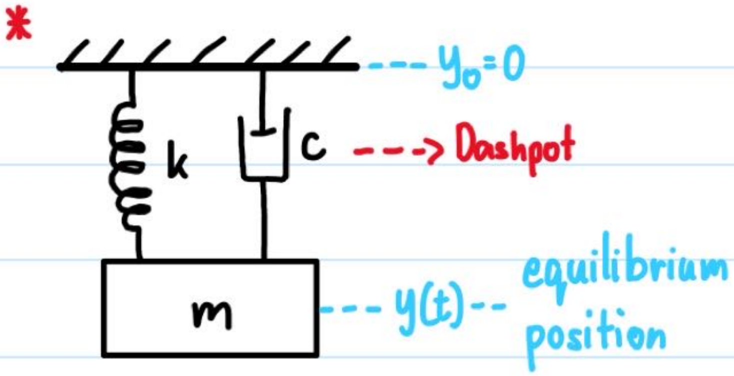
VIBRATION ENGINEERING

LONG EXAM 2 NOTES

One Degree of Freedom System with Viscous Damping

Tuesday, January 30, 2024 4:43 PM



* where:

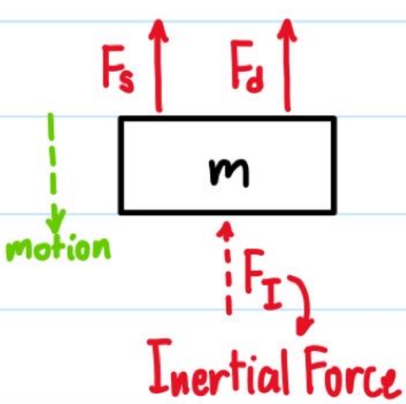
k = spring constant

c = damping coefficient

m = mass

~ occurs only for a system with a dashpot.

~ use Newton's 2nd Law of Motion:



* F_s = force due to spring

$$= k(y - \cancel{y_0}^{0}) = ky$$

F_d = force due to damping

$$= c(\dot{y} - \cancel{\dot{y}_0}^{0}) = c\dot{y}$$

$$F_I = ma = m\ddot{y}$$

$$\sum F_y = 0$$

$$F_I + F_d + F_s = 0$$

$$m\ddot{y} + c\dot{y} + ky = 0$$ ① Differential Eq. of Motion

~ evaluating eq. ①:

$$\left[m\ddot{y} + c\dot{y} + ky = 0\right]\frac{1}{m}$$

$$\boxed{\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = 0} \quad ②$$

* where:

Natural Frequency

$$\omega_n^2 = \frac{k}{m} \quad ③$$

$$\zeta = \frac{c}{2m\omega_n} \quad ④$$

"zeta" = viscous damping factor

~ Combining eqns. ②, ③, & ④:

$$\boxed{\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0}$$

Differential Eq. of Motion w/ Zeta

~ the solution yields:

$$y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad ⑤$$

~ where A is a constant and s is a quantity to be determined.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

~ use quadratic formula:

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4(1)\omega_n^2}}{2(1)}$$

$$= \frac{-\cancel{2}\zeta\omega_n \pm \cancel{2}\omega_n\sqrt{\zeta^2 - 1}}{\cancel{2}}$$

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \quad ⑥$$

~ combining ⑤ & ⑥: (eliminating s_1 & s_2)

$$y(t) = A_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + A_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

$$\boxed{y(t) = e^{-\zeta\omega_n t}\left(A_1 e^{\sqrt{\zeta^2 - 1}\,\omega_n t} + A_2 e^{-\sqrt{\zeta^2 - 1}\,\omega_n t}\right)}$$

* $\omega_d = \omega_n\sqrt{\zeta^2 - 1}$

* **NOTE:**

~ if $\zeta < 1$, then: under damping case

~ if $\zeta = 1$, then: critical damping case

~ if $\zeta > 1$, then: overdetermined case

[OR]

~ use L.E.M.: (let $q_i = y$)

forcing function

$$\frac{\partial}{\partial t}\left[\frac{\partial T}{\partial \dot{y}}\right] - \frac{\partial T}{\partial y} + \frac{\partial V}{\partial y} = Q_i \quad ⓐ$$

* not sure here

$$* \; Q_i = F_I - \frac{\partial \mathcal{F}}{\partial \dot{y}} \; ; \; \mathcal{F} = \frac{1}{2} \sum C_{rs} \delta \dot{y}^2 \quad ⓒ$$

ⓑ

* where:

$$T = \frac{1}{2} m \dot{y}^2$$

$$V = \frac{1}{2} k (y - \cancel{y_0})^2 = \frac{1}{2} k y^2$$

$$\mathcal{F} = \frac{1}{2} c (\dot{y} - \cancel{\dot{y}_0})^2 = \frac{1}{2} c \dot{y}^2$$

$$\frac{\partial}{\partial t}\left[\frac{\partial T}{\partial \dot{y}}\right] = \frac{\partial}{\partial t}\left[\frac{1}{2}(2) m \dot{y}\right] = m \ddot{y}$$

$$\frac{\partial V}{\partial y} = \frac{1}{2}(2) k y = k y$$

$$\frac{\partial \mathcal{F}}{\partial \dot{y}} = \frac{1}{2}(2) c \dot{y} = c \dot{y}$$

$$\frac{\partial T}{\partial y} = 0 \, , \; F_I = 0$$

~ thus,

$$m\ddot{y} - 0 + ky = -c\dot{y}$$

$$\left[m\ddot{y} + c\dot{y} + ky = 0\right]\frac{1}{m}$$

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = 0 \quad ①$$

~ let, $\omega_n^2 = \frac{k}{m}$ (2a), zeta $= \zeta =$ viscous damping factor

$$\frac{c\cdot(2\omega_n)}{m\cdot(2\omega_n)} = \frac{2\omega_n c}{2m\omega_n} \quad (2b)$$

~ thus,

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0 \quad ③$$

~ the complementary function:

$$y(t) = A_1e^{s_1t} + A_2e^{s_2t} \quad ④$$

~ where A is a constant and S is a quantity to be determined.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad ⑤$$

~ use quadratic formula:

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4(1)\omega_n^2}}{2(1)}$$

$$= \frac{-\cancel{2}\zeta\omega_n \pm \cancel{2}\omega_n\sqrt{\zeta^2 - 1}}{\cancel{2}}$$

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \quad ⑥$$

~ combining ④ & ⑥ : (eliminating s_1 & s_2)

$$y(t) = A_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + A_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

$$\boxed{y(t) = e^{-\zeta\omega_n t}\left(A_1 e^{\sqrt{\zeta^2-1}\,\omega_n t} + A_2 e^{-\sqrt{\zeta^2-1}\,\omega_n t}\right)} \quad (7)$$

General Solution for Single Degree of Freedom System w/ Viscous Damping

* $\underline{\omega_d = \omega_n\sqrt{\zeta^2-1}}$

3 Cases:

1.) if $\zeta < 1$, then: under damping case

2.) if $\zeta = 1$, then: critical damping case

3.) if $\zeta > 1$, then: overdetermined case

3 Cases of Viscous Damping

Thursday, February 22, 2024 8:21 AM

CASE 1: Underdamped Free Vibration (if $\zeta < 1$)

Analytical Method:

~ using eq. 6:

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

* if $\zeta < 1$,

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = -\zeta\omega_n \pm \omega_n\sqrt{(1-\zeta)^2(-1)}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{(1-\zeta)^2}\, i \quad (8)$$

~ thus,

$$y(t) = e^{-\zeta\omega_n t}\left[C_1\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + C_2\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right] \quad (9)$$

~ Differentiating eq. 9:

$$\begin{aligned}\dot{y}(t) &= e^{-\zeta\omega_n t}\left[C_1\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + C_2\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right](-\zeta\omega_n) \\ &\quad + e^{-\zeta\omega_n t}\left[-C_1\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + C_2\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right]\left(\omega_n\sqrt{1-\zeta^2}\right)\end{aligned} \quad (10)$$

~ when $t = 0$,

Eq. 9: $y(0) = \cancelto{1}{e^0}\left[C_1\cancelto{1}{\cos(0)} + \cancel{C_2\cancelto{0}{\sin(0)}}\right] = y_0$

$y_0 = C_1$ (11)

Eq. 10: $\dot{y}(0) = \cancelto{1}{e^0}\left[C_1\cancelto{1}{\cos(0)} + \cancel{C_2\cancelto{0}{\sin(0)}}\right](-\zeta\omega_n) + \cancelto{1}{e^0}\left[\cancel{-C_1\cancelto{0}{\sin(0)}} + C_2\cancelto{1}{\cos(0)}\right]\left(\omega_n\sqrt{1-\zeta^2}\right) = \dot{y}_0$

$\dot{y}_0 = -\zeta\omega_n C_1 + \omega_n\sqrt{1-\zeta^2}\,C_2$ (12)

~ combine eq. 11 & 12:

$$\dot{y}_0 = -\zeta\omega_n y_0 + \omega_n\sqrt{1-\zeta^2}\,C_2$$

$$C_2 = \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} \quad (13)$$

~ combine eq. 9, 11, & 13: (eliminating C_1 & C_2)

$$y(t) = e^{-\zeta\omega_n t}\left[y_0\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right]$$

~ but,

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$ Frequency of Damped Free Vibration

* Displacement Response:

$$\boxed{y(t) = e^{-\zeta\omega_n t}\left[y_0\cos(\omega_d t) + \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right]}$$

* Velocity Response:

$$\dot{y}(t) = e^{-\zeta\omega_n t}\left[y_0\cos(\omega_d t) + \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right](-\zeta\omega_n)$$

$$+\ e^{-\zeta\omega_n t}\left[-y_0\sin(\omega_d t) + \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_d}\cos(\omega_d t)\right]\omega_d$$

$$\therefore\ \boxed{\begin{aligned}\dot{y}(t) = e^{-\zeta\omega_n t}\Bigg[&\left(y_0\cos(\omega_d t) + \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right)(-\zeta\omega_n)\\ &+\left(-y_0\sin(\omega_d t) + \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_d}\cos(\omega_d t)\right)\omega_d\Bigg]\end{aligned}}$$

~ evaluating eq. 9:

$$y(t) = e^{-\zeta\omega_n t}\left[C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right]$$

* let, $C_1 = A\sin(\phi_d)$

$C_2 = A\cos(\phi_d)$

~ then:

$$y(t) = e^{-\zeta\omega_n t}\left[A\sin(\phi_d)\cos(\omega_d t) + A\cos(\right.$$

$$\boxed{y(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d + \phi_d)}$$

--> (Damped) Phase Angle

Amplitude

Frequency of Damped Free Vibration

~ let us find A:

$$c_1^2 + c_2^2 = A^2\left[\cancel{\sin^2(\phi_d) + \cos^2(\phi_d)}^{\,1}\right]$$

$$A = \sqrt{y_0^2 + \left(\frac{\dot{y}_0 + y_0\zeta\omega_n^2}{\omega_n\sqrt{1-\zeta^2}}\right)}$$

$$\therefore \boxed{y(t) = \sqrt{y_0^2 + \left(\frac{\dot{y}_0 + y_0\zeta\omega_n^2}{\omega_n\sqrt{1-\zeta^2}}\right)}\, e^{-\zeta\omega_n t}\sin(\omega_d + \phi_d)}$$

RKN Method:

~ let: $k = 1.0$ when $t=0$:

$c = 1.0$ $y(0) = y_0 = 2.0$

$m = 1.0$ $\dot{y}(0) = \dot{y}_0 = 1.0$

$h = 0.5$

also: $\omega_n^2 = \frac{k}{m} = 1.0$

$$\zeta = \frac{c}{2m\omega_n} = \frac{1.0}{2(1)(1)} = 0.5$$

$$\therefore \underline{\zeta = 0.5 < 1}$$

~ the Differential Equation of Motion:

$$m\ddot{y} + c\dot{y} + ky = 0$$

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = 0$$

$$\boxed{\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0} \quad (1)$$

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4(1)\omega_n^2}}{2(1)}$$

$$= \frac{-\cancel{2}\zeta\omega_n \pm \cancel{2}\omega_n\sqrt{1-\zeta^2}\,i}{\cancel{2}}$$

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,i \quad (2)$$

~ then the complementary function:

$$y(t) = e^{-\zeta\omega_n t}\left[C_1\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + C_2\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right] \quad (3)$$

~ differentiating eqn. 3:

$$\dot{y}(t) = e^{-\zeta\omega_n t}\left[C_1\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + C_2\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right](-\zeta\omega_n)$$
$$+ e^{-\zeta\omega_n t}\left[-C_1\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + C_2\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right]\left(\omega_n\sqrt{1-\zeta^2}\right)$$

* **Recall:**

$$C_1 = y_0 = \underline{2.0}$$

$$C_2 = \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{1 + 2(0.5)(1)}{(1)\sqrt{1-(0.5)^2}} = \underline{2.3094}$$

~ use excel *

~ In RKN use 6 auxiliary eqns:

- $k_1 = \frac{1}{2}h \cdot f(t_n, y_n, \dot{y}_n)$
- $k_2 = \frac{1}{2}h \cdot f(t_n + \frac{1}{2}h, y_n + K, \dot{y}_n + k_1)$; $K = \frac{1}{2}h(\dot{y}_n + \frac{1}{2}k_1)$
- $k_3 = \frac{1}{2}h \cdot f(t_n + \frac{1}{2}h, y_n + K, \dot{y}_n + k_2)$
- $k_4 = \frac{1}{2}h \cdot f(t_n + \frac{1}{2}h, y_n + L, \dot{y}_n + 2k_3)$; $L = h(\dot{y}_n + k_3)$

- $y_{n+1} = y_n + h\left[\dot{y}_n + \frac{1}{3}(k_1 + k_2 + k_3)\right]$ ~ for displacement
- $\dot{y}_{n+1} = \dot{y}_n + \frac{1}{3}(k_1 + 2(k_2 + k_3) + k_4)$ ~ for velocity

~ using eq. 1:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0$$

$$\ddot{y} = -2\zeta\omega_n\dot{y} - \omega_n^2 y = f(t_n, y_n, y'_n)$$

~ then when n=0,

$$k_1 = \frac{1}{2}h\cdot f(t_0, y_0, \dot{y}_0) = \frac{1}{2}h(-2\zeta\omega_n\dot{y}_0 - \omega_n^2 y_0)$$

$$= \frac{1}{2}(0.5)(-2(0.5)(1)(1) - (1)^2(2)) = \underline{-0.75}$$

$$* \; K = \frac{1}{2}h\left(\dot{y}_0 + \frac{1}{2}k_1\right) = \frac{1}{2}(0.5)\left(1 + \frac{1}{2}(-0.75)\right)$$

$$= \underline{0.15625}$$

$$k_2 = \frac{1}{2}h\cdot f\left(t_0 + \frac{1}{2}h, y_0 + K, \dot{y}_0 + k_1\right)$$

$$= \frac{1}{2}(0.5)\left(-2(0.5)(1)(1 - 0.75) - (1)^2(2 + 0.15625)\right)$$

$$= \underline{-0.60156}$$

$$k_3 = \frac{1}{2}h\cdot f\left(t_0 + \frac{1}{2}h, y_0 + K, \dot{y}_0 + k_2\right)$$

$$= \frac{1}{2}(0.5)\left(-2(0.5)(1)(1 - 0.60156) - (1)^2(2 + 0.15625)\right)$$

$$= \underline{-0.63867}$$

$$* \; L = h(\dot{y}_0 + k_3) = 0.5(1 - 0.63867) = \underline{0.18066}$$

$$k_4 = \frac{1}{2}h\cdot f\left(t_0 + \frac{1}{2}h, y_0 + L, y'_0 + 2k_3\right)$$

$$= \frac{1}{2}(0.5)\left(-2(0.5)(1)(1 + 2(-0.63867) - (1)^2(2 + 0.18066)\right)$$

$$= \underline{-0.47583}$$

~ then at n=1,

$$y_1 = y_0 + h\left[\dot{y}_0 + \frac{1}{3}(k_1 + k_2 + k_3)\right]$$

$$= 2 + 0.5\left[1 + \frac{1}{3}(-0.75 - 0.60156 - 0.63867)\right]$$

$$\boxed{y_1 = 2.18629\,m}$$

$$\dot{y}_1 = \dot{y}_0 + \frac{1}{3}(k_1 + 2(k_2 + k_3) + k_4)$$

$$= 1 + \frac{1}{3}(-0.75 + 2(-0.60156 - 0.63867) - 0.47583)$$

$$\boxed{\dot{y}_1 = -0.23543 \text{ m/s}}$$

* if you want to decrease ζ, increase mass.

if stepsize is decreased, accuracy increases & reduce errors.

CASE 2: Critically Damped Free Vibration ($\zeta = 1$)

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0 \quad (1)$$

$$\downarrow$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad (2)$$

~ using quadratic formula:

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = -\zeta\omega_n \pm \omega_n\underbrace{\sqrt{\zeta^2 - 1}}_{=0} \; ; \text{ if } \zeta = 1 \quad (3)$$

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = -\zeta\omega_n \text{ (repeated roots)} \quad (4)$$

~ the complementary function:

$$y(t) = e^{-\zeta\omega_n t}(c_1 + c_2 t) \quad (5) \text{ General Solution for Displacement}$$

~ differentiating eq. 5:

$$\dot{y}(t) = e^{-\zeta\omega_n t}(c_2) + (c_1 + c_2 t)e^{-\zeta\omega_n t}(-\zeta\omega_n)$$

$$\dot{y}(t) = e^{-\zeta\omega_n t}\left[-\zeta\omega_n c_1 + c_2(1 - \zeta\omega_n t)\right] \quad (6) \text{ General Solution for Velocity}$$

* When $t = 0$, let initial values be: $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$

~ using eqn. 5:

$$y_0 = \cancel{e^0}^{\,1}\left[c_1 + \cancel{c_2(0)}^{\,0}\right]$$

$$\therefore c_1 = y_0 \quad (7)$$

~ using eqn. 6:

$$\dot{y}_0 = \cancel{e^0}^{\,1}\left[-\zeta\omega_n y_0 + C_2(1-\cancel{\zeta\omega_n(0)}^{\,0})\right]$$

$$\therefore \underline{C_2 = \dot{y}_0 + y_0\zeta\omega_n} \quad (8)$$

~ then,

Displacement: $$y(t) = e^{-\zeta\omega_n t}\left[y_0 + (\dot{y}_0 + y_0\zeta\omega_n)t\right] \quad (9)$$

Velocity: $$\dot{y}(t) = e^{-\zeta\omega_n t}\left[-\zeta\omega_n y_0 + (\dot{y}_0 + y_0\zeta\omega_n)(1-\zeta\omega_n t)\right] \quad (10)$$

Ex.

1.) let $m = 1.0$ $\qquad \omega_n = \sqrt{\dfrac{k}{m}} = 1.0$

$k = 1.0$

$C = 2.0$ $\qquad \zeta = \dfrac{C}{2m\omega_n} = 1.0 \rightarrow (\zeta = 1)$

$\therefore$ $\boxed{\text{Case 2}}$

~ let $y_0 = 2.00$

$\dot{y}_0 = 1.00$

[excel demonstration]

~ analytical & RKN are the same

CASE 3: Overdamped Free Vibration $(\zeta > 1)$

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0$$

$$\downarrow$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

~ using quadratic formula:

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} \quad (11)$$

; if $\zeta > 1$, roots are real & distinct

~ then the complementary function is:

$$y(t) = c_1 e^{[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}]t} + c_2 e^{[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}]t}$$

$$y(t) = e^{-\zeta\omega_n t}\left[c_1 e^{\omega_n\sqrt{\zeta^2-1}]t} + c_2 e^{-\omega_n\sqrt{\zeta^2-1}]t}\right]$$ 12 for Displacement

~ differentiating eq. 12:

$$\dot{y}(t) = e^{-\zeta\omega_n t}\left[(\omega_n\sqrt{\zeta^2-1})c_1 e^{\omega_n\sqrt{\zeta^2-1}]t} + (-\omega_n\sqrt{\zeta^2-1})c_2 e^{-\omega_n\sqrt{\zeta^2-1}]t}\right] + e^{-\zeta\omega_n t}(-\zeta\omega_n)\left[c_1 e^{\omega_n\sqrt{\zeta^2-1}]t} + c_2 e^{-\omega_n\sqrt{\zeta^2-1}]t}\right]$$

$$\dot{y}(t) = e^{-\zeta\omega_n t}(\omega_n\sqrt{\zeta^2-1})\left[c_1 e^{\omega_n\sqrt{\zeta^2-1}]t} - c_2 e^{-\omega_n\sqrt{\zeta^2-1}]t}\right] + e^{-\zeta\omega_n t}(-\zeta\omega_n)\left[c_1 e^{\omega_n\sqrt{\zeta^2-1}]t} + c_2 e^{-\omega_n\sqrt{\zeta^2-1}]t}\right]$$ 13 for velocity

*When $t=0$, let initial values be: $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$

~ using eqn. 12:

$$y_0 = \cancel{e}^0\left[c_1\cancel{e}^0 + c_2\cancel{e}^0\right]$$

$$y_0 = c_1 + c_2$$

$$\therefore c_1 = y_0 - c_2$$ 14

~ using eqn. 13:

$$\dot{y}_0 = \cancel{e}^0(\omega_n\sqrt{\zeta^2-1})\left[c_1\cancel{e}^0 - c_2\cancel{e}^0\right]$$ 15

$$+ \cancel{e}^0(-\zeta\omega_n)\left[c_1\cancel{e}^0 + c_2\cancel{e}^0\right]$$ * $c_1 = y_0 - c_2$, $y_0 = c_1 + c_2$

$$\dot{y}_0 = \omega_n\sqrt{\zeta^2-1}\,(y_0 - c_2 - c_2) - \zeta\omega_n y_0$$

$$c_2 = \frac{1}{2}\left[y_0 - \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{\zeta^2-1}}\right]$$ 16

~ then,

$$c_1 = y_0 - \frac{1}{2}\left[y_0 - \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{\zeta^2-1}}\right] \dashrightarrow = y_0 - \frac{1}{2}y_0 + \left(\frac{1}{2}\right)\frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{\zeta^2-1}} = \boxed{\frac{1}{2}y_0 + \frac{\dot{y}_0 + y_0\zeta\omega_n}{2\omega_n\sqrt{\zeta^2-1}}}$$

$$c_2 = \frac{1}{2}\left[y_0 - \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{\zeta^2-1}}\right]$$

~ Finally,

Displacement

$$y(t) = e^{-\zeta\omega_n t}\left[\left(\frac{1}{2}y_0 + \frac{\dot{y}_0 + y_0\zeta\omega_n}{2\omega_n\sqrt{\zeta^2-1}}\right)e^{\left[\omega_n\sqrt{\zeta^2-1}\right]t} + \frac{1}{2}\left(y_0 - \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{\zeta^2-1}}\right)e^{\left[-\omega_n\sqrt{\zeta^2-1}\right]t}\right] \quad (18)$$

Velocity

$$\dot{y}(t) = e^{-\zeta\omega_n t}\left(\omega_n\sqrt{\zeta^2-1}\right)\left[\left(\frac{1}{2}y_0 + \frac{\dot{y}_0 + y_0\zeta\omega_n}{2\omega_n\sqrt{\zeta^2-1}}\right)e^{\left[\omega_n\sqrt{\zeta^2-1}\right]t} - \frac{1}{2}\left(y_0 - \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{\zeta^2-1}}\right)e^{\left[-\omega_n\sqrt{\zeta^2-1}\right]t}\right] + e^{-\zeta\omega_n t}\left(-\zeta\omega_n\right)\left[\left(\frac{1}{2}y_0 + \frac{\dot{y}_0 + y_0\zeta\omega_n}{2\omega_n\sqrt{\zeta^2-1}}\right)e^{\left[\omega_n\sqrt{\zeta^2-1}\right]t} + \frac{1}{2}\left(y_0 - \frac{\dot{y}_0 + y_0\zeta\omega_n}{\omega_n\sqrt{\zeta^2-1}}\right)e^{\left[-\omega_n\sqrt{\zeta^2-1}\right]t}\right] \quad (19)$$

~ Plot & compare with RKN.