Stochastic Differential Equations

August 25, 2021

1 Numerical Simulation of Stochastic Differential Equations

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import fsolve
from tqdm.notebook import trange
import scipy.stats as stats

%matplotlib inline
```

[2]:
```
plt.style.use(['science', 'notebook', 'grid'])
```

1.1 CONSTANTS

[3]:
```
m = 1
l = 1
U_B = 4
R = 0.1
T = 0.5
eta = 1
d = 10
I_0 = 0.0002
T_e = 0.1
e = 5
V = 5 ## IMP ## Overflow at V = 20

## Precalculated
eta_m = eta/m
sqrt_2Teta = np.sqrt(2*T*eta)
```

1.2 Helper Functions

[4]:
```
def defplot_1D(f, plotrange):
    x = np.linspace(plotrange[0],plotrange[1], 200)
    y = [f(i) for i in x]
    plt.plot(x,y)
    return
```

1.3 Implementation 2 (Euler-Maruyama)

1. $$C_0(V) = \frac{-4V + 49}{9}$$

2. $$C(x, V) = \frac{C_0(V)}{1 + \frac{x}{d}}$$

3. $$U(x) = U_B \left(\frac{x^4}{4l^4} - \frac{x^2}{2l^2} \right)$$

4. $$\mathcal{H}(x, p, q) = \frac{p^2}{2m} + U(x) + \frac{q^2}{2C(x)} - \frac{C_0(V)V^2}{2d}x + qV$$

5. $$\frac{\partial \mathcal{H}}{\partial q} = V + \frac{q}{C(x)}$$

6. $$\frac{\partial \mathcal{H}}{\partial x} = U'(x) - \frac{C_0(V)V^2}{2d} + \frac{q^2}{2} \left[\frac{C_0(V)d}{(C(x, V))^2(d + x)^2} \right] d$$

7. $$u + 2I_0 Rsinh \left(\frac{u}{T_e} \right) - V - \frac{q}{C} = 0$$

8. $$\mathcal{R} = R + R_E$$

1.3.1 Functions

[5]:
```
def C_0(V):
    '''Base Capacitance'''
    return (-4*V + 49)/9

def C(x,V):
    '''Final Capacitance'''
    return C_0(V)/(1+(x/d))

def U(x):
    '''Potential'''
```

```python
    return U_B*((x**4/(4*l**4))-(x**2/(2*l**2)))


def H(x,p,q,V):
    '''Hamiltonian'''
    return p**2/(2*m) + U(x) + q**2/(2*C(x,V)) - C_0(V)*V**2*x/(2*d)+q*V


def dHdq(q,x,V):
    '''dH/dq partial'''
    return V + q/C(x,V)

def dHdx(x,p,q,V):
    '''dH/dx partial'''
    uprime = U_B*(x**3/l**4 - x/l**2)
    C0 = C_0(V)
    Cx  = C(x,V)
    C1 = C0*V**2/(2*d)
    C2 = (q**2/2)*(d*C0/(Cx**2*(d+x)**2))
    return uprime - C1 + C2

def get_u(x,q,V):
    '''Potential Drop across Diodes'''
    sub1 = 2*I_0*R
    sub2 = V+q/C(x,V)
    def func_u(u):
        return u + sub1*np.sinh(u/T_e)-sub2
    u = fsolve(func_u,1)
    return u

def get_R_E(x,q,V,u = None):
    ''' Equivalent Resistance of Diodes'''
    if u == None:
        u = get_u(x,q,V)
    return u/(2*I_0*np.sinh(u/T_e))
#     return R*u/(V + q/C(x,V) - u)

def dT_Rdq(x,q,V,R_E):
    '''d(T/R)/dq partial'''
    delta = 1e-7
    T1 = -T/(R+R_E)**2
    R_E2 = get_R_E(x, q+delta, V)
    return T1*(R_E2 - R_E)/delta
```

1.3.2 Simulation Parameters

- Parameters used in paper :

1. $N = 1$ million (using 20)
2. $n = 10$ million (using 1 million)
3. Time Horizon = 5000

[6]:
```
N = 1 # Separate Instances
n = 10*10**5 # steps in an instance
T_H = 5000 # Time Horizon for an instance
times = np.linspace(0,T_H,n)
dt = times[1] - times[0]
```

1.3.3 Simulation

1. Constant Bias Potential (V = 5V)

[7]:
```
x_values = np.zeros((n,N))
p_values = np.zeros((n,N))
q_values = np.zeros((n,N))
I_values = np.zeros((n,N))
R_net_values = np.zeros((n,N))
u_values = np.zeros((n,N))

dwp = np.sqrt(dt)*np.random.normal(size = (n,N))
dwq = np.sqrt(dt)*np.random.normal(size = (n,N))

dwp2 = sqrt_2Teta*dwp #Precalculated

# dWdt = np.zeros((n,N))

for i in trange(N, desc='Instances Completed'):

    for j in trange(n-1, desc=f'Completion of Instance {i+1}'):

        # update x
        x_values[j+1,i] = x_values[j,i] + p_values[j,i]/m*dt

        # update p
        p_values[j+1,i] = p_values[j,i] - (eta_m*p_values[j,i] +␣
↪dHdx(x_values[j,i], p_values[j,i], q_values[j,i],V))*dt + dwp2[j,i]


        # get u : Potential Drop across diodes
        u = get_u(x_values[j,i],q_values[j,i],V)
        u_values[j,i] = u

        # getting net resistance
        R_E = get_R_E(x_values[j,i], q_values[j,i], V, u)
        R_net = R + R_E
        R_net_values[j,i] = R_net
```

```
        # storing total current
        I_values[j+1,i] = -(1/R_net)*(V + q_values[j,i]/C(x_values[j,i],V))

        # update q
        q_values[j+1,i] = q_values[j,i] + (dT_Rdq(x_values[j,i],␣
↪q_values[j,i],V, R_E)-1/R_net*(V + q_values[j,i]/C(x_values[j,i],V)))*dt +␣
↪np.sqrt(2*T/R_net)*dwq[j,i]

        #update dWdt

# np.save("10instances.npy", np.stack([x_values, p_values, q_values, I_values,␣
↪R_net_values, u_values], axis = 2))
```

```
Instances Completed:   0%|          | 0/1 [00:00<?, ?it/s]

Completion of Instance 1:   0%|          | 0/999999 [00:00<?, ?it/s]
```

1.4 Plots

[12]:
```
# Loader

z = np.load("20instances.npy")
z2 = np.load("20instances_newer.npy")
z3 = z2[:,:,:6]
final = np.concatenate([z,z3], axis = 1)
x_values = final[:,:,0]
p_values = final[:,:,1]
q_values = final[:,:,2]
I_values = final[:,:,3]
R_net_values = final[:,:,4]
u_values = final[:,:,5]
```

[13]:
```
x_values.shape
```

[13]:
```
(1000000, 40)
```

[14]:
```
N = 40
```

[15]:
```
# Choose which instance to plot
import random
instance = None
if instance is None:
    instance = random.choice(list(range(N)))
print(f"Plotting for instance {instance + 1} of {N}")
```

```
Plotting for instance 20 of 40
```

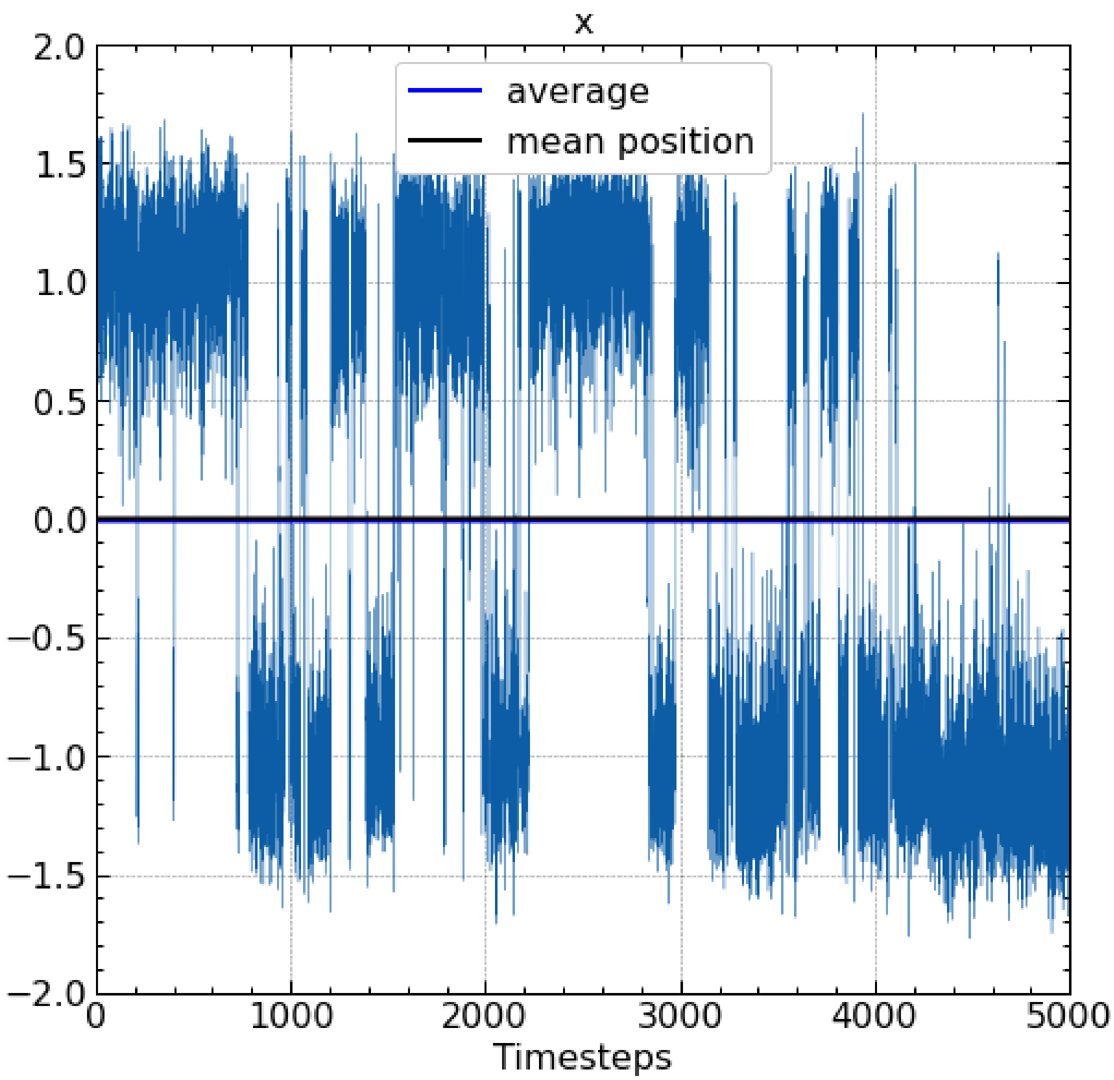
1.5 x v/s t

[43]:
```
instance = 13
plt.figure(figsize = (8,8))
plt.xlabel("Timesteps")
plt.xlim(0,T_H)

plt.plot(times, x_values[:, instance], linewidth = 0.3)
plt.plot(times, np.ones((n,1))*np.mean(x_values[:,instance]), color = "blue",␣
 ↪label = "average")
plt.plot(times, np.zeros((n,1)), color = "black", label = "mean position")
plt.title("x")
plt.ylim(-2,2)
plt.legend(loc = "upper center")

# plt.plot(times, p_values[:, instance], linewidth = 0.25)
# plt.plot(times, R_net_values[:, instance], linewidth = 0.25)
# plt.plot(times, u_values[:, instance], linewidth = 0.25)
# plt.title("$\mathcal{R}$")
```

[43]: <matplotlib.legend.Legend at 0x189b8dc6448>



1.6 q v/s t

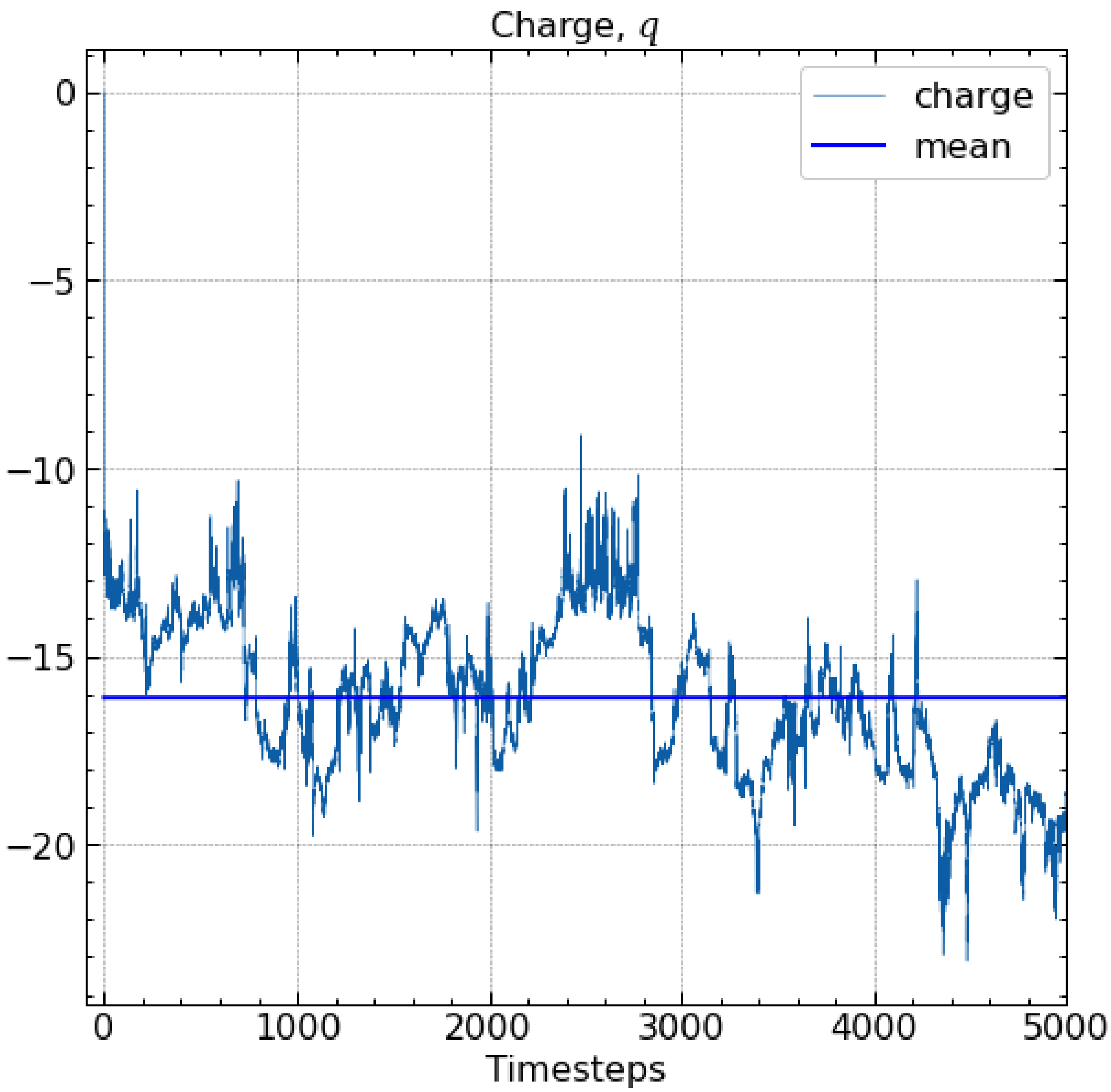
[32]:
```
instance = 13

plt.figure(figsize = (8,8))
plt.xlabel("Timesteps")
plt.xlim(-100,T_H)

# plt.plot(times, q_values[:, instance]/3, linewidth = 0.25)
plt.plot(times, q_values[:, instance], linewidth = 0.5,label = "charge")
plt.plot(times, np.ones((n,1))*np.mean(q_values[:,instance]), color = 'blue', 
 ↪label = "mean")
# plt.ylim(-25,0)
# plt.xlim(-100,T_H)
```

```
plt.title("Charge, $q$")
plt.legend()
```

[32]: <matplotlib.legend.Legend at 0x18953cc8888>



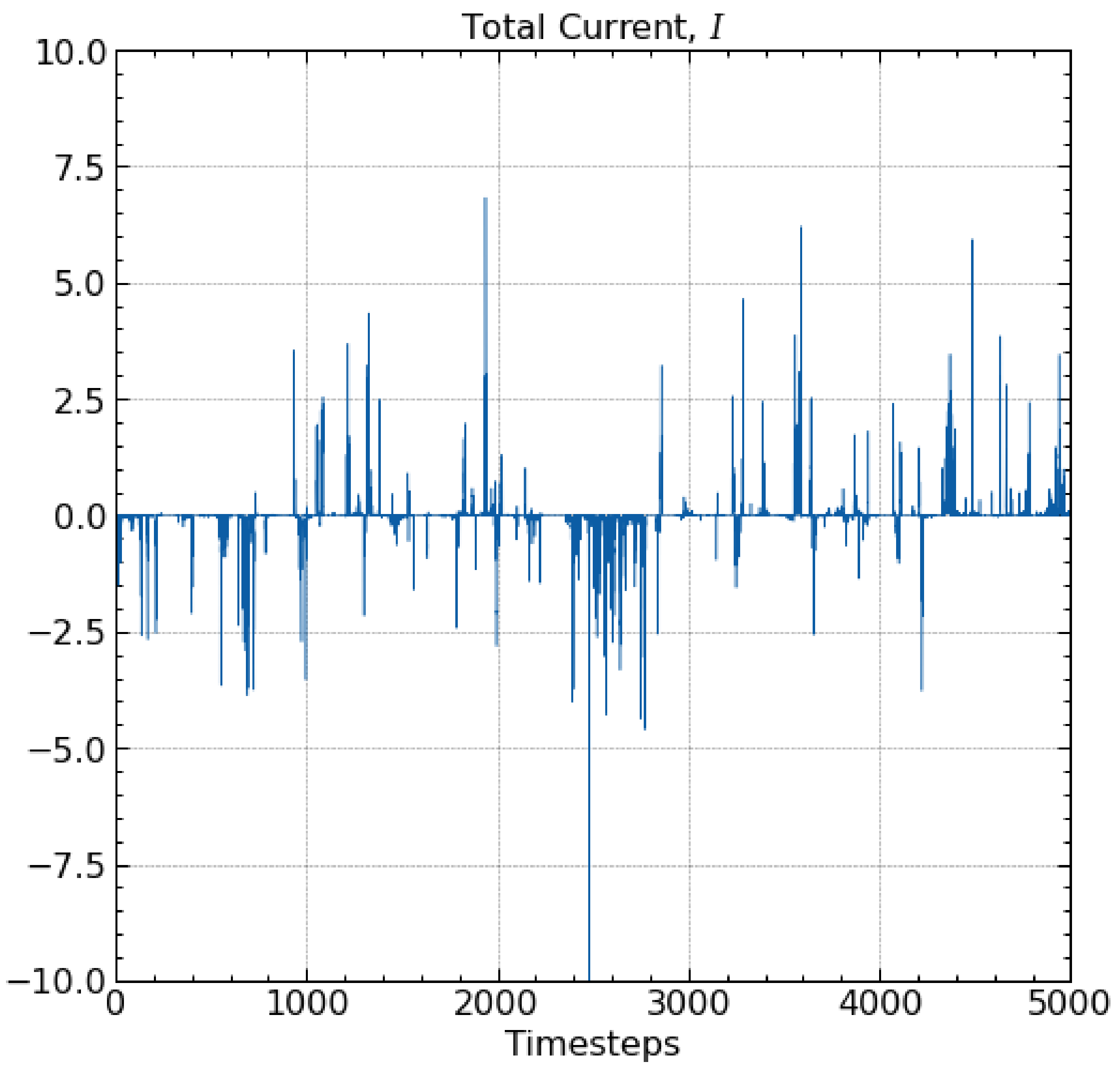
1.7 I v/s t

[33]:
```
instance = 13

plt.figure(figsize = (8,8))
plt.xlabel("Timesteps")
plt.xlim(0,T_H)
```

```
plt.plot(times, I_values[:, instance], linewidth = 0.5)
plt.title("Total Current, $I$")
plt.ylim(-10,10)
```

[33]: (-10.0, 10.0)

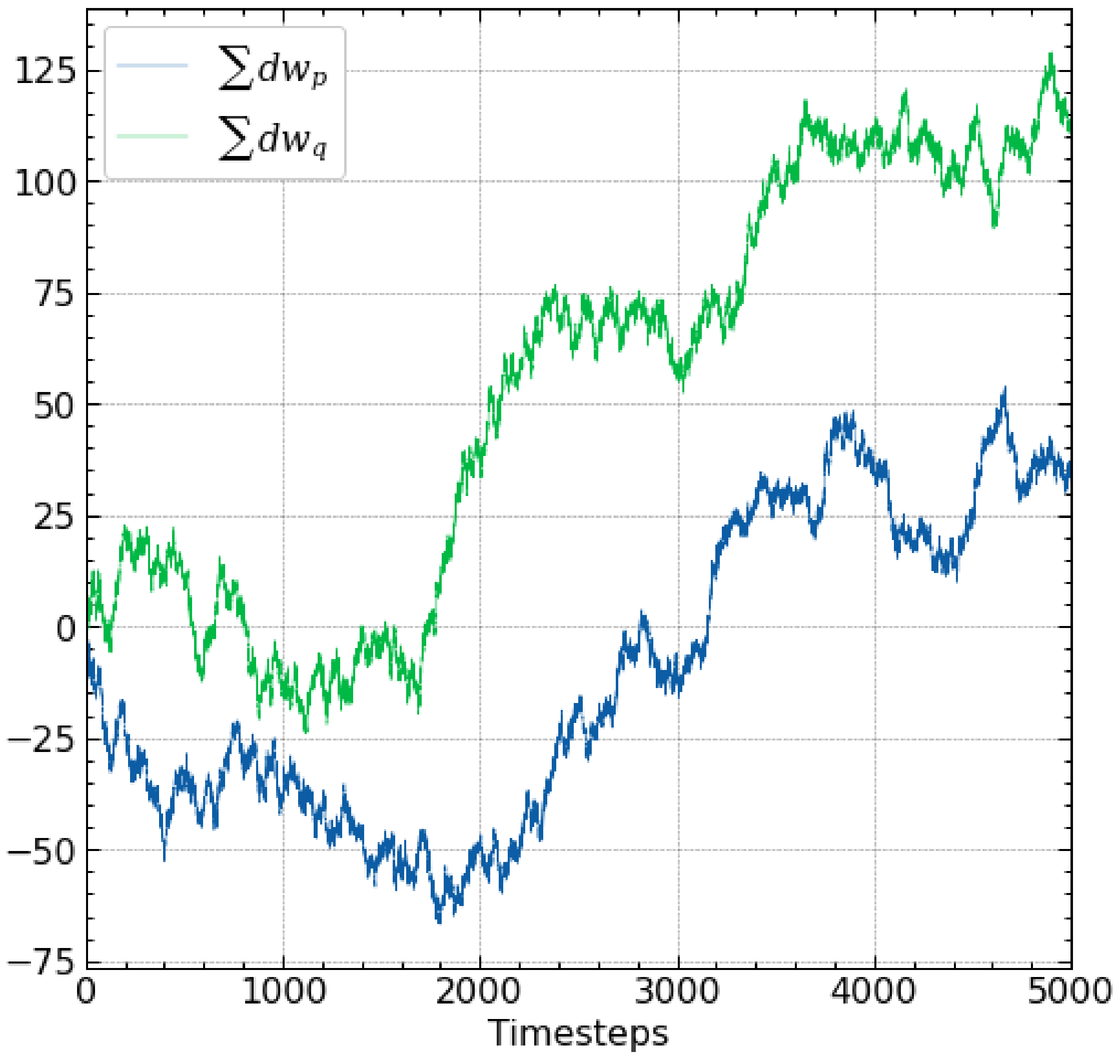


1.8 Brownian Noise

[34]:
```
instance1 = 0
plt.figure(figsize = (8,8))
plt.xlabel("Timesteps")
plt.xlim(0,T_H)
```

```
plt.plot(times, np.cumsum(dwp[:,instance1]) , label = "$\sum\, dw_p$",␣
↪linewidth = 0.4)
plt.plot(times, np.cumsum(dwq[:,instance1]), label = "$\sum \,dw_q$", linewidth␣
↪= 0.4)
plt.legend()
```

[34]: <matplotlib.legend.Legend at 0x189787e15c8>



1.9 Average Calculation

1.9.1 1.

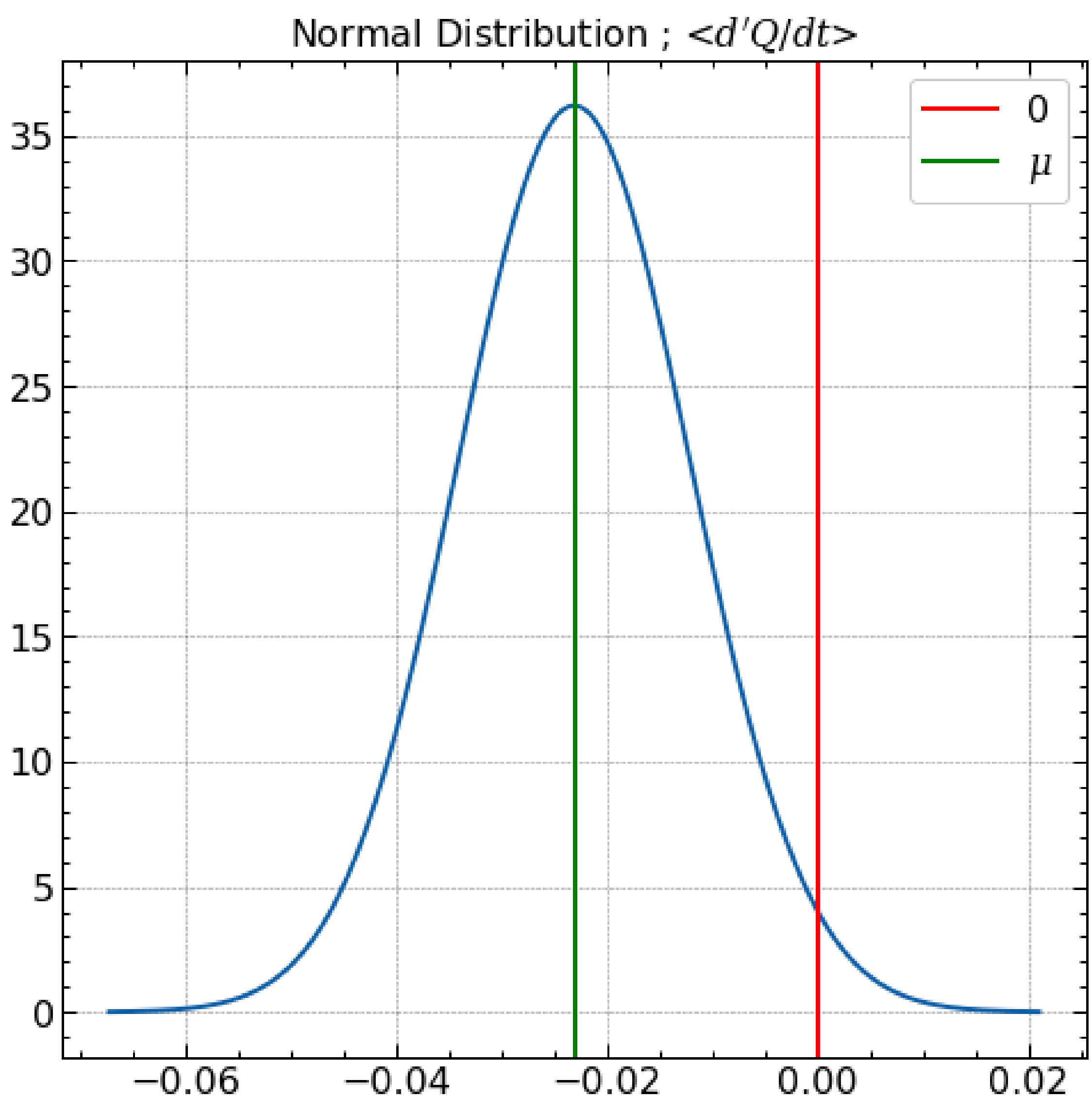
$$\left\langle \frac{d'Q}{dt} \right\rangle = \frac{\eta}{m}\left(T - \left\langle \frac{p^2}{m} \right\rangle\right)$$

```
[35]: dQdt = np.array([eta_m*(T - np.average(p_values[:,i]**2/m)) for i in range(N)])
mean = np.mean(dQdt)
std = np.std(dQdt)
print("Mean = ", mean)
print("Std  = ", std)
```

```
Mean =  -0.023091763665147344
Std  =  0.011019708953354636
```

```
[36]: plt.figure(figsize = (8,8))
x = np.linspace(mean + 4*std, mean - 4*std, 200)
plt.plot(x, stats.norm.pdf(x, mean, std))
plt.axvline(0, color='red', label = "0")
plt.axvline(mean, color = 'green', label = "$\mu$")
plt.legend()
plt.title("Normal Distribution ; <$d'Q/dt$>")
```

```
[36]: Text(0.5, 1.0, "Normal Distribution ; <$d'Q/dt$>")
```



1.9.2 2.

$$\left\langle \frac{d'W}{dt} \right\rangle = \left\langle \frac{\partial}{\partial q}\left(\frac{T}{\mathcal{R}}\frac{\partial \mathcal{H}}{\partial q}\right)\right\rangle - \left\langle \frac{1}{\mathcal{R}}\left(\frac{\partial \mathcal{H}}{\partial q}\right)^2 \right\rangle$$

[37]:
```
dWdt = []

for i in trange(N, desc='Instances Completed'):
    T1 = []
    T2 = []
    precalc = 2*e*I_0
    for j in trange(n-1, desc=f'Completion of Instance {i+1}'):
        Cx = C(x_values[j,i],V)
        t3 = V + q_values[j,i]/Cx
        R_net = R_net_values[j,i]
        if Cx == 0 or t3==0 or R_net==0:
            continue
        t1 = precalc*np.cosh(u_values[j,i]/T_e)/(Cx*(1 + 2*I_0*R*np.
 ↪cosh(u_values[j,i]/T_e)/T_e))
        t2 = 1/R_net*(t3**2)
        T1.append(t1)
        T2.append(t2)
#          if t1 == np.inf or t1 == 0:
#              print(j)
    dWdt.append(np.mean(T1) - np.mean(T2))

mean2 = np.mean(dWdt)
std2 = np.std(dWdt)
print("Mean = ", mean2)
print("Std  = ", std2)
```

```
Instances Completed:   0%|          | 0/40 [00:00<?, ?it/s]
Completion of Instance 1:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 2:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 3:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 4:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 5:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 6:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 7:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 8:   0%|          | 0/999999 [00:00<?, ?it/s]
```

```
Completion of Instance 9:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 10:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 11:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 12:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 13:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 14:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 15:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 16:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 17:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 18:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 19:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 20:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 21:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 22:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 23:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 24:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 25:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 26:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 27:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 28:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 29:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 30:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 31:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 32:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 33:   0%|          | 0/999999 [00:00<?, ?it/s]
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Completion of Instance 37:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 38:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 39:   0%|          | 0/999999 [00:00<?, ?it/s]
Completion of Instance 40:   0%|          | 0/999999 [00:00<?, ?it/s]
```
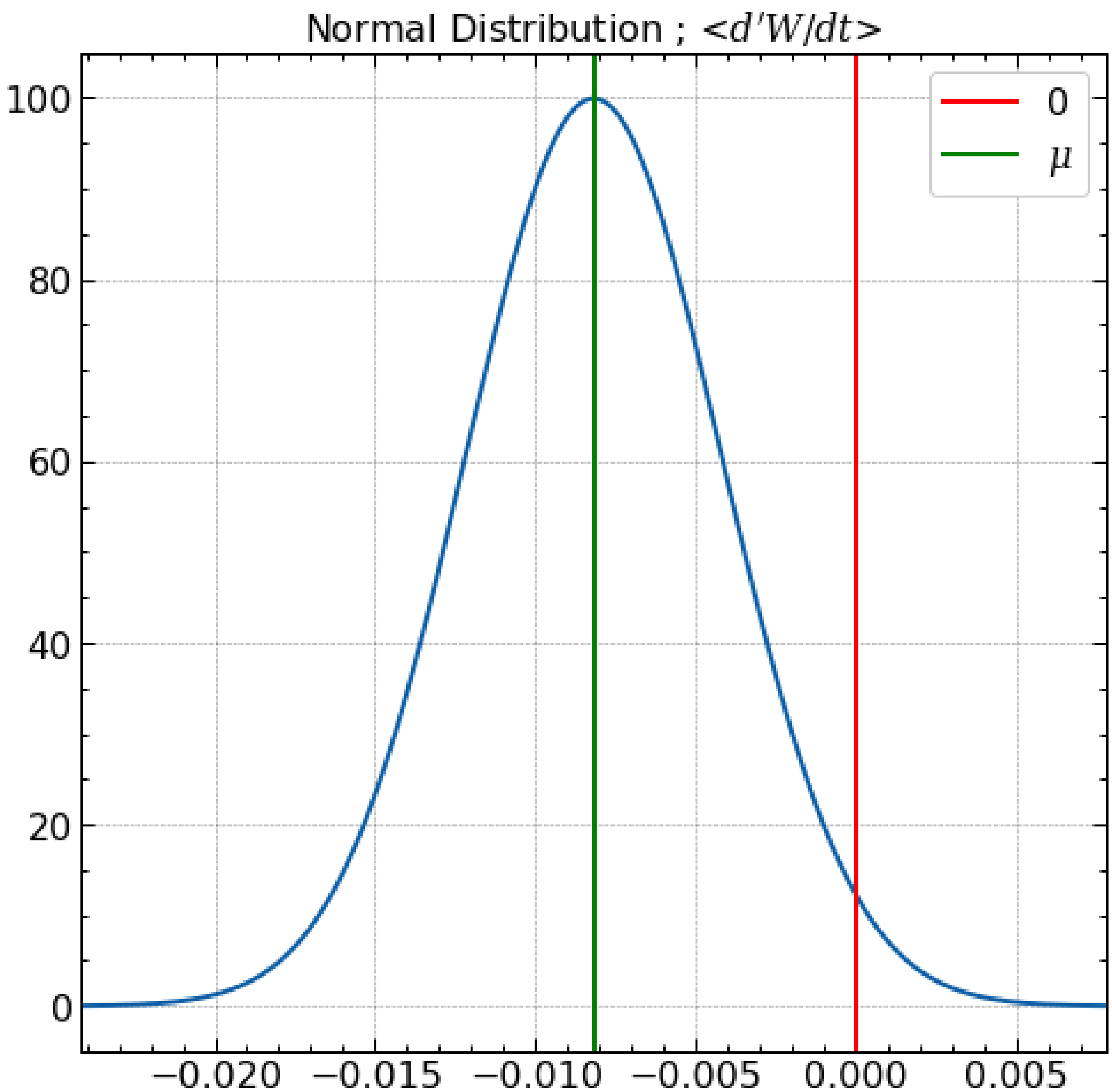

```
Mean =  -0.008184436196814558
Std  =  0.003992071322733361
```

[23]:
```
plt.figure(figsize = (8,8))
x = np.linspace(mean2 + 4*std2, mean2 - 4*std2, 200)
plt.plot(x, stats.norm.pdf(x, mean2, std2))
plt.xlim(mean2 - 4*std2, mean2 + 4*std2)
plt.axvline(0, color='red', label = "0")
plt.axvline(mean2, color = 'green', label = "$\mu$")
plt.legend()
plt.title("Normal Distribution ; <$d'W/dt$>")

# plt.xlabel("Timesteps")
# plt.ylabel("Power")
```

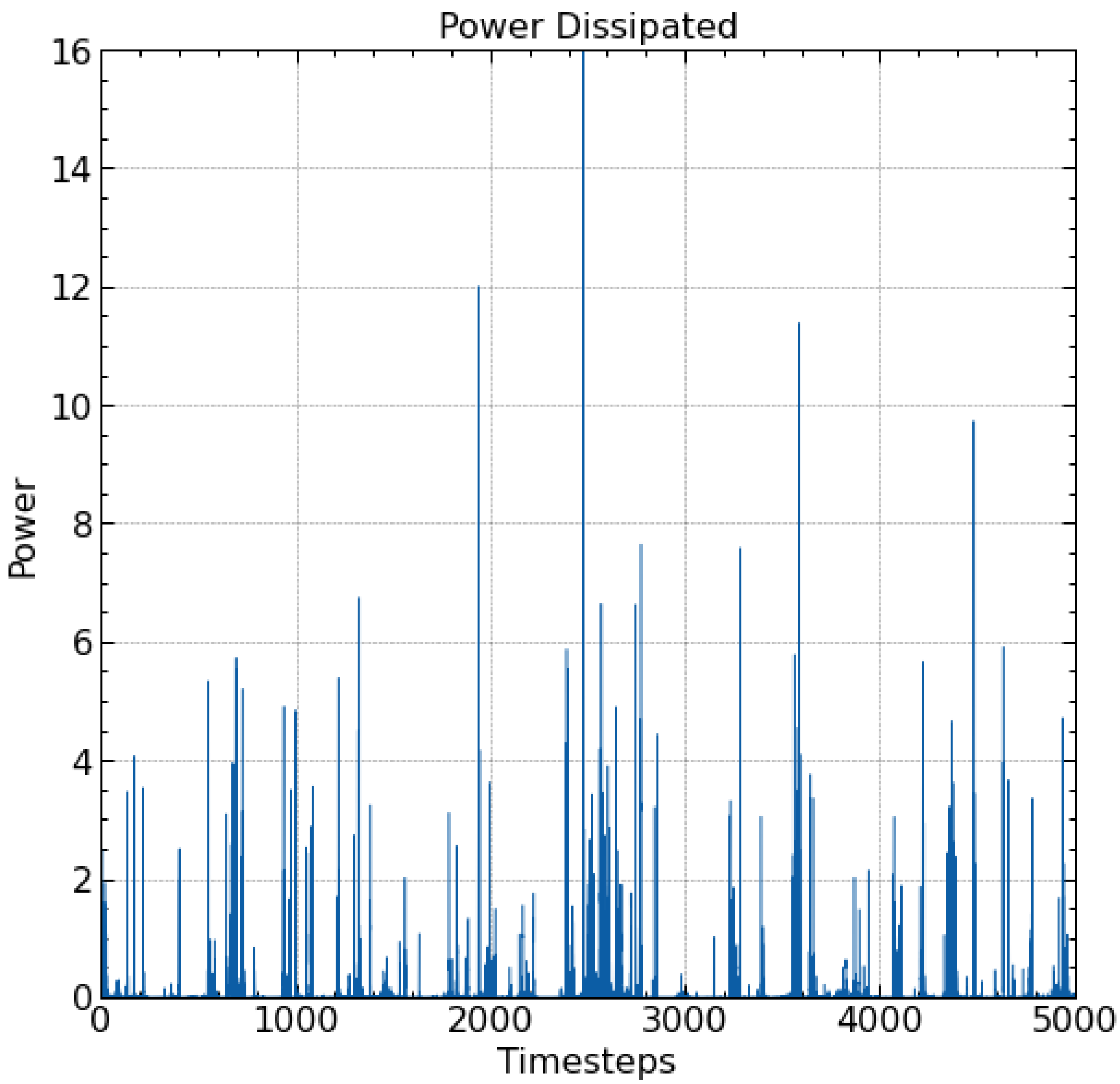
[23]:
```
Text(0.5, 1.0, "Normal Distribution ; <$d'W/dt$>")
```



1.10 Power Dissipated

[39]:
```
plt.figure(figsize = (8,8))
# instance = 0
Instantaneous_Power = I_values[:,instance]**2*R_net_values[:,instance]
plt.title("Power Dissipated")
# Instantaneous_Power2 = u_values[:,instance]**2/(R_net_values[:,instance] - R)
# plt.title("Dissipated in D2")

plt.plot(times, Instantaneous_Power, linewidth = 0.5)
# plt.plot(times, Instantaneous_Power2, linewidth = 0.2)

plt.ylim(0,16)
plt.xlim(0,5000)
plt.xlabel("Timesteps")
plt.ylabel("Power")
```

[39]:
```
Text(0, 0.5, 'Power')
```



[34]:
```
# np.save("20instances_newer.npy", np.stack([x_values, p_values, q_values,
↪I_values, R_net_values, u_values, dwp, dwq], axis = 2))
```