CDE Method Notes

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Introduction

Standard Model is solid but we have evidences that is not the definite theory. It is possible that there is new physics at energy scales not achievable for us now and we can only see indirect effects of this new physics. Assuming that this new physics occur in an energy scale much bigger than the SM masses, what we observe can be described by an Effective Field Theory, which comes from integrating out the heavy particles of a complete UV theory that consists in an extension for SM.

Chapter 1

Covariant Derivative Expansion Method (CDE)

For modern physics, the ultimate goal is to understand every observed phenomenon in terms of fundamental dynamics among the basic constituents of nature. The so-called "theory of everything" would unify all known interactions. However, even if such a theory were available, performing quantitative analyses directly at the most fundamental level would be of limited practical use for describing nature across all physical scales. In fact, even with the knowledge we currently possess, we often encounter similar challenges.

For example, to understand the essential features of atomic behavior in a simple way, we typically use a simplified description involving non-relativistic electrons moving in the Coulomb potential of a nucleus. This approach is far more practical than starting from the fundamental interactions of quarks inside the proton and the electron, which are described by Quantum Electrodynamics (QED). In a first approximation, atomic behavior, can be described in terms of the electron mass m_e and the fine-structure constant $\alpha \approx 1/137$, with the proton mass m_p being necessary only to estimate leading corrections. However, if we attempt to extend this simplified atomic model to obtain a useful understanding of condensed matter phenomena, we quickly run into difficulties. The complexity increases dramatically, requiring another layer of effective description.

To analyze a particular physical system embedded in a complex environment, it is essential to isolate the relevant parameters, enabling a simplified description without requiring a complete understanding of the entire system. In situations involving widely separated energy scales, it is often possible to study the low-energy dynamics independently of the high-energy details. The key idea is to identify parameters that are very large or very small compared to the characteristic energy scale of interest and to take appropriate limits. This yields a well-defined approximation that can be systematically improved by including corrections arising from the neglected scales, treated as perturbations.

In the context of elementary particle physics described by Quantum Field Theory (QFT), such low-energy approximations, with respect to some energy scale Λ, are formalized through the framework of Effective Field Theory (EFT). An EFT retains only the degrees of freedom relevant at energies well below Λ and must reproduce the same physical observables of the UV theory in within this low-energy regime. To illustrate this, consider a UV theory containing heavy and light particles associated with fields of masses $m \ll \Lambda$ and $M = \Lambda$, respectively. In physical processes occurring at energy scales much smaller than M, heavy particles do not appear in the initial or final states, as their on-shell production is energetically forbidden. Therefore, at such energy scales, the light fields constitute the relevant degrees of freedom, and the EFT is constructed entirely in terms of them. However, heavy particle still can contribute virtually in the processes trough loops, and their effects must be encoded in the coefficients of the resulting EFT Lagrangian.

Given a UV theory, an EFT can be systematically derived by integrating out the irrelevant (heavy or high-energy) degrees of freedom using the path integral formalism, followed by the imposition of matching conditions to ensure that the EFT reproduces the low-energy behavior of the full theory. In most cases, this procedure cannot be carried out exactly, and so it is implemented perturbatively. The result is an EFT Lagrangian expressed as an infinite series of higher-dimensional, non-renormalizable operators:[1] these are built from the light fields and are organized as an expansion in powers of E/Λ, where E is the characteristic energy of the process under consideration.

In this chapter, we present a method known as the Covariant Derivative Expansion (CDE) for com-

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[1] Although EFTs generally include an infinite number of terms, this does not obstruct renormalization. At any given order in the low-energy expansion, only a finite number of operators contribute, enabling a consistent order-by-order renormalization.

puting the one-loop Effective Field Theory (EFT) Lagrangian starting from a given ultraviolet (UV) theory, while explicitly preserving covariance in the final result. The discussion is primarily based on Refs.[1, 2] with additional insights drawn from Refs. [3, 4, 5, 6].

We begin the chapter with a brief review of essential elements of the path integral formalism that form the foundation for the derivation. We then introduce the core ideas behind the CDE approach, focusing on the procedure for integrating out heavy real scalar fields and the concept of matching. To illustrate the method and reinforce the main concepts, we apply it to a simple toy model that allows for a transparent, step-by-step computation.

In the fourth section, we extend the formalism to include Dirac fields thereby generalizing the computation of the EFT Lagrangian, up to one-loop order, to theories involving fermions. Finally, we apply the full framework to a more realistic setting by constructing an EFT for a simple extension of the Standard Model

1.1 Brief review on essential elements of path integral formalism

Theoretical predictions of physical observables play a crucial role in physics, as they provide the primary means of experimentally testing the validity of a theory. In the context of Quantum Field Theory, observable quantities such as scattering cross sections and decay rates are directly related to the elements of the S-matrix, which can be computed from the n-point correlation functions of the theory using the LSZ reduction procedure [7]. Within the path integral formalism, all such n-point correlation functions are encoded in the generating functional[8, 9],

$$Z[J] = \int D\phi \, e^{i \int d^d x [\mathcal{L}(\phi) + J_r(x)\phi^r(x)]},$$ (1.1.1)

and they can be obtained by taking functional derivatives with respect to the sources $J_r(x)$:

$$G^{(n)}(x_1, ..., x_n) = \frac{\int D\phi \, \phi^r(x_1)...\phi^s(x_n) e^{i \int d^d x \mathcal{L}(\phi)}}{\int D\phi \, e^{i \int d^d x \mathcal{L}(\phi)}} = \frac{(-i)^n}{Z[0]} \frac{\delta^n}{\delta J_r(x_1)...\delta J_s(x_n)} Z[J] \bigg|_{J=0}$$ (1.1.2)

where is $\mathcal{L}(\phi)$ is the Lagrangian density of the theory and $\phi^r(x)$ are the fields. These fields are not necessarily scalars, they might even be Dirac fields.

The generating functional corresponds to the sum of all vacuum-vacuum amplitudes in the presence of the source J. In terms of Feynman diagrams, this sum includes contributions from both connected and disconnected diagrams. However, diagrams that differ only by permutations of vertices within the same or across different connected subdiagrams are not counted as distinct. For a generic disconnected diagram composed of N connected components, its contribution to $Z[J]$ is given by the product of the contributions from each connected component, divided by the number of permutations of the connected components, $N!$. As a result, the sum of all diagrams simplifies to[8]:

$$Z[J] = \sum_{N=0}^{\infty} \frac{1}{N!} (iW[J])^N = e^{iW[J]} \implies W[J] = -i \ln Z[J]$$ (1.1.3)

where $W[J]$ is the generating functional for all fully connected correlation functions. In other words, $W[J]$ represents the sum of all fully connected Feynman diagrams contributing to the vacuum-to-vacuum amplitude.

For many applications, such as renormalization and matching, it is more convenient go one step further and to work with the sum of all connected one-particle-irreducible (1PI) diagrams. These diagrams are connected Feynman diagrams that cannot be separated into disconnected parts by cutting a single internal propagator. To proceed and designate the expression for the generating functional of the 1PI correlation functions, we first introduce the vacuum expectation value (VEV) of the field $\phi^r(x)$ in the presence of the external source $J_r(x)$. This VEV, commonly referred to as the classical field, is defined as the first functional derivative of $W[J]$ with respect to the source:

$$\phi_c^r(x) \equiv \frac{\delta W[J]}{J_r(x)} = -i \frac{\delta \ln Z[J]}{\delta J_r(x)}$$ (1.1.4)

Using the classical field $\phi_c^r(x)$, we define the generating functional for 1PI correlation functions, also known as the 1PI quantum action, through a Legendre transformation of $W[J]$, with $J_r(x)$ and $\phi_c^r(x)$ treated as conjugate variables[10, 8]:

$$\Gamma[\phi_c] = W[J] - \int d^dx\phi_c^r(x)J_r(x) = -i\ln Z[J] - \int d^dx\phi_c^r(x)J_r(x) \qquad (1.1.5)$$

1.1.1 Background field method

Although we previously defined the one-particle irreducible (1PI) generating functional using the vacuum expectation value of the field, this is not the only viable approach for computing it. In particular, to derive the EFT Lagrangian via matching in the upcoming sections, we will employ the Background Field Method (see [7, 11, 12]) to evaluate the 1PI quantum effective action.

In the Background Field Method, the field is split into a classical background field ϕ_b and a quantum fluctuation ϕ' component, as follows:

$$\phi^r(x) = \phi_b^r(x) + \phi'^r(x) \qquad (1.1.6)$$

where background field is defined to satisfy the classical equation of motion in the presence of the external source:

$$\left.\frac{\delta\mathcal{L}}{\delta\phi^r}\right|_{\phi^r=\phi_b^r} + J_r(x) = 0 \qquad (1.1.7)$$

We now expand the Lagrangian, including the source term, around the background field:

$$\mathcal{L}[\phi] + J_r\phi^r = \mathcal{L}[\phi_b] + J_r\phi_b^r - \frac{1}{2}\phi'^r\left.\frac{\delta^2\mathcal{L}}{\delta\phi^r\delta\phi^l}\right|_{\phi=\phi_b}\phi'^l + ... \qquad (1.1.8)$$

where there is no linear term in ϕ' because the background field satisfies the equation of motion. Substituting this expansion into the generating functional yields:

$$Z[J] = e^{i\int d^dx(\mathcal{L}[\phi_b]+J_r\phi_b^r)}\int D\phi \exp\left\{-\frac{i}{2}\int d^dx\phi'^r\left.\frac{\delta^2\mathcal{L}}{\delta\phi^r\delta\phi^l}\right|_{\phi=\phi_b}\phi'^l + ...\right\} \qquad (1.1.9)$$

In this work, we are interested only in the one-loop contributions, which correspond to the terms quadratic in the quantum fluctuations. This can be understood diagrammatically: the background field enters as tree-level lines, while the quantum fluctuation propagates internally and forms loops. Thus, terms higher than quadratic in ϕ' contribute only at two-loop order or beyond. Truncating at one-loop order, we obtain:

$$Z[J] \approx e^{i\int d^dx(\mathcal{L}[\phi_b]+J_r\phi_b^r)}\int D\phi \exp\left\{-\frac{i}{2}\int d^dx\phi'^r\left.\frac{\delta^2\mathcal{L}}{\delta\phi^r\delta\phi^l}\right|_{\phi=\phi_b}\phi'^l\right\}$$
$$= Ne^{i\int d^dx(\mathcal{L}[\phi_b]+J_r\phi_b^r)}\left[\det\left(-\left.\frac{\delta^2\mathcal{L}}{\delta\phi^r\delta\phi^l}\right|_{\phi=\phi_b}\right)\right]^{-c_s} \qquad (1.1.10)$$

where we have used the standard result for a Gaussian path integral [8]. Here, $c_s = \frac{1}{2}$ for bosonic fields and $c_s = -1$ for fermionic fields, and N is an irrelevant normalization constant.

With this result we can calculate the generating functional of all connected n-point correlation functions using Eq. (1.1.3):

$$W[J] = \int d^dx\left(\mathcal{L}[\phi_b] + J_r\phi_b^r\right) + ic_s\ln\det\left(-\left.\frac{\delta^2\mathcal{L}}{\delta\phi^r\delta\phi^l}\right|_{\phi=\phi_b}\right) \qquad (1.1.11)$$

Subtracting $\int d^dxJ_r\phi_b^r$ at each side and defining:

$$\Gamma[\phi_b] \equiv W[J] - \int d^dxJ_r\phi_b^r = \int d^dx\mathcal{L}[\phi_b] + ic_s\ln\det\left(-\left.\frac{\delta^2\mathcal{L}}{\delta\phi^r\delta\phi^l}\right|_{\phi=\phi_b}\right) \qquad (1.1.12)$$

This object is equivalent, up to one-loop order, to the $1PI$ generating functional defined by Eq. (1.1.5). To see this, consider the relationship between the vacuum expectation value ϕ_c^r the background field ϕ_b^r, obtained by functional differentiating Eq. (1.1.11) with respect to the source:

$$\frac{\delta W[J]}{\delta J_r} = \phi_b^r + ic_s \frac{\delta}{\delta J} \ln \det \left(-\frac{\delta^2 \mathcal{L}}{\delta \phi^r \delta \phi^l} \bigg|_{\phi=\phi_b} \right) \tag{1.1.13}$$

By Eq. (1.1.4), the left-hand side equals ϕ_c^r, leading to:

$$\phi_c^r = \phi_b^r + (\text{quantum corrections}) \tag{1.1.14}$$

This result shows that the vacuum expectation value ϕ_c^r differs from the background field ϕ_b^r by quantum corrections that start at one-loop order. Therefore, to leading order, we can identify $\phi_b^r \approx \phi_c^r$. Moreover, since the background field satisfies the classical equation of motion, the action $S[\phi_b] = \int d^d x, \mathcal{L}[\phi_b]$ is stationary at ϕ_b. As discussed in [7], this implies:

$$S[\phi_c] + J_r \phi_c^r - S[\phi_b] - J_r \phi_b^r = \text{Second order loop corrections} \tag{1.1.15}$$

As a result, since we are only interested in the 1PI quantum action up to one-loop order, functional $\Gamma[\phi_b]$ defined in Eq.(1.1.12), is equivalent to the $1PI$ generating functional defined in Eq. (1.1.5). Consequently, we can use the Background Field Method to compute the $1PI$ generating functional to this order.

1.2 Matching for real scalar fields

Consider a UV-complete theory with a set of light (ϕ) and heavy (Φ) scalar fields whose interactions are described by the $\mathcal{L}_{UV}$ Lagrangian density. As discussed in the previous section, every correlation function of this theory are encoded in generating functional:

$$Z_{UV}[J_\Phi, J_\phi] = \int D\Phi D\phi e^{i \int d^d x [\mathcal{L}_{UV}(\phi,\Phi) + J_\phi \phi + J_\Phi \Phi]}. \tag{1.2.1}$$

At energy scales below the heavy fields mass M, the light fields are the only relevant degrees of freedom. Consequently, all physical processes at these scales must be described by correlation functions involving only light external fields, which are generated by setting $J_\Phi = 0$ in the UV generating functional, i.e.,

$$Z_{UV}[J_\Phi = 0, J_\phi] = \int D\Phi D\phi e^{i \int d^d x [\mathcal{L}_{UV}(\phi,\Phi) + J_\phi \phi]}. \tag{1.2.2}$$

On the other hand, at low energies, the same physics can be described using an EFT, where the heavy fields Φ are integrated out. In this approach, the interactions of the light fields are described by a local Lagrangian density $\mathcal{L}_{EFT}(\phi)$, with the corresponding EFT generating functional given by:

$$Z_{EFT}[J_\phi] = \int D\phi e^{i \int d^d x [\mathcal{L}_{EFT}(\phi) + J_\phi \phi]}. \tag{1.2.3}$$

Since both the UV theory and its EFT describe the same low-energy physics, all observable predictions must agree. A natural first attempt at ensuring this agreement is to impose the matching condition:

$$Z_{EFT}[J_\phi] = Z_{UV}[J_\Phi = 0, J_\phi] \tag{1.2.4}$$

However, this condition is stronger than necessary, as it requires equality not only for on-shell but also for off-shell correlation functions. In practice, we only need to ensure that the S-matrix elements match between the two theories, meaning that on-shell correlation functions must be identical. Furthermore, these generating functionals are highly complex objects, making direct matching impractical.

To address these issues, the matching condition is instead imposed on the one-light-particle-irreducible (1LPI) generating functional in the UV theory and the 1PI generating functional of the EFT[1]:

$$\Gamma_{EFT}[\phi_b] = \Gamma_{L,UV}[\phi_b] \tag{1.2.5}$$

where ϕ_b is the classical background field. However, in practice, it is generally not possible to find an effective Lagrangian $\mathcal{L}_{EFT}(\phi)$ that exactly satisfies this equation. Instead, we impose the matching condition perturbatively, order-by-order in the loop expansion. In particular, throughout this chapter, we will restrict the matching condition to one-loop order, which leads to the conditions:

$$\Gamma_{EFT}^{\text{tree}}[\phi_b] = \Gamma_{L,UV}^{\text{tree}}[\phi_b], \tag{1.2.6}$$

$$\Gamma_{EFT}^{\text{1-loop}}[\phi_b] = \Gamma_{L,UV}^{\text{1-loop}}[\phi_b] \tag{1.2.7}$$

where this is imposed for a renormalization scale $\mu = M$.

1.2.1 Calculating the 1LPI generating functional

To compute $\Gamma_{L,UV}[\phi_b]$, we begin with the generating functional of the UV-complete theory, setting the source for the heavy field to zero:

$$Z_{UV}[J_\Phi = 0, J_\phi] = \int D\Phi D\phi\, e^{i \int d^d x [\mathcal{L}_{UV}(\phi,\Phi) + J_\phi \phi]}. \tag{1.2.8}$$

Next, we decompose each field into a classical background configuration, labeled by a subscript "b", and a quantum fluctuation, labeled by a prime. That is, we write:

$$\phi = \phi_b + \phi', \quad \Phi = \Phi_b + \Phi' \tag{1.2.9}$$

The background fields satisfy the classical equations of motion (EOM) in the presence of the sources, which are given by [1]:

$$\left.\frac{\delta \mathcal{L}_{UV}(\phi,\Phi)}{\delta \phi}\right|_{\phi=\phi_b} + J_\phi = 0, \quad \left.\frac{\delta \mathcal{L}_{UV}(\phi,\Phi)}{\delta \Phi}\right|_{\Phi=\Phi_b} = 0 \tag{1.2.10}$$

where we already set $J_\Phi = 0$ due our interest in energy scales smaller than the heavy fields mass.

Diagrammatically, the background fields correspond to tree lines in Feynman diagrams, while internal lines within loops arise from quantum fluctuations. As a result, terms higher than quadratic in the quantum fields generate interaction vertices that contribute only at higher-loop orders. Since we aim to perform the matching up to a specific order in the loop expansion, we can isolate the relevant contributions to $Z_{UV}[J_\Phi = 0, J_\phi]$ by counting the powers the quantum fluctuations fields in the argument of the exponential.

A systematic approach to achieving this is to Taylor expand $\mathcal{L}_{UV}[\Phi,\phi] + J_\phi \phi$ around $\phi = \phi_b$, decompose the fields into background and fluctuation components, and truncate the expansion at the desired order in the quantum fluctuations, or equivalently in the desired loop order. Following this procedure, we obtain:

$$\mathcal{L}_{UV}[\Phi,\phi] + J_\phi \phi = \mathcal{L}_{UV}[\Phi_b,\phi_b] + J_\phi \phi_b - \frac{1}{2} \begin{pmatrix} \Phi'^T & \phi'^T \end{pmatrix} \mathcal{Q}_{UV}[\Phi_b,\phi_b] \begin{pmatrix} \Phi' \\ \phi' \end{pmatrix} + \dots \tag{1.2.11}$$

where the absence of a linear term in ϕ' or Φ' follows from the fact that the background fields satisfy the equations of motion (Eq. (1.2.10)). Furthermore, the quadratic operator $\mathcal{Q}_{UV}[\Phi_b,\phi_b]$, also know as fluctuation operator, is defined as:

$$\mathcal{Q}_{UV}[\Phi_b,\phi_b] = \begin{pmatrix} \Delta_H & X_{HL} \\ X_{LH} & \Delta_L \end{pmatrix} = \begin{pmatrix} -\frac{\delta^2 \mathcal{L}_{UV}}{\Phi'^2}[\Phi_b,\phi_b] & -\frac{\delta^2 \mathcal{L}_{UV}}{\Phi'\phi'}[\Phi_b,\phi_b] \\ -\frac{\delta^2 \mathcal{L}_{UV}}{\phi'\Phi'}[\Phi_b,\phi_b] & -\frac{\delta^2 \mathcal{L}_{UV}}{\phi'^2}[\Phi_b,\phi_b] \end{pmatrix} \tag{1.2.12}$$

At zeroth order, the expansion depends only on the background fields and contributes to the tree-level generating functional. Therefore, truncating the series in the first term and substituting in Eq. (1.2.8) give us:

$$Z_{UV}^{\text{tree}}[J_\Phi = 0, J_\phi] = Ne^{i \int d^d x [\mathcal{L}_{UV}[\Phi_b, \phi_b] + J_\phi \phi_b]} \tag{1.2.13}$$

where N is some renormalization constant which is irrelevant in practical applications.

To include one-loop effects, we must retain the quadratic term in the fluctuation fields. Substituting Eq. (1.2.11) into Eq. (1.2.8) and truncating at quadratic order, we obtain:

$$Z_{UV}[J_\Phi = 0, J_\phi] = Z_{UV}^{\text{tree}}[J_\Phi = 0, J_\phi] \int D\phi' D\Phi' \exp\left\{ -\frac{i}{2} \int d^d x \left[\begin{pmatrix} \Phi'^T & \phi'^T \end{pmatrix} \mathcal{Q}_{UV}[\Phi_b, \phi_b] \begin{pmatrix} \Phi' \\ \phi' \end{pmatrix} \right] \right\} \tag{1.2.14}$$

Although the path integral could, in principle, be evaluated using Gaussian integration, it is useful to first bring the fluctuation operator into a block-diagonal form, decoupling the heavy and light path integrals. This can be achieved by performing shifts, with unit Jacobian determinant, in the quantum fields. Specifically, we can choose a field transformation which transfers the information in the mixed components[2] (X_{LR} and X_{RL}) to the heavy fields block (Δ_H), while leaving Δ_L untouched. This approach simplifies the matching procedure by ensuring that all heavy particle effects at one-loop order are encapsulated within the result of the heavy field integral. Additionally, since pure light loops contribution must trivially appears in both the UV theory and the EFT, it cancels out when imposing the matching conditions. Furthermore, this decoupling is particularly crucial when the heavy and light fields obey different statistics, ensuring a consistent treatment of their fluctuations.

A suitable choice for the fields transformation that brings the fluctuation operator into the desired block-diagonal form is:

$$V = \begin{pmatrix} \mathbb{1} & 0 \\ -\Delta_L^{-1} X_{LH} & \mathbb{1} \end{pmatrix} \tag{1.2.15}$$

Applying this transformation to the fluctuation operator, we obtain:

$$V^\dagger \mathcal{Q}_{UV}[\hat{\Phi}_c[\phi_b], \phi_b] V = \begin{pmatrix} \Delta_H - X_{HL} \Delta_L^{-1} X_{LH} & 0 \\ 0 & \Delta_L \end{pmatrix} \tag{1.2.16}$$

The functional integrals in Eq. (1.2.14) can now be carried out easily[1]:

$$Z_{UV}[J_\Phi = 0, J_\phi] = Z_{UV}^{\text{tree}}[J_\Phi = 0, J_\phi] \int D\Phi' \exp\left\{ -\frac{i}{2} \int d^d x \Phi'^T \left(\Delta_H - X_{HL} \Delta_L^{-1} X_{LH} \right) \Phi' \right\}$$

$$\int D\phi' \exp\left\{ -\frac{i}{2} \int d^d x \phi'^T \Delta_L \phi' \right\}$$

$$= N' Z_{UV}^{\text{tree}}[J_\Phi = 0, J_\phi] \left[\det \left(\Delta_H - X_{HL} \Delta_L^{-1} X_{LH} \right) \right]^{-\frac{1}{2}} \left[\det \Delta_L \right]^{-\frac{1}{2}} \tag{1.2.17}$$

where N' is another renormalization constant with irrelevant physical effects.

From $Z_{UV}[J_\Phi = 0, J_\phi]$, we obtain the generating functional of connected correlation functions, $W_{UV}[J_\Phi = 0, J_\phi]$, by taking its natural logarithm and multiplying by $-i$. Finally, we obtain the 1LPI quantum action by performing a Legendre transform on W_{UV}, treating ϕ_b and J_ϕ as conjugate variables. Up to an irrelevant constant, this yields:

$$\Gamma_{L,UV}[\phi_b] = -i \ln Z_{UV}[J_\Phi = 0, J_\phi] - \int d^d x J_\phi(x) \phi_b(x)$$

$$= -i \ln Z_{UV}^{\text{tree}}[J_\Phi = 0, J_\phi] - \int d^d x J_\phi(x) \phi_b(x) + \frac{i}{2} \ln \det \left(\Delta_H - X_{HL} \Delta_L^{-1} X_{LH} \right) + \frac{i}{2} \ln \det \Delta_L \tag{1.2.18}$$

To explicitly separate contributions at different loop orders, we define:

$$\Gamma_{L,UV}[\phi_b] \equiv \Gamma_{L,UV}^{\text{tree}}[\phi_b] + \Gamma_{L,UV}^{\text{1-loop}}[\phi_b] \tag{1.2.19}$$

where:

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[2]Equivalently, one-loop diagrams containing both heavy and light lines.

$$\Gamma^{\text{tree}}_{L,UV}[\phi_b] = -i \ln Z^{\text{tree}}_{UV}[J_\Phi = 0, J_\phi] - \int d^d x J_\phi(x)\phi_b(x) = \int d^d x \mathcal{L}_{UV}[\Phi_b, \phi_b], \tag{1.2.20}$$

$$\Gamma^{\text{1-loop}}_{L,UV}[\phi_b] = \frac{i}{2} \ln \det \left(\Delta_H - X_{HL}\Delta_L^{-1}X_{LH} \right) + \frac{i}{2} \ln \det \Delta_L \tag{1.2.21}$$

Notice that on the right-hand side of the 1LPI equation, the functional dependence is explicitly in ϕ_b. whereas on the left-handed side, it appears to depend also on Φ_b. However, since we have set the source for the heavy field to zero, Φ_b satisfies the equation of motion given by Eq. (1.2.10), making it a function of ϕ_b alone. This allows us to eliminate the explicit dependence on Φ_b in the 1LPI quantum action by solving its equation of motion with ϕ_b treated as the background.

The solution $\Phi_b|_{J_\Phi=0}$ may be non-local due to the presence of inverse differential operators, whereas the Lagrangian density must consist in a sum of local operators. Therefore, this substitution must be handled carefully. When the solution is non-local, we expand it in inverse powers of the heavy mass and truncate at a finite order to ensure locality. Additionally, to clearly distinguish the heavy field from its solution in the absence of the source, we define:

$$\Phi_c[\phi_b] \equiv \Phi_b|_{J_\Phi=0}. \tag{1.2.22}$$

After performing the expansion in $1/M$, we denote the resulting local approximation of $\Phi_c[\phi_b]$ with a hat to emphasize its locality. Therefore, using the EOM of the heavy fields in Eqs. (1.2.20) and (1.2.21):

$$\Gamma^{\text{tree}}_{L,UV}[\phi_b] = \int d^d x \mathcal{L}_{UV}[\hat{\Phi}_c[\phi_b], \phi_b], \tag{1.2.23}$$

$$\Gamma^{\text{1-loop}}_{L,UV}[\phi_b] = \frac{i}{2} \ln \det \left(\Delta_H - X_{HL}\Delta_L^{-1}X_{LH} \right) + \frac{i}{2} \ln \det \Delta_L \tag{1.2.24}$$

where we omitted the arguments $[\hat{\Phi}_c[\phi_b], \phi_b]$ of the operators in the second equation in order to keep a clear notation.

1.2.2 Calculating the 1PI generating functional of the EFT

Having obtained the 1LPI quantum action of the UV theory, $\Gamma_{L,UV}[\phi_b]$, up to one-loop order, we now turn to the calculation of the 1PI generating functional of the EFT to impose the matching condition. The procedure closely follows the steps from the previous section. We begin with the generating functional of the EFT:

$$Z_{EFT}[J_\phi] = \int D\phi e^{i \int d^d x [\mathcal{L}_{EFT}[\phi] + J_\phi \phi]} \tag{1.2.25}$$

Here, we are working within the EFT framework and consequently only the light fields ϕ remains as relevant degrees of freedom. Our goal is to determine $\mathcal{L}_{EFT}[\phi]$ such that the matching condition (Eq. (1.2.5)) holds up to a given loop order. To systematically organize the contributions, we write the EFT Lagrangian distinguishing each loop order:

$$\mathcal{L}_{EFT}[\phi] = \mathcal{L}^{\text{tree}}_{EFT}[\phi] + \mathcal{L}^{\text{1-loop}}_{EFT}[\phi] + ... \tag{1.2.26}$$

where $\mathcal{L}^{\text{tree}}_{EFT}[\phi]$ and $\mathcal{L}^{\text{1-loop}}_{EFT}[\phi]$ contain effective operators at tree and one-loop level, respectively. Furthermore, we also split the fields ϕ into background and quantum fluctuations, Taylor expand the Lagrangian plus source around $\phi = \phi_b$, truncate the series at quadratic order in the quantum fluctuations, and substitute in the generating functional (Eq. (1.2.25)). Up to one-loop order, we obtain:

$$Z_{EFT}[J_\phi] = e^{i \int d^d x [\mathcal{L}^{\text{tree}}_{EFT}[\phi_b] + \mathcal{L}^{\text{1-loop}}_{EFT}[\phi_b] + J_\phi \phi_b]} \int D\phi' \exp\left\{ -\frac{i}{2} \int d^d x \left[\phi'^T \mathcal{Q}_{EFT}[\phi_b] \phi' \right] \right\}$$

$$= e^{i \int d^d x [\mathcal{L}^{\text{tree}}_{EFT}[\phi_b] + \mathcal{L}^{\text{1-loop}}_{EFT}[\phi_b] + J_\phi \phi_b]} \left(\det \mathcal{Q}_{EFT}[\phi_b] \right)^{-\frac{1}{2}} \tag{1.2.27}$$

where the quadratic operator $\mathcal{Q}_{EFT}[\phi_b]$ is:

$$\mathcal{Q}^{\text{tree}}_{EFT}[\phi_b] \equiv -\frac{\delta^2 \mathcal{L}^{\text{tree}}_{EFT}[\phi]}{\delta \phi^2}\Big|_{\phi=\phi_b} \tag{1.2.28}$$

Observe that, up to one-loop order, only $\mathcal{L}_{EFT}^{\text{tree}}[\phi]$ is considered in the computation of the quadratic operator, as the terms from $\mathcal{L}_{EFT}^{\text{1-loop}}[\phi]$ enter the calculation at two-loop order after performing the Gaussian integration.

Given the generating functional of the EFT, the 1PI quantum action can be calculated by the same Legendre transform of the UV case, resulting in:

$$\Gamma_{EFT}[\phi_b] = \int d^dx \mathcal{L}_{EFT}^{tree}[\phi_b] + \int d^dx \mathcal{L}_{EFT}^{1-loop}[\phi_b] + \frac{i}{2}\ln\det\mathcal{Q}_{EFT}[\phi_b] \tag{1.2.29}$$

Since we have carefully kept track of the contributions at each loop order, we can now explicitly identify the tree-level and one-loop terms in the 1PI quantum action:

$$\Gamma_{EFT}^{\text{tree}}[\phi_b] = \int d^dx \mathcal{L}_{EFT}^{tree}[\phi_b], \tag{1.2.30}$$

$$\Gamma_{EFT}^{\text{1-loop}}[\phi_b] = \int d^dx \mathcal{L}_{EFT}^{1-loop}[\phi_b] + \frac{i}{2}\ln\det\mathcal{Q}_{EFT}[\phi_b] \tag{1.2.31}$$

Notice that the one-loop contribution to the 1PI quantum action in the EFT consists of two terms, whereas in the UV theory, there is only one. This distinction arises because the EFT includes effective operators that are already of one-loop order and contribute at tree level (first term), in addition to the one-loop corrections generated from the tree-level EFT operators (second term).

1.2.3 Matching $\Gamma_{L,UV}[\phi_b]$ and $\Gamma_{EFT}[\phi_b]$

With the tree and one-loop contributions to the 1LPI generating functional of the UV theory and the 1PI quantum action of the EFT at hand, we can now impose the matching condition order by order in the loop expansion. At tree level, substituting Eqs. (1.2.23) and (1.2.30) into Eq. (1.2.6) yields:

$$\mathcal{L}_{EFT}^{tree}[\phi_b] = \mathcal{L}_{UV}[\hat{\Phi}_c[\phi_b], \phi_b] \tag{1.2.32}$$

Thus, the tree-level EFT Lagrangian is simply the UV Lagrangian evaluated with the light fields set to their classical background values and the heavy fields replaced by the local expansion of the EOM solution in the absence of a heavy source.

At one-loop level, substituting Eqs. (1.2.24) and (1.2.31) into Eq. (1.2.7) and rearranging, we obtain:

$$\int d^dx \mathcal{L}_{EFT}^{1-loop}[\phi_b] = \frac{i}{2}\ln\det\left(\Delta_H - X_{HL}\Delta_L^{-1}X_{LH}\right) + \frac{i}{2}\ln\det\Delta_L - \frac{i}{2}\ln\det\mathcal{Q}_{EFT}[\phi_b] \tag{1.2.33}$$

To proceed, we must rewrite the quadratic operator of the EFT in term of elements in the first and second term in the equation above. Recall that only the tree-level part of the EFT Lagrangian contributes to $\mathcal{Q}_{EFT}^{\text{tree}}[\phi_b]$ at one-loop. Therefore, substituting Eq.(1.2.32) into (1.2.28) and applying the chain rule:

$$\mathcal{Q}_{EFT}^{\text{tree}}[\phi_b] = -\left[\frac{\delta}{\delta\phi}\left(\frac{\delta\mathcal{L}_{UV}[\hat{\Phi}_c[\phi], \phi]}{\delta\phi}\right)\right]_{\phi=\phi_b} = -\left[\frac{\delta}{\delta\phi}\left(\frac{\delta\mathcal{L}_{UV}[\hat{\Phi}_c[\phi], \phi]}{\delta\phi} + \frac{\delta\hat{\Phi}_c[\phi]}{\delta\phi}\frac{\delta\mathcal{L}_{UV}[\hat{\Phi}_c[\phi], \phi]}{\delta\Phi}\right)\right]_{\phi=\phi_b} \tag{1.2.34}$$

Since $\Phi_c[\phi]$ satisfies the heavy-field equation of motion (Eq. (1.2.10)), the second term on the right-hand side vanishes. Applying the chain rule again to the first term, we obtain:

$$\mathcal{Q}_{EFT}^{\text{tree}}[\phi_b] = -\left[\frac{\delta^2\mathcal{L}_{UV}[\hat{\Phi}_c[\phi], \phi]}{\delta\phi^2} + \frac{\delta\hat{\Phi}_c[\phi]}{\delta\phi}\frac{\delta^2\mathcal{L}_{UV}[\hat{\Phi}_c[\phi], \phi]}{\delta\Phi\phi'}\right]_{\phi=\phi_b} = \Delta_L + \left.\frac{\delta\hat{\Phi}_c[\phi]}{\delta\phi}\right|_{\phi=\phi_b}X_{HL} \tag{1.2.35}$$

To further simplify, we use again the equation of motion for the heavy fields:

$$0 = \left[\frac{\delta}{\delta\phi}\left(\frac{\delta\mathcal{L}_{UV}[\hat{\Phi}_c[\phi]]}{\delta\Phi}\right)\right]_{\phi=\phi_b} = \left[\frac{\delta^2\mathcal{L}_{UV}[\hat{\Phi}_c[\phi], \phi]}{\delta\phi\delta\Phi} + \frac{\delta\hat{\Phi}_c[\phi]}{\delta\phi}\frac{\delta^2\mathcal{L}_{UV}[\hat{\Phi}_c[\phi], \phi]}{\delta\Phi^2}\right]_{\phi=\phi_b} = -X_{LH} - \left.\frac{\delta\hat{\Phi}_c[\phi]}{\delta\phi}\right|_{\phi=\phi_b}\Delta_H \tag{1.2.36}$$

Solving for $\left.\frac{\delta\hat{\Phi}_c[\phi]}{\delta\phi}\right|_{\phi=\phi_b}$:

$$\left.\frac{\delta\hat{\Phi}_c[\phi]}{\delta\phi}\right|_{\phi=\phi_b} = -X_{LH}\hat{\Delta}_H^{-1} \tag{1.2.37}$$

where $\hat{\Delta}_H^{-1}$ represents the local operator expansion of Δ_H^{-1}. Substituting this result into Eq. (1.2.35), we obtain:

$$\mathcal{Q}_{EFT}^{\text{tree}}[\phi_b] = \Delta_L - X_{LH}\hat{\Delta}_H^{-1}X_{HL} \tag{1.2.38}$$

Taking the natural logarithm of the determinant:

$$\begin{aligned}
-i\ln\det\mathcal{Q}_{EFT}^{\text{tree}}[\phi_b] &= -\frac{i}{2}\ln\det\left(\Delta_L - X_{LH}\hat{\Delta}_H^{-1}X_{HL}\right)\\
&= -i\left[\ln\det\Delta_L + \ln\det\left(\mathbb{1} - \Delta_L^{-1}X_{LH}\hat{\Delta}_H^{-1}X_{HL}\right)\right]\\
&= -i\left[\ln\det\Delta_L + \ln\det\left(\mathbb{1} - \hat{\Delta}_H^{-1}X_{HL}\Delta_L^{-1}X_{LH}\right)\right]\\
&= -i\left[\ln\det\Delta_L - \ln\det\hat{\Delta}_H + \ln\det\left(\hat{\Delta}_H - X_{HL}\Delta_L^{-1}X_{LH}\right)\right]
\end{aligned} \tag{1.2.39}$$

Now, substituting Eq. (1.2.39) into (1.2.33), we obtain the one-loop contribution to the effective Lagrangian:

$$\int d^dx\,\mathcal{L}_{EFT}^{1-loop}[\phi_b] = \frac{i}{2}\ln\det\hat{\Delta}_H + \frac{i}{2}\left[\ln\det\left(\Delta_H - X_{HL}\Delta_L^{-1}X_{LH}\right) - \ln\det\left(\hat{\Delta}_H - X_{HL}\Delta_L^{-1}X_{LH}\right)\right] \tag{1.2.40}$$

Notice that the contributions from pure light loops, i.e., $\ln\det\Delta_L$, cancels out. This is expected, as the one-loop effective operators should only encode the indirect contributions from mixed and pure heavy field loops.

1.2.4 Application of the expansion by regions method

Although Eq. (1.2.40) provides a formal expression for the one-loop effective Lagrangian, directly computing it is often impractical. The presence of natural logarithms of determinants leads to d-dimensional loop integrals, which become difficult to evaluate exactly using standard dimensional regularization, especially when multiple mass and kinematic scales are involved. A more effective approach is the expansion by regions method [13, 14, 15], which reconstructs the loop integral result in an expanded form. This method proceeds as follows:

1. Divide the space of the loop momenta into regions according with the masses or kinematical scales and, in every region expand the integrand into a Taylor series with respect to the parameters that are small there.

2. Use dimensional regularization to integrate the integrand, expanded in the appropriate way in every region, over the whole integration domain of the loop momenta.

3. Set to zero any scaleless integral.

4. Sum the result of the integral in each region.

When integrating out heavy fields of a UV theory, we have two distinct mass scales: the light fields mass m and the heavy fields mass M. It is therefore natural to assume that the two regions of the loop momenta space relevant to this problem are

- the hard region, where $q \sim M$,

- and the soft region where $q \sim m$.

Here, $q \sim M$ mean that all components of the loop momentum q are of the order of the mass M, and similarly for $q \sim m$. As a consequence of the expansion by regions method, any "ln det" can be decomposed as[14]:

$$\ln \det X = \ln \det X|_{\text{hard}} + \ln \det X|_{\text{soft}}, \qquad (1.2.41)$$

where hard and the soft region contributions are obtained by first expanding the integrand for $|q^2| \sim M^2 \gg m^2$ and $|q^2| \sim m^2 \ll M^2$, respectively, and then integrating over the full loop momentum space, as specified by the method steps. Applying this decomposition to the logarithms of determinants appearing in Eq. (1.2.40), we obtain:

$$\begin{cases} \ln \det \left(\Delta_H - X_{HL} \Delta_L^{-1} X_{LH} \right) = \ln \det \left(\Delta_H - X_{HL} \Delta_L^{-1} X_{LH} \right)_{\text{hard}} + \ln \det \left(\Delta_H - X_{HL} \Delta_L^{-1} X_{LH} \right)_{\text{soft}} \\ \ln \det \left(\hat{\Delta}_H - X_{HL} \Delta_L^{-1} X_{LH} \right) = \ln \det \left(\hat{\Delta}_H - X_{HL} \Delta_L^{-1} X_{LH} \right)_{\text{soft}} \\ \ln \det \det \hat{\Delta}_H = \ln \det \hat{\Delta}_H|_{\text{soft}} = 0 \end{cases}$$

$$(1.2.42)$$

In the second and third equation there is no hard region contribution once the replacement of Δ_H by its local expansion $\hat{\Delta}_H$ in powers of $1/M$ is equivalent to isolating only the soft region. Furthermore, $\ln \det \hat{\Delta}_H|_{\text{soft}}$ vanishes because, for pure heavy loops, expanding in the soft region produces scaleless integrals $(M^{-c} \int d^d q, q^n = 0)$, which are set to zero according to the third step of the expansion by regions method.

Substituting these results into the one-loop EFT Lagrangian expression (Eq. (1.2.40)), we obtain:

$$\int d^d x \mathcal{L}_{EFT}^{1-loop}[\phi_b] = \frac{i}{2} \ln \det \left(\Delta_H - X_{HL} \Delta_L^{-1} X_{LH} \right)|_{\text{hard}} \qquad (1.2.43)$$

Since pure light loops do not receive contributions from the hard region, i.e., $\ln \det \Delta_L|_{\text{hard}} = 0$, the right-hand side of the equation above corresponds to the hard contribution of the one-loop 1LPI generating functional of the UV theory:

$$\int d^d x \mathcal{L}_{EFT}^{1-loop}[\phi_b] = \Gamma_{L,UV}^{\text{1-loop}}|_{\text{hard}} \qquad (1.2.44)$$

This result demonstrates that the one-loop EFT Lagrangian is entirely determined by the hard contribution, i.e., by highly virtual loops with momenta outside the EFT regime ($q \sim M \gg m$). Physically, this means that the effective operators in $\mathcal{L}_{\text{EFT}}^{\text{1-loop}}[\phi_b]$ encode the short-distance behavior of the full theory, while the long-distance effects, governed by the soft region, are already captured by the tree-level EFT Lagrangian.

1.2.5 Evaluating functional determinants

To evaluate the one-loop EFT Lagrangian via Eq. (1.2.43), it is convenient to transform the determinant into a trace using a standard linear algebra identity[8]:

$$\ln \det X = \text{Tr} \ln X. \qquad (1.2.45)$$

Applying this identity, we obtain:

$$\int d^d x, \mathcal{L}_{\text{EFT}}^{\text{1-loop}}[\phi_b] = \frac{i}{2} \text{Tr} \ln \left(\Delta_H - X_{HL} \Delta_L^{-1} X_{LH} \right) \Big|_{\text{hard}}, \qquad (1.2.46)$$

where Tr denotes the trace over both the function space in which the operator acts and all relevant internal indices (e.g., spin, color, and flavor) of the quantum fields involved.

The computation of this trace requires choosing a basis, which determines a specific representation for the operator. A natural choice would be the eigenstate basis of $\ln \left(\Delta_H - X_{HL} \Delta_L^{-1} X_{LH} \right)$, which we simplify as $\ln \Delta$. However, in general, this operator depends on arbitrary background fields, preventing its diagonalization and direct computation of its spectrum. Instead, a practical alternative is to work in the position or momentum eigenstate basis, where we can express

$$\langle x| \ln \Delta = \ln \Delta(x, i\partial_x) \langle x| \quad or \quad \langle q| \ln \Delta = \ln \Delta(-i\partial_q, q) \langle q|, \qquad (1.2.47)$$

once Δ derives from the UV Lagrangian density, which is typically represented in one of these spaces. For convenience, we start computing the trace in momentum eigenstate basis and insert an identity in position representation:

$$\text{Tr}\ln\Delta = \int\frac{d^dq}{(2\pi)^d}\langle q|\,\text{tr}\ln\Delta\,|q\rangle = \int d^dx\int\frac{d^dq}{(2\pi)^d}\langle q|x\rangle\langle x|\,\text{tr}\ln\Delta\,|q\rangle = \int d^dx\int\frac{d^dq}{(2\pi)^d}e^{iq\cdot x}\,\text{tr}\ln\Delta(x,i\partial_x)e^{-iq\cdot x}$$

$$(1.2.48)$$

where we used that $\langle q|x\rangle = e^{iq\cdot x}$, and "tr" represent the trace over internal indices only. The derivatives in $\Delta(x,i\partial_x)$ yields factors of q upon acting on the exponentials, leading to:

$$\text{Tr}\ln\Delta = \int d^dx\int\frac{d^dq}{(2\pi)^d}\,\text{tr}\ln\Delta(x,i\partial_x+q)\mathbb{1} \tag{1.2.49}$$

Notice that even if $\Delta(x,i\partial_x)$ contains transpose derivatives ∂_x^T acting on the left, it can be replaced by $-\partial_x$ using integration by parts, since the difference is a total derivative. Therefore, the result above remains general. Furthermore, the identity $\mathbb{1}$ is a reminder that any derivative at rightmost disappear after acting on the exponential.

Since the integral runs over the entire momentum space, it remains invariant over the variable change $q \to -q$, which gives:

$$\text{Tr}\ln\Delta = \int d^dx\int\frac{d^dq}{(2\pi)^d}\,\text{tr}\ln\Delta(x,i\partial_x-q) \tag{1.2.50}$$

Substituting this result into Eq. (1.2.46):

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] = \frac{i}{2}\int\frac{d^dq}{(2\pi)^d}\text{tr}\ln\left(\Delta_H - X_{HL}\Delta_L^{-1}X_{LH}\right)_{\text{hard}}|_{P\to P-q} \tag{1.2.51}$$

where P is the momentum representation of the covariant derivative and q is the shift coming from the trace calculation that we discussed along this section.

Some references [5, 16] apply an additional transformation to Eq. (1.2.51), which we do not adopt here. However, it is worth briefly discussing. The idea is to introduce $e^{P\cdot\partial_q}$ and $e^{-P\cdot\partial_q}$, sandwiching the integrand of Eq. (1.2.51)[3] and use the Baker-Campbell-Hausdorff formula to put all covariant derivatives into commutators. Although, is advantageous to deal with P_μ in intermediate steps or in the final result through commutators, the use of Baker-Campbell-Hausdorff formula generate a lot of terms with a notation that is hard to follow and rearranging them.

1.2.6 Covariant derivative expansion (CDE)

Unfortunately, it is not possible to compute exactly the right-handed side of Eq. (1.2.51). However, we can still develop a perturbative expansion. In this work, we employ the Covariant Derivative Expansion (CDE), a method that, as the name suggests, involves a power series expansion in the covariant derivative (P_μ or D_μ), rather than splitting it into the partial derivative and gauge fields.

In order to obtain a CDE for the one-loop EFT Lagrangian, we start by splitting the kinetic and interaction contributions in Δ_H assuming that it takes the generic form:

$$\Delta_H = -P^2 + M^2 + X_H(\phi_b,P) \tag{1.2.52}$$

with $X_H(\phi,P)$ containing the interaction contributions. Substituting this into Eq. (1.2.51) and already performing the shift $P \to P-q$ in Δ_H:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] = \frac{i}{2}\int\frac{d^dq}{(2\pi)^d}\text{tr}\ln\left(-P^2 + 2q\cdot P - q^2 + \left(X_H - X_{HL}\Delta_L^{-1}X_{LH}\right)_{P\to P-q}\right)_{\text{hard}}$$

$$= \frac{i}{2}\int\frac{d^dq}{(2\pi)^d}\text{tr}\left\{\ln\left[M^2-q^2\right] + \ln\left[1 - \frac{\left(2q\cdot P - P^2 + \left(X_H - X_{HL}\Delta_L^{-1}X_{LH}\right)_{P\to P-q}\right)}{q^2-M^2}\right]\right\}_{\text{hard}}$$

$$(1.2.53)$$

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[3]This is equivalent to inserting identities, given that the derivatives ∂_q in the exponentials act on unity to the right (for $e^{-P\cdot\partial_q}$) and, by integration by parts, can be made to act on unity to the left (for $e^{P\cdot\partial_q}$).

The first term on the right-hand side is independent of ϕ_b or any gauge field. Consequently, it can be dropped, as the Lagrangian is invariant under constant shifts. Thus:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] = \frac{i}{2} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \ln \left[1 - \frac{\left(2q \cdot P - P^2 + \left(X_H - X_{HL} \Delta_L^{-1} X_{LH} \right)_{P \to P-q} \right)}{q^2 - M^2} \right]_{\text{hard}} \tag{1.2.54}$$

In the hard region, the fraction inside the logarithm is small, allowing us to expand it as a Taylor series. To see why, let us analyze each term. The denominator, $q^2 - M^2$, is of order $O(M^2)$ since $q \sim M$ in this region. Meanwhile, in the numerator, X_H arises from interaction terms in the UV Lagrangian that involve at least three fields. Since we are dealing with bosons, it may include dimension-4 operators containing either a dimensionful parameter of order M or a derivative, X_H is at most of order $\mathcal{O}(M)$. The terms in $X_{HL} \Delta_L^{-1} X_{LH}$ appear from the product of two interaction terms and a light-field propagator and hence they generate terms of order $\mathcal{O}(M^0)$[3].

The remaining two terms contain covariant derivatives that acts on ϕ_b or generates field strength tensors. Decomposing the covariant derivatives using its definition implies that those terms are composed by background light fields (gauge and others) and ordinary derivatives of them, or equivalently momentum powers (p) related to this fields. Therefore, since the EFT validity is for energy scales $\Lambda \ll M$, these momentum should satisfy $p \ll M$ which also means that the fields in the EFT need to be slowly varying on distance scales of order M^{-1}[5]. Trivially, these conditions extends to the whole covariant derivative, i.e, $P \ll M$, which also implies that $2q \cdot P$ is $O(M)$, once $q \sim M$ in the hard region.

With all these considerations we can generally state that the numerator is at most $O(M)$ while the denominator is $O(M^2)$ in the hard region. Therefore:

$$\frac{2q \cdot P - P^2 + \left(X_H - X_{HL} \Delta_L^{-1} X_{LH} \right)_{P \to P-q}}{q^2 - M^2} << 1 \tag{1.2.55}$$

and we can Taylor expand the logarithm in Eq. 1.2.54:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] = -\frac{i}{2} \sum_{n=1}^{+\infty} \frac{1}{n} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \left[\frac{\left(2q \cdot P - P^2 + \left(X_H - X_{HL} \Delta_L^{-1} X_{LH} \right)_{P \to P-q} \right)}{q^2 - M^2} \right]^n_{\text{hard}} \tag{1.2.56}$$

This result demonstrates that, for the case of real scalars, the one-loop effective Lagrangian is an infinite series of local operators generated by evaluating momentum integrals. However, in practice, we typically truncate this series at a specific operator dimension or, equivalently, a given power of $1/M$, based on the desired level of accuracy.

Although this may not be immediately apparent, the final result of the expansion can always be rearranged so that every covariant derivative appears within a commutator structure, i.e., as $[P_\mu, \cdot]$. This property follows directly from the use of the CDE, which manifestly preserves gauge invariance at every step. This structure would be evident if we had inserted the exponential operators $e^{P \cdot \partial_q}$ and $e^{-P \cdot \partial_q}$ to the left and right of the integrand, respectively, and then applied the Baker-Campbell-Hausdorff (BCH) formula in Eq. (1.2.51), which systematically organizes terms in commutator form. To illustrate this feature of the CDE explicitly, we now consider a simple example where the terms are directly rearranged so that all covariant derivatives appear inside commutators.

Consider the case where the loop only contains heavy fields, i.e, there are no mixed heavy-light contributions in the one-loop matching, so $X_{LH} = X_{HL} = 0$[4]. In this case, the one-loop Lagrangian becomes[5]:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] = -\frac{i}{2} \sum_{n=1}^{+\infty} \frac{1}{n} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \left[\frac{\left(2q \cdot P - P^2 + X_H \right)}{q^2 - M^2} \right]^n_{\text{hard}} \tag{1.2.57}$$

In order to demonstrate the commutator structure, we focus on operators containing two X_H and two covariant derivatives. Analyzing the expression above, we see that those terms arises from the $n = 3$ and $n = 4$ contributions. Starting with $n = 3$ term, we have:

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[4]This occurs in UV models where heavy fields do not couple linearly to the light degrees of freedom and $\Phi_c = 0$.
[5]For simplicity, we suppress the shift $P \to P - q$ in X_H.

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$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] \supset -\frac{i}{6} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr}\left[\frac{(2q \cdot P - P^2 + X_H)}{q^2 - M^2}\right]^3$$

$$\supset -\frac{i}{6} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr}\left[\frac{(2q \cdot P - P^2 + X_H)(2q \cdot P - P^2 + X_H)(2q \cdot P - P^2 + X_H)}{(q^2 - M^2)^3}\right] \quad (1.2.58)$$

Assuming that X_H does not contain any covariant derivatives and consequently no q dependence:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] \supset -\frac{i}{6} \operatorname{tr}\left(-X_H P^2 X_H - P^2 X_H^2\right) \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - M^2)^3}$$

$$\supset -\frac{i}{6} \operatorname{tr}\left(-3X_H P^2 X_H - 2P_\mu X_H P^\mu X_H\right) \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - M^2)^3}$$

$$(1.2.59)$$

Using dimensional regularization[17]:

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - M^2)^3} = -\frac{i}{32\pi^2 M^2} \quad (1.2.60)$$

we find:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] \supset \frac{1}{192\pi^2 M^2} \operatorname{tr}\left(3X_H P^2 X_H + 2P_\mu X_H P^\mu X_H\right) \quad (1.2.61)$$

Now consider the $n = 4$ term. The operators involving two insertions of X_H and two covariant derivatives arise from the following expression:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] \supset -\frac{i}{8} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr}\left[\frac{(2q \cdot P - P^2 + X_H)}{q^2 - M^2}\right]^4$$

$$\supset -\frac{i}{8} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr}\left[\frac{(2q \cdot P - P^2 + X_H)(2q \cdot P - P^2 + X_H)(2q \cdot P - P^2 + X_H)(2q \cdot P - P^2 + X_H)}{(q^2 - M^2)^4}\right]$$

$$\supset -\frac{i}{2} \operatorname{tr}\left(X_H P_\mu P_\nu X_H + P_\mu P_\nu X_H^2 + P_\mu X_H P_\nu X_H\right) \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{(q^2 - M^2)^4} \quad (1.2.62)$$

By Lorentz symmetry, in dimensional regularization we replace $q^\mu q_\nu \to \frac{g^{\mu\nu}}{d} q^2$, yielding:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] \supset -\frac{i}{2} \operatorname{tr}\left(X_H P^2 X_H + P_2 X_H^2 + P_\mu X_H P_\mu X_H\right) \int \frac{d^d q}{(2\pi)^d} \frac{1}{d} \frac{q^2}{(q^2 - M^2)^4}$$

$$\supset -\frac{i}{2} \operatorname{tr}\left(3X_H P^2 X_H + 3P_\mu X_H P_\mu X_H\right) \int \frac{d^d q}{(2\pi)^d} \frac{1}{d} \frac{q^2}{(q^2 - M^2)^4}$$

$$(1.2.63)$$

Using the known integral identity from dimensional regularization[17]:

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{d} \frac{q^2}{(q^2 - M^2)^4} = -\frac{i}{192\pi^2 M^2} \quad (1.2.64)$$

we find:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] \supset -\frac{1}{128\pi^2 M^2} \operatorname{tr}\left(X_H P^2 X_H + P_\mu X_H P_\mu X_H\right) \quad (1.2.65)$$

Adding this to the $n = 3$, we obtain:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] \supset \frac{1}{128\pi^2 M^2}\operatorname{tr}\!\left(X_H P^2 X_H\right) + \frac{1}{384\pi^2 M^2}\operatorname{tr}(P_\mu X_H P_\mu X_H) \tag{1.2.66}$$

By integration by parts, we have that $P_\mu X_H P_\mu X_H = -X_H P^2 X_H$ plus a total derivative. Therefore, we can rewrite the expression above as:

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] \supset \frac{1}{128\pi^2 M^2}\operatorname{tr}\left(P_\mu X_H P_\mu X_H - X_H P^2 X_H\right) \tag{1.2.67}$$

$$\supset \frac{1}{128\pi^2 M^2}\operatorname{tr}\left([P_\mu, X_H][P^\mu, X_H]\right) \tag{1.2.68}$$

Thus, all contributions involving two covariant derivatives and two X_H and two P_μ adds up to an operator with every covariant derivative in a commutator. Now that we know that, we can anticipate the form of the final result before computing all the terms. Since there is only one independent operator involving two X_H and two covariant derivative, we know that all relevant terms in our CDE must add up to:

$$c_{ij}\operatorname{tr}\left\{[P_\mu,(X_H)_{ij}][P^\mu,(X_H)_{ji}]\right\} = c_{ij}\operatorname{tr}\left\{P_\mu(X_H)_{ij}P_\mu(X_H)_{ji} - (X_H)_{ij}P^2(X_H)_{ji}\right\}$$
$$= 2c_{ij}P\operatorname{tr}(P_\mu(X_H)_{ij}P_\mu(X_H)_{ji}) - (c_{ij}+c_{ji})\operatorname{tr}\left(P^2(X_H)_{ij}(X_H)_{ji}\right) \tag{1.2.69}$$

where the first and second line follows from integration by parts and relabeling indices. To determine c_{ij}, it is sufficient to isolate a single structure, such as $P_\mu(X_H)_{ij}P_\mu(X_H)_{ji}$, and match it with the corresponding term in Eq. (1.2.67). Doing so yields:

$$c_{ij} = \frac{1}{128\pi^2 M^2} \tag{1.2.70}$$

which is in perfect agreement with the full result in Eq. (1.2.68).

More generally, this illustrates an important point: in order to determine the coefficients of effective operators in the final Lagrangian, it is often sufficient to compute only a subset of all possible contributions in the CDE. This is because the number of independent structures one can write down with open covariant derivative is greater than the number of independent operators with the covariant derivatives appearing only in commutators. Although a fully general algorithm for identifying the minimal required subset is not known, a useful practical rule is to discard all0 terms involving adjacent contracted covariant derivatives, namely those with $\operatorname{tr}\!\left(...P^2...\right)$, since they do not contribute new information to the final operator coefficients. In the example shown here (Eq. (1.2.67)), such terms were indeed redundant for determining c_{ij}

Even after following the procedure outlined in this section to compute the EFT Lagrangian up to one-loop order and to the desired operator dimension, there is often still room for further simplification. This can be achieved by identifying and eliminating redundant operators, which are those that can be rewritten in terms of others without affecting the S-matrix elements and, consequently, any physical observables. The objective is to construct a minimal, non-redundant set of operators that accurately captures the same physics as the unsimplified Lagrangian. This process is analogous to finding a basis in linear algebra, but in this context, it applies to the space of higher-dimensional operators.

Simplification procedure is decomposed in two steps. First, we apply exact identities, such as integration by parts, group identities, and commutation relations, to relate operators linearly and reduce the Lagrangian to what is known as a Green's basis. In this context, the Lagrangian can be seen as an element of a vector space spanned by all possible operators consistent with the symmetries of the theory. The exact identities serve as constraints that reduce this space, eliminating linear dependencies and leaving a smaller, but physically equivalent, basis.

The second step involves field redefinitions, where we rewrite the field variables as nonlinear but local functions of a new set of field variables to eliminate redundant operators. According to the equivalence theorem [18, 19, 20], such field redefinitions do not alter the on-shell S-matrix elements, ensuring that the simplified Lagrangian is physically equivalent to the original.

1.3 Example: Integrating out a real scalar in a toy model

In the previous section, we established a systematic approach to perform matching, up to one-loop order, between a UV theory and its corresponding EFT, where a multiplet of heavy and real scalar fields

are integrated out. To illustrate this method concretely, we will now apply it to a simple toy model consisting of two scalar fields: one heavy and one light. This example will provide a clear and practical demonstration of the matching procedure, highlighting the essential steps and techniques discussed earlier.

Consider a renormalizable UV model with a $\mathcal{Z}_2$ symmetry, described by the following Lagrangian:

$$\mathcal{L}_{UV}[\phi, \Phi] = \frac{1}{2} \left[\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 + \partial_\mu \Phi \partial^\mu \Phi - M^2 \Phi^2 \right] - \frac{\lambda_0}{4!} \phi^4 - \frac{\lambda_1}{4} \phi^2 \Phi^2 \tag{1.3.1}$$

where ϕ and Φ are light and heavy real scalar fields with mass m and $M \gg M$, respectively.

For energy scales $\Lambda \ll M$, we wish to derive the EFT obtained by integrating out the heavy field Φ. Applying the systematic procedure discussed earlier, the resulting EFT Lagrangian will be composed of local effective operators constructed solely from the light field ϕ. These operators can be separated into kinetic, mass and interaction terms, which are organized as an infinite series of operators with increasing dimension. We expect that the general structure of the EFT Lagrangian is given by:

$$\mathcal{L}_{EFT}[\phi] = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{m_L^2}{2} \phi^2 - \sum_{\mathcal{D}=4}^{\infty} \frac{C_\mathcal{D}}{M^{\mathcal{D}-4}} O_\mathcal{D} \tag{1.3.2}$$

where $O_\mathcal{D}$ are operators of dimension $\mathcal{D}$, $C_\mathcal{D}$ are dimensionless Wilson coefficients, and m_L is the effective mass of the light field, potentially modified by virtual heavy field contributions. For clarity, we have omitted the subscript b from the background fields of the EFT to simplify the notation.

To make the calculation more tractable while still capturing the essential features of the matching procedure, we truncate the series at dimension six, corresponding to terms suppressed by $1/M^2$. We now proceed with the explicit matching calculation, beginning with the tree-level contribution before moving on to one-loop corrections.

1.3.1 Tree level

At tree level, the EFT Lagrangian is obtained by integrating out the heavy field using its classical equation of motion. From our earlier discussion, this is given by Eq. (1.2.32):

$$\mathcal{L}_{EFT}^{\text{tree}}[\phi] = \mathcal{L}_{UV}[\phi, \hat{\Phi}_c[\phi]] \tag{1.3.3}$$

The first step in evaluating this expression is to find the solution to the classical equation of motion for the heavy field Φ:

$$\left. \frac{\delta \mathcal{L}_{UV}[\phi, \Phi]}{\delta \Phi} \right|_{\Phi = \Phi_c} = 0 \implies \left. \left(\partial^2 + M^2 + \frac{\lambda_1}{2} \phi^2 \right) \Phi \right|_{\Phi = \Phi_c} = 0 \implies \Phi_c[\phi] = 0 \tag{1.3.4}$$

Since the solution is already local, there is no need to perform an expansion. Substituting this result into the UV Lagrangian (1.3.1) and then into the tree-level EFT expression, we obtain:

$$\mathcal{L}_{EFT}^{\text{tree}}[\phi] = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda_0}{4!} \phi^4 \tag{1.3.5}$$

As we can see, the tree-level EFT Lagrangian does not contain higher-dimensional operators.

1.3.2 One-loop level

Having obtained the tree-level EFT Lagrangian, we now move on to the one-loop level matching. As derived in the previous section, the resulting one-loop contribution to the EFT Lagrangian is given by (Eq. (1.3.6)):

$$\mathcal{L}_{EFT}^{1-loop}[\phi_b] = -\frac{i}{2} \sum_{n=1}^{+\infty} \frac{1}{n} \int \frac{d^d q}{(2\pi)^d} \left[\frac{\left(2q \cdot P - P^2 + \left(X_H - X_{HL} \Delta_L^{-1} X_{LH} \right)_{P \to P-q} \right)}{q^2 - M^2} \right]_{\text{hard}}^n \tag{1.3.6}$$

where X_H, X_{LH}, X_{HL} and Δ_L are components of the fluctuation operator of the UV theory. Using the example UV Lagrangian expression into the definition of theses components (Eq. (1.2.12)) we find:

$$X_H = -\frac{\delta^2 \mathcal{L}_{UV}}{\Phi'^2}\bigg|_{\Phi=0} = \frac{\lambda_1}{2}\phi^2 \tag{1.3.7}$$

$$X_{LH} = -\frac{\delta^2 \mathcal{L}_{UV}}{\phi'\Phi'}\bigg|_{\Phi=0} = 0 \tag{1.3.8}$$

$$X_{HL} = -\frac{\delta^2 \mathcal{L}_{UV}}{\Phi'\phi'}\bigg|_{\Phi=0} = 0 \tag{1.3.9}$$

$$\Delta_L = -\frac{\delta^2 \mathcal{L}_{UV}}{\phi'^2}\bigg|_{\Phi=0} = \left(\partial^2 + m^2 + \frac{\lambda_0}{2}\phi\right)\phi \tag{1.3.10}$$

The absence of heavy-light mixed terms (X_{LH} and X_{HL}) arises due to two factors: the $\mathcal{Z}_2$ symmetry of the theory, which prohibits linear couplings in at least one of the fields, and the fact that the classical solution for the heavy field is zero.

$$\mathcal{L}_{EFT}^{1-loop}[\phi] = -\frac{i}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int\frac{d^dq}{(2\pi)^d}\left[\frac{\left(2q\cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)}{q^2 - M^2}\right]^n_{\text{hard}} \tag{1.3.11}$$

In the above expression, the subscript "hard" denotes that the loop momenta are in the high-energy regime, $q \sim M$. Normally, this would allow us to Taylor expand small parameters in the integrand, i.e, low-energy scales such as light field mass or derivatives acting on the light field with the appropriate factors of $1/M$ to adjust dimension. However, since the integrand already is a polynomial in low-energy scales, the hard subscript becomes redundant and can be omitted. Consequently, we have:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] = -\frac{i}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int\frac{d^dq}{(2\pi)^d}\left[\frac{\left(2q\cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)}{q^2 - M^2}\right]^n \tag{1.3.12}$$

The infinite series above generates effective operators of all possible dimensions. However, since we aim to retain only operators up to dimension six, we must extract the relevant contributions accordingly. This can be accomplished by analyzing the initial terms of the series, selecting those that contain the appropriate number of light fields and derivatives to match the target dimensionality. Alternatively, we can filter contributions by focusing on terms suppressed by factors up to $\mathcal{O}(1/M^2)$.

Since heavy mass contributions always arise from the loop integral, it is more practical in this scenario to count the fields and derivatives directly to determine the operator's dimension. Recall that in four-dimensional field theory, a scalar field ϕ and each derivative ∂_μ have mass dimension 1.

We begin by examining the term with $n = 1$:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{2}\int\frac{d^dq}{(2\pi)^d}\left[\frac{\left(2q\cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)}{q^2 - M^2}\right] \tag{1.3.13}$$

In this simple UV theory without Gauge symmetries, the covariant derivative reduces to the ordinary derivative. Since derivatives act on fields placed to their left, they do not contribute here, simplifying the analysis. Thus, the only surviving term corresponds to a dimension-2 operator:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{2}\frac{\lambda_1}{2}\phi^2\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2 - M^2} \tag{1.3.14}$$

The loop integral is evaluated using dimensional regularization[1]:

$$\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2 - M^2} = \frac{i}{16\pi^2}M^2\left[\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M^2}\right) + 1\right] \tag{1.3.15}$$

where $\frac{2}{\bar{\epsilon}} = \frac{2}{\epsilon} - \gamma_E + \ln 4\pi$. In the $\overline{MS}$ scheme, is understood that $\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M^2}\right)$ is replaced by $ln\left(\frac{\mu^2}{M^2}\right)$ in the final result. Applying this, we obtain:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset \frac{\lambda_1 M^2}{32\pi^2}\left[1 + \ln\left(\frac{\mu^2}{M^2}\right)\right]\frac{\phi^2}{2} \tag{1.3.16}$$

We now examine the contribution from the $n = 2$ term:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{4} \int \frac{d^d q}{(2\pi)^d} \left[\frac{\left(2q \cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)}{q^2 - M^2} \right] \left[\frac{\left(2q \cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)}{q^2 - M^2} \right]$$

$$\supset -\frac{i}{4} \int \frac{d^d q}{(2\pi)^d} \frac{\left(2q \cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)\left(\frac{\lambda_1}{2}\phi^2\right)}{\left(q^2 - M^2\right)^2} \quad (1.3.17)$$

In the above expression, the first two terms in the numerator involve total derivatives. Since total derivative terms in the Lagrangian do not contribute to the physical dynamics, they can be discarded. As a result, the surviving contribution is a dimension-four operator:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{4}\frac{\lambda_1^2}{4}\phi^4 \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left(q^2 - M^2\right)^2} \quad (1.3.18)$$

To evaluate the loop integral, we use dimensional regularization[1]:

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{\left(q^2 - M^2\right)^2} = \frac{i}{16\pi^2}\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{M^2}\right)\right] \quad (1.3.19)$$

In the $\overline{\text{MS}}$ scheme, where $\frac{2}{\epsilon}$ is subtracted, we obtain:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset \frac{3\lambda_1^2}{32\pi^2} \ln\left(\frac{\mu^2}{M^2}\right)\frac{\phi^4}{4!} \quad (1.3.20)$$

Let's examine the contribution from the $n = 3$ term in the series:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{6} \int \frac{d^d q}{(2\pi)^d} \left[\frac{\left(2q \cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)}{q^2 - M^2} \right]^3$$

$$\supset -\frac{i}{6} \int \frac{d^d q}{(2\pi)^d} \frac{\left(2q \cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)\left(2q \cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)\left(\frac{\lambda_1}{2}\phi^2\right)}{\left(q^2 - M^2\right)^3} \quad (1.3.21)$$

To understand the possible effective operators generated, we analyze the operator dimensions. The minimal dimension possible is four. However:

- The only dimension-four operator with two derivatives and two scalar fields is a total derivative. Since total derivatives do not affect the physical dynamics, these terms can be ignored.

- Dimension-five operators, containing three derivatives and two scalar fields, are forbidden by the $\mathbb{Z}_2$ symmetry.

- Thus, the leading non-trivial contributions arise from dimension-six operators. These can take the form of either six scalar fields or four scalar fields with two derivatives.

Therefore, the relevant operators emerging from this term are those with six scalar fields or four fields with two derivatives:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{6}\left[\frac{\lambda_1^3}{8}\phi^6 - \frac{\lambda_1^2}{4}\left(\phi^2 P^2 \phi^2 + P^2 \phi^4\right)\right] \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left(q^2 - M^2\right)^3} \quad (1.3.22)$$

Applying $P_\mu = i\partial_\mu$ to make the derivatives explicit, we have:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{6}\left[\frac{\lambda_1^3}{8}\phi^6 + \frac{\lambda_1^2}{4}\left(\phi^2 \Box \phi^2 + \Box \phi^4\right)\right] \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left(q^2 - M^2\right)^3} \quad (1.3.23)$$

To further simplify, we use the identities:

$$\Box \phi^2 = 2(\partial^\mu \phi \partial_\mu \phi + \phi \Box \phi), \quad \Box \phi^4 = 12\phi^2 \partial^\mu \phi \partial_\mu \phi + 4\phi^3 \Box \phi \quad (1.3.24)$$

Substituting these, the Lagrangian becomes:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{6}\left[\frac{\lambda_1^3}{8}\phi^6 + \frac{\lambda_1^2}{2}\left(3\phi^3\Box\phi + 7\phi^2\partial^\mu\phi\partial_\mu\phi\right)\right]\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2-M^2)^3} \tag{1.3.25}$$

Using integration by parts to eliminate the redundant derivative term, we can replace $\phi^2\partial^\mu\phi\partial_\mu\phi$ with $-\frac{1}{3}\phi^3\Box\phi$, yielding:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{6}\left[\frac{\lambda_1^3}{8}\phi^6 + \frac{\lambda_1^2}{3}\phi^3\Box\phi\right]\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2-M^2)^3} \tag{1.3.26}$$

Using dimensional regularization, the momentum integral evaluates to[1]:

$$\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2-M^2)^3} = -\frac{i}{32\pi^2 M^2} \tag{1.3.27}$$

Thus, the final contribution from the $n=3$ term is:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{15\lambda_1^3}{32\pi^2}\frac{\phi^6}{6!M^2} - \frac{\lambda_1^2}{24\pi^2}\frac{\phi^3\Box\phi}{4!M^2} \tag{1.3.28}$$

Term with $n=4$:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{8}\int\frac{d^dq}{(2\pi)^d}\left[\frac{\left(2q\cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)}{q^2-M^2}\right]^4$$
$$\supset -\frac{i}{8}\int\frac{d^dq}{(2\pi)^d}\frac{\left(2q\cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)\left(2q\cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)\left(2q\cdot P - P^2 + \frac{\lambda_1}{2}\phi^2\right)\left(\frac{\lambda_1}{2}\phi^2\right)}{(q^2-M^2)^4} \tag{1.3.29}$$

Here the minimal operator dimension is five, which are those with three derivative and two fields. The relevant operator to us will be only those with dimension six, which in these case are those involving four fields and two derivatives. The rest of the contributions are of dimension higher than six, which does not interest us. Therefore, the $n=4$ will contribute only with dimension six operators:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset -\frac{i}{8}\lambda_1^2\left(\phi^2 P_\mu P_\nu\phi^2 + P_\mu P_\nu\phi^4 + P_\mu\phi^2 P_\nu\phi^2\right)\int\frac{d^dq}{(2\pi)^d}\frac{q^\mu q^\nu}{(q^2-M^2)^4}$$
$$\supset \frac{i}{8}\lambda_1^2\left(\phi^2\partial_\mu\partial_\nu\phi^2 + \partial_\mu\partial_\nu\phi^4 + \partial_\mu\phi^2\partial_\nu\phi^2\right)\int\frac{d^dq}{(2\pi)^d}\frac{q^\mu q^\nu}{(q^2-M^2)^4} \tag{1.3.30}$$

We can simplify a little bit by using the following identities:

$$\begin{cases} \phi^2\partial_\mu\partial_\nu\phi^2 = 2\phi^2\partial_\mu\phi\partial_\nu\phi + 2\phi^3\partial_\mu\partial_\nu\phi \\ \partial_\mu\partial_\nu\phi^4 = 12\phi^2\partial_\mu\phi\partial_\nu\phi + 4\phi^3\partial_\mu\partial_\nu\phi \\ \partial_\mu\phi^2\partial_\nu\phi^2 = 6\phi^2\partial_\mu\phi\partial_\nu\phi + 2\phi^3\partial_\mu\partial_\nu\phi \end{cases} \tag{1.3.31}$$

Substituting into the Lagrangian:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset \frac{i}{2}\lambda_1^2\left(2\phi^3\partial_\mu\partial_\nu\phi + 5\phi^2\partial_\mu\phi\partial_\nu\phi\right)\int\frac{d^dq}{(2\pi)^d}\frac{q^\mu q^\nu}{(q^2-M^2)^4} \tag{1.3.32}$$

Performing the loop integral with dimensional regularization and also discarding the $\frac{2}{\epsilon}$ term because we are using the $\overline{MS}$ scheme:

$$\int\frac{d^dq}{(2\pi)^d}\frac{q^\mu q^\nu}{(q^2-M^2)^4} = -\frac{i}{192\pi^2 M^2}g^{\mu\nu} \tag{1.3.33}$$

Substituting this into the Lagrangian expression:

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset \frac{\lambda_1^2}{384\pi^2 M^2}\left(2\phi^3\Box\phi + 5\phi^2\partial_\mu\phi\partial^\mu\phi\right) \tag{1.3.34}$$

Using the integration by parts to replace $\phi^2\partial_\mu\phi\partial^\mu$ by $-\frac{1}{3}\phi^3\Box\phi$: '

$$\mathcal{L}_{EFT}^{1-loop}[\phi] \supset \frac{\lambda_1^2}{1152\pi^2 M^2}\phi^3\Box\phi \tag{1.3.35}$$

The other terms in $n > 4$ contributes from operator of higher dimension, or equivalently to order $1/M^4$. Therefore they are not interesting to us.

Now we sum Eq. (1.3.16), (1.3.20), 1.3.28), 1.3.35) and 1.3.5) to obtain the EFT Lagrangian up to one-loop order and up to dimension six operators:

$$\mathcal{L}_{EFT}^{\text{tree}}[\phi] + \mathcal{L}_{EFT}^{1-loop}[\phi] \supset \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - m_L^2\frac{\phi^2}{2} - c_4\frac{\phi^4}{4!} - c_6\frac{\phi^6}{6!} - c_6'\frac{\phi^3\Box\phi}{6!} \tag{1.3.36}$$

where:

$$m_L^2 = m^2 - \frac{\lambda_1 M^2}{32\pi^2}\left(\ln\left(\frac{\mu^2}{M^2}\right) + 1\right) \tag{1.3.37}$$

$$c_4 = \lambda_0 - \frac{3\lambda_1^2}{32\pi^2}\ln\left(\frac{\mu^2}{M^2}\right) \tag{1.3.38}$$

$$c_6 = \frac{15\lambda_1^3}{32\pi^2 M^2} \tag{1.3.39}$$

$$c_6' = \frac{5\lambda_1^2}{8\pi^2 M^2} \tag{1.3.40}$$

As discussed in the previous section, we can simplify this result by eliminating redundant operators. Actually, we already implemented the first step of simplification using exact identities when we used integration by parts to replace $\phi^2\partial^\mu\phi\partial_\mu\phi$ by $-\frac{1}{3}\phi^3\Box\phi$. Therefore, we can jump to the second step and use the following field redefinition:

$$\phi \to \phi\left(1 - c_6'\frac{\phi^2}{6!}\right) \tag{1.3.41}$$

This will eliminate the $\phi^3\Box\phi$ operator by adding contributions to ϕ^4 and ϕ^6 operators:

$$\mathcal{L}_{EFT}^{\text{tree}}[\phi] + \mathcal{L}_{EFT}^{1-loop}[\phi] \supset \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - m_L^2\frac{\phi^2}{2} - \left[c_4 - \frac{m_L^2}{30}c_6'\right]\frac{\phi^4}{4!} - \left[c_6 + \frac{m_L^2}{2}\frac{c_6'^2}{6!} - \frac{c_4 c_6'}{6}\right]\frac{\phi^6}{6!} \tag{1.3.42}$$

Recall that m_L^2 and c_4 have leading order contributions from tree-level matching, along with quantum corrections from one-loop matching, whereas c_6 and c_6' arises purely form quantum effects. Consequently, the products $m_L^2 c_6'$ and $c_4 c_6'$ consist of two types of terms: one involving a leading-order contribution multiplied by a quantum correction, and another resulting from the product of two quantum corrections. Since we are working within a perturbative framework, quantum corrections are expected to be small, and terms involving the product of two such corrections are even smaller. Thus, retain only the leading-order times quantum correction contributions and neglecting the subleading terms is a good approximation:

$$m_L^2 c_6' \approx m^2 c_6', \quad c_4 c_6' \approx \lambda_0 c_6' \tag{1.3.43}$$

Furthermore, since c_6 and c_6' are of order $1/M^2$ and $M^2 \gg m_L^2$ we can neglect the quadratic contribution of $m_L c_6'$ that appears in the ϕ^6 coefficient:

$$c_6 - \frac{c_4 c_6'}{6} \gg \frac{m_L^2}{2}\frac{c_6'^2}{6!} \implies c_6 + \frac{m_L^2}{2}\frac{c_6'^2}{6!} - \frac{c_4 c_6'}{6} \approx c_6 - \frac{c_4 c_6'}{6} \tag{1.3.44}$$

Taking all these approximations into account and redefining the coefficients, we can rewrite the EFT Lagrangian up to one-loop order and up to dimension six as

$$\mathcal{L}_{EFT}^{\text{tree}}[\phi] + \mathcal{L}_{EFT}^{1-loop}[\phi] \supset \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - m_L^2\frac{\phi}{2} - C_4\frac{\phi^4}{4!} - \frac{C_6}{M^2}\frac{\phi^6}{6!} \tag{1.3.45}$$

where the effective coefficients are given by:

$$m_L^2 = m^2 - \frac{\lambda_1 M^2}{32\pi^2}\left(\ln\left(\frac{\mu^2}{M^2}\right) + 1\right) \tag{1.3.46}$$

$$C_4 = c_4 - \frac{m_L^2}{30}c_6' \approx \lambda_0 - \frac{3\lambda_1^2}{32\pi^2}\ln\left(\frac{\mu^2}{M^2}\right) - \frac{\lambda_1^2 m^2}{48\pi^2 M^2} \tag{1.3.47}$$

$$C_6 = c_6 + \frac{m_L^2}{2}\frac{c_6'^2}{6!} - \frac{c_4 c_6'}{6} \approx \frac{15\lambda_1^2}{32\pi^2 M^2} - \frac{5\lambda_0\lambda_1^2}{48\pi^2} \tag{1.3.48}$$

Despite its simplicity, this example allowed us to apply the matching method developed in the previous section, practice identifying relevant operators up to the desired dimension, and develop a better understanding of simplification techniques used to recognize and eliminate redundant operators.

1.4 Generalization of the matching method for fermions

So far, we have applied the matching procedure to derive an effective theory in the simplest case: integrating out a heavy real scalar from a UV theory composed solely of real scalar fields. In this section, we extend this framework to UV theories that include both real and complex scalars, as well as fermions. While the fundamental principles of matching in the path integral formalism remain unchanged, new features arise due to the distinct statistical properties of bosons and fermions. In particular, scalar fields are ordinary commuting variables, whereas fermionic fields are Grassmann variables, leading to key differences in their integration and the structure of the resulting effective operators.

Consider a UV theory containing both heavy and light scalar and fermionic fields, organized into multiplets as shown in Table 1.1. Our goal is to derive the tree-level and one-loop effective Lagrangian by integrating out the heavy fields. Since the tree-level matching procedure applied to the real scalar case did not rely on the field statistics, the same steps can be followed here. As a result, the tree-level EFT Lagrangian is obtained simply by replacing the heavy fields in the UV Lagrangian with their local classical equations of motion solution, expressed in terms of the background light fields:

$$\mathcal{L}_{EFT}^{\text{tree}}[\psi, \phi] = \mathcal{L}_{UV}[\hat{\Psi}_c[\psi, \phi], \hat{\Phi}_c[\psi, \phi], \psi, \phi] \tag{1.4.1}$$

where, for simplicity, we omit the subscript b, assuming that the light fields are treated as background fields.

Multiplet	Components	Description
Ψ	$\left(\Omega, \Omega^C, \Xi\right)^T$	Ω, Ω^C: heavy Dirac fermions Ξ: heavy Majorana fermions
Φ	$\left(\Sigma, \Sigma^*, \Theta\right)^T$	Σ, Σ^*: heavy complex scalars Θ: heavy real scalars
ψ	$\left(\omega, \omega^C, \xi\right)^T$	ω, ω^C: light Dirac fermions ξ: light Majorana fermions
ϕ	$\left(\sigma, \sigma^*, \theta\right)^T$	σ, σ^*: light complex scalars θ: light real scalars

Table 1.1: Composition of the multiplets involved in the calculation. The superscript on the Dirac fields indicates charge conjugation, i.e., $\omega^C = \mathcal{C}\bar{\Omega}^T$, where $\mathcal{C}$ is the charge conjugation operator.

At the one-loop level, the first and most significant difference when integrating out fields with different statistics, compared to the purely scalar case, appears in the evaluation of the Gaussian integral over quantum fluctuations. This step occurs when computing the one-loop perturbative interaction (1LPI) and one-particle-irreducible (1PI) generating functionals for the UV and EFT theories, respectively. The key distinction lies in how functional determinants arise based on field statistics. Specifically, bosonic fields lead to determinants in the denominator, while fermionic fields, being Grassmann variables, contribute determinants in the numerator:

$$\int D\Phi \exp\left\{\int d^4x \Phi^\dagger \mathcal{O}\Phi\right\} = N\left(\det\mathcal{O}\right)^{-\frac{1}{2}} \qquad\qquad \text{(Bosonic scalar multiplet)}$$

$$\int D\Psi D\bar\Psi \exp\left\{\int d^4x \bar\Psi\mathcal{O}\Psi\right\} = N\det\mathcal{O} \qquad\qquad \text{(Fermionic multiplet)}$$

Because the Gaussian integral depends on field statistics, and typical UV theories contain both scalar and Dirac fields, the fluctuation operator often introduces mixed terms that couple fields with different statistics. As a result, the Gaussian integral cannot be evaluated directly as in the purely scalar case. Instead, we first transform the fluctuation operator into a block-diagonal form to decouple bosonic and fermionic contributions.

Rather than working directly with the fluctuation operator, we examine the second variation of the UV Lagrangian. This approach makes it easier to identify mixed terms, particularly given the presence of four distinct multiplets. Given the multiplet definitions in Table 1.1, the second variation of the UV Lagrangian is[6]:

$$
\begin{aligned}
\delta^2\mathcal{L}_{UV} = \delta^2\mathcal{L}_{\mathrm{S}} &+ \frac{1}{2}\Psi'^T\Delta_\Psi\Psi' - \frac{1}{2}\Psi'^T\tilde{\mathbf{X}}_{\Psi\Phi}\Phi' + \frac{1}{2}\Phi'^T\tilde{\mathbf{X}}_{\Phi\Psi}\Psi' - \frac{1}{2}\Psi'^T\tilde{\mathbf{X}}_{\Psi\phi}\phi' + \frac{1}{2}\phi'^T\tilde{\mathbf{X}}_{\phi\Psi}\Psi' \\
&+ \frac{1}{2}\psi'^T\tilde{\mathbf{X}}_{\psi\Psi}\Psi' + \frac{1}{2}\Psi'^T\tilde{\mathbf{X}}_{\Psi\psi}\psi' + \frac{1}{2}\psi'^T\Delta_\psi\psi' - \frac{1}{2}\psi'^T\tilde{\mathbf{X}}_{\psi\Phi}\Phi' + \frac{1}{2}\Phi'^T\tilde{\mathbf{X}}_{\Phi\psi}\psi' \\
&- \frac{1}{2}\psi'^T\tilde{\mathbf{X}}_{\psi\phi}\phi' + \frac{1}{2}\phi'^T\tilde{\mathbf{X}}_{\phi\psi}\psi',
\end{aligned}
\tag{1.4.2}
$$

where the pure scalar part is given by:

$$\delta^2\mathcal{L}_S = -\frac{1}{2}\Phi'\Delta_\Phi\Phi' - \frac{1}{2}\phi'\Delta_\phi\phi' - \frac{1}{2}\Phi'^T\tilde{\mathbf{X}}_{\Phi\phi}\phi' - \frac{1}{2}\phi'^T\tilde{\mathbf{X}}_{\phi\Phi}\Phi' \tag{1.4.3}$$

To simplify the visualization of Eqs. (1.4.2) and (1.4.3), we introduce some compact notation. For the mixed terms, we defined matrices $\tilde{\mathbf{X}}$ with elements:

$$(X_{AB})_{ij} \equiv -\frac{\delta^2\mathcal{L}_{UV,\mathrm{int}}}{\delta A_i \delta B_j}\bigg|_{A=A_b, B=B_b} \tag{1.4.4}$$

where A and B represent arbitrary scalar or fermionic fields, and $\mathcal{L}_{UV,\mathrm{int}}$ denotes the interaction part of the UV Lagrangian. The indices i and j collectively account for all indices carried by the fields A and B. Using that $\mathcal{C}$ as the charge conjugation operator, some representative examples of these matrices are:

$$
\tilde{\mathbf{X}}_{\Psi\Phi} = \begin{pmatrix} X_{\Omega\Sigma} & X_{\Omega\Sigma^*} & X_{\Omega\Theta} \\ \mathcal{C}X_{\bar\Omega\Sigma} & \mathcal{C}X_{\bar\Omega\Sigma^*} & \mathcal{C}X_{\bar\Omega\Theta} \\ X_{\Xi\Sigma} & X_{\Xi\Sigma^*} & X_{\Xi\Theta} \end{pmatrix}, \qquad
\tilde{\mathbf{X}}_{\Phi\Psi} = \begin{pmatrix} X_{\Sigma\omega} & X_{\Sigma\bar\Omega}\mathcal{C}^{-1} & X_{\Sigma\Xi} \\ X_{\Sigma^*\Omega} & X_{\Sigma^*\bar\Omega}\mathcal{C}^{-1} & X_{\Sigma^*\Xi} \\ X_{\Theta\Omega} & X_{\Theta\bar\Omega}\mathcal{C}^{-1} & X_{\Theta\Xi} \end{pmatrix}, \tag{1.4.5}
$$

$$
\tilde{\mathbf{X}}_{\Psi\psi} = \begin{pmatrix} X_{\Omega\omega} & X_{\Omega\bar\omega}\mathcal{C}^{-1} & X_{\Omega\xi} \\ \mathcal{C}X_{\bar\Omega\omega} & \mathcal{C}X_{\bar\Omega\bar\omega}\mathcal{C}^{-1} & \mathcal{C}X_{\bar\Omega\xi} \\ X_{\Xi\omega} & X_{\xi\bar\omega}\mathcal{C}^{-1} & X_{\Xi\xi} \end{pmatrix}, \qquad
\tilde{\mathbf{X}}_{\Phi\phi} = \begin{pmatrix} X_{\Sigma\sigma} & X_{\Sigma\sigma^*} & X_{\Sigma\theta} \\ X_{\Sigma^*\sigma} & \mathcal{C}X_{\Sigma^*\sigma^*} & X_{\Sigma^*\theta} \\ X_{\Theta\Sigma} & X_{\Theta\sigma^*} & X_{\Theta\theta} \end{pmatrix}, \tag{1.4.6}
$$

The remaining $\tilde{\mathbf{X}}$ matrices follows similar definitions for substitutions $\Phi \rightarrow \phi$ and $\Psi \rightarrow \psi$. For the quadratic terms, we also introduce a compact notation:

$$
\Delta_\Psi = \begin{pmatrix} X_{\Omega\Omega} & \mathcal{C}\left(\slashed{P}_{\Omega^C} - M_\Omega + \mathcal{C}^{-1}X_{\Omega\bar\Omega}\mathcal{C}^{-1}\right) & X_{\Omega\Xi} \\ \mathcal{C}\left(\slashed{P}_\Omega - M_\Omega + X_{\bar\Omega\Omega}\right) & \mathcal{C}X_{\bar\Omega\bar\Omega}\mathcal{C}^{-1} & \mathcal{C}X_{\bar\Omega\Xi} \\ X_{\Xi\Omega} & X_{\Xi\bar\Omega}\mathcal{C}^{-1} & \mathcal{C}\left(\slashed{P}_\Xi - M_\Xi + \mathcal{C}^{-1}X_{\Xi\Xi}\right) \end{pmatrix}, \tag{1.4.7}
$$

$$
\Delta_\Phi = \begin{pmatrix} X_{\Sigma\Sigma} & -P_{\Sigma^*}^2 + M_\Sigma^2 + X_{\Sigma\Sigma^*} & X_{\Sigma\Theta} \\ -P_\Sigma^2 + M_\Sigma^2 + X_{\Sigma^*\Sigma} & X_{\Sigma^*\Sigma^*} & X_{\Sigma\Theta} \\ X_{\Theta\Sigma} & X_{\Theta\Sigma^*} & -P_\Theta^2 + M_\Theta^2 + X_{\Theta\Theta} \end{pmatrix}, \tag{1.4.8}
$$

where P_Ω^μ contains the generators T_r^a of a representation r of the group symmetry and $P_{\Omega^C}^\mu$ contains the generators $T_{\bar r}^a$ of a the conjugate representation $\bar r$. The same holds for the generators of the complex scalar fields.

---

[6]The difference of signs in fermionic terms result from using ait-commutation relation between fermions and derivatives with respect to fermions

For a more intuitive form, we rewrite these as:

$$\Delta_\Psi = \mathcal{C}\tilde{\mathbb{1}}\left(\slashed{P} - M_\Psi\right) + \tilde{\mathbf{X}}_{\Psi\Psi}, \quad \Delta_\Phi = \tilde{\mathbb{1}}\left(-P^2 + M_\Phi^2\right) + \tilde{\mathbf{X}}_{\Phi\Phi}, \tag{1.4.9}$$

where:

$$\slashed{P} - M_\Psi = \begin{pmatrix} \slashed{P}_\Omega - M_\Omega & 0 & 0 \\ 0 & \slashed{P}_{\Omega^C} - M_\Omega & 0 \\ 0 & 0 & \slashed{P}_\Xi - M_\Xi \end{pmatrix}, \qquad \tilde{\mathbf{X}}_{\Psi\Psi} = \begin{pmatrix} X_{\omega\omega} & X_{\Omega\bar{\Omega}}\mathcal{C}^{-1} & X_{\Omega\Xi} \\ \mathcal{C}X_{\bar{\Omega}\Omega} & \mathcal{C}X_{\bar{\omega}\omega}\mathcal{C}^{-1} & \mathcal{C}X_{\bar{\Omega}\Xi} \\ X_{\Xi\Omega} & X_{\Xi\bar{\Omega}}\mathcal{C}^{-1} & X_{\Xi\Xi} \end{pmatrix}, \tag{1.4.10}$$

$$-P^2 + M_\Phi^2 = \begin{pmatrix} -P_\Sigma^2 + M_\Sigma^2 & 0 & 0 \\ 0 & -P_{\Sigma^*}^2 + M_\Sigma^2 & 0 \\ 0 & 0 & -P_\Theta^2 + M_\Theta^2 \end{pmatrix}, \qquad \tilde{\mathbf{X}}_{\Phi\Phi} = \begin{pmatrix} X_{\Sigma\Sigma} & X_{\Sigma\Sigma^*} & X_{\Sigma\Theta} \\ X_{\Sigma^*\Sigma} & X_{\Sigma^*\Sigma^*} & X_{\Sigma^*\Theta} \\ X_{\Theta\Sigma} & X_{\Theta\Sigma^*} & X_{\Theta\Theta} \end{pmatrix}, \tag{1.4.11}$$

Finally, for convenience, we introduced:

$$\tilde{\mathbb{1}} = \begin{pmatrix} 0 & \mathbb{1} & 0 \\ \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \tag{1.4.12}$$

The quadratic terms for the light field multiplets follow the same structure and can be trivially obtained by substituting $\Psi \to \psi$ and $\Phi \to \phi$.

In order to write the fluctuation operator in a block diagonal form, our strategy will be to eliminate the mixed scalar and fermionic terms by redefining the fermionic fields. We start working in the terms of the second variation (Eq. (1.4.2)) with at least one variation in the light fermionic multiplet:

$$\delta^2\mathcal{L}_\psi = \frac{1}{2}\psi'^T\tilde{\mathbf{X}}_{\psi\Psi}\Psi' + \frac{1}{2}\delta\Psi'^T\tilde{\mathbf{X}}_{\Psi\psi}\psi' + \frac{1}{2}\psi'^T\Delta_\psi\psi' - \frac{1}{2}\psi'^T\tilde{\mathbf{X}}_{\psi\Phi}\Phi' + \frac{1}{2}\Phi'^T\tilde{\mathbf{X}}_{\Phi\psi}\psi'$$
$$- \frac{1}{2}\psi'^T\tilde{\mathbf{X}}_{\psi\phi}\phi' + \frac{1}{2}\phi'^T\tilde{\mathbf{X}}_{\phi\psi}\psi' \tag{1.4.13}$$

Defining the matrix-valued Green's function of Δ_ψ

$$\Delta_\psi(x)\Delta_\psi^{-1}(x,y) = \delta^{(d)}(x-y) \tag{1.4.14}$$

such it is understood that the product of these matrix contains an implicit integral:

$$\Delta_\psi^{-1}f(x) \equiv \int d^d y \Delta_\psi^{-1}(x,y)f(y) \tag{1.4.15}$$

Similar to $\Delta_\psi^{-1}(x)$ we define $\overleftarrow{\Delta}_\psi^{-1}$ in such a way that:

$$\int d^d y f(y)\overleftarrow{\Delta}_\psi^{-1}(x,y)\overleftarrow{\Delta}_\psi(x) = f(x) \tag{1.4.16}$$

where the left arrow indicates where the operator is acting. With these definitions of matrix-valued Green's function, the light fermionic part of the second variation can be expressed as:

$$\delta^2\mathcal{L}_\psi = \frac{1}{2}\left[\psi'^T + \left(\Psi'^T\tilde{\mathbf{X}}_{\Psi\psi} + \Phi'^T\tilde{\mathbf{X}}_{\Phi\psi} + \phi'^T\tilde{\mathbf{X}}_{\phi\psi}\right)\overleftarrow{\Delta}_\psi^{-1}\right]\Delta_\psi\left[\psi' + \Delta_\psi^{-1}\left(\tilde{\mathbf{X}}_{\psi\Psi}\Psi' - \tilde{\mathbf{X}}_{\psi\phi}\Phi' - \tilde{\mathbf{X}}_{\psi\phi}\phi'\right)\right]$$
$$- \frac{1}{2}\left[\Psi'^T\tilde{\mathbf{X}}_{\Psi\psi} + \Phi'^T\tilde{\mathbf{X}}_{\Phi\psi} + \phi'^T\tilde{\mathbf{X}}_{\phi\psi}\right]\Delta_\psi^{-1}\left[\tilde{\mathbf{X}}_{\psi\Psi}\Psi' - \tilde{\mathbf{X}}_{\psi\Phi}\Phi' - \tilde{\mathbf{X}}_{\psi\phi}\phi'\right] \tag{1.4.17}$$

Next, we shift the light fermion multiplet as:

$$\psi'' = \psi' + \Delta_\psi^{-1}\left(\tilde{\mathbf{X}}_{\psi\Psi}\Psi' - \tilde{\mathbf{X}}_{\psi\phi}\Phi' - \tilde{\mathbf{X}}_{\psi\phi}\phi'\right) \tag{1.4.18}$$

$$\psi''^T = \psi'^T + \left(\Psi'^T\tilde{\mathbf{X}}_{\Psi\psi} + \Phi'^T\tilde{\mathbf{X}}_{\Phi\psi} + \phi'^T\tilde{\mathbf{X}}_{\phi\psi}\right)\overleftarrow{\Delta}_\psi^{-1} \tag{1.4.19}$$

under with the path integral measure is invariant. Since ψ is a multiplet with Majorana-like spinors, the two shifts above are not independent. As proven in Ref. [2], it is required that:

$$\left[\Delta_\psi^{-1}\left(\tilde{\mathbf{X}}_{\psi\Psi}\Psi' - \tilde{\mathbf{X}}_{\psi\Phi}\Phi' - \tilde{\mathbf{X}}_{\psi\phi}\phi'\right)\right]^T = \left(\Psi'^T\tilde{\mathbf{X}}_{\Psi\psi} + \Phi'^T\tilde{\mathbf{X}}_{\Phi\psi} + \phi'^T\tilde{\mathbf{X}}_{\phi\psi}\right)\overleftarrow{\Delta}_\psi^{-1} \tag{1.4.20}$$

After perform this shift in the light fermionic part of the second variation we obtain:

$$\delta^2\mathcal{L}_\psi = \frac{1}{2}\psi''^T\Delta_\psi\psi'' - \frac{1}{2}\Psi'^T\tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi}\Psi' + \frac{1}{2}\Psi'^T\tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi}\Phi' + \frac{1}{2}\Psi'^T\tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\phi}\phi'$$
$$- \frac{1}{2}\Delta\Phi^T\tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi}\Psi' + \frac{1}{2}\Delta\Phi^T\tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi}\Phi' + \frac{1}{2}\Delta\Phi^T\tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\phi}\phi'$$
$$- \frac{1}{2}\phi'^T\tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi}\Psi' + \frac{1}{2}\phi'^T\tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi}\Phi' + \frac{1}{2}\phi'^T\tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\phi}\phi' \tag{1.4.21}$$

Now we can substitute this into Eq. (1.4.2), which will lead us to several terms. To keep the expression compact, we dfefine

$$\bar{\Delta}_A \equiv \Delta_A - \tilde{\mathbf{X}}_{A\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi A} \tag{1.4.22}$$
$$\bar{\mathbf{X}}_{AB} = \tilde{\mathbf{X}}_{AB} - \tilde{\mathbf{X}}_{A\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi B} \tag{1.4.23}$$

where A and B can be any fermionic of scalar multiplet of the theory. Using these definitions, the second variation of the UV Lagrangian takes the form:

$$\delta^2\mathcal{L}_{UV} = \delta^2\bar{\mathcal{L}}_S + \frac{1}{2}\psi''^T\Delta_\psi\psi'' + \frac{1}{2}\Psi'^T\bar{\Delta}_\Psi\Psi' - \frac{1}{2}\Psi'^T\bar{\mathbf{X}}_{\Psi\Phi}\Phi' + \frac{1}{2}\Phi'^T\bar{\mathbf{X}}_{\Phi\Psi}\Psi'$$
$$- \frac{1}{2}\Psi'^T\bar{\mathbf{X}}_{\Psi\phi}\phi' + \frac{1}{2}\phi'^T\bar{\mathbf{X}}_{\phi\Psi}\Psi' \tag{1.4.24}$$

where:

$$\delta^2\bar{\mathcal{L}}_S = -\frac{1}{2}\Phi'^T\bar{\Delta}_\Phi\Phi' - \frac{1}{2}\phi'\bar{\Delta}_\phi\phi' - \frac{1}{2}\Phi'^T\bar{\mathbf{X}}_{\Phi\phi}\phi' - \frac{1}{2}\phi'^T\bar{\mathbf{X}}_{\phi\Phi}\Phi' \tag{1.4.25}$$

Next, we proceed to eliminate the mixed scalar-heavy fermion terms. We first introduce Δ_Ψ^{-1} as the matrix-valued Green's function, analogous to the light fermions. This allows us to rewrite the second variation of the UV Lagrangian as:

$$\delta^2\mathcal{L}_{UV} = \delta^2\bar{\mathcal{L}}_S + \frac{1}{2}\psi''^T\Delta_\psi\psi'' + \frac{1}{2}\left[\Psi'^T + \left(\Phi'^T\bar{\mathbf{X}}_{\Phi\Psi} + \phi'^T\bar{\mathbf{X}}_{\phi\Psi}\right)\overleftarrow{\bar{\Delta}}_\Psi^{-1}\right]\bar{\Delta}_\Psi\left[\Psi' - \bar{\Delta}_\Psi^{-1}\left(\bar{\mathbf{X}}_{\Psi\Phi}\Phi' + \bar{\mathbf{X}}_{\Psi\phi}\phi'\right)\right]$$
$$+ \frac{1}{2}\left[\Phi'^T\bar{\mathbf{X}}_{\Phi\Psi} + \phi'^T\bar{\mathbf{X}}_{\phi\Psi}\right]\bar{\Delta}_\Psi^{-1}\left[\bar{\mathbf{X}}_{\Psi\Phi}\Phi' + \bar{\mathbf{X}}_{\Psi\phi}\phi'\right] \tag{1.4.26}$$

From this expression, we identify an appropriate shift for the heavy fermionic multiplet:[7]

$$\Psi'' = \Psi' + \bar{\Delta}_\Psi^{-1}\left(\bar{\mathbf{X}}_{\Psi\Phi}\Phi' + \bar{\mathbf{X}}_{\Psi\phi}\phi'\right) \tag{1.4.27}$$
$$\Psi''^T = \Psi'^T + \left(\Phi'^T\bar{\mathbf{X}}_{\Phi\Psi} + \phi'^T\bar{\mathbf{X}}_{\phi\Psi}\right)\overleftarrow{\bar{\Delta}}_\Psi^{-1} \tag{1.4.28}$$

Applying this shift to the second variation of the UV Lagrangian yields:

$$\delta^2\mathcal{L}_{UV} = \delta^2\bar{\mathcal{L}}_S + \frac{1}{2}\psi''^T\Delta_\psi\psi'' + \frac{1}{2}\Psi''\bar{\Delta}_\Psi\Psi'' + \Phi'^T\bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi}\Phi' + \Phi'^T\bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\phi'$$
$$+ \phi'^T\bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi}\Phi' + \phi'^T\bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\phi' \tag{1.4.29}$$

Explicitly writing the scalar terms:

_____

[7]These shifts are not independent and satisfy a relation analogous to that for the light fermionic multiplet.

$$\delta^2 \mathcal{L}_{UV} = -\frac{1}{2}\Phi'^T \left[\bar{\Delta}_\Phi - \bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi}\right]\Phi' - \frac{1}{2}\phi'\left[\bar{\Delta}_\phi - \bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\right]\phi' - \frac{1}{2}\Phi'^T\left[\bar{\mathbf{X}}_{\Phi\phi} - \bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\right]\phi'$$

$$- \frac{1}{2}\phi'^T\left[\bar{\mathbf{X}}_{\phi\Phi} - \bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi}\right]\Phi' + \frac{1}{2}\psi''^T\Delta_\psi\psi'' + \frac{1}{2}\Psi''\bar{\Delta}_\Psi\Psi''$$

$$= -\frac{1}{2}\begin{pmatrix}\Phi'^T & \phi'^T\end{pmatrix}\begin{pmatrix}\bar{\Delta}_\Phi - \bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi} & \bar{\mathbf{X}}_{\Phi\phi} - \bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi} \\ \bar{\mathbf{X}}_{\phi\Phi} - \bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi} & \bar{\Delta}_\phi - \bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\end{pmatrix}\begin{pmatrix}\Phi' \\ \phi'\end{pmatrix} + \frac{1}{2}\psi''^T\Delta_\psi\psi'' + \frac{1}{2}\Psi''\bar{\Delta}_\Psi\Psi''$$

$$= -\frac{1}{2}\begin{pmatrix}\Phi'^T & \phi'^T\end{pmatrix}\mathcal{Q}_S\begin{pmatrix}\Phi' \\ \phi'\end{pmatrix} + \frac{1}{2}\psi''^T\Delta_\psi\psi'' + \frac{1}{2}\Psi''\bar{\Delta}_\Psi\Psi'' \tag{1.4.30}$$

where we introduce the folowing compact notation for the scalar fluctuation operator:

$$\mathcal{Q}_S \equiv \begin{pmatrix}\hat{\Delta}_\Phi & \hat{\mathbf{X}}_{\Phi\phi} \\ \hat{\mathbf{X}}_{\phi\Phi} & \hat{\Delta}_\phi\end{pmatrix} \equiv \begin{pmatrix}\bar{\Delta}_\Phi - \bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi} & \bar{\mathbf{X}}_{\Phi\phi} - \bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi} \\ \bar{\mathbf{X}}_{\phi\Phi} - \bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi} & \bar{\Delta}_\phi - \bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\end{pmatrix} \tag{1.4.31}$$

Notice that all mixed fermionic-bosonic fluctuation terms have been eliminated. As a result, the fluctuation operator is now block-diagonal, allowing us to evaluate the path integrals separately according to the statistics of each field. The first term corresponds to the bosonic sector, while the second and third terms belong to the fermionic sector. Therefore, after splitting the fields into classical backgrounds and quantum fluctuations, and truncating the expansion at quadratic order in the fluctuations, the generating functional for the light degrees of freedom in the UV theory becomes:

$$Z_{L,UV}[J_\phi, J_\psi] = Z_{UV}^{\text{tree}}[J_\phi, J_\psi]\int D\Phi' D\phi' \exp\left\{-\frac{i}{2}\int d^d x \begin{pmatrix}\Phi'^T & \phi'^T\end{pmatrix}\mathcal{Q}_S\begin{pmatrix}\Phi' \\ \phi'\end{pmatrix}\right\}$$

$$\int D\Psi'' \exp\left\{\frac{i}{2}\int d^d x\,\psi''^T\Delta_\psi\psi''\right\}\int D\Psi'' \exp\left\{\frac{i}{2}\int d^d x\,\Psi''\bar{\Delta}_\Psi\Psi''\right\}$$

$$\propto Z_{UV}^{\text{tree}}(\det \mathcal{Q}_S)^{-\frac{1}{2}}\det \Delta_\psi \det \bar{\Delta}_\Psi \tag{1.4.32}$$

Performing the Legendre transformation to obtain the $1LPI$ generating functional and already separating the one-loop contributions:

$$\Gamma_{L,UV}^{\text{1-loop}}[\phi_b, \psi_b] = \frac{i}{2}\ln\det Q_S - i\ln\det \Delta_\psi - i\ln\det \bar{\Delta}_\Psi \tag{1.4.33}$$

By the procedure used in the real scalar case, the next step is to compute the $1PI$ quantum action for the EFT and then perform the one-loop matching. However, recall that when we applied the expansion-by-regions method in one-loop matching, the role of the EFT was to cancel the non-vanishing soft contributions of the UV theory. This reflects the physical insight that the one-loop EFT Lagrangian captures the short-distance behavior of the full theory, while the long-distance effects are already encoded in the tree-level EFT. Therefore, by analogy with the real scalar case, the one-loop spacetime integral of the EFT Lagrangian must match the hard contribution of the one-loop $1PI$ action of the UV theory [3]:

$$\int d^d x \mathcal{L}_{EFT}^{\text{1-loop}}[\phi_b, \psi_b] = \left[\frac{i}{2}\ln\det Q_S - i\ln\det \Delta_\psi - i\ln\det \bar{\Delta}_\Psi\right]_{\text{hard}} = \left[\frac{i}{2}\ln\det Q_S - i\ln\det \bar{\Delta}_\Psi\right]_{\text{hard}} \tag{1.4.34}$$

where the contribution $\ln\det \Delta_\psi$ vanishes in the hard region, since it involves only light fields.

To make the computation as transparent as possible, we separate the one-loop EFT Lagrangian into two distinct contributions: one from the bosonic path integral and another from the fermionic path integral,

$$\mathcal{L}_{EFT}^{\text{1-loop}}[\phi_b, \psi_b] = \mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] + \mathcal{L}_{EFT,F}^{\text{1-loop}}[\phi_b, \psi_b] \tag{1.4.35}$$

with:

$$\int d^d x \mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] = \frac{i}{2}\ln\det \mathcal{Q}_S\big|_{\text{hard}} \tag{1.4.36}$$

$$\int d^d x \mathcal{L}_{EFT,F}^{\text{1-loop}}[\phi_b, \psi_b] = -i\ln\det \bar{\Delta}_\Psi\big|_{\text{hard}} \tag{1.4.37}$$

Let us first compute the bosonic contribution. Following the same steps as in the real scalar case, we begin by rewriting $\mathcal{Q}_S$ in block-diagonal form via field redefinitions:

$$\mathcal{Q}'_S = \begin{pmatrix} \hat{\Delta}_\Phi - \hat{\mathbf{X}}_{\Phi\phi}\hat{\Delta}_\phi^{-1}\hat{\mathbf{X}}_{\phi\Phi} & 0 \\ 0 & \hat{\Delta}_\phi \end{pmatrix} \tag{1.4.38}$$

Therefore, the bosonic one-loop contribution becomes:

$$\int d^d x \mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] = \frac{i}{2}\left[\ln\det\left(\hat{\Delta}_\Phi - \hat{\mathbf{X}}_{\Phi\phi}\hat{\Delta}_\phi^{-1}\hat{\mathbf{X}}_{\phi\Phi}\right) + \ln\det\hat{\Delta}_\phi\right]_{\text{hard}} \tag{1.4.39}$$

Following the exact same steps as in section 1.2.5, we write:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] = \frac{i}{2}\int \frac{d^d q}{(2\pi)^d}\left[\operatorname{tr}\ln\left(\hat{\Delta}_\Phi - \hat{\mathbf{X}}_{\Phi\phi}\hat{\Delta}_\phi^{-1}\hat{\mathbf{X}}_{\phi\Phi}\right) + \operatorname{tr}\ln\hat{\Delta}_\phi\right]\Bigg|_{\text{hard}}^{P\to P-q} \tag{1.4.40}$$

Note that, in contrast to the real scalar case, $\hat{\Delta}\phi$ now includes contributions from heavy fermions. This is a consequence of the field redefinitions used to block-diagonalize the fluctuation operator and separate fields according to their statistics. As a result, $\operatorname{tr}\ln\hat{\Delta}\phi$ do not vanishes entirely in the hard region. However, we can isolate the non-vanishing part by unpacking the hatted operator definition:

$$\operatorname{tr}\ln\hat{\Delta}_\phi = \operatorname{tr}\ln\left(\bar{\Delta}_\phi - \bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\right) = \operatorname{tr}\ln\bar{\Delta}_\phi + \operatorname{tr}\ln\left(1 - \bar{\Delta}_\phi^{-1}\bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\right) \tag{1.4.41}$$

Here, the first term on the right-hand side contains only light-field contributions (both fermionic and bosonic) and thus vanishes in the hard region. The second term, however, contains heavy fermions terms and remains non-zero. Therefore:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] = \frac{i}{2}\int \frac{d^d q}{(2\pi)^d}\left[\operatorname{tr}\ln\left(\hat{\Delta}_\Phi - \hat{\mathbf{X}}_{\Phi\phi}\hat{\Delta}_\phi^{-1}\hat{\mathbf{X}}_{\phi\Phi}\right) + \operatorname{tr}\ln\left(1 - \bar{\Delta}_\phi^{-1}\bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\right)\right]\Bigg|_{\text{hard}}^{P\to P-q} \tag{1.4.42}$$

In principle, one can proceed further by expressing the hatted and barred operators in terms of the original second variation quantities and applying the CDE method, as done in section (1.2.6). However, this generates several terms without offering significant insight. Instead, we will illustrate the procedure explicitly through a concrete example.

We now proceed to evaluate the contribution from the fermionic path integral:

$$\int d^d x \mathcal{L}_{EFT,F}^{\text{1-loop}}[\phi_b, \psi_b] = -i\ln\det\bar{\Delta}_\Psi\big|_{\text{hard}} = -i\operatorname{Tr}\ln\left(\Delta_\Psi - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi}\right)\big|_{\text{hard}} \tag{1.4.43}$$

Following the steps of section 1.2.5 on how to calculate functional determinants:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}}[\phi_b, \psi_b] = -i\int \frac{d^d q}{(2\pi)^d}\operatorname{tr}\ln\left(\Delta_\Psi - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi}\right)\Bigg|_{\text{hard}}^{P\to P-q} \tag{1.4.44}$$

Using the structure of Δ_Ψ (Eq. (1.4.9)), we rewrite the integrand as:

$$
\begin{aligned}
\mathcal{L}_{EFT,F}^{\text{1-loop}}[\phi_b, \psi_b] = & -i\int \frac{d^d q}{(2\pi)^d}\operatorname{tr}\ln\left(\mathcal{C}\tilde{\mathbb{1}}\left(\slashed{P} - M_\Psi\right) + \tilde{\mathbf{X}}_{\Psi\Psi} - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi}\right)\Bigg|_{\text{hard}}^{P\to P-q} \\
= & -i\int \frac{d^d q}{(2\pi)^d}\Bigg\{\operatorname{tr}\ln\left(\mathcal{C}\tilde{\mathbb{1}}(-\slashed{q} - M_\Psi)\right) \\
& + \operatorname{tr}\ln\left[1 - (-\slashed{q} - M_\Psi)^{-1}\left(-\slashed{P} - \tilde{\mathbb{1}}\mathcal{C}^{-1}\left(\tilde{\mathbf{X}}_{\Psi\Psi} - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi}\right)^{P\to P-q}\right)\right]\Bigg\}_{\text{hard}}
\end{aligned} \tag{1.4.45}
$$

The first term is a field-independent constant and can be absorbed into the normalization of the path integral. Before proceeding with the CDE, it is convenient to eliminate explicit factors of $\tilde{\mathbb{1}}$ and $\mathcal{C}$ by defining:

$$\mathbf{X}_{\Psi A} = \tilde{\mathbb{1}} \mathcal{C}^{-1} \tilde{\mathbf{X}}_{\Psi A}, \quad A \in \{\Psi, \psi\} \tag{1.4.46}$$

With this, the expression becomes:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}}[\phi_b, \psi_b] = -i \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \ln \left[1 - (-\slashed{q} - M_\Psi)^{-1} \left(-\slashed{P} - \left(\mathbf{X}_{\Psi\Psi} - \mathbf{X}_{\Psi\psi} \Delta_\psi^{-1} \tilde{\mathbf{X}}_{\psi\Psi} \right)^{P \to P-q} \right) \right]_{\text{hard}} \tag{1.4.47}$$

Although Δ_ψ still contains $\mathcal{C}$ and $\tilde{\mathbb{1}}$ explicitly, these can also be absorbed. To see this, we apply Eq. (1.4.9) with $\Psi \to \psi$ and expand:

$$\Delta_\psi^{-1}\big|_{\text{hard}} = \left[\mathcal{C} \tilde{\mathbb{1}} (\slashed{P} - \slashed{q} - M_\psi) + \tilde{\mathbf{X}}_{\psi\psi} \right]^{-1} \bigg|_{\text{hard}}$$

$$= \left[1 - (-\slashed{q} - M_\psi)^{-1} \left(\tilde{\mathbb{1}} \mathcal{C}^{-1} (-\mathcal{C} \tilde{\mathbb{1}} \slashed{P} - \tilde{\mathbf{X}}_{\psi\psi}) \right) \right]^{-1} (-\slashed{q} - M_\psi)^{-1} \tilde{\mathbb{1}} \mathcal{C}^{-1} \bigg|_{\text{hard}} \tag{1.4.48}$$

In the hard region, the loop momentum scales as $q \sim M_\Psi, M_\Phi$, which is much larger than the light fermion mass M_ψ, any external momenta, or the operators in $\tilde{\mathbf{X}}_{\psi\psi}$. The latter is at most of zeroth order in the heavy mass expansion, since it contains two light fermion fields and does not correspond to a mass term. As a result, the term involving these light quantities appears suppressed relative to the dominant hard scale. This hierarchy justifies expanding the relevant expression as a Taylor series in powers of small ratios involving light fields and derivatives over the large momentum q:

$$\Delta_\psi^{-1}\big|_{\text{hard}} = \sum_{n=0}^{+\infty} \left[(-\slashed{q} - M_\psi)^{-1} \tilde{\mathbb{1}} \mathcal{C}^{-1} (-\mathcal{C} \tilde{\mathbb{1}} \slashed{P} - \tilde{\mathbf{X}}_{\psi\psi}) \right]^n (-\slashed{q} - M_\psi)^{-1} \tilde{\mathbb{1}} \mathcal{C}^{-1} \bigg|_{\text{hard}}$$

$$= \sum_{n=0}^{+\infty} \left[(-\slashed{q} - M_\psi)^{-1} (-\slashed{P} - \mathbf{X}_{\psi\psi}) \right]^n (-\slashed{q} - M_\psi)^{-1} \tilde{\mathbb{1}} \mathcal{C}^{-1} \bigg|_{\text{hard}} \tag{1.4.49}$$

where we have defined:

$$\mathbf{X}_{\psi\psi} \equiv \tilde{\mathbb{1}} \mathcal{C}^{-1} \tilde{\mathbf{X}}_{\psi\psi} \tag{1.4.50}$$

Therefore, Δ_ψ^{-1} in the hard region always appears with a rightmost factor of $\tilde{\mathbb{1}} \mathcal{C}^{-1}$. As a result, in any product involving Δ_ψ^{-1} and some $\tilde{\mathbf{X}}_{AB}$, these factors can be absorbed via redefinitions. For example:

$$\Delta_\psi^{-1} \tilde{\mathbf{X}}_{\psi\Psi} = \sum_{n=0}^{+\infty} \left[(-\slashed{q} - M_\psi)^{-1} \tilde{\mathbb{1}} \mathcal{C}^{-1} (-\mathcal{C} \tilde{\mathbb{1}} \slashed{P} - \tilde{\mathbf{X}}_{\psi\psi}) \right]^n (-\slashed{q} - M_\psi)^{-1} \mathbf{X}_{\psi\Psi} \bigg|_{\text{hard}}, \quad \mathbf{X}_{\psi\Psi} \equiv \tilde{\mathbb{1}} \mathcal{C}^{-1} \tilde{\mathbf{X}}_{\psi\Psi} \tag{1.4.51}$$

The same reasoning applies to insertions of $\bar{\Delta}\psi^{-1}$. A similar property holds for the operators $\tilde{\mathbf{X}}_{\Phi B}$ and $\tilde{\mathbf{X}}_{\phi B}$, which always appear as $\mathbf{X}_{\Phi B} = \tilde{\mathbb{1}} \tilde{\mathbf{X}}_{\Phi B}$ and $\mathbf{X}_{\phi B} = \tilde{\mathbb{1}} \tilde{\mathbf{X}}_{\phi B}$. Hence, the full result can ultimately be expressed in terms of the redefined operators $\mathbf{X}_{AB}$, without any explicit factors of $\tilde{\mathbb{1}}$ or $\mathcal{C}$ appearing in the final operator expressions.

Returning to the computation of the one-loop EFT Lagrangian from the fermionic path integral (Eq. (1.4.47)), we can now proceed with the covariant derivative expansion (CDE). As in the case of the real scalar field and in Eq. (1.4.48), the argument of the logarithm contains a small fraction due to the fact that the loop momentum lies in the hard region. Consequently, by the same reasoning presented earlier, we can expand the logarithm as a Taylor series:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}}[\phi_b, \psi_b] = i \sum_{n=0}^{\infty} \frac{1}{n} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \left[(-\slashed{q} - M_\Psi)^{-1} \left(-\slashed{P} - \left(\mathbf{X}_{\Psi\Psi} - \mathbf{X}_{\Psi\psi} \Delta_\psi^{-1} \tilde{\mathbf{X}}_{\psi\Psi} \right)^{P \to P-q} \right) \right]_{\text{hard}}^n \tag{1.4.52}$$

This expression, combined with Eq. (1.4.42), gives the full one-loop EFT Lagrangian resulting from integrating out heavy fermions and scalars. We will further develop some of these expressions and elaborate on the details in the context of an explicit example.

It is worth noting that this is not the most general result for integrating out heavy fields. In particular, one can also include contributions from heavy vector bosons. Although we will not consider those here, the relevant discussion can be found in Ref. [21].

1.5   DM toy model

In the previous section, we extended the matching procedure used to derive an effective theory to also include the integration of heavy fermions and complex scalar fields from the UV model. To illustrate this method, and especially to explore the CDE of the scalar path integral's contribution to the one-loop effective Lagrangian, we now apply it to a toy model based on an extension to a simplified version of the Standard Model[8] (SM). This model introduces a vector-like fermionic top-quark partner (T) and a scalar singlet under the SM Gauge group (s).

Upon imposing a $\mathcal{Z}_2$ symmetry, the scalar singlet can serve as a dark matter (DM) candidate, while the top partner mediates interactions between the DM and SM sectors. The Lagrangian describing the interactions of these new particles, as well as their couplings to the SM fields, is given by:

$$\mathcal{L}_{UV} = \mathcal{L}_{SSM} + \bar{\psi}_T \left(i\gamma^\mu D_\mu - M_T \right) \psi_T + \frac{1}{2}(\partial_\mu s)^2 - \frac{1}{2}m_s^2 s^2 - y_{DM} \left(\bar{\psi}_T t_R + \bar{t}_R \psi_T \right) s - V(s) \quad (1.5.1)$$

where:

$$V(s) = \frac{1}{4}y_{Hs}h^2 s^2 + \frac{\lambda_s}{4!}s^4 \tag{1.5.2}$$

$$\mathcal{L}_{SSM} = -\frac{1}{4}G^{A\mu\nu}G^A_{\mu\nu} + i\bar{q}\gamma^\mu D_\mu q + \bar{t}\left(i\gamma^\mu D_\mu - m_t \right) t + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h h^2 - \frac{\lambda_h}{4!}h^4 - y_{Ht}h\left(\bar{t}_L t_R + \bar{t}_R t_L \right) \tag{1.5.3}$$

Here $G^{\mu\nu}$ denotes the field strength tensor associated with the $SU(3)_c$ Gauge group, q represents the fields of lighter quarks than the top, t is te top quark field, t_R is his right handed component. Furthermore, T is assumed to be a singlet under $SU(2)_L$.

For sufficient large values of M_T and M_s, we can apply the procedure outlined in the previous section to integrate out the BSM fields and construct the corresponding (EFT), which includes higher-dimensional operators capturing their low-energy effects.

To begin the calculation of the EFT Lagrangian, our first step is to identify the heavy and light multiplets among the fields appearing in the UV Lagrangian:

$$\Phi = s, \quad \phi = h, \quad \Psi = \begin{pmatrix} \psi_T \\ \psi_T^C \end{pmatrix}, \quad \psi = \begin{pmatrix} t \\ q \end{pmatrix} \tag{1.5.4}$$

where Φ and Ψ denote the heavy scalar and fermionic multiplets, respectively, and ϕ and ψ represent the light SM fields.

With this identification, we are now ready to construct the effective Lagrangian, which consists of tree-level and one-loop contributions:

$$\mathcal{L}_{EFT} = \mathcal{L}_{EFT}^{\text{tree}} + \mathcal{L}_{EFT}^{\text{1-loop}} \tag{1.5.5}$$

which can be computed separately.

1.5.1   Tree-level matching

At tree level, according to our matching procedure (Eq. 1.4.1), the EFT Lagrangian is given by:

$$\mathcal{L}_{EFT}^{\text{tree}} = \mathcal{L}_{UV}[\hat{\Psi}_c[\psi, \phi], \hat{\Phi}_c[\psi, \phi], \psi, \phi] \tag{1.5.6}$$

where $\hat{\Psi}_c[\psi, \phi]$ and $\hat{\Phi}_c[\psi, \phi]$ are the classical solutions to the equations of motion for the heavy fields. Therefore, to proceed we solve these classical EOMs, which are derived from the Euler-Lagrangian equations of the UV Lagrangian $\mathcal{L}_{UV}$. There are:

$$\left[\Box + M_s^2 + \frac{1}{2}y_{Hs}h^2 + \frac{\lambda_s}{6}s_c^2 \right] s_c = -y_{DM} \left(\bar{\psi}_T t_R + \bar{t}_R \psi_T \right) \tag{1.5.7}$$

$$\left(i\slashed{D}^{\psi_T} - M_T \right) \psi_{T,c} = y_{DM} t_R s_c \tag{1.5.8}$$

$$\left(i\slashed{D}^{\psi_T} + M_T \right) \bar{\psi}_{T,c} = -y_{DM} \bar{t}_R s_c \tag{1.5.9}$$

The classical solutions that contribute to the EFT up to dimension-six operators are:

$$s_c = 0, \quad \psi_{T,c} = 0 \quad \text{and} \quad \bar{\psi}_{T,c} = 0 \tag{1.5.10}$$

---

[8]By "simplified version of the Standard Model," we mean that the Higgs field is treated as a real scalar singlet with a mass of 125 GeV, corresponding to the observed resonance peak.

The derivation of these solutions can be found in Appendix A.1. Substituting these classical solutions into the UV Lagrangian (Eq. 1.5.1), we obtain the tree-level EFT Lagrangian:

$$\mathcal{L}_{EFT}^{\text{tree}} = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2 h^2 + \bar{t}\left(i\slashed{D} - m_t\right)t + i\bar{q}\slashed{D}q - y_{Ht}h\bar{t}t - \frac{\lambda_h}{4!}h^4 \tag{1.5.11}$$

It is interesting to note that, since the solutions to the equations of motion for the heavy fields vanish at leading order, there are no tree-level contributions to the EFT involving higher-dimensional operators.

1.5.2   One-loop matching

Having completed the tree-level matching, we now proceed to compute the one-loop corrections to the effective Lagrangian. As discussed previously (Eq. 1.4.35), the full one-loop EFT Lagrangian splits into scalar and fermionic contributions:

$$\mathcal{L}_{EFT}^{\text{1-loop}} = \mathcal{L}_{EFT,SF}^{\text{1-loop}} + \mathcal{L}_{EFT,F}^{\text{1-loop}} \tag{1.5.12}$$

According to Eqs. (1.4.42) and (1.4.52) the scalar (SF) and fermionic (F) components are given respectively by:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} = \frac{i}{2}\int \frac{d^d q}{(2\pi)^d}\left[\text{tr}\ln\left(\hat{\Delta}_\Phi - \hat{\mathbf{X}}_{\Phi\phi}\hat{\Delta}_\phi^{-1}\hat{\mathbf{X}}_{\phi\Phi}\right) + \text{tr}\ln\left(1 - \bar{\Delta}_\phi^{-1}\bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\right)\right]\Bigg|_{\text{hard}}^{P\to P-q}$$

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} = i\sum_{n=0}^{\infty}\frac{1}{n}\int \frac{d^d q}{(2\pi)^d}\,\text{tr}\left[(-\slashed{q} - M_\Psi)^{-1}\left(-\slashed{P} - \left(\mathbf{X}_{\Psi\Psi} - \mathbf{X}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi}\right)^{P\to P-q}\right)\right]_{\text{hard}}^n \tag{1.5.13}$$

Before proceeding to compute these loop contributions, we first determine the interaction matrices $\tilde{\mathbf{X}}$ using Eqs. (1.4.5) and (1.4.6), based on their definition:

$$(X_{AB})_{ij} \equiv -\frac{\delta^2\mathcal{L}_{UV,\text{int}}}{\delta A_i \delta B_j}\Bigg|_{A=A_b, B=B_b} \tag{1.5.14}$$

The non-zero interaction matrices are:

$$\tilde{\mathbf{X}}_{\Psi\Phi} = y_{DM}\begin{pmatrix} 0 & 0 & \bar{t}_R \\ 0 & 0 & \mathcal{C}t_R \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{X}}_{\Phi\Psi} = y_{DM}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{t}_R & t_R\mathcal{C}^{-1} & 0 \end{pmatrix}, \quad \tilde{\mathbf{X}}_{\psi\psi} = y_{Ht}\begin{pmatrix} 0 & h\mathcal{C}^{-1} & 0 \\ \mathcal{C}h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.5.15}$$

$$\tilde{\mathbf{X}}_{\psi\phi} = y_{Ht}\begin{pmatrix} 0 & 0 & \bar{t} \\ 0 & 0 & \mathcal{C}t \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{X}}_{\phi\psi} = y_{Ht}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{t} & t\mathcal{C}^{-1} & 0 \end{pmatrix}, \quad \tilde{\mathbf{X}}_{\Phi\Phi} = \frac{1}{2}y_{Hs}h^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.5.16}$$

$$\tilde{\mathbf{X}}_{\phi\phi} = \frac{\lambda_h}{2}h^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.5.17}$$

To decouple the bosonic and fermionic degrees of freedom, we performed shifts that redefine the propagators and interaction matrices. This leads to the barred notation:

$$\bar{\Delta}_A \equiv \Delta_A - \tilde{\mathbf{X}}_{A\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi A} \tag{1.5.18}$$

$$\bar{\mathbf{X}}_{AB} = \tilde{\mathbf{X}}_{AB} - \tilde{\mathbf{X}}_{A\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi B} \tag{1.5.19}$$

and the only non-vanishing are:

$$\bar{\mathbf{X}}_{\Psi\Phi} = \tilde{\mathbf{X}}_{\Psi\Phi} - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi} = \tilde{\mathbf{X}}_{\Psi\Phi} \tag{1.5.20}$$

$$\bar{\mathbf{X}}_{\Phi\Psi} = \tilde{\mathbf{X}}_{\Phi\Psi} - \tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi} = \tilde{\mathbf{X}}_{\Phi\Psi} \tag{1.5.21}$$

$$\bar{\mathbf{X}}_{\psi\psi} = \tilde{\mathbf{X}}_{\psi\psi} - \tilde{\mathbf{X}}_{\psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\psi} \tag{1.5.22}$$

$$\bar{\mathbf{X}}_{\phi\psi} = \tilde{\mathbf{X}}_{\phi\psi} - \tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\psi} \tag{1.5.23}$$

$$\bar{\mathbf{X}}_{\Phi\Phi} = \tilde{\mathbf{X}}_{\Phi\Phi} - \tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi} = \tilde{X}_{\Phi\Phi} \tag{1.5.24}$$

$$\bar{\mathbf{X}}_{\phi\phi} = \tilde{\mathbf{X}}_{\phi\phi} - \tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\phi} \tag{1.5.25}$$

$$\bar{\Delta}_\Psi = \Delta_\Psi - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi} = \Delta_\Psi \tag{1.5.26}$$

$$\bar{\Delta}_\psi = \Delta_\psi - \tilde{\mathbf{X}}_{\psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\psi} \tag{1.5.27}$$

$$\bar{\Delta}_\Phi = \Delta_\Phi - \tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi} = \Delta_\Phi \tag{1.5.28}$$

$$\bar{\Delta}_\phi = \Delta_\phi - \tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\phi} \tag{1.5.29}$$

The definition and calculation of all the interaction matrices in detail can be found in Appendix A.2. Using the simplifications obtained above, we now return to the scalar one-loop contribution $\mathcal{L}_{EFT,SF}^{\text{1-loop}}$. By applying the definition of the hatted matrices (Eq. (1.4.31)) and the relations above, we obtain:

$$\hat{\Delta}_\Phi = \bar{\Delta}_\Phi - \bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi} = \Delta_\Phi - \tilde{\mathbf{X}}_{\Phi\Psi}\Delta_\Psi^{-1}\tilde{\mathbf{X}}_{\Psi\Phi}$$

$$\bar{\Delta}_\phi^{-1}\bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi} = 0$$

$$\hat{\mathbf{X}}_{\Phi\phi}\hat{\Delta}_\phi^{-1}\hat{\mathbf{X}}_{\phi\Phi} = \left(\bar{\mathbf{X}}_{\Phi\phi} - \bar{\mathbf{X}}_{\Phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\right)\left(\bar{\Delta}_\phi - \bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\phi}\right)^{-1}\left(\bar{\mathbf{X}}_{\phi\Phi} - \bar{\mathbf{X}}_{\phi\Psi}\bar{\Delta}_\Psi^{-1}\bar{\mathbf{X}}_{\Psi\Phi}\right) = 0 \quad (1.5.30)$$

Thus, Eq. (1.5.13) simplifies to:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] = \frac{i}{2}\int \frac{d^d q}{(2\pi)^d}\left[\text{tr}\ln\left(\Delta_\Phi - \tilde{\mathbf{X}}_{\Phi\Psi}\Delta_\Psi^{-1}\tilde{\mathbf{X}}_{\Psi\Phi}\right)\right]\Bigg|_{\text{hard}}^{P\to P-q} \quad (1.5.31)$$

Recalling the structure of Δ_Φ:

$$\Delta_\Phi = \tilde{\mathbb{1}}(-P_\Phi^2 + M_\Phi^2) + \tilde{\mathbf{X}}_{\Phi\Phi} \quad (1.5.32)$$

we rewrite the scalar contribution as:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] = \frac{i}{2}\int \frac{d^d q}{(2\pi)^d}\left[\text{tr}\ln\left(\tilde{\mathbb{1}}(-P_\Phi^2 + M_\Phi^2) + \tilde{\mathbf{X}}_{\Phi\Phi} - \tilde{\mathbf{X}}_{\Phi\Psi}\Delta_\Psi^{-1}\tilde{\mathbf{X}}_{\Psi\Phi}\right)\right]\Bigg|_{\text{hard}}^{P\to P-q} \quad (1.5.33)$$

This structure matches the form obtained in the real scalar case before we did the CDE, allowing us to proceed with an analogous expansion. Following the same steps as in Section 1.2.6, we expand the logarithm to obtain:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] = -\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int \frac{d^d q}{(2\pi)^d}\,\text{tr}\left[\frac{2q\cdot P_\Phi - P_\Phi^2 + \left(\mathbf{X}_{\Phi\Phi} - \tilde{\mathbb{1}}\tilde{\mathbf{X}}_{\Phi\Psi}\Delta_\Psi^{-1}\tilde{\mathbf{X}}_{\Psi\Phi}\right)_{P\to P-q}}{q^2 - M_\Phi^2}\right]^n\Bigg|_{\text{hard}} \quad (1.5.34)$$

Notice that, performing an expansion in the hard region as we did in Eq. (1.4.51):

$$\tilde{\mathbb{1}}\tilde{\mathbf{X}}_{\Phi\Psi}\Delta_\Psi^{-1}\tilde{\mathbf{X}}_{\Psi\Phi}\Big|_{\text{hard}} = \sum_{l=0}^{+\infty}\tilde{\mathbb{1}}\tilde{\mathbf{X}}_{\Phi\Psi}\left[(-\slashed{q} - M_\Psi)(-\slashed{P}_\Psi - \mathbf{X}_{\Psi\Psi})\right]^l(-\slashed{q} - M_\Psi)^{-1}\mathbf{X}_{\Psi\Phi}\Big|_{\text{hard}}$$

$$= \sum_{l=0}^{+\infty}\tilde{\mathbb{1}}\tilde{\mathbf{X}}_{\Phi\Psi}\frac{\left[(-\slashed{q} + M_\Psi)(-\slashed{P}_\Psi - \mathbf{X}_{\Psi\Psi})\right]^l(-\slashed{q} + M_\Psi)}{(q^2 - M_\Psi^2)^{l+1}}\mathbf{X}_{\Psi\Phi}\Big|_{\text{hard}}$$

$$= y_{DM}^2\sum_{l=0}^{+\infty}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \bar{t}_R & t_R^T\mathcal{C}^{-1} & 0\end{pmatrix}\frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi)\right]^l(-\slashed{q} + M_T)}{(q^2 - M_T^2)^{l+1}}\begin{pmatrix}0 & 0 & t_R\\ 0 & 0 & \mathcal{C}^{-1}\bar{t}_R^T\\ 0 & 0 & 0\end{pmatrix}\Bigg|_{\text{hard}}$$

$$= y_{DM}^2\sum_{l=0}^{+\infty}\begin{pmatrix}\bar{t}_R & t_R^T\mathcal{C}^{-1}\end{pmatrix}\frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi)\right]^l(-\slashed{q} + M_T)}{(q^2 - M_T^2)^{l+1}}\begin{pmatrix}t_R\\ \mathcal{C}^{-1}\bar{t}_R^T\end{pmatrix}\Bigg|_{\text{hard}} \quad (1.5.35)$$

where we used the fact that covariant derivative matrix is proportional to the identity and:

$$\mathbf{X}_{\Psi\Phi} = \tilde{\mathbb{1}}\mathcal{C}^{-1}\tilde{\mathbf{X}}_{\Psi\Phi} = y_{DM}\tilde{\mathbb{1}}\mathcal{C}^{-1}\begin{pmatrix}0 & 0 & \bar{t}_R\\ 0 & 0 & \mathcal{C}t_R\\ 0 & 0 & 0\end{pmatrix} = y_{DM}'\begin{pmatrix}0 & 0 & t_R\\ 0 & 0 & \mathcal{C}^{-1}\bar{t}_R\\ 0 & 0 & 0\end{pmatrix} \quad (1.5.36)$$

$$\tilde{\mathbb{1}}\tilde{\mathbf{X}}_{\Phi\Psi} = y_{DM}\tilde{\mathbb{1}}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \bar{t}_R & t_R\mathcal{C}^{-1} & 0\end{pmatrix} = y_{DM}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \bar{t}_R & t_R\mathcal{C}^{-1} & 0\end{pmatrix} \quad (1.5.37)$$

$$\mathbf{X}_{\Psi\Psi} = 0_{3\times 3} \quad (1.5.38)$$

Substituting the result (1.5.35) into Eq. (1.5.34) and using that:

$$\tilde{\mathbf{X}}_{\Phi\Phi} = \frac{1}{2}y_{Hs}h^2\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \quad (1.5.39)$$

we get:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] = -\frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr}\left[\frac{2q \cdot P_\Phi - P_\Phi^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_\Phi^2} \right.$$
$$\left. - \frac{y_{DM}^2}{q^2 - M_\Phi^2} \sum_{l=3}^{+\infty} \left(\bar{t}_R \quad t_R^T \mathcal{C}^{-1}\right) \frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi)\right]^l (-\slashed{q} + M_T)}{(q^2 - M_T^2)^{l+1}} \begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R^T \end{pmatrix} \right]^n \Bigg|_{\text{hard}}$$

$$(1.5.40)$$

Each spinor field has dimension 3/2, while each covariant derivative has dimension 1. As a consequence the $l=1$ term has dimension 4, the $l=2$ has dimension 5 and $l=3$ has dimension 6. Therefore, for our purpose, truncate the sum at $l=3$.

Before we proceed and calculate the terms in the l summation, we need to clarify some aspects that will reduce the number of terms. Let's analyze the first values of n.

$n=1$ contribution

Setting $n=1$ into Eq. (1.5.40) give us:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] \supset$$
$$-\frac{i}{2}\int \frac{d^d q}{(2\pi)^d} \operatorname{tr}\left[\frac{\frac{1}{2}y_{Hs}h^2}{q^2 - M_\Phi^2} - \frac{y_{DM}^2}{q^2 - M_\Phi^2}\sum_{l=3}^{+\infty} \left(\bar{t}_R \quad t_R^T \mathcal{C}^{-1}\right) \frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi)\right]^l (-\slashed{q} + M_T)}{(q^2 - M_T^2)^{l+1}} \begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R^T \end{pmatrix} \right]\Bigg|_{\text{hard}}$$

$$(1.5.41)$$

where the P_Φ are ordinary derivatives and, as a consequence, these terms does not contributes since they act on the identity on the right. Furthermore, we can see that all terms will give us operators up to dimension six.

The operators coming from the l summation are:

- $l=0$:

$$\left(\bar{t}_R \quad t_R^T \mathcal{C}^{-1}\right) \frac{(-\slashed{q} + M_T)}{(q^2 - M_T^2)} \begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R^T \end{pmatrix} = \bar{t}_R \frac{(-\slashed{q} + M_T)}{(q^2 - M_T^2)} t_R + t_R^T \mathcal{C}^{-1} \frac{(-\slashed{q} + M_T)}{(q^2 - M_T^2)} \mathcal{C}^{-1}\bar{t}_R^T = 0 \quad (1.5.42)$$

- $l=1$:

$$\left(\bar{t}_R \quad t_R^T \mathcal{C}^{-1}\right) \frac{(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)}{(q^2 - M_T^2)^2} \begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R^T \end{pmatrix} =$$
$$i\frac{\left(\left(\frac{1}{2} + \frac{\epsilon}{8}\right)q^2 - M_T^2\right)}{(q^2 - M_T^2)^2}\left[\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t\right]$$

$$(1.5.43)$$

- $l=2$:

$$\left(\bar{t}_R \quad t_R \mathcal{C}^{-1}\right) \frac{(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)}{(q^2 - M_T^2)^3} \begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R \end{pmatrix} = 0 \quad (1.5.44)$$

- $l=3$

$$\left(\bar{t}_R \quad t_R \mathcal{C}^{-1}\right) \frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi)\right]^3 (-\slashed{q} + M_T)}{(q^2 - M_T^2)^4} \begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R \end{pmatrix} =$$
$$\frac{-i}{(q^2 - M_T^2)^4}\left[\left(\frac{2}{3}q^4 - 2M_\Psi^2 q^2 + M_\Psi^4\right)(\bar{t}\gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t}\gamma^\mu P_R t)\right.$$
$$+ \left(-\frac{1}{3}q^4 + M_\Psi^2 - M_\Psi^4\right)(\bar{t}\gamma^\mu P_R D_\mu D^2 t - \bar{t}\gamma^\mu P_R D_\mu D^2 t \bar{t}\gamma^\mu P_R t)$$
$$+ \left(-\frac{1}{3}q^4 + M_\Psi^2 - M_\Psi^4\right)(\bar{t}\gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t}\gamma^\mu P_R t)$$
$$\left. + \left(M^2 q^2 - M^4\right)(\bar{t}\,\Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t}\,\Gamma^{\mu\nu\rho} P_R t)\right] \quad (1.5.45)$$

where the calculation of this terms can be found in details in Appendix A.3.1. To obtain the $n = 1$ contribution, we substitute (1.5.42), (1.5.43), (1.5.44), (1.5.45) into Eq. (1.5.41):

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset$$

$$-\frac{i}{2} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \left\{ \frac{\frac{1}{2} y_{Hs} h^2}{q^2 - M_s^2} - i y_{DM}^2 \frac{\left((\frac{1}{2} + \frac{\epsilon}{8}) q^2 - M_T^2 \right)}{(q^2 - M_s^2)(q^2 - M_T^2)^2} \left(\bar{t} \gamma^\mu P_R D_\mu t - D_\mu \bar{t} \gamma^\mu P_R t \right) \right.$$

$$- \frac{i y_{DM}^2}{(q^2 - M_s^2)(q^2 - M_T^2)^4} \left[\left(\frac{2}{3} q^4 - 2 M_\Psi^2 q^2 + M_\Psi^4 \right) \left(\bar{t} \gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t} \gamma^\mu P_R t \right) \right.$$

$$+ \left(-\frac{1}{3} q^4 + M_T^2 - M_T^4 \right) \left(\bar{t} \gamma^\mu P_R D_\mu D^2 t - D_\mu D^2 \bar{t} \gamma^\mu P_R t \right)$$

$$+ \left(-\frac{1}{3} q^4 + M_T^2 - M_T^4 \right) \left(\bar{t} \gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t} \gamma^\mu P_R t \right)$$

$$\left. + \left(M_T^2 q^2 - M_T^4 \right) \left(\bar{t} \Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t} \Gamma^{\mu\nu\rho} P_R t \right) \right. \tag{1.5.46}$$

We can compact the loop integrals that appears using the following notation:

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^{\mu_1} q^{\mu_2} \ldots q^{\mu_{2n_c}}}{(q^2 - M_i^2)^{n_i} (q^2 - M_j^2)^{n_j} (q^2)^{n_L}} \equiv \frac{i}{16\pi^2} g^{\mu_1 \mu_2 \cdots \mu_{2n_c}} \tilde{\mathcal{I}}[q^{2n_c}]_{ij\ldots 0}^{n_i, n_j, \ldots, n_L} \tag{1.5.47}$$

where $g^{\mu_1 \mu_2 \cdots \mu_{2n_c}}$ is the completely symmetric combination of metric tensor with $2n_C$ indices. For $n_c = 0$ we define the shorthand notation $\tilde{\mathcal{I}}[q^0]_{ij\ldots 0}^{n_i, n_j, \ldots, n_L} \equiv \tilde{\mathcal{I}}_{ij\ldots 0}^{n_i, n_j, \ldots, n_L}$.

Using this definition, the $n = 1$ contribution can be written as:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset \frac{1}{64\pi^2} y_{Hs} \tilde{\mathcal{I}}_s^{1,0} h^2 - i \frac{y_{DM}^2}{16\pi^2} \left(\left(\frac{1}{4} + \frac{\epsilon}{16} \right) \tilde{\mathcal{I}}_{T,s}^{2,1,-1} - \frac{1}{2} M_T^2 \tilde{\mathcal{I}}_{T,s}^{2,1,0} \right) \left(\bar{t} \gamma^\mu P_R D_\mu t - D_\mu \bar{t} \gamma^\mu P_R t \right)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(\frac{2}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} - 2 M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} + M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) \left(\bar{t} \gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t} \gamma^\mu P_R t \right)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(-\frac{1}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} + M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) \left(\bar{t} \gamma^\mu P_R D_\mu D^2 t - D_\mu D^2 \bar{t} \gamma^\mu P_R t \right)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(-\frac{1}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} + M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) \left(\bar{t} \gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t} \gamma^\mu P_R t \right)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) \left(\bar{t} \Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t} \Gamma^{\mu\nu\rho} P_R t \right) \tag{1.5.48}$$

Notice that since:

$$\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{2}{\epsilon} + \frac{M_S^2}{M_s^2 - M_T^2} - \frac{(M_T^4 - 2 M_T^2 M_s^2)}{(M_T^2 - M_s^2)^2} \ln \left(\frac{M_\Psi^2}{M_s^2} \right) + \ln \left(\frac{\mu^2}{M_s^2} \right) \tag{1.5.49}$$

the product $\frac{\epsilon}{16} \tilde{\mathcal{I}}_{T,s}^{2,1,-1}$ generates a finite contribution of $\frac{1}{8}$. Therefore:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset \frac{1}{64\pi^2} y_{Hs} \tilde{\mathcal{I}}_s^{1,0} h^2 - i \frac{y_{DM}^2}{16\pi^2} \left(\frac{1}{8} + \frac{1}{4} \tilde{\mathcal{I}}_{T,s}^{2,1,-1} - \frac{1}{2} M_T^2 \tilde{\mathcal{I}}_{T,s}^{2,1,0} \right) \left(\bar{t} \gamma^\mu P_R D_\mu t - D_\mu \bar{t} \gamma^\mu P_R t \right)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(\frac{2}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} - 2 M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} + M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) \left(\bar{t} \gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t} \gamma^\mu P_R t \right)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(-\frac{1}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} + M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) \left(\bar{t} \gamma^\mu P_R D_\mu D^2 t - D_\mu D^2 \bar{t} \gamma^\mu P_R t \right)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(-\frac{1}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} + M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) \left(\bar{t} \gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t} \gamma^\mu P_R t \right)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) \left(\bar{t} \Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t} \Gamma^{\mu\nu\rho} P_R t \right) \tag{1.5.50}$$

In order to simplify this expression, we can use this reduction formula to loop integrals:

$$\tilde{\mathcal{I}}[q^{2n_c}]_{ij\ldots 0}^{n_i, n_j, \ldots, n_L} = \frac{1}{M_i^2} \left(\tilde{\mathcal{I}}[q^{2n_c}]_{ij\ldots 0}^{n_i, n_j, \ldots, n_L - 1} - \tilde{\mathcal{I}}[q^{2n_c}]_{ij\ldots 0}^{n_i - 1, n_j, \ldots, n_L} \right) \tag{1.5.51}$$

Using this, we get:

$$
\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset \frac{1}{64\pi^2} y_{Hs} \tilde{\mathcal{I}}_s^{1,0} h^2 + i \frac{y_{DM}^2}{16\pi^2} \left(-\frac{1}{8} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} \right) (\bar{t}\,\gamma^\mu\, P_R\, D_\mu t - D_\mu \bar{t}\,\gamma^\mu\, P_R\, t)
$$

$$
+ i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0} \right)(\bar{t}\,\gamma^\mu\, P_R\, D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t}\,\gamma^\mu\, P_R\, t)
$$

$$
+ i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,,s}^{2,1,0} \right)(\bar{t}\,\gamma^\mu\, P_R\, D_\mu D^2 t - D_\mu D^2 \bar{t}\,\gamma^\mu\, P_R\, t)
$$

$$
+ i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0} \right)(\bar{t}\,\gamma^\mu\, P_R\, D^2 D_\mu t - D^2 D_\mu \bar{t}\,\gamma^\mu\, P_R\, t)
$$

$$
+ i\frac{y_{DM}^2}{16\pi^2}\left(-\frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0} \right)(\bar{t}\,\Gamma^{\mu\nu\rho}\, P_R\, D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t}\,\Gamma^{\mu\nu\rho}\, P_R\, t) \tag{1.5.52}
$$

Now we can proceed to the $n=2$ calculations

$n=2$ contribution

Setting $n=2$ into Eq. (1.5.40), lead us to:

$$
\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset
$$

$$
-\frac{i}{4}\int \frac{d^d q}{(2\pi)^d}\, \text{tr}\left[\frac{-P_s^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} - \frac{y_{DM}^2}{q^2 - M_s^2}\sum_{l=3}^{+\infty}\begin{pmatrix}\bar{t}_R & t_R \mathcal{C}^{-1}\end{pmatrix}\frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi)\right]^l(-\slashed{q}+M_T)}{(q^2 - M_T^2)^{l+1}}\begin{pmatrix}t_R \\ \mathcal{C}^{-1}\bar{t}_R\end{pmatrix}\right] \tag{1.5.53}
$$

where total derivative terms and those involving derivatives acting directly on the identity operator to the right are omitted. Furthermore, we could have a term proportional to $P_s^2 h^2$, however it vanishes if we apply integration by parts.

Since the operators in the l sum starts with dimension 4 and we are interest only in operators up to dimension six, only the term with $l=1$ (Eq. (1.5.43)) contributes to $n=2$, i.e:

$$
\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset
$$

$$
\supset -\frac{i}{4}\int\frac{d^d q}{(2\pi)^d}\,\text{tr}\left[\frac{\frac{1}{4}y_{Hs}^2 h^4}{(q^2 - M_\Phi^2)^2} - \frac{y_{DM}^2 y_{Hs}h^2}{(q^2 - M_\Phi^2)^2}\begin{pmatrix}\bar{t}_R & t_R\mathcal{C}^{-1}\end{pmatrix}\frac{(-\slashed{q}+M_T)(-\slashed{P}_\Psi)(-\slashed{q}+M_T)}{(q^2 - M_T^2)^2}\begin{pmatrix}t_R \\ \mathcal{C}^{-1}\bar{t}_R\end{pmatrix}\right]_{\text{hard}}
$$

$$
\supset -\frac{i}{4}\int\frac{d^d q}{(2\pi)^d}\,\text{tr}\left[\frac{\frac{1}{4}y_{Hs}^2 h^4}{(q^2 - M_\Phi^2)^2} - i\frac{y_{DM}^2 y_{Hs}h^2}{(q^2 - M_\Phi^2)^2}\frac{(\frac{1}{2}q^2 - M_T^2)}{(q^2 - M_T^2)^2}\left[\bar{t}\,\gamma^\mu\, P_R\, D_\mu t - D_\mu \bar{t}\,\gamma^\mu\, P_R\, t\right]\right]_{\text{hard}} \tag{1.5.54}
$$

Applying the notation to loop integrals leads to:

$$
\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset \frac{y_{Hs}^2}{256\pi^2}\tilde{\mathcal{I}}_s^{2,0}h^4 - i\frac{y_{DM}^2 y_{Hs}}{16\pi^2}\left(\frac{1}{8}\tilde{\mathcal{I}}_{T,s}^{2,2,-1} - \frac{1}{4}M_T^2 \tilde{\mathcal{I}}_{T,s}^{2,2,0} \right)h^2 (\bar{t}\,\gamma^\mu\, P_R\, D_\mu t - D_\mu \bar{t}\,\gamma^\mu\, P_R\, t) \tag{1.5.55}
$$

Using the reduction formula (Eq. (1.5.51)):

$$
\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset \frac{y_{Hs}^2}{256\pi^2}\tilde{\mathcal{I}}_s^{2,0}h^4 - i\frac{y_{DM}^2 y_{Hs}}{16\pi^2}\left(\frac{1}{8}\tilde{\mathcal{I}}_{T,s}^{2,2,-1} - \frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,0} \right)h^2 (\bar{t}\,\gamma^\mu\, P_R\, D_\mu t - D_\mu \bar{t}\,\gamma^\mu\, P_R\, t) \tag{1.5.56}
$$

This conclude the $n=2$ contribution to the one-loop EFT Lagrangian. We now turn to the $n=3$ computation.

$n = 3$ contribution

Setting $n = 3$ in Eq. (1.5.40), we obtain:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset -\frac{i}{6} \int \frac{d^d q}{(2\pi)^d}$$

$$\text{tr}\left[\frac{2q \cdot P_s - P_s^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} - \frac{y_{DM}^2}{q^2 - M_s^2} \sum_{l=3}^{+\infty} \left(\bar{t}_R \quad t_R \mathcal{C}^{-1} \right) \frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi) \right]^l (-\slashed{q} + M_T)}{(q^2 - M_T^2)^{l+1}} \begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R \end{pmatrix} \right]$$

$$\left[\frac{2q \cdot P_s - P_s^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} - \frac{y_{DM}^2}{q^2 - M_s^2} \sum_{l=3}^{+\infty} \left(\bar{t}_R \quad t_R \mathcal{C}^{-1} \right) \frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi) \right]^l (-\slashed{q} + M_T)}{(q^2 - M_T^2)^{l+1}} \begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R \end{pmatrix} \right]$$

$$\left[\frac{\frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} - \frac{y_{DM}^2}{q^2 - M_s^2} \sum_{l=3}^{+\infty} \left(\bar{t}_R \quad t_R \mathcal{C}^{-1} \right) \frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi) \right]^l (-\slashed{q} + M_T)}{(q^2 - M_T^2)^{l+1}} \begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R \end{pmatrix} \right]\Bigg|_{\text{hard}} \quad (1.5.57)$$

At this point, the l sum no longer contributes, as every operator generated from it has dimension greater than six. Therefore, the expression above simplifies to:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset -\frac{i}{6} \int \frac{d^d q}{(2\pi)^d} \text{tr}\left[\frac{2q \cdot P_s - P_s^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} \right]\left[\frac{2q \cdot P_s - P_s^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} \right]\left[\frac{\frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} \right]\Bigg|_{\text{hard}}$$

$$\supset -\frac{1}{384\pi^2} y_{Hs}^2 \tilde{\mathcal{I}}_s^{3,0} \left(P_s^2 h^4 + h^2 P_s^2 h^2 \right) + \frac{1}{768\pi^2} y_{Hs}^3 \tilde{\mathcal{I}}_s^{3,0} h^6$$

$$\supset \frac{1}{384\pi^2} y_{Hs}^2 \tilde{\mathcal{I}}_s^{3,0} \left(12h^2 \partial_\mu h \partial^\mu h + 4h^3 \Box h + 2h^3 \Box h + 2h^2 \partial_\mu h \partial^\mu h \right) - \frac{1}{768\pi^2} y_{Hs}^3 \tilde{\mathcal{I}}_s^{3,0} h^6$$

$$\supset \frac{7}{192\pi^2} y_{Hs}^2 \tilde{\mathcal{I}}_s^{3,0} h^2 \partial_\mu h \partial^\mu h + \frac{1}{64\pi^2} y_{Hs}^2 \tilde{\mathcal{I}}_s^{3,0} h^3 \Box h - \frac{1}{768\pi^2} y_{Hs}^3 \tilde{\mathcal{I}}_s^{3,0} h^6 \quad (1.5.58)$$

where in the second to third line we used that $P_s = i\partial$.

$n = 4$ contribution

Similarly to the $n = 3$ contribution, the l sum does not contributes. Therefore, the only operators contribution at dimension six or lower are:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}}$$

$$\supset -\frac{i}{8} \int \frac{d^d q}{(2\pi)^d} \text{tr}\left[\frac{2q \cdot P_s - P_s^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} \right]\left[\frac{2q \cdot P_s - P_s^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} \right]\left[\frac{2q \cdot P_s - P_s^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} \right]\left[\frac{\frac{1}{2}y_{Hs}h^2}{q^2 - M_s^2} \right]\Bigg|_{\text{hard}}$$

$$(1.5.59)$$

Expanding and isolating the relevant dimension-6 operators yields:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset -\frac{i}{8} \int \frac{d^d q}{(2\pi)^d} \text{tr}\left[\frac{y_{Hs}^2 h^2 (q \cdot P_s)(q \cdot P_s)h^2 + y_{Hs}^2 (q \cdot P_s)h^2(q \cdot P_s)h^2 + y_{Hs}^2 (q \cdot P_s)(q \cdot P_s)h^2 h^2}{(q^2 - M_s^2)^4} \right]$$

$$(1.5.60)$$

Using the loop integration symmetry substitution $q^\mu q^\nu \to \frac{g^{\mu\nu}}{4} q^2$, we obtain:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset -\frac{i}{32} y_{Hs}^2 \left(h^2 P_s^2 h^2 + P_s h^2 P_s h^2 + P_s h^2 \right) \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{(q^2 - M_s^2)^4}$$

$$\supset \frac{1}{512\pi^2} y_{Hs}^2 \tilde{\mathcal{I}}_s^{4,-1} \left(2h^3 P_s^2 h + 2h^2 P_{s,\mu} h P_s^\mu h + 6h^2 P_s^2 h + 2h^3 P_s^2 h + 12h^2 P_{s,\mu} h P_s^\mu h + 4h^3 P_s^2 h \right)$$

$$\supset \frac{1}{512\pi^2} y_{Hs}^2 \tilde{\mathcal{I}}_s^{4,-1} \left(8h^3 P_s^2 h + 20h^2 P_{s,\mu} h P_s^\mu h \right)$$

$$\supset -\frac{1}{64\pi^2} y_{Hs}^2 \tilde{\mathcal{I}}_s^{4,-1} h^3 \Box h - \frac{5}{128\pi^2} y_{Hs}^2 \tilde{\mathcal{I}}_s^{4,-1} h^2 \partial_\mu h \partial^\mu h \quad (1.5.61)$$

This completes the set of relevant contributions, since all terms with $n \geq 5$ yield operators of dimension greater than six and are thus beyond the scope of our analysis.

All one-loop contribution up to dimension six for scalar path integral

Having computed all relevant operators up to dimension six, we can now assemble the complete expression. Summing the results from equations (1.5.50), (1.5.56), (1.5.58), and (1.5.60), we obtain the scalar path integral contribution to one-loop effective Lagrangian:

$$
\begin{aligned}
\mathcal{L}^{\text{1-loop}}_{EFT,SF} \supset\; & \frac{1}{64\pi^2} y_{Hs} \tilde{\mathcal{I}}_s^{1,0} h^2 + i\frac{y_{DM}^2}{16\pi^2}\left(-\frac{1}{8} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,-1}\right)(\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t) \\
& + i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t}\gamma^\mu P_R t) \\
& + i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D_\mu D^2 t - D_\mu D^2\bar{t}\gamma^\mu P_R t) \\
& + i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t}\gamma^\mu P_R t) \\
& + i\frac{y_{DM}^2}{16\pi^2}\left(-\frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t}\Gamma^{\mu\nu\rho} P_R t) \\
& + \frac{y_{Hs}^2}{256\pi^2}\tilde{\mathcal{I}}_s^{2,0} h^4 + i\frac{y_{DM}^2 y_{Hs}}{16\pi^2}\left(\frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,0} - \frac{1}{8}\tilde{\mathcal{I}}_{T,s}^{2,2,-1}\right) h^2 (\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t) \\
& + \frac{1}{64\pi^2}y_{Hs}^2\left(\tilde{\mathcal{I}}_s^{3,0} - \tilde{\mathcal{I}}_s^{4,-1}\right)h^3\Box h + \frac{1}{192\pi^2}y_{Hs}^2\left(7\tilde{\mathcal{I}}_s^{3,0} - \frac{15}{2}\tilde{\mathcal{I}}_s^{4,-1}\right)h^2\partial_\mu h\partial^\mu h - \frac{1}{768\pi^2}y_{Hs}^3\tilde{\mathcal{I}}_s^{3,0} h^6
\end{aligned}
$$

(1.5.62)

For future reference, the loop functions for operators containing only the Higgs field are evaluated as follows:

$$
\begin{aligned}
\tilde{\mathcal{I}}_s^{1,0} &= M_s^2\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{M_s^2}\right)\right] \\
\tilde{\mathcal{I}}_s^{2,0} &= \frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{M_s^2}\right) \\
\tilde{\mathcal{I}}_s^{3,0} &= -\frac{1}{2M_s^2} \\
7\tilde{\mathcal{I}}_s^{3,0} - \frac{15}{2}\tilde{\mathcal{I}}_s^{4,-1} &= -\frac{1}{M_s^2} \\
\tilde{\mathcal{I}}_s^{3,0} - \tilde{\mathcal{I}}_s^{4,-1} &= -\frac{1}{6M_s^2} \\
\frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} &= \frac{1}{4}\frac{2}{\epsilon} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,-1}
\end{aligned}
$$

(1.5.63)

Here, the bolded $\tilde{\mathcal{I}}$ indicates that we have extracted the divergent term of the loop functions. These loop functions were evaluated using Package X [17]. The complete notebook containing these calculations is available on GitHub. Incorporating these evaluated loop functions, the Lagrangian simplifies to:

$$
\begin{aligned}
\mathcal{L}^{\text{1-loop}}_{EFT,SF} \supset\; & \frac{1}{64\pi^2} y_{Hs} M_s^2\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{M_s^2}\right)\right] h^2 \\
& + i\frac{y_{DM}^2}{16\pi^2}\left(-\frac{1}{8} + \frac{1}{2}\frac{1}{\epsilon} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,-1}\right)(\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t) \\
& + i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t}\gamma^\mu P_R t) \\
& + i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D_\mu D^2 t - D_\mu D^2\bar{t}\gamma^\mu P_R t) \\
& + i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t}\gamma^\mu P_R t) \\
& + i\frac{y_{DM}^2}{16\pi^2}\left(-\frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t}\Gamma^{\mu\nu\rho} P_R t) \\
& + \frac{y_{Hs}^2}{256\pi^2}\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{M_s^2}\right)\right] h^4 + i\frac{y_{DM}^2 y_{Hs}}{16\pi^2}\left(\frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,0} - \frac{1}{8}\tilde{\mathcal{I}}_{T,s}^{2,2,-1}\right) h^2 (\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t) \\
& - \frac{1}{384\pi^2}\frac{y_{Hs}^2}{M_s^2} h^3\Box h - \frac{1}{192\pi^2}\frac{y_{Hs}^2}{M_s^2} h^2\partial_\mu h\partial^\mu h + \frac{1}{1536\pi^2}\frac{y_{Hs}^3}{M_s^2} h^6
\end{aligned}
$$

(1.5.64)

While the computation involved specific algebraic manipulations of Dirac matrices within the summation terms and additional initial steps preceding the CDE, the overall methodology for deriving this result closely resembled the approach taken in the scalar toy model.

Computing one-loop contribution up to dimension six for fermionic path integral

With the scalar path integral's contribution now established, we shift our focus to the fermionic path integral. The relevant fermionic contribution, as detailed in Section 1.4, is provided by:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} = i \sum_{n=0}^{\infty} \frac{1}{n} \int \frac{d^d q}{(2\pi)^d} \, \text{tr} \left[(-\slashed{q} - M_\Psi)^{-1} \left(-\slashed{P} - \left(\mathbf{X}_{\Psi\Psi} - \mathbf{X}_{\Psi\psi} \Delta_\psi^{-1} \tilde{\mathbf{X}}_{\psi\Psi} \right)^{P \to P-q} \right) \right]_{\text{hard}}^{n} \tag{1.5.65}$$

Based on our previous calculations for the interaction matrices in this toy model (Equations (1.5.15)-(1.5.17)), we found that $\mathbf{X}_{\Psi\Psi} = \mathbf{X}_{\Psi\psi} = 0_{3\times3}$. Therefore, this expression simplifies significantly to:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} = i \sum_{n=0}^{\infty} \frac{1}{n} \int \frac{d^d q}{(2\pi)^d} \, \text{tr} \left[(-\slashed{q} - M_T)^{-1} (-\slashed{P}) \right]_{\text{hard}}^{n} = i \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \int \frac{d^d q}{(2\pi)^d} \, \text{tr} \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right]^{n} \tag{1.5.66}$$

rearranged such that all covariant derivatives are within commutators will not vanish, as there will be no open derivative acting on the identity from the right. These non-vanishing terms will therefore generate pure gauge contributions to the one-loop EFT Lagrangian.

Additionally, the terms corresponding to $n = 0$ and $n = 1$ o not contribute to the effective Lagrangian. The $n = 0$ term yields an infinite constant, while the $n = 1$ term, being a single open covariant derivative that cannot be expressed as a commutator, consequently vanishes.

Given these considerations, we now proceed to compute the contribution from the n=2 term in Eq. (1.5.66):

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{2} \int \frac{d^d q}{(2\pi)^d} \, \text{tr} \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \tag{1.5.67}$$

Since loop integrals with odd powers of q vanishes:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{2} \int \frac{d^d q}{(2\pi)^d} \, \text{tr} \left[\frac{\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2 \slashed{P}\slashed{P}}{(q^2 - M_T^2)^2} \right] \tag{1.5.68}$$

Using the substitution $q^\mu q^\nu \to \frac{g^{\mu\nu}}{4} q^2$ in the loop integral and that $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{2} \int \frac{d^d q}{(2\pi)^d} \, \text{tr} \left[\frac{-\frac{1}{2}\gamma^\mu \gamma^\nu P_\mu P_\nu q^2 + M_T^2 \slashed{P}\slashed{P}}{(q^2 - M_T^2)^2} \right] \tag{1.5.69}$$

The trace over spinor indices is $\text{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$. Therefore:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{2} \int \frac{d^d q}{(2\pi)^d} \, \text{tr} \left[\frac{-2P^2 q^2 + 4M_T^2 P^2}{(q^2 - M_T^2)^2} \right] \tag{1.5.70}$$

Since the commutator $[P^\mu, P_\mu] = 0$, there will be no contribution for Gauge operators coming from $n = 2$. Our next step is to consider the $n = 3$ contribution from Eq. (1.5.66)Eq. (1.5.66), which is given by:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{3} \int \frac{d^d q}{(2\pi)^d} \, \text{tr} \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \tag{1.5.71}$$

When computing the product in the numerator, ignoring terms with odd powers of q, we find that only terms with an odd number of Dirac matrices remain[9]. Consequently, upon taking the trace over spinor indices, every such term will vanish. This implies that the $n = 3$ contribution yields no terms for pure gauge operators.

Proceeding to the contribution $n = 4$ of Eq. (1.5.66), we have:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{4} \int \frac{d^d q}{(2\pi)^d} \, \text{tr} \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \tag{1.5.72}$$

---

[9] This calculations can be found in detail in Appendix A.4

Simplifying the numerator of the integrand and calculating its trace over the spinor indices involves extensive algebra. Therefore, we present the detailed calculation in Appendix A.4. The result obtained is:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset -\frac{1}{64\pi^2}\left[\left(-\frac{8}{3}+\frac{5}{9}\epsilon\right)\tilde{\mathcal{I}}_T^{4,-2} + 8M_T^2\tilde{\mathcal{I}}_T^{4,-1} - 4M_T^4\tilde{\mathcal{I}}_T^{4,0}\right]\text{tr}(P^\mu P^\nu P_\mu P_\nu)$$
$$-\frac{1}{64\pi^2}\left[\left(\frac{4}{3}-\frac{4}{9}\epsilon\right)\tilde{\mathcal{I}}_T^{4,-2} - 4M_T^2\tilde{\mathcal{I}}_T^{4,-1} + 4M_T^4\tilde{\mathcal{I}}_T^{4,0}\right]\text{tr}\left(P^\mu P_\mu P^\nu P_\nu + P^\mu P^\nu P_\nu P_\mu\right) \quad (1.5.73)$$

Using Package X [17][10], the loop functions are evaluated as follows:

$$\tilde{\mathcal{I}}_T^{4,-2} = -\frac{5}{6} + \frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_T^2}\right) \quad (1.5.74)$$

$$\tilde{\mathcal{I}}_T^{4,-1} = -\frac{1}{3M_T^2} \quad (1.5.75)$$

$$\tilde{\mathcal{I}}_T^{4,0} = \frac{1}{6M_T^4} \quad (1.5.76)$$

Therefore, substituting these values, we obtain:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{1}{24\pi^2}\left[\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]\text{tr}(P^\mu P^\nu P_\mu P_\nu) - \frac{1}{48\pi^2}\left[\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]\text{tr}\left(P^\mu P_\mu P^\nu P_\nu + P^\mu P^\nu P_\nu P_\mu\right)$$
$$(1.5.77)$$

Using the cyclic property of the trace:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{1}{24\pi^2}\left[\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]\text{tr}(P^\mu P^\nu P_\mu P_\nu - P^\mu P^\nu P_\nu P_\mu)$$
$$\supset \frac{1}{48\pi^2}\left[\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]\text{tr}([P^\mu, P^\nu][P_\mu, P_\nu]) \quad (1.5.78)$$

Given that $[P_\mu, P_\nu] = ig_s G_{\mu\nu}^a t_R^a$, the expression further simplifies to:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset -\frac{g_s^2}{48\pi^2}\left[\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]G_{\mu\nu}^a G^{b\mu\nu}\,\text{tr}\left(t_R^a t_R^b\right) \quad (1.5.79)$$

Recall that for $SU(N)$, $\text{tr}\left(t_R^a t_R^b\right) = \frac{1}{2}\delta^{ab}$. Therefore:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset -\frac{g_s^2}{96\pi^2}\left[\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]G_{\mu\nu}^a G^{a\mu\nu} \quad (1.5.80)$$

Thus, the $n = 4$ contribution provides a correction to the gauge field strength kinetic term.
We now turn our attention to the n=5 term in the expansion given by Equation (1.5.66).

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{5}\int\frac{d^dq}{(2\pi)^d}\,\text{tr}\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]$$
$$(1.5.81)$$

Similar to the $n = 3$ case, after simplifying the numerator of the integrand and discarding all terms with odd powers of q, only terms with an odd number of Dirac matrices remain. Consequently, when we compute the trace over the spinor indices, all such terms will vanish. Detailed calculations for this can be found in Appendix A.4.

Proceeding to the $n = 6$ contribution of Eq. (1.5.66), we have:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{6}\int\frac{d^dq}{(2\pi)^d}\,\text{tr}\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]$$
$$\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right] \quad (1.5.82)$$

---

[10]The notebook holding this loop functions evaluations can be found in GitHub

Simplifying the numerator of the integrand and calculating its trace over the spinor indices requires extensive algebra, the details of which are provided in Appendix A.4. The result obtained is:

$$
\begin{aligned}
\mathcal{L}_{EFT,F}^{\text{1-loop}} = -\frac{1}{16\pi^2}\frac{1}{6}\,\mathrm{tr}\Bigg\{ & \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\mu P_\nu P_\rho \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 8M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\mu P_\rho P_\nu \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\rho P_\mu P_\nu \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 8M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\nu P_\mu P_\rho \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\nu P_\rho P_\mu \\
& + \left[\frac{8}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\rho P_\nu P_\mu \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 8M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\mu P^\rho P_\nu P_\rho \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 4M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\nu P^\rho P_\rho P_\mu \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 4M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P_\mu P^\nu P_\nu P^\rho P_\rho \\
& + \left[\frac{8}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\nu P_\mu P^\rho P_\rho \\
& + \left[\frac{8}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P_\mu P^\nu P^\rho P_\rho P_\nu \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\nu P^\rho P_\mu P_\rho \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\mu P^\rho P_\rho P_\nu \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P_\mu P^\nu P^\rho P_\nu P_\rho \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\mu P_\nu P^\rho P_\rho \Bigg\}
\end{aligned}
\tag{1.5.83}
$$

Using Package X [17] to compute the loop integrals[11]:

$$
\begin{aligned}
\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset -\frac{1}{16\pi^2}\frac{1}{M_T^2}\frac{1}{6}\,\mathrm{tr}\Bigg\{ & -\frac{2}{15}P^\mu P^\nu P^\rho P_\mu P_\nu P_\rho + \frac{6}{5}P^\mu P^\nu P^\rho P_\mu P_\rho P_\nu - \frac{17}{15}P^\mu P^\nu P^\rho P_\rho P_\mu P_\nu \\
& + \frac{6}{5}P^\mu P^\nu P^\rho P_\nu P_\mu P_\rho - \frac{17}{15}P^\mu P^\nu P^\rho P_\nu P_\rho P_\mu + \frac{8}{15}P^\mu P^\nu P^\rho P_\rho P_\nu P_\mu \\
& + \frac{6}{5}P^\mu P^\nu P_\mu P^\rho P_\nu P_\rho + \frac{13}{15}P^\mu P^\nu P_\nu P^\rho P_\rho P_\mu + \frac{13}{15}P^\mu P_\mu P^\nu P_\nu P^\rho P_\rho \\
& + \frac{8}{15}P^\mu P^\nu P_\nu P_\mu P^\rho P_\rho + \frac{8}{15}P^\mu P_\mu P^\nu P^\rho P_\rho P_\nu - \frac{17}{15}P^\mu P^\nu P_\nu P^\rho P_\mu P_\rho \\
& - \frac{17}{15}P^\mu P^\nu P_\mu P^\rho P_\rho P_\nu - \frac{17}{15}P^\mu P_\mu P^\nu P^\rho P_\nu P_\rho - \frac{17}{15}P^\mu P^\nu P_\mu P_\nu P^\rho P_\rho \Bigg\}
\end{aligned}
\tag{1.5.84}
$$

Exploiting the cyclic property of the trace, the expression can be written as:

$$
\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset -\frac{1}{16\pi^2}\frac{1}{M_T^2}\frac{1}{6}\Bigg\{ -\frac{2}{15}\,\mathrm{tr}(P^\mu P^\nu P^\rho P_\mu P_\nu P_\rho) + \frac{18}{5}\,\mathrm{tr}(P^\mu P^\nu P^\rho P_\nu P_\mu P_\rho) + ... \Bigg\}
\tag{1.5.85}
$$

---

[11]The Mathematica notebook holding this computations can be found in GitHub

These are the only relevant combinations for the matching that we'll perform with operators involving only commutators.

A key feature of CDE is that final operators can always be expressed with all covariant derivatives appearing in commutators. For pure gauge dimension-six operators, there are only two independent forms [1]:

$$\text{tr}([P^\mu, [P_\mu, P_\nu]][P_\rho, [P^\rho, P^\nu]]) \quad \text{and} \quad \text{tr}([P^\mu, P_\nu][P^\nu, P_\rho][P^\rho, P_\mu]) \tag{1.5.86}$$

As discussed in Section 1.2.6, we can match the coefficients of the operators involving only commutators with the ones we've found. By utilizing the cyclic property of the trace, we find[12]:

$$f_1 \text{tr}([P^\mu, [P_\mu, P_\nu]][P_\rho, [P^\rho, P^\nu]]) - f_2 \text{tr}([P^\mu, P_\nu][P^\nu, P_\rho][P^\rho, P_\mu]) \supset f_2 \text{tr}(P^\mu P^\nu P^\rho P_\mu P_\nu P_\rho)$$
$$+ (4f_1 - 3f_2) \text{tr}(P^\mu P^\nu P^\rho P_\nu P_\mu P_\rho) \tag{1.5.87}$$

Matching the result (1.5.84) with the equation above implies:

$$\begin{cases} f_2 = \frac{1}{16\pi^2} \frac{1}{M_T^2} \frac{1}{45} \\ 4f_1 - 3f_2 = -\frac{1}{16\pi^2} \frac{1}{M_T^2} \frac{3}{5} \end{cases} \implies \begin{cases} f_2 = \frac{1}{16\pi^2} \frac{1}{M_T^2} \frac{1}{45} \\ f_1 = -\frac{1}{16\pi^2} \frac{1}{M_T^2} \frac{2}{15} \end{cases} \tag{1.5.88}$$

Therefore, the dimension six pure Gauge operators are:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{1}{16\pi^2} \frac{1}{M_T^2} \left\{ -\frac{2}{15} \text{tr}([P^\mu, [P_\mu, P_\nu]][P_\rho, [P^\rho, P^\nu]]) - \frac{1}{45} \text{tr}([P^\mu, P_\nu][P^\nu, P_\rho][P^\rho, P_\mu]) \right\} \tag{1.5.89}$$

Recall that $[P_\mu, P_\nu] = ig_s G_{\mu\nu}^a t_R^a$. Then:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{1}{16\pi^2} \frac{g_s^2}{M_T^2} \left\{ \frac{2}{15} \text{tr}(t_R^a t_R^b)[P^\mu, G_{\mu\nu}^a][P_\rho, G_{\rho\nu}^b] + \frac{ig_s}{45} \text{tr}(t_R^a t_R^b t_R^c) G^{a\mu}_{\ \ \nu} G^{b\nu}_{\ \ \rho} G^{c\rho}_{\ \ \mu} \right\} \tag{1.5.90}$$

For $SU(3)$, we have[22]:

$$\text{tr}(t_R^a t_R^b) = \frac{1}{2} \delta^{ab} \quad \text{and} \quad \text{tr}(t_R^a t_R^b t_R^c) = \frac{1}{4}(d^{abc} + if^{abc}) \tag{1.5.91}$$

Therefore:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{1}{16\pi^2} \frac{g_s^2}{M_T^2} \left\{ \frac{1}{15}[P^\mu, G_{\mu\nu}^a][P_\rho, G_{\rho\nu}^b] - \frac{g_s}{180} f^{abc} G^{a\mu}_{\ \ \nu} G^{b\nu}_{\ \ \rho} G^{c\rho}_{\ \ \mu} \right\} \tag{1.5.92}$$

Using Bianchi identity and integration by parts, $\text{tr}(A(P_\mu B)) = -\text{tr}((P_\mu A)B) + \text{total derivative}$, the above terms can be arranged into just these two dimension-six operators[5]:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{1}{16\pi^2} \frac{g_s^2}{M_T^2} \left\{ \frac{1}{15}(P^\mu G_{\mu\nu}^a)^2 - \frac{g_s}{180} f^{abc} G^{a\mu}_{\ \ \nu} G^{b\nu}_{\ \ \rho} G^{c\rho}_{\ \ \mu} \right\}$$
$$\frac{1}{16\pi^2} \frac{g_s^2}{M_T^2} \left\{ -\frac{1}{15}(D^\mu G_{\mu\nu}^a)^2 - \frac{g_s}{180} f^{abc} G^{a\mu}_{\ \ \nu} G^{b\nu}_{\ \ \rho} G^{c\rho}_{\ \ \mu} \right\} \tag{1.5.93}$$

where we used that $P^\mu = iD^\mu$ in the last line.

Terms with $n \geq 7$ contributes only to pure Gauge operators of dimension greater than six. Therefore, having computed all relevant contributions up to dimension six, we can now collect the total result for the fermionic path integral's contribution to the one-loop effective Lagrangian.

All one-loop contribution up to dimension six for fermionic path integral

Adding the non-vanishing contributions that we computed (Equations (1.5.80) and (1.5.93)), the fermionic one-loop effective Lagrangian is:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset -\frac{g_s^2}{96\pi^2} \left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{M_T^2}\right) \right] G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{16\pi^2} \frac{g_s^2}{M_T^2} \left\{ \frac{1}{15}(D^\mu G_{\mu\nu}^a)^2 + \frac{g_s}{180} f^{abc} G^{a\mu}_{\ \ \nu} G^{b\nu}_{\ \ \rho} G^{c\rho}_{\ \ \mu} \right\} \tag{1.5.94}$$

This result shows that the fermionic path integral contributes a correction to the gauge field strength kinetic term and a higher-dimensional term representing gluon self-interactions.

---

[12]The explicit expression for this terms with open covariant derivatives can be found in Appendix A.4

One-loop Lagrangian for the EFT toy model

Having calculated the individual contributions from both the scalar and fermionic path integrals to the one-loop effective Lagrangian, we can now combine them, as $\mathcal{L}_{EFT}^{\text{1-loop}}$ is simply their sum. Summing the results from Equation (1.5.64) and Equation (1.5.94) leads to:

$$
\begin{aligned}
\mathcal{L}_{EFT}^{\text{1-loop}} \supset\ & \frac{1}{64\pi^2} y_{Hs} M_s^2 \left[\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_s^2}\right)\right] h^2 - \frac{g_s^2}{96\pi^2}\left[\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right] G_{\mu\nu}^a G^{a\mu\nu} \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(-\frac{1}{8} + \frac{1}{2}\frac{1}{\bar{\epsilon}} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{4}\tilde{\boldsymbol{\mathcal{I}}}_{T,s}^{\mathbf{2,1,-1}}\right)(\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t) \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t}\gamma^\mu P_R t) \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D_\mu D^2 t - D_\mu D^2 \bar{t}\gamma^\mu P_R t) \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t}\gamma^\mu P_R t) \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(-\frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t}\Gamma^{\mu\nu\rho} P_R t) \\
&+ \frac{y_{Hs}^2}{256\pi^2}\left[\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_s^2}\right)\right] h^4 + i\frac{y_{DM}^2 y_{Hs}}{16\pi^2}\left(\frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,0} - \frac{1}{8}\tilde{\mathcal{I}}_{T,s}^{2,2,-1}\right) h^2 (\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t) \\
&- \frac{1}{384\pi^2}\frac{y_{Hs}^2}{M_s^2} h^3 \Box h - \frac{1}{192\pi^2}\frac{y_{Hs}^2}{M_s^2} h^2 \partial_\mu h \partial^\mu h + \frac{1}{1536\pi^2}\frac{y_{Hs}^3}{M_s^2} h^6 \\
&- \frac{1}{16\pi^2}\frac{g_s^2}{M_T^2}\left\{\frac{1}{15}(D^\mu G_{\mu\nu}^a)^2 + \frac{g_s}{180} f^{abc} G^{a\mu}{}_\nu G^{b\nu}{}_\rho G^{c\rho}{}_\mu\right\}
\end{aligned}
\tag{1.5.95}
$$

Lagrangian for the EFT toy model up to one-loop order

To obtain the full EFT Lagrangian up to one-loop order, we simply add the tree-level contribution (Equation (1.5.11)) to the combined one-loop results presented above. Organizing by dimension, the complete Lagrangian is:

$$
\begin{aligned}
\mathcal{L}_{EFT}[h,t,q,G] \supset\ & \left[-\frac{1}{4} - \frac{g_s^2}{96\pi^2}\left(\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right)\right] G_{\mu\nu}^a G^{a\mu\nu} + \frac{1}{2}(\partial_\mu h)^2 \\
&+ \left[-\frac{1}{2}m_h^2 + \frac{1}{64\pi^2}y_{Hs}M_s^2\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_s^2}\right)\right)\right] h^2 + i\bar{q}\slashed{D}q + \bar{t}\left(i\slashed{D} - m_t\right)t \\
&+ \left[-\frac{\lambda_h}{4!} + \frac{y_{Hs}^2}{256\pi^2}\left(\frac{2}{\bar{\epsilon}} + \ln\left(\frac{\mu^2}{M_s^2}\right)\right)\right] h^4 - y_{Ht}h\bar{t}t \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(-\frac{1}{8} + \frac{1}{2}\frac{1}{\bar{\epsilon}} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{4}\tilde{\boldsymbol{\mathcal{I}}}_{T,s}^{\mathbf{2,1,-1}}\right)(\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t) \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t}\gamma^\mu P_R t) \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D_\mu D^2 t - D_\mu D^2 \bar{t}\gamma^\mu P_R t) \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(\frac{1}{6}\tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t}\gamma^\mu P_R t) \\
&+ i\frac{y_{DM}^2}{16\pi^2}\left(-\frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,0}\right)(\bar{t}\Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t}\Gamma^{\mu\nu\rho} P_R t) \\
&+ i\frac{y_{DM}^2 y_{Hs}}{16\pi^2}\left(\frac{1}{4}\tilde{\mathcal{I}}_{T,s}^{2,1,0} - \frac{1}{8}\tilde{\mathcal{I}}_{T,s}^{2,2,-1}\right) h^2 (\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t) \\
&- \frac{1}{384\pi^2}\frac{y_{Hs}^2}{M_s^2} h^3 \Box h - \frac{1}{192\pi^2}\frac{y_{Hs}^2}{M_s^2} h^2 \partial_\mu h \partial^\mu h + \frac{1}{1536\pi^2}\frac{y_{Hs}^3}{M_s^2} h^6 \\
&- \frac{1}{16\pi^2}\frac{g_s^2}{M_T^2}\left\{\frac{1}{15}(D^\mu G_{\mu\nu}^a)^2 + \frac{g_s}{180} f^{abc} G^{a\mu}{}_\nu G^{b\nu}{}_\rho G^{c\rho}{}_\mu\right\}
\end{aligned}
\tag{1.5.96}
$$

1.6 Warsaw Basis

Appendix A

Calculations of the toy model EFT

A.1 Solving the classical equation of motion for the heavy fields

In chapter 1, we found the following equations of motion for the heavy fields of the toy model:

$$\left[\Box + M_s^2 + \frac{1}{2}y_{Hs}h^2 + \frac{\lambda_s}{6}s_c^2\right]s_c = -y_{DM}\left(\bar{\psi}_T t_R + \bar{t}_R \psi_T\right) \tag{A.1.1}$$

$$\left(i\slashed{D}^{\psi_T} - M_T\right)\psi_{T,c} = y_{DM}t_R s_c \tag{A.1.2}$$

$$\left(i\slashed{D}^{\psi_T} + M_T\right)\bar{\psi}_{T,c} = -y_{DM}\bar{t}_R s_c \tag{A.1.3}$$

In order to solve them, we start by rewriting the last two equations:

$$\psi_{T,c} = y_{DM}\left(i\slashed{D}^{\psi_T} - M_T\right)^{-1}t_R s_c \tag{A.1.4}$$

$$\bar{\psi}_{T,c} = -y_{DM}\bar{t}_R\left(i\overleftarrow{\slashed{D}}^{\psi_T} + M_T\right)^{-1}s_c \tag{A.1.5}$$

Since we are interested in operators up to dimension six, we can expand the expression above up to order $\mathcal{O}(\frac{1}{M_T^2})$:

$$\psi_{T,c} = -\frac{y_{DM}}{M_T}\left(1 + \frac{i\slashed{D}^{\psi_T}}{M_T}\right)t_R s_c \tag{A.1.6}$$

$$\bar{\psi}_{T,c} = -\frac{y_{DM}}{M_T}\bar{t}_R\left(1 - \frac{i\overleftarrow{\slashed{D}}^{\psi_T}}{M_T}\right)s_c \tag{A.1.7}$$

Substituting this into Eq. (A.1.1):

$$\left[\Box + M_s^2 + \frac{1}{2}y_{Hs}h^2 + \frac{\lambda_s}{6}s_c^2\right]s_c = \frac{y_{DM}^2}{M_T}\left[\bar{t}_R\left(1 - \frac{i\overleftarrow{\slashed{D}}^{\psi_T}}{M_T}\right)s_c t_R + \bar{t}_R\left(1 + \frac{i\slashed{D}^{\psi_T}}{M_T}\right)t_R s_c\right]$$

$$\left[\Box + M_s^2 + \frac{1}{2}y_{Hs}h^2 + \frac{\lambda_s}{6}s_c^2 + i\frac{y_{DM}^2}{M_T^2}\bar{t}_R\overleftarrow{\slashed{D}}^{\psi_T}t_R - i\frac{y_{DM}^2}{M_T^2}\bar{t}_R\slashed{D}^{\psi_T}t_R\right]s_c = 0 \tag{A.1.8}$$

To solve this equation order by order, we write the solution s_c as:

$$s_c = s_c^{(2)} + s_c^{(4)} + s_c^{(6)} \tag{A.1.9}$$

where $s_c^{(n)}$ contains operators with mass dimension n multiplied by prefactors that scale as M^{1-n}, where M represents a heavy mass scale such as M_s and M_T. Grouping terms in the equation of motion by operator dimension, we obtain:

$$M_s^2 s_c^{(2)} = 0$$

$$\left[\Box + \frac{1}{2}y_{Hs}h^2\right]s_c^{(2)} + M^2 s_c^{(4)} = 0$$

$$\left[\Box + \frac{1}{2}y_{Hs}h^2\right]s_c^{(4)} + M^2 s_c^{(6)} + \frac{\lambda_s}{6}[s_c^{(2)}]^3 + i\frac{y_{DM}^2}{M_T^2}\bar{t}_R\overleftarrow{\slashed{D}}^{\psi_T}t_R s_c^{(2)} - i\frac{y_{DM}^2}{M_T^2}\bar{t}_R\slashed{D}^{\psi_T}t_R s_c^{(2)} = 0 \tag{A.1.10}$$

From the first equation, we immediately see that $s_c^{(2)} = 0$. his result eliminates the source term in the second equation, implying $s_c^{(4)} = 0$ as well. With both $s_c^{(2)}$ and $s_c^{(4)}$ all remaining terms in the third equation also vanish, leading to $s_c^{(6)} = 0$. Thus, up to and including dimension-6 contributions, the classical solution is simply:

$$s_c = 0 \tag{A.1.11}$$

Substituting this into Eq. (A.1.4) and (A.1.5) implies that:

$$\psi_{T,c} = 0 \quad \text{and} \quad \bar{\psi}_{T,c} = 0 \tag{A.1.12}$$

Therefore, the classical solution for the heavy fields it is zero.

A.2 List of interaction matrices

When generalizing the matching method to include the effects of integrating out fermions, we introduced interaction matrices, as defined in Equations (1.4.5) and (1.4.6). The elements of these matrices take the general form:

$$(X_{AB})_{ij} \equiv -\left.\frac{\delta^2 \mathcal{L}_{UV,\text{int}}}{\delta A_i \delta B_j}\right|_{A=A_b, B=B_b} \tag{A.2.1}$$

Here, A and Bepresent arbitrary scalar or fermionic fields, and $\mathcal{L}_{UV,\text{int}}$ denotes the interaction part of the UV Lagrangian. The indices i and j collectively account for all indices carried by the fields A and B.

For our specific toy model, the interaction Lagrangian is given by:

$$\mathcal{L}_{UV,\text{int}} = y_{DM}\left(\bar{\psi}_T t_R + \bar{t}_R \psi_T\right)s - y_{Ht}h\left(\bar{t}_L t_R + \bar{t}_R t_L\right) - \frac{1}{4}y_{Hs}h^2 s^2 - \frac{\lambda_s}{4!}s^4 - \frac{\lambda_h}{4!}h^4 \tag{A.2.2}$$

Following these definitions, we can now derive the specific forms of the interaction matrices for our

model:

$$\tilde{\mathbf{X}}_{\Psi\Psi} = \begin{pmatrix} X_{\psi_T\psi_T} & X_{\psi_T\bar{\psi}_T}\mathcal{C}^{-1} & 0 \\ \mathcal{C}X_{\bar{\psi}_T\psi_T} & \mathcal{C}X_{\bar{\psi}_T\bar{\psi}_T}\mathcal{C}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_{3\times3} \tag{A.2.3}$$

$$\tilde{\mathbf{X}}_{\Psi\psi} = \begin{pmatrix} X_{\psi_T\psi_T} & X_{\psi_T\bar{t}}\mathcal{C}^{-1} & 0 \\ \mathcal{C}X_{\bar{\psi}_T t} & \mathcal{C}X_{\bar{\psi}_T\bar{t}}\mathcal{C}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = y_{DM}s_c \begin{pmatrix} 0 & P_L\mathcal{C}^{-1} & 0 \\ \mathcal{C}P_R & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_{3\times3} \tag{A.2.4}$$

$$\tilde{\mathbf{X}}_{\psi\Psi} = \begin{pmatrix} X_{t\psi_T} & X_{t\bar{\psi}_T}\mathcal{C}^{-1} & 0 \\ \mathcal{C}X_{\bar{t}\psi_T} & \mathcal{C}X_{\bar{t}\bar{\psi}_T}\mathcal{C}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = y_{DM}s_c \begin{pmatrix} 0 & P_L\mathcal{C}^{-1} & 0 \\ \mathcal{C}P_R & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_{3\times3} \tag{A.2.5}$$

$$\tilde{\mathbf{X}}_{\Psi\phi} = \begin{pmatrix} 0 & 0 & X_{\psi_T h} \\ 0 & 0 & \mathcal{C}X_{\bar{\psi}_T h} \\ 0 & 0 & 0 \end{pmatrix} = 0_{3\times3} \tag{A.2.6}$$

$$\tilde{\mathbf{X}}_{\phi\Psi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_{h\psi_T} & X_{h\bar{\psi}_T}\mathcal{C}^{-1} & 0 \end{pmatrix} = 0_{3\times3} \tag{A.2.7}$$

$$\tilde{\mathbf{X}}_{\Psi\Phi} = \begin{pmatrix} 0 & 0 & X_{\psi_T s} \\ 0 & 0 & \mathcal{C}X_{\bar{\psi}_T s} \\ 0 & 0 & 0 \end{pmatrix} = y_{DM} \begin{pmatrix} 0 & 0 & \bar{t}_R \\ 0 & 0 & \mathcal{C}t_R \\ 0 & 0 & 0 \end{pmatrix} \tag{A.2.8}$$

$$\tilde{\mathbf{X}}_{\Phi\Psi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_{s\psi_T} & X_{s\bar{\psi}_T}\mathcal{C}^{-1} & 0 \end{pmatrix} = y_{DM} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{t}_R & t_R\mathcal{C}^{-1} & 0 \end{pmatrix} \tag{A.2.9}$$

$$\tilde{\mathbf{X}}_{\psi\psi} = \begin{pmatrix} X_{tt} & X_{t\bar{t}}\mathcal{C}^{-1} & 0 \\ \mathcal{C}X_{\bar{t}t} & \mathcal{C}X_{\bar{t}\bar{t}}\mathcal{C}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = y_{Ht} \begin{pmatrix} 0 & h\mathcal{C}^{-1} & 0 \\ \mathcal{C}h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.2.10}$$

$$\tilde{\mathbf{X}}_{\psi\Phi} = \begin{pmatrix} 0 & 0 & X_{ts} \\ 0 & 0 & \mathcal{C}X_{\bar{t}s} \\ 0 & 0 & 0 \end{pmatrix} = y_{DM} \begin{pmatrix} 0 & 0 & \bar{\psi}_{T,c}P_R \\ 0 & 0 & \mathcal{C}P_L\psi_{T,c} \\ 0 & 0 & 0 \end{pmatrix} = 0_{3\times3} \tag{A.2.11}$$

$$\tilde{\mathbf{X}}_{\Phi\psi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_{st} & X_{s\bar{t}}\mathcal{C}^{-1} & 0 \end{pmatrix} = y_{DM} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{\psi}_{T,c}P_R & P_L\psi_{T,c}\mathcal{C}^{-1} & 0 \end{pmatrix} = 0_{3\times3} \tag{A.2.12}$$

$$\tilde{\mathbf{X}}_{\psi\phi} = \begin{pmatrix} 0 & 0 & X_{th} \\ 0 & 0 & \mathcal{C}X_{\bar{t}h} \\ 0 & 0 & 0 \end{pmatrix} = y_{Ht} \begin{pmatrix} 0 & 0 & \bar{t} \\ 0 & 0 & \mathcal{C}t \\ 0 & 0 & 0 \end{pmatrix} \tag{A.2.13}$$

$$\tilde{\mathbf{X}}_{\phi\psi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_{ht} & X_{h\bar{t}}\mathcal{C}^{-1} & 0 \end{pmatrix} = y_{Ht} \begin{pmatrix} \bar{t} & t\mathcal{C}^{-1} \end{pmatrix} \tag{A.2.14}$$

$$\tilde{\mathbf{X}}_{\Phi\phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & X_{sh} \end{pmatrix} = y_{Hs}hs_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0_{3\times3} \tag{A.2.15}$$

$$\tilde{\mathbf{X}}_{\Phi\Phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & X_{ss} \end{pmatrix} = \left(\frac{1}{2}y_{Hs}h^2 + \frac{\lambda_s}{2}s_c^2\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}y_{Hs}h^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.2.16}$$

$$\tilde{\mathbf{X}}_{\phi\phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & X_{hh} \end{pmatrix} = \left(\frac{1}{2}y_{Hs}s_c^2 + \frac{\lambda_h}{2}h^2\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{\lambda_h}{2}h^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.2.17}$$

After computing this matrices, we must use these results to calculate the following shifts:

$$\bar{\Delta}_A \equiv \Delta_A - \tilde{\mathbf{X}}_{A\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi A} \tag{A.2.18}$$

$$\bar{\mathbf{X}}_{AB} = \tilde{\mathbf{X}}_{AB} - \tilde{\mathbf{X}}_{A\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi B} \tag{A.2.19}$$

As a result, we obtain:

$$\bar{\mathbf{X}}_{\Psi\Psi} = \tilde{\mathbf{X}}_{\Psi\Psi} - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi} = 0_{3\times3} \tag{A.2.20}$$

$$\bar{\mathbf{X}}_{\Psi\psi} = \tilde{\mathbf{X}}_{\Psi\psi} - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\psi} = 0_{3\times3} \tag{A.2.21}$$

$$\bar{\mathbf{X}}_{\psi\Psi} = \tilde{\mathbf{X}}_{\psi\Psi} - \tilde{\mathbf{X}}_{\psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi} = 0_{3\times3} \tag{A.2.22}$$

$$\bar{\mathbf{X}}_{\Psi\phi} = \tilde{\mathbf{X}}_{\Psi\phi} - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\phi} = 0_{3\times3} \tag{A.2.23}$$

$$\bar{\mathbf{X}}_{\phi\Psi} = \tilde{\mathbf{X}}_{\phi\Psi} - \tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi} = 0_{3\times3} \tag{A.2.24}$$

$$\bar{\mathbf{X}}_{\Psi\Phi} = \tilde{\mathbf{X}}_{\Psi\Phi} - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi} = \tilde{\mathbf{X}}_{\Psi\Phi} \tag{A.2.25}$$

$$\bar{\mathbf{X}}_{\Phi\Psi} = \tilde{\mathbf{X}}_{\Phi\Psi} - \tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi} = \tilde{\mathbf{X}}_{\Phi\Psi} \tag{A.2.26}$$

$$\bar{\mathbf{X}}_{\psi\psi} = \tilde{\mathbf{X}}_{\psi\psi} - \tilde{\mathbf{X}}_{\psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\psi} \tag{A.2.27}$$

$$\bar{\mathbf{X}}_{\psi\Phi} = \tilde{\mathbf{X}}_{\psi\Phi} - \tilde{\mathbf{X}}_{\psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi} = 0_{3\times3} \tag{A.2.28}$$

$$\bar{\mathbf{X}}_{\Phi\psi} = \tilde{\mathbf{X}}_{\Phi\psi} - \tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\psi} = 0_{3\times3} \tag{A.2.29}$$

$$\bar{\mathbf{X}}_{\phi\psi} = \tilde{\mathbf{X}}_{\phi\psi} - \tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\psi} \tag{A.2.30}$$

$$\bar{\mathbf{X}}_{\Phi\phi} = \tilde{\mathbf{X}}_{\Phi\phi} - \tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\phi} = 0_{3\times3} \tag{A.2.31}$$

$$\bar{\mathbf{X}}_{\Phi\Phi} = \tilde{\mathbf{X}}_{\Phi\Phi} - \tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi} = \tilde{X}_{\Phi\Phi} \tag{A.2.32}$$

$$\bar{\mathbf{X}}_{\phi\phi} = \tilde{\mathbf{X}}_{\phi\phi} - \tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\phi}$$

$$\bar{\Delta}_\Psi = \Delta_\Psi - \tilde{\mathbf{X}}_{\Psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Psi} = \Delta_\Psi \tag{A.2.33}$$

$$\bar{\Delta}_\psi = \Delta_\psi - \tilde{\mathbf{X}}_{\psi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\psi} \tag{A.2.34}$$

$$\bar{\Delta}_\Phi = \Delta_\Phi - \tilde{\mathbf{X}}_{\Phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\Phi} = \Delta_\Phi \tag{A.2.35}$$

$$\bar{\Delta}_\phi = \Delta_\phi - \tilde{\mathbf{X}}_{\phi\psi}\Delta_\psi^{-1}\tilde{\mathbf{X}}_{\psi\phi} \tag{A.2.36}$$

A.3 Scalar path integral contribution

As we saw in the calculation of the toy model one-loop EFT Lagrangian, the scalar path integral contribution is given by (Eq. 1.5.40):

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} = -\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int\frac{d^dq}{(2\pi)^d}\,\text{tr}\left[\frac{2q\cdot P_\Phi - P_\Phi^2 + \frac{1}{2}y_{Hs}h^2}{q^2 - M_\Phi^2}\right.$$
$$\left.-\frac{y_{DM}^2}{q^2 - M_\Phi^2}\sum_{l=3}^{+\infty}\begin{pmatrix}\bar{t}_R & t_R^T\mathcal{C}^{-1}\end{pmatrix}\frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi)\right]^l(-\slashed{q} + M_T)}{(q^2 - M_T^2)^{l+1}}\begin{pmatrix}t_R \\ \mathcal{C}^{-1}\bar{t}_R^T\end{pmatrix}\right]^n\Bigg|_{\text{hard}} \tag{A.3.1}$$

By computing the first values of n in the expression above we will all operators up to dimension six. In order to keep the calculation as clear as possible, we will compute the contribution of each value of n separately and then summing all at the end. Starting with $n = 1$ contribution.

A.3.1 Computing the $n = 1$ contribution coming from the scalar path integral

Setting $n = 1$ into Eq. (1.5.40) give us:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}}[\phi_b, \psi_b] \supset$$
$$-\frac{i}{2}\int\frac{d^dq}{(2\pi)^d}\,\text{tr}\left[\frac{\frac{1}{2}y_{Hs}h^2}{q^2 - M_\Phi^2} - \frac{y_{DM}^2}{q^2 - M_\Phi^2}\sum_{l=3}^{+\infty}\begin{pmatrix}\bar{t}_R & t_R^T\mathcal{C}^{-1}\end{pmatrix}\frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi)\right]^l(-\slashed{q} + M_T)}{(q^2 - M_T^2)^{l+1}}\begin{pmatrix}t_R \\ \mathcal{C}^{-1}\bar{t}_R^T\end{pmatrix}\right]\Bigg|_{\text{hard}} \tag{A.3.2}$$

where the P_Φ are ordinary derivatives and, as a consequence, these terms does not contributes since they act on the identity on the right. Furthermore, we can see that all terms will give us operators up to dimension six.

To proceed we need to compute the contributions coming from the l summation in the expression above. Beginning with $l = 0$, we have:

$$\begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)}{(q^2 - M_T^2)} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} = \bar{t}_R \frac{(-\slashed{q} + M_T)}{(q^2 - M_T^2)} t_R + t_R^T \mathcal{C}^{-1} \frac{(-\slashed{q} + M_T)}{(q^2 - M_T^2)} \mathcal{C}^{-1} \bar{t}_R^T \qquad (A.3.3)$$

Since we have right-hand projectors in the fields, only the terms with an odd number of gamma matrices between the fields will not vanish. To see why, consider a generic term with m gamma matrices:

$$\bar{t}_R \underbrace{\gamma^\mu ... \gamma^\nu}_{m \text{ times}} t_R = (P_R t)^\dagger \gamma^0 \gamma^\mu ... \gamma^\nu P_R t = t^\dagger P_R \gamma^0 \gamma^\mu ... \gamma^\nu P_R t \qquad (A.3.4)$$

Using that $P_{R,L}\gamma^\mu = \gamma^\mu P_{L,R}$:

$$\bar{t}_R \gamma^\mu ... \gamma^\nu t_R = \bar{t} P_L \gamma^\mu ... \gamma^\nu P_R t = \begin{cases} \bar{t}\gamma^\mu ... \gamma^\nu P_R^2 t, & \text{for odd } m \\ \bar{t}\gamma^\mu ... \gamma^\nu P_L P_R t, & \text{for an even } m \end{cases}$$
$$= \begin{cases} \bar{t}\gamma^\mu ... \gamma^\nu P_R t, & \text{for odd } m \\ 0, & \text{for even } m \end{cases} \qquad (A.3.5)$$

Therefore, the non vanishing contributions in $l = 0$ are:

$$\begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)}{(q^2 - M_T^2)} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} = -\bar{t}_R \frac{\slashed{q}}{(q^2 - M_T^2)} t_R - t_R^T \mathcal{C}^{-1} \frac{\slashed{q}}{(q^2 - M_T^2)} \mathcal{C}^{-1} \bar{t}_R^T \qquad (A.3.6)$$

Using that $\mathcal{C}^{-1} = -\mathcal{C}$, $\mathcal{C}\gamma^\mu \mathcal{C}^{-1} = -(\gamma^\mu)^T$ and when contracting spinor indices we have $\psi^T \Gamma^T \bar{\psi}^T = -\bar{\psi}\Gamma\psi$:

$$\begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)}{(q^2 - M_T^2)} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} = -\bar{t}_R \frac{\slashed{q}}{(q^2 - M_T^2)} t_R - t_R^T \frac{(\gamma^\mu)^T q_\mu}{(q^2 - M_T^2)} \bar{t}_R^T$$
$$= -\bar{t}_R \frac{\slashed{q}}{(q^2 - M_T^2)} t_R + \bar{t}_R \frac{\slashed{q}}{(q^2 - M_T^2)} t_R$$
$$= 0 \qquad (A.3.7)$$

This result show us that there is no contribution for $l = 0$.
For $l = 1$, the contribution is:

$$l = 1 \implies \begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)}{(q^2 - M_T^2)^2} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} \qquad (A.3.8)$$

Since terms with an even number of gamma matrices vanishes, the non-null contributions for the numerator of the expression above are:

$$(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T) = -\slashed{q}\slashed{P}_\Psi\slashed{q} - M_T^2 \slashed{P}_\Psi \qquad (A.3.9)$$

Using the anticommutation relation of the gamma matrices:

$$(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T) = -2q \cdot P_\Psi \slashed{q} + \slashed{P}_\Psi \slashed{q}\slashed{q} - M_T^2 \slashed{P}_\Psi$$
$$= -2q \cdot P_\Psi \slashed{q} + \slashed{P}_\Psi q^2 - M_T^2 \slashed{P}_\Psi \qquad (A.3.10)$$

In the $n = 1$ contribution for the effective Lagrangian, the loop momenta appears only in the l summation. Therefore, we can anticipate some steps that are done when using dimensional regularization to compute the loop integral. Due to symmetry, we substitute $q^\mu q^\nu \to \frac{g^{\mu\nu}}{d} q^2$ when computing the loop integral. Then, if we already do this:

$$(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T) = -\frac{2}{d}\slashed{P}_\Psi q^2 + \slashed{P}_\Psi q^2 - M_T^2 \slashed{P}_\Psi \qquad (A.3.11)$$

Rewriting d as $4 - \epsilon$ and expanding up to order $\mathcal{O}(\epsilon)$:

$$(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T) = \left(-\frac{1}{2} + \frac{\epsilon}{8}\right)\slashed{P}_\Psi q^2 + \slashed{P}_\Psi q^2 - M_T^2 \slashed{P}_\Psi$$
$$= \left(\frac{1}{2} + \frac{\epsilon}{8}\right)\slashed{P}_\Psi q^2 - M_T^2 \slashed{P}_\Psi \qquad (A.3.12)$$

Substituting this in the $l = 1$ contribution for the summation and using:

$$\slashed{P}_\Psi = \begin{pmatrix} \slashed{P}_{\psi_T} & 0 \\ 0 & \slashed{P}^{\mathcal{C}}_{\psi_T} \end{pmatrix} \tag{A.3.13}$$

we get:

$$\begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)(-\slashed{P}_{\psi_T})(-\slashed{q} + M_T)}{(q^2 - M_T^2)^2} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} = \frac{\left(\left(\frac{1}{2} + \frac{\epsilon}{8}\right) q^2 - M_T^2\right)}{(q^2 - M_T^2)^2} \left[\bar{t}_R \slashed{P}_\Psi t_R + t_R^T \mathcal{C}^{-1} \slashed{P}_{\psi_T} \mathcal{C}^{-1} \bar{t}_R^T\right] \tag{A.3.14}$$

Rewriting the second term using the same identities of the $l = 0$ case:

$$\begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)}{(q^2 - M_T^2)^2} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} = \frac{\left(\left(\frac{1}{2} + \frac{\epsilon}{8}\right) q^2 - M_T^2\right)}{(q^2 - M_T^2)^2} \left[\bar{t}_R \slashed{P}_{\psi_T} t_R + t_R^T (\gamma^\mu)^T P_{\psi_T, \mu} \bar{t}_R^T\right]$$

$$= \frac{\left(\left(\frac{1}{2} + \frac{\epsilon}{8}\right) q^2 - M_T^2\right)}{(q^2 - M_T^2)^2} \left[\bar{t}_R \slashed{P}_{\psi_T} t_R - \bar{t}_R \gamma^\mu P_{\psi_T, \mu}^T t_R\right] \tag{A.3.15}$$

Here the transpose in the derivative means that it acts on the left. Therefore, to keep this clear we can introduce an arrow to indicate the action of the covariant derivative:

$$\begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)}{(q^2 - M_T^2)^2} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} = \frac{\left(\left(\frac{1}{2} + \frac{\epsilon}{8}\right) q^2 - M_T^2\right)}{(q^2 - M_T^2)^2} \left[\bar{t}_R \slashed{P}_{\psi_T} t_R - \bar{t}_R \overleftarrow{\slashed{P}}_{\psi_T} t_R\right] \tag{A.3.16}$$

Using that $P_\mu = i D_\mu$ and keeping the chirality projectors explicitly:

$$\begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)}{(q^2 - M_T^2)^2} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} = i \frac{\left(\left(\frac{1}{2} + \frac{\epsilon}{8}\right) q^2 - M_T^2\right)}{(q^2 - M_T^2)^2} \left[\bar{t} \gamma^\mu P_R D_\mu t - D_\mu \bar{t} \gamma^\mu P_R t\right] \tag{A.3.17}$$

For $l = 2$, we have:

$$l = 2 \implies \begin{pmatrix} \bar{t}_R & t_R \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)}{(q^2 - M_T^2)^3} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R \end{pmatrix} \tag{A.3.18}$$

Focusing in the numerator of the fraction above:

$$(-\slashed{q} + M_\Psi)(-\slashed{P}_\Psi)(-\slashed{q} + M_\Psi)(-\slashed{P}_\Psi)(-\slashed{q} + M_\Psi) = (\slashed{q}\slashed{P}_\Psi - M_\Psi \slashed{P}_\Psi)(\slashed{q}\slashed{P}_\Psi - M_\Psi \slashed{P})(-\slashed{q} + M_\Psi)$$

$$= (\slashed{q}\slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi - M_\Psi \slashed{q}\slashed{P}_\Psi \slashed{P}_\Psi - M_\Psi \slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi + M_\Psi^2 \slashed{P}_\Psi \slashed{P}_\Psi)(-\slashed{q} + M_\Psi)$$

$$= -\slashed{q}\slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi \slashed{q} + M_\Psi \slashed{q}\slashed{P}_\Psi \slashed{P}_\Psi \slashed{q} + M_\Psi \slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi \slashed{q} - M_\Psi^2 \slashed{P}_\Psi \slashed{P}_\Psi \slashed{q}$$

$$+ M_\Psi \slashed{q}\slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi - M_\Psi^2 \slashed{q}\slashed{P}_\Psi \slashed{P}_\Psi - M_\Psi^2 \slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi + M_\Psi^3 \slashed{P}_\Psi \slashed{P}_\Psi \tag{A.3.19}$$

Next, we discard terms containing an even number of gamma matrices, as we shown at the $l = 0$ term calculation:

$$(-\slashed{q} + M_\Psi)(-\slashed{P}_\Psi)(-\slashed{q} + M_\Psi)(-\slashed{P}_\Psi)(-\slashed{q} + M_\Psi) = -\slashed{q}\slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi \slashed{q} - M_\Psi^2 \slashed{P}_\Psi \slashed{P}_\Psi \slashed{q} - M_\Psi^2 \slashed{q}\slashed{P}_\Psi \slashed{P}_\Psi - M_\Psi^2 \slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi \tag{A.3.20}$$

Since all loop momenta dependence in the numerator arises from these terms above, and integrals integrals with odd powers of q vanish by symmetry, all contributions in $l = 2$ will be zero when we substitute this result back in Eq. **??**. Therefore, we can go to the $l = 3$ computation, which is given by:

$$l = 3 \implies \begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{(-\slashed{q} + M_T)(-\slashed{P}_\Psi(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)(-\slashed{P}_\Psi)(-\slashed{q} + M_T)}{(q^2 - M_T^2)^4} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} \tag{A.3.21}$$

Focusing in the numerator of the fraction above:

$$(-\slashed{q} + M_\Psi)(-\slashed{P}_\Psi)(-\slashed{q} + M_\Psi)(-\slashed{P}_\Psi)(-\slashed{q} + M_\Psi)(-\slashed{P}_\Psi)(-\slashed{q} + M_\Psi) = (\slashed{q}\slashed{P}_\Psi - M_\Psi \slashed{P}_\Psi) \times$$

$$(-\slashed{q}\slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi \slashed{q} + M_\Psi \slashed{q}\slashed{P}_\Psi \slashed{P}_\Psi \slashed{q} + M_\Psi \slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi \slashed{q} - M_\Psi^2 \slashed{P}_\Psi \slashed{P}_\Psi \slashed{q} + M_\Psi \slashed{q}\slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi - M_\Psi^2 \slashed{q}\slashed{P}_\Psi \slashed{P}_\Psi - M_\Psi^2 \slashed{P}_\Psi \slashed{q}\slashed{P}_\Psi$$

$$+ M_\Psi^3 \slashed{P}_\Psi \slashed{P}_\Psi) \tag{A.3.22}$$

Again, since we know that only terms with an even powers of q do not vanish:

$$(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi) =$$
$$- \not{q}\not{P}_\Psi\not{q}\not{P}_\Psi\not{q}\not{P}_\Psi\not{q} - M_\Psi^2\not{q}\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi\not{q} - M_\Psi^2\not{q}\not{P}_\Psi\not{q}\not{P}_\Psi\not{P}_\Psi - M_\Psi^2\not{q}\not{P}_\Psi\not{P}_\Psi\not{q}\not{P}_\Psi - M_\Psi^2\not{P}_\Psi\not{q}\not{P}_\Psi\not{P}_\Psi\not{q}$$
$$- M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{q}\not{P}_\Psi\not{q} - M_\Psi^2\not{P}_\Psi\not{q}\not{P}_\Psi\not{q}\not{P}_\Psi - M_\Psi^4\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi$$
$$\text{(A.3.23)}$$

Keeping only the terms with an even number of q:

$$(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi) =$$
$$- \not{q}\not{P}_\Psi\not{q}\not{P}_\Psi\not{q}\not{P}_\Psi\not{q} - M_\Psi^2\not{q}\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi\not{q} - M_\Psi^2\not{q}\not{P}_\Psi\not{q}\not{P}_\Psi\not{P}_\Psi - M_\Psi^2\not{q}\not{P}_\Psi\not{P}_\Psi\not{q}\not{P}_\Psi - M_\Psi^2\not{P}_\Psi\not{q}\not{P}_\Psi\not{P}_\Psi\not{q}$$
$$- M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{q}\not{P}_\Psi\not{q} - M_\Psi^2\not{P}_\Psi\not{q}\not{P}_\Psi\not{q}\not{P}_\Psi - M_\Psi^4\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi$$
$$\text{(A.3.24)}$$

Anticipating the substitution $q^\mu q^\nu \to \frac{g^{\mu\nu}}{4}q^2$ and $q^\mu q^\nu q^\rho q^\sigma \to \frac{1}{24}(g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})q^4$:

$$(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi) =$$
$$- \frac{1}{24}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\rho\gamma^\lambda\gamma^\sigma(g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})P_{\Psi,\alpha}P_{\Psi,\beta}P_{\Psi,\lambda}q^4$$
$$- \frac{1}{4}M_\Psi^2\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu P_{\Psi,\nu}P_{\Psi,\rho}P_{\Psi,\sigma}q^2 - \frac{1}{4}M_\Psi^2\gamma^\mu\gamma^\nu\gamma_\mu P_{\Psi,\nu}\not{P}_\Psi\not{P}_\Psi q^2 - \frac{1}{4}M_\Psi^2\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu P_{\Psi,\nu}P_{\Psi,\rho}\not{P}_\Psi q^2$$
$$- \frac{1}{4}M_\Psi^2\not{P}_\Psi\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu P_{\Psi,\nu}P_{\Psi,\rho}q^2 - \frac{1}{4}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\gamma^\mu\gamma^\nu\gamma_\mu P_{\Psi,\nu}q^2 - \frac{1}{4}M_\Psi^2\not{P}_\Psi\gamma^\mu\gamma^\nu\gamma_\mu P_{\Psi,\nu}\not{P}_\Psi q^2 - M_\Psi^4\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi$$
$$\text{(A.3.25)}$$

Note that the only powers of q appearing in this expression are 0, 2 and 4. Since the loop integrals involve propagators with powers $(q^2 - M_T^2)^4$ and $(q^2 - M_s^2)$ for $n = 1$ and $l = 3$, we can anticipate that the resulting loop functions multiplying the operator structures will :

$$\tilde{\mathcal{I}}_{T,s}^{4,1,-2}, \quad \tilde{\mathcal{I}}_{T,s}^{4,1,-1}, \quad \tilde{\mathcal{I}}_{T,s}^{4,1,0} \tag{A.3.26}$$

All of these loop functions are finite, so we are free to set $d = 4$ from the outset. There is no poles $\frac{1}{\epsilon}$ to cancel possible $\mathcal{O}(\epsilon)$ terms and generate finite contributions.

$$(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi) =$$
$$- \frac{1}{24}(\gamma^\mu\gamma^\alpha\gamma_\mu\gamma^\beta\gamma^\rho\gamma^\lambda\gamma_\rho + \gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma_\mu\gamma^\lambda\gamma_\nu + \gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma_\nu\gamma^\lambda\gamma_\mu)P_{\Psi,\alpha}P_{\Psi,\beta}P_{\Psi,\lambda}q^4$$
$$- \frac{1}{4}M_\Psi^2\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu P_{\Psi,\nu}P_{\Psi,\rho}P_{\Psi,\sigma}q^2 - \frac{1}{4}M_\Psi^2\gamma^\mu\gamma^\nu\gamma_\mu P_{\Psi,\nu}\not{P}_\Psi\not{P}_\Psi q^2 - \frac{1}{4}M_\Psi^2\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu P_{\Psi,\nu}P_{\Psi,\rho}\not{P}_\Psi q^2$$
$$- \frac{1}{4}M_\Psi^2\not{P}_\Psi\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu P_{\Psi,\nu}P_{\Psi,\rho}q^2 - \frac{1}{4}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\gamma^\mu\gamma^\nu\gamma_\mu P_{\Psi,\nu}q^2 - \frac{1}{4}M_\Psi^2\not{P}_\Psi\gamma^\mu\gamma^\nu\gamma_\mu P_{\Psi,\nu}\not{P}_\Psi q^2 - M_\Psi^4\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi$$
$$\text{(A.3.27)}$$

Using that:

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{A.3.28}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} \tag{A.3.29}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{A.3.30}$$

we get:

$$(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi) =$$
$$- \frac{1}{24}(4\gamma^\alpha\gamma^\beta\gamma^\lambda - 8g^{\alpha\lambda}\gamma^\beta + 4\gamma^\lambda\gamma^\beta\gamma^\alpha)P_{\Psi,\alpha}P_{\Psi,\beta}P_{\Psi,\lambda}q^4$$
$$+ \frac{1}{2}M_\Psi^2\gamma^\sigma\gamma^\rho\gamma^\nu P_{\Psi,\nu}P_{\Psi,\rho}P_{\Psi,\sigma}q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 - M_\Psi^2 P_\Psi^2\not{P}_\Psi q^2$$
$$- M_\Psi^2\not{P}_\Psi P_\Psi^2 q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 - M_\Psi^4\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi \tag{A.3.31}$$

$$(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)$$

$$= -\frac{1}{24}(4\gamma^\alpha\gamma^\beta\gamma^\lambda - 8g^{\alpha\lambda}\gamma^\beta + 8g^{\lambda\beta}\gamma^\alpha - 8g^{\lambda\alpha}\gamma^\beta + 8g^{\alpha\beta}\gamma^\lambda - 4\gamma^\alpha\gamma^\beta\gamma^\lambda)P_{\Psi,\alpha}P_{\Psi,\beta}P_{\Psi,\lambda}q^4$$

$$+ \frac{1}{2}M_\Psi^2(2g^{\sigma\rho}\gamma^\nu - 2g^{\sigma\nu}\gamma^\rho + 2g^{\nu\rho}\gamma^\sigma - \gamma^\nu\gamma^\rho\gamma^\sigma)P_{\Psi,\nu}P_{\Psi,\rho}P_{\Psi,\sigma}q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 - M_\Psi^2 P_\Psi^2\not{P}_\Psi q^2$$

$$- M_\Psi^2\not{P}_\Psi P_\Psi^2 q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 - M_\Psi^4\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi$$

$$= -\frac{1}{24}(-16g^{\alpha\lambda}\gamma^\beta + 8g^{\lambda\beta}\gamma^\alpha + 8g^{\alpha\beta}\gamma^\lambda)P_{\Psi,\alpha}P_{\Psi,\beta}P_{\Psi,\lambda}q^4$$

$$+ \frac{1}{2}M_\Psi^2(2g^{\sigma\rho}\gamma^\nu - 2g^{\sigma\nu}\gamma^\rho + 2g^{\nu\rho}\gamma^\sigma - \gamma^\nu\gamma^\rho\gamma^\sigma)P_{\Psi,\nu}P_{\Psi,\rho}P_{\Psi,\sigma}q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 - M_\Psi^2 P_\Psi^2\not{P}_\Psi q^2$$

$$- M_\Psi^2\not{P}_\Psi P_\Psi^2 q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 - M_\Psi^4\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi$$

$$= \frac{2}{3}P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}q^4 - \frac{1}{3}\not{P}_\Psi P_\Psi^2 q^4 - \frac{1}{3}P_\Psi^2\not{P}_\Psi q^4$$

$$+ M_\Psi^2\not{P}_\Psi P_\Psi^2 q^2 - M_\Psi^2 P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}q^2 + M_\Psi^2 P_\Psi^2\not{P}_\Psi q^2 - \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 - M_\Psi^2 P_\Psi^2\not{P}_\Psi q^2$$

$$- M_\Psi^2\not{P}_\Psi P_\Psi^2 q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 + \frac{1}{2}M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 - M_\Psi^4\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi$$

$$= \frac{2}{3}P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}q^4 - \frac{1}{3}\not{P}_\Psi P_\Psi^2 q^4 - \frac{1}{3}P_\Psi^2\not{P}_\Psi q^4 - M_\Psi^2 P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}q^2 + M_\Psi^2\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi q^2 - M_\Psi^4\not{P}_\Psi\not{P}_\Psi\not{P}_\Psi$$

$$\tag{A.3.32}$$

Using that the antisymmetrized and normalized product of three Dirac matrices is:

$$\gamma^\mu\gamma^\nu\gamma^\rho = \Gamma^{\mu\rho\nu} + g^{\nu\rho}\gamma^\mu + g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu \tag{A.3.33}$$

we get:

$$(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)(-\not{P}_\Psi)(-\not{q} + M_\Psi)$$

$$= \frac{2}{3}P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}q^4 - \frac{1}{3}\not{P}_\Psi P_\Psi^2 q^4 - \frac{1}{3}P_\Psi^2\not{P}_\Psi q^4 - M_\Psi^2 P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}q^2$$

$$+ M_\Psi^2\not{P}_\Psi P^2\Psi q^2 + M_\Psi^2 P^2\Psi\not{P}_\Psi - M_\Psi^2 P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}q^2 - M_\Psi^4\not{P}_\Psi P_\Psi^2 - M_\Psi^4 P_\Psi^2\not{P}_\Psi + M_\Psi^4 P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}$$

$$+ M_\Psi^2\Gamma^{\mu\nu\rho}P_{\Psi,\mu}P_{\Psi,\nu}P_{\Psi,\rho}q^2 - M_\Psi^4\Gamma^{\mu\nu\rho}P_{\Psi,\mu}P_{\Psi,\nu}P_{\Psi,\rho}$$

$$= \frac{2}{3}P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}q^4 - \frac{1}{3}\not{P}_\Psi P_\Psi^2 q^4 - \frac{1}{3}P_\Psi^2\not{P}_\Psi q^4 - 2M_\Psi^2 P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}q^2$$

$$+ M_\Psi^2\not{P}_\Psi P_\Psi^2 q^2 + M_\Psi^2 P_\Psi^2\not{P}_\Psi - M_\Psi^4\not{P}_\Psi P_\Psi^2 - M_\Psi^4 P_\Psi^2\not{P}_\Psi + M_\Psi^4 P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu}$$

$$+ M_\Psi^2\Gamma^{\mu\nu\rho}P_{\Psi,\mu}P_{\Psi,\nu}P_{\Psi,\rho}q^2 - M_\Psi^4\Gamma^{\mu\nu\rho}P_{\Psi,\mu}P_{\Psi,\nu}P_{\Psi,\rho}$$

$$= \left(\frac{2}{3}q^4 - 2M_\Psi^2 q^2 + M_\Psi^4\right)P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu} + \left(-\frac{1}{3}q^4 + M_\Psi^2 - M_\Psi^4\right)\not{P}_\Psi P_\Psi^2$$

$$\left(-\frac{1}{3}q^4 + M_\Psi^2 - M_\Psi^4\right)P_\Psi^2\not{P}_\Psi + \left(M^2 q^2 - M^4\right)\Gamma^{\mu\nu\rho}P_{\Psi,\mu}P_{\Psi,\nu}P_{\Psi,\rho} \tag{A.3.34}$$

Substituting this in the $l = 3$ contribution:

$$\left(\bar{t}_R \quad t_R^T\mathcal{C}^{-1}\right)\frac{\left[(-\not{q} + M_T)(-\not{P}_\Psi)\right]^3(-\not{q} + M_T)}{(q^2 - M_T^2)^4}\begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R^T \end{pmatrix} =$$

$$\frac{1}{(q^2 - M_T^2)^4}\left(\bar{t}_R \quad t_R^T\mathcal{C}^{-1}\right)\left[\left(\frac{2}{3}q^4 - 2M_T^2 q^2 + M_T^4\right)P_\Psi^\mu\not{P}_\Psi P_{\Psi,\mu} + \left(-\frac{1}{3}q^4 + M_T^2 - M_T^4\right)\not{P}_\Psi P_\Psi^2 \right.$$

$$\left. + \left(-\frac{1}{3}q^4 + M_T^2 - M_T^4\right)P_\Psi^2\not{P}_\Psi + \left(M_T^2 q^2 - M_T^4\right)\Gamma^{\mu\nu\rho}P_{\Psi,\mu}P_{\Psi,\nu}P_{\Psi,\rho}\right]\begin{pmatrix} t_R \\ \mathcal{C}^{-1}\bar{t}_R^T \end{pmatrix}$$

$$\tag{A.3.35}$$

By repeating the exact same steps as in the $l = 0$ and $l = 1$ cases, we find that the terms involving charge conjugation operators are equal to those without them, up to an overall minus sign and with the

covariant derivatives acting on the barred fields instead. Therefore:

$$\begin{pmatrix} \bar{t}_R & t_R^T \mathcal{C}^{-1} \end{pmatrix} \frac{\left[(-\slashed{q} + M_T)(-\slashed{P}_\Psi) \right]^3 (-\slashed{q} + M_T)}{(q^2 - M_T^2)^4} \begin{pmatrix} t_R \\ \mathcal{C}^{-1} \bar{t}_R^T \end{pmatrix} =$$

$$\frac{-i}{(q^2 - M_T^2)^4} \left[\left(\frac{2}{3} q^4 - 2M_\Psi^2 q^2 + M_\Psi^4 \right) (\bar{t} \gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t} \gamma^\mu P_R t) \right.$$

$$+ \left(-\frac{1}{3} q^4 + M_\Psi^2 - M_\Psi^4 \right) (\bar{t} \gamma^\mu P_R D_\mu D^2 t - \bar{t} \gamma^\mu P_R D_\mu D^2 \bar{t} \gamma^\mu P_R t)$$

$$+ \left(-\frac{1}{3} q^4 + M_\Psi^2 - M_\Psi^4 \right) (\bar{t} \gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t} \gamma^\mu P_R t)$$

$$\left. + \left(M^2 q^2 - M^4 \right) (\bar{t} \Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t} \Gamma^{\mu\nu\rho} P_R t) \right] \qquad (A.3.36)$$

To obtain the $n = 1$ contribution, we substitute (A.3.7), (A.3.17), (A.3.20), (A.3.36) into Eq. (A.3.2):

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset$$

$$-\frac{i}{2} \int \frac{d^d q}{(2\pi)^d} \text{tr} \left\{ \frac{\frac{1}{2} y_{Hs} h^2}{q^2 - M_s^2} - i y_{DM}^2 \frac{\left((\frac{1}{2} + \frac{\epsilon}{8}) q^2 - M_T^2 \right)}{(q^2 - M_s^2)(q^2 - M_T^2)^2} (\bar{t} \gamma^\mu P_R D_\mu t - D_\mu \bar{t} \gamma^\mu P_R t) \right.$$

$$- \frac{i y_{DM}^2}{(q^2 - M_s^2)(q^2 - M_T^2)^4} \left[\left(\frac{2}{3} q^4 - 2M_\Psi^2 q^2 + M_\Psi^4 \right) (\bar{t} \gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t} \gamma^\mu P_R t) \right.$$

$$+ \left(-\frac{1}{3} q^4 + M_T^2 - M_T^4 \right) (\bar{t} \gamma^\mu P_R D_\mu D^2 t - D_\mu D^2 \bar{t} \gamma^\mu P_R t)$$

$$+ \left(-\frac{1}{3} q^4 + M_T^2 - M_T^4 \right) (\bar{t} \gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t} \gamma^\mu P_R t)$$

$$\left. + \left(M_T^2 q^2 - M_T^4 \right) (\bar{t} \Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t} \Gamma^{\mu\nu\rho} P_R t) \qquad (A.3.37)$$

We can compact the loop integrals that appears using the following notation:

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^{\mu_1} q^{\mu_2} ... q^{\mu_{2n_c}}}{(q^2 - M_i^2)^{n_i}(q^2 - M_j^2)^{n_j}(q^2)^{n_L}} \equiv \frac{i}{16\pi^2} g^{\mu_1 \mu_2 ... \mu_{2n_c}} \tilde{\mathcal{I}}[q^{2n_c}]_{ij...0}^{n_i, n_j, ..., n_L} \qquad (A.3.38)$$

where $g^{\mu_1 \mu_2 ... \mu_{2n_c}}$ is the completely symmetric combination of metric tensor with $2n_C$ indices. For $n_c = 0$ we define the shorthand notation $\tilde{\mathcal{I}}[q^0]_{ij...0}^{n_i, n_j, ..., n_L} \equiv \tilde{\mathcal{I}}_{ij...0}^{n_i, n_j, ..., n_L}$.

Using this definition, the $n = 1$ contribution can be written as:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset \frac{1}{64\pi^2} y_{Hs} \tilde{\mathcal{I}}_s^{1,0} h^2 - i \frac{y_{DM}^2}{16\pi^2} \left(\left(\frac{1}{4} + \frac{\epsilon}{16} \right) \tilde{\mathcal{I}}_{T,s}^{2,1,-1} - \frac{1}{2} M_T^2 \tilde{\mathcal{I}}_{T,s}^{2,1,0} \right) (\bar{t} \gamma^\mu P_R D_\mu t - D_\mu \bar{t} \gamma^\mu P_R t)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(\frac{2}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} - 2M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} + M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) (\bar{t} \gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t} \gamma^\mu P_R t)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(-\frac{1}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} + M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) (\bar{t} \gamma^\mu P_R D_\mu D^2 t - D_\mu D^2 \bar{t} \gamma^\mu P_R t)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(-\frac{1}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} + M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) (\bar{t} \gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t} \gamma^\mu P_R t)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) (\bar{t} \Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t} \Gamma^{\mu\nu\rho} P_R t) \qquad (A.3.39)$$

Notice that since:

$$\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{2}{\epsilon} + \frac{M_S^2}{M_s^2 - M_T^2} - \frac{(M_T^4 - 2M_T^2 M_s^2)}{(M_T^2 - M_s^2)^2} \ln \left(\frac{M_\Psi^2}{M_s^2} \right) + \ln \left(\frac{\mu^2}{M_s^2} \right) \qquad (A.3.40)$$

the product $\frac{\epsilon}{16} \tilde{\mathcal{I}}_{T,s}^{2,1,-1}$ generates a finite contribution of $\frac{1}{8}$. Therefore:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset \frac{1}{64\pi^2} y_{Hs} \tilde{\mathcal{I}}_s^{1,0} h^2 - i \frac{y_{DM}^2}{16\pi^2} \left(\frac{1}{8} + \frac{1}{4} \tilde{\mathcal{I}}_{T,s}^{2,1,-1} - \frac{1}{2} M_T^2 \tilde{\mathcal{I}}_{T,s}^{2,1,0} \right) (\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(\frac{2}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} - 2 M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} + M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) (\bar{t}\gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t}\gamma^\mu P_R t)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(-\frac{1}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} + M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) (\bar{t}\gamma^\mu P_R D_\mu D^2 t - D_\mu D^2 \bar{t}\gamma^\mu P_R t)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(-\frac{1}{3} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} + M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) (\bar{t}\gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t}\gamma^\mu P_R t)$$

$$- i \frac{y_{DM}^2}{32\pi^2} \left(M_T^2 \tilde{\mathcal{I}}_{T,s}^{4,1,-1} - M_T^4 \tilde{\mathcal{I}}_{T,s}^{4,1,0} \right) (\bar{t}\Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t}\Gamma^{\mu\nu\rho} P_R t) \qquad \text{(A.3.41)}$$

In order to simplify this expression, we can use this reduction formula to loop integrals:

$$\tilde{\mathcal{I}}[q^{2n_c}]_{ij...0}^{n_i,n_j,...,n_L} = \frac{1}{M_i^2} \left(\tilde{\mathcal{I}}[q^{2n_c}]_{ij...0}^{n_i,n_j,...,n_L-1} - \tilde{\mathcal{I}}[q^{2n_c}]_{ij...0}^{n_i-1,n_j,...,n_L} \right) \qquad \text{(A.3.42)}$$

Using this, we get:

$$\mathcal{L}_{EFT,SF}^{\text{1-loop}} \supset \frac{1}{64\pi^2} y_{Hs} \tilde{\mathcal{I}}_s^{1,0} h^2 + i \frac{y_{DM}^2}{16\pi^2} \left(-\frac{1}{8} + \frac{1}{2} \tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{4} \tilde{\mathcal{I}}_{T,s}^{2,1,-1} \right) (\bar{t}\gamma^\mu P_R D_\mu t - D_\mu \bar{t}\gamma^\mu P_R t)$$

$$+ i \frac{y_{DM}^2}{16\pi^2} \left(\frac{1}{6} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2} \tilde{\mathcal{I}}_{T,s}^{2,1,0} \right) (\bar{t}\gamma^\mu P_R D_\nu D_\mu D_\nu t - D_\nu D_\mu D_\nu \bar{t}\gamma^\mu P_R t)$$

$$+ i \frac{y_{DM}^2}{16\pi^2} \left(\frac{1}{6} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2} \tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2} \tilde{\mathcal{I}}_{T,,s}^{2,1,0} \right) (\bar{t}\gamma^\mu P_R D_\mu D^2 t - D_\mu D^2 \bar{t}\gamma^\mu P_R t)$$

$$+ i \frac{y_{DM}^2}{16\pi^2} \left(\frac{1}{6} \tilde{\mathcal{I}}_{T,s}^{4,1,-2} - \frac{1}{2} \tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2} \tilde{\mathcal{I}}_{T,s}^{2,1,0} \right) (\bar{t}\gamma^\mu P_R D^2 D_\mu t - D^2 D_\mu \bar{t}\gamma^\mu P_R t)$$

$$+ i \frac{y_{DM}^2}{16\pi^2} \left(-\frac{1}{2} \tilde{\mathcal{I}}_{T,s}^{3,1,-1} + \frac{1}{2} \tilde{\mathcal{I}}_{T,s}^{2,1,0} \right) (\bar{t}\Gamma^{\mu\nu\rho} P_R D_\mu D_\nu D_\rho t - D_\mu D_\nu D_\rho \bar{t}\Gamma^{\mu\nu\rho} P_R t) \qquad \text{(A.3.43)}$$

A.4 Pure Gauge operators computation

The pure gauge operators in the one-loop effective Lagrangian of our toy model originate from the fermionic path integral contribution, which is given by:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} = i \sum_{n=0}^{\infty} \frac{1}{n} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \left[(-\slashed{q} - M_T)^{-1}(-\slashed{P}) \right]_{\text{hard}}^n = i \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right]^n \tag{A.4.1}$$

discussed in the DM toy model EFT Lagrangian computation, the $n = 0$ and $n = 1$ terms do not yield any contributions. The $n = 0$ term is an infinite constant, while the $n = 1$ term corresponds to a single open covariant derivative that cannot be written as a commutator. The $n = 2$ contribution is explicitly derived in the main text. Therefore, we now proceed to the $n = 3$ contribution.

A.4.1 $n = 3$

Setting $n = 3$ in Eq. (A.4.1), the relevant term is:

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{3} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2} \right] \tag{A.4.2}$$

Focusing on the numerator, we expand the product:

$$\left[(-\slashed{q} + M_T)\slashed{P} \right]^3 = (-\slashed{q} + M_T)\slashed{P}(-\slashed{q} + M_T)\slashed{P}(-\slashed{q} + M_T)\slashed{P}$$

$$= (-\slashed{q} + M_T)\slashed{P}(\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{P} - M_T\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{P})$$

$$= (-\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$

$$+ M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{P}) \qquad \text{(A.4.3)}$$

Recall that loop integrals with odd powers of q vanish due to symmetry. Therefore, discarding this kind of terms:

$$\left[(-\slashed{q} + M_T)\slashed{P}\right]^3 \supset (+M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{P}) \tag{A.4.4}$$

After retaining only the terms that will yield even powers of q in the loop integral, we observe that the remaining expression always contains an odd number of Dirac matrices. Consequently, when taking the trace over spinor indices, all such terms will vanish:

$$\mathrm{tr}\left[(-\slashed{q} + M_T)\slashed{P}\right]^3 = 0 \tag{A.4.5}$$

Therefore, the $n = 3$ term does not contribute to the effective Lagrangian.

A.4.2 $n = 4$

To proceed, we set n=4 in Equation (A.4.1):

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} \supset \frac{i}{4} \int \frac{d^d q}{(2\pi)^d} \mathrm{tr} \left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2}\right]\left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2}\right]\left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2}\right]\left[\frac{(-\slashed{q} + M_T)\slashed{P}}{q^2 - M_T^2}\right] \tag{A.4.6}$$

Let's first focus on the numerator. Expanding the product $\left[(-\slashed{q} + M_T)\slashed{P}\right]^4$:

$$
\begin{aligned}
\left[(-\slashed{q} + M_T)\slashed{P}\right]^4 &= (-\slashed{q} + M_T)\slashed{P}(-\slashed{q} + M_T)\slashed{P}(-\slashed{q} + M_T)\slashed{P}(-\slashed{q} + M_T)\slashed{P} \\
&= (-\slashed{q} + M_T)\slashed{P}(-\slashed{q} + M_T)\slashed{P}(\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{P} - M_T\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{P}) \\
&= (-\slashed{q} + M_T)\slashed{P}(-\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P} \\
&\qquad + M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{P}) \\
&= \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P} \\
&\quad - M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P} \\
&\quad - M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P} \\
&\quad + M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^3\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{P}
\end{aligned} \tag{A.4.7}
$$

Since loop integrals with odd powers of q in the numerator vanishes, we can discard them:

$$
\begin{aligned}
\left[(-\slashed{q} + M_T)\slashed{P}\right]^4 &\supset \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} \\
&\quad + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{P}
\end{aligned} \tag{A.4.8}
$$

Next, we apply the standard substitutions for loop integrals, replacing products of loop momenta: $q^\mu q^\nu \to \frac{g^{\mu\nu}}{d}q^2$ and $q^\mu q^\nu q^\rho q^\sigma \to \frac{1}{d(d+2)}(g^{\mu\nu}g^{\sigma\rho} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})q^4$:

$$
\begin{aligned}
\left[(-\slashed{q} + M_T)\slashed{P}\right]^4 \supset &\left[\frac{1}{d(d+2)}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\rho\gamma^\lambda\gamma^\sigma\gamma^\theta(g_{\mu\nu}g_{\sigma\rho} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})q^4 \right.\\
&\frac{1}{d}M_T^2 q^2\left(\gamma^\mu\gamma^\alpha\gamma_\mu\gamma^\beta\gamma^\lambda\gamma^\theta + \gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu\gamma^\lambda\gamma^\theta + \gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma_\mu\gamma^\theta \right.\\
&\left. + \gamma^\alpha\gamma^\mu\gamma^\beta\gamma_\mu\gamma^\lambda\gamma^\theta + \gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\lambda\gamma_\mu\gamma^\theta + \gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\lambda\gamma_\mu\gamma^\theta \right) + M_T^4\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta\right]P_\alpha P_\beta P_\lambda P_\theta
\end{aligned} \tag{A.4.9}
$$

When substituting this back into Equation (1.5.72), terms proportional to q^4, q^2 and 1 give rise to $\tilde{\mathcal{I}}_T^{4,-2}$, $\tilde{\mathcal{I}}_T^{4,-1}$, and $\tilde{\mathcal{I}}_T^{4,0}$, respectively. Furthermore, we know that only $\tilde{\mathcal{I}}_T^{4,-2}$ has a divergent contribution. Therefore, we only need to keep track of the dimension d (or ϵ) in terms proportional to q^4, as it's possible for $\mathcal{O}(\epsilon)$ contribution to cancel the $\frac{1}{\epsilon}$ pole. For the other terms, we can set $d = 4$, or equivalently $\epsilon = 0$:

$$
\begin{aligned}
\left[(-\slashed{q} + M_T)\slashed{P}\right]^4 \supset &\left[\frac{q^4}{d(d+2)}\left(\gamma^\mu\gamma^\alpha\gamma_\mu\gamma^\beta\gamma^\rho\gamma^\lambda\gamma_\rho\gamma^\theta + \gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma_\mu\gamma^\lambda\gamma_\nu\gamma^\theta + \gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma_\nu\gamma^\lambda\gamma_\mu\gamma^\theta\right) \right.\\
&\frac{1}{4}M_T^2 q^2\left(\gamma^\mu\gamma^\alpha\gamma_\mu\gamma^\beta\gamma^\lambda\gamma^\theta + \gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu\gamma^\lambda\gamma^\theta + \gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma_\mu\gamma^\theta \right.\\
&\left. + \gamma^\alpha\gamma^\mu\gamma^\beta\gamma_\mu\gamma^\lambda\gamma^\theta + \gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\lambda\gamma_\mu\gamma^\theta + \gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\lambda\gamma_\mu\gamma^\theta \right) + M_T^4\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta\right]P_\alpha P_\beta P_\lambda P_\theta
\end{aligned} \tag{A.4.10}
$$

We use the following Dirac matrix identities [8]:

$$\gamma^\mu \gamma^\nu \gamma_\mu = (-2+\epsilon)\gamma^\nu \tag{A.4.11}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho} - \epsilon\gamma^\nu\gamma^\rho \tag{A.4.12}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + \epsilon\gamma^\nu\gamma^\rho\gamma^\sigma \tag{A.4.13}$$

Applying these identities and retaining terms up to $\mathcal{O}(\epsilon)$, specifically keeping track of ϵ only in terms proportional to q^4:

$$
\begin{aligned}
[(-\slashed{q}+M_T)\slashed{P}]^4 \supset &\left[\frac{q^4}{d(d+2)}\left((-2+\epsilon)^2\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta - 2\gamma^\beta\gamma^\nu\gamma^\alpha\gamma^\lambda\gamma_\nu\gamma^\theta + \epsilon\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\lambda\gamma_\nu\gamma^\theta \right.\right. \\
&\left. + (-2+\epsilon)\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma_\mu\gamma^\theta \right) \\
&\frac{1}{4}M_T^2 q^2\left(-2\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta + 4g^{\alpha\beta}\gamma^\lambda\gamma^\theta - 2\gamma^\lambda\gamma^\beta\gamma^\alpha\gamma^\theta \right. \\
&\left.\left. - 2\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta + 4g^{\beta\lambda}\gamma^\alpha\gamma^\theta - 2\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right) + M_T^4\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right] P_\alpha P_\beta P_\lambda P_\theta
\end{aligned} \tag{A.4.14}
$$

Further simplifying terms:

$$
\begin{aligned}
[(-\slashed{q}+M_T)\slashed{P}]^4 \supset &\left[\frac{q^4}{d(d+2)}\left(4(1-\epsilon)\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta - 8g^{\alpha\lambda}\gamma^\beta\gamma^\theta + 2\epsilon\gamma^\beta\gamma^\alpha\gamma^\lambda\gamma^\theta \right.\right. \\
&\left. + 4\epsilon g^{\beta\lambda}\gamma^\alpha\gamma^\theta - 2(-2+\epsilon)\gamma^\lambda\gamma^\beta\gamma^\alpha\gamma^\theta - 2\epsilon\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right) \\
&\left. \frac{1}{4}M_T^2 q^2\left(-6\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta + 4g^{\alpha\beta}\gamma^\lambda\gamma^\theta - 2\gamma^\lambda\gamma^\beta\gamma^\alpha\gamma^\theta + 4g^{\beta\lambda}\gamma^\alpha\gamma^\theta \right) + M_T^4\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right] P_\alpha P_\beta P_\lambda P_\theta
\end{aligned} \tag{A.4.15}
$$

Organizing the terms with four gamma matrices that are not in the correct order by using:

$$\gamma^\lambda\gamma^\beta\gamma^\alpha = (2g^{\lambda\beta}\gamma^\alpha - 2g^{\lambda\alpha}\gamma^\beta + 2g^{\alpha\beta}\gamma^\lambda - \gamma^\alpha\gamma^\beta\gamma^\lambda) \tag{A.4.16}$$

we obtain:

$$
\begin{aligned}
[(-\slashed{q}+M_T)\slashed{P}]^4 \supset &\left[\frac{q^4}{d(d+2)}\left(4(1-\epsilon)\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta - 8g^{\alpha\lambda}\gamma^\beta\gamma^\theta + 4\epsilon g^{\alpha\beta}\gamma^\lambda\gamma^\theta - 2\epsilon\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right.\right. \\
&\left. + 4\epsilon g^{\beta\lambda}\gamma^\alpha\gamma^\theta - 2(-2+\epsilon)(2g^{\lambda\beta}\gamma^\alpha - 2g^{\lambda\alpha}\gamma^\beta + 2g^{\alpha\beta}\gamma^\lambda - \gamma^\alpha\gamma^\beta\gamma^\lambda)\gamma^\theta - 2\epsilon\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right) \\
&+ \frac{1}{4}M_T^2 q^2\left(-6\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta + 4g^{\alpha\beta}\gamma^\lambda\gamma^\theta - 2(2g^{\lambda\beta}\gamma^\alpha - 2g^{\lambda\alpha}\gamma^\beta + 2g^{\alpha\beta}\gamma^\lambda - \gamma^\alpha\gamma^\beta\gamma^\lambda)\gamma^\theta + 4g^{\beta\lambda}\gamma^\alpha\gamma^\theta \right) \\
&\left. + M_T^4\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right] P_\alpha P_\beta P_\lambda P_\theta \\[2ex]
\supset &\left[\frac{q^4}{d(d+2)}\left(4(1-\epsilon)\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta - 8g^{\alpha\lambda}\gamma^\beta\gamma^\theta + 4\epsilon g^{\alpha\beta}\gamma^\lambda\gamma^\theta - 2\epsilon\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right.\right. \\
&\left. + 4\epsilon g^{\beta\lambda}\gamma^\alpha\gamma^\theta - 4(-2+\epsilon)g^{\lambda\beta}\gamma^\alpha\gamma^\theta + 4(-2+\epsilon)g^{\lambda\alpha}\gamma^\beta\gamma^\theta - 4(-2+\epsilon)g^{\alpha\beta}\gamma^\lambda\gamma^\theta \right. \\
&\left. + 2(-2+\epsilon)\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta - 2\epsilon\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right) \\
&+ \frac{1}{4}M_T^2 q^2\left(-6\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta + 4g^{\alpha\beta}\gamma^\lambda\gamma^\theta - 4g^{\lambda\beta}\gamma^\alpha\gamma^\theta + 4g^{\lambda\alpha}\gamma^\beta\gamma^\theta - 4g^{\alpha\beta}\gamma^\lambda\gamma^\theta \right. \\
&\left. + 2\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta + 4g^{\beta\lambda}\gamma^\alpha\gamma^\theta \right) + M_T^4\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right] P_\alpha P_\beta P_\lambda P_\theta \\[2ex]
\supset &\left[\frac{q^4}{d(d+2)}\left(-6\epsilon\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta - 4(4-\epsilon)g^{\alpha\lambda}\gamma^\beta\gamma^\theta + 8g^{\lambda\beta}\gamma^\alpha\gamma^\theta + 8g^{\alpha\beta}\gamma^\lambda\gamma^\theta \right) \right. \\
&\left. + \frac{1}{4}M_T^2 q^2\left(-4\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta + 4g^{\lambda\alpha}\gamma^\beta\gamma^\theta \right) + M_T^4\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\theta \right] P_\alpha P_\beta P_\lambda P_\theta
\end{aligned} \tag{A.4.17}
$$

Now, we compute the trace of these gamma matrices. We need the following trace identities[8]:

$$\text{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu} \tag{A.4.18}$$

$$\text{tr}\big(\gamma^\alpha \gamma^\beta \gamma^\lambda \gamma^\theta\big) = 4(g^{\alpha\beta}g^{\lambda\theta} - g^{\alpha\lambda}g^{\beta\theta} + g^{\alpha\theta}g^{\beta\lambda}) \tag{A.4.19}$$

Applying these identities to the expression for $\text{tr}\left[(-\slashed{q} + M_T)\slashed{P}\right]^4$:

$$
\begin{aligned}
\text{tr}\left[(-\slashed{q} + M_T)\slashed{P}\right]^4 \supset & \left[\frac{q^4}{d(d+2)}\Big(-24\epsilon(g^{\alpha\beta}g^{\lambda\theta} - g^{\alpha\lambda}g^{\beta\theta} + g^{\alpha\theta}g^{\beta\lambda}) - 16(4-\epsilon)g^{\alpha\lambda}g^{\beta\theta}\right. \\
& + 32g^{\lambda\beta}g^{\alpha\theta} + 32g^{\alpha\beta}g^{\lambda\theta}\Big) + \frac{1}{4}M_T^2 q^2\Big(-16(g^{\alpha\beta}g^{\lambda\theta} - g^{\alpha\lambda}g^{\beta\theta} + g^{\alpha\theta}g^{\beta\lambda}) + 16g^{\lambda\alpha}g^{\beta\theta}\Big) \\
& \left. + 4M_T^4(g^{\alpha\beta}g^{\lambda\theta} - g^{\alpha\lambda}g^{\beta\theta} + g^{\alpha\theta}g^{\beta\lambda})\right]P_\alpha P_\beta P_\lambda P_\theta
\end{aligned} \tag{A.4.20}
$$

Rewriting $d(d+2)$ as $(4-\epsilon)(6-\epsilon)$:

$$
\begin{aligned}
\text{tr}\left[(-\slashed{q} + M_T)\slashed{P}\right]^4 \supset & \left[q^4\left(4\frac{(-16+10\epsilon)}{(4-\epsilon)(6-\epsilon)}g^{\alpha\lambda}g^{\beta\theta} + 8\frac{(4-3\epsilon)}{(4-\epsilon)(6-\epsilon)}(g^{\alpha\beta}g^{\lambda\theta} + g^{\alpha\theta}g^{\beta\lambda})\right)\right. \\
& + M_T^2 q^2\left(8g^{\alpha\lambda}g^{\beta\theta} - 4(g^{\alpha\beta}g^{\lambda\theta} + g^{\alpha\theta}g^{\beta\lambda})\right) \\
& \left. + 4M_T^4(g^{\alpha\beta}g^{\lambda\theta} - g^{\alpha\lambda}g^{\beta\theta} + g^{\alpha\theta}g^{\beta\lambda})\right]P_\alpha P_\beta P_\lambda P_\theta
\end{aligned} \tag{A.4.21}
$$

Expanding around $\epsilon = 0$ and keeping terms up to first order:

$$
\begin{aligned}
\text{tr}\left[(-\slashed{q} + M_T)\slashed{P}\right]^4 \supset & \left[q^4\left(\left(-\frac{8}{3} + \frac{5}{9}\epsilon\right)g^{\alpha\lambda}g^{\beta\theta} + \left(\frac{4}{3} - \frac{4}{9}\epsilon\right)(g^{\alpha\beta}g^{\lambda\theta} + g^{\alpha\theta}g^{\beta\lambda})\right)\right. \\
& \left. + M_T^2 q^2\left(8g^{\alpha\lambda}g^{\beta\theta} - 4(g^{\alpha\beta}g^{\lambda\theta} + g^{\alpha\theta}g^{\beta\lambda})\right) + 4M_T^4(g^{\alpha\beta}g^{\lambda\theta} - g^{\alpha\lambda}g^{\beta\theta} + g^{\alpha\theta}g^{\beta\lambda})\right]P_\alpha P_\beta P_\lambda P_\theta \\
\supset & \left[\left(-\frac{8}{3} + \frac{5}{9}\epsilon\right)q^4 + 8M_T^2 q^2 - 4M_T^4\right]P^\mu P^\nu P_\mu P_\nu \\
& + \left[\left(\frac{4}{3} - \frac{4}{9}\epsilon\right)q^4 - 4M_T^2 q^2 + 4M_T^4\right](P^\mu P_\mu P^\nu P_\nu + P^\mu P^\nu P_\nu P_\mu)
\end{aligned} \tag{A.4.22}
$$

Substituting this back into Eq. (1.5.72) and using the definition of the loop functions:

$$
\begin{aligned}
\mathcal{L}^{\text{1-loop}}_{EFT,F}[\phi_b, \psi_b] \supset & -\frac{1}{64\pi^2}\left[\left(-\frac{8}{3} + \frac{5}{9}\epsilon\right)\tilde{\mathcal{I}}_T^{4,-2} + 8M_T^2\tilde{\mathcal{I}}^{4,-1} - 4M_T^4\tilde{\mathcal{I}}_T^{4,0}\right]\text{tr}(P^\mu P^\nu P_\mu P_\nu) \\
& -\frac{1}{64\pi^2}\left[\left(\frac{4}{3} - \frac{4}{9}\epsilon\right)\tilde{\mathcal{I}}_T^{4,-2} - 4M_T^2\tilde{\mathcal{I}}^{4,-1} + 4M_T^4\tilde{\mathcal{I}}_T^{4,0}\right]\text{tr}\left(P^\mu P_\mu P^\nu P_\nu + P^\mu P^\nu P_\nu P_\mu\right)
\end{aligned} \tag{A.4.23}
$$

Using Package X [17][1] we have that:

$$\tilde{\mathcal{I}}_T^{4,-2} = -\frac{5}{6} + \frac{2}{\bar\epsilon} + \ln\left(\frac{\mu^2}{M_T^2}\right) \tag{A.4.24}$$

$$\tilde{\mathcal{I}}_T^{4,-1} = -\frac{1}{3M_T^2} \tag{A.4.25}$$

$$\tilde{\mathcal{I}}_T^{4,0} = \frac{1}{6M_T^4} \tag{A.4.26}$$

Substituting these values, we get:

$$
\begin{aligned}
\mathcal{L}^{\text{1-loop}}_{EFT,F}[\phi_b, \psi_b] \supset & \frac{1}{24\pi^2}\left[\frac{2}{\bar\epsilon} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]\text{tr}(P^\mu P^\nu P_\mu P_\nu) - \frac{1}{48\pi^2}\left[\frac{2}{\bar\epsilon} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]\text{tr}\left(P^\mu P_\mu P^\nu P_\nu + P^\mu P^\nu P_\nu P_\mu\right) \\
\supset & \frac{1}{24\pi^2}\left[\frac{2}{\bar\epsilon} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]\text{tr}(P^\mu P^\nu P_\mu P_\nu - P^\mu P^\nu P_\nu P_\mu) \\
\supset & \frac{1}{24\pi^2}\left[\frac{2}{\bar\epsilon} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right](P^\mu P^\nu P_\mu P_\nu - P^\mu P^\nu P_\nu P_\mu) \\
\supset & \frac{1}{48\pi^2}\left[\frac{2}{\bar\epsilon} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right]\text{tr}([P^\mu, P^\nu][P_\mu, P_\nu])
\end{aligned} \tag{A.4.27}
$$

---

[1]The notebook holding this loop functions evaluations can be found in GitHub

Using that $[P_\mu, P_\nu] = ig_s G^a_{\mu\nu} t_R^a$:

$$\mathcal{L}^{\text{1-loop}}_{EFT,F}[\phi_b, \psi_b] \supset -\frac{g_s^2}{48\pi^2}\left[\frac{2}{\bar\epsilon} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right] G^a_{\mu\nu} G^{b\mu\nu}\, \text{tr}\!\left(t_R^a t_R^b\right) \tag{A.4.28}$$

Recall that for $SU(N)$, we have $\text{tr}\!\left(t_R^a t_R^b\right) = \frac{1}{2}\delta^{ab}$. Therefore:

$$\mathcal{L}^{\text{1-loop}}_{EFT,F}[\phi_b, \psi_b] \supset -\frac{g_s^2}{96\pi^2}\left[\frac{2}{\bar\epsilon} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right] G^a_{\mu\nu} G^{a\mu\nu} \tag{A.4.29}$$

A.4.3 $\quad n = 5$

Considering the $n = 5$ term from Equation (A.4.1), the expression for it is:

$$\mathcal{L}^{\text{1-loop}}_{EFT,F}[\phi_b, \psi_b]$$
$$= \frac{i}{5}\int \frac{d^d q}{(2\pi)^d}\, \text{tr}\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]$$
$$\tag{A.4.30}$$

Expanding the numerator, $\left[(-\slashed{q}+M_T)\slashed{P}\right]^5$:

$$\left[(-\slashed{q}+M_T)\slashed{P}\right]^5 = (-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}$$
$$= (-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{P} - M_T\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{P})$$
$$= (-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$\qquad + M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{P})$$
$$= (-\slashed{q}+M_T)\slashed{P}(\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^3\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{P})$$
$$= -\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^4\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^3\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^5\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P} \tag{A.4.31}$$

Discarding terms with odd powers of q:

$$\left[(-\slashed{q}+M_T)\slashed{P}\right]^5 = +M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ M_T^3\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^5\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P} \tag{A.4.32}$$

After discarding terms with odd powers of q, we find that every remaining term in the numerator contains an odd number of Dirac matrices. Consequently, when taking the trace over the spinor indices, all such terms will vanish:

$$\text{tr}\left[(-\slashed{q}+M_T)\right]^5 = 0 \tag{A.4.33}$$

Therefore, the $n = 5$ term does not contribute to the effective Lagrangian.

A.4.4 $n = 6$

Setting $n = 6$ in Eq. (A.4.1):

$$\mathcal{L}_{EFT,F}^{\text{1-loop}} =$$

$$\frac{i}{6}\int \frac{d^d q}{(2\pi)^d}\,\text{tr}\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]$$

$$\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right]\left[\frac{(-\slashed{q}+M_T)\slashed{P}}{q^2-M_T^2}\right] \qquad (A.4.34)$$

Expanding the numerator, $\left[(-\slashed{q}+M_T)\slashed{P}\right]^6$:

$$\left[(-\slashed{q}+M_T)\slashed{P}\right]^6 = (-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}$$

$$= (-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{P} - M_T\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{P})$$

$$= (-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(-\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{P})$$

$$= (-\slashed{q}+M_T)\slashed{P}(-\slashed{q}+M_T)\slashed{P}(\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^3\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{P})$$

$$= (-\slashed{q}+M_T)\slashed{P}(-\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^4\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^3\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^5\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P})$$

$$= \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^3\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^5\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^3\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^5\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^3\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^5\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$- M_T^3\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} - M_T^5\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} - M_T^5\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} - M_T^5\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^6\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P} \qquad (A.4.35)$$

Discarding terms with odd powers of q:

$$\left[(-\slashed{q} + M_T)\slashed{P}\right]^6 = \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^4\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ M_T^2\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P} + M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^4\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ M_T^2\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + M_T^4\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + M_T^6\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P} \quad \text{(A.4.36)}$$

After discarding terms with odd powers of q, the remaining terms in the numerator can be organized by their dependence on M_T. This results in:

$$\left[(-\slashed{q} + M_T)\slashed{P}\right]^6 \supset A_6 + M_T^2 A_4 + M_T^4 A_2 + M_T^6 A_0 \quad \text{(A.4.37)}$$

where A_6, A_4, A_2, and A_0 represent terms with q^6, q^4, q^2 and q^0 dependence, respectively. Their explicit expressions are:

$$A_6 = \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}$$
$$A_4 = \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P} + \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}$$
$$+ \slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + \slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + \slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}$$
$$\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + \slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + \slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ \slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + \slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P} + \slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}$$
$$A_2 = \slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P} + \slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P} + \slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}$$
$$+ \slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + \slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + \slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}$$
$$+ \slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P} + \slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + \slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ \slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P} + \slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + \slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}$$
$$+ \slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}\slashed{P} + \slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{P}\slashed{q}\slashed{P} + \slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{q}\slashed{P}\slashed{q}\slashed{P}$$
$$A_0 = \slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P}\slashed{P} \quad \text{(A.4.38)}$$

The powers of q in this expression, combined awith the denominator of the $n = 6$ contribution, ehich is $1/(q^2 - M_T^2)^6$, indicate that the terms will generate loop functions $\tilde{\mathcal{I}}_T^{6,-3}$, $\tilde{\mathcal{I}}_T^{6,-2}$, $\tilde{\mathcal{I}}_T^{6,-1}$, and $\tilde{\mathcal{I}}_T^{6,0}$. Since these loop functions are finite, we don't need to track $\mathcal{O}(\epsilon)$ contributions, allowing us to set $d = 4$ in our following calculations.

To apply the substitutions of the loop integral calculations and compute the traces of this components above, we used Mathematica Package X [17][2]. As a result we got:

- For A_6:

$$\text{tr}\, A_6 = q^6 \Bigg(-\frac{4}{3} g^{\eta\tau} g^{\theta\omega} g^{\kappa\lambda} - \frac{4}{3} g^{\eta\omega} g^{\theta\lambda} g^{\kappa\tau} - \frac{4}{3} g^{\eta\lambda} g^{\theta\tau} g^{\kappa\omega} + \frac{4}{3} g^{\eta\omega} g^{\theta\kappa} g^{\lambda\tau}$$
$$+ \frac{4}{3} g^{\eta\kappa} g^{\theta\omega} g^{\lambda\tau} - \frac{4}{3} g^{\eta\theta} g^{\kappa\omega} g^{\lambda\tau} + \frac{4}{3} g^{\eta\tau} g^{\theta\kappa} g^{\lambda\omega} - \frac{4}{3} g^{\eta\kappa} g^{\theta\tau} g^{\lambda\omega}$$
$$+ \frac{4}{3} g^{\eta\theta} g^{\kappa\tau} g^{\lambda\omega} - \frac{4}{3} g^{\eta\lambda} g^{\theta\kappa} g^{\tau\omega} + \frac{4}{3} g^{\eta\kappa} g^{\theta\lambda} g^{\tau\omega} + \frac{4}{3} g^{\eta\theta} g^{\kappa\lambda} g^{\tau\omega} \Bigg) P_\lambda P_\theta P_\omega P_\eta P_\tau P_\kappa$$

$$\frac{4}{3} q^6 \Bigg(-P^\mu P^\nu P_\nu P^\rho P_\rho P_\mu - P^\mu P_\mu P^\nu P_\nu P^\rho P_\rho - P^\mu P^\nu P^\rho P_\mu P_\nu P_\rho + P^\mu P^\nu P^\rho P_\rho P_\mu P_\nu$$
$$+ P^\mu P^\nu P_\nu P^\rho P_\mu P_\rho - P^\mu P^\nu P^\rho P_\nu P_\mu P_\rho + P^\mu P^\nu P_\mu P^\rho P_\rho P_\nu - P^\mu P^\nu P_\mu P^\rho P_\nu P_\rho$$
$$+ P^\mu P^\nu P_\mu P_\nu P^\rho P_\rho - P^\mu P^\nu P^\rho P_\mu P_\rho P_\nu + P^\mu P_\mu P^\nu P^\rho P_\nu P_\rho + P^\mu P^\nu P^\rho P_\nu P_\rho P_\mu \Bigg) \quad \text{(A.4.39)}$$

[2]The notebook used for this calculation can be found in GitHub

- For A_4:

$$\operatorname{tr} A_4 = q^4 \bigg(\frac{8}{3} \, g^{\eta\omega} g^{\theta\tau} g^{\kappa\lambda} + 4 \, g^{\eta\tau} g^{\theta\omega} g^{\kappa\lambda} + 4 \, g^{\eta\omega} g^{\theta\lambda} g^{\kappa\tau} + \frac{8}{3} \, g^{\eta\lambda} g^{\theta\omega} g^{\kappa\tau}$$

$$+ \frac{8}{3} \, g^{\eta\tau} g^{\theta\lambda} g^{\kappa\omega} - \frac{20}{3} \, g^{\eta\omega} g^{\theta\kappa} g^{\lambda\tau} - \frac{20}{3} \, g^{\eta\kappa} g^{\theta\omega} g^{\lambda\tau} + 8 \, g^{\eta\theta} g^{\kappa\omega} g^{\lambda\tau}$$

$$- \frac{20}{3} \, g^{\eta\tau} g^{\theta\kappa} g^{\lambda\omega} + 8 \, g^{\eta\kappa} g^{\theta\tau} g^{\lambda\omega} - \frac{20}{3} \, g^{\eta\theta} g^{\kappa\tau} g^{\lambda\omega} + 8 \, g^{\eta\lambda} g^{\theta\kappa} g^{\tau\omega}$$

$$- \frac{20}{3} \, g^{\eta\kappa} g^{\theta\lambda} g^{\tau\omega} - \frac{20}{3} \, g^{\eta\theta} g^{\kappa\lambda} g^{\tau\omega} \bigg) P_\lambda P_\theta P_\omega P_\eta P_\tau P_\kappa$$

$$= \frac{4}{3} q^4 \bigg(2 P^\mu P^\nu P^\rho P_\rho P_\nu P_\mu + 3 P^\mu P^\nu P_\nu P^\rho P_\rho P_\mu + 3 P^\mu P_\mu P^\nu P_\nu P^\rho P_\rho + 2 P^\mu P^\nu P_\nu P_\mu P^\rho P_\rho$$

$$+ 2 P^\mu P_\mu P^\nu P^\rho P_\rho P_\nu - 5 P^\mu P^\nu P^\rho P_\rho P_\mu P_\nu - 5 P^\mu P^\nu P_\nu P^\rho P_\mu P_\rho + 6 P^\mu P^\nu P^\rho P_\nu P_\mu P_\rho$$

$$- 5 P^\mu P^\nu P_\mu P^\rho P_\rho P_\nu + 6 P^\mu P^\nu P_\mu P^\rho P_\nu P_\rho - 5 P^\mu P^\nu P_\mu P_\nu P^\rho P_\rho + 6 P^\mu P^\nu P^\rho P_\mu P_\rho P_\nu$$

$$- 5 P^\mu P_\mu P^\nu P^\rho P_\nu P_\rho - 5 P^\mu P^\nu P^\rho P_\nu P_\rho P_\mu \bigg) \tag{A.4.40}$$

- For A_2:

$$\operatorname{tr} A_2 = q^2 \bigg(- 6 \, g^{\eta\omega} g^{\theta\tau} g^{\kappa\lambda} - 6 \, g^{\eta\tau} g^{\theta\omega} g^{\kappa\lambda} - 6 \, g^{\eta\omega} g^{\theta\lambda} g^{\kappa\tau} - 6 \, g^{\eta\lambda} g^{\theta\omega} g^{\kappa\tau} - 6 \, g^{\eta\tau} g^{\theta\lambda} g^{\kappa\omega} + 6 \, g^{\eta\lambda} g^{\theta\tau} g^{\kappa\omega}$$

$$+ 10 \, g^{\eta\omega} g^{\theta\kappa} g^{\lambda\tau} + 10 \, g^{\eta\kappa} g^{\theta\omega} g^{\lambda\tau} - 10 \, g^{\eta\theta} g^{\kappa\omega} g^{\lambda\tau} + 10 \, g^{\eta\tau} g^{\theta\kappa} g^{\lambda\omega} - 10 \, g^{\eta\kappa} g^{\theta\tau} g^{\lambda\omega}$$

$$+ 10 \, g^{\eta\theta} g^{\kappa\tau} g^{\lambda\omega} - 10 \, g^{\eta\lambda} g^{\theta\kappa} g^{\tau\omega} + 10 \, g^{\eta\kappa} g^{\theta\lambda} g^{\tau\omega} + 10 \, g^{\eta\theta} g^{\kappa\lambda} g^{\tau\omega} \bigg) P_\lambda P_\theta P_\omega P_\eta P_\tau P_\kappa$$

$$= q^2 \bigg(- 6 P^\mu P^\nu P^\rho P_\rho P_\nu P_\mu - 6 P^\mu P^\nu P_\nu P^\rho P_\rho P_\mu - 6 P^\mu P_\mu P^\nu P_\nu P^\rho P_\rho - 6 P^\mu P^\nu P_\nu P_\mu P^\rho P_\rho$$

$$- 6 P^\mu P_\mu P^\nu P^\rho P_\rho P_\nu + 6 P^\mu P^\nu P^\rho P_\mu P_\nu P_\rho + 10 P^\mu P^\nu P^\rho P_\rho P_\mu P_\nu + 10 P^\mu P^\nu P_\nu P^\rho P_\mu P_\rho$$

$$- 10 P^\mu P^\nu P^\rho P_\nu P_\mu P_\rho + 10 P^\mu P^\nu P_\mu P^\rho P_\rho P_\nu - 10 P^\mu P^\nu P_\mu P^\rho P_\nu P_\rho + 10 P^\mu P^\nu P_\mu P_\nu P^\rho P_\rho$$

$$- 10 P^\mu P^\nu P^\rho P_\mu P_\rho P_\nu + 10 P^\mu P_\mu P^\nu P^\rho P_\nu P_\rho + 10 P^\mu P^\nu P^\rho P_\nu P_\rho P_\mu \bigg) \tag{A.4.41}$$

- For A_0:

$$\operatorname{tr} A_0 = \bigg(4 \, g^{\eta\omega} g^{\theta\tau} g^{\kappa\lambda} + 4 \, g^{\eta\tau} g^{\theta\omega} g^{\kappa\lambda} + 4 \, g^{\eta\omega} g^{\theta\lambda} g^{\kappa\tau} + 4 \, g^{\eta\lambda} g^{\theta\omega} g^{\kappa\tau} + 4 \, g^{\eta\tau} g^{\theta\lambda} g^{\kappa\omega}$$

$$- 4 \, g^{\eta\lambda} g^{\theta\tau} g^{\kappa\omega} - 4 \, g^{\eta\omega} g^{\theta\kappa} g^{\lambda\tau} - 4 \, g^{\eta\kappa} g^{\theta\omega} g^{\lambda\tau} + 4 \, g^{\eta\theta} g^{\kappa\omega} g^{\lambda\tau} - 4 \, g^{\eta\tau} g^{\theta\kappa} g^{\lambda\omega}$$

$$+ 4 \, g^{\eta\kappa} g^{\theta\tau} g^{\lambda\omega} - 4 \, g^{\eta\theta} g^{\kappa\tau} g^{\lambda\omega} + 4 \, g^{\eta\lambda} g^{\theta\kappa} g^{\tau\omega} - 4 \, g^{\eta\kappa} g^{\theta\lambda} g^{\tau\omega} - 4 \, g^{\eta\theta} g^{\kappa\lambda} g^{\tau\omega} \bigg) P_\lambda P_\theta P_\omega P_\eta P_\tau P_\kappa$$

$$= 4 \bigg(P^\mu P^\nu P^\rho P_\rho P_\nu P_\mu + P^\mu P^\nu P_\nu P^\rho P_\rho P_\mu + P^\mu P_\mu P^\nu P_\nu P^\rho P_\rho + P^\mu P^\nu P_\nu P_\mu P^\rho P_\rho$$

$$+ P^\mu P_\mu P^\nu P^\rho P_\rho P_\nu - P^\mu P^\nu P^\rho P_\mu P_\nu P_\rho - P^\mu P^\nu P^\rho P_\rho P_\mu P_\nu - P^\mu P^\nu P_\nu P^\rho P_\mu P_\rho$$

$$+ P^\mu P^\nu P^\rho P_\nu P_\mu P_\rho - P^\mu P^\nu P_\mu P^\rho P_\rho P_\nu + P^\mu P^\nu P_\mu P^\rho P_\nu P_\rho - P^\mu P^\nu P_\mu P_\nu P^\rho P_\rho$$

$$+ P^\mu P^\nu P^\rho P_\mu P_\rho P_\nu - P^\mu P_\mu P^\nu P^\rho P_\nu P_\rho - P^\mu P^\nu P^\rho P_\nu P_\rho P_\mu \bigg) \tag{A.4.42}$$

Combining these intermediate results from Equation (A.4.37) with Equation (A.4.34) and applying

the loop integral notation, the contribution to the effective Lagrangian simplifies to:

$$
\begin{aligned}
\mathcal{L}_{EFT,F}^{\text{1-loop}} = -\frac{1}{16\pi^2}\frac{1}{6}\operatorname{tr}\bigg\{ & \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\mu P_\nu P_\rho \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 8M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\mu P_\rho P_\nu \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\rho P_\mu P_\nu \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 8M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\nu P_\mu P_\rho \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\nu P_\rho P_\mu \\
& + \left[\frac{8}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P^\rho P_\rho P_\nu P_\mu \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 8M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\mu P^\rho P_\nu P_\rho \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 4M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\nu P^\rho P_\rho P_\mu \\
& + \left[-\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} + 4M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P_\mu P^\nu P_\nu P^\rho P_\rho \\
& + \left[\frac{8}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\nu P_\mu P^\rho P_\rho \\
& + \left[\frac{8}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} - 6M_T^4\tilde{\mathcal{I}}_T^{6,-1} + 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P_\mu P^\nu P^\rho P_\rho P_\nu \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\nu P^\rho P_\mu P_\rho \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\mu P^\rho P_\rho P_\nu \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P_\mu P^\nu P^\rho P_\nu P_\rho \\
& + \left[\frac{4}{3}\tilde{\mathcal{I}}_T^{6,-3} - \frac{20}{3}M_T^2\tilde{\mathcal{I}}_T^{6,-2} + 10M_T^4\tilde{\mathcal{I}}_T^{6,-1} - 4M_T^6\tilde{\mathcal{I}}_T^{6,0}\right]P^\mu P^\nu P_\mu P_\nu P^\rho P_\rho \bigg\}
\end{aligned}
$$

$$(A.4.43)$$

A.5   Dimension six commutator expansion

In this section we are interested in the expansion of:

$$\operatorname{tr}([P^\mu,[P_\mu,P_\nu]][P_\rho,[P^\rho,P^\nu]]) \quad\text{and}\quad \operatorname{tr}([P^\mu,P_\nu][P^\nu,P_\rho][P^\rho,P_\mu]) \tag{A.5.1}$$

in terms of explicit open covariant derivative.

A.5.1   Expansion of $\operatorname{tr}([P^\mu,[P_\mu,P_\nu]][P_\rho,[P^\rho,P^\nu]])$

Starting with the first operator, we obtain:

$$
\begin{aligned}
\operatorname{tr}([P^\mu,[P_\mu,P_\nu]][P_\rho,[P^\rho,P^\nu]]) &= \operatorname{tr}\{(P^\mu[P_\mu,P_\nu] - [P_\mu,P_\nu]P^\mu)(P_\rho[P^\rho,P^\nu] - [P^\rho,P^\nu]P_\rho)\} \\
&= \operatorname{tr}\{(P^\mu P_\mu P_\nu - P^\mu P_\nu P_\mu - P_\mu P_\nu P^\mu + P_\nu P_\mu P^\mu) \\
&\qquad\qquad (P_\rho P^\rho P^\nu - P_\rho P^\nu P^\rho - P^\rho P^\nu P_\rho + P^\nu P^\rho P_\rho)\} \\
&= \operatorname{tr}\{(P^\mu P_\mu P_\nu - 2P^\mu P_\nu P_\mu + P_\nu P_\mu P^\mu) \\
&\qquad\qquad (P_\rho P^\rho P^\nu - 2P_\rho P^\nu P^\rho + P^\nu P^\rho P_\rho)\} \\
&= \operatorname{tr}\{P^\mu P_\mu P_\nu P_\rho P^\rho P^\nu - 2P^\mu P_\mu P_\nu P_\rho P^\nu P^\rho + P^\mu P_\mu P_\nu P^\nu P^\rho P_\rho \\
&\quad - 2P^\mu P_\nu P_\mu P_\rho P^\rho P^\nu + 4P^\mu P_\nu P_\mu P_\rho P^\nu P^\rho - 2P^\mu P_\nu P_\mu P^\nu P^\rho P_\rho \\
&\quad + P_\nu P_\mu P^\mu P_\rho P^\rho P^\nu - 2P_\nu P_\mu P^\mu P_\rho P^\nu P^\rho + P_\nu P_\mu P^\mu P^\nu P^\rho P_\rho\} \tag{A.5.2}
\end{aligned}
$$

By applying the cyclic properties of the trace, we can rearrange and simplify terms:

$$
\begin{aligned}
\mathrm{tr}([P^\mu,[P_\mu,P_\nu]][P_\rho,[P^\rho,P^\nu]]) = \mathrm{tr}\{&4P^\mu P_\nu P_\rho P_\nu P^\mu P^\rho + P^\mu P_\mu P_\nu P_\rho P^\rho P^\nu - 2P^\mu P_\mu P_\nu P_\rho P^\nu P^\rho + P^\mu P_\mu P_\nu P^\nu P^\rho P_\rho \\
&- 2P^\mu P_\nu P_\mu P_\rho P^\rho P^\nu - 2P^\mu P_\nu P_\mu P^\nu P^\rho P_\rho \\
&+ P_\nu P_\mu P^\mu P_\rho P^\rho P^\nu - 2P_\nu P_\mu P^\mu P_\rho P^\nu P^\rho + P_\nu P_\mu P^\mu P^\nu P^\rho P_\rho\} \qquad \text{(A.5.3)}
\end{aligned}
$$

A.5.2 Expansion of $\mathrm{tr}([P^\mu,P_\nu][P^\nu,P_\rho][P^\rho,P_\mu])$

Next, we expand the second operator:

$$
\begin{aligned}
\mathrm{tr}([P^\mu,P_\nu][P^\nu,P_\rho][P^\rho,P_\mu]) &= \mathrm{tr}\{(P^\mu P_\nu - P_\nu P^\mu)(P^\nu P_\rho - P_\rho P^\nu)(P^\rho P_\mu - P_\mu P^\rho)\} \\
&= \mathrm{tr}\{(P^\mu P_\nu - P_\nu P^\mu)(P^\nu P_\rho P^\rho P_\mu - P^\nu P_\rho P_\mu P^\rho - P_\rho P^\nu P^\rho P_\mu + P_\rho P^\nu P_\mu P^\rho)\} \\
&= \mathrm{tr}\{P^\mu P_\nu P^\nu P_\rho P^\rho P_\mu - P^\mu P_\nu P^\nu P_\rho P_\mu P^\rho - P^\mu P_\nu P_\rho P^\nu P^\rho P_\mu + P^\mu P_\nu P_\rho P^\nu P_\mu P^\rho \\
&\quad - P_\nu P^\mu P^\nu P_\rho P^\rho P_\mu + P_\nu P^\mu P^\nu P_\rho P_\mu P^\rho + P_\nu P^\mu P_\rho P^\nu P^\rho P_\mu - P_\nu P^\mu P_\rho P^\nu P_\mu P^\rho\}
\end{aligned}
$$
$$\text{(A.5.4)}$$

Again, by using the cyclic properties of the trace, we simplify the expression to:

$$
\begin{aligned}
\mathrm{tr}([P^\mu,P_\nu][P^\nu,P_\rho][P^\rho,P_\mu]) = \mathrm{tr}\{&3P^\mu P_\nu P^\rho P_\nu P^\mu P_\rho - P^\mu P_\nu P^\rho P_\mu P^\nu P_\rho + P^\mu P_\nu P^\nu P_\rho P^\rho P_\mu \\
&- P^\mu P_\nu P^\nu P_\rho P_\mu P^\rho - P^\mu P_\nu P_\rho P^\nu P^\rho P_\mu - P_\nu P^\mu P^\nu P_\rho P^\rho P_\mu\} \qquad \text{(A.5.5)}
\end{aligned}
$$

A.6 Checking dimensional regularization

$$
\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} \tag{A.6.1}
$$

Using that:

$$
\tilde{\mathcal{I}}_{ij}^{n_i,n_j,n_l} = \frac{1}{(M_i^2 - M_j^2)}(\tilde{\mathcal{I}}_i^{n_i,n_j-1,n_l}j - \tilde{\mathcal{I}}_{ij}^{n_i-1,n_j,n_l}) \tag{A.6.2}
$$

we get:

$$
\begin{aligned}
\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} &= \frac{1}{(M_T^2 - M_s^2)}\left(\tilde{\mathcal{I}}_T^{1,0} - \tilde{\mathcal{I}}_s^{1,0}\right) - \frac{1}{2}\frac{1}{(M_T^2 - M_s^2)}\left(\tilde{\mathcal{I}}_T^{2,-1} - \tilde{\mathcal{I}}_{T,s}^{1,1,-1}\right) \\
&= \frac{1}{(M_T^2 - M_s^2)}\left(\tilde{\mathcal{I}}_T^{1,0} - \tilde{\mathcal{I}}_s^{1,0}\right) - \frac{1}{2}\frac{1}{(M_T^2 - M_s^2)}\tilde{\mathcal{I}}_T^{2,-1} + \frac{1}{2}\frac{1}{(M_T^2 - M_s^2)^2}\left(\tilde{\mathcal{I}}_T^{1,-1} - \tilde{\mathcal{I}}_s^{1,-1}\right)
\end{aligned}
$$
$$\text{(A.6.3)}$$

Now we use that:

$$
\tilde{\mathcal{I}}_i^{1,0} = M_i^2\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_i^2}\right)\right) \tag{A.6.4}
$$

$$
\tilde{\mathcal{I}}_i^{1,-1} = M_i^4\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_i^2}\right)\right) \tag{A.6.5}
$$

$$
\tilde{\mathcal{I}}_i^{2,-1} = 2M_i^2\left(\frac{2}{\bar{\epsilon}} + \frac{1}{2} + \ln\left(\frac{\mu^2}{M_i^2}\right)\right) \tag{A.6.6}
$$

and then:

$$
\begin{aligned}
\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} &= \frac{1}{(M_T^2 - M_s^2)}\left(M_T^2\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_T^2}\right)\right) - M_s^2\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_s^2}\right)\right)\right) \\
&\quad - \frac{1}{(M_T^2 - M_s^2)}M_T^2\left(\frac{2}{\bar{\epsilon}} + \frac{1}{2} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right) \\
&\quad + \frac{1}{2}\frac{1}{(M_T^2 - M_s^2)^2}\left(M_T^4\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_T^2}\right)\right) - M_s^4\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_s^2}\right)\right)\right) \quad \text{(A.6.7)}
\end{aligned}
$$

$$\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{2}{\bar{\epsilon}} + 1 + \frac{\left[M_T^2 \ln\left(\frac{\mu^2}{M_T^2}\right) - M_s^2 \ln\left(\frac{\mu^2}{M_s^2}\right)\right]}{(M_T^2 - M_s^2)}$$
$$- \frac{1}{(M_T^2 - M_s^2)} M_T^2 \left(\frac{2}{\bar{\epsilon}} + \frac{1}{2} + \ln\left(\frac{\mu^2}{M_T^2}\right)\right)$$
$$+ \frac{1}{2}\frac{1}{(M_T^2 - M_s^2)^2}\left(M_T^4\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_T^2}\right)\right) - M_s^4\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_s^2}\right)\right)\right) \quad \text{(A.6.8)}$$

$$\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{2}{\bar{\epsilon}} + 1 - (M_T^2 - M_s^2)\frac{\left[M_s^2 \ln\left(\frac{\mu^2}{M_s^2}\right)\right]}{(M_T^2 - M_s^2)^2}$$
$$- \frac{(M_T^2 - M_s^2)}{(M_T^2 - M_s^2)^2} M_T^2 \left(\frac{2}{\bar{\epsilon}} + \frac{1}{2}\right)$$
$$+ \frac{1}{2}\frac{1}{(M_T^2 - M_s^2)^2}\left(M_T^4\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_T^2}\right)\right) - M_s^4\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_s^2}\right)\right)\right) \quad \text{(A.6.9)}$$

$$\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{2}{\bar{\epsilon}} + 1 + \frac{1}{2(M_T^2 - M_s^2)^2}\left[-2(M_T^2 - M_s^2)M_s^2 \ln\left(\frac{\mu^2}{M_s^2}\right) - 2(M_T^2 - M_s^2)M_T^2\left(\frac{2}{\bar{\epsilon}} + \frac{1}{2}\right)\right.$$
$$\left. + M_T^4\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_T^2}\right)\right) - M_s^4\left(\frac{2}{\bar{\epsilon}} + 1 + \ln\left(\frac{\mu^2}{M_s^2}\right)\right)\right] \quad \text{(A.6.10)}$$

$$\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{2}{\bar{\epsilon}} + 1 + \frac{1}{2(M_T^2 - M_s^2)^2}\left[\frac{2}{\bar{\epsilon}}\left(-M_T^4 + 2M_s^2 M_T^2 - M_s^4\right)\right.$$
$$\left. + M_T^4 \ln\left(\frac{\mu^2}{M_T^2}\right) - (2M_T^2 - M_s^2)M_s^2 \ln\left(\frac{\mu^2}{M_s^2}\right) + M_s^2 M_T^2 - M_s^4\right] \quad \text{(A.6.11)}$$

$$\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{1}{2}\frac{2}{\bar{\epsilon}} + \frac{1}{2(M_T^2 - M_s^2)^2}\left[M_T^4 \ln\left(\frac{\mu^2}{M_T^2}\right)\right.$$
$$\left. - (2M_T^2 - M_s^2)M_s^2 \ln\left(\frac{\mu^2}{M_s^2}\right) + M_s^2 M_T^2 - M_s^4 + 2M_T^4 - 4M_s^2 M_T^2 + 2M_s^4\right] \quad \text{(A.6.12)}$$

$$\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{1}{2}\frac{2}{\bar{\epsilon}} + \frac{1}{2(M_T^2 - M_s^2)^2}\left[M_T^4 \ln\left(\frac{\mu^2}{M_T^2}\right)\right.$$
$$\left. - (2M_T^2 - M_s^2)M_s^2 \ln\left(\frac{\mu^2}{M_s^2}\right) - M_s^4 + 2M_T^4 - 3M_s^2 M_T^2 + M_s^4\right] \quad \text{(A.6.13)}$$

$$\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{1}{2}\frac{2}{\bar{\epsilon}} + \frac{1}{2(M_T^2 - M_s^2)^2}\left[M_T^4 \ln\left(\frac{\mu^2}{M_T^2}\right)\right.$$
$$\left. - (2M_T^2 - M_s^2)M_s^2 \ln\left(\frac{\mu^2}{M_s^2}\right) + (2M_T^2 - M_s)(M_T^2 - M_s^2)\right] \quad \text{(A.6.14)}$$

$$\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = \frac{1}{2}\frac{2}{\bar{\epsilon}} + \frac{M_T^4 \ln\left(\frac{\mu^2}{M_T^2}\right)}{2(M_T^2 - M_s^2)^2} + \frac{(2M_T^2 - M_s)}{2(M_T^2 - M_s^2)^2}\left[M_T^2 - M_s^2 - M_s^2 \ln\left(\frac{\mu^2}{M_s^2}\right)\right] \quad \text{(A.6.15)}$$

Recall that $\frac{2}{\bar{\epsilon}} = \frac{2}{\epsilon} - \gamma_E$. Therefore:

$$\tilde{\mathcal{I}}_{T,s}^{1,1,0} - \frac{1}{2}\tilde{\mathcal{I}}_{T,s}^{2,1,-1} = -\frac{1}{2}\gamma_E + \frac{1}{2}\frac{2}{\epsilon} + \frac{M_T^4 \ln\left(\frac{\mu^2}{M_T^2}\right)}{2(M_T^2 - M_s^2)^2} + \frac{(2M_T^2 - M_s)}{2(M_T^2 - M_s^2)^2}\left[M_T^2 - M_s^2 - M_s^2 \ln\left(\frac{\mu^2}{M_s^2}\right)\right]$$
$$\text{(A.6.16)}$$

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