4.30).

Special Cases: When $\phi = 0$ (no nanofluid) and $D_s \to \infty$ (rigid boundary). In this case the results are exactly agrees with Zanchini [58].

4.4. Results analysis

When dealing with the issue of constant mixed convection flow using a vertical tube filled with a Al2O3 nanofluid, the analytical solutions of Eqs. (4.26) and (4.31) with the boundary conditions (4.27) - (4.28) are found by using semi-numerical-analytical method called Differential Transform method and velocity and temperature fields are plotted. It is very important to observe the effects of different solid volume fraction of the nanofluid, on the thermal performances of the flow. Also, we regard the mixed convection parameter Λ, as measuring the free convection effects relative to those of forced convection, Brinkman number Br which is a measure of the importance of the viscous heating with relation to conduction heat transfer, slip-Darcy number D_s.

We represent velocity u(y) and temperature (y) for a variety of mixed convection parameter values to observe the velocity and temperature patterns in the channel. Λ from 0 to 1000, $Br = 0$ to 0.1, slip-Darcy number $D_s = 1$ to 10^8 and following Oztop and Abu-Nada

[77], the value of the nanofluid volume fraction is from $\phi = 0$ to 0.3 in which $\phi = 0$ corresponds to the base fluid.

In order to bring out the salient features of the flow and heat transfer characteristics in the case of asymmetric heating ($R_\theta = 1$), the numerical results are presented in Figures 4.2-4.6. The temperature profiles and velocity profiles are shown in these figures. It is important to note that we used the information from Oztop and Abu-Nada [77] for the thermo-physical characteristics of the fluid and the nanofluids. Water in its purest form is used as the basis fluid for all calculations.

The flow of Al2O3 nanofluid for different values of mixed convection parameter Λ and volume fraction ϕ is shown in Figure 4.2. It is observed that the effect of mixed convection parameter Λ on velocity is more effective near the hot wall and less effective near the cold wall. The flow reversal for assisting flow ($\Lambda > 0$) is occurs near the cold wall. It is noted from Figure 4.2 that velocity is a parabolic in nature. It is also observed that velocity for base fluid is more when compared to nanofluid. The flow reversal for nanofluid is occurs exactly at middle of the channel. Physically the mixed convection parameter $\Lambda > 0$ denotes either surface cooling or fluid heating. A convection current is absent when the temperature of the fluid is $\Lambda < 0$ or the surface is heated to $\Lambda = 0$. Due to the fact that a rise in causes an increase in the temperature difference($\theta_2 - \theta_1$), which in turn causes an increase in velocity in the flow direction, an increase in the mixed convection parameter leads to an increase in velocity near the hot wall.

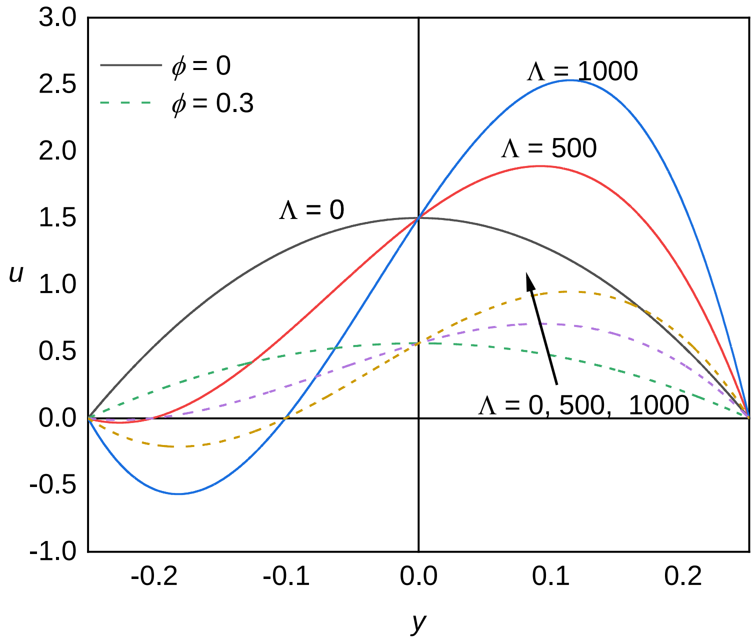

Figure 4.2 Variations of velocity for different values of mixed parameter Λ and volume fraction ϕ =0.1 and 0.3 for fixed values of $R_\theta = 1, D_s = 10^8, s = 1, Br = 0, Bi_1 = Bi_2 = 10$.

Figures 4.3 explain the Brinkman number's effects and nanofluid volume fraction on temperature field. The effect of Brinkman number is to increase the temperature for the both base fluid and nanofluid as seen in Figure 4.3. It is also observed from Figure 4.3 that the effect of Brinkman number for base fluid is rapidly increases when compared to nanofluid. In the absence of Brinkman number the temperature field is independent of governing parameters for $R_\theta = 1$ (see equation (4.26)). Hence the temperature profile for has a linear relationship with y. It should be noted that the dimensionless parameter Br measures the effect of heat dissipation in the medium, and that high values of Br indicate that the medium dissipates more heat than low values do. Heat dissipation can function as a heat source in the medium and raise the temperature of the fluid. It is evident that raising the Br value raises fluid temperature for both conventional fluid and nanofluid movement.

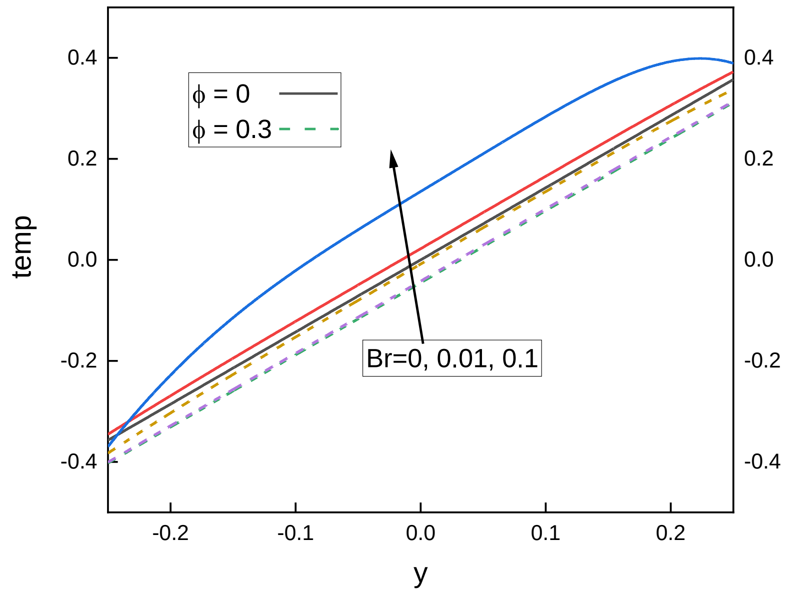

Figure 4.3 Variations of temperature for different values of Br and volume fraction ϕ with fixed values of $R_\theta = 1, D_s = 10^8, s = 1, Br = 0,\ \Lambda = 100\ , Bi_1 = 10 = Bi_2$.

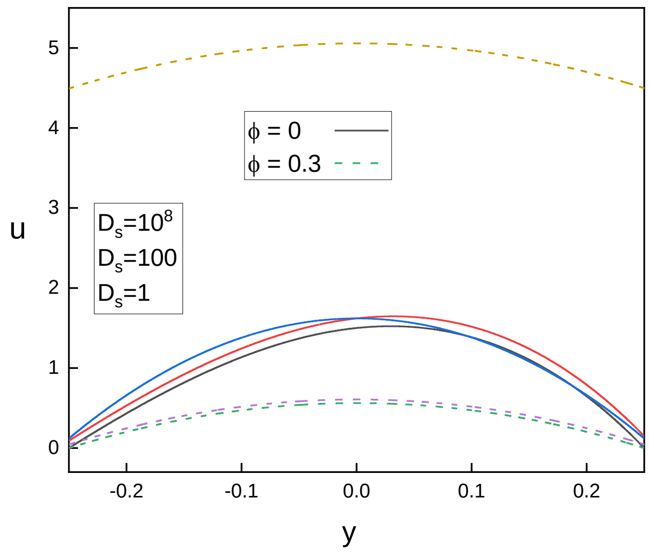

Figure 4.4 Variations of velocity for different values of slip-Darcy number D_s and volume fraction ϕ with fixed values of $R_\theta = 1, Br = 0, s = 1, \Lambda = 100, Bi_1 = 10 = Bi_2$.

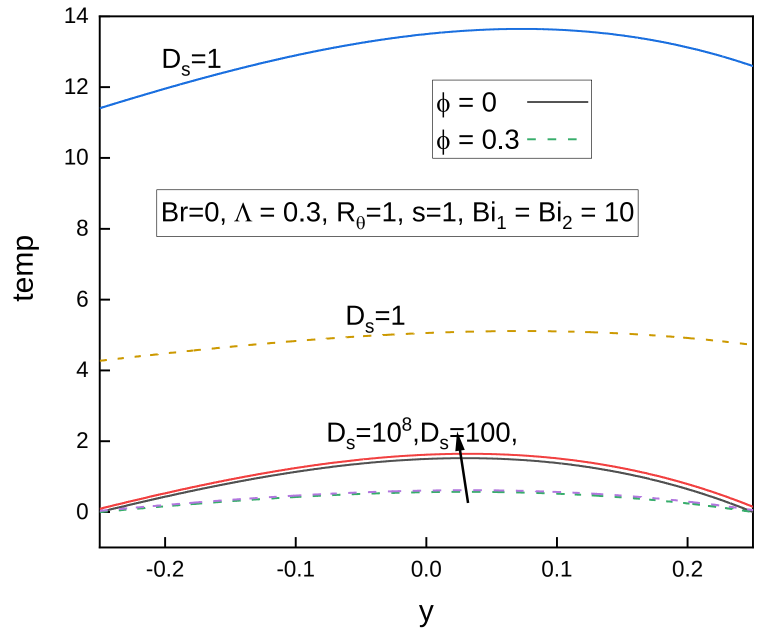

Figure 4.5 Variations of temperature for different values of silp-Darcy number $\boldsymbol{D_s}$ and volume fraction ϕ with fixed parameters $R_\theta = 1, Br = 0, s = 1, \Lambda = 100, Bi_1 = 10 = Bi_2$.

Figures 4.4 and 4.5 depict the effect of slip-Darcy number D_s on the velocity and temperature. It is observed that for different values of D_s, velocity decreases. However, the rate of decrease becomes smaller as $D_s \to \infty$. As D_s increases, porous region becomes less permeable and this decrease in permeability allows lesser fluid motion in the porous region. Effect of slip-Darcy number is very small on nano fluid for both velocity and temperature fields.

The effect of Biot numbers on velocity and temperature field is shown in Figures 4.6 and 4.7 respectively. As the Biot number on the cold wall increases, velocity and temperature decreases. It is observed that the temperature has less effect on the cold wall compared to hot wall. That is, temperature increases more at the hot wall which has the smaller external convection coefficient. The effect of Biot number on nano fluid is negligible small compared to base fluid.

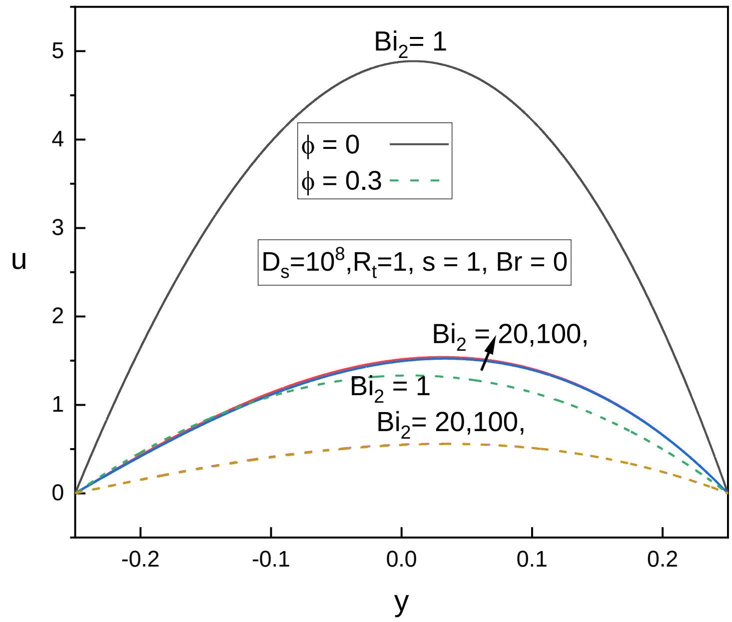

Figure 4.6 Variations of velocity for different values of Bi_2 and volume fraction ϕ with fixed values of $R_\theta = 1, Br = 0.01, s = 1,\ \Lambda = 100, Bi_1 = 10$.

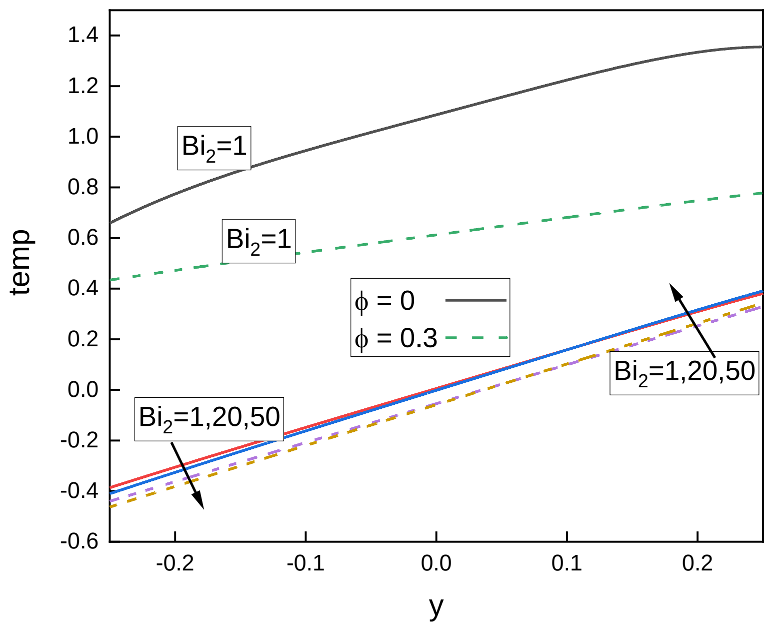

Figure 4.7 Variations of temperature for different values of Bi_2 and volume fraction ϕ for fixed values of $R_\theta = 1, Br = 0.01, s = 1, \Lambda = 100, Bi_1 = 10$.

4.5. Conclusion

In this work, mixed-convective flow and heat transfer of nanofluids in a parallel-plate channel are thoroughly investigated has been studied numerically for Beveris-Joseph condition and third kind temperature boundary condition. When the effects of the viscous dissipation term are negligible, good agreement between numerical and analytical findings is found. The following were the main conclusions:

1. As the mixed convection parameter and Brinkman number expand the nanofluid temperature and velocity increase. As the mixed convection parameter increases, the flow reverse.
2. The temperature and velocity of the nanofluid decrease as the slip-Darcy number increases.
3. The presence of nanoparticles in the base fluid causes a noticeable decrease in the velocity and the temperature field.

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