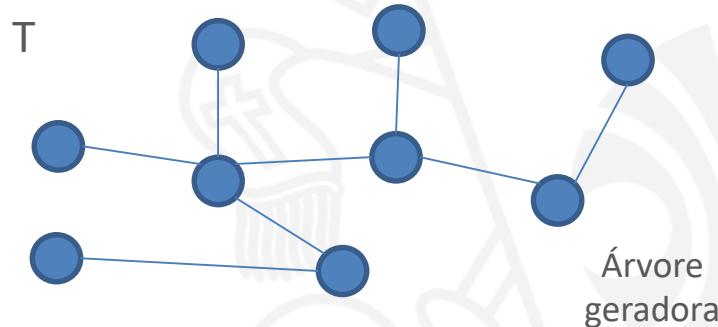
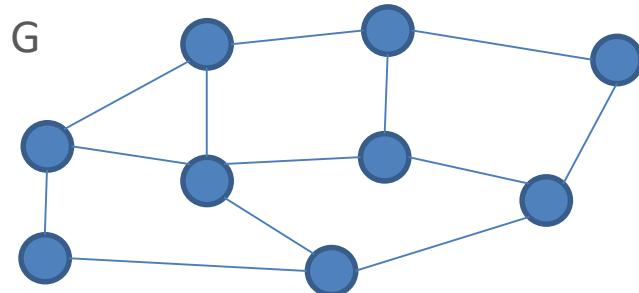


Árvore Geradora Mínima

Zenilton Patrocínio

Árvore Geradora – Lembrete

Uma **árvore geradora** $T = (V, E_T)$ de um grafo não direcionado $G = (V, E)$ é um subgrafo gerador de G que também é uma árvore, isto é , $E_T \subseteq E$, $|E_T| = |V| - 1$ e T é conexo.



Árvore Geradora de Custo Mínimo

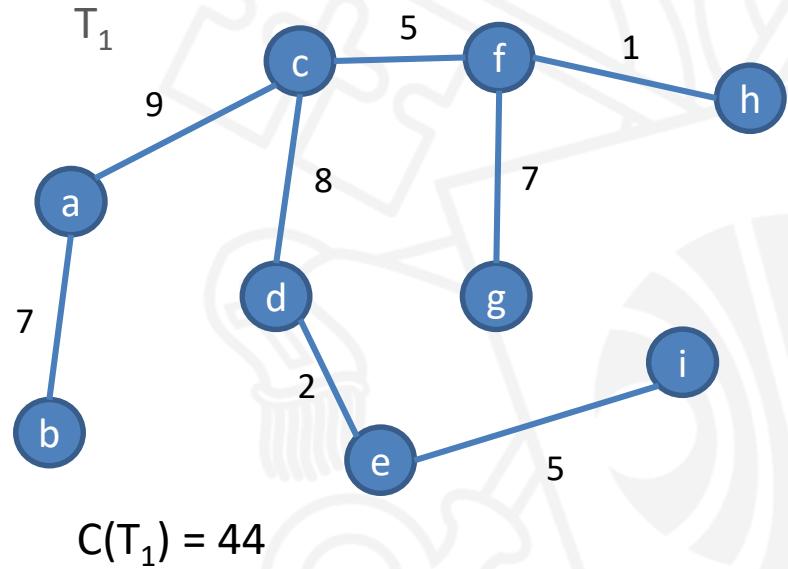
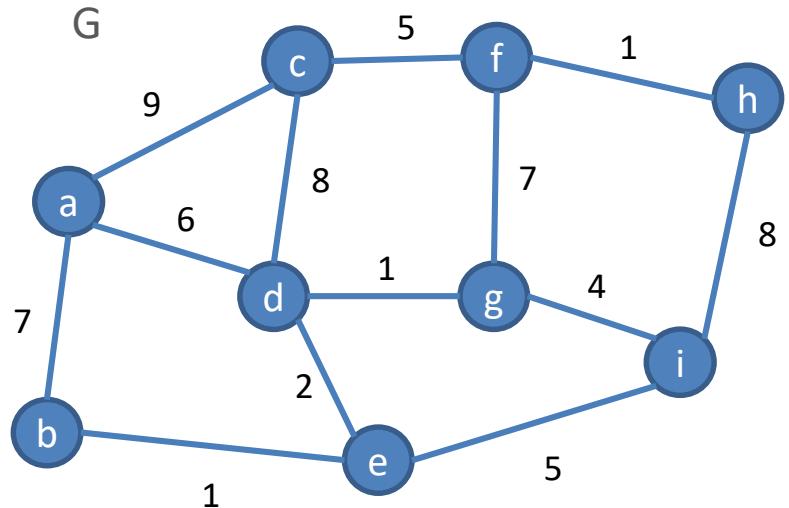
Seja um grafo não direcionado e ponderado $G = (V, E)$ em que um valor de custo $c_e > 0$ é associado a cada aresta $e \in E$.

O custo total $C(T)$ de uma árvore geradora $T = (V, E_T)$ de G pode ser obtido pela soma dos custos de todas as arestas de T , isto é, $C(T) = \sum_{e \in E_T} c_e$.

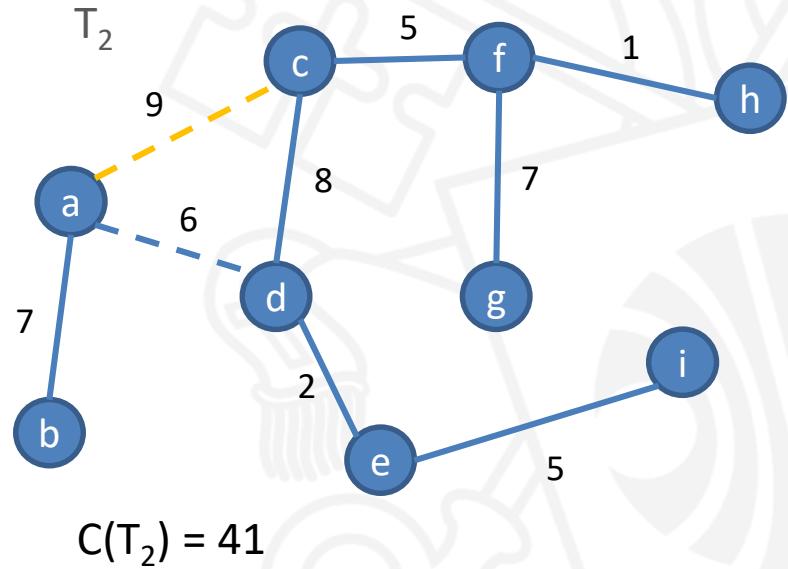
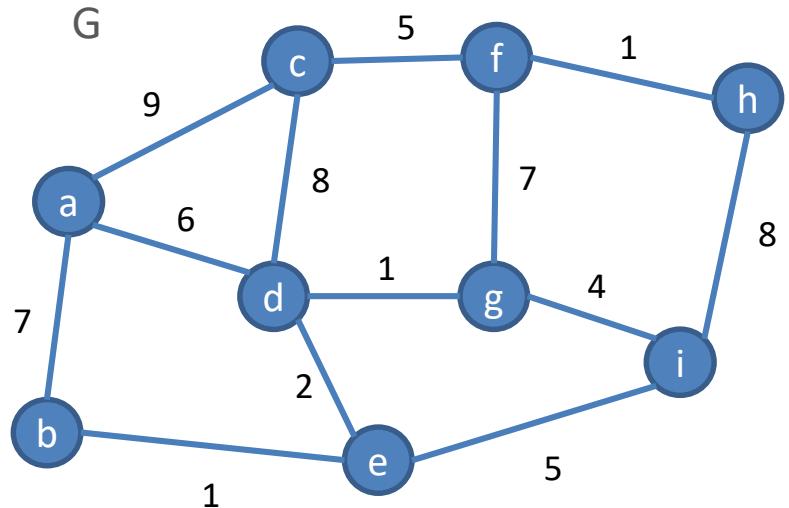
A árvore geradora de custo mínimo (AGM) é a árvore geradora de menor custo total dentre todas as possíveis árvores geradoras de um grafo.

A determinação de uma AGM pode ser feita em tempo polinomial.

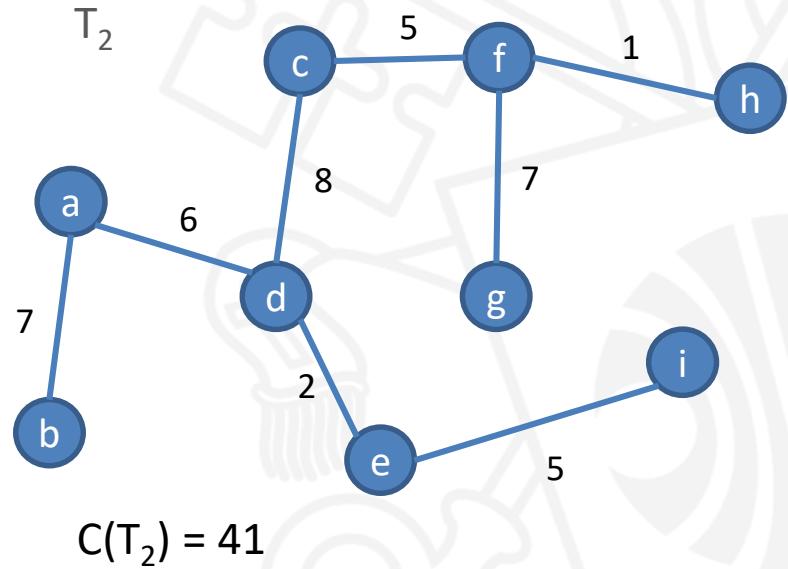
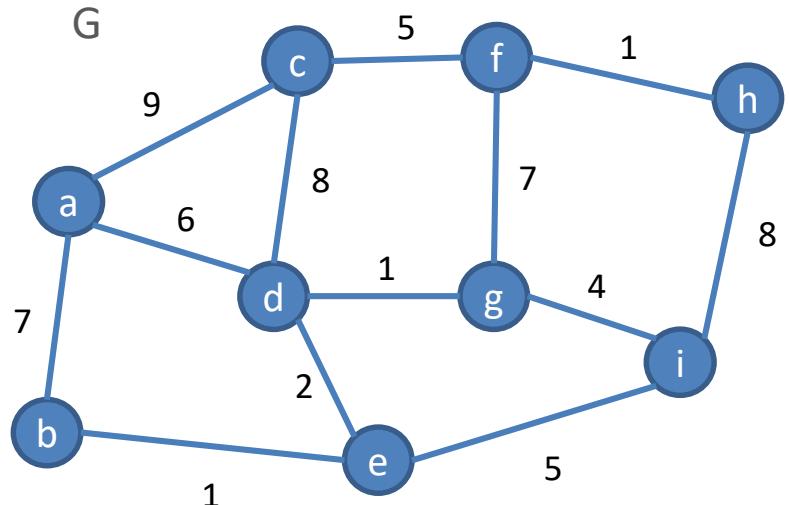
Árvore Geradora de Custo Mínimo



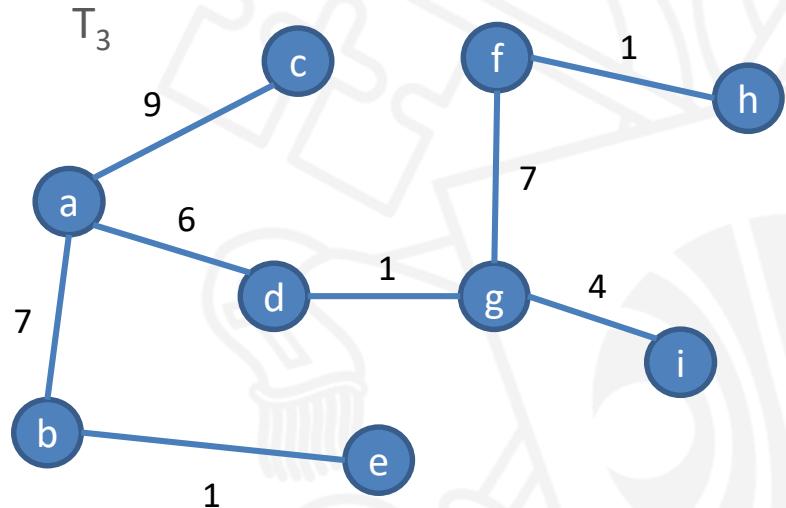
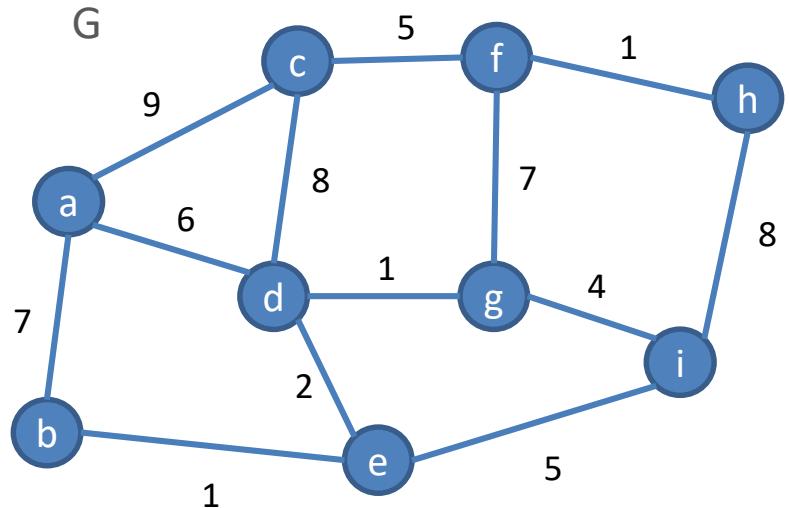
Árvore Geradora de Custo Mínimo



Árvore Geradora de Custo Mínimo



Árvore Geradora de Custo Mínimo



$$C(T_3) = 36$$

Método de Prim



Método de Prim

Este algoritmo foi proposto originalmente em 1930 por um matemático tcheco, posteriormente pelo americano Robert Prim em 1957 e redescoberto por Edsger Dijkstra em 1959.

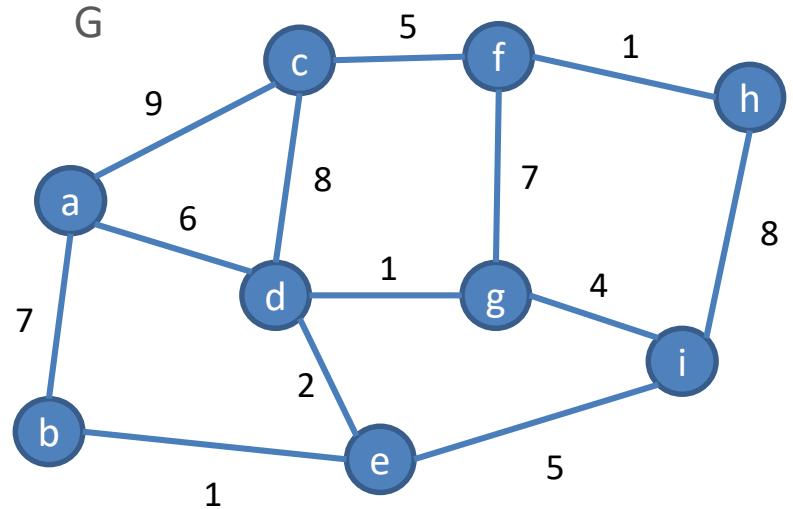
Incluir, de forma gulosa, um a um, os vértices da árvore geradora mínima.

A partir de um vértice qualquer do grafo (raiz), a cada passo, acrescenta-se a aresta de menor peso incidente ao conjunto de vértices que já foram selecionados e que possui uma extremidade em vértices no conjunto de não selecionados.

Método de Prim – Algoritmo

1. Escolher um vértice qualquer $r \in V(G)$ // Selecionar uma raiz
2. $V(T) \leftarrow \{ r \};$ // Inicializar conj. de vértices selecionados
3. $E(T) \leftarrow \emptyset;$ // Inicializar conj. arestas da AGM
4. enquanto $V(T) \neq V(G)$ efetuar // Se houver vértice não selecionado?
 - a. Encontrar a aresta $\{v, w\}$ de menor custo tal que $v \in V(T)$ e $w \notin V(T)$
// Isto é, a aresta de menor custo entre selecionados e não selecionados
 - b. Acrescentar w a $V(T)$ // Adicionar novo vértice à AGM
 - c. Acrescentar $\{v, w\}$ a $E(T)$ // Adicionar nova aresta à AGM

Método de Prim – Exemplo

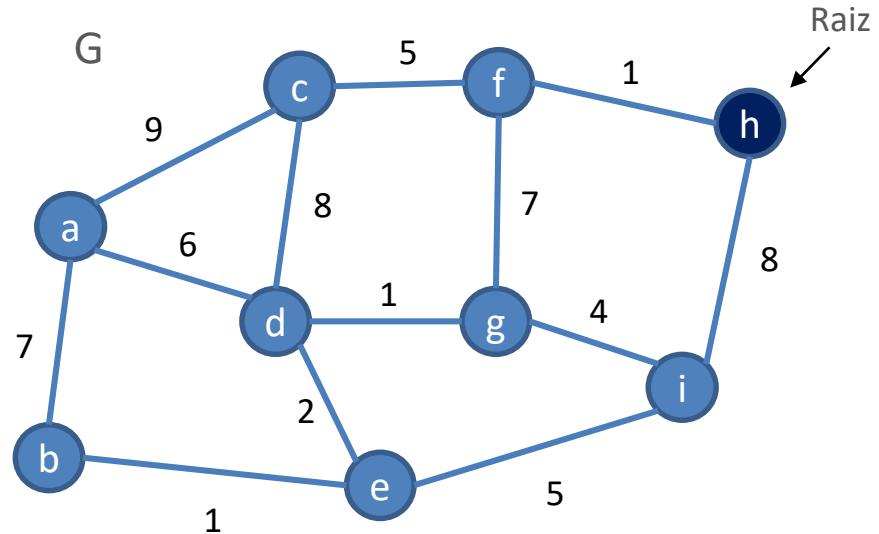


● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
··· Aresta em análise

— Aresta inserida na AGM

Método de Prim – Exemplo



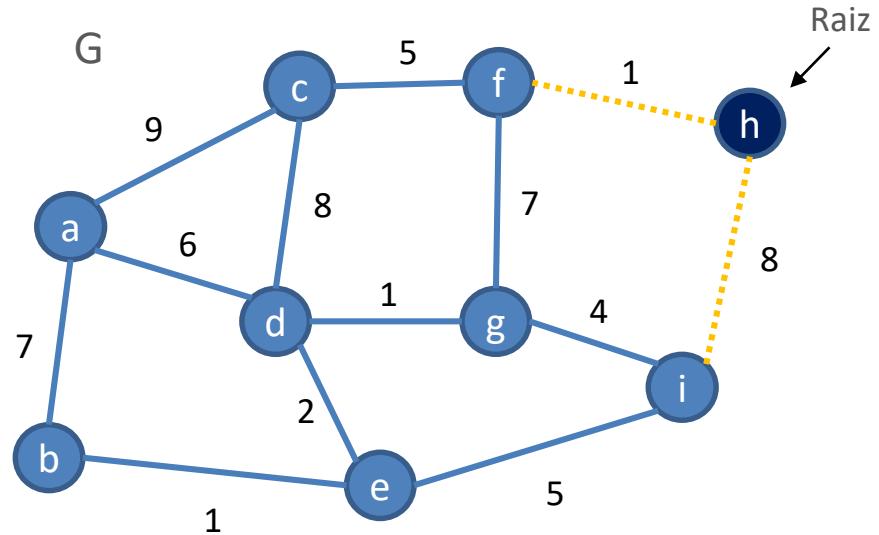
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— Aresta não explorada

T

● Vértice inserido na AGM
- - - Aresta em análise

— Aresta inserida na AGM

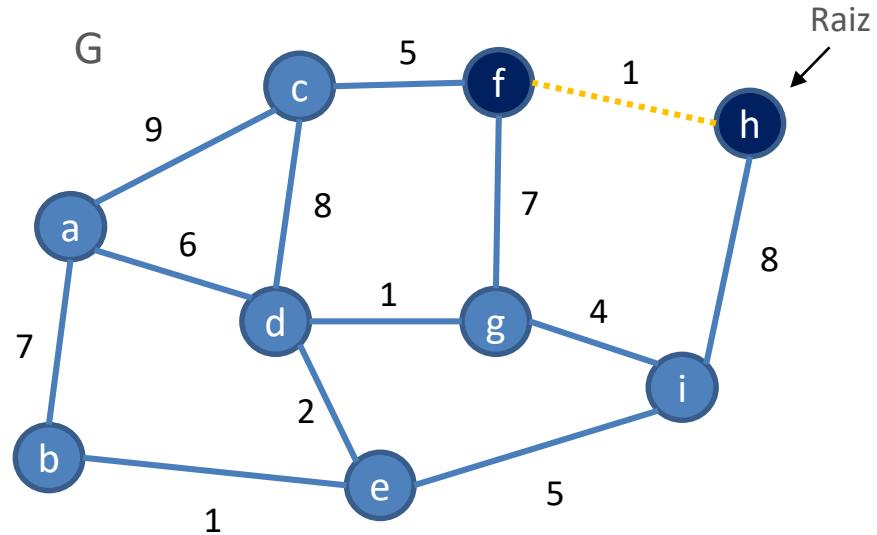
Método de Prim – Exemplo



T

- Vértice não explorado
- Aresta não explorada
- Vértice inserido na AGM
- Aresta em análise
- Aresta inserida na AGM

Método de Prim – Exemplo



T

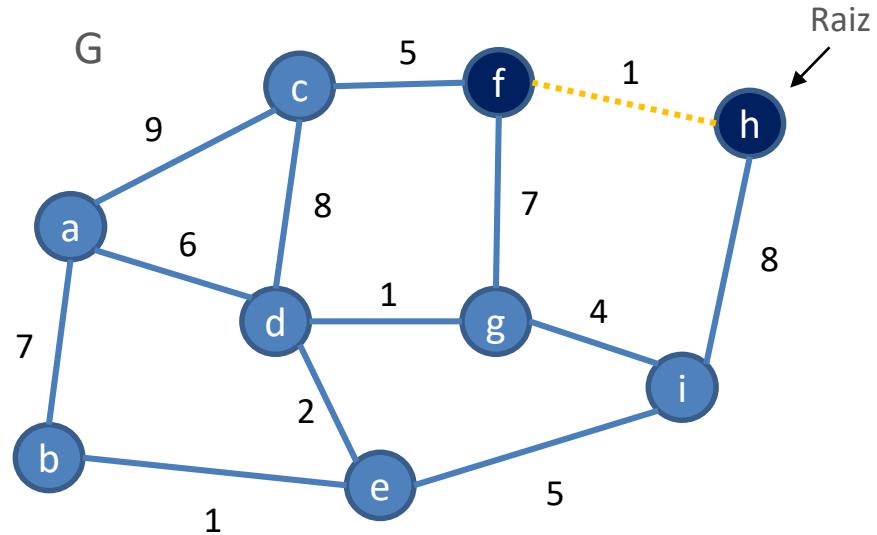


● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise

— Aresta inserida na AGM

Método de Prim – Exemplo

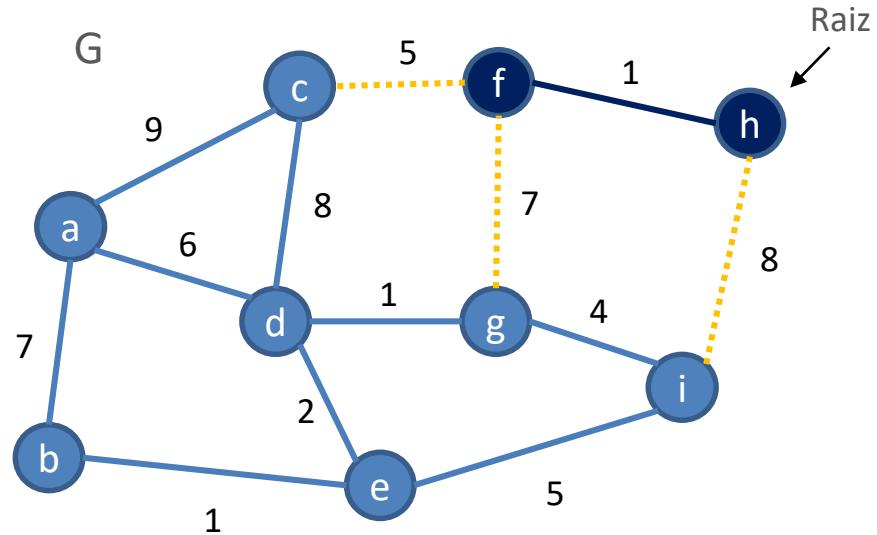


● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise

— Aresta inserida na AGM

Método de Prim – Exemplo

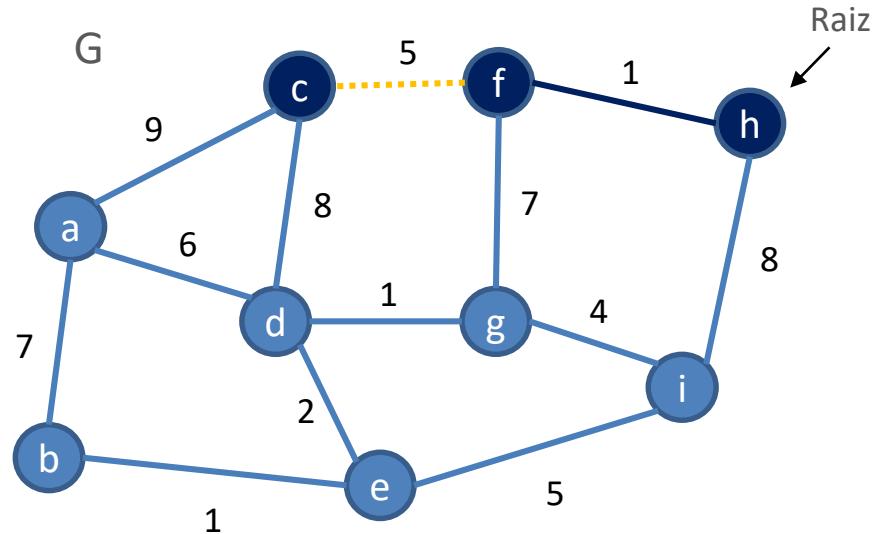


● Vértice não explorado
— Aresta não explorada

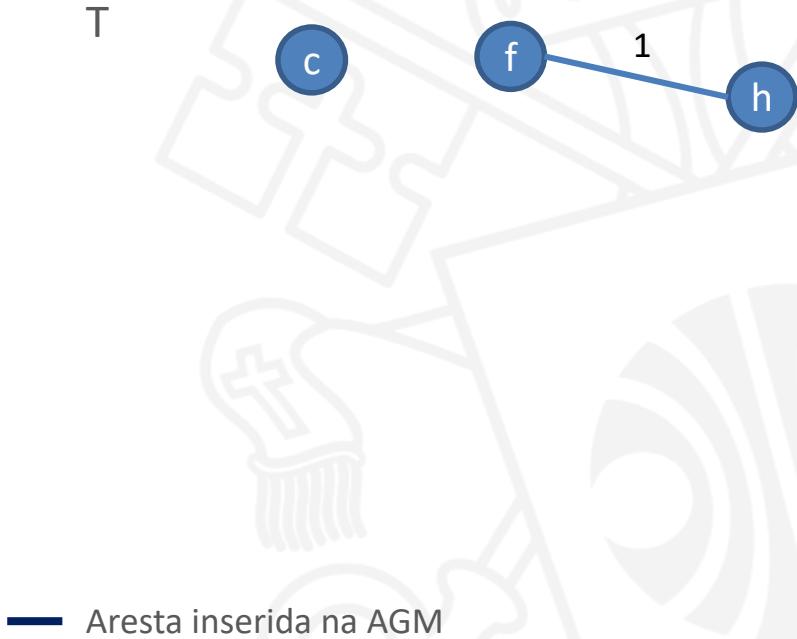
● Vértice inserido na AGM
— Aresta em análise

— Aresta inserida na AGM

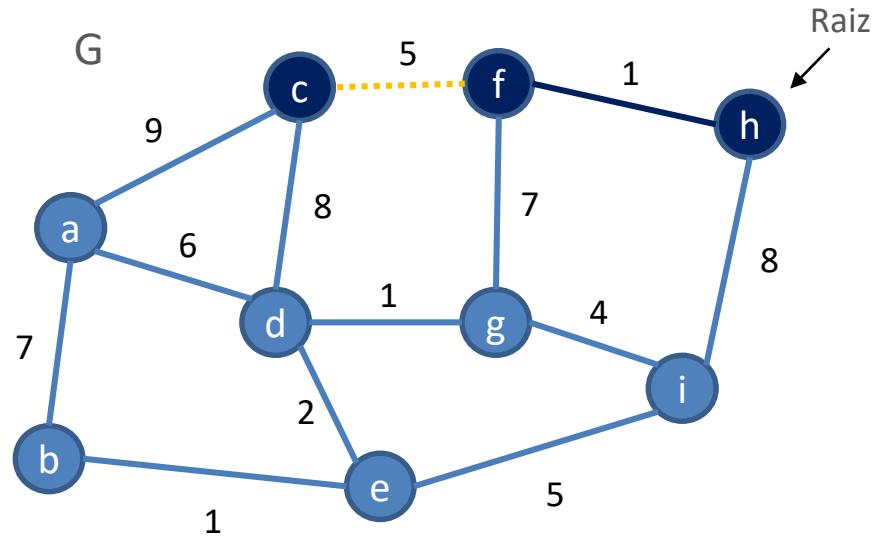
Método de Prim – Exemplo



● Vértice não explorado ● Vértice inserido na AGM
— Aresta não explorada ····· Aresta em análise



Método de Prim – Exemplo



● Vértice não explorado
— Aresta não explorada

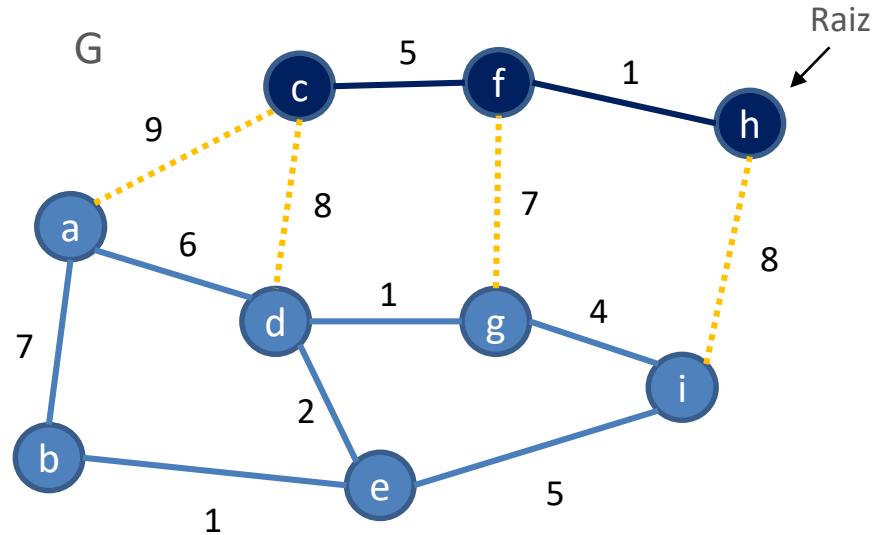
● Vértice inserido na AGM
— Aresta em análise

T



— Aresta inserida na AGM

Método de Prim – Exemplo



● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise

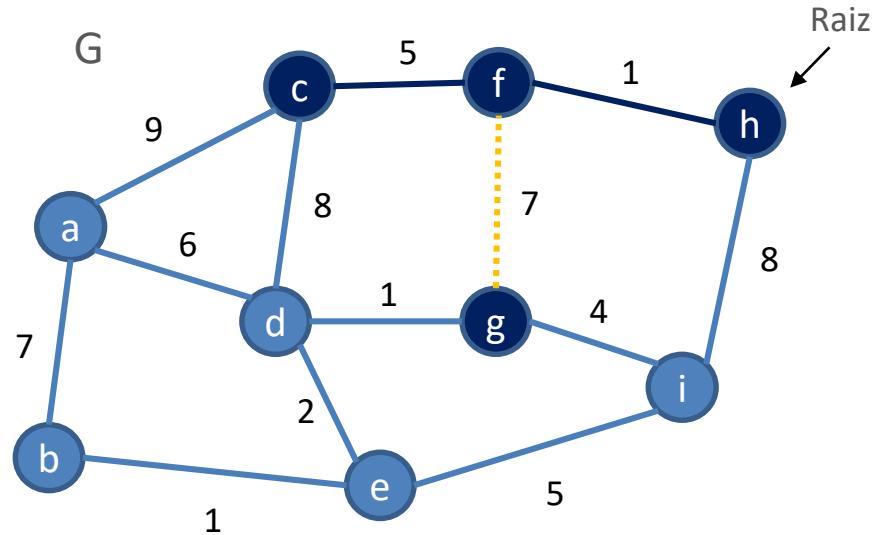
T

A tree T representing the Minimum Spanning Tree (MST) found by Prim's algorithm. It consists of vertices c, f, and h, and the edges (c, f) with weight 5 and (f, h) with weight 1.

Aresta	Peso
(c, f)	5
(f, h)	1

— Aresta inserida na AGM

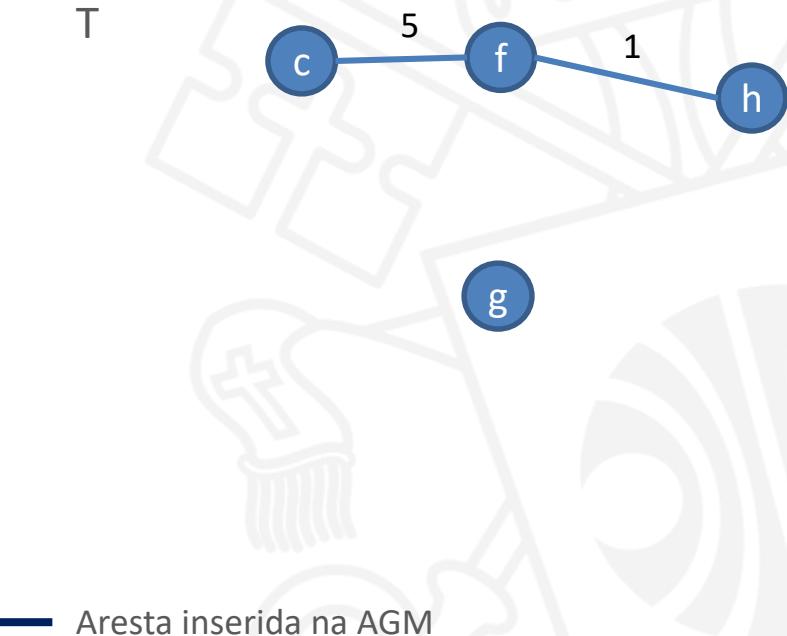
Método de Prim – Exemplo



● Vértice não explorado
— Aresta não explorada

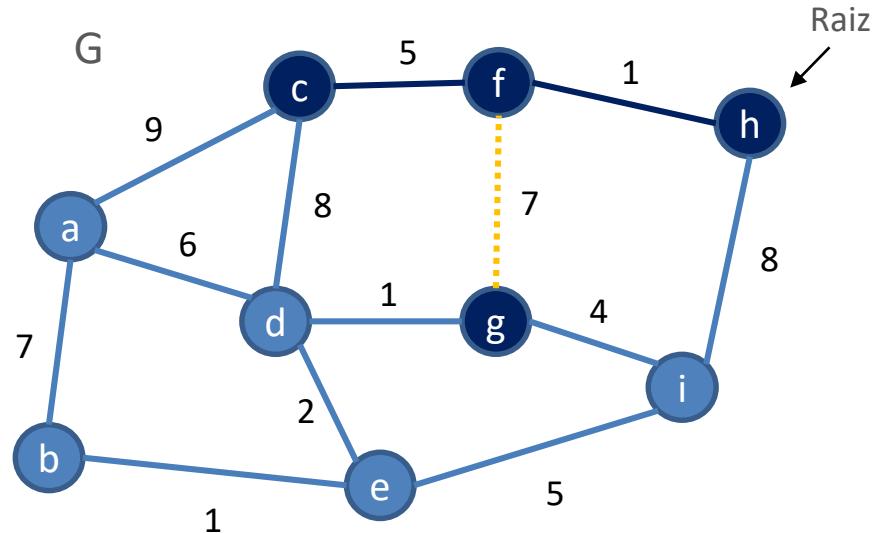
● Vértice inserido na AGM
··· Aresta em análise

T



— Aresta inserida na AGM

Método de Prim – Exemplo



● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
··· Aresta em análise

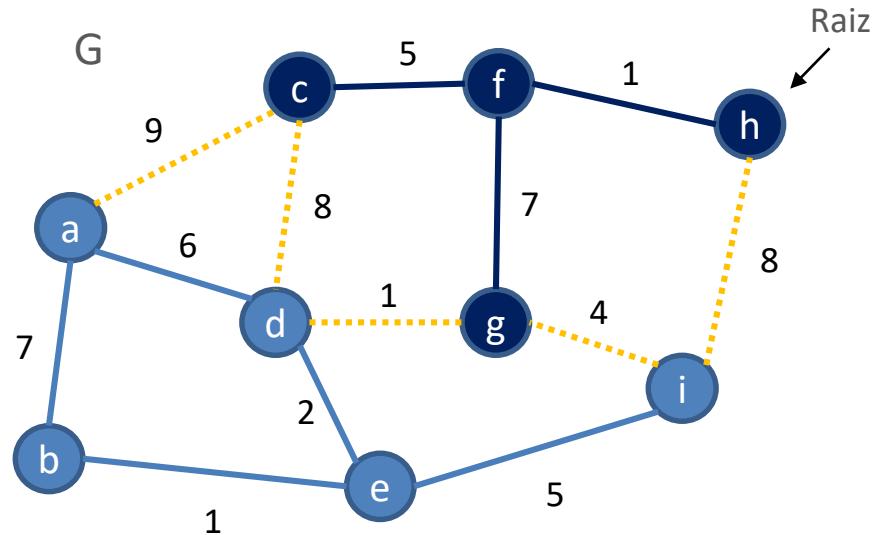
T

The subgraph T shows the Minimum Spanning Tree (MST) found by Prim's algorithm. It consists of vertices c, f, h, and g, connected by edges (c,f) = 5, (f,h) = 1, (f,g) = 7, and (h,i) = 8. Vertex f is the root (Raiz).

Aresta	Peso
(c, f)	5
(f, h)	1
(f, g)	7
(h, i)	8

— Aresta inserida na AGM

Método de Prim – Exemplo



● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise

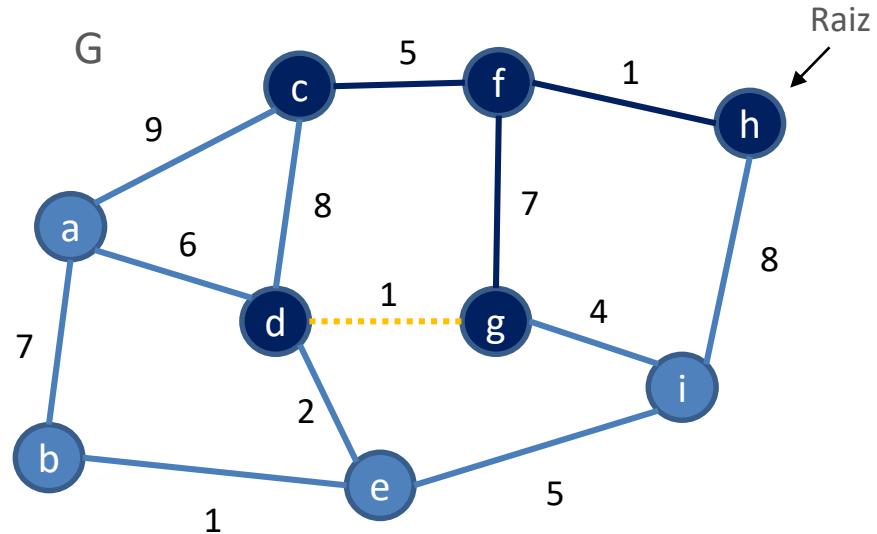
T

The Minimum Spanning Tree (T) is shown on the right, containing vertices c, f, g, and h, connected by edges (c, f) = 5, (f, g) = 7, and (f, h) = 1.

Aresta	Peso
(c, f)	5
(f, g)	7
(f, h)	1

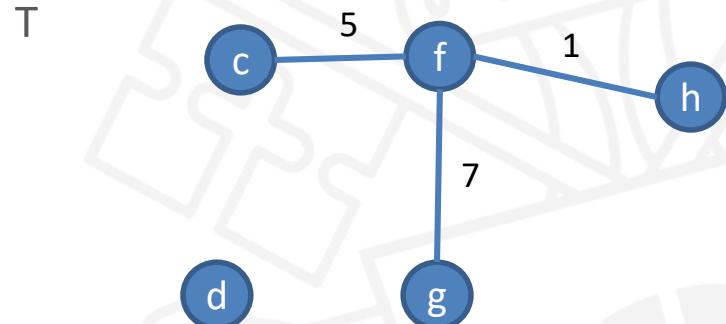
— Aresta inserida na AGM

Método de Prim – Exemplo



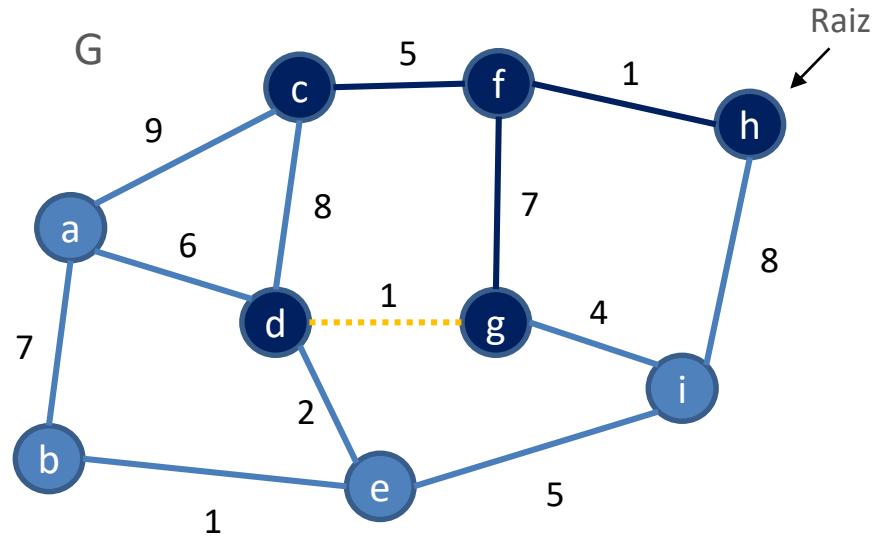
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— Aresta não explorada

● Vértice inserido na AGM
— Aresta inserida na AGM



— Aresta inserida na AGM

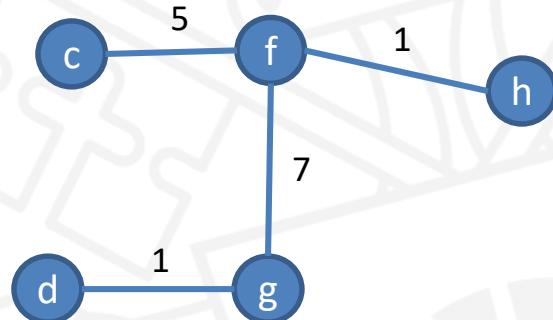
Método de Prim – Exemplo



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— Aresta não explorada

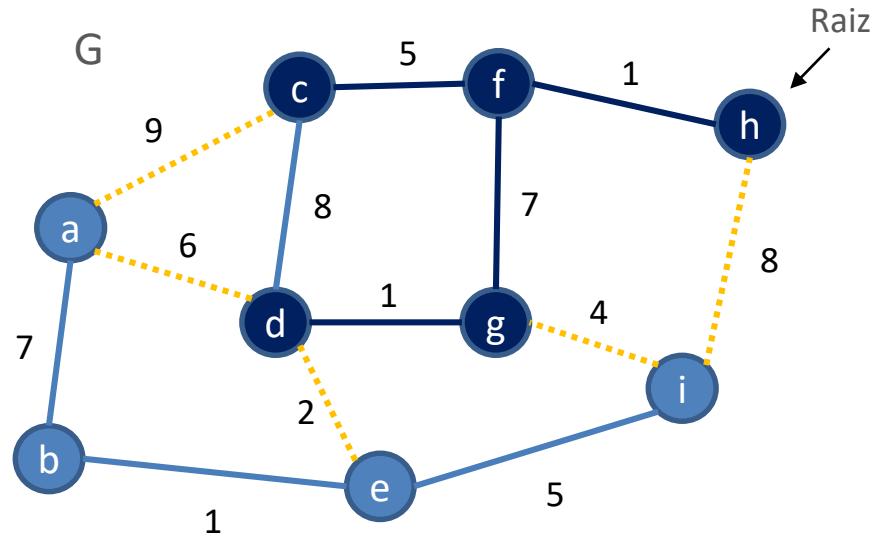
● Vértice inserido na AGM
— Aresta inserida na AGM

T



— Aresta inserida na AGM

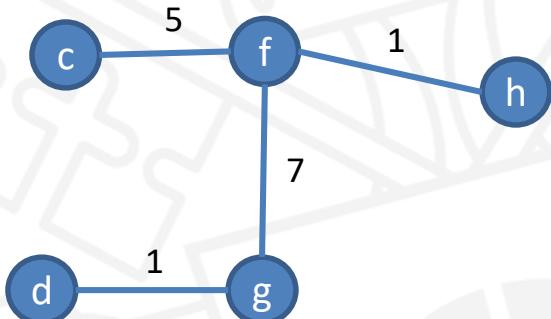
Método de Prim – Exemplo



● Vértice não explorado
— Aresta não explorada

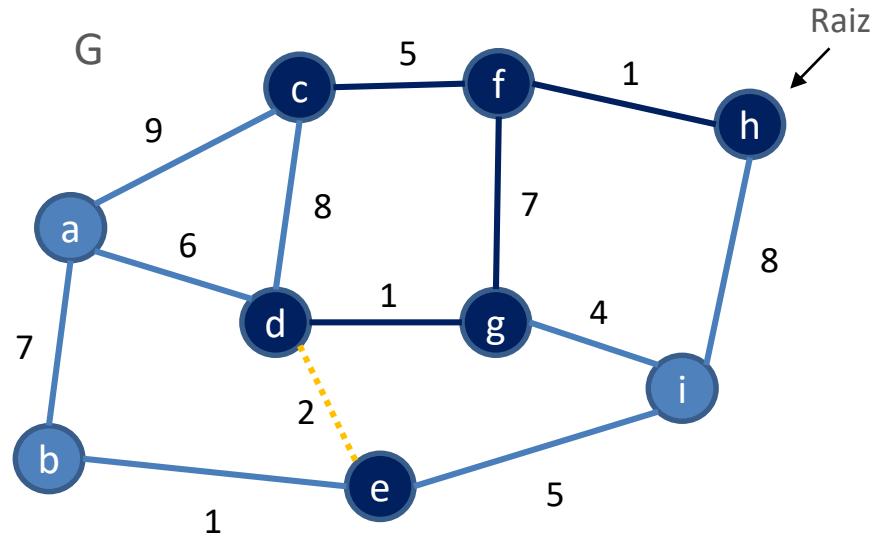
● Vértice inserido na AGM
— Aresta em análise

T



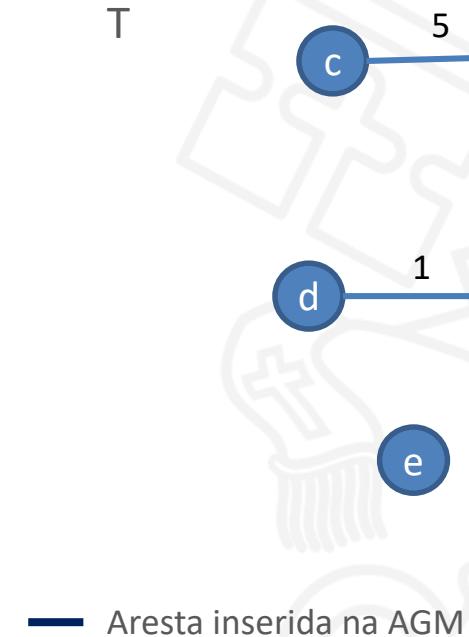
— Aresta inserida na AGM

Método de Prim – Exemplo



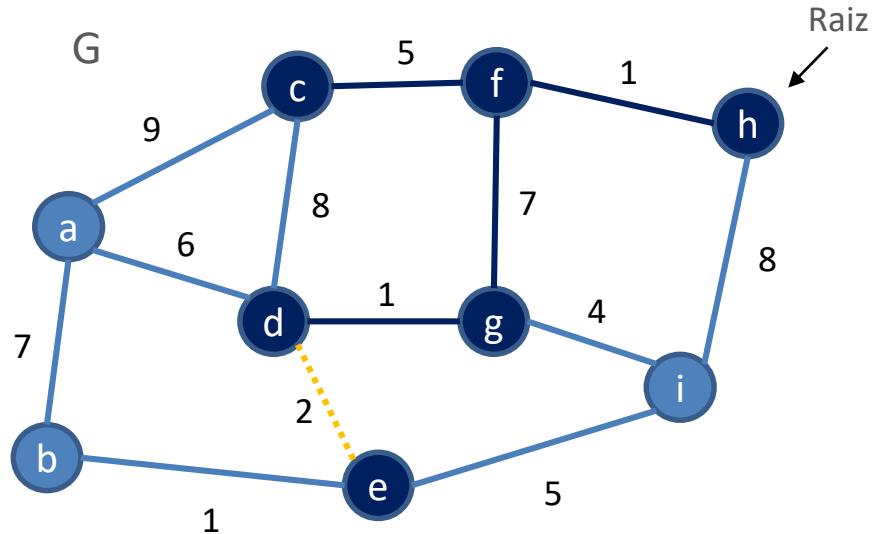
● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise



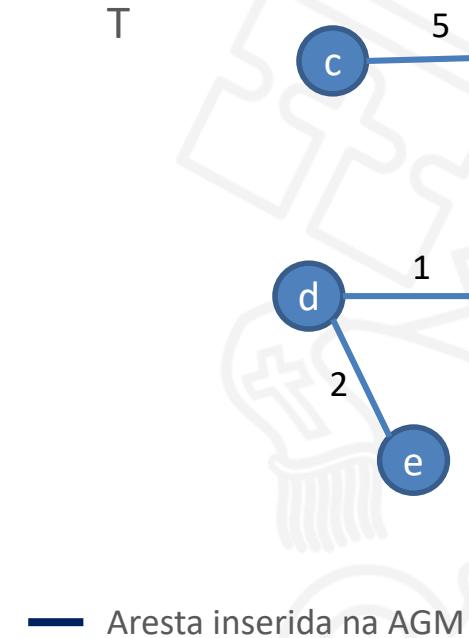
— Aresta inserida na AGM

Método de Prim – Exemplo



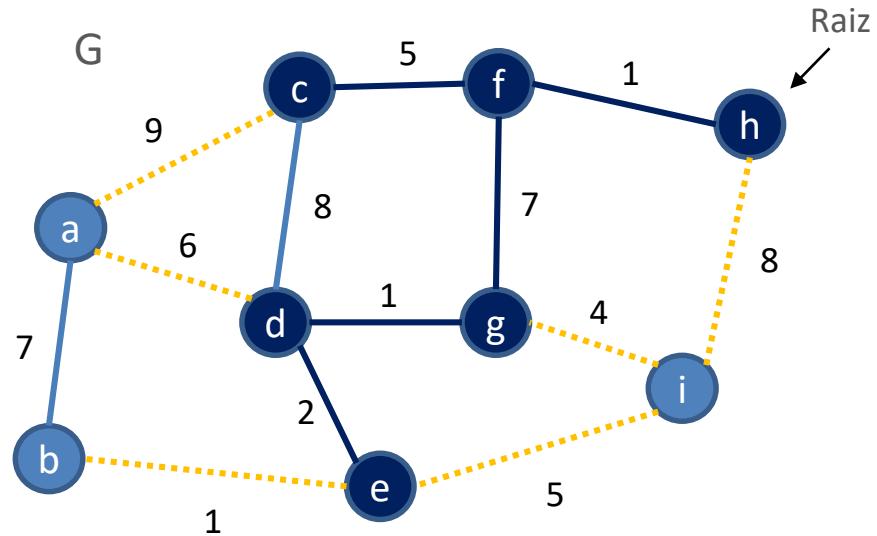
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— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise



— Aresta inserida na AGM

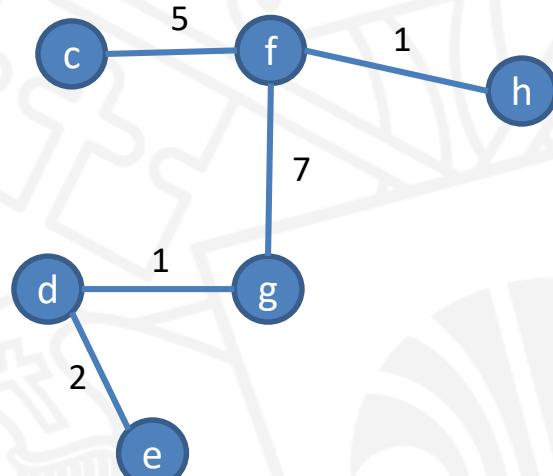
Método de Prim – Exemplo



● Vértice não explorado
— Aresta não explorada

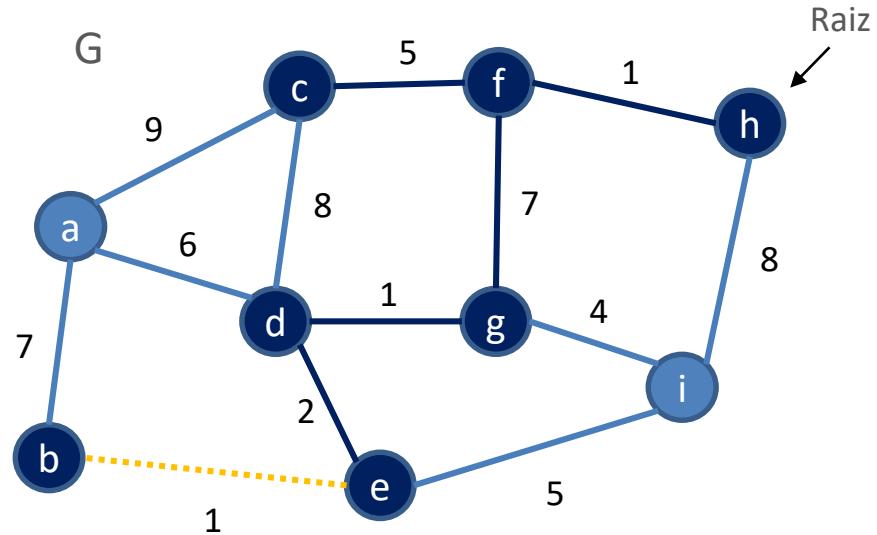
● Vértice inserido na AGM
---- Aresta em análise

T



— Aresta inserida na AGM

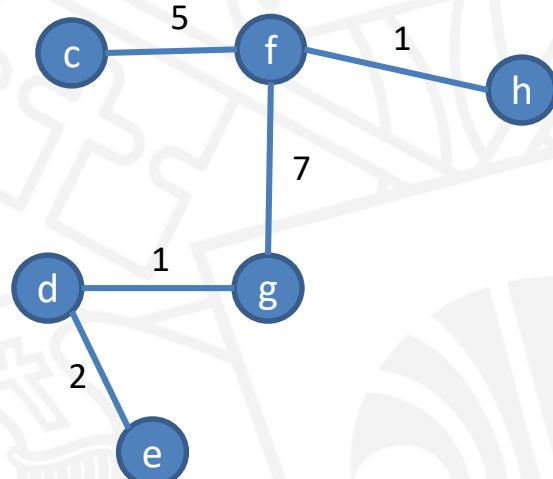
Método de Prim – Exemplo



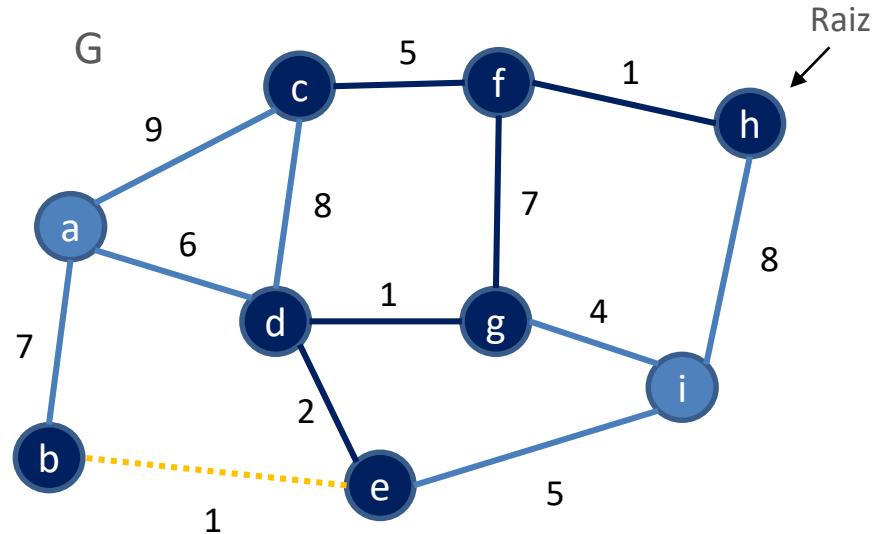
● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise

T
● b
— Aresta inserida na AGM

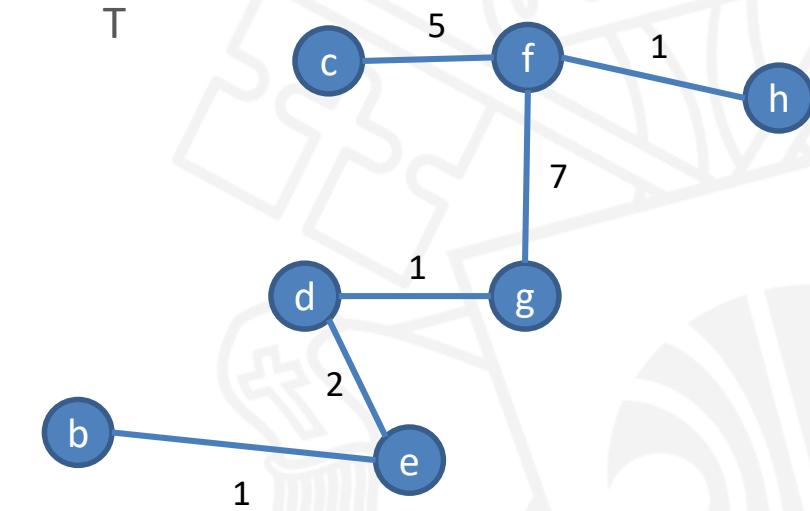


Método de Prim – Exemplo



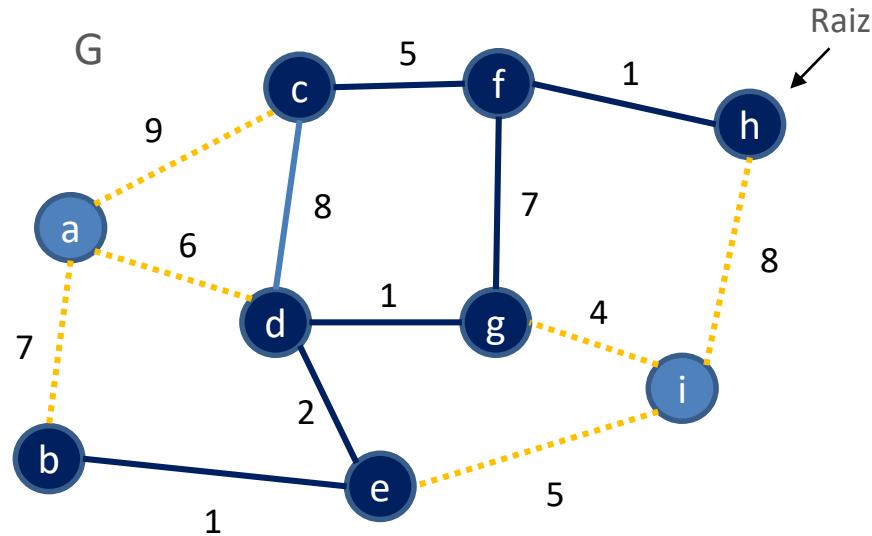
● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise



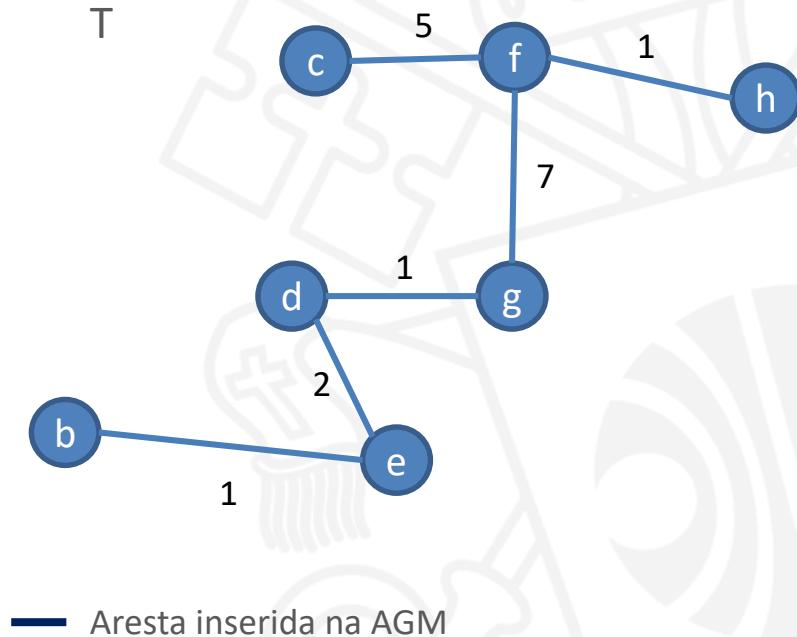
— Aresta inserida na AGM

Método de Prim – Exemplo



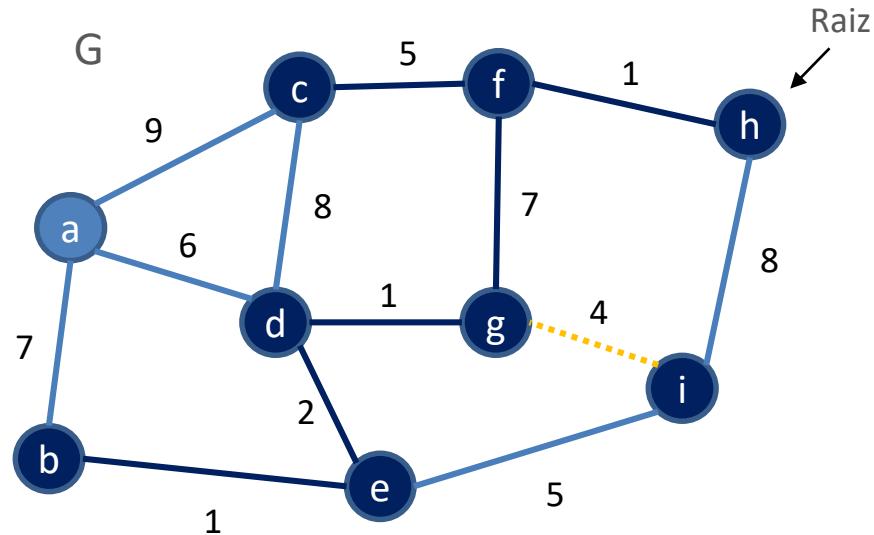
● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise



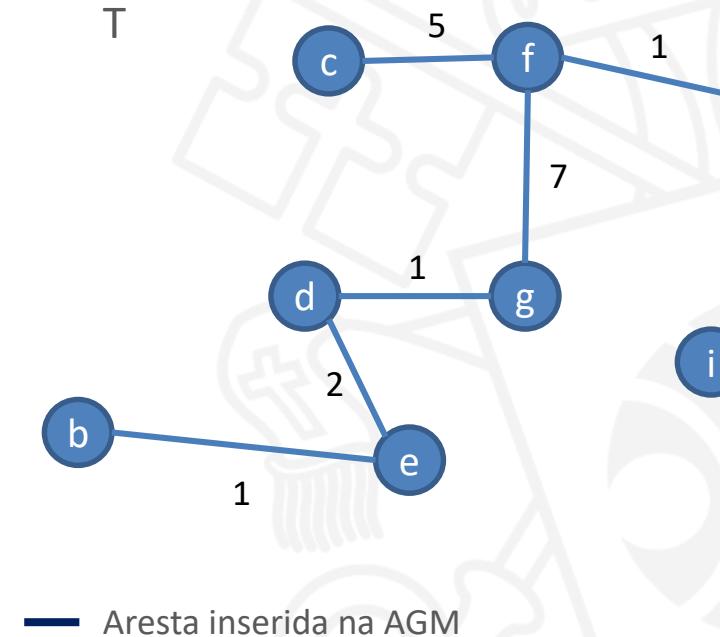
— Aresta inserida na AGM

Método de Prim – Exemplo



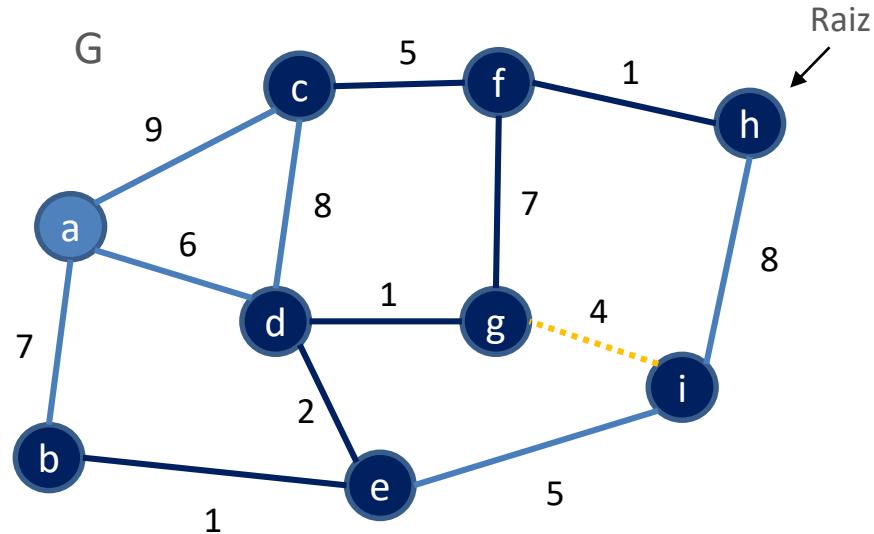
● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta inserida na AGM



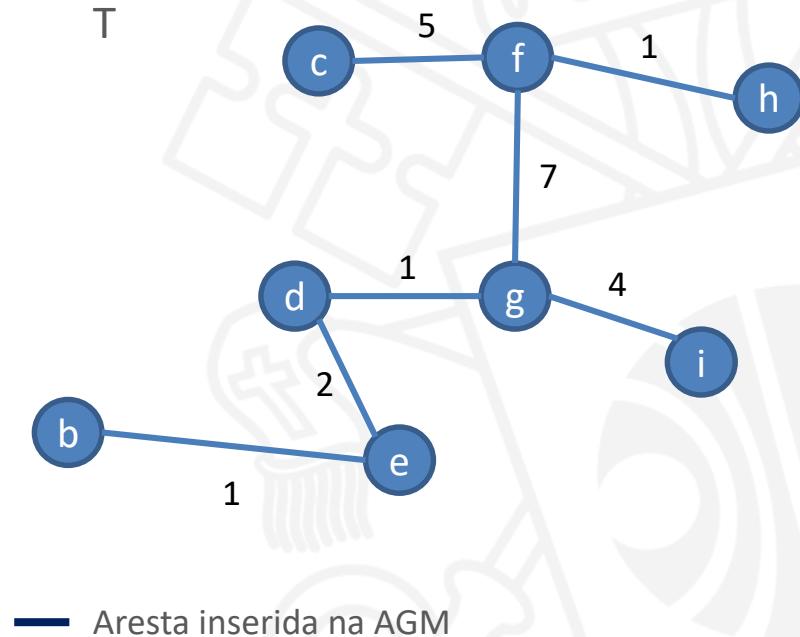
— Aresta em análise

Método de Prim – Exemplo



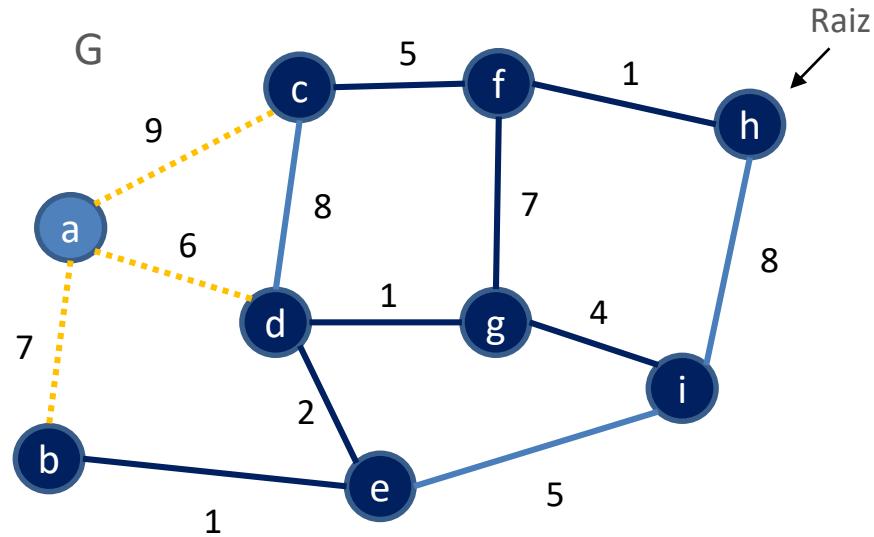
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— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise



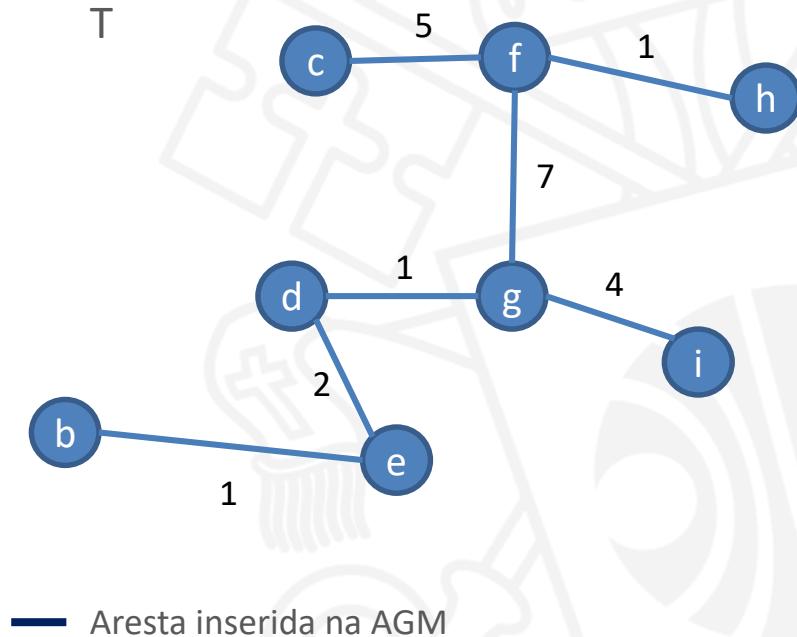
— Aresta inserida na AGM

Método de Prim – Exemplo



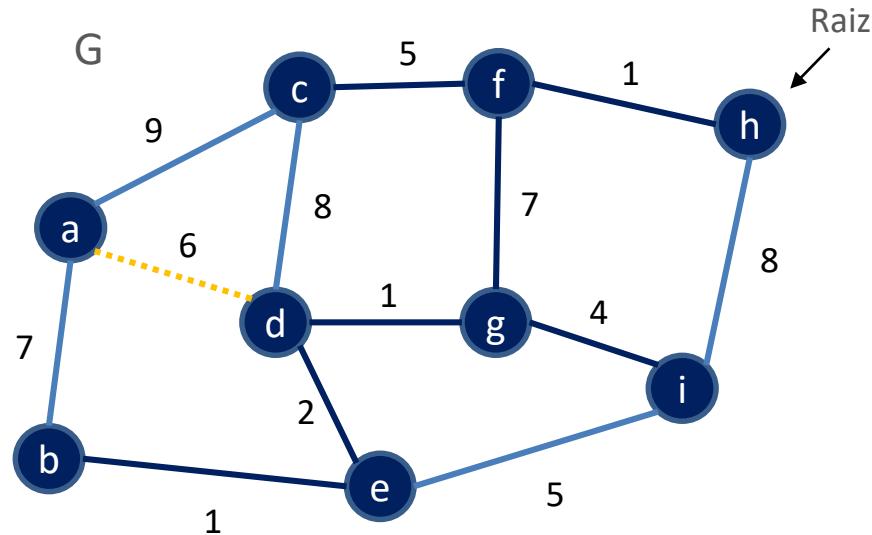
● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta em análise



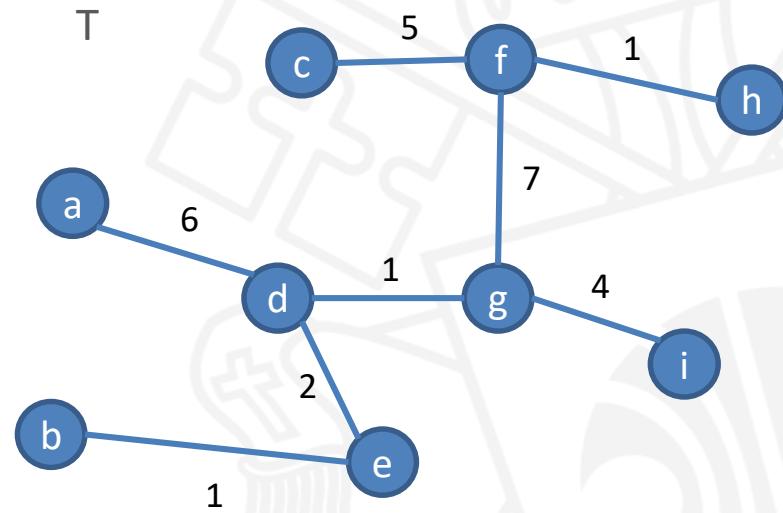
— Aresta inserida na AGM

Método de Prim – Exemplo



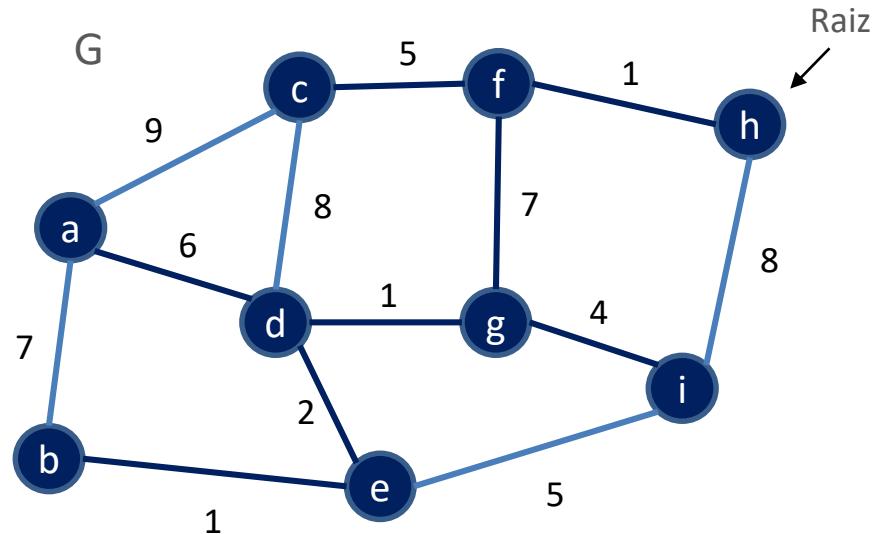
● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
— Aresta inserida na AGM



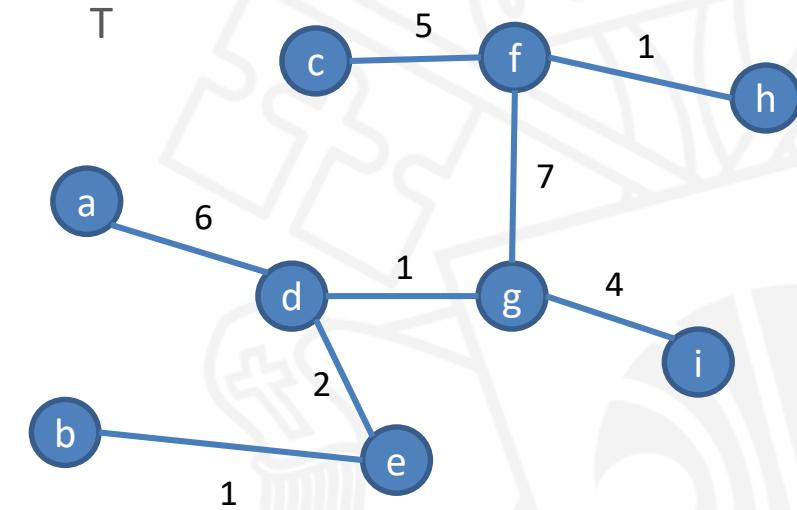
— Aresta inserida na AGM

Método de Prim – Exemplo



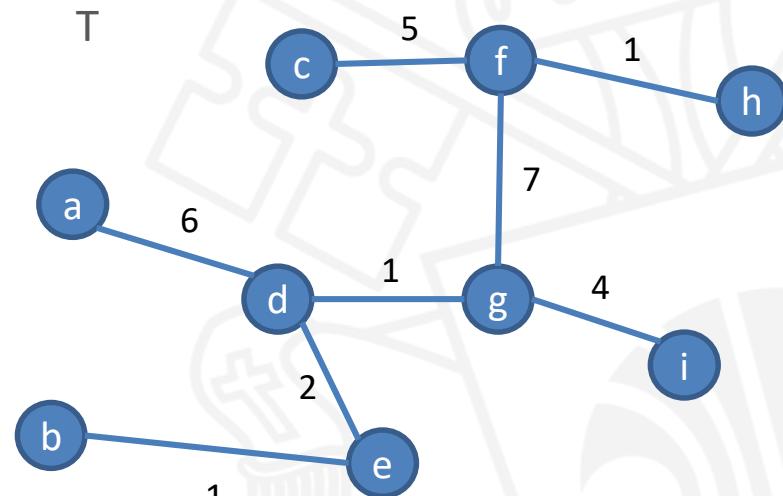
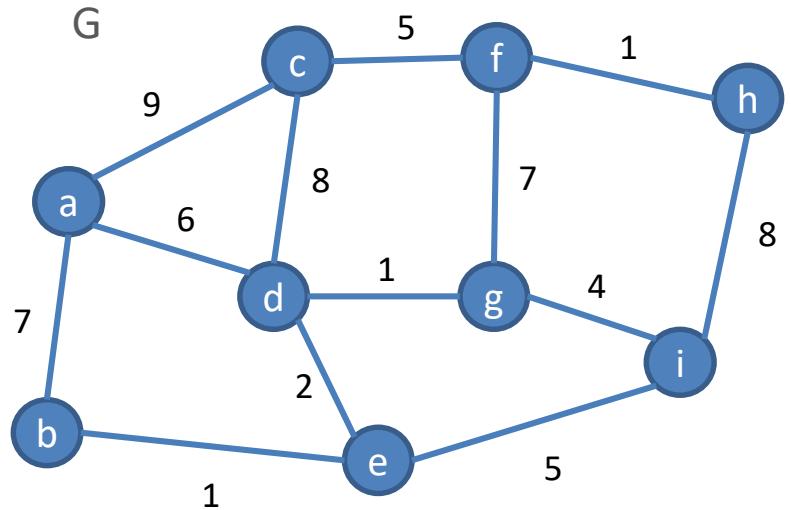
● Vértice não explorado
— Aresta não explorada

● Vértice inserido na AGM
··· Aresta em análise



— Aresta inserida na AGM

Método de Prim – Exemplo



AGM $\rightarrow C(T) = 27$

Método de Kruskal

Método de Kruskal

Este algoritmo foi proposto por Kruskal em 1956 e ele propõe incluir na árvore, a cada iteração, a aresta de menor custo que não forme um ciclo com as inseridas anteriormente.

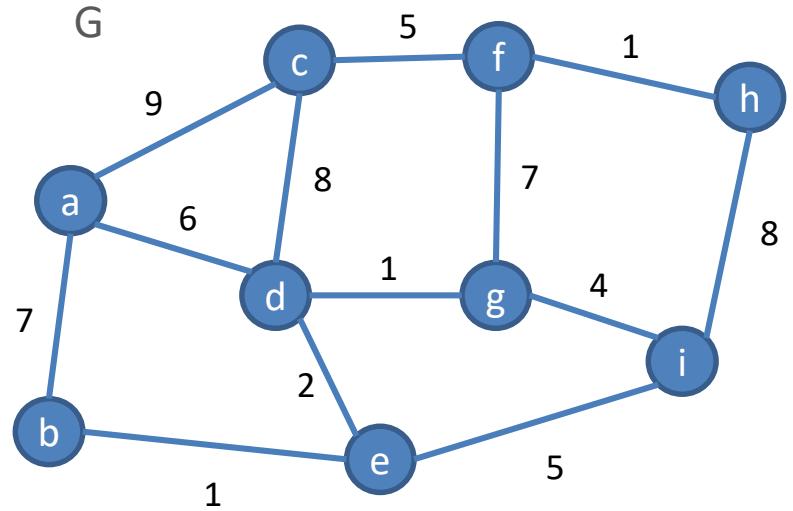
Dessa forma são necessárias pelo menos $n - 1$ iterações (em que n representa o número vértices do grafo). Contudo, pode ser necessário realizar mais iterações devido a necessidade de se ignorar arestas que formem ciclos.

O raciocínio está voltado para a formação da árvore a partir da inclusão de arestas, e não de vértices, como no algoritmo de Prim.

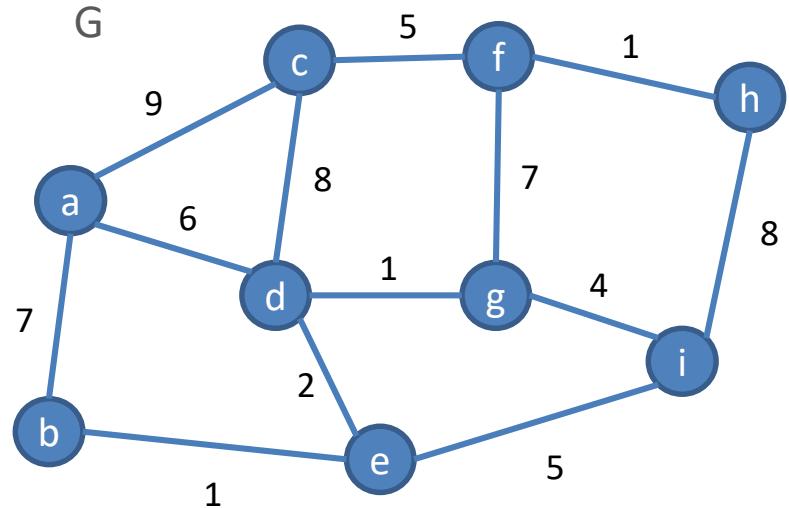
Método de Kruskal – Algoritmo

1. Ordenar arestas em ordem não decrescente e_1, e_2, e_3, \dots
2. $V(T) \leftarrow V(G);$ // Adiciona todos vértices à AGM
3. $E(T) \leftarrow \{ e_1 \};$ // Inicializar arestas da AGM
4. $j \leftarrow 2;$ // Indica aresta a ser analisada
5. enquanto $|E(T)| < |V(T)| - 1$ efetuar // Se ainda não for conexo?
 - a. se aresta e_j não forma ciclo com as arestas em $E(T)$ então
 - i. Acrescentar $\{v, w\}$ a $E(T)$ // Adicionar nova aresta à AGM
 - b. $j \leftarrow j + 1;$ // Indica próxima aresta

Método de Kruskal – Exemplo

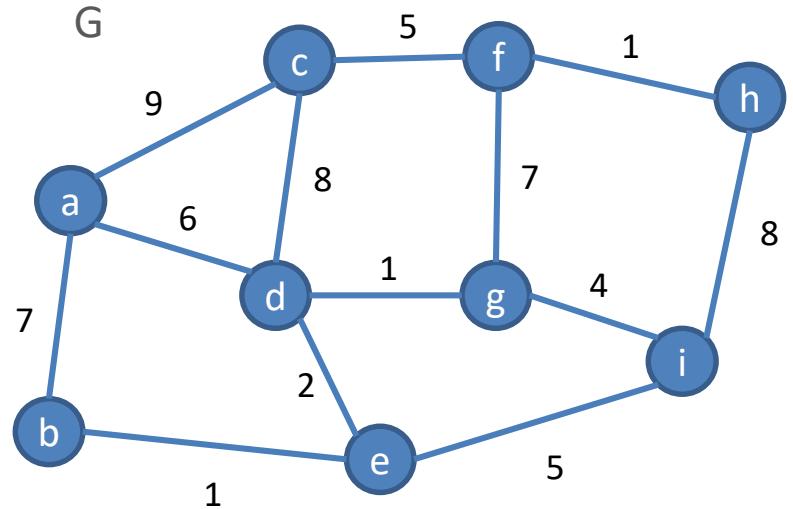


Método de Kruskal – Exemplo

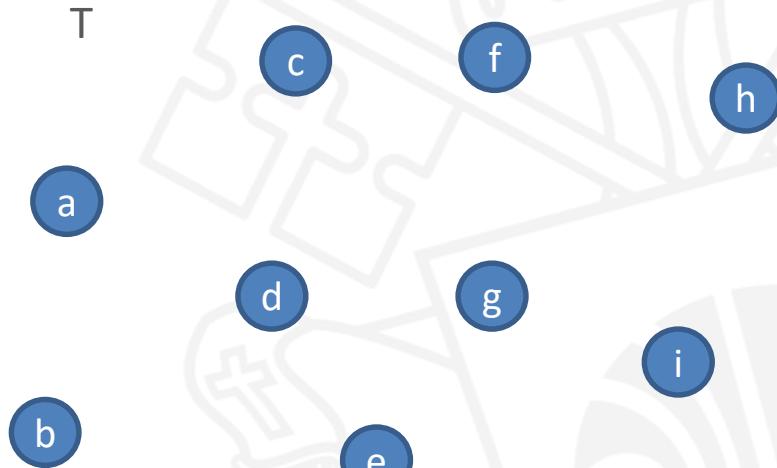


Lista ordenada de arestas: {b,e}, {d,g}, {f,h}, {d,e}, {g,i}, {e,i},
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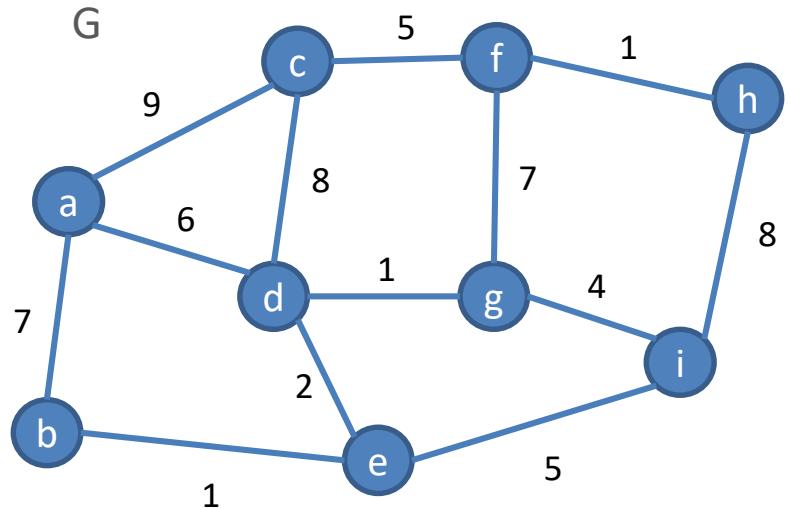
Método de Kruskal – Exemplo



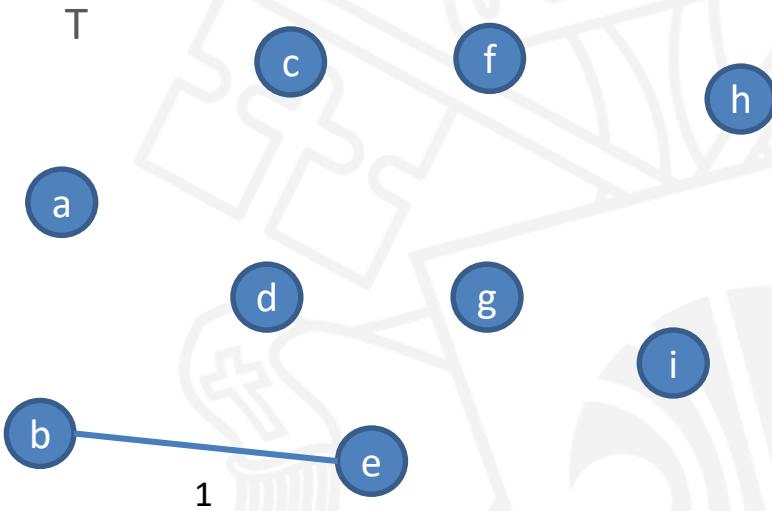
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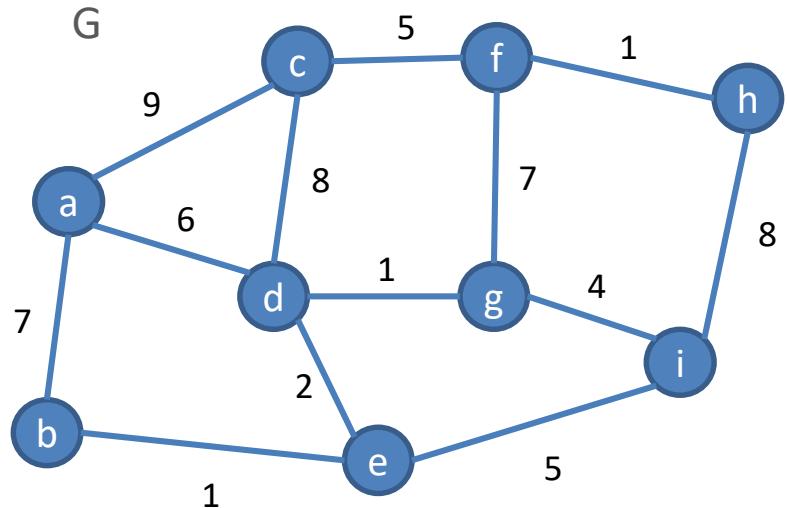
Método de Kruskal – Exemplo



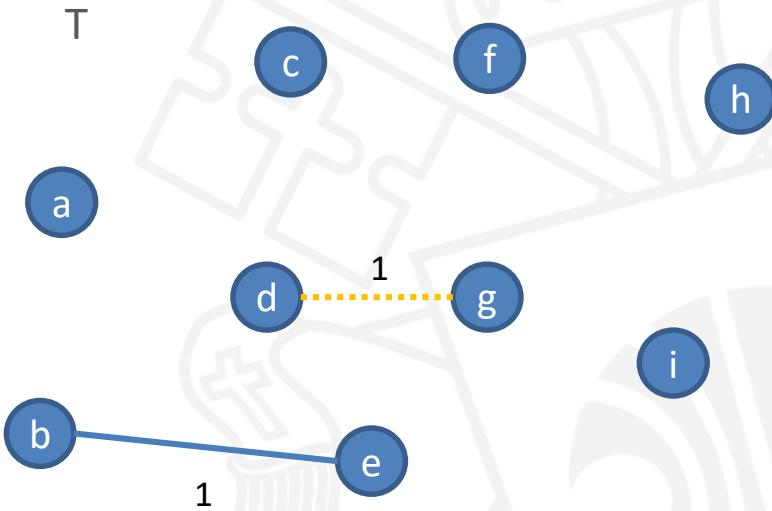
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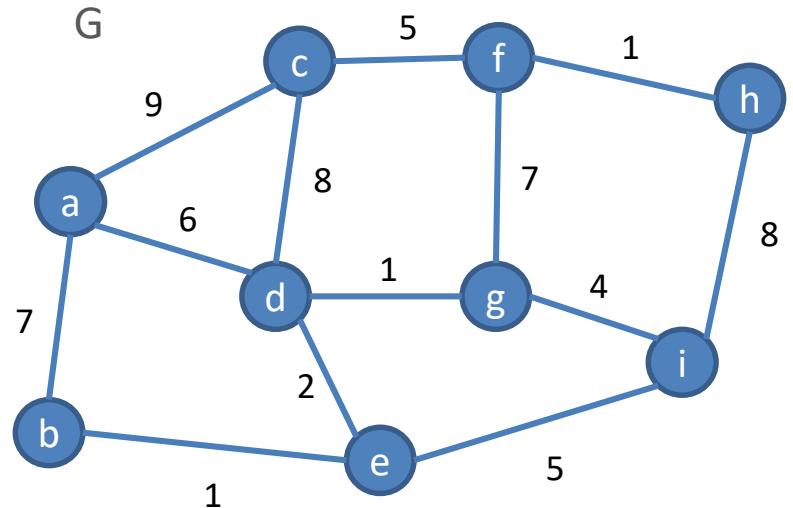
Método de Kruskal – Exemplo



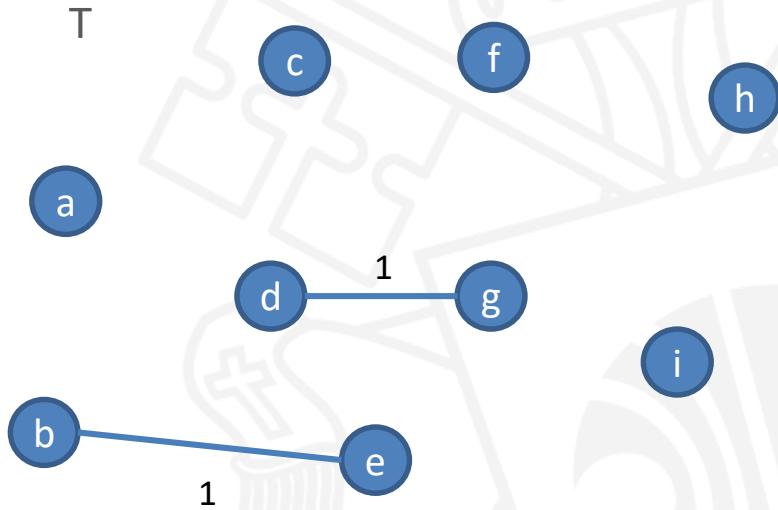
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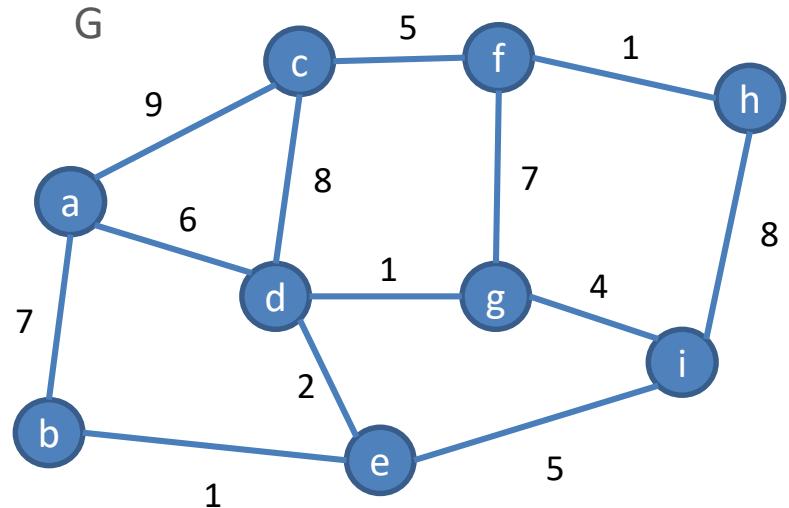
Método de Kruskal – Exemplo



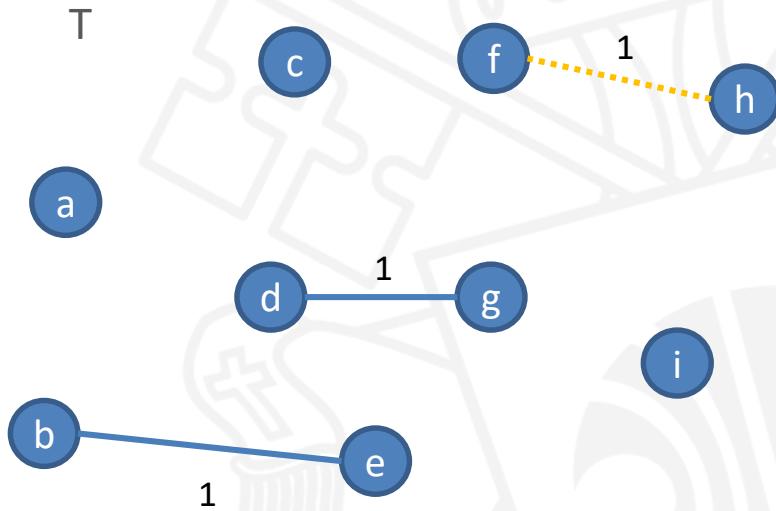
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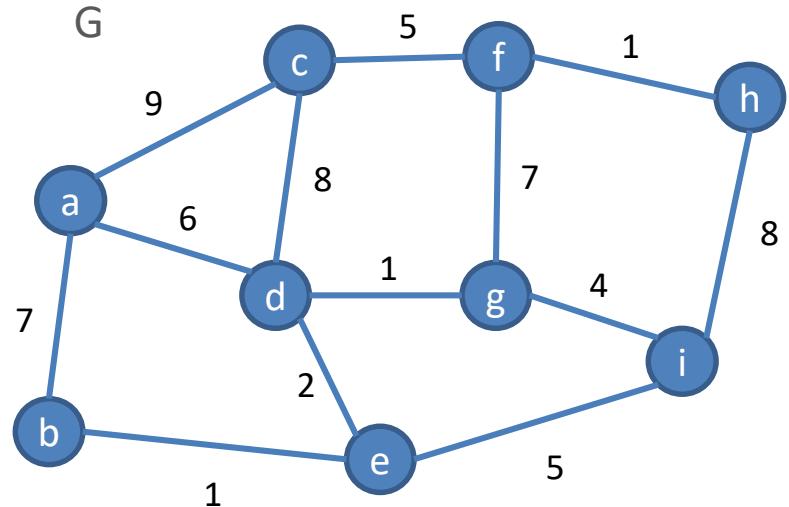
Método de Kruskal – Exemplo



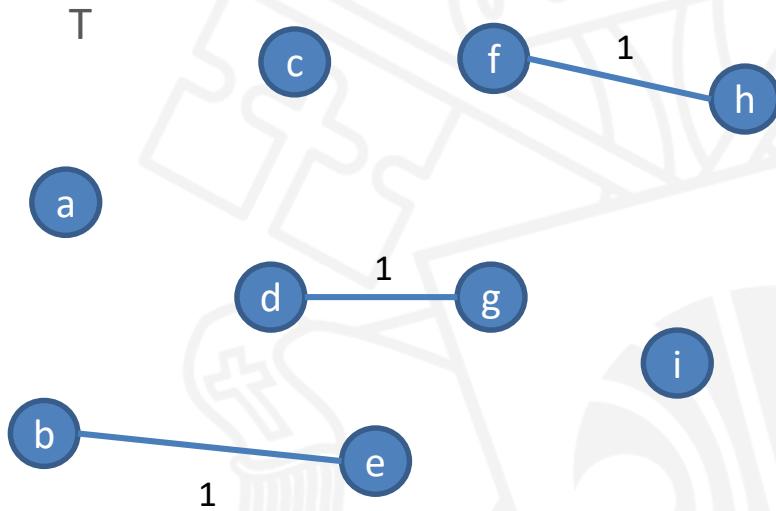
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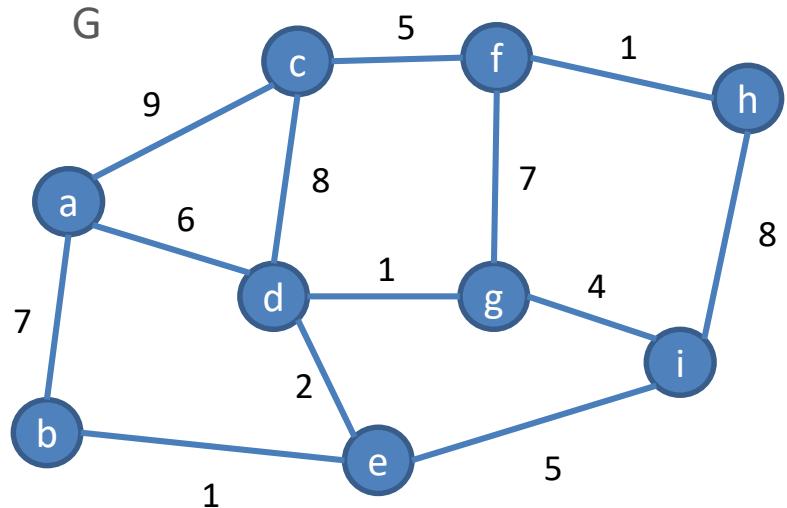
Método de Kruskal – Exemplo



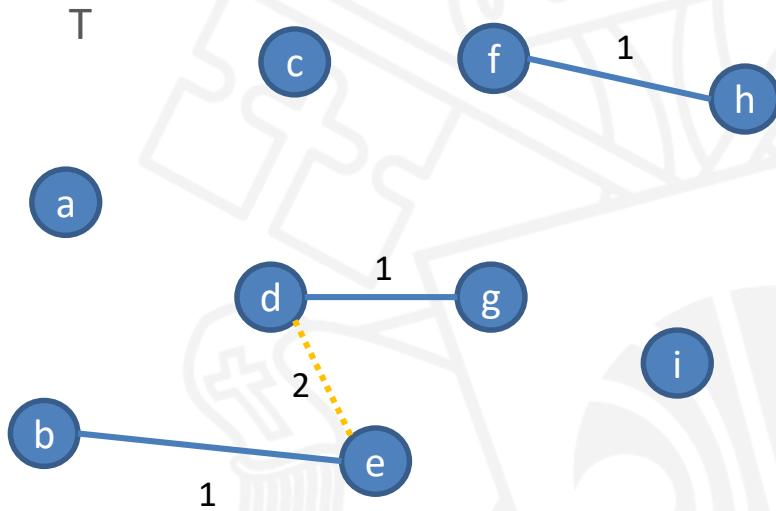
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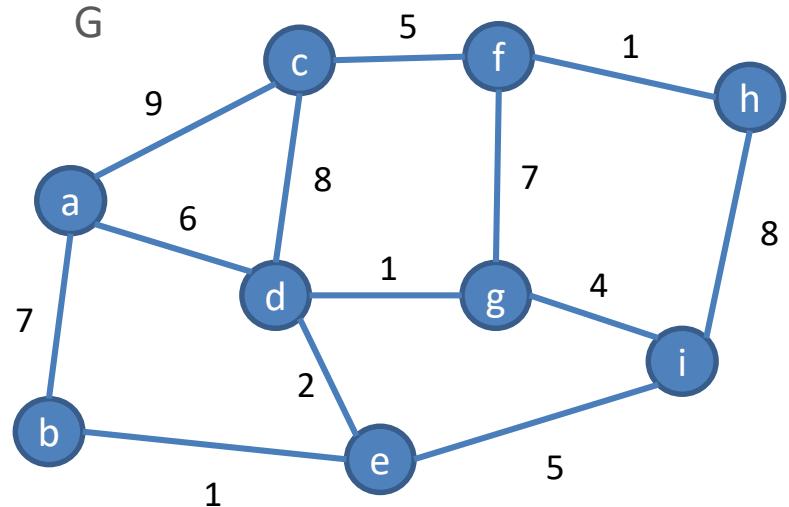
Método de Kruskal – Exemplo



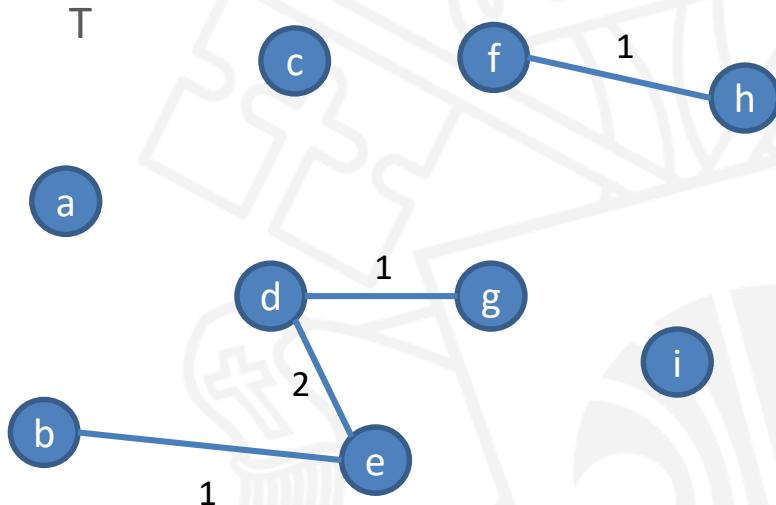
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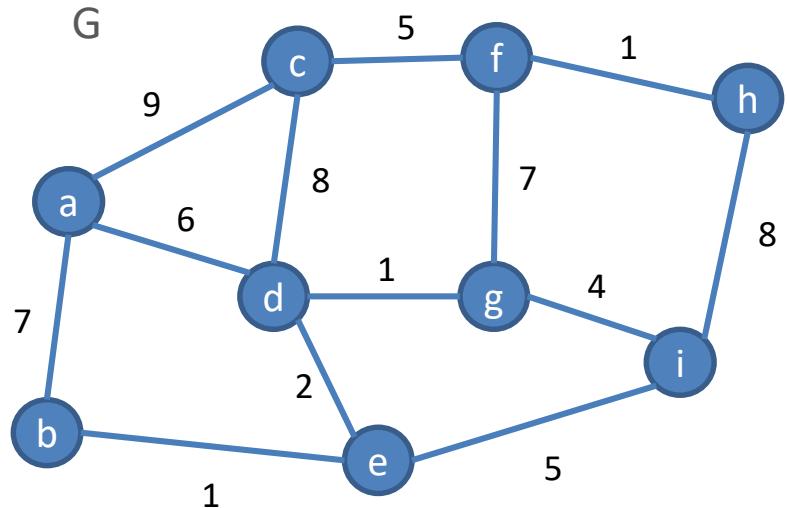
Método de Kruskal – Exemplo



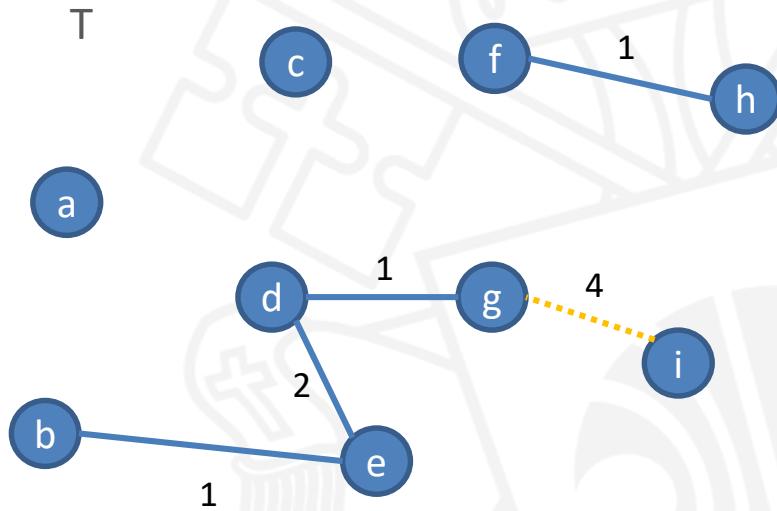
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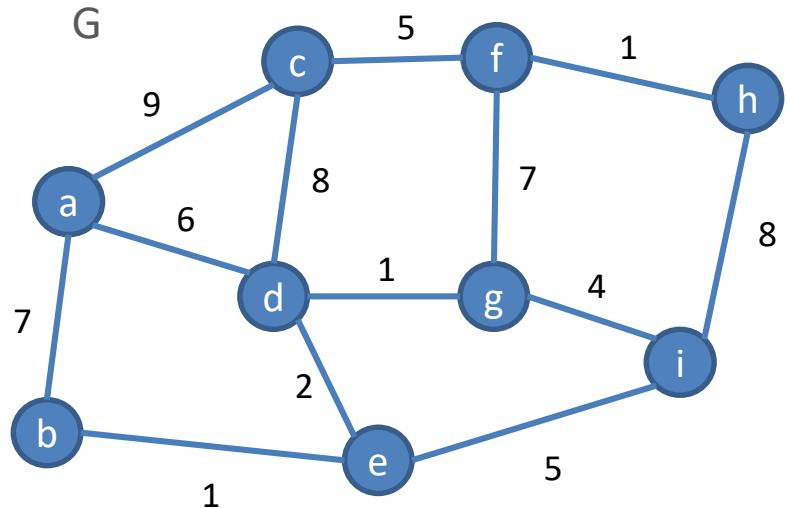
Método de Kruskal – Exemplo



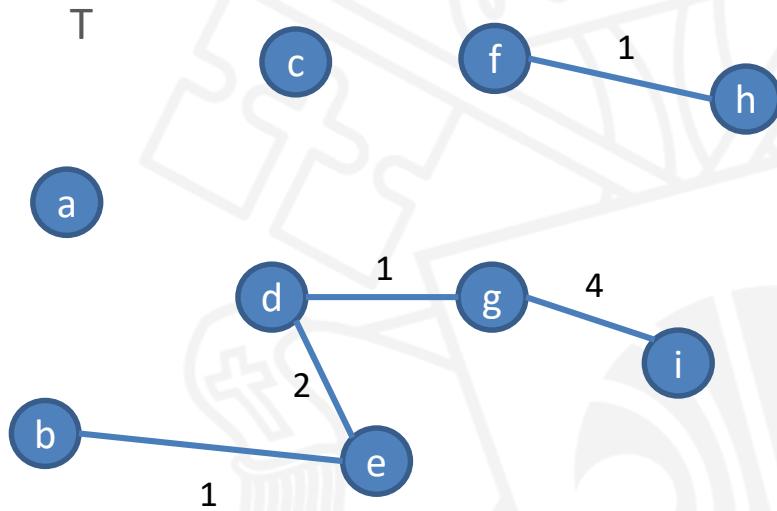
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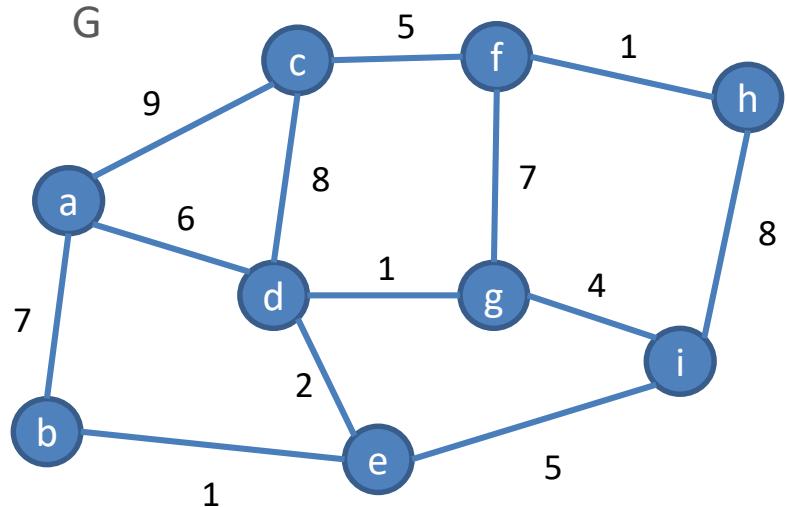
Método de Kruskal – Exemplo



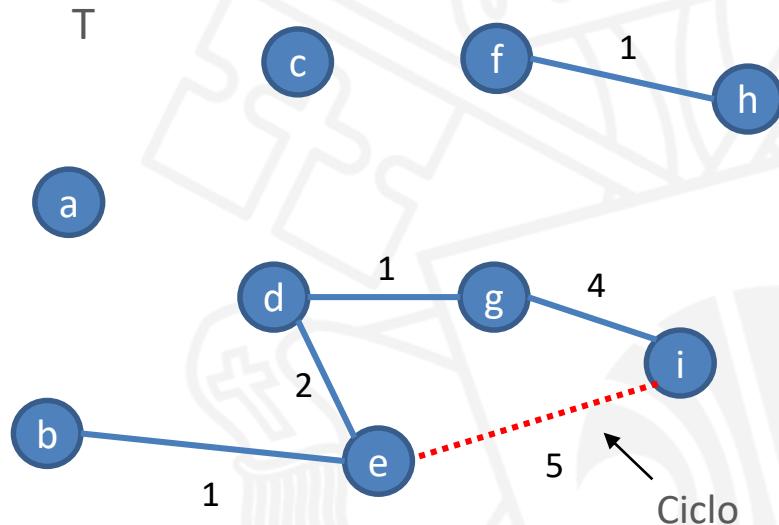
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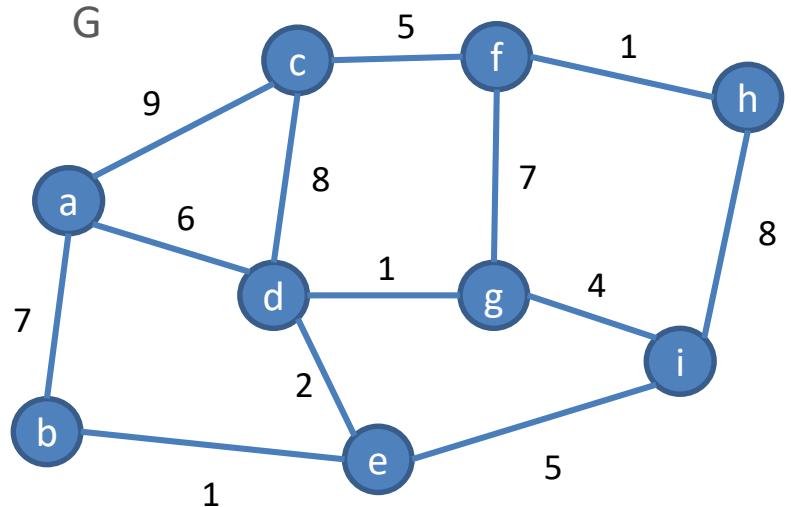
Método de Kruskal – Exemplo



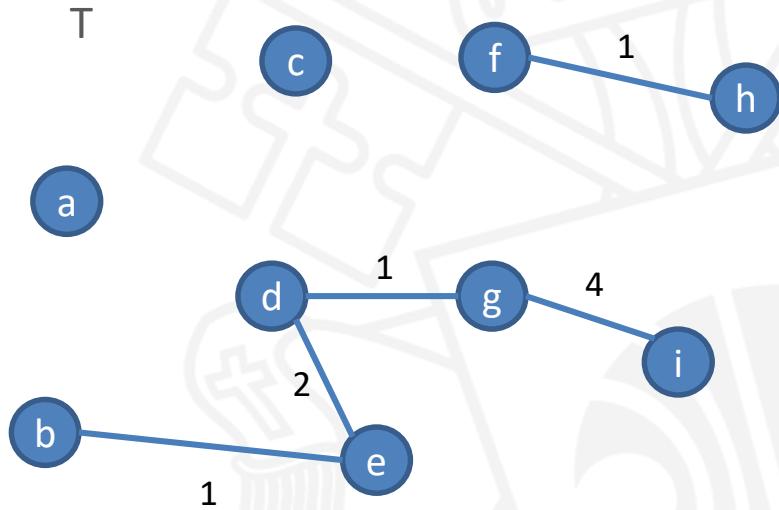
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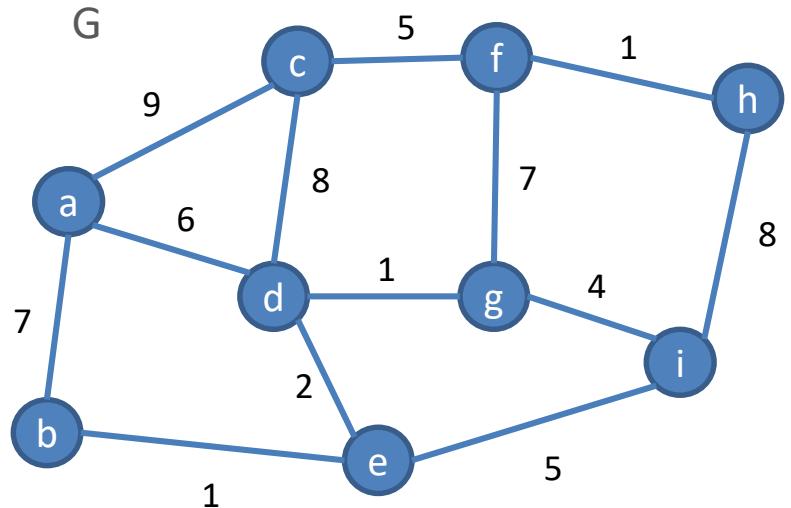
Método de Kruskal – Exemplo



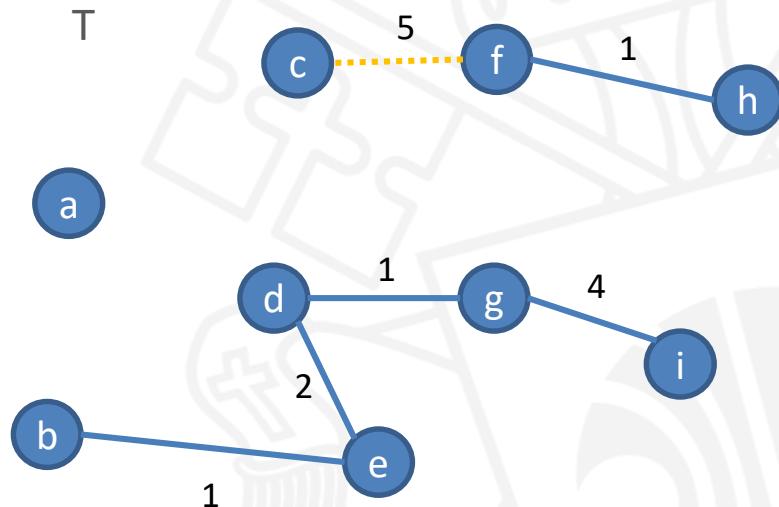
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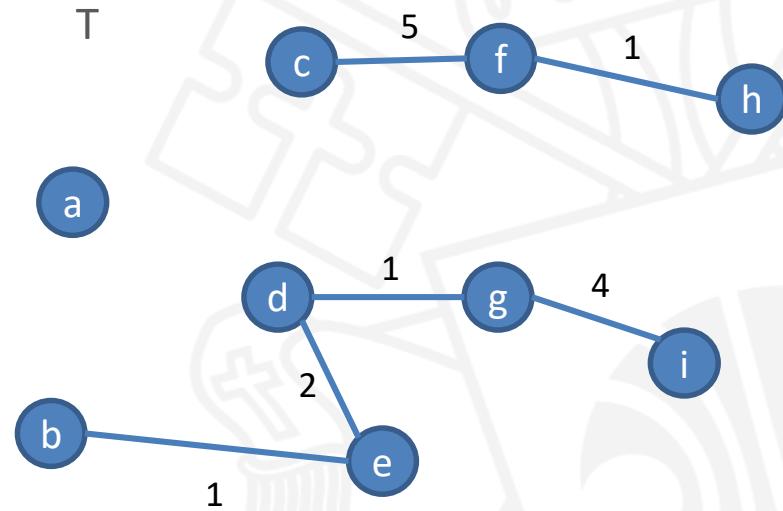
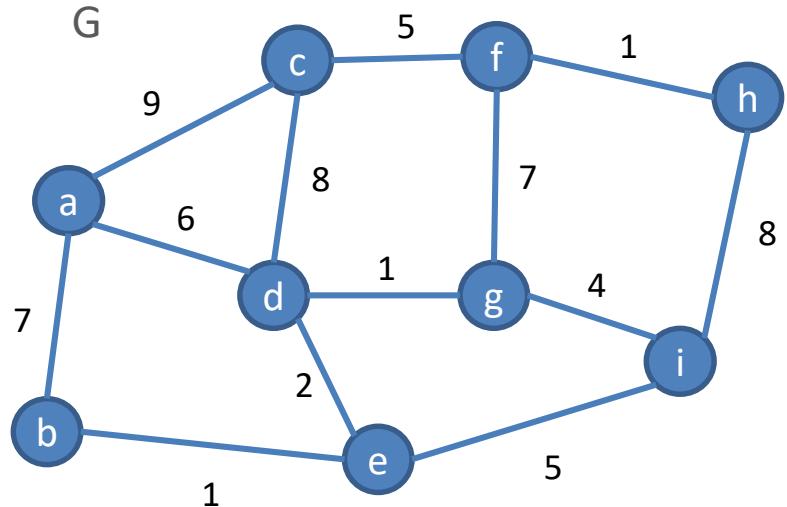
Método de Kruskal – Exemplo



Lista ordenada de arestas: $\{b,e\}$, $\{d,g\}$, $\{f,h\}$, $\{d,e\}$, $\{g,i\}$, $\{e,i\}$,
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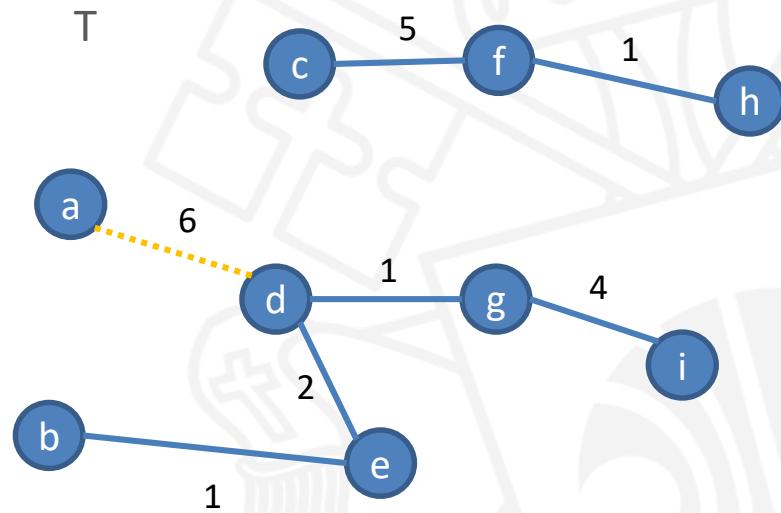
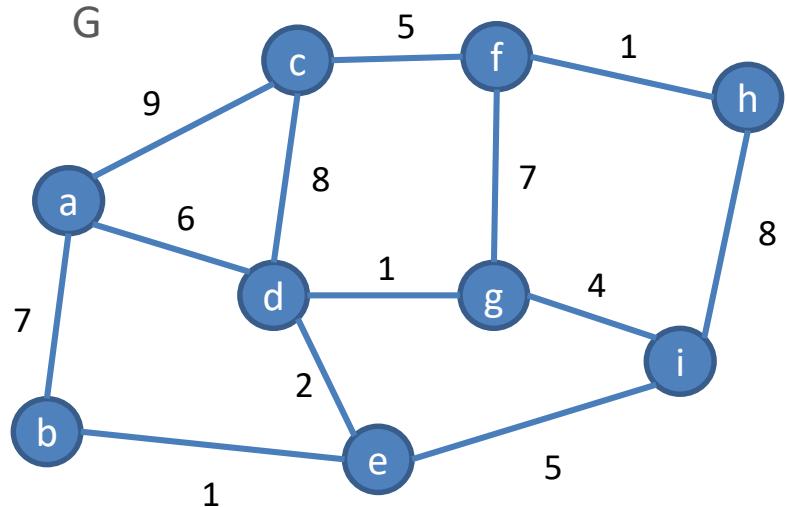


Método de Kruskal – Exemplo



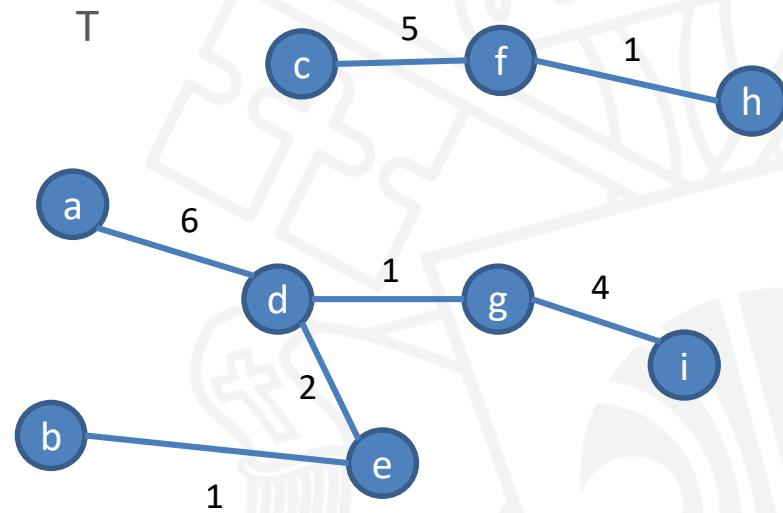
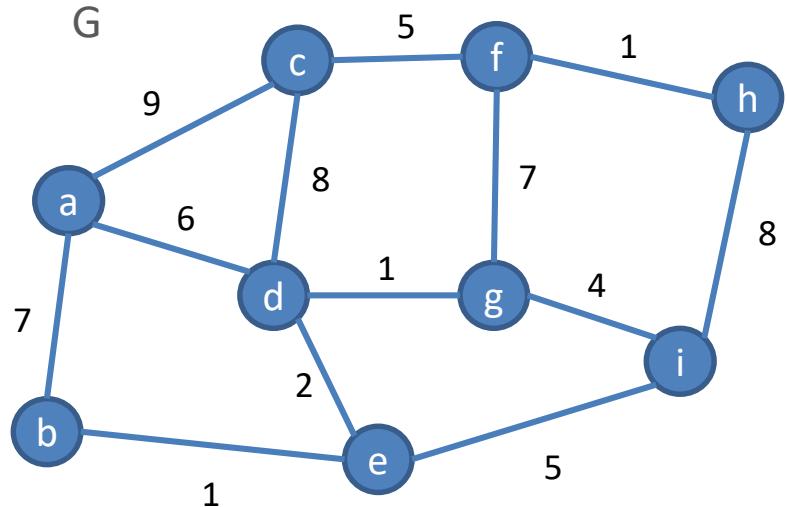
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Método de Kruskal – Exemplo



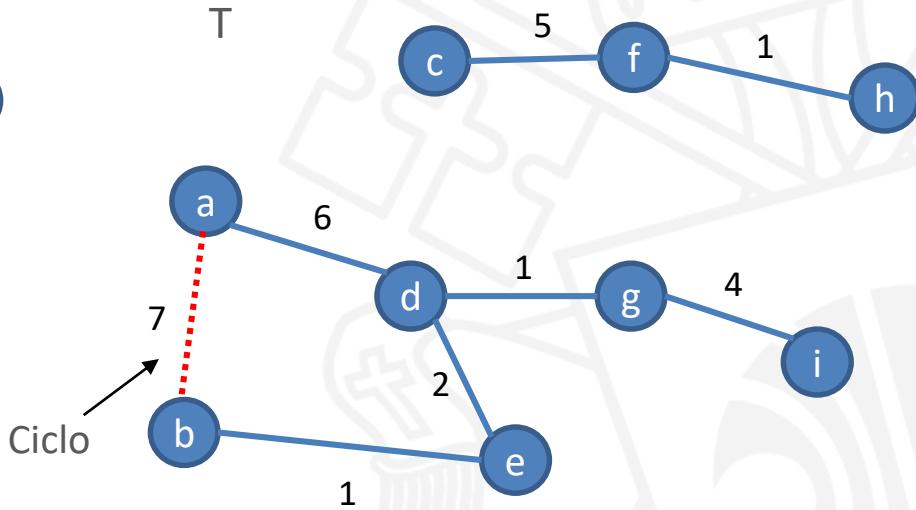
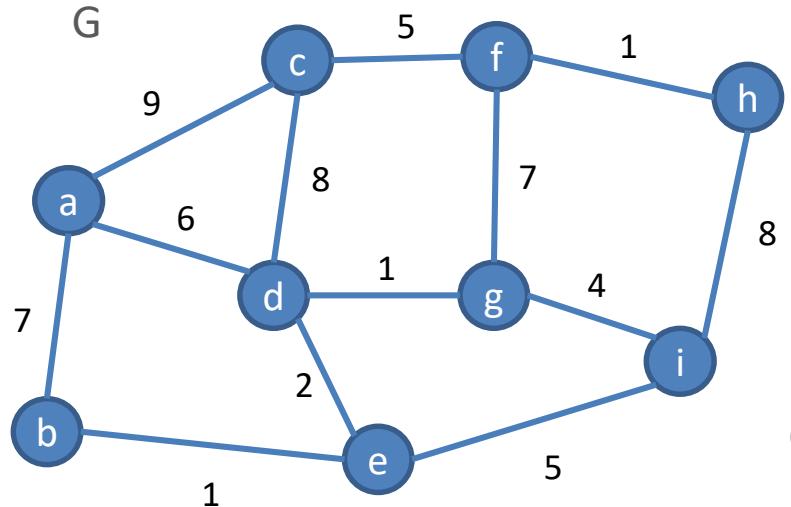
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Método de Kruskal – Exemplo



Lista ordenada de arestas: $\{\underline{b,e}\}, \{\underline{d,g}\}, \{\cancel{f,h}\}, \{\cancel{d,e}\}, \{\cancel{g,i}\}, \{\cancel{e,i}\}, \{\cancel{c,f}\}, \{\cancel{a,d}\}, \{\cancel{a,b}\}, \{\cancel{f,g}\}, \{\cancel{c,d}\}, \{\cancel{h,i}\}, \{\cancel{a,c}\}$

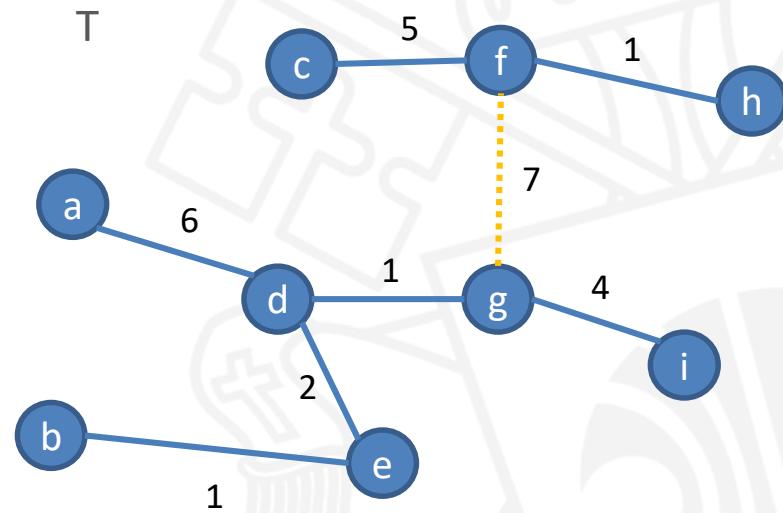
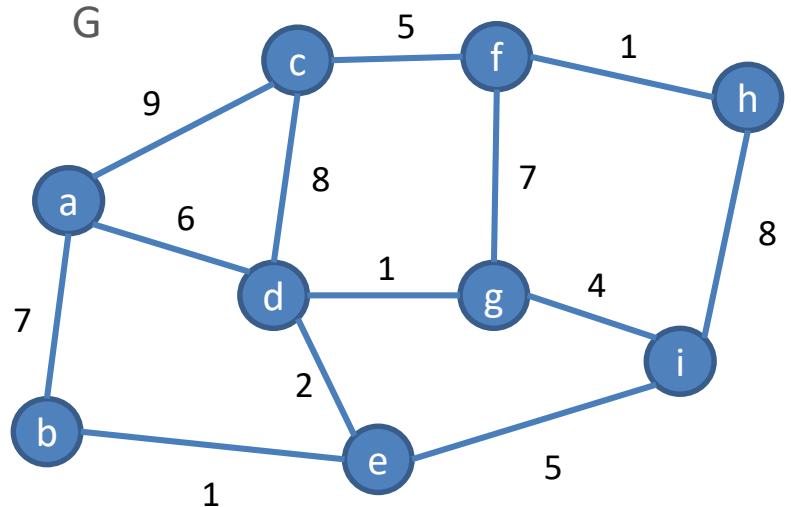
Método de Kruskal – Exemplo



Ciclo

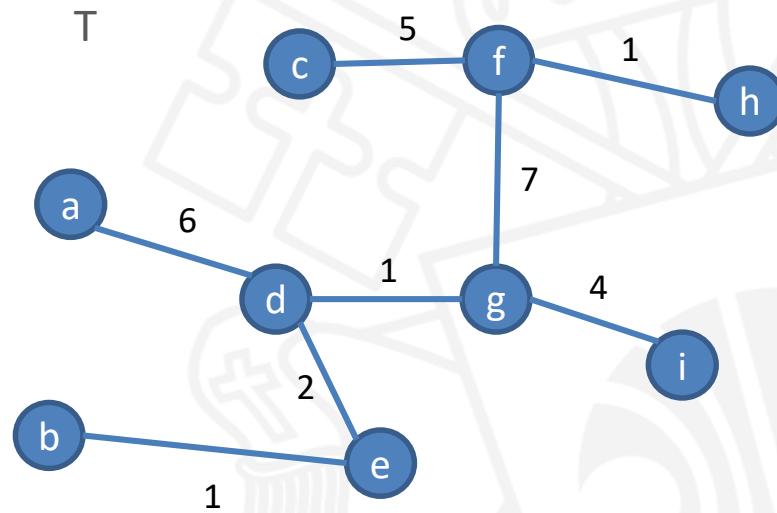
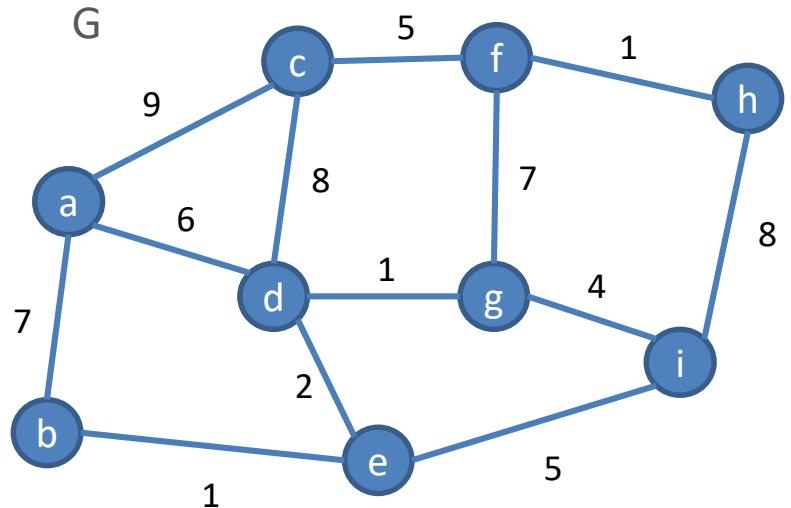
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Método de Kruskal – Exemplo



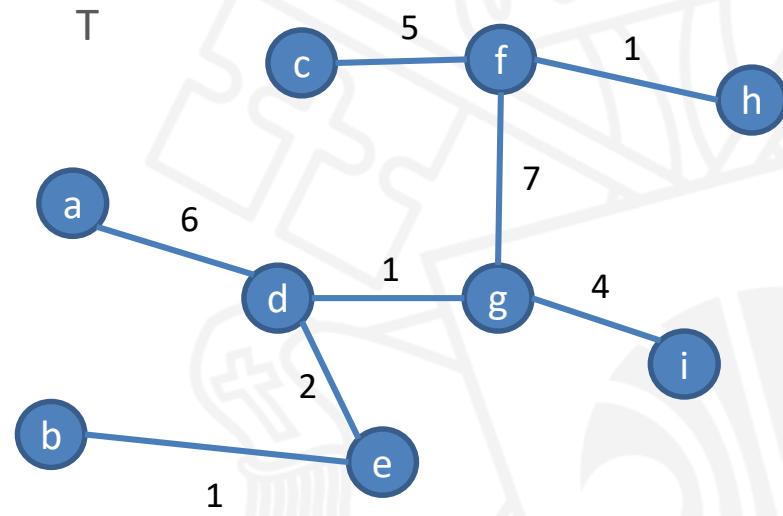
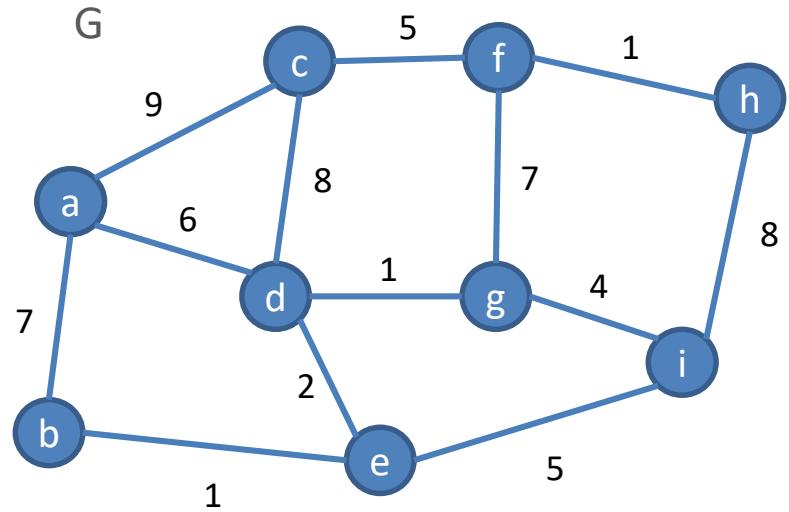
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Método de Kruskal – Exemplo



Lista ordenada de arestas: {b,e}, {d,g}, {f,h}, {d,e}, {g,i}, {e,i},
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Método de Kruskal – Exemplo



AGM $\rightarrow C(T) = 27$

