XXII Brazilian Congress of Mechanical Engineering Students – 19 to 23/10/2015 – Campos dos Goytacazes – RJ

BOUNDARY ELEMENT METHOD APPLIED TO KIRCHHOFF PLATE MODELING

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ABSTRACT: The present paper dwells on the structural static analysis of thin plates subjected to bending, via the Boundary Element Method. A MATLAB© program has been developed using quadratic isoparametric boundary elements. The numerical results have been compared with available analytical solutions, showing excellent convergence. Several integral kernels, not easily found in the literature (e.g. hipersingular kernels) have been independently derived in the present work.

Keywords: Plates, Boundary Element Method

INTRODUCTION

The Boundary Element Method is an alternative to the well-known Finite Element Method. Both are numerical methods used for simulating physical problems, including structural analysis. While in the Finite Element Method the domain of the problem is discretized into small elements, in the Boundary Element Method only the boundary is discretized, thereby reducing the dimensionality of the problem. In the case of analysis of three-dimensional parts, the region to be discretized is the two-dimensional surface of the body, in the case of analysis of 2D structures, such as plates, the region to be discretized is the edges of it, 1D. This reduction in the dimensionality of the problem brings significant gains in time and cost of computational analysis, in addition to the fact that, as errors are computed in the contour, greater precision is obtained.

The price paid for these benefits is the treatment of more complex integral equations, including integrals of functions with singularities.

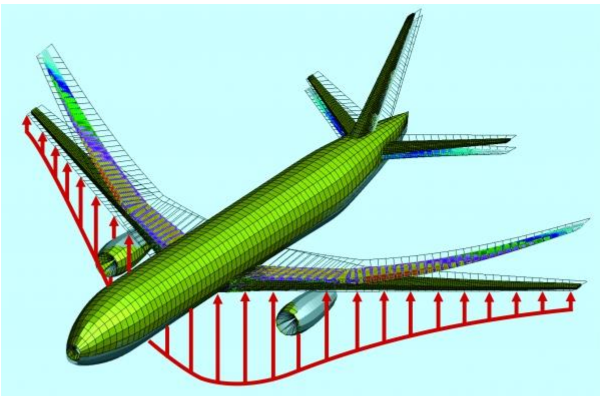

Figure 1. Airplane model showing pressure loading on the wing
(German Aerospace Center Portal - Institut für Aeroelastik)

The idea of solving problems by boundary integrals is not new, but the Boundary Element Method (MEC), as it is known today, only emerged when Rizzo (1967) presented the formulation of singular integral equations in the direct form of the method.

Plates are structural elements in which one of the dimensions, the thickness, is much smaller than the other two, so that for purposes of mathematical model, they are considered two-dimensional components. This consideration is a simplifying hypothesis since all real bodies are three-dimensional. However, when the thickness is sufficiently thin, plate models show deflection results compatible with real plates.

Figure 1 shows an airplane model in which the wing, in a first structural analysis, could be considered as a flat plate. The fuselage of an airplane, in turn, can be treated by means of shell elements. Shell elements have curvature, unlike flat plate elements. Figure 2 shows the suspension bridge of São Vicente - SP, in this type of bridge the deck could be calculated using the plate theory. Finally, in figure 3, a ship with a helipad is shown. In this case, the helipad could be calculated using the plate theory and the ship's hull using shell elements.

Figure 2. Suspension bridge in São Vicente – SP (Forúm Terceira Idade Praia Grande)

Figure 3. Ship with helipad (Heliport Systems Inc.)

There are two most common plate theories, the first, known as Classical Plate Theory, was developed by Kirchhoff and Love in 1888, and offers accurate results for thin plates. According to Chavez (1997), depending on the ratio (h/L), between the thickness (h) and the smallest dimension (L) measured in the middle plane, the plate can be classified as: very thin ($h/L < 1/80$), thin ($1/80 \leq h/L \leq 1/5$) or thick ($h/L > 1/5$).

The second theory was developed independently by Reissner in 1945 and Mindlin in 1951. This theory considers effects of shear deformation on the plate thickness, thus it is possible to obtain more accurate results for thick plates. There is a correspondence between plate and beam theories, in fact, Kirchhoff plates are a 2D generalization of Euler-Bernoulli beams, and Reissner-Mindlin plates are analogous to Timoshenko's beams.

The Kirchhoff's plate bending analysis problem, mathematically, corresponds to a biharmonic differential equation:

$$D\nabla^4 w = b(x, y) \tag{1}$$

Where $w = w(x, y)$ is the vertical displacement field of the plate, $b(x, y)$ is the loading function per unit area of the plate and D is the flexural rigidity of the plate, given by:

$$D = \frac{Eh^3}{12(1 - \nu^2)} \tag{2}$$

In Eq. (2), E is the Young's modulus, h is the thickness of the plate and ν is the Poisson's ratio of the material. In the International System of units (SI), the units of D are Pa.m³.

For the evaluation of the plate problem using the Boundary Elements Method (BEM), it is necessary to know a fundamental solution to the biharmonic equation. A possible fundamental solution for Eq. (1) is:

$$w^* = \frac{1}{8\pi D} r^2 lnr \tag{3}$$

Where r is a distance given by:

$$r = \sqrt{x^2 + y^2} \tag{4}$$

Forbes & Robinson (1969) were the first to address the plate problem through BEM. In this description, the plates had a smooth outline, without any assessment of corner forces. Bézine (1978) applied the BEM to polygonal plates, but without placing nodes in the corners. Stern (1979) presented a complete description of plates through the BEM dealing with free corner terms.

Figure 4 shows an infinitesimal plate element, with dimensions dx by dy. This element shows the shearing forces Q_x and Q_y, bending moments M_x and M_y, and the torsional moments M_{xy} and M_{yx}. The variation of these quantities along the differentials dx and dy is done through expansion of a 1st order Taylor series. Note that in the plate convention the positive z axis points downwards. Loading $b(x, y)$ and deflection w are also positive downwards.

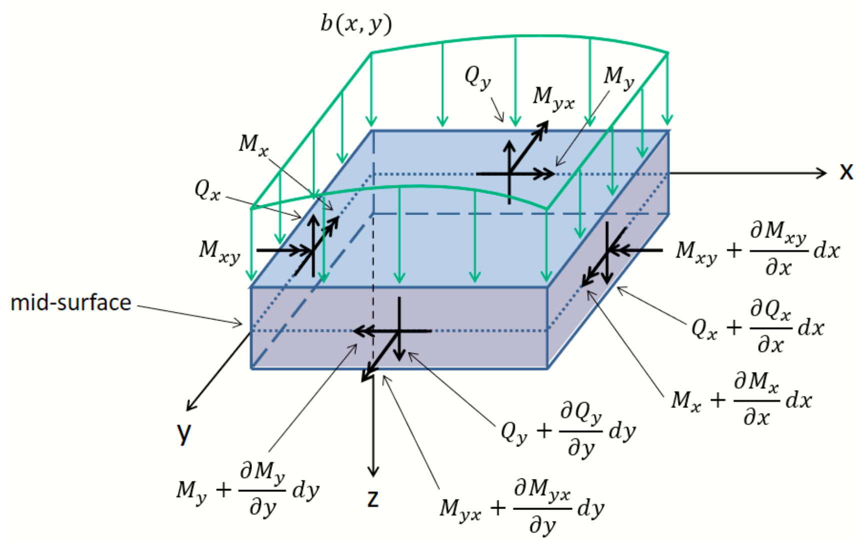

Figure 4. Plate element

The acting efforts are derived from the plate deflection function $w(x, y)$:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \quad , \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \tag{5}$$

$$M_{xy} = M_{yx} = -(1-\nu) D \frac{\partial^2 w}{\partial x \partial y} \tag{6}$$

$$Q_x = -D \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \quad , \quad Q_y = -D \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{7}$$

GOVERNING EQUATIONS

From the plate theory and the Betti Theorem or the Weighted Residuals Method it is possible to write the integral boundary equation for plates, as obtained by Santana (2008):

$$Kw(x_i, y_i) + \oint_{\Gamma} \left[V_n^*(x_i, y_i, x, y) w(x, y) - M_n^*(x_i, y_i, x, y) \frac{\partial w(x, y)}{\partial n} \right] d\Gamma(x, y) + \sum_{j=1}^{Nc} R_{cj}^*(x_i, y_i, x, y) w_{cj}(x, y)$$
$$= \oint_{\Gamma} \left[V_n(x, y) w^*(x_i, y_i, x, y) - M_n(x, y) \frac{\partial w^*}{\partial n}(x_i, y_i, x, y) \right] d\Gamma(x, y) + \sum_{j=1}^{Nc} R_{cj}(x, y) w_{cj}^*(x_i, y_i, x, y)$$
$$+ \int_{\Omega} b(x, y) w^*(x_i, y_i, x, y) d\Omega \qquad (8)$$

Applying Eq. (8) to each of the N_i boundary source points, there is a lack of N_i equations for system resolution since each node has two unknowns. In this way, the derivative of Eq. (8) is performed, resulting in a hypersingular equation:

$$\frac{1}{2} \frac{\partial w(x_i, y_i)}{\partial n_i}$$
$$+ \oint_{\Gamma} \left[\frac{\partial V_n^*}{\partial n_i}(x_i, y_i, x, y) w(x, y) - \frac{\partial M_n^*}{\partial n_i}(x_i, y_i, x, y) \frac{\partial w(x, y)}{\partial n} \right] d\Gamma(x, y) + \sum_{j=1}^{Nc} \frac{\partial R_{cj}^*}{\partial n_i}(x_i, y_i, x, y) w_{cj}(x, y)$$
$$= \oint_{\Gamma} \left[V_n(x, y) \frac{\partial w^*}{\partial n_i}(x_i, y_i, x, y) - M_n(x, y) \frac{\partial}{\partial n_i} \left(\frac{\partial w^*(x_i, y_i, x, y)}{\partial n} \right) \right] d\Gamma(x, y) + \sum_{j=1}^{Nc} R_{cj}(x, y) \frac{\partial w_{cj}^*}{\partial n_i}(x_i, y_i, x, y)$$
$$+ \int_{\Omega} b(x, y) \frac{\partial w^*}{\partial n_i}(x_i, y_i, x, y) d\Omega \qquad (9)$$

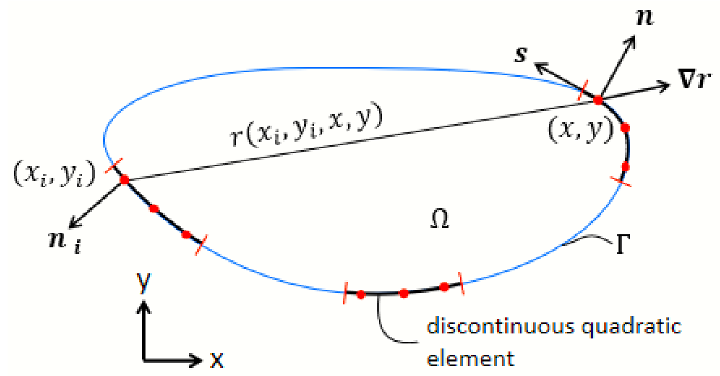

Figure 5. Plate with definition of vectors and boundary elements

In equations (8) and (9) K is the free term, V_n is the shear force acting on the edges, M_n is the bending moment, $\partial w(\xi)/\partial n$ is the angle of inclination of the plate at the edges and $d\Gamma$ is the boundary differential of the plate. In summation terms, R_{cj} and w_{cj} are the corner force and corner deflection, respectively, acting on corner j of the plate. The integral term in Ω is the domain integral, with Ω representing the plate's domain and $d\Omega$ representing the domain differential. The functions with derivatives in relation to n_i indicate that this derivative is performed in relation to the source node. In Fig. (5) you can see a schematic representation of a plate with definitions of: source point (x_i, y_i), normal vector in relation to source point n_i, the field point (x, y), the distance r, the vector gradient of r: ∇r, the normal vector in relation to the field point n, the tangent vector at the field point s, the boundary Γ and the domain Ω.

Eq. (4) written for the source placed in the position (x_i, y_i) is:

$$r = \sqrt{(x - x_i)^2 + (y - y_i)^2} \qquad (10)$$

Note that although ∇r is a vector, which points out of the plate in the direction defined by (x_i, y_i) and (x, y), r is not, being a scalar distance. In fact, for a fixed position (x_i, y_i), r takes the form of a scalar field.

The functions that appear with an asterisk (*) are the kernels of the integral equations. These functions are given by:

$$M_n^* = a_{nx}^2 M_x^* + 2a_{nx}a_{ny}M_{xy}^* + a_{ny}^2 M_y^* \qquad (11)$$

$$V_n^* = Q_n^* + \frac{\partial M_{ns}^*}{\partial s} \tag{12}$$

$$R_{cj}^* = {M_{ns}^*}_j^+ - {M_{ns}^*}_j^- \tag{13}$$

Where:

$$M_{ns}^* = a_{nx} a_{ny} \left(M_y^* - M_x^* \right) + \left(a_{nx}^2 - a_{ny}^2 \right) M_{xy}^* \tag{14}$$

$$Q_n^* = a_{nx} Q_x^* + a_{ny} Q_y^* \tag{15}$$

The functions M_x^*, M_{xy}^*, M_y^*, Q_x^* and Q_y^* are obtained from derivations, Eqs. (5), (6) and (7), of the fundamental solution, Eq. (3). a_{nx} and a_{ny} are the direction cosines of the transformation of coordinates from system *x-y* to system *n-s* as shown in Fig. (5). The symbols (+) and (-) as indices indicate the element immediately before and after the corner *j* in the contour path, following a rotation given by the right hand rule applied in +z.

By making the derivative and grouping the terms, the kernels are obtained explicitly, for example:

$$\frac{\partial V_n^*}{\partial n_i} = -\frac{(\nabla \mathbf{r} \cdot \boldsymbol{n})(\nabla \mathbf{r} \cdot \boldsymbol{n}_i)}{4\,\pi\, r^2} \{[4 + 2(1-\nu)(2(\nabla \mathbf{r} \cdot \boldsymbol{n})^2 - 1)] + 4(1-\nu)(\nabla \mathbf{r} \cdot \boldsymbol{n})^2\} + \frac{(\mathbf{n} \cdot \boldsymbol{n}_i)}{4\,\pi\, r^2} \{[2 + (1-\nu)(2(\nabla \mathbf{r} \cdot \boldsymbol{n})^2 - 1)] + 4(1-\nu)(\nabla \mathbf{r} \cdot \boldsymbol{n})^2\} \tag{16}$$

$$\frac{\partial M_n^*}{\partial n_i} = \frac{1}{4\,\pi\, r} \left[(1+\nu)(\nabla \mathbf{r} \cdot \boldsymbol{n}_i) - 2(1-\nu)\left((\nabla \mathbf{r} \cdot \boldsymbol{n})^2 (\nabla \mathbf{r} \cdot \boldsymbol{n}_i) - (\nabla \mathbf{r} \cdot \boldsymbol{n})(\mathbf{n} \cdot \boldsymbol{n}_i) \right) \right] \tag{17}$$

$$\frac{\partial R_{cj}^*}{\partial n_i} = -\frac{(1-\nu)}{4\,\pi\, r} \langle \{ (\nabla \mathbf{r} \cdot \boldsymbol{s}_j^+) [(\nabla \mathbf{r} \cdot \boldsymbol{n}_j^+)(\nabla \mathbf{r} \cdot \boldsymbol{n}_i) - (\boldsymbol{n}_i \cdot \boldsymbol{n}_j^+)] + (\nabla \mathbf{r} \cdot \boldsymbol{n}_j^+) [(\nabla \mathbf{r} \cdot \boldsymbol{s}_j^+)(\nabla \mathbf{r} \cdot \boldsymbol{n}_i) - (\boldsymbol{n}_i \cdot \boldsymbol{s}_j^+)] \} - \{ (\nabla \mathbf{r} \cdot \boldsymbol{s}_j^-) [(\nabla \mathbf{r} \cdot \boldsymbol{n}_j^-)(\nabla \mathbf{r} \cdot \boldsymbol{n}_i) - (\boldsymbol{n}_i \cdot \boldsymbol{n}_j^-)] + (\nabla \mathbf{r} \cdot \boldsymbol{n}_j^-) [(\nabla \mathbf{r} \cdot \boldsymbol{s}_j^-)(\nabla \mathbf{r} \cdot \boldsymbol{n}_i) - (\boldsymbol{n}_i \cdot \boldsymbol{s}_j^-)] \} \rangle \tag{18}$$

The Eq. (16) kernel has singularity of the type $1/r^2$, that is, it is hypersingular. Eqs. (17) and (18) kernels have strong singularity, $1/r$. The other kernels not shown have a weak singularity, lnr, or are regular. Eqs. (16), (17) and (18) were derived independently in this work, as these kernels are not easily found in the literature.

In equations (8) and (9), (x_i, y_i) are the coordinates of the source point, where the load is applied, and (x, y) are the coordinates of the field point, where the effect of the applied load is observed. In these same equations, the field points (x, y) have a continuous variation in the boundary. The boundary element method, however, employs a nodal solution methodology, where the solution is obtained only at the N_i boundary points, known as nodes. In other points, the solution is obtained through a polynomial approximation. In the present work, it was used the polynomial quadratic approximation with discontinuous elements. The choice of quadratic elements is due to the fact that they are the simplest elements capable of representing curvature. The discontinuous elements are used due to the Holder continuity condition, necessary to obtain the hypersingular integrals, in addition, the exchange of columns in the resulting matrix system is facilitated.

Writing x and V_n, for example, in terms of nodal values we have:

$$x(\xi) = x_1 N_{c1}(\xi) + x_2 N_{c2}(\xi) + x_3 N_{c3}(\xi) \tag{19}$$

$$V_n(\xi) = V_{n1} N_{d1}(\xi) + V_{n2} N_{d2}(\xi) + V_{n3} N_{d3}(\xi) \tag{20}$$

Where the N functions are the interpolator polynomials, known as shape functions. $N_{c1}(\xi)$, $N_{c2}(\xi)$ and $N_{c3}(\xi)$ are the continuous shape functions, while $N_{d1}(\xi)$, $N_{d2}(\xi)$ and $N_{d3}(\xi)$ are the discontinuous shape functions:

$$N_{c1}(\xi) = \frac{1}{2}\xi(\xi - 1) \qquad N_{d1}(\xi) = \xi\left(\frac{9}{8}\xi - \frac{3}{4}\right)$$

$$N_{c2}(\xi) = 1 - \xi^2 \qquad N_{d2}(\xi) = \left(1 + \frac{3}{2}\xi\right)\left(1 - \frac{3}{2}\xi\right) \tag{21}$$

$$N_{c3}(\xi) = \frac{1}{2}\xi(\xi + 1) \qquad N_{d3}(\xi) = \xi\left(\frac{9}{8}\xi + \frac{3}{4}\right)$$

The same procedure from Eq. (19) is used for writing $y(\xi)$, and from Eq. (20) for writing $M_n(\xi)$, $w(\xi)$ and $\partial w(\xi)/\partial n$ in terms of their respective nodal values.

Once the 4 variables of Eqs. (8) and (9) have been written according to Eqs. (19) and (20), the nodal values can leave the integrals, leaving only the shape functions. Thus the integral equations are written as matrix systems, where in the matrices are the integrals of the multiplication of the fundamental solutions by the shape functions, and in the vectors are the nodal values. The nodal values can be unknowns or input data, depending on the set of boundary conditions (BC) considered.

$$g_{11} = \int_{-1}^{+1} w^*(x_i, y_i, x, y) N_{d1}(\xi) \frac{d\Gamma}{d\xi} d\xi \tag{22}$$

All fundamental solutions are written combined with the 3 discontinuous shape functions, thus creating the elements of matrices H and G. The values obtained according to Eq. (22) are known as influence functions, and do not depend on the boundary conditions of the problem, only on its geometry.

Each integral like that of Eq. (22) is evaluated numerically using the Gaussian quadrature. For that, it is necessary to rewrite the variables of the integral in the dimensionless variable ξ, which varies from -1 to +1 according to Gauss points. This procedure requires the use of jacobian, $\partial\Gamma/\partial\xi$, which makes the scale correction of the interval -1 to +1 to the limits of integration of the boundary element in turn.

Thus the assembly of the matrix system is:

$$\begin{bmatrix} H1 & R1 \\ H2 & R2 \end{bmatrix} \begin{Bmatrix} W \\ \frac{\partial w}{\partial n} \\ W_c \end{Bmatrix} = \begin{bmatrix} G1 & C1 \\ G2 & C2 \end{bmatrix} \begin{Bmatrix} V_n \\ M_n \\ R_c \end{Bmatrix} + \begin{Bmatrix} B \\ B_c \end{Bmatrix} \tag{23}$$

This system can be rewritten in a more concise way as:

$$Hu = Gq + P \tag{24}$$

Where:

$$H = \begin{bmatrix} H1 & R1 \\ H2 & R2 \end{bmatrix} \qquad e \qquad G = \begin{bmatrix} G1 & C1 \\ G2 & C2 \end{bmatrix} \tag{25}$$

H1, *H2*, *G1*, *G2*, *R1*, *R2*, *C1* and *C2* are submatrices whose elements are functions of influence such as that of Eq. (22).

RESULTS AND DISCUSSION

To validate the program, two cases of boundary conditions were considered and then the number of boundary elements was varied until the deflection results obtained converged to the known analytical values.

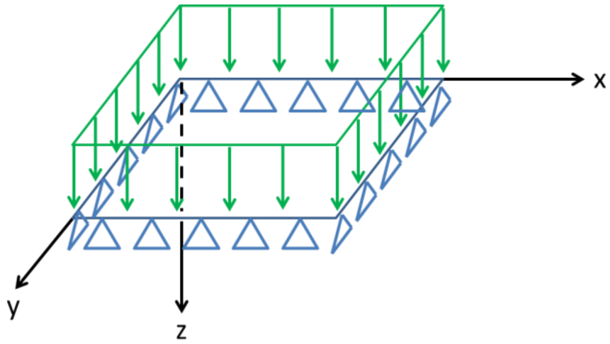

Figure 6. Plate with BC case SSSS

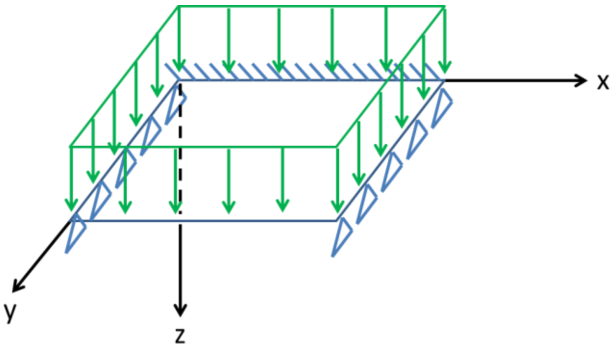

Figure 7. Plate with BC case CSFS

Figure 6 shows the schematic drawing of the plate with boundary conditions (BC) in which the four edges are simply supported (SSSS) and the loading is uniform. In figure 7 you can see the illustration of the second case of BC considered: Clamped – simply-Supported – Free – simply-Supported (CSFS) and uniform loading.

Both cases tested were square plates with dimensions 1 x 1, thickness 0.01. Flexural stiffness is unitary. The loading value applied was $b(x, y) = 1$ (constant). The problem was treated in a dimensionless way, aiming at the generality of the method, and any system of compatible units could be used. The dimensionless deflection is:

$$w_{adi} = \frac{w\, D}{b\, L^4} \tag{26}$$

In Tab. 1 and Figs. (8) to (11) it is presented the results obtained and the errors related to analytical solutions available in Timoshenko & Krieger (1959).

Table 1. Numerical values and relative errors obtained with the program

BC	Position (x,y)	Quantity	1 Elem./Side	2 Elem./Side	4 Elem./Side	6 Elem./Side
SSSS	x/L = 0,5 y/L = 0,1	$\frac{wD}{bL^4}$	0,00132848	0,00132142	0,00131638	0,00131550
		Relative error	0,991%	0,455%	0,0718%	0,00522%
SSSS	x/L = 0,5 y/L = 0,5	$\frac{wD}{bL^4}$	0,00410724	0,00407717	0,00406485	0,00406267
		Relative error	1,11%	0,365%	0,0615%	0,00749%
CSFS	x/L = 0,5 y/L = 1	$\frac{wD}{bL^4}$	0,0112593	0,0112646	0,0112110	0,0112229
		Relative error	0,209%	0,257%	0,221%	0,115%
CSFS	x/L = 0,5 y/L = 0,5	$\frac{wD}{bL^4}$	0,00580238	0,00568383	0,00566837	0,00566666
		Relative error	2,39%	0,294%	0,0209%	0,00926%

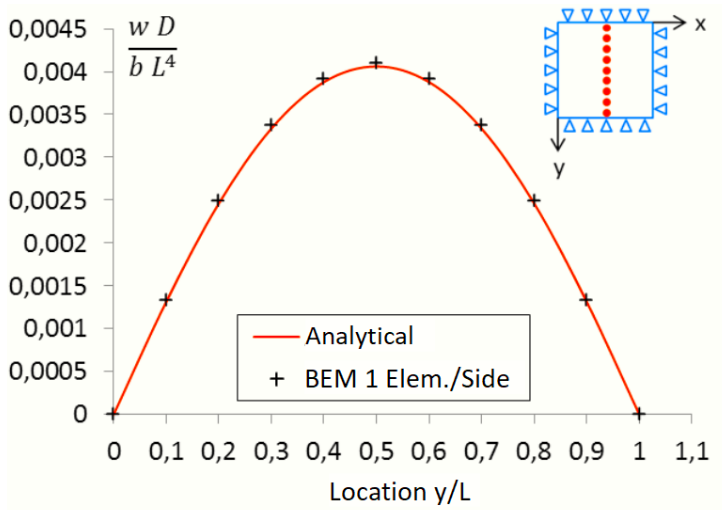

Figure 8. Deflection profile comparison for case SSSS with 1 Element/Side

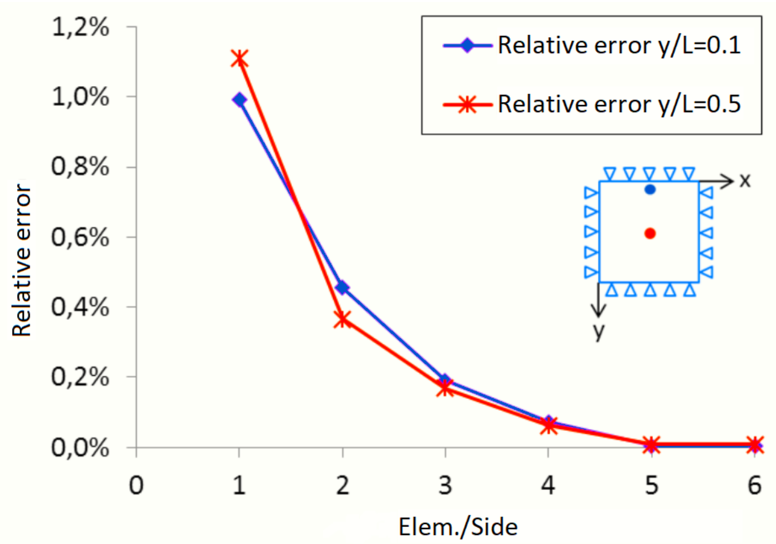

Figure 9. Relative deflection errors for two points of the SSSS plate

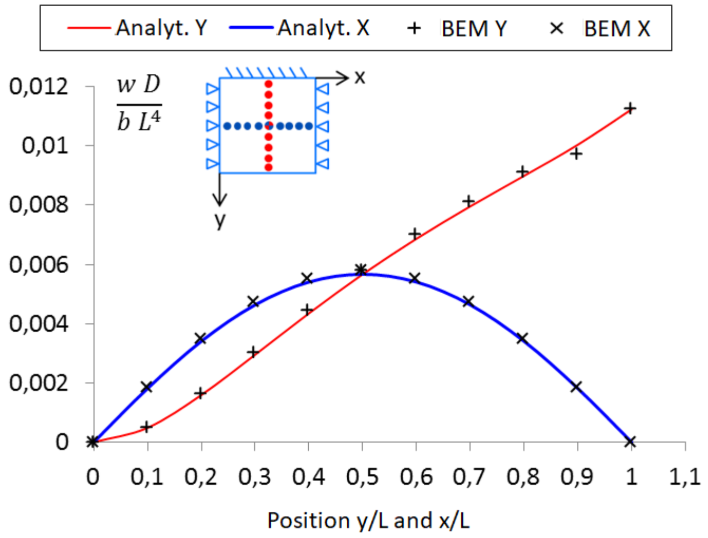

Figure 10. Deflection profile comparison for the CSFS case with 1 Element/Side

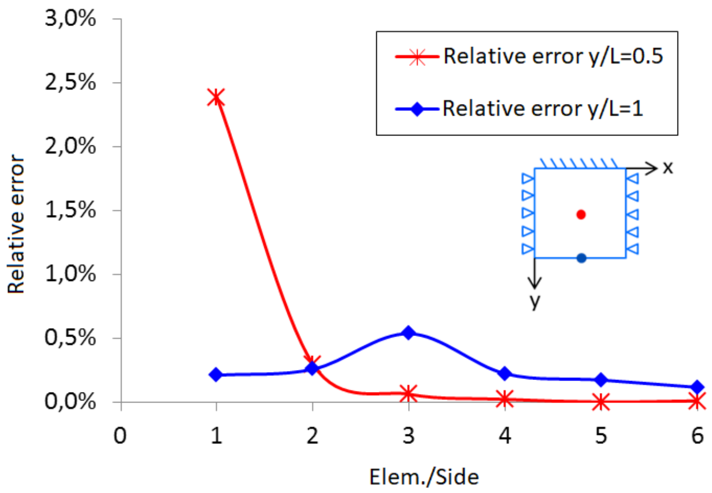

Figure 11. Relative deflection errors of two points of the CSFS plate with convergence curves

Both analyzed cases showed convergence, considered as error relative to analytical solution of less than 1%, with only 2 elements per side. In the plate with BC SSSS, errors fall faster with the increase in the number of elements than in the case of BC CSFS. This greater difficulty in convergence occurs because the clamped side of the plate is equivalent to a simply supported side in which bending moments were applied to cancel the rotations $\partial w/\partial n$. The presence of these moments causes stress concentration.

CONCLUSION

The boundary element program, developed in MATLAB©, presented excellent convergence results, when compared to available analytical solutions. The use of the Boundary Element Method for plate analysis allows an important reduction in processing time and memory usage in structural analysis. For future work, it is suggested the application for plates with curved contours and shells.

ACKNOWLEDGMENTS

The authors thank CAPES for the financial assistance provided.

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DECLARATION OF RESPONSIBILITY