Second Quantum Computing School
ICTP - SAIFR

Hackathon

Problems solutions

Group: Manezinhos + 1

Members: Cesar Augusto do Amaral
João Pedro Engster
Leticia Bertuzzi
Matheus Eiji Ohno Bezerra
Vinícius Luz Oliveira

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Chapter 1

Introduction to Applications of Quantum Computing to Quantum Chemistry

1.1 Exercise: obtaining expectation value of a Hamiltonian

The answer is in the notebook.

1.2 Challenge: ground state energy for molecule and spin system with variational quantum algorithms and trotterization

The answer is in the notebook.

Chapter 2

Bose-Einstein Condensates and the Involvement in Advances for New Technologies

2.1 Exercise 1: interactions between atoms and the low-energy limit

(a) In the low energy limit we set the wave vector as $k \to 0$. For the s-wave we consider a zero angular momentum ($l = 0$), which is interpreted due to the spherical symmetry of the problem. That is, the path does not depend on the direction of scattering and is uniform in all directions. Indeed, in the low energy regime, the wavefunction of a single particle scattered is given by

$$\psi(\mathbf{r}) = \mathrm{e}^{ikz} + f(\theta)\frac{\mathrm{e}^{ikr}}{r} \to \psi(\mathbf{r}) = 1 - \frac{a}{r} \tag{2.1}$$

where the first term is the incident wave, the second the scattered wave, $f(\theta)$ the scattering amplitude and a is the scattering length.

(b) In the low energy limit, the scattering length provides a measure of the strength of interaction in a scattering process, as shown in Eq.(2.1).

(c) For the two particle system, we hae the following Schrodinger equation:

$$\left[-\frac{\hbar^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right) + V(\mathbf{r}_1, \mathbf{r}_2)\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}_1, \mathbf{r}_2) \tag{2.2}$$

where we can consider the coordinates of the center of mass of one particle and look only for the scattered particle. Then, considering the Laplacian in spherical coordinates and a separable solution for the wave function, we have the following differential equation for the radial part [1]:

$$\frac{d^2}{dr^2}R_{kl}(r) = \frac{2}{r}\frac{d}{dr}R_{kl}(r) + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2}V(r)\right]R_{kl}(r) = 0 \tag{2.3}$$

and considering the low energy regime, i.e. $k \to 0$, $l \to 0$, we have:

$$\frac{d^2}{dr^2}R_{kl}(r) = \frac{2}{r}\frac{d}{dr}R_{kl}(r) - \frac{2m}{\hbar^2}V(r)R_{kl}(r) = 0 \tag{2.4}$$

and then using Eq.(2.1) in the previous equation and considering the scattering length a from the exercise set, ...

2.2 Exercise 2: the Gross-Pitaevskii equation

(a) We can write the Hamiltonian of the problem in a more convenient way:

$$H = \sum_{i=1}^{N} \underbrace{\left[\frac{\mathbf{p}_i^2}{2m} + V(\mathbf{r}_i)\right]}_{H_i} + \sum_{i<j} \underbrace{U_0\delta(\mathbf{r}_i - \mathbf{r}_j)}_{H_{ij}^{(int)}}, \tag{2.5}$$

where first term H_i is the non-interacting Hamiltonian of each particle and the second term H_{int} is the Hamiltonian of the interactions between particles. Then, considering a state of the form:

$$\Psi(\mathbf{r}_1, ..., \mathbf{r}_N) = \langle \mathbf{r}_1, ..., \mathbf{r}_N | \Psi \rangle = \prod_{i=1}^{N} \phi(\mathbf{r}_i), \tag{2.6}$$

we can find each term of the expectation value as follows:

- In order to calculate the first term, the following relation will be useful:

$$\phi^*(\mathbf{r}_i)\nabla^2\phi(\mathbf{r}_i) = \nabla_i\left(\phi^*(\mathbf{r}_i)\nabla_i\phi(\mathbf{r}_i)\right) - |\nabla_i\phi(\mathbf{r}_i)|^2, \tag{2.7}$$

  and finally, since the integral of the first term in the right side vanishes, we have:

$$\int d^3r_i\ \phi^*(\mathbf{r}_i)\nabla^2\phi(\mathbf{r}_i) = -\int d^3r_i\ |\nabla_i\phi(\mathbf{r}_i)|^2. \tag{2.8}$$

[1] It is a straightforward derivation, found in many textbooks of quantum mechanics, in the context of the Hydrogen atom, for example

Then, using Eqs.(2.6),(2.7) we can calculate the expectation value of the individual Hamiltonian H_i of Eq.(2.5) as follows:

$$\begin{aligned}\langle H_i\rangle &= \left[\prod_{k\neq i}\int d^3r_k|\phi(\mathbf{r}_k)|^2\right]\times\\ &\quad\times\int d^3r_i\left[-\frac{\hbar^2}{2m}\phi^*(\mathbf{r}_i)\nabla_i^2\phi(\mathbf{r}_i)+V(\mathbf{r}_i)|\phi(\mathbf{r}_i)|^2\right]\\ &= \int d^3r_i\left[\frac{\hbar^2}{2m}|\nabla_i\phi(\mathbf{r}_i)|^2+V(\mathbf{r}_i)|\phi(\mathbf{r}_i)|^2\right].\end{aligned}\tag{2.9}$$

Finally, since the expectation value of each Hamiltonian are the same, i.e. $\langle H_i\rangle=\langle H_j\rangle$, the expectation value of the total Hamiltonian is given by:

$$\begin{aligned}\sum_{i=1}^{N}\langle H_i\rangle &= \sum_{i=1}^{N}\int d^3r_i\left[\frac{\hbar^2}{2m}|\nabla_i\phi(\mathbf{r}_i)|^2+V(\mathbf{r}_i)|\phi(\mathbf{r}_i)|^2\right]\\ &= N\int d^3r\left[\frac{\hbar^2}{2m}|\nabla\phi(\mathbf{r})|^2+V(\mathbf{r})|\phi(\mathbf{r})|^2\right].\end{aligned}\tag{2.10}$$

- The second term is calculated in a similar way:

$$\begin{aligned}\langle H_{ij}^{(int)}\rangle &= U_0\left[\prod_{k\neq i,j}\int d^3r_k|\phi(\mathbf{r}_k)|^2\right]\int d^3r_i\,d^3r_j\,|\phi(\mathbf{r}_i)|^2|\phi(\mathbf{r}_j)|^2\,\delta(\mathbf{r}_i-\mathbf{r}_j)\\ &= U_0\int d^3r_i\,|\phi(\mathbf{r}_i)|^4,\end{aligned}\tag{2.11}$$

and then, after the summation:

$$\begin{aligned}\sum_{i<j}\langle H_{ij}^{(int)}\rangle &= \sum_{i<j}U_0\int d^3r_i\,|\phi(\mathbf{r}_i)|^4\\ &= \frac{N(N-1)}{2}U_0\int d^3r\,|\phi(\mathbf{r})|^4.\end{aligned}\tag{2.12}$$

Finally, replacing Eqs.(2.10),(2.12) in the complete Hamiltonian given by Eq.(2.5) and denoting $\psi(\mathbf{r})=\sqrt{N}\phi(\mathbf{r})$, we have:

$$E:=\langle H\rangle=\int d^3r\,\left[\frac{\hbar^2}{2m}|\nabla\psi(\mathbf{r})|^2+V(\mathbf{r})|\psi(\mathbf{r})|^2+\frac{U_0}{2}\left(1-\frac{1}{N^2}\right)|\psi(\mathbf{r})|^4\right].\tag{2.13}$$

(b) Considering variations with respect to $\psi^*(\mathbf{r})\to\psi^*(\mathbf{r})+\delta\psi^*(\mathbf{r})$ we have the following perturbations:

$$\psi^*(\mathbf{r})\nabla^2\psi(\mathbf{r})\to\psi^*(\mathbf{r})\nabla^2\psi(\mathbf{r})+\left[\delta\psi^*(\mathbf{r})\right]\nabla^2\psi(\mathbf{r})\tag{2.14}$$

$$|\psi(\mathbf{r})|^2 \rightarrow |\psi(\mathbf{r})|^2 + [\delta\psi^*(\mathbf{r})]\,\psi(\mathbf{r}) \tag{2.15}$$

$$|\psi(\mathbf{r})|^4 \rightarrow |\psi(\mathbf{r})|^4 + 2\,[\delta\psi^*(\mathbf{r})]\,|\psi(\mathbf{r})|^2\psi(\mathbf{r}) + \mathcal{O}\left(\delta^2\right) \tag{2.16}$$

Then, remembering the relation given by Eq.(2.7) we can find the perturbation on $\langle H \rangle$ from the three perturbations in the Eq.(2.13), obtaining:

$$\begin{aligned}\delta E = \int d^3r \Bigg[\delta\psi^*(\mathbf{r})\left(\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + U_0|\psi(\mathbf{r})|^2\right)\psi(\mathbf{r}) \qquad + \\ +\frac{\hbar^2}{2m}\psi^*(\mathbf{r})\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})|\psi(\mathbf{r})|^2 + \frac{U_0}{2}|\psi(\mathbf{r})|^4 + \mathcal{O}\left(\delta^2\right)\Bigg]\end{aligned} \tag{2.17}$$

In a similar way, we can perturbate the number of particles as follows:

$$\delta N = \delta \int d^3r|\psi(\mathbf{r})|^2 = \delta \int d^3r\,\left[[\delta\psi^*(\mathbf{r})]\,\psi(\mathbf{r}) + |\psi(\mathbf{r})|^2\right] \tag{2.18}$$

Finally, using the proper Lagrange multiplier, we have from the two previous equations:

$$\begin{aligned}\delta E - \mu\delta N = \int d^3r \Bigg[\delta\psi^*(\mathbf{r})\left(\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + U_0|\psi(\mathbf{r})|^2 - \mu\right)\psi(\mathbf{r}) \qquad + \\ +\frac{\hbar^2}{2m}\psi^*(\mathbf{r})\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})|\psi(\mathbf{r})|^2 + \frac{U_0}{2}|\psi(\mathbf{r})|^4 - \mu|\psi(\mathbf{r})|^2 + \mathcal{O}\left(\delta^2\right)\Bigg]\end{aligned} \tag{2.19}$$

and then, since $\delta E - \mu\delta N = 0$ the integrand should be zero, leading us in to expected Gross-Pitaevskii equation:

$$\left[\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + U_0|\psi(\mathbf{r})|^2\right]\psi(\mathbf{r}) = \mu\,\psi(\mathbf{r}) \tag{2.20}$$

2.3 Challenge: computational project

The answer is in the notebook.

Chapter 3

Prospects and Challenges for Quantum Machine Learning

3.1 Exercise 1

Take the set $G = \{\mathbf{1}, X\}$, where $\mathbf{1}$ is the identity matrix and X is the Pauli-X matrix, on which we choose the following representation:

We want to show G is a group (with the usual matrix product)

3.1.1 Closure

We can easily verify closure by taking the products (or building the Cayley Table):

$$\mathbf{1}X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \in G$$

$$X\mathbf{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \in G$$

$$\mathbf{1}\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} \in G$$

$$XX = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} \in G$$

3.1.2 Associativity

Matrix product is associative by definition.

3.1.3 Identity element

As one can see from the products taken when showing, closure, 1 is an identity element, i.e, $1g = g, \forall g \in G$:

$$\mathbf{1}X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$X\mathbf{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$\mathbf{1}\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}$$

3.1.4 Inverse element

Again, it follows naturally from the Cayley Table that every element in G has an inverse. The products:

$$\mathbf{1}\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} \in G$$

$$XX = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1} \in G$$

Therefore, the set $\{\mathbf{1}, X\}$ equipped with usual matrix product satisfies the definition of a group.

3.2 Exercise 2

Take $U = e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_1 Y}$

One can determine the set S of generators of the Lie algebra by taking the derivative with respect to each parameter (applying Theorem 1 of Lecture notes of Lie group reps inducing Lie algebra reps):

$$S_1 = \frac{\partial}{\partial \phi_1} (U) \bigg|_{\phi_1 = \phi_2 \phi_3 = 0} = -iY$$

$$S_2 = \frac{\partial}{\partial \phi_2} (U) \bigg|_{\phi_1 = \phi_2 \phi_3 = 0} = -iX$$

$$S_3 = \frac{\partial}{\partial \phi_3} (U) \bigg|_{\phi_1 = \phi_2 \phi_3 = 0} = -iY$$

Notice that $S_1 = S_3$. Taking the commutator one can get the Lie closure $< S >_{\text{Lie}}$ of this set by evaluating the commutators:

$$[S_1, S_2] = [S_1, S_3] = [-iY, -iX] = [X, Y] = 2iZ$$
$$[S_2, S_3] = 0$$

By taking further commutators (and using already known Pauli matrices commutation relations) :

$$[-iX, -iZ] = 2iY$$
$$[-iY, -iZ] = 2iX$$

That is, no other elements can be obtained to the closure $< S >_{\text{Lie}}$ that is not already in the set ($\{-iX, -iY, -iZ\}$). By inspecting the calculated Lie brackets, the structure is the same of the $\mathfrak{su}(2)$, i.e,

$$[S_i, S] = 2\epsilon_{ijk} S_j$$

Therefore, the Lie algebra induced by $e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_1 Y}$ guarantees it is a representation of $SU(2)$.

Chapter 4

High Dimensional Quantum Communication with Structured Light

4.1 Exercise 1: Hermite-Gauss mode

First of all, the Hermite-Gauss polynomials are given by:

$$\begin{aligned} HG_{mn}(x,y) &= \frac{A_{mn}}{w(z)} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) \times \\ &\times \exp\left[-(x^2+y^2)\left(\frac{1}{w^2(z)} - \frac{ik}{r(z)}\right) - i\varphi z\right] \end{aligned} \tag{4.1}$$

where $H_n(...)$ are the Hermite polynomials of order n. This set of polynomials forms an orthonormal set:

$$\int dx\, dy\, HG^*_{m'n'}(x,y)\, HG_{mn}(x,y) = \delta_{nn'}\delta_{mm'} \tag{4.2}$$

In addition, since we are considering the first order Hermite-Gauss polynomials, i.e. $m+n=1$, only matters the first order Hermite polynomials, given by:

$$H_0(\lambda) = 1 \ \ , \ \ H_1(\lambda) = 2\lambda \tag{4.3}$$

- Let us denote our position vector as $\mathbf{r} = (x, y)$, which are affected by a rotation θ from the action of a orthogonal matrix R_θ, given by:

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.4}$$

Then, after the rotation, the position vector is transformed to $\mathbf{r}' = R_\theta \mathbf{r}$, with coordinates:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta\ x + \sin\theta\ y \\ \cos\theta\ y - \sin\theta\ x \end{pmatrix} \tag{4.5}$$

In addition, it is easy to see, that the rotation preserves the norm of the vector:

$$(x')^2 + (y')^2 = (\mathbf{r}')^T \mathbf{r}' = (R_\theta \mathbf{r})^T (R_\theta \mathbf{r}) = \mathbf{r}^T (\underbrace{R_\theta^T R_\theta}_{1}) \mathbf{r} = x^2 + y^2 \tag{4.6}$$

Then, the first order Hermite-Gauss mode rotated counterclockwise are given by:

$$\begin{aligned} G_\theta(x,y) &= G_{10}(x',y') \\ &= \frac{A_{10}}{w(z)} H_1 \left(\frac{\sqrt{2}x'}{w(z)} \right) H_0 \left(\frac{\sqrt{2}y'}{w(z)} \right) \times \\ &\quad \times \exp \left[- \left((x')^2 + (y')^2 \right) \left(\frac{1}{w^2(z)} - \frac{ik}{r(z)} \right) - i\varphi z \right] \end{aligned} \tag{4.7}$$

and then, from Eq.(4.3), Eq.(4.5) we can rewrite the Hermite polynomial as follows:

$$\begin{aligned} H_1 \left(\frac{\sqrt{2}x'}{w(z)} \right) H_0 \left(\frac{\sqrt{2}y'}{w(z)} \right) &= \frac{2\sqrt{2}}{w(z)} \Big(\cos\theta\ x + \sin\theta\ y \Big) = \\ &= \cos\theta\ H_1 \left(\frac{\sqrt{2}x}{w(z)} \right) H_0 \left(\frac{\sqrt{2}y}{w(z)} \right) + \\ &\quad + \sin\theta\ H_0 \left(\frac{\sqrt{2}x}{w(z)} \right) H_1 \left(\frac{\sqrt{2}y}{w(z)} \right). \end{aligned} \tag{4.8}$$

Finally, since the rotation preserves the norm, according to Eq.(4.6), we can just replace the previous equation in Eq.(4.7), arriving in the following expression:

$$\begin{aligned} G_\theta(x,y) &= \left[\cos\theta\ \frac{A_{10}}{w(z)}\ H_1 \left(\frac{\sqrt{2}x}{w(z)} \right) H_0 \left(\frac{\sqrt{2}y}{w(z)} \right) + \right. \\ &\quad \left. + \sin\theta\ \frac{A_{01}}{w(z)}\ H_0 \left(\frac{\sqrt{2}x}{w(z)} \right) H_1 \left(\frac{\sqrt{2}y}{w(z)} \right) \right] \times \\ &\quad \times \exp \left[- \left(x^2 + y^2 \right) \left(\frac{1}{w^2(z)} - \frac{ik}{r(z)} \right) - i\varphi z \right] \\ &= \cos\theta\ HG_{10}(x,y) + \sin\theta\ HG_{01}(x,y). \end{aligned} \tag{4.9}$$

where we considered the definition of the Hermite-Gauss polynomials given by Eq.(4.1) and $A_{10} = A_{10}$ (which is valid due to the cylindrical symmetry of these normalization factors).

- Using the result given by Eq.(4.9) and the orthonormalization condition given by Eq.(4.2), we have:

$$\begin{aligned}
\int dx\ dy\ HG^*_\theta(x,y)\ HG_\theta(x,y) &= \int dx\ dy \Big(\cos\theta\ HG^*_{10}(x,y) + \sin\theta\ HG^*_{01}(x,y)\Big) \times \\
&\times \Big(\cos\theta\ HG_{10}(x,y) + \sin\theta\ HG_{01}(x,y)\Big) \\
&= \cos\theta\ \sin\theta\ \Big(\underbrace{\int dx\ dy\ HG^*_{10}(x,y)\ HG_{01}(x,y)}_{0} + c.c.\Big) + \\
&+ \cos^2\theta \underbrace{\int dx\ dy\ |\ HG_{10}(x,y)|^2}_{1} + \sin^2\theta \underbrace{\int dx\ dy\ |\ HG_{01}(x,y)|^2}_{1} \\
&= \cos^2\theta + \sin^2\theta \\
&= 1.
\end{aligned} \tag{4.10}$$

- Similarly to the previous item, using the result given by Eq.(4.9), the orthonormalization condition given by Eq.(4.2) and the trigonometric relations $\cos(\theta + \pi/2) = -\sin\theta$ and $\sin(\theta + \pi/2) = \cos\theta$, we have:

$$\begin{aligned}
\int dx\ dy\ HG^*_\theta(x,y)\ HG_{\theta+\pi/2}(x,y) &= \int dx\ dy \Big(\cos\theta\, HG^*_{10}(x,y) + \sin\theta\ HG^*_{01}(x,y)\Big) \times \\
&\times \Big(-\sin\theta\ HG_{10}(x,y) + \cos\theta\ HG_{01}(x,y)\Big) \\
&= \cos^2\theta \Big(\underbrace{\int dx\ dy\ HG^*_{10}(x,y)\ HG_{01}(x,y)}_{0}\Big) + \sin^2\theta\ \Big(c.c.\Big) + \\
&+ \sin\theta\cos\theta \Big(\underbrace{\int dx\ dy\ |\ HG_{10}(x,y)|^2}_{1} - \underbrace{\int dx\ dy\ |\ HG_{01}(x,y)|^2}_{1}\Big) \\
&= 0.
\end{aligned} \tag{4.11}$$

4.2 Exercise 2: square integrable functions

Since $\{|u_{n,m}\rangle\}$ is a basis, we have:

$$\sum_{n,m} |u_{n,m}\rangle \langle u_{n,m}| = \mathbb{I}, \tag{4.12}$$

and then, calculating the matrix elements in the basis $\{|x,y\rangle\}$, we arrive in:

$$\sum_{n,m} \langle x,y|u_{n,m}\rangle \langle u_{n,m}|x',y'\rangle = \langle x,y|x',y'\rangle \tag{4.13}$$

and finally:

$$\sum_{n,m} u_{n,m}(x,y)u^*_{n,m}(x',y') = \delta(x-x')\delta(y-y') \tag{4.14}$$

4.3 Challenge

- First of all, from Eq.(4.9) we recover the expression for the Hermite-Gauss mode rotated, as following:

$$\begin{aligned} HG_\theta(x,y) &= \cos\theta\ HG_{10}(x,y) + \sin\theta\ HG_{01}(x,y) \\ HG_{\theta+\pi/2}(x,y) &= -\sin\theta\ HG_{10}(x,y) + \cos\theta\ HG_{01}(x,y) \end{aligned} \tag{4.15}$$

In addition, in the problem set where defined the following polarization vectors:

$$\begin{aligned} \mathbf{e}_\theta &= \cos\theta\ \mathbf{e}_H + \sin\theta\ \mathbf{e}_V \\ \mathbf{e}_{\theta+\pi/2} &= -\sin\theta\ \mathbf{e}_H + \cos\theta\ \mathbf{e}_V \end{aligned} \tag{4.16}$$

Then, for the first state, we have:

$$\begin{aligned} \Psi^{(1)}_\theta(x,y) &= HG_\theta(x,y)\ \mathbf{e}_\theta + HG_{\theta+\pi/2}(x,y)\ \mathbf{e}_{\theta+\pi/2} \\ &= \Big(\cos\theta\ HG_{10}(x,y) + \sin\theta\ HG_{01}(x,y)\Big)\Big(\cos\theta\ \mathbf{e}_H + \sin\theta\ \mathbf{e}_V\Big) + \\ &\quad + \Big(-\sin\theta\ HG_{10}(x,y) + \cos\theta\ HG_{01}(x,y)\Big)\Big(-\sin\theta\ \mathbf{e}_H + \cos\theta\ \mathbf{e}_V\Big) \\ &= \Big(\underbrace{\sin^2\theta + \cos^2\theta}_{1}\Big)\Big(HG_{10}(x,y)\ \mathbf{e}_H + HG_{01}(x,y)\ \mathbf{e}_V\Big) + \\ &\quad \Big(\underbrace{\sin\theta\ \cos\theta - \sin\theta\ \cos\theta}_{0}\Big)\Big(HG_{10}(x,y)\ \mathbf{e}_V + HG_{01}(x,y)\ \mathbf{e}_H\Big) + \\ &= HG_{10}(x,y)\ \mathbf{e}_H + HG_{01}(x,y)\ \mathbf{e}_V \end{aligned} \tag{4.17}$$

that does not depends on the arbitrary phase rotation θ and therefore is a rotation invariant.

Finally, for the second state, we have:

$$\begin{aligned}\Psi_\theta^{(2)}(x,y) &= HG_{\theta+\pi/2}(x,y)\ \mathbf{e}_\theta + HG_\theta(x,y)\ \mathbf{e}_{\theta+\pi/2} \\ &= \Big(\cos\theta\ HG_{10}(x,y) + \sin\theta\ HG_{01}(x,y)\Big)\Big(-\sin\theta\ \mathbf{e}_H + \cos\theta\ \mathbf{e}_V\Big) + \\ &\quad +\Big(-\sin\theta\ HG_{10}(x,y) + \cos\theta\ HG_{01}(x,y)\Big)\Big(\cos\theta\ \mathbf{e}_H + \sin\theta\ \mathbf{e}_V\Big) \\ &= -\sin(2\theta)\ HG_{10}(x,y)\ \mathbf{e}_H + \cos(2\theta)\ HG_{01}(x,y)\ \mathbf{e}_V + \\ &\quad +\cos(2\theta)\ HG_{10}(x,y)\ \mathbf{e}_V + \sin(2\theta)\ HG_{01}(x,y)\ \mathbf{e}_H \\ &= \ldots \end{aligned} \tag{4.18}$$

- Let an electromagnetic field given by:

$$\mathbf{E} = E_H\ \mathbf{e}_H + E_V\ \mathbf{e}_V \tag{4.19}$$

where is $\{\mathbf{e}_H, \mathbf{e}_V\}$ the polarization basis. Then we define the polarization Stokes parameters as follows:

$$\begin{aligned} S_1 &= |E_H|^2 - |E_V|^2, \\ S_2 &= 2\,\mathrm{Re}\{E_H\ E_V^*\}, \\ S_3 &= -2\,\mathrm{Im}\{E_H\ E_V^*\}. \end{aligned} \tag{4.20}$$

Considering the state Eq.(4.17) and using the the orthonormalization condition given by Eq.(4.9) as well as detectors with large area (i.e. integral over the entire $\mathbb{R}^2$), we have:

$$\begin{aligned} \langle S_1^{(1)}\rangle &= \underbrace{\int dx\ dy\ \ |HG_{10}(x,y)|^2}_{1} - \underbrace{\int dx\ dy\ |HG_{01}(x,y)|^2}_{1} = 0 \\ \langle S_2^{(1)}\rangle &= 2\,\mathrm{Re}\left\{\underbrace{\int dx\ dy\ \ HG_{10}(x,y)\ HG_{01}^*(x,y)}_{0}\right\} = 0 \\ \langle S_3^{(1)}\rangle &= 2\,\mathrm{Im}\left\{\underbrace{\int dx\ dy\ \ HG_{10}(x,y)\ HG_{01}^*(x,y)}_{0}\right\} = 0 \end{aligned} \tag{4.21}$$