Basic of Rigid Body Kinematics

Review of some basic tools

Eric Rohmer

material is a courtesy of Prof. Marco Hutter of ETHZ

Objective

- Learn or refresh some of the basic tools such as:
  - Translation
  - Rotation
  - Homogeneous transformation of multi-body systems
  - …
- Being able to later describe the motion and constraints of mobile platforms, such as wheeled or legged vehicles.

Agenda

- Some definitions and type of motion
- Vectors and coordinates transformation
- Vector differentiation
- Recapitulation

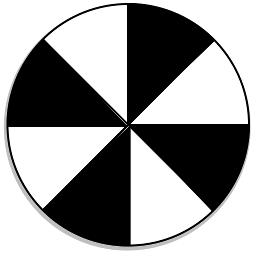
Introduction

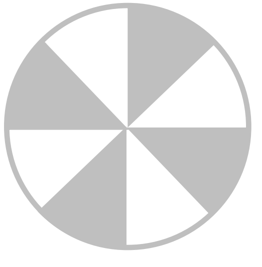
- Definition of kinematics

motion = positions, velocities and accelerations

  - "Description of the motion of *points*, *bodies*, or *systems of bodies*"
  - … without consideration of the causes of motion (=> dynamics)
  - Required for kinematic simulation and control

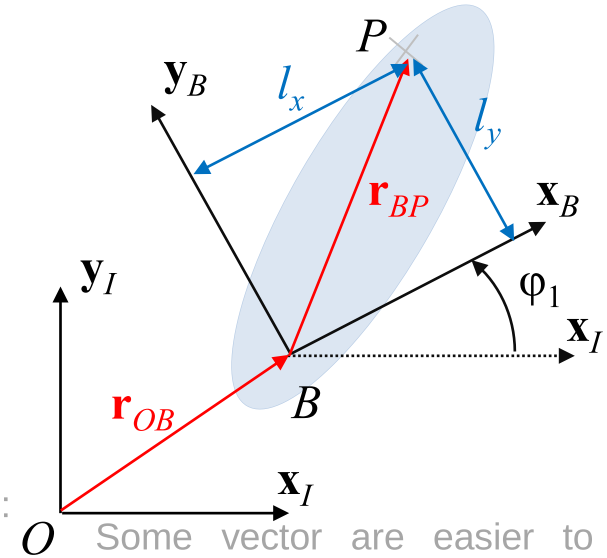
- Type of motion of single body
  - **Translation :**all the points of the body perform parallel trajectories
  - **Rotation:** one point is always fixed and other points rotate around it
  - **Combined motion:** instantaneous rotation around a single point with ICR moving over time. Ex: if Rolling=> ICR coincides always with the contact point





ICR: Instantaneous Center of Rotation

Vectors and coordinate transformation



To describe such motion:

- Reference frame
  - Coordinate system **I** (inertial, not moving)
  - Coordinate system **B** (body-fixed, moving)
- Translation The position of P expressed in the world coord. system:
  - Vector from *O* to *P*, expressed in **I**: $_I\mathbf{r}_{OP} \in \mathbb{R}^{3\text{x}1}$

Some vector are easier to describe in a certain frame

we can't add vect. expressed in different frames: 1st have to rotate them into the appropriate coord. sys.

- Rotation
  - Matrix from frame ***B*** to frame **I**: $\mathbf{R}_{IB} \in \mathbb{R}^{3\text{x}1}$

$\mathbf{R}_{IB}$ is the rot. matrix that rotates a vect expressed in frame B into target frame I

- Homegeneous transformation
  - Combination of translation and rotation

The transf from P in B to P in I is a trans & a rot

We can write it in its matrix form called homogeneous transformation

$$_I\mathbf{r}_{OP} = {}_I\mathbf{r}_{OB} + {}_I\mathbf{r}_{BP} \quad = {}_I\mathbf{r}_{OB} + \mathbf{R}_{IB}\,{}_B\mathbf{r}_{BP}$$

Only if vectors expressed in the same coordinate system — *When vectors expressed in the different frames*

$$\begin{pmatrix} _I\mathbf{r}_{OP} \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R}_{IB} & _I\mathbf{r}_{OB} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} _B\mathbf{r}_{BP} \\ 1 \end{pmatrix}$$

Rotation

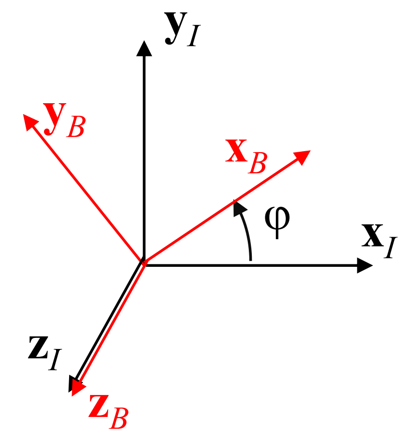
- Reference frame **B** to **I** such that ${}_I\mathbf{r} = \mathbf{R}_{IB}\,{}_B\mathbf{r}$

$$\begin{array}{cc} & \begin{array}{ccc} {}_I\mathbf{x}_B & {}_I\mathbf{y}_B & {}_I\mathbf{z}_B \end{array} \\ \mathbf{R}_{IB} = & \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The x-axis of coordinate system **B** expressed in **I**

The y-axis of coordinate system **B** expressed in **I**

Rotational axis



- Inversion ${}_I\mathbf{r} = \mathbf{R}_{IB}\,{}_B\mathbf{r}, \quad {}_B\mathbf{r} = \mathbf{R}_{BI}\,{}_I\mathbf{r}, \;\; \mathbf{R}_{BI} = \mathbf{R}_{IB}^T$
- Composition of rotations

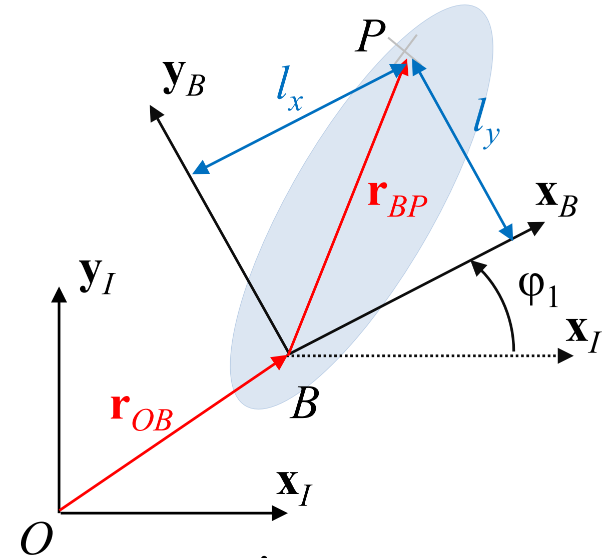
$${}_C\mathbf{r} = \mathbf{R}_{CI}\,{}_I\mathbf{r}\,, \quad \mathbf{R}_{CI} = \mathbf{R}_{CB}\,\mathbf{R}_{BI}$$

Consecutive rot. are the multiplication of corresponding matrices

Orientation in 3D

- Euler angles (z-x-z)
- Tait-Bryan/Cardan angles (z-y-x)

Vectors and coordinate transformation



To describe such motion:

- Reference frame
  - Coordinate system **I** (inertial, not moving)
  - Coordinate system **B** (body-fixed, moving)
- Translation
  - Vector from *O* to *P*, expressed in **I**: $\;{}_I\mathbf{r}_{OP} \in \mathbb{R}^{3\text{x}1} \longrightarrow {}_I\,\mathbf{v}_P = {}_I\,\dot{\mathbf{r}}_{OP} \in \mathbb{R}^{3\text{x}1}$

    ${}_I\mathbf{v}_P$: velocity of point P (i.e. time derivative of the position vector)
- Rotation
  - Matrix from frame ***B*** to frame **I**: $\;\mathbf{R}_{IB} \in \mathbb{R}^{3\text{x}1} \longrightarrow {}_I\,\boldsymbol{\omega}_{IB} \in \mathbb{R}^{3\text{x}1}$

    ${}_I\boldsymbol{\omega}_{IB}$: angular velocity of the body-fixed frame **B** with respect to **I**

    ${}_I\boldsymbol{\omega}_{IB}$ is the same at any points of the body
- Homegeneous transformation
  - Combination of translation and rotation

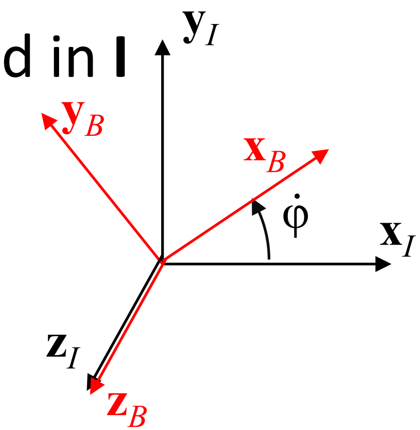
As ${}_I\boldsymbol{\omega}_{IB}$ is invariant, knowing the velocity and angular velocity at a single point => we know the full motion of the rigid body (i.e. determine the velocity at any point) using:

Rigid body kinematic formulation: $\;{}_I\,\mathbf{v}_P = {}_I\,\dot{\mathbf{r}}_{OP} = {}_I\,\dot{\mathbf{r}}_{OB} + {}_I\,\boldsymbol{\omega}_{IB} \times {}_I\,\mathbf{r}_{BP}$

It is possible to determine the velocity at any point using the rigid body kinematic formulation, which says that P has the velocity of another point on the same rigid body (B here), plus the cross product of the ang. vel. with the relative offset

Angular velocity

- Angular velocity of frame **B** with respect to **I** expressed in **I**



$$_I\boldsymbol{\omega}_{IB} = \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix}$$

$_I\boldsymbol{\omega}_{IB}$ here is a an angular velocity in z direction of frame B with respect of the inertial frame.

- Relation to rotation matrix: skew-symmetric matrix $\hat{\boldsymbol{\omega}}$

$$_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{R}}_{IB}\mathbf{R}_{IB}^T, \quad _I\hat{\boldsymbol{\omega}}_{IB} = \begin{bmatrix} 0 & -\omega_{IB}^z & \omega_{IB}^y \\ \omega_{IB}^z & 0 & -\omega_{IB}^x \\ -\omega_{IB}^y & \omega_{IB}^x & 0 \end{bmatrix}, \quad \hat{\boldsymbol{\omega}}_{IB} = \begin{pmatrix} \omega_{IB}^x \\ \omega_{IB}^y \\ \omega_{IB}^z \end{pmatrix}$$

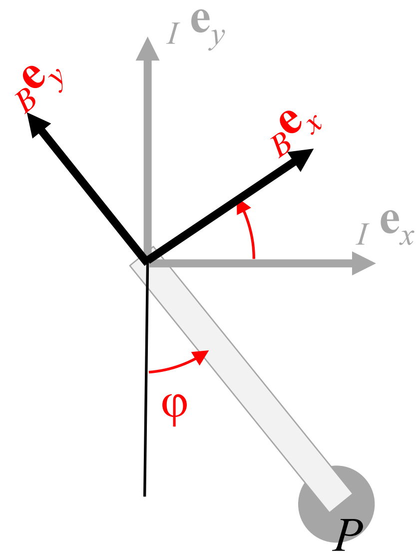
The skew-symmetric matrix $_I\hat{\boldsymbol{\omega}}_{IB}$ is used to get the angular velocity around each axis from the rotation matrices

- Transformation $\quad _I\boldsymbol{\omega}_{IB} = \mathbf{R}_{IB}\ {}_B\boldsymbol{\omega}_{IB}$

$_I\boldsymbol{\omega}_{IB}$ can be rotated from one to another coordinate frame using rotation matrix

- Consecutive rotations $\quad _I\boldsymbol{\omega}_{IC} = {}_I\boldsymbol{\omega}_{IB} + {}_I\boldsymbol{\omega}_{BC}$

Vector differentiation



- Position and velocity in different frames

How to calculate velocities from position vectors

- Inertial frame $_I\mathbf{r} \Rightarrow (\dot{\mathbf{r}}) = \dfrac{d\,_I\mathbf{r}}{dt}$,

is a non moving coordinate system

- Moving frame $_B\mathbf{r} \Rightarrow {}_B(\dot{\mathbf{r}}) = \underbrace{\dfrac{d\,_B\mathbf{r}}{dt}}_{\text{Change of the vector in its local frame}} + \underbrace{_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}}_{\text{Contribution of the relative rotation of the coordinate system itself}}$

- Example: single pendulum, $_B\mathbf{v}_P$ = ?

1st approach:

$$_I\mathbf{r}_{OP} = \begin{pmatrix} l.\sin(\varphi) \\ -l.\cos(\varphi) \\ 0 \end{pmatrix} \Rightarrow {}_I\mathbf{v}_P = \frac{d\,_I\mathbf{r}_{OP}}{dt} = \begin{pmatrix} l.\cos(\varphi)\dot{\varphi} \\ l.\sin(\varphi)\dot{\varphi} \\ 0 \end{pmatrix}$$

_determine the end-effector position as a function of the angle φ expressed in the inertial frame.

_transform it from inertial frame to body-fixed frame **B**

_this yields to:

$$_B\mathbf{v}_P = \mathbf{R}_{BI}\,_I\mathbf{v}_P = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} l.\cos(\varphi)\dot{\varphi} \\ l.\sin(\varphi)\dot{\varphi} \\ 0 \end{pmatrix} = \begin{pmatrix} l\dot{\varphi} \\ 0 \\ 0 \end{pmatrix}$$

2nd approach: determine the position vector in body-fixed coordinates **B**

$$_B\mathbf{r}_{OP} = \begin{pmatrix} 0 \\ -l \\ 0 \end{pmatrix}$$

_ differentiation with a moving frame:

$$_B\mathbf{v}_P = \frac{d\,_B\mathbf{r}_{OP}}{dt} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{OP}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \times \begin{pmatrix} 0 \\ -l \\ 0 \end{pmatrix} = \begin{pmatrix} l\dot{\varphi} \\ 0 \\ 0 \end{pmatrix}$$

Recapitulation

- We learned the basics for describing the motion
  - Translation $_I\mathbf{r}_{OP} = {}_I\mathbf{r}_{OB} + {}_I\mathbf{r}_{BP}$
  - Rotations $_B\mathbf{r}_{OP} = \mathbf{R}_{BI}\,{}_I\mathbf{r}_{BP}$
  - Homogeneous transformation $\begin{pmatrix} {}_I\mathbf{r}_{OP} \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R}_{IB} & {}_I\mathbf{r}_{OB} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} {}_B\mathbf{r}_{BP} \\ 1 \end{pmatrix}$
  - Angular velocities $_I\boldsymbol{\omega}_{IC} = {}_I\boldsymbol{\omega}_{IB} + {}_I\boldsymbol{\omega}_{BC}$
  - Differentiation of (position) vectors $_B\mathbf{r} \Rightarrow {}_B(\dot{\mathbf{r}}) = \frac{d\,{}_B\mathbf{r}}{dt} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}$