IE616 Decision Analysis & Game Theory

A Project Report On

Chaos and Game Theory analysis of Stackelberg Duopoly Model

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1 Stackelberg Duopoly Model[1]

The proposed novel duopoly Stackelberg model assumes that two rival firms produce a similar good in a market, with one serving as the leader and the other as the follower. Both firms produce a certain quantity of the good within a given time frame ($q_i(t)$) and have a planned production amount ($Q_i(t)$). As these are production quantities, it is easy to see that $q_i(t) > 0, Q_i(t) > 0, i = 1, 2, t = 0, 1, 2, ...$ In reality, there is often a difference between the actual production amount and the ideal announced production amount, which acts as a cost function for factories in this proposed model. As the actual production amount approaches the planned amount, the cost function value decreases. The inconsistency in production can be influenced by various factors, but for simplicity, they can be summarized by c_i, which is a positive shift parameter. Moreover, to better represent economic dynamics, a marginal cost (d_i) is incorporated into the cost function in this model. In this model the cost function is an quadratic function.

$$C_i(q_i) = c_i(q_i(t) - Q_i(t))^2 + d_i q_i(t), i = 1, 2, t = 0, 1, 2, ... \tag{1}$$

When the value of d_i is set to zero, the standard model cost function is obtained. In a market, the price of a particular good is influenced by the maximum price and the total quantity of the good produced by all firms. The demand function ($f(q)$) is used to compute the total supply $q(t)$ and the corresponding price $p(t)$ of the good.

$$q(t) = q_1(t) + q_2(t), t = 0, 1, 2, ... \tag{2}$$

$$p(t) = f(q) = a - bq(t), t = 0, 1, 2, ... \tag{3}$$

here, a and b are positive constants.

In a competitive market, the primary goal of each firm is to maximize its profit. This profit is determined by a combination of the firm's income and its cost function. Therefore, the profit of each firm can be expressed as follows:

$$\pi_i(q_1, q_2) = q_i p - C_i(q_i) = q_i(a - bq) - c_i(q_i - Q_i)^2 - d_i q_i, i = 1, 2 \tag{4}$$

To achieve the goal of maximizing profit, firms must determine the optimal level of production that results in the highest profit. This involves calculating the derivative of Equation (4) with respect to the quantity of production, q_i, and this should be set equal to 0, i.e., $\frac{\partial \pi_i(q_1,q_2)}{\partial q_i} = 0$, this leads to a set of equations.

$$\frac{\partial \pi_i(q_1, q_2)}{\partial q_i} = a - b(2q_i - q_j) + 2c_i(Q_i - q_i) - d_i = 0, i = 1, 2, j = 1, 2, i \neq j \tag{5}$$

This set of equations need to be solved simultaneously. On doing this we get the optimal production quantities for both firm, which are given by:

$$q_1 = \frac{a(b + 2c_2) + b(4c_1Q_1 - 2c_2Q_2 - 2d_1 + d_2) - 2c_2(d_1 - 2c_1Q_1)}{3b^2 + 4b(c_1 + c_2) + 4c_1c_2} \tag{6}$$

$$q_2 = \frac{a(b + 2c_1) + b(4c_2Q_2 - 2c_1Q_1 - 2d_2 + d_1) - 2c_1(d_2 - 2c_2Q_2)}{3b^2 + 4b(c_1 + c_2) + 4c_1c_2} \tag{7}$$

Aside from the model parameters, Equations (6) and (7) is also dependent on the planned production quantities of each firm. Once the optimal amounts of announced production are determined, the optimal amount of Equation (7) can also be obtained. In a duopoly Stackelberg game, the firms plan their production sequentially but produce their goods simultaneously in the market. This means that the leader firm announces its production plan first, and then the follower firm tries to determine its own production plan based on the leader's plan, in order to maximize its profit in the market. Therefore, the leader firm must be able to anticipate the follower's response to its announced production plan, and adjust its own plan accordingly to remain competitive.

Overall, the leader's production plan is dependent on forecasting the follower's strategy after observing the leader's actions. This process makes finding the optimal planned production quantities somewhat challenging, but it can be tackled using the backward induction method. In this technique, the solutions obtained in Equations (6) and (7) are substituted into Equation (4). To find the optimal production quantity for the follower firm, the derivative of the follower's profit with respect to its planned production must be set to zero, i.e., $\frac{\partial \pi_2(q_1,q_2)}{\partial Q_2} = 0$. This will result in an equation that provides the optimal value of Q2 in terms of Q1, which is expressed as follows:

$$Q_2 = \frac{4(b+c_1)(b+c_2)(a(b+2c_1)) + b(d_1 - 2(c_1Q_1 + d_2)) - 2c_1d_2}{b(8c_2(b+c_1)(b+2c_1) + b(3b+4c_1)^2)} \tag{8}$$

After obtaining the optimal value of the follower firm's production quantity (Q2), it can be substituted into the profit expression of the leader firm, i.e., $\frac{\partial \pi_1(q_1,q_2)}{\partial Q_1} = 0$. This will ultimately result in an equation that provides the optimal value of Q1 in terms of the model parameters, which is expressed below

$$Q_1 = \frac{4(b+c_1)(\alpha - bc_2)(a(\alpha - 2bc_1)) + \alpha d_2 - 2d_1(\alpha - bc_2)}{b\beta} \tag{9}$$

where,

$$\alpha = 3b^2 + 4b(c_1 + c_2) + 4c_1c_2 \tag{10}$$

$$\beta = (9b + 8c_1)\alpha^2 + 8bc_2(b+c_1)(2bc_2 - 3\alpha) \tag{11}$$

Using the optimal value of Q1 obtained from Equation (9), it can be substituted into Equation (8). This will result in a final equation that provides the optimal value of Q2 solely in terms of the model parameters, which is expressed as follows

$$Q_2 = \frac{(\alpha + b^2)(\alpha(3b(a + d_1 - 2d_2) + 2c_1(a + 2d_1 - 3d_2)) - 4bc_2(b+c_1)(a + d_1 - 2d_2))}{b\beta} \tag{12}$$

By substituting Equations (9) and (12) into the expressions of Equation (7), the optimal production quantities for both firms can be determined, which is expressed as follows:

$$q_1^* = \frac{(\alpha(3b + 4c_1) - 4bc_2(b+c_1))(\alpha(a + d_2 - 2d_1) - 2bc_2(a - d_1))}{b\beta} \tag{13}$$

$$q_2^* = \frac{\alpha(\alpha(3ab + 2a + d_1(3b + 4c_1) - 6d_2(b+c_1)) - 4bc_2(b+c_1)(a + d_1 - 2d_2))}{b\beta} \tag{14}$$

The optimal production quantities for the two firms, denoted as (q_1^*, q_2^*), represent the Nash equilibrium of the duopoly Stackelberg game.

After deriving all of the necessary equations and expressions, the final form of the duopoly Stackelberg game with marginal costs and heterogeneous players can be expressed as follows:

$$q_1(t+1) = q_1(t)(1 + v_1(a + 2c_1Q_1 - d_1 - 2(b+c_1)q_1(t) - bq_2(t))) \tag{15}$$

$$q_2(t+1) = q_2(t) + \frac{v_2}{2(b+c_2)}(a + 2c_2Q_2 - d_2 - 2(b+c_2)q_2(t) - bq_1(t)) \tag{16}$$

In this duopoly Stackelberg game with marginal costs and heterogeneous players, the model is represented as a two-dimensional nonlinear map, where the production level of each firm in the next step is dependent on its current production level. The leader firm is assumed to be boundedly rational, meaning that it only has partial knowledge of market interactions. On the other hand, the follower firm has adaptable expectations. Therefore, the updating rule for each firm is different from its rival. Both firms update their production level based on their current production level and a portion of their profit variation, which is derived from Equation (5). The production levels are adjusted at a constant rate, represented by v_i in Equations (15) and (16). If vi is zero, the firm does not pay attention to market trends that could potentially lead to a loss.

1.1 Equilibrium Points and stability analysis

The identification of equilibrium points in the proposed economic model of Equation (15) and (16), is crucial for market analysis. If the Nash equilibrium of the economic model is unstable, the market may experience undesired and uncontrolled variations that can negatively impact both firms and customers. The proposed model is a map, and its equilibrium points can be determined by setting $q_i(t+1) = q_i(t)$, with $i = 1, 2$. In other words, when a firm's production remains unchanged over time, it is in a settled equilibrium point, which can either be stable or unstable. By applying this method to the model presented in equation (15) and (16), two distinct solutions can be obtained. The first solution is the production amount that satisfies $q_i(t+1) = q_i(t)$, with $i = 1, 2$, and the corresponding equilibrium point is called $E_0 = \left(0, \frac{a+2c_2Q_2-d_2}{2(b+c_2)}\right)$. The second equilibrium state, $E_1 = (q_1^*, q_2^*)$, is obtained by solving this set of equations

$$a + 2c_1Q_1 - d_1 - 2(b + c_1)q_1(t) - bq_2(t) = 0 \tag{17}$$
$$a + 2c_2Q_2 - d_2 - 2(b + c_2)q_2(t) - bq_1(t) = 0 \tag{18}$$

The set of equations (17) and (18) are similar to equation (5), and thus, the resulting equilibrium point is the previously mentioned Nash equilibrium, E1. The values of the entities of the equilibrium point E1 are the same as those in equation (13) and (14). Therefore, the proposed model has two distinct equilibrium points. To determine their stability, the Jacobian matrix of equation equation (15) and (16) needs to be calculated, which is given by:

$$\mathbf{J}(q_1, q_2) = \begin{bmatrix} 1 + v_1(a + 2c_1Q_1 - d_1 - 4(b + c_1)q_1 - bq_2) & -bv_1q_1 \\ \frac{-bv_2}{2(b+c_2)} & 1 - v_2 \end{bmatrix} \tag{19}$$

The next step is to obtain the characteristic equation of the Jacobian matrix. The characteristic equation is given by equation (20), where $Tr(\mathbf{J})$ and $Det(\mathbf{J})$ represent the trace and determinant of the Jacobian matrix, respectively. The eigenvalues of the equilibrium point are represented by λ. Using the Jacobian matrix derived earlier, the characteristic equation can be written as:

$$\lambda^2 - \mathrm{Tr}(\mathbf{J})\lambda + \mathrm{Det}(\mathbf{J}) = 0 \tag{20}$$

The characteristic equation is obtained by replacing the state variables q_1 and q_2 with equilibrium points in the Jacobian matrix and solving for the eigenvalues. If the absolute value of all eigenvalues is less than 1, indicating they lie inside the unit circle, then the equilibrium point is stable. On the other hand, if at least one eigenvalue is outside the unit circle, the equilibrium point is unstable. If there is an eigenvalue precisely on the unit circle, it requires in-depth experiments to determine the stability of the equilibrium point. The stability analysis involves replacing (q_1, q_2) in the Jacobian matrix with E_0, and then solving for the eigenvalues for E_0.

$$\lambda_1 = 1 - v_2 \tag{21}$$
$$\lambda_2 = 1 + v_1[a - d_1 + 2Q_1c_1 - \frac{b(a - d_2 + 2Q_2c_2)}{2(b + c_2)} \tag{22}$$

As $v_i > 0$, λ_1 is always less than 1. Also, as $q_i(t) > 0$, $i = 1, 2$, $lambda_2$ is always greater than 1. Due to which, E_0 is a saddle node and is always unstable. On Substitiuing E_1 into equation (19) results in the following Jacobian Matrix.

$$\mathbf{J}(E_1) = \begin{bmatrix} 1 - 2v_1(b + c_1)q_1^* & -bv_1q_1^* \\ \frac{-bv_2}{2(b+c_2)} & 1 - v_2 \end{bmatrix} \tag{23}$$

The trace and determinant of this matrix are:

$$Tr(\mathbf{J}) = 2 - v_2 - 2v_1(b + c_1)q_1^*, \tag{24}$$
$$Det(\mathbf{J}) = 1 - v_2 - v_1(2(1 - v_2)(b + c_1) + \frac{b^2v_2}{2(b + c_2)})q_1^* \tag{25}$$

If $Tr(\mathbf{J})^2 - 4Det(\mathbf{J}) \geq 0$ for a equilibrium point, then the stability conditions for that equilibrium points are determined using the Jury Conditions. Using equartion (24) and (25), $Tr(\mathbf{J})^2 - 4Det(\mathbf{J})$ is:

$$Tr(\mathbf{J})^2 - 4Det(\mathbf{J}) = 4(v_2 - 1)(b(1 - 2v_1(c_1 + c_2)) + c_2(1 - 2c_1v_1) - b^2v_1) + (2b^2v_1v_2 + 4(b + c_2)(v_2 - 1))q_1^* \geq 0 \tag{26}$$

On simplifying equation (26), $Tr(\mathbf{J})^2 - 4Det(\mathbf{J})$ leads to a non-negative expression, therefore the Jury Conditions can used to determine the stability of the Nash equilibrium. Jury Conditions can specify the stability region of the equilibrium points using the model parameters as,

$$\begin{gathered} \mathrm{Det}(\mathbf{J}) < 1 \\ 1 - \mathrm{Tr}(\mathbf{J}) + \mathrm{Det}(\mathbf{J}) > 0 \\ 1 + \mathrm{Tr}(\mathbf{J}) + \mathrm{Det}(\mathbf{J}) > 0 \end{gathered} \tag{27}$$

If all of the conditions shown in equation (27) are satisfied, then the equilibrium point is known to be stable, as these three conditions bound the parameter in such a way that the Nash equilibrium becomes stable. The inequalities of (27) result in:

$$2v_2(b + c_2) + v_1(\alpha(1 - v_2) + b^2)q_1^* > 0 \tag{28}$$
$$v_1(\alpha + b^2) - (b^2v_1 + 2(b + c_2))q_1^* > 0 \tag{29}$$
$$(2 - v_2)(4(b + c_2) - v_1(\alpha + b^2)) + (2(b + c_2)(2 - v_2) - b^2v_1v_2)q_1^* > 0 \tag{30}$$

2 Analysis of Chaotic Behaviour

2.1 Chaotic Behaviour

The dynamics of fundamental financial markets and businesses are characterized by non-stationary and non-predictable behaviour, which can be attributed to the complex interactions among the market actors such as consumers, suppliers, and producers. The lack of stability in these markets can be a result of various factors, such as technological advancements, government policies, and market psychology. Thus, an economic model that incorporates a stable Nash equilibrium and can account for unpredictable circumstances is crucial for understanding and navigating these markets.

One way to model such unpredictable behaviour in markets is by examining the bounded variations that resemble chaotic behaviour. These variations are not completely random, but they exhibit a degree of predictability within a certain range of values. This chaotic behaviour is an essential feature of many real-world financial systems, and it requires sophisticated mathematical tools to model and analyze. Therefore, researchers and practitioners in the field of finance and economics are constantly seeking to develop new models and methods that can capture the dynamics of these complex and unpredictable systems.

2.2 Analysis

2.2.1 Sensitivity to initial conditions

In the Figure below, the x-axis of each panel corresponds to the time step or iteration number, while the y-axis represents the production amount of the firms. The simulation is conducted over a period of 1000 iterations, and the final 100 iterations are selected for plotting after discarding the initial transient phase. The proposed model is capable of exhibiting chaotic behaviour, as evidenced by the time series pattern. The model parameters are assigned specific values, namely a=6, b=0.5, c1=2, c2=1, d1=0.5, d2=1, v1=0.1, and v2=1. To explore the model's sensitivity to initial conditions, four panels are created, each with different initial conditions. Panel (a) has q1(0) = q2(0) = 1, panel (b) has q1(0) = 1.0001, q2(0) = 1, panel (c) has q1(0) = 1, q2(0) = 1.0001, and panel (d) has q1(0) = 1.0001, q2(0) = 1.0001.

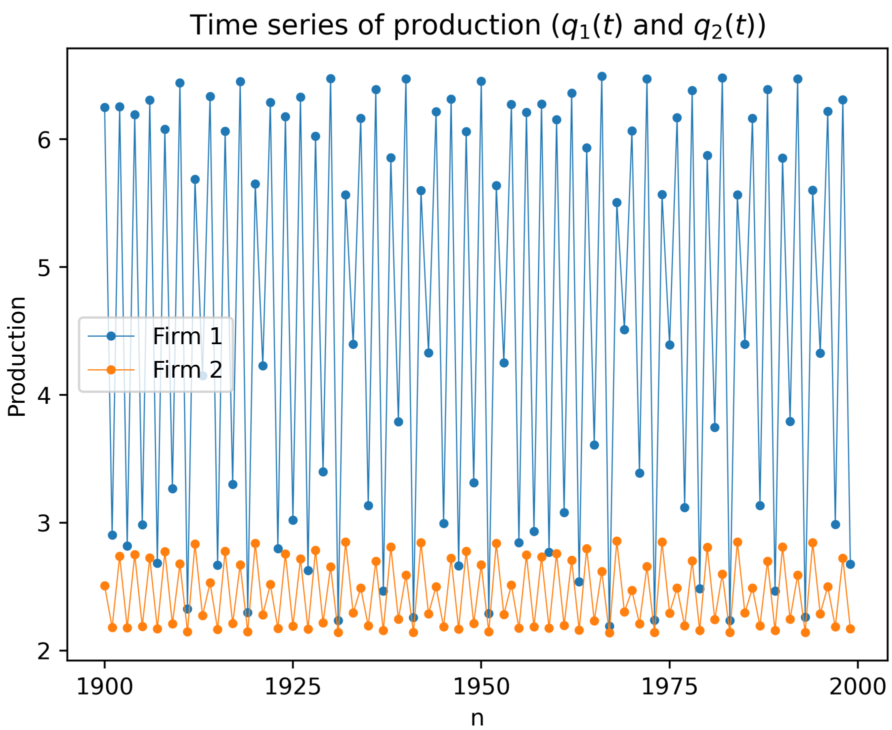

Figure 1: Production Timeseries

The results plotted in the figure reveal that a slight change in either or both of the initial conditions by an order of 10^{-4} can lead to a completely different model time series. This sensitivity to initial conditions is a hallmark of chaotic systems and underscores the need for accurate and precise measurements in the initial stages of modelling.

2.2.2 Chaotic Attractors

When the attractors of the proposed model are plotted on the q1 - q2 plane, a distinctive pattern emerges that sheds light on the behaviour of the model. The model can generate periodic and chaotic attractors, as demonstrated in the figure below, for various sets of parameters. A constant set of parameters, including c1=2, c2=1, d1=0.5, and v1=0.1, is used for all simulations, while the initial conditions are set as q1(0) = q2(0) = 1.

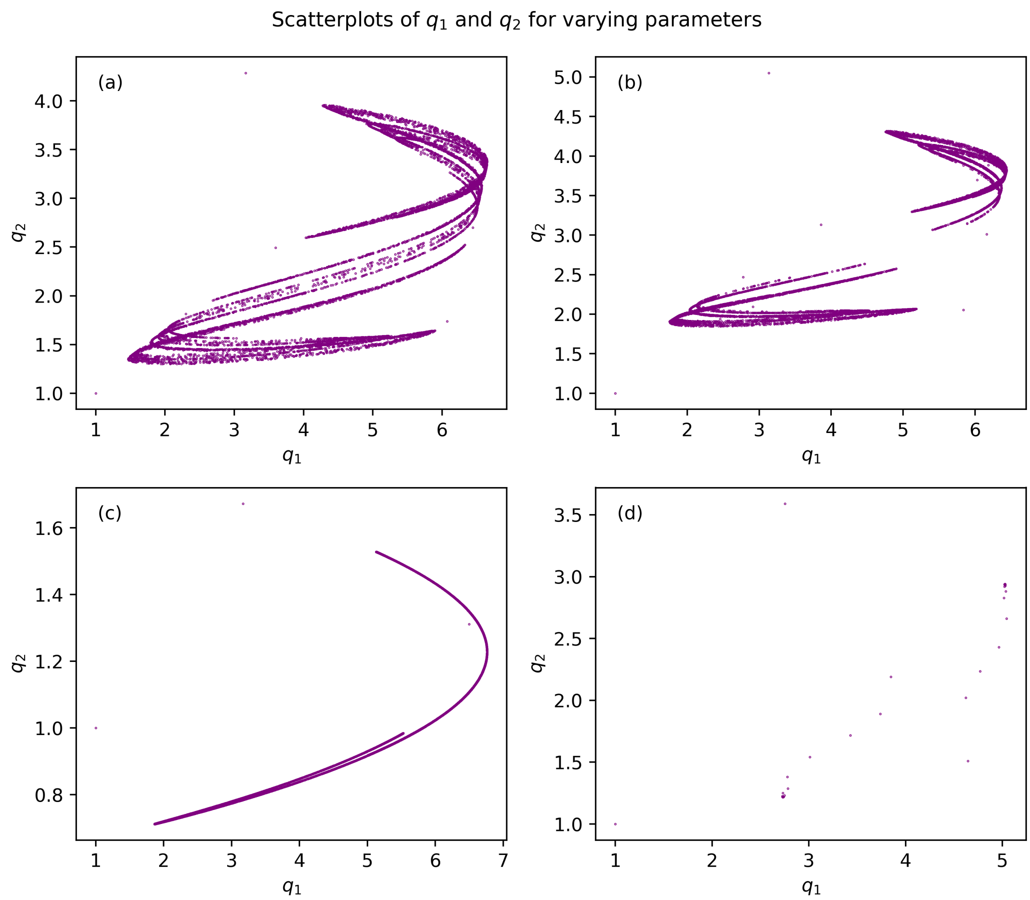

Figure 2: Chaotic Attractors and Cycles

Panel (a) represents a strange attractor with a = 6, b = 0.5, d2 = 1, and v2 = 1.6. Panel (b) demonstrates another strange attractor with a = 6, b = 0.5, d2 = 0.5, and v2 = 1.6. Also, panel (c) depicts a new chaotic attractor with a = 6, b = 0.5, d2 = 1, and v2 = 1. Finally, a period-8 attractor is shown in panel (d) with a = 6, b = 0.6, d2 = 1, and v2 = 1.6.

2.2.3 Variation of a single parameter

The model has eight different parameters; some are related to the pricing, and some are associated with the leader and follower firms. Up to now, all parameters have been kept unchanged in the investigations. Now we study the role of those parameters in the model's dynamics. The parameters are considered as the bifurcation parameter one by one.

Bifurcation diagram for the dynamics concerning parameter a varying in the [0, 6.4] interval with b=0.5, c1 =2, c2 =1, d1 =0.5, d2 =1, v1 =0.1, v2 =1 and initial conditions q1(0) = q2(0) = 1.

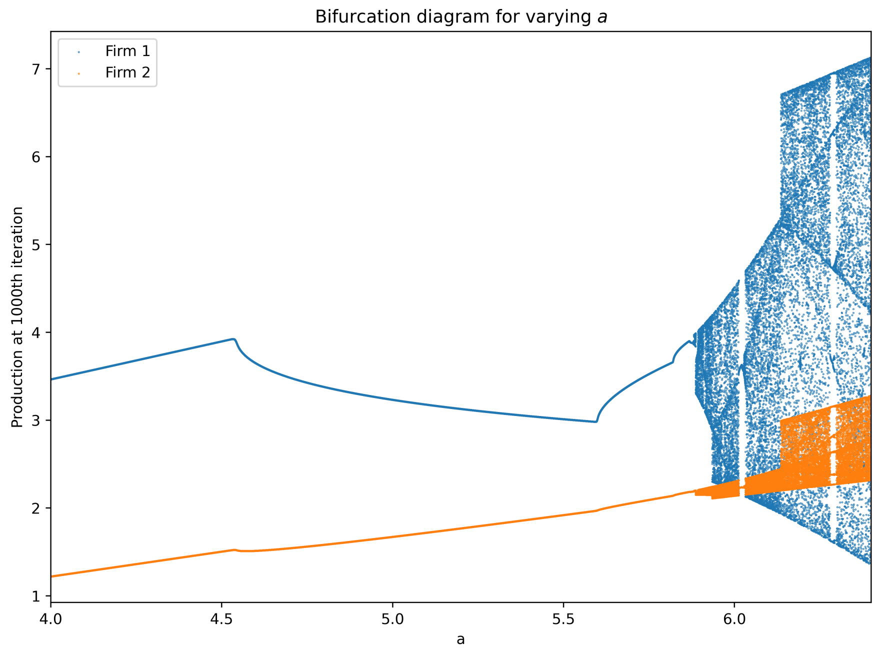

Figure 3: varying a

Bifurcation diagram for the dynamics concerning parameter b varying in the [0.5, 4.6] interval with a=6, c1 =2, c2 =1, d1 =0.5, d2 =1, v1 =0.1, v2 =1 and initial conditions q1(0) = q2(0) = 1.

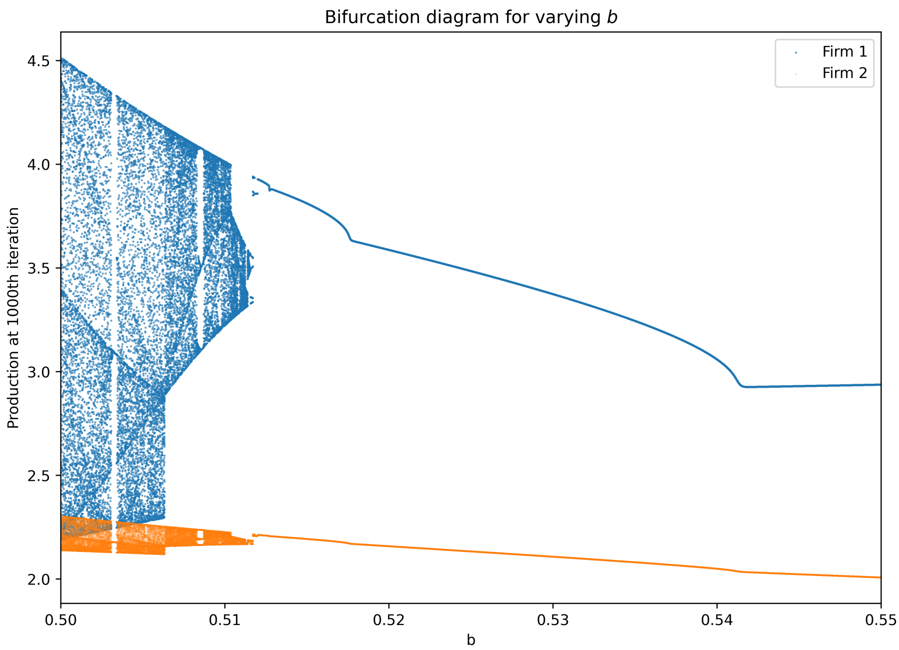

Figure 4: varying b

By appropriately selecting the value of parameter b in the demand function, the unpredictable behaviour of the market is avoidable, and the market experiences stability.

Bifurcation diagram for the dynamics concerning parameter c1 varying with a=6, b=0.5, c2 =1, d1 =0.5, d2 =1, v1 =0.1, v2 =1 and initial conditions q1(0) = q2(0) = 1.

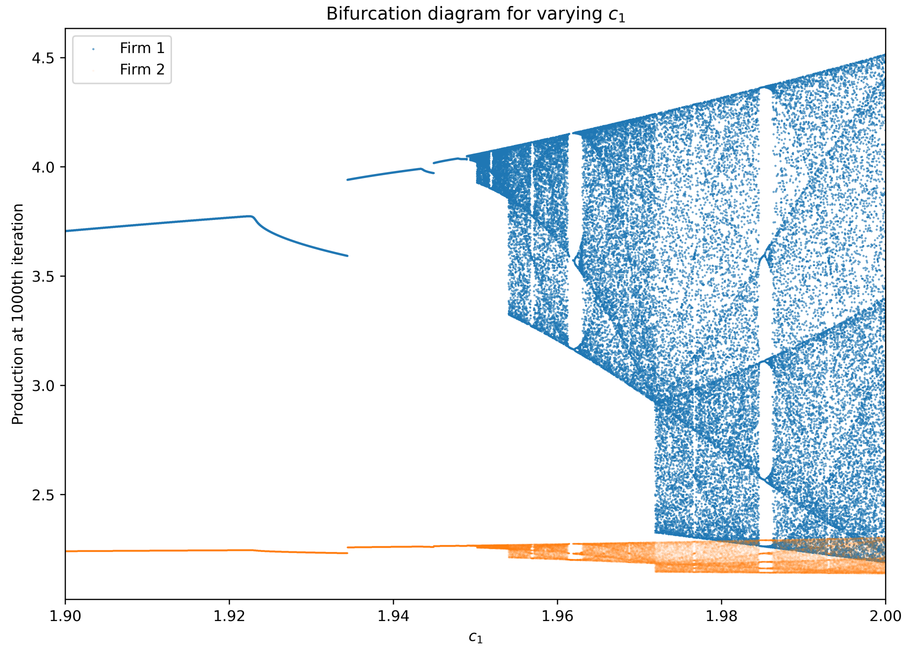

Figure 5: varying c_1

By increasing the value of c1, the importance of the difference between the announced production amount and the current production amount becomes crucial.

Bifurcation diagram for the dynamics concerning parameter c2 varying in the [0,10] interval with a=6, b=0.5, c1 =2, d1 =0.5, d2 =1, v1 =0.1, v2 =1 and initial conditions q1(0) = q2(0) = 1.

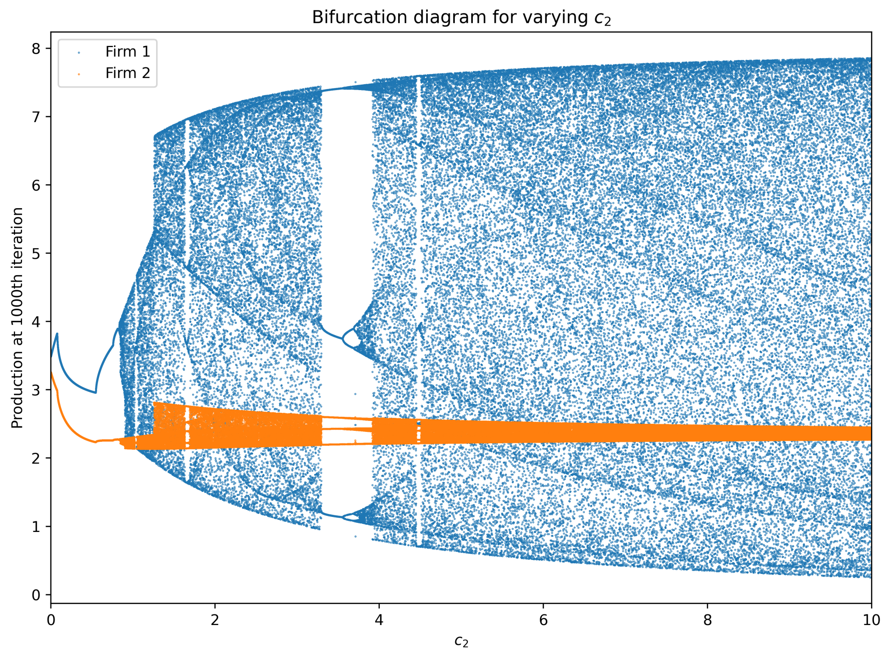

Figure 6: varying c_2

Nevertheless, there is a notable difference between the leader's behaviour and the follower firm; the leader firm's oscillation ranges are roughly constant, but the follower firm is falling.

Bifurcation diagram for the dynamics concerning parameter d1 varying in the [0.3, 3.4] interval with a=6, b=0.5, c1 =2, c2 =1, d2 =1, v1 =0.1, v2 =1 and initial conditions q1(0) = q2(0) = 1.

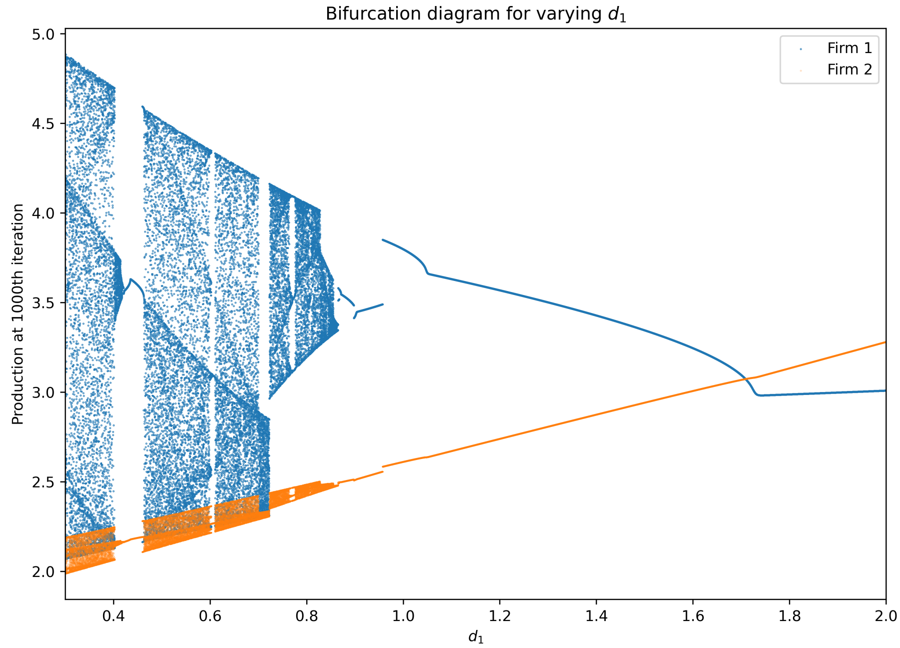

Figure 7: varying d_1

Increasing this marginal cost results in a period halving route to chaos.

Bifurcation diagram for the dynamics concerning parameter d2 varying in the [0, 1.4] interval with a=6, b=0.5, c1 =2, c2 =1, d1 =0.5, v1 =0.1, v2 =1 and initial conditions q1(0) = q2(0) = 1.

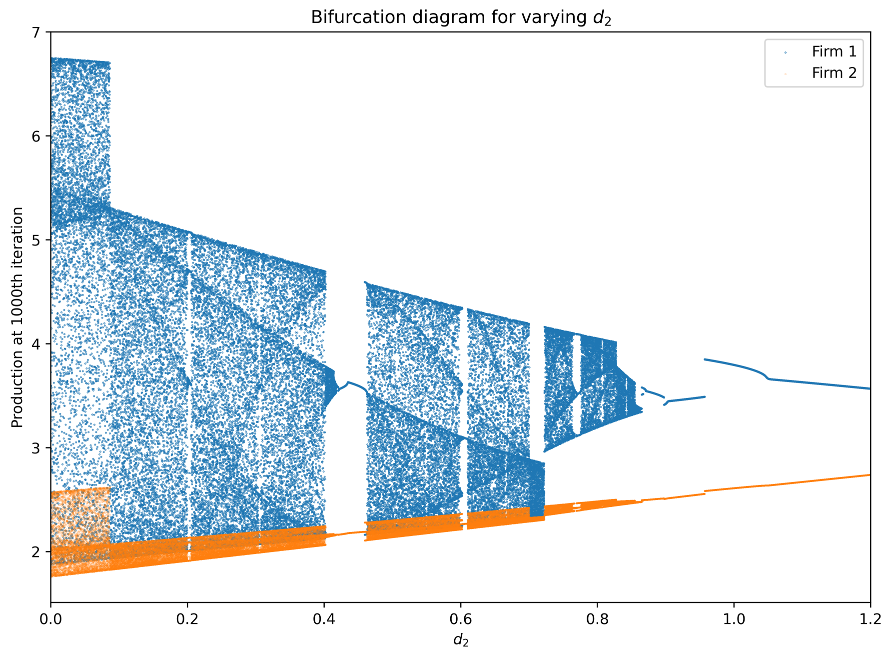

Figure 8: varying d_2

For the follower firm, smaller values of the marginal cost play a better role in converging to the Nash equilibrium.

Bifurcation diagram for the dynamics concerning parameter v1 varying interval with a=6, b=0.5, c1 =2, c2 =1, d1 =0.5, d2 =1, v2 =1 and initial conditions q1(0) = q2(0) = 1.

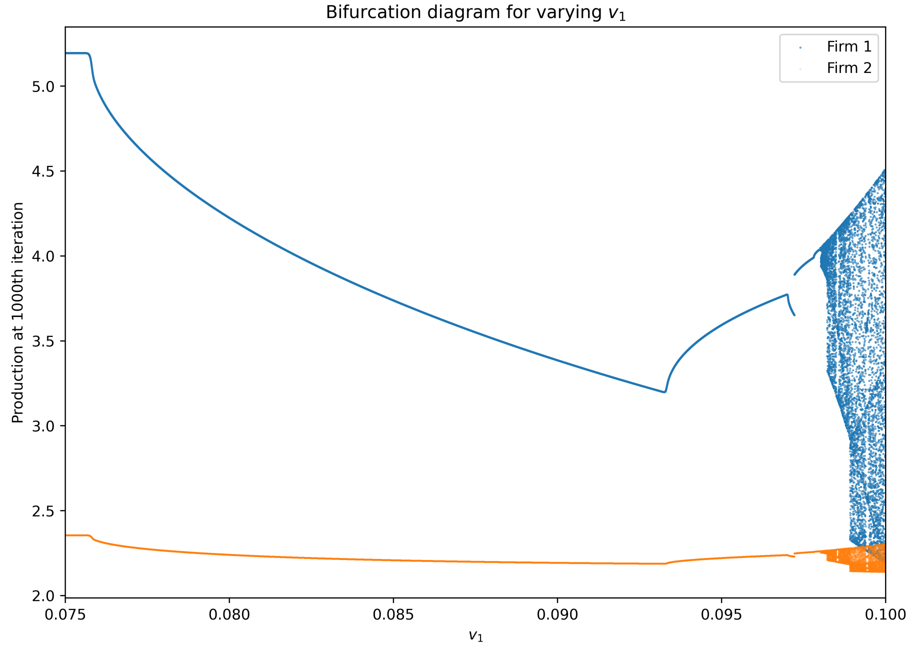

Figure 9: varying v_1

Bifurcation diagram for the dynamics concerning parameter v2 varying interval with a=6, b=0.5, c1 =2, c2 =1, d1 =0.5, d2 =1, v1 =0.1 and initial conditions q1(0) = q2(0) = 1.

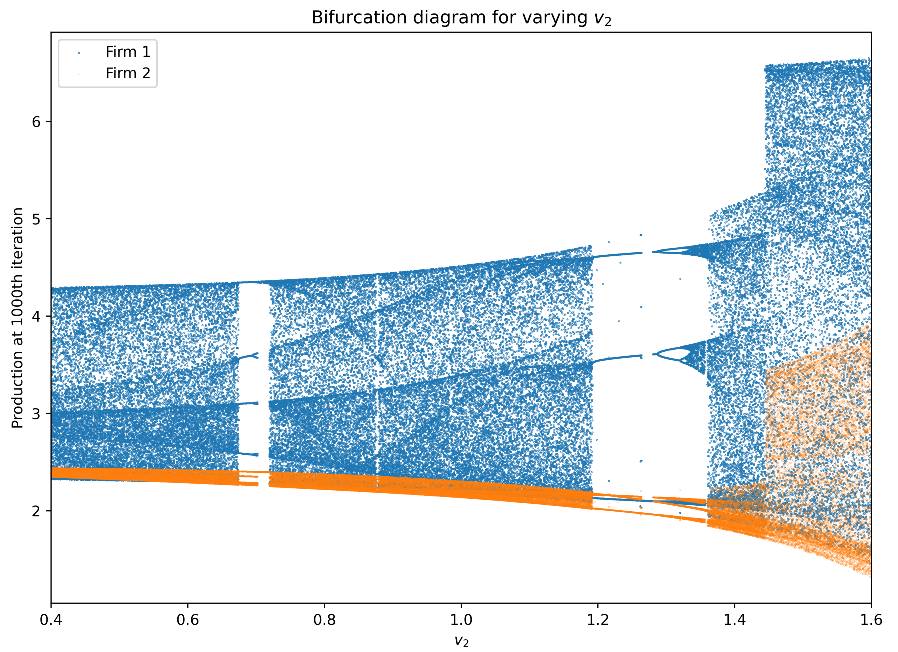

Figure 10: varying v_2

3 Conclusion

We have derived the equilibrium points for a novel Stackelberg Duopoly Model and also analyzed the parameter dependence and its chaotic behaviour as described in the reference paper [1]. We have noticed how chaotic behaviour causes a shift in the Nash Equilibrium and how a model-maker can navigate these chaotic waters to create a more realistic model to articulate market behaviour. Further avenues of research might include multi-agent modelling of market entities, inclusion of more real-world parameters and deep-diving into Chaos Theory for deeper insights.

4 Plotting and Simulation Code

For brevity, rather than including the code in this document, the Jupyter Notebook created to simulate the scenario and create plots is hosted on the following GitHub repository :
https://github.com/vinitdoke/IE616_Stackelberg_Duopoly_Model

References

[1] Atefeh Ahmadi, Sourav Roy, Mahtab Mehrabbeik, Dibakar Ghosh, Sajad Jafari, and Matjaž Perc. The dynamics of a duopoly stackelberg game with marginal costs among heterogeneous players. *PLOS ONE*, 18(4):e0283757, April 2023.