ECE355 - Homework IV

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2

2.22

For each of the following pairs of waveforms, use the convolution integral to find the response $y(t)$ and sketch the results.

(a)

$$\begin{cases} x(t) & = & e^{-\alpha t}u(t) \\ h(t) & = & e^{-\beta t}u(t) \end{cases}$$

Use both cases: $\alpha = \beta$, and $\alpha \neq \beta$.

$$\begin{aligned}
& y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)d\tau \\
& \Rightarrow y(t) = e^{-\beta t}\int_{-\infty}^{+\infty} e^{-\alpha\tau}u(\tau)e^{\beta\tau}u(t-\tau)d\tau \\
& \Rightarrow y(t) = e^{-\beta t}\int_{0}^{+\infty} e^{-\alpha\tau}e^{\beta\tau}u(t-\tau)d\tau \\
& \Rightarrow y(t) = e^{-\beta t}\int_{0}^{t} e^{-\alpha\tau}e^{\beta\tau}d\tau = e^{-\beta t}\int_{0}^{t} e^{\tau(\beta-\alpha)}d\tau
\end{aligned} \tag{1}$$

From (1), taking $\alpha = \beta$, we have:

$$y(t) = e^{-\beta t}\int_{0}^{t} e^{\tau(\beta-\alpha)}d\tau = e^{-\beta t}\int_{0}^{t} d\tau = te^{-\beta t}u(t)$$

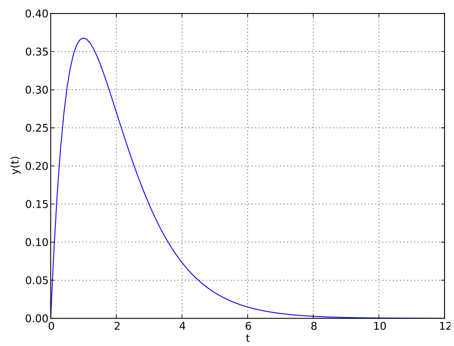
We multiply $y(t)$ by $u(t)$ because this signal only exists for $t > 0$, since $x(t)$ and $h(t)$ only exist for $t > 0$. Now, going back to (1), and taking

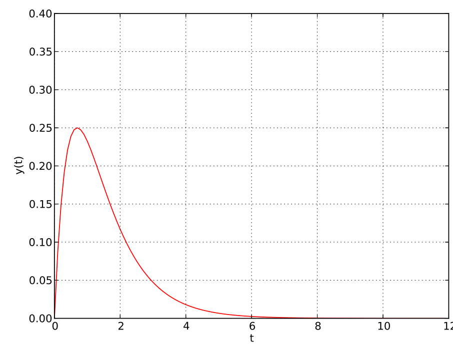
$\alpha \neq \beta$:

$$y(t) = e^{-\beta t}\int_0^t e^{\tau(\beta-\alpha)}d\tau = e^{-\beta t}\frac{e^{\tau(\beta-\alpha)}}{\beta-\alpha}\Bigg|_{\tau=0}^{t} = e^{-\beta t}\frac{e^{t(\beta-\alpha)}}{\beta-\alpha} = e^{-\beta t}\left(\frac{e^{t(\beta-\alpha)}-1}{\beta-\alpha}\right)u(t)$$

$$\therefore y(t) = \begin{cases} te^{-\beta t}u(t) & \text{, if } \alpha = \beta \\ e^{-\beta t}\left(\frac{e^{t(\beta-\alpha)}-1}{\beta-\alpha}\right)u(t) & \text{, if } \alpha \neq \beta \end{cases}$$

$\alpha = \beta$ $\qquad\qquad$ $\alpha \neq \beta$





(b)

$$\begin{cases} x(t) &= u(t) - 2u(t-2) + u(t-5) \\ h(t) &= e^{2t}u(1-t) \end{cases}$$

So we can calculate $y(t)$:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty}[u(\tau) - 2u(\tau-2) + u(\tau-5)]e^{2(t-\tau)}u(1-t+\tau)d\tau$$
$$\Rightarrow y(t) = \int_{t-1}^{+\infty}[u(\tau) - 2u(\tau-2) + u(\tau-5)]e^{2(t-\tau)}d\tau$$
$$\Rightarrow y(t) = \int_{t-1}^{+\infty} e^{2(t-\tau)}u(\tau)d\tau - 2\int_{t-1}^{+\infty} e^{2(t-\tau)}u(\tau-2)d\tau + \int_{t-1}^{+\infty} e^{2(t-\tau)}u(\tau-5)d\tau$$
$$\Rightarrow y(t) = e^{2t}[\int_{t-1}^{+\infty} e^{-2\tau}u(\tau)d\tau - 2\int_{t-1}^{+\infty} e^{-2\tau}u(\tau-2)d\tau + \int_{t-1}^{+\infty} e^{-2\tau}u(\tau-5)d\tau]$$

Now we need to break it into different cases (pieces).

For $\tau < 0 \Leftrightarrow t < 1$:

$$\begin{aligned}
&y(t) = e^{2t}[\int_{t-1}^{+\infty} e^{-2\tau}u(\tau)d\tau - 2\int_{t-1}^{+\infty} e^{-2\tau}u(\tau-2)d\tau + \int_{t-1}^{+\infty} e^{-2\tau}u(\tau-5)d\tau] \\
&\Rightarrow y(t) = e^{2t}[\int_{0}^{+\infty} e^{-2\tau}d\tau - 2\int_{2}^{+\infty} e^{-2\tau}d\tau + \int_{5}^{+\infty} e^{-2\tau}d\tau] \\
&\Rightarrow y(t) = e^{2t}[\int_{0}^{2} e^{-2\tau}d\tau + \int_{2}^{+\infty} e^{-2\tau}d\tau - \int_{2}^{+\infty} e^{-2\tau}d\tau - \int_{2}^{+\infty} e^{-2\tau}d\tau + \int_{5}^{+\infty} e^{-2\tau}d\tau] \\
&\Rightarrow y(t) = e^{2t}[\int_{0}^{2} e^{-2\tau}d\tau - \int_{2}^{5} e^{-2\tau}d\tau - \int_{5}^{+\infty} e^{-2\tau}d\tau + \int_{5}^{+\infty} e^{-2\tau}d\tau] \\
&\Rightarrow y(t) = e^{2t}[\int_{0}^{2} e^{-2\tau}d\tau - \int_{2}^{5} e^{-2\tau}d\tau] = e^{2t}\left[\frac{e^{-2\tau}}{-2}\Bigg|_{\tau=0}^{2} - \frac{e^{-2\tau}}{-2}\Bigg|_{\tau=2}^{5}\right] \\
&\Rightarrow y(t) = \frac{e^{2t}}{2}[1 - 2e^{-4} + e^{-10}]
\end{aligned} \tag{2}$$

For $1 < t < 3$:

$$\begin{aligned}
&y(t) = e^{2t}[\int_{t-1}^{+\infty} e^{-2\tau}u(\tau)d\tau - 2\int_{t-1}^{+\infty} e^{-2\tau}u(\tau-2)d\tau + \int_{t-1}^{+\infty} e^{-2\tau}u(\tau-5)d\tau] \\
&\Rightarrow y(t) = e^{2t}[\int_{t-1}^{+\infty} e^{-2\tau}d\tau - 2\int_{2}^{+\infty} e^{-2\tau}d\tau + \int_{5}^{+\infty} e^{-2\tau}d\tau] \\
&\Rightarrow y(t) = e^{2t}[\int_{t-1}^{2} e^{-2\tau}d\tau + \int_{2}^{+\infty} e^{-2\tau}d\tau - 2\int_{2}^{+\infty} e^{-2\tau}d\tau + \int_{5}^{+\infty} e^{-2\tau}d\tau] \\
&\Rightarrow y(t) = e^{2t}[\int_{t-1}^{2} e^{-2\tau}d\tau - \int_{2}^{5} e^{-2\tau}d\tau - \int_{5}^{+\infty} e^{-2\tau}d\tau + \int_{5}^{+\infty} e^{-2\tau}d\tau] \\
&\Rightarrow y(t) = e^{2t}[\int_{t-1}^{2} e^{-2\tau}d\tau - \int_{2}^{5} e^{-2\tau}d\tau] \\
&\Rightarrow y(t) = \frac{e^{2t}}{2}[e^{2(1-t)} - 2e^{-4} + e^{-10}]
\end{aligned} \tag{3}$$

For $3 < t < 6$:

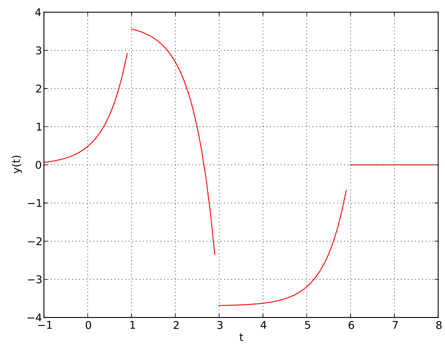
$$\begin{aligned}
&y(t) = e^{2t}[\int_{t-1}^{+\infty} e^{-2\tau}u(\tau)d\tau - 2\int_{t-1}^{+\infty} e^{-2\tau}u(\tau-2)d\tau + \int_{t-1}^{+\infty} e^{-2\tau}u(\tau-5)d\tau] \\
&\Rightarrow y(t) = e^{2t}[\int_{t-1}^{+\infty} e^{-2\tau}d\tau - 2\int_{t-1}^{+\infty} e^{-2\tau}d\tau + \int_{5}^{+\infty} e^{-2\tau}d\tau] \\
&\Rightarrow y(t) = e^{2t}[-\int_{t-1}^{+\infty} e^{-2\tau}d\tau + \int_{5}^{+\infty} e^{-2\tau}d\tau] \\
&\Rightarrow y(t) = e^{2t}[-\int_{t-1}^{5} e^{-2\tau}d\tau - \int_{5}^{+\infty} e^{-2\tau}d\tau + \int_{5}^{+\infty} e^{-2\tau}d\tau] = e^{2t}(\int_{t-1}^{5} -e^{-2\tau}d\tau) \\
&\Rightarrow y(t) = \frac{e^{2t}}{2}e^{-2\tau}\Bigg|_{\tau=t-1}^{5} = \frac{e^{2t}}{2}[e^{-10} - e^{2(1-t)}]
\end{aligned} \tag{4}$$

For $t > 6$:

$$\begin{aligned}
&y(t) = e^{2t}[\int_{t-1}^{+\infty} e^{-2\tau}u(\tau)d\tau - 2\int_{t-1}^{+\infty} e^{-2\tau}u(\tau-2)d\tau + \int_{t-1}^{+\infty} e^{-2\tau}u(\tau-5)d\tau] \\
&y(t) = e^{2t}[\int_{t-1}^{+\infty} e^{-2\tau}d\tau - 2\int_{t-1}^{+\infty} e^{-2\tau}d\tau + \int_{t-1}^{+\infty} e^{-2\tau}d\tau] = 0
\end{aligned} \tag{5}$$

Putting (6), (7), (4) and (5) together:

$$y(t) = \begin{cases} \frac{e^{2t}}{2}[1 - 2e^{-4} + e^{-10}] & \text{, if } t < 1 \\ \frac{e^{2t}}{2}[e^{2(1-t)} - 2e^{-4} + e^{-10}] & \text{, if } 1 < t < 3 \\ \frac{e^{2t}}{2}[e^{-10} - e^{2(1-t)}] & \text{, if } 3 < t < 6 \\ 0 & \text{, if } t > 6 \end{cases}$$



(c)

$$\begin{cases} x(t) &= sin(\pi t)[u(t) - u(t-2)] \\ h(t) &= 2[u(t-1) - u(t-3)] \end{cases}$$

So we can calculate $y(t)$:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} sin(\pi\tau)[u(\tau) - u(\tau-2)] \cdot 2 \cdot [u(t-\tau-1) - u(t-\tau-3)]d\tau$$

From $[u(t-\tau-1) - u(t-\tau-3)]$, we have that $t-3 < \tau < t-1$. In addition, from $[u(\tau) - u(\tau-2)]$, we have that $0 < \tau < 2$. So:

$$y(t) = 2\int_{t-3}^{t-1} sin(\pi\tau)d\tau$$

If $t < 1$, we will have the upper limit of integration less then 0, but τ must be in the interval $(0, 2)$, so $y(t) = 0$. In a similar way, if $t > 5$, $y(t) = 0$ because the lower limit would be greater then 2.

There are two possible situations remaining. The first situation is if $t-3 < 0$, i.e., $t \in (1, 3)$, the other one is if $t-3 > 0$, i.e., $t \in (3, 5)$.

For the first case:

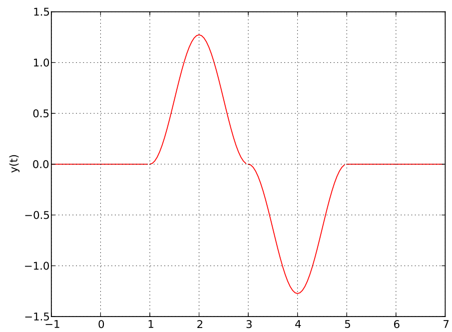
$$\begin{aligned}
&y(t) = 2\int_0^{t-1} sin(\pi\tau)d\tau = -2\frac{cos(\pi\tau)}{\pi}\Bigg|_{\tau=0}^{t-1} = -\frac{2}{\pi}[cos(\pi(t-1)) - 1] \\
&\Rightarrow y(t) = \frac{2}{\pi}[1 - cos(\pi(t-1))]
\end{aligned} \tag{6}$$

For the second case:

$$\begin{aligned}
&y(t) = 2\int_{t-3}^{2} sin(\pi\tau)d\tau = -2\frac{cos(\pi\tau)}{\pi}\Bigg|_{\tau=t-3}^{2} = -\frac{2}{\pi}[1 - cos(\pi(t-3))] \\
&y(t) = \frac{2}{\pi}[cos(\pi(t-3)) - 1]
\end{aligned} \tag{7}$$

Putting 6 and 7 together with the previous results:

$$y(t) = \begin{cases} 0 & \text{, if } t < 1 \\ \frac{2}{\pi}[1 - cos(\pi(t-1))] & \text{, if } 1 < t < 3 \\ \frac{2}{\pi}[cos(\pi(t-3)) - 1] & \text{, if } 3 < t < 5 \\ 0 & \text{, if } t > 5 \end{cases}$$



2.29

Given $h(t)$, determinate if the system is either causal or stable.

(a)

$$h(t) = e^{-4t}u(t-2)$$

The system is causal because $h(t) = 0\ \forall t < 0$. In addition, the system is stable as well because $\int_{-\infty}^{+\infty} |h(t)|dt = \frac{e^{-8}}{4} < +\infty$.

(b)

$$h(t) = e^{-6t}u(3-t)$$

The system is not causal because $h(t) \neq 0$ for some values of $t < 0$. In addition, the system is not stable because $\int_{-\infty}^{+\infty} |h(t)|dt = -\frac{e^{-6t}}{4}\Big|_{-\infty}^{3} = +\infty$.

(c)

$$h(t) = e^{-2t}u(t+50)$$

The system is not causal because $h(t) \neq 0$ for some values of $t < 0$. However, the system is stable because $\int_{-\infty}^{+\infty} |h(t)|dt = -\frac{e^{-2t}}{4}\Big|_{-50}^{+\infty} = \frac{e^{100}}{2} < +\infty$.

(d)

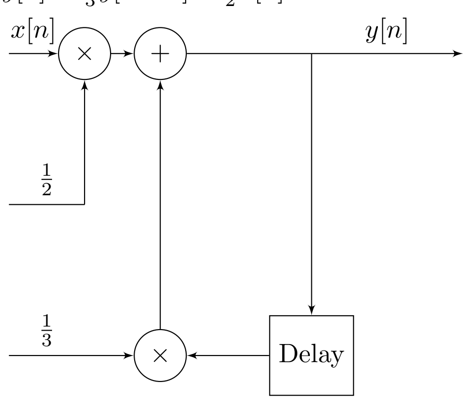
$$h(t) = e^{2t}u(-1-t)$$

The system is not causal because $h(t) \neq 0$ for some values of $t < 0$. However, the system is stable because $\int_{-\infty}^{+\infty} |h(t)|dt = -\frac{e^{2t}}{2}\Big|_{-\infty}^{-1} = \frac{e^{-2}}{2} < +\infty$.

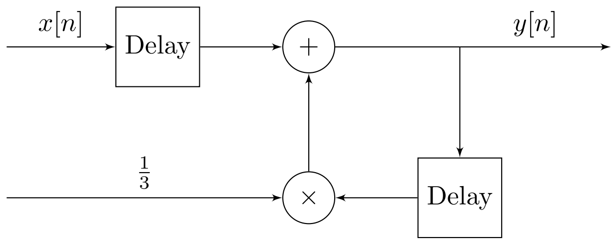
2.38

Draw a block diagram representation for the following systems

(a) $y[n] = \frac{1}{3}y[n-1] + \frac{1}{2}x[n]$



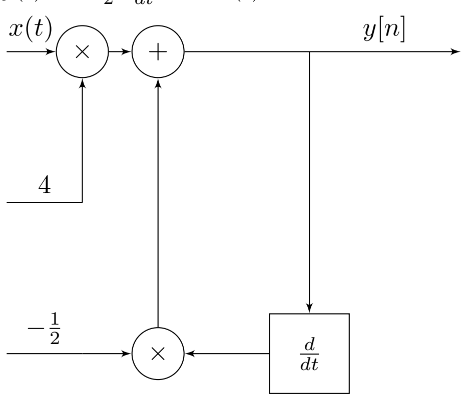
(b) $y[n] = \frac{1}{3}y[n-1] + x[n-1]$



2.39

Draw a block diagram representation for the following systems

(a) $y(t) = -\frac{1}{2}\frac{dy(t)}{dt} + 4x(t)$



(b) $y(t) = \frac{1}{3}[x(t) - \frac{dy(t)}{dt}]$