# Application of Rigid Body Kinematics

General coordinates, Jacobians and kinematics

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material is a courtesy of Prof. Marco Hutter of ETHZ

# Objective

- Learn about general coordinates and Jacobians
- Introduction of the concepts of inverse kinematics and differential kinematics
- Discussion of the kinematic structure of mobile platforms

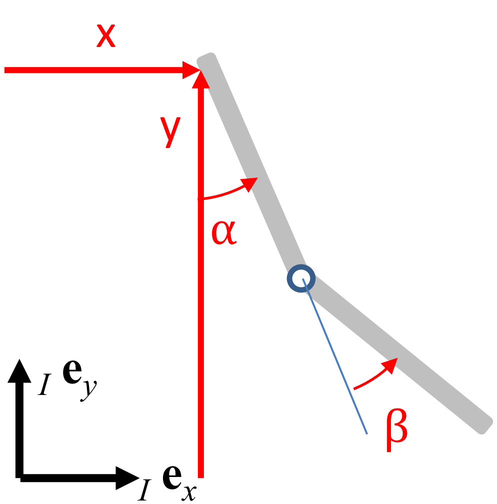
# Agenda

- Generalized coordinates
- Jacobians
- Inverse kinematics
- Differential kinematics
- Redundancy and singularity
- Kinematic structure of mobile robots

# Genralized coordinates

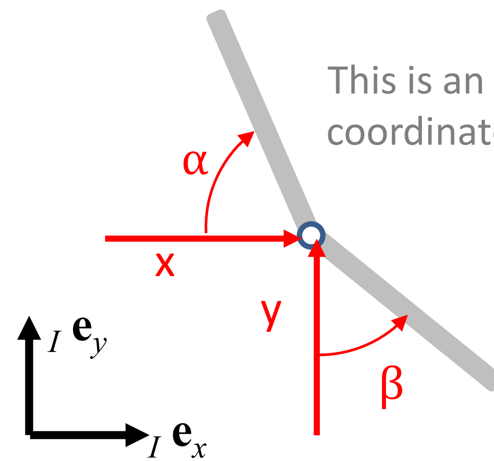
- A set of **independent** variables (general coordinates **q**) that **uniquely** describes the robot’s configuration (motion of the system)
  (could be using the famous Denavit-Hartenberg parameters in case of industrial manipulators)

Ex: double inverted pendulum that can freely move on a plane




$$\mathbf{q} = \begin{pmatrix} x \\ y \\ \alpha \\ \beta \end{pmatrix}$$

This **q** not a unique choice




This is an other set of general coordinates that also works.

x, y: local reference point

α,β: angles that describe the rotation of both links

However, in most cases the generalized coordinates are chosen so that they directly correspond to the joint or actuator degrees of freedom (DoF).

- Express all quantities as a function of generalized coordinates, e.g. $_I\mathbf{r}_{OP} = {_I\mathbf{r}_{OP}}(\mathbf{q})$

For us, any **q** will be composed of a unique relative angle or displacement at the joint or actuator’s DoF.

# Jacobians

- Partial derivative of position vector $\mathbf{J}_P = \frac{\partial \mathbf{r}_{OP}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial r_1}{\partial q_1} & \cdots & \frac{\partial r_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial q_1} & \cdots & \frac{\partial r_m}{\partial q_n} \end{bmatrix}$

- Jacobian is important for kinematics

Jacobian are used to map changes, (deltas or velocities which are changes per time) from generalized coordinates to Cartesian space or any other space the vector was represented in.

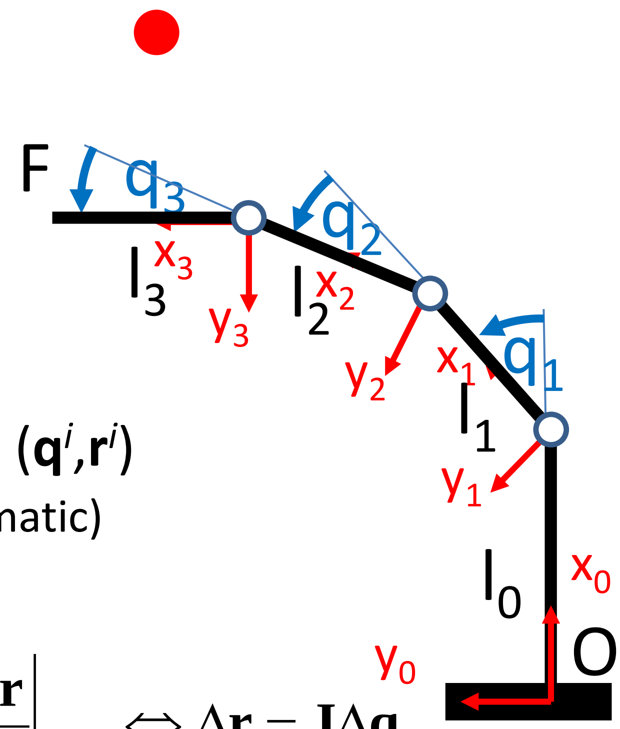
  - Linear mapping of Cartesian velocities to generalizes velocities: $\dot{\mathbf{r}}_P = \frac{\partial \mathbf{r}_{OP}}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{q}}$
    - Express contact constraints
    - Trajectory control

  - Linear mapping of changes in generalized coordinates to Cartesian space:
    - Forward and inverse kinematics

$$\Delta \mathbf{r}_P = \frac{\partial \mathbf{r}_{OP}}{\partial \mathbf{q}} \Delta \mathbf{q} = \mathbf{J}\Delta \mathbf{q}$$

# Inverse kinematics

- Problem
  - Given a desired end-effector position $\mathbf{r}_{OF}(\mathbf{q}) = \mathbf{r}_{OF}^{goal}$
  - Determine generalized coordinates $\mathbf{q}$




  - $\mathbf{r}_{OF}(\mathbf{q})$ often not easily invertible in closed form

- Approach: Iteratively perform the following steps $(\mathbf{q}^i, \mathbf{r}^i)$
  1. Start from known solution (e.g. given by forward kinematic)
  2. Linearize the function $\mathbf{r}_{OF}(\mathbf{q})$ around the solution resulting in the *Jacobian* that maps deltas in generalized coord. to deltas in Cartesian space: $\mathbf{J} = \left.\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right|_{\mathrm{q}=\mathrm{q}^i} \Leftrightarrow \Delta\mathbf{r} = \mathbf{J}\Delta\mathbf{q}$
  3. Invert the Jacobian to obtain $\Delta\mathbf{q} = \mathbf{J}^{+}\Delta\mathbf{r}$ Such that given a certain Δr we can determine the corresponding Δq
  4. Update generalized coordinates $\mathbf{q}^{i+1} = \mathbf{q}^{i} + \mathbf{J}^{+}(\mathbf{r}^{goal} - \mathbf{r}^{i})$

We iterate through these steps until the end-effector position has converged to the goal location. So we evaluate the **J** at updated **q** and update **q** again.

# Differential kinematics

To control the robot, the goal is often not only to find the gen. coord. for the desired end-effector position but to bring the end-effector from its current position on a smooth trajectory to the target.

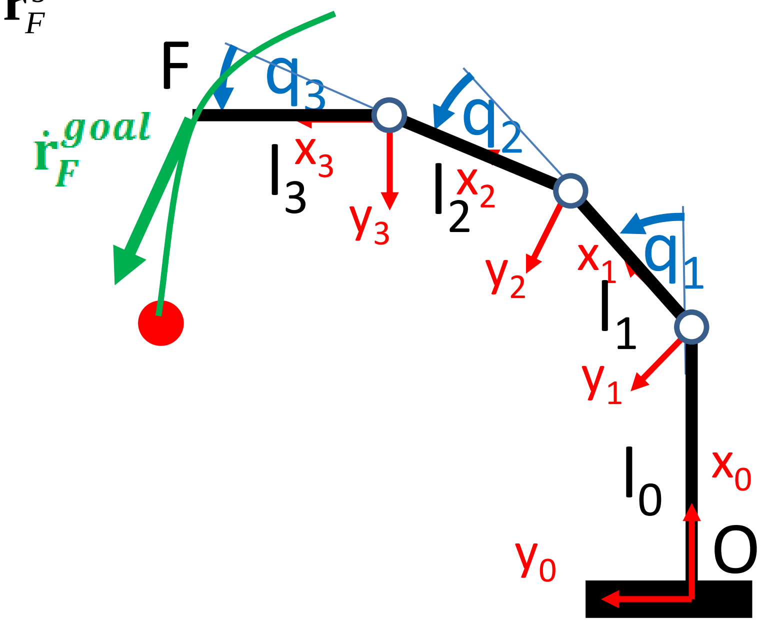
- Relation between Cartesian velocity and generalized velocities

$$\dot{\mathbf{r}}_F = \mathbf{J}_F \dot{\mathbf{q}}$$

Differential kinematics says: the Jacobian linearly maps the generalized velocities on to end-effector velocities

- Inverse differential kinematics
  - Given the end-effector velotcity $\dot{\mathbf{r}}$
  - Determine the generalized velocities $\dot{\mathbf{q}} = \mathbf{J}^{+} \dot{\mathbf{r}}_F^{goal}$

So to follow a predefined trajectory, we can determine the desired velocities by multiplying the desired end-effector velocity by the pseudo-inverse of J. This is the inverse differential kinematics approach



# Redundancy and singularity

As we have seen in these inverse kinematics examples we always need to take the inverse of the Jacobian. However the matrix can be rank deficient (i.e. not invertible), so we need pseudo-inversion

- Jacobian is often not invertible => use the Moore-Penrose pseudo inverse $\mathbf{J}^+$

There are two cases we need to differ:

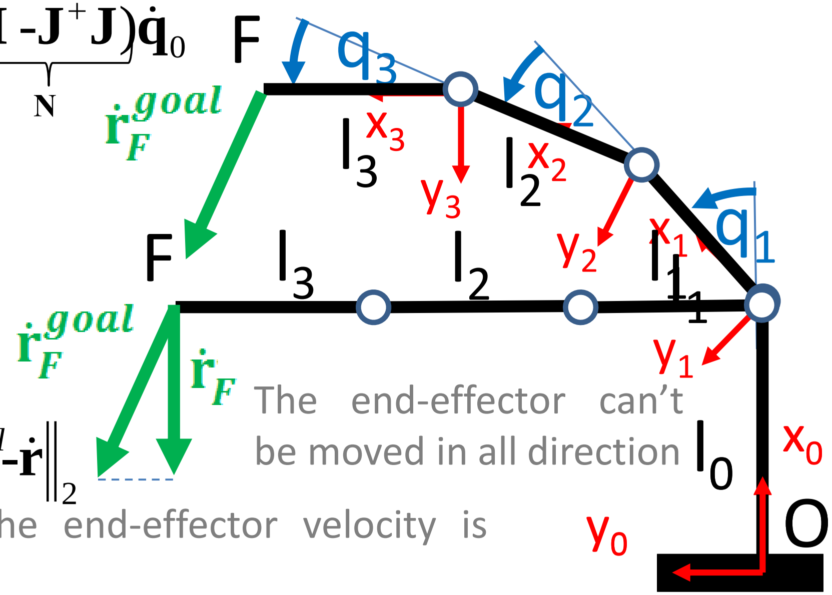
- Redundancy
  - Jacobian is column-rank deficient
  - Pseudo inverse minimizes $\|\dot{\mathbf{q}}\|_2$
  - Can have multiple solution $\dot{\mathbf{q}} = \mathbf{J}^+\dot{\mathbf{r}}^{goal} + \underbrace{(\mathbf{I} - \mathbf{J}^+\mathbf{J})}_{\mathbf{N}}\dot{\mathbf{q}}_0$

<u>Ex</u>: the planar three-link robot arm: 3 gen. coord. But only two DoF to control=> multiple solutions of gen. vel. that satisfy the desired end-effector velocities

And out of the possible solution space, the pseudo inversion takes the least square minimal q dot

Mathematically, the infinite number of solutions can be written by an additional **null space protector matrix N** which ensures that whatever $\dot{q}_0$ we choose, there is no influence on the end-effector velocity $\dot{r}_{goal}$

- Singularity
  - Jacobian is row-rank deficient
  - Pseudo inverse minimizes the error: $\|\dot{\mathbf{r}}^{goal} - \dot{\mathbf{r}}\|_2$

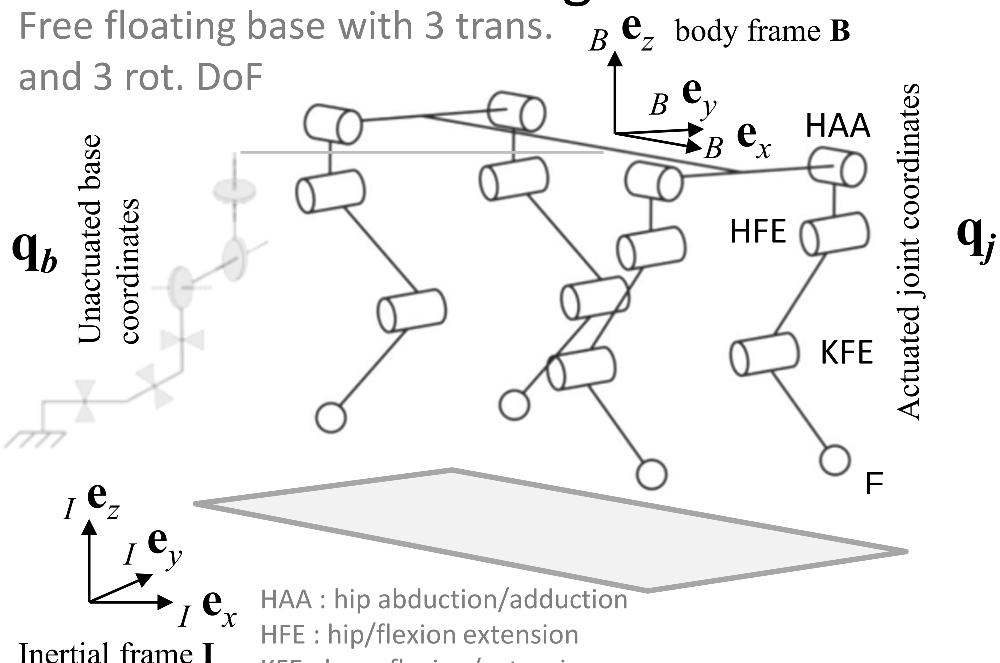



However taking the pseudo-inverse of J ensures that the end-effector velocity is achieved as good as possible in a least squares sense

# Kinematic structure of mobile robots

- Mobile plateform are not rigidly connected to the ground

Free floating base with 3 trans. and 3 rot. DoF




HAA : hip abduction/adduction
HFE : hip/flexion extension
KFE : knee flexion/extension

- Generalized coordinates

As a result the gen. coord. is the stack vector of :

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_b \\ \mathbf{q}_j \end{pmatrix} \begin{matrix} \text{Un-actuated base} \\ \text{Actuated joints} \end{matrix}$$

- Contact contraints
  - Footpoint is not allowed to move

$$\dot{\mathbf{r}}_F = \mathbf{J}_F \dot{\mathbf{q}} = \mathbf{0}$$

$\mathbf{J}_F$ is the contact Jacobian

  - Choose the joint speed such that this constraint is satisfied

$$\dot{\mathbf{q}} = \cancel{\mathbf{J}_F^+ \overset{=0}{\dot{\mathbf{r}}_F}} + \mathbf{N}_F \dot{\mathbf{q}}_0$$

  - Move body upward ensuring the feet are not moving:

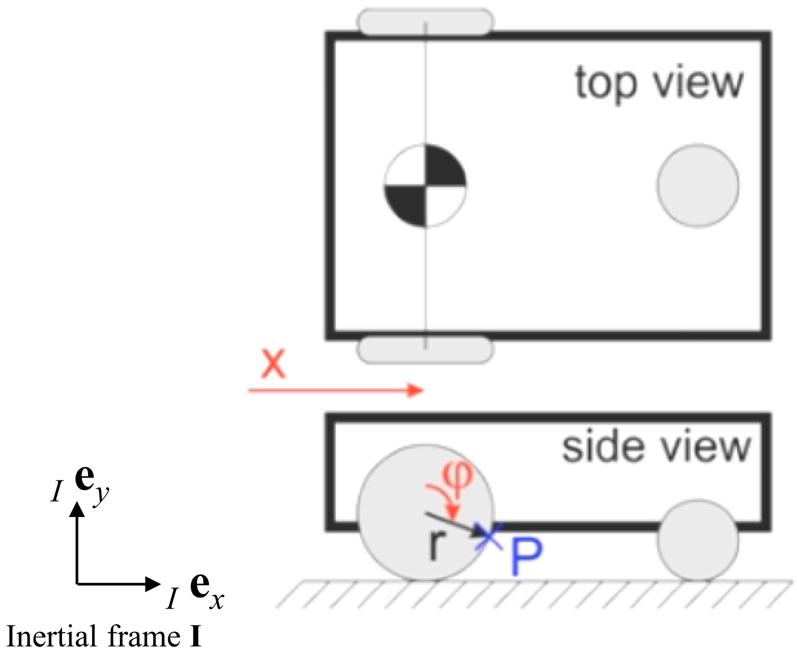
$$\dot{\mathbf{r}}_{body} = \mathbf{J}_{body} \dot{\mathbf{q}} = \mathbf{J}_{body} \mathbf{N}_F \dot{\mathbf{q}}_0$$

$$\dot{\mathbf{q}} = \cancel{\mathbf{J}_F^+ \overset{=0}{\dot{\mathbf{r}}_F}} + \mathbf{N}_F \dot{\mathbf{q}}_0$$

$$= \mathbf{N}_F (\mathbf{J}_{body} \mathbf{N}_F)^+ \dot{\mathbf{r}}_{body}$$

Equation to get the joint velocities that satisfy the contact constraint and at the same time, move the body upward

# Kinematic structure of mobile robots




- Generalized coordinates

$$\mathbf{q} = \begin{pmatrix} x \\ \varphi \end{pmatrix} \quad \begin{matrix} \text{Un-actuated base} \\ \text{Actuated joints} \end{matrix}$$

- Contact contraints
  - Point P on wheel

Description of the motion of P

$$_I\mathbf{r}_{OP} = \begin{pmatrix} x + r\sin(\phi) \\ r + r\cos(\varphi) \\ 0 \end{pmatrix}$$

  - Jacobian

Then we take the partial derivative to get the Jacobian

$$_I\mathbf{J}_P = \begin{bmatrix} 1 & r\cos(\varphi) \\ 0 & -r\sin(\varphi) \\ 0 & 0 \end{bmatrix}$$

  - Contact constraints

$$_I\mathbf{v}_P\Big|_{\varphi=\pi} = {_I\mathbf{J}_P}\Big|_{\varphi=\pi}\dot{\mathbf{q}} = \begin{bmatrix} 1 & -r \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} = 0$$

=>Rolling condition $\dot{x} - r\dot{\varphi} = 0$