Computation with Hyperexponential Functions

Ziming Li and Dabin Zheng
Chinese Academy of Sciences
Beijing 100080, China

Abstract

A multivariate hyperexponential function is a function whose "logarithmic derivatives" are rational. Examples of hyperexponential functions include rational functions, exponential functions, and hypergeometric terms. Hyperexponential functions play an important role in the handling of analytic and combinatorial objects. We present a few algorithms applicable to the manipulation of hyperexponential functions in an uniform way.

Let F be a field of characteristic zero, on which derivation operators $\delta_1, \ldots, \delta_\ell$ and difference operators (automorphisms) $\sigma_{\ell+1}, \ldots, \sigma_m$ act. Let E be an F-algebra. Assume that the δ_i for $1 \le i \le \ell$ and σ_j for $\ell+1 \le j \le m$ can be extended to E as derivation and difference operators. Moreover, these operators commute with each other on E. A hyperexponential element of E over F is defined to be a nonzero element $h \in E$ such that

$$\delta_1(h) = r_1 h, \ldots, \delta_\ell(h) = r_\ell h,\ \sigma_{\ell+1}(h) = r_{\ell+1} h, \ldots, \sigma_m(h) = r_m h$$

for some $r_1, \ldots, r_m \in F$. These rational functions are called (rational) certificates for h.

Problem 1 *Given the certificates of a hyperexponential function $h \in E$, decide whether there exists a hyperexponential function $g \in E$ such that:*

1. *the certificates of g are equal to those of h;*
2. *g is algebraic over F.*

Roughly speaking, we want to decide whether h is algebraic over F. The statement in Problem 1 is more precise, since two hyperexponential elements with the same certificates may differ by a multiplicative constant.

Our algorithm makes essential use of differential and shift canonical forms of rational functions (see [1, 2, 3]), and requires only operations in the field of rational functions in $x_1, \ldots, x_n$.

Example 1 *Let $F = \mathbb{C}(x,y,n,m)$ where x, y are differential variables and n, m are shift variables. Let h be a hyperexponential function in x, y, n and m whose respective certificates are*

$$\frac{4x^3 - 2nx^2 + 9x^2y - 12nxy + 6xy^2 - 3ny^2}{3x(x+3y)(n+x)(2x+y)}, -\frac{6x^2 - 5mx + 16xy + 9y^2}{3(m+y)(x+3y)(2x+y)},$$
$$-\frac{n+1+x}{n+x}, \quad \frac{(-1+\sqrt{-3})(m+y)}{2(m+1+y)}.$$

The algorithm shows $h = c\,\frac{n+x}{m+y}\,(-1)^n \left(\frac{-1+\sqrt{-3}}{2}\right)^m \left(\frac{x+3y}{2x^2+xy}\right)^{\frac{1}{3}}$ for some c in $\mathbb{C}$. Therefore, h satisfies a polynomial of degree 6 over F.

Problem 2 *Given a list of certificates for hyperexponential functions $h_1, \ldots, h_n$, decide whether these functions are linearly dependent over the field C of constants of F.*

We present two algorithms to solve Problem 2. The first is based on a generalization of partial Wronskians ([5]) for hyperexponential elements. Note that partial Wronskians do not work well for arbitrary elements in a difference ring ([4]). The second is based on the notion of similarity for hyperexponential elements ([6]). Since dissimilar hyperexponential elements are linearly independent over F, we first decide whether $h_1, \ldots, h_n$ are similar, which amounts to finding rational solutions of a linear homogeneous differential-difference system of order one. Then we decide whether several similar hyperexponential elements are linearly dependent over C, which amounts to deciding whether several rational functions are linearly dependent over C.

Intuitively, the second algorithm appears to be more favorable than the first, because, in the worst case, one has to compute many partial Wronskians in order to decide whether several multivariate hyperexponential elements are linearly dependent over C.

The study of the last problem is motivated by looking for a back-substitution process for computing hyperexponential solutions of finite-rank ideals in an orthogonal Ore ring [7, 6].

Problem 3 *Given rational functions $r_1(x,y), \ldots, r_k(x,y)$, and $h(x,y)$, which is hyperexponential with respect to x, decide whether there exists a function $H(x,y)$ hyperexponential with respect to both x and y such that*

$$H(x,y) = \left(\sum_i^k c_i(y) r_i(x,y)\right) h(x,y)$$

for some $c_1(y),\ldots,c_k(y)$, which are constants with respect to x.

It is shown that a desired $H(x,y)$ exists if and only if

$$H(x,y) = \left(\sum_{i=1}^{k} b_i(y) r_i(x,y)\right) g(x,y), \tag{3}$$

where $b_1(y),\ldots,b_k(y) \in \mathbb{C}(y)$ and $g(x,y)$ is hyperexponential with respect to both x and y. Moreover, h and g have the same certificate with respect to x, denote by u. Then the compatible condition implies that the certificate of g with respect to y must satisfy one of the following equations:

1. $\delta_x(Z) = \delta_y u$ (differential case);
2. $\sigma_x(Z) = \frac{\sigma_y(u)}{u} Z$ (difference case);
3. $\delta_x(Z) = \big(\sigma_y(u) - u\big) Z$ (differential-difference case);
4. $\sigma_x(Z) - Z = \frac{\delta_y(u)}{u}$ (difference-differential case)

Thus, $g(x,y)$ in (3) can be determined up to a multiplicative constant $c(y)$ by finding rational solutions of one of the equations listed above.

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Rational and Replacement Invariants of a Group Action

Evelyne Hubert
INRIA Sophia Antipolis
`http://www.inria.fr/cafe/Evelyne.Hubert`

Irina Kogan
North Carolina State University
`http://www.math.ncsu.edu/~iakogan`

Abstract

Group actions are ubiquitous in mathematics. They arise in diverse areas of applications, from classical mechanics to computer vision. A classical but central problem is to compute a generating set of invariants.

We consider a rational group action on the affine space and propose a construction of a finite set of rational invariants and a simple algorithm to rewrite any rational invariant in terms of those generators. The construction comes into two variants both consisting in computing a reduced Gröbner basis of a polynomial ideal. That polynomial ideal is of dimension zero in the second variant that relies on the choice of a cross-section, a variety that intersects generic orbits in a finite number of points. A generic linear space of complementary dimension to the orbits can be chosen for cross-section.