Chapter 11

Radiation

11.1 Dipole Radiation

11.1.1 What is Radiation?

In Chapter 9 we discussed the propagation of plane electromagnetic waves through various media, but I did not tell you how the waves got started in the first place. Like all electromagnetic fields, their source is some arrangement of electric charge. But a charge at rest does not generate electromagnetic waves; nor does a steady current. It takes *accelerating* charges, and *changing* currents, as we shall see. My purpose in this chapter is to show you how such configurations produce electromagnetic waves—that is, how they **radiate**.

Once established, electromagnetic waves in vacuum propagate out "to infinity," carrying energy with them; the *signature* of radiation is this irreversible flow of energy away from the source. Throughout this chapter I shall assume the source is *localized*[1] near the origin. Imagine a gigantic spherical shell, out at radius r (Fig. 11.1); the total power passing out through this surface is the integral of the Poynting vector:

$$P(r) = \oint \mathbf{S} \cdot d\mathbf{a} = \frac{1}{\mu_0} \oint (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}. \tag{11.1}$$

The power *radiated* is the limit of this quantity as r goes to infinity:

$$P_{\text{rad}} \equiv \lim_{r \to \infty} P(r). \tag{11.2}$$

This is the energy (per unit time) that is transported out to infinity, and never comes back.

Now, the area of the sphere is $4\pi r^2$, so for radiation to occur the Poynting vector must decrease (at large r) no faster than $1/r^2$ (if it went like $1/r^3$, for example, then $P(r)$ would go like $1/r$, and P_{rad} would be zero). According to Coulomb's law, electro*static* fields fall off like $1/r^2$ (or even faster, if the total charge is zero), and the Biot-Savart law says

[1]For *non*localized sources, such as infinite planes, wires, or solenoids, the whole concept of "radiation" must be reformulated—see Prob. 11.24.

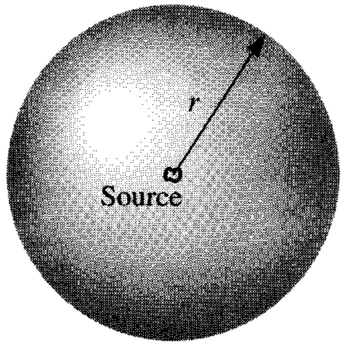

Figure 11.1

that magneto*static* fields go like $1/r^2$ (or faster), which means that $S \sim 1/r^4$, for static configurations. So *static* sources do not radiate. But Jefimenko's equations (10.29 and 10.31) indicate that *time-dependent* fields include terms (involving $\dot{\rho}$ and $\dot{\mathbf{J}}$) that go like $1/r$; it is *these* terms that are responsible for electromagnetic radiation.

The study of radiation, then, involves picking out the parts of **E** and **B** that go like $1/r$ at large distances from the source, constructing from them the $1/r^2$ term in **S**, integrating over a large spherical[2] surface, and taking the limit as $r \to \infty$. I'll carry through this procedure first for oscillating electric and magnetic dipoles; then, in Sect. 11.2, we'll consider the more difficult case of radiation from an accelerating point charge.

11.1.2 Electric Dipole Radiation

Picture two tiny metal spheres separated by a distance d and connected by a fine wire (Fig. 11.2); at time t the charge on the upper sphere is $q(t)$, and the charge on the lower sphere is $-q(t)$. Suppose that we drive the charge back and forth through the wire, from one end to the other, at an angular frequency ω:

$$q(t) = q_0 \cos(\omega t). \tag{11.3}$$

The result is an oscillating electric dipole:[3]

$$\mathbf{p}(t) = p_0 \cos(\omega t)\,\hat{\mathbf{z}}, \tag{11.4}$$

where

$$p_0 \equiv q_0 d$$

is the the maximum value of the dipole moment.

[2]It doesn't have to be a sphere, of course, but this makes the calculations a lot easier.

[3]It might occur to you that a more natural model would consist of equal and opposite charges mounted on a spring, say, so that q is constant while d oscillates, instead of the other way around. Such a model would lead to the same result, but there is a subtle problem in calculating the retarded potentials of a moving point charge, which I would prefer to save for Sect. 11.2.

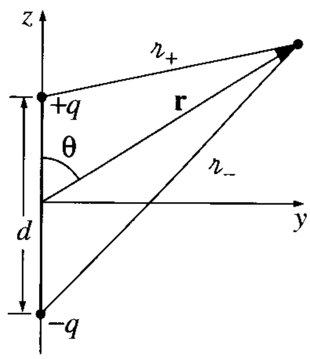

Figure 11.2

The retarded potential (Eq. 10.19) is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-}\right\}, \tag{11.5}$$

where, by the law of cosines,

$$\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}. \tag{11.6}$$

Now, to make this *physical* dipole into a *perfect* dipole, we want the separation distance to be extremely small:

$$\textbf{approximation 1}: \ d \ll r. \tag{11.7}$$

Of course, if d is *zero* we get no potential at all; what we want is an expansion carried to *first order* in d. Thus

$$\mathscr{r}_\pm \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right). \tag{11.8}$$

It follows that

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right), \tag{11.9}$$

and

$$\cos[\omega(t - \mathscr{r}_\pm/c)] \cong \cos\left[\omega(t - r/c) \pm \frac{\omega d}{2c}\cos\theta\right]$$

$$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin[\omega(t - r/c)]\sin\left(\frac{\omega d}{2c}\cos\theta\right).$$

In the perfect dipole limit we have, further,

$$\textbf{approximation 2}: \ d \ll \frac{c}{\omega}. \tag{11.10}$$

(Since waves of frequency ω have a wavelength $\lambda = 2\pi c/\omega$, this amounts to the requirement $d \ll \lambda$.) Under these conditions

$$\cos[\omega(t - \imath_\pm/c)] \cong \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c}\cos\theta \sin[\omega(t - r/c)]. \tag{11.11}$$

Putting Eqs. 11.9 and 11.11 into Eq. 11.5, we obtain the potential of an oscillating perfect dipole:

$$V(r, \theta, t) = \frac{p_0 \cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t - r/c)] + \frac{1}{r}\cos[\omega(t - r/c)]\right\}. \tag{11.12}$$

In the static limit ($\omega \to 0$) the second term reproduces the old formula for the potential of a stationary dipole (Eq. 3.99):

$$V = \frac{p_0 \cos\theta}{4\pi\epsilon_0 r^2}.$$

This is not, however, the term that concerns us now; we are interested in the fields that survive at *large distances from the source*, in the so-called **radiation zone**:[4]

$$\textbf{approximation 3}: \quad r \gg \frac{c}{\omega}. \tag{11.13}$$

(Or, in terms of the wavelength, $r \gg \lambda$.) In this region the potential reduces to

$$\boxed{V(r, \theta, t) = -\frac{p_0 \omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)].} \tag{11.14}$$

Meanwhile, the *vector* potential is determined by the current flowing in the wire:

$$\mathbf{I}(t) = \frac{dq}{dt}\,\hat{\mathbf{z}} = -q_0\omega\sin(\omega t)\,\hat{\mathbf{z}}. \tag{11.15}$$

Referring to Fig. 11.3,

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q_0\omega \sin[\omega(t - \imath/c)]\,\hat{\mathbf{z}}}{\imath}\,dz. \tag{11.16}$$

Because the integration itself introduces a factor of d, we can, to first order, replace the integrand by its value at the center:

$$\boxed{\mathbf{A}(r, \theta, t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - r/c)]\,\hat{\mathbf{z}}.} \tag{11.17}$$

(Notice that whereas I implicitly used approximations 1 and 2, in keeping only the first order in d, Eq. 11.17 is *not* subject to approximation 3.)

[4]Note that approximations 2 and 3 subsume approximation 1; all together, we have $d \ll \lambda \ll r$.

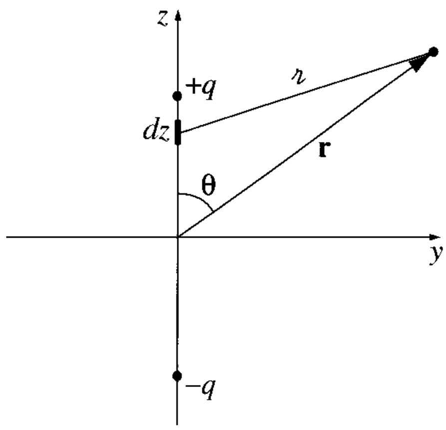

Figure 11.3

From the potentials, it is a straightforward matter to compute the fields.

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}}$$

$$= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin[\omega(t-r/c)] - \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\hat{\mathbf{r}} - \frac{\sin\theta}{r^2}\sin[\omega(t-r/c)]\hat{\boldsymbol{\theta}}\right\}$$

$$\cong \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\mathbf{r}}.$$

(I dropped the first and last terms, in accordance with approximation 3.) Likewise,

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}),$$

and therefore

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}.} \tag{11.18}$$

Meanwhile

$$\nabla \times \mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)] + \frac{\sin\theta}{r}\sin[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}.$$

The second term is again eliminated by approximation 3, so

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\phi}}.} \tag{11.19}$$

Equations 11.18 and 11.19 represent monochromatic waves of frequency ω traveling in the radial direction at the speed of light. **E** and **B** are in phase, mutually perpendicular, and transverse; the ratio of their amplitudes is $E_0/B_0 = c$. All of which is precisely what we expect for electromagnetic waves in free space. (These are actually *spherical* waves, not plane waves, and their amplitude decreases like $1/r$ as they progress. But for large r, they are approximately plane over small regions—just as the surface of the earth is reasonably flat, locally.)

The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right\}^2 \hat{\mathbf{r}}. \tag{11.20}$$

The intensity is obtained by averaging (in time) over a complete cycle:

$$\langle\mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}. \tag{11.21}$$

Notice that there is no radiation along the *axis* of the dipole (here $\sin\theta = 0$); the intensity profile[5] takes the form of a donut, with its maximum in the equatorial plane (Fig. 11.4). The total power radiated is found by integrating $\langle\mathbf{S}\rangle$ over a sphere of radius r:

$$\langle P\rangle = \int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta\,d\phi = \frac{\mu_0 p_0^2\omega^4}{12\pi c}. \tag{11.22}$$

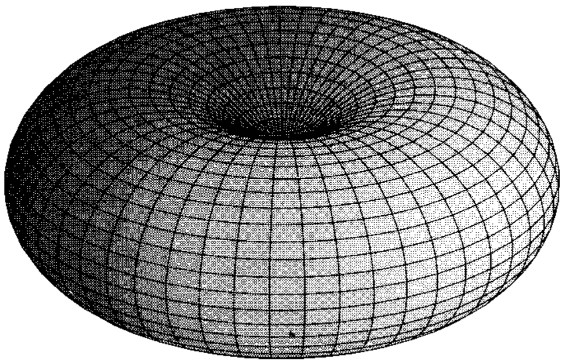

Figure 11.4

[5]The "radial" coordinate in Fig. 11.4 represents the magnitude of $\langle\mathbf{S}\rangle$ (at fixed r), as a function of θ and ϕ.

It is independent of the radius of the sphere, as one would expect from conservation of energy (with approximation 3 we were anticipating the limit $r \to \infty$).

Example 11.1

The sharp frequency dependence of the power formula is what accounts for the blueness of the sky. Sunlight passing through the atmosphere stimulates atoms to oscillate as tiny dipoles. The incident solar radiation covers a broad range of frequencies (white light), but the energy absorbed and reradiated by the atmospheric dipoles is stronger at the higher frequencies because of the ω^4 in Eq. 11.22. It is more intense in the blue, then, than in the red. It is this reradiated light that you see when you look up in the sky—unless, of course, you're staring directly at the sun.

Because electromagnetic waves are transverse, the dipoles oscillate in a plane orthogonal to the sun's rays. In the celestial arc perpendicular to these rays, where the blueness is most pronounced, the dipoles oscillating along the line of sight send no radiation to the observer (because of the $\sin^2\theta$ in equation Eq. 11.21); light received at this angle is therefore polarized perpendicular to the sun's rays (Fig. 11.5).

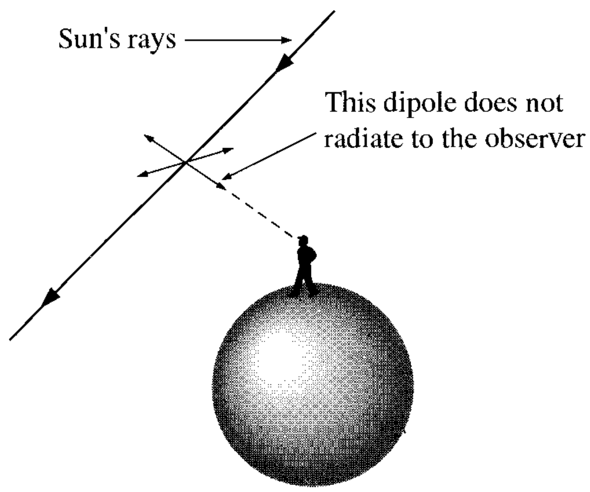

Figure 11.5

The redness of sunset is the other side of the same coin: Sunlight coming in at a tangent to the earth's surface must pass through a much longer stretch of atmosphere than sunlight coming from overhead (Fig. 11.6). Accordingly, much of the blue has been *removed* by scattering and what's left is red.

Problem 11.1 Check that the retarded potentials of an oscillating dipole (Eqs. 11.12 and 11.17) satisfy the Lorentz gauge condition. Do *not* use approximation 3.

Problem 11.2 Equation 11.14 can be expressed in "coordinate-free" form by writing $p_0 \cos\theta = \mathbf{p}_0 \cdot \hat{\mathbf{r}}$. Do so, and likewise for Eqs. 11.17, 11.18, 11.19, and 11.21.

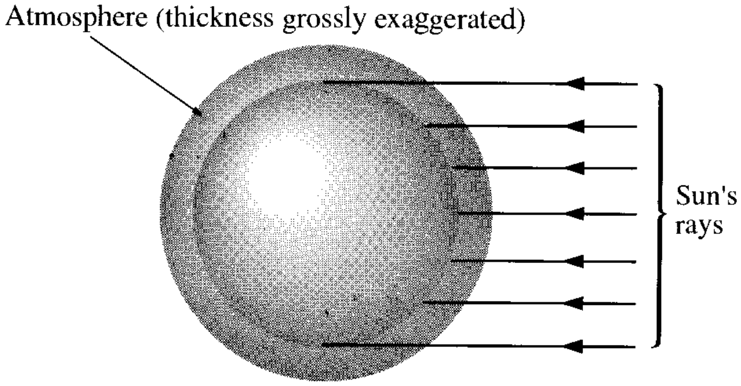

Figure 11.6

Problem 11.3 Find the **radiation resistance** of the wire joining the two ends of the dipole. (This is the resistance that would give the same average power loss—to heat—as the oscillating dipole in *fact* puts out in the form of radiation.) Show that $R = 790\,(d/\lambda)^2\ \Omega$, where λ is the wavelength of the radiation. For the wires in an ordinary radio (say, $d = 5$ cm), should you worry about the radiative contribution to the total resistance?

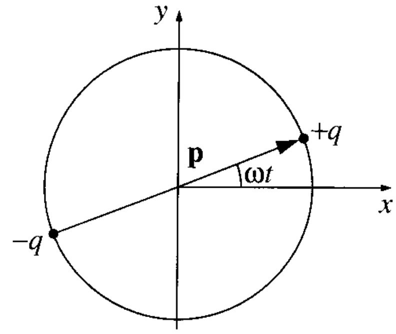

Figure 11.7

! **Problem 11.4** A *rotating* electric dipole can be thought of as the superposition of two *oscillating* dipoles, one along the x axis, and the other along the y axis (Fig. 11.7), with the latter out of phase by 90°:

$$\mathbf{p} = p_0[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}].$$

Using the principle of superposition and Eqs. 11.18 and 11.19 (perhaps in the form suggested by Prob. 11.2), find the fields of a rotating dipole. Also find the Poynting vector and the intensity of the radiation. Sketch the intensity profile as a function of the polar angle θ, and calculate the total power radiated. Does the answer seem reasonable? (Note that power, being *quadratic* in the fields, does *not* satisfy the superposition principle. In this instance, however, it *seems* to. Can you account for this?)

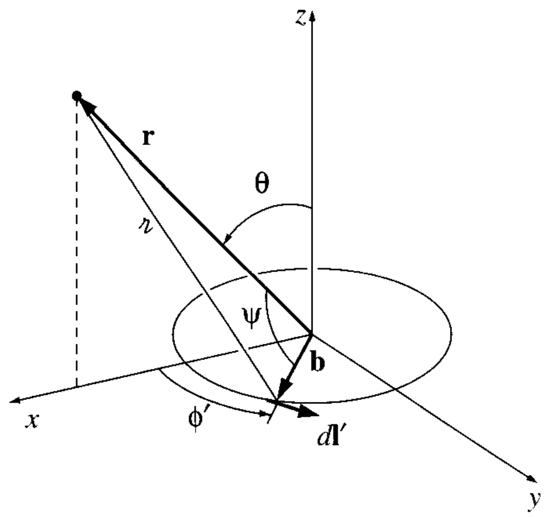

Figure 11.8

11.1.3 Magnetic Dipole Radiation

Suppose now that we have a wire loop of radius b (Fig. 11.8), around which we drive an alternating current:

$$I(t) = I_0 \cos(\omega t). \tag{11.23}$$

This is a model for an oscillating *magnetic* dipole,

$$\mathbf{m}(t) = \pi b^2 I(t)\,\hat{\mathbf{z}} = m_0 \cos(\omega t)\,\hat{\mathbf{z}}, \tag{11.24}$$

where

$$m_0 \equiv \pi b^2 I_0 \tag{11.25}$$

is the maximum value of the magnetic dipole moment.

The loop is uncharged, so the scalar potential is zero. The retarded vector potential is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}}\, d\mathbf{l}'. \tag{11.26}$$

For a point $\mathbf{r}$ directly above the x axis (Fig. 11.8), $\mathbf{A}$ must aim in the y direction, since the x components from symmetrically placed points on either side of the x axis will cancel. Thus

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 I_0 b}{4\pi}\,\hat{\mathbf{y}} \int_0^{2\pi} \frac{\cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}} \cos\phi'\, d\phi' \tag{11.27}$$

($\cos\phi'$ serves to pick out the y-component of $d\mathbf{l}'$). By the law of cosines,

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\cos\psi}\,,$$

where ψ is the angle between the vectors $\mathbf{r}$ and $\mathbf{b}$:

$$\mathbf{r} = r\sin\theta\,\hat{\mathbf{x}} + r\cos\theta\,\hat{\mathbf{z}}, \quad \mathbf{b} = b\cos\phi'\,\hat{\mathbf{x}} + b\sin\phi'\,\hat{\mathbf{y}}.$$

So $rb\cos\psi = \mathbf{r}\cdot\mathbf{b} = rb\sin\theta\cos\phi'$, and therefore

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi'}. \tag{11.28}$$

For a "perfect" dipole, we want the loop to be extremely small:

$$\textbf{approximation 1}: \quad b \ll r. \tag{11.29}$$

To first order in b, then,

$$\mathscr{r} \cong r\left(1 - \frac{b}{r}\sin\theta\cos\phi'\right),$$

so

$$\frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi'\right) \tag{11.30}$$

and

$$\cos[\omega(t - \mathscr{r}/c)] \cong \cos\left[\omega(t - r/c) + \frac{\omega b}{c}\sin\theta\cos\phi'\right]$$

$$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right) - \sin[\omega(t - r/c)]\sin\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right).$$

As before, we also assume the size of the dipole is small compared to the wavelength radiated:

$$\textbf{approximation 2}: \quad b \ll \frac{c}{\omega}. \tag{11.31}$$

In that case,

$$\cos[\omega(t - \mathscr{r}/c)] \cong \cos[\omega(t - r/c)] - \frac{\omega b}{c}\sin\theta\cos\phi'\sin[\omega(t - r/c)]. \tag{11.32}$$

Inserting Eqs. 11.30 and 11.32 into Eq. 11.27, and dropping the second-order term:

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0 I_0 b}{4\pi r}\,\hat{\mathbf{y}} \int_0^{2\pi} \left\{\cos[\omega(t - r/c)]\right.$$

$$\left. + b\sin\theta\cos\phi'\left(\frac{1}{r}\cos[\omega(t - r/c)] - \frac{\omega}{c}\sin[\omega(t - r/c)]\right)\right\}\cos\phi'\,d\phi'.$$

The first term integrates to zero:

$$\int_0^{2\pi} \cos\phi'\,d\phi' = 0.$$

The second term involves the integral of cosine squared:

$$\int_0^{2\pi} \cos^2 \phi' \, d\phi' = \pi.$$

Putting this in, and noting that in general **A** points in the $\hat{\boldsymbol{\phi}}$-direction, I conclude that the vector potential of an oscillating perfect magnetic dipole is

$$\mathbf{A}(r, \theta, t) = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\cos[\omega(t - r/c)] - \frac{\omega}{c}\sin[\omega(t - r/c)]\right\}\hat{\boldsymbol{\phi}}. \tag{11.33}$$

In the static limit ($\omega = 0$) we recover the familiar formula for the potential of a magnetic dipole (Eq. 5.85)

$$\mathbf{A}(r, \theta) = \frac{\mu_0}{4\pi}\frac{m_0 \sin\theta}{r^2}\hat{\boldsymbol{\phi}}.$$

In the radiation zone,

$$\textbf{approximation 3}: \; r \gg \frac{c}{\omega}, \tag{11.34}$$

the first term in **A** is negligible, so

$$\boxed{\mathbf{A}(r, \theta, t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t - r/c)]\hat{\boldsymbol{\phi}}.} \tag{11.35}$$

From **A** we obtain the fields at large r:

$$\boxed{\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}},} \tag{11.36}$$

and

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\hat{\boldsymbol{\theta}}.} \tag{11.37}$$

(I used approximation 3 in calculating **B**.) These fields are in phase, mutually perpendicular, and transverse to the direction of propagation ($\hat{\mathbf{r}}$), and the ratio of their amplitudes is $E_0/B_0 = c$, all of which is as expected for electromagnetic waves. They are, in fact, remarkably similar in structure to the fields of an oscillating *electric* dipole (Eqs. 11.18 and 11.19), only this time it is **B** that points in the $\hat{\boldsymbol{\theta}}$ direction and **E** in the $\hat{\boldsymbol{\phi}}$ direction, whereas for electric dipoles it's the other way around.

The energy flux for magnetic dipole radiation is

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\mathbf{r}}, \tag{11.38}$$

the intensity is

$$\langle \mathbf{S} \rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}, \tag{11.39}$$

and the total radiated power is

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}. \tag{11.40}$$

Once again, the intensity profile has the shape of a donut (Fig. 11.4), and the power radiated goes like ω^4. There is, however, one important difference between electric and magnetic dipole radiation: For configurations with comparable dimensions, the power radiated electrically is enormously greater. Comparing Eqs. 11.22 and 11.40,

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{m_0}{p_0 c}\right)^2, \tag{11.41}$$

where (remember) $m_0 = \pi b^2 I_0$, and $p_0 = q_0 d$. The amplitude of the current in the electrical case was $I_0 = q_0\omega$ (Eq. 11.15). Setting $d = \pi b$, for the sake of comparison, I get

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{\omega b}{c}\right)^2. \tag{11.42}$$

But $\omega b/c$ is precisely the quantity we assumed was very small (approximation 2), and here it appears *squared*. Ordinarily, then, one should expect electric dipole radiation to dominate. Only when the system is carefully contrived to exclude any electric contribution (as in the case just treated) will the magnetic dipole radiation reveal itself.

Problem 11.5 Calculate the electric and magnetic fields of an oscillating magnetic dipole *without* using approximation 3. [Do they look familiar? Compare Prob. 9.33.] Find the Poynting vector, and show that the intensity of the radiation is exactly the same as we got using approximation 3.

Problem 11.6 Find the radiation resistance (Prob. 11.3) for the oscillating magnetic dipole in Fig. 11.8. Express your answer in terms of λ and b, and compare the radiation resistance of the *electric* dipole. [*Answer:* $3 \times 10^5\ (b/\lambda)^4\ \Omega$]

Problem 11.7 Use the "duality" transformation of Prob. 7.60, together with the fields of an oscillating *electric* dipole (Eqs. 11.18 and 11.19), to determine the fields that would be produced by an oscillating "Gilbert" *magnetic* dipole (composed of equal and opposite magnetic charges, instead of an electric current loop). Compare Eqs. 11.36 and 11.37, and comment on the result.

11.1.4 Radiation from an Arbitrary Source

In the previous sections we studied the radiation produced by two specific systems: oscillating electric dipoles and oscillating magnetic dipoles. Now I want to apply the same procedures to a configuration of charge and current that is entirely arbitrary, except that it is localized within some finite volume near the origin (Fig. 11.9). The retarded scalar potential is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t - \imath/c)}{\imath}\, d\tau', \tag{11.43}$$

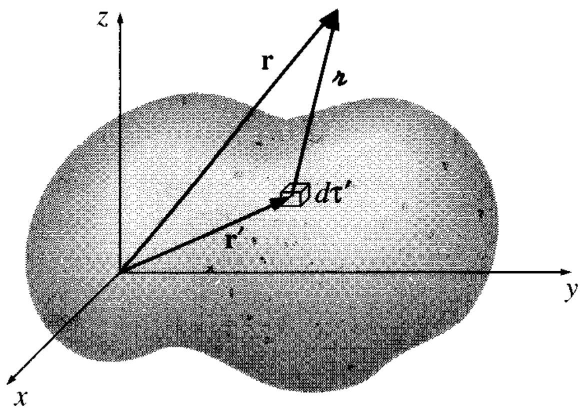

Figure 11.9

where

$$\mathscr{r} = \sqrt{r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}'}. \tag{11.44}$$

As before, we shall assume that the field point $\mathbf{r}$ is far away, in comparison to the dimensions of the source:

$$\textbf{approximation 1}: \quad r' \ll r. \tag{11.45}$$

(Actually, r' is a variable of integration; approximation 1 means that the *maximum* value of r', as it ranges over the source, is much less than r.) On this assumption,

$$\mathscr{r} \cong r\left(1 - \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2}\right), \tag{11.46}$$

so

$$\frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2}\right) \tag{11.47}$$

and

$$\rho(\mathbf{r}', t - \mathscr{r}/c) \cong \rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right).$$

Expanding ρ as a Taylor series in t about the retarded time at the origin,

$$t_0 \equiv t - \frac{r}{c}, \tag{11.48}$$

we have

$$\rho(\mathbf{r}', t - \mathscr{r}/c) \cong \rho(\mathbf{r}', t_0) + \dot{\rho}(\mathbf{r}', t_0)\left(\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) + \dots \tag{11.49}$$

where the dot signifies differentiation with respect to time. The next terms in the series would be

$$\frac{1}{2}\ddot{\rho}\left(\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right)^2, \quad \frac{1}{3!}\dddot{\rho}\left(\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right)^3, \dots .$$

We can afford to drop them, provided

$$\textbf{approximation 2}: \; r' \ll \frac{c}{|\ddot{\rho}/\dot{\rho}|}, \; \frac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}}, \; \frac{c}{|\ddddot{\rho}/\dot{\rho}|^{1/3}}, \; \ldots \tag{11.50}$$

For an oscillating system each of these ratios is c/ω, and we recover the old approximation 2. In the general case it's more difficult to interpret Eq. 11.50, but as a *procedural* matter approximations 1 and 2 amount to *keeping only the first-order terms in r'.*

Putting Eqs. 11.47 and 11.49 into the formula for V (Eq. 11.43), and again discarding the second-order term:

$$V(\mathbf{r}, t) \cong \frac{1}{4\pi\epsilon_0 r}\left[\int \rho(\mathbf{r}', t_0)\, d\tau' + \frac{\hat{\mathbf{r}}}{r}\cdot\int \mathbf{r}'\rho(\mathbf{r}', t_0)\, d\tau' + \frac{\hat{\mathbf{r}}}{c}\cdot\frac{d}{dt}\int \mathbf{r}'\rho(\mathbf{r}', t_0)\, d\tau'\right].$$

The first integral is simply the total charge, Q, at time t_0. Because charge is conserved, however, Q is actually *independent* of time. The other two integrals represent the electric dipole moment at time t_0. Thus

$$V(\mathbf{r}, t) \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}(t_0)}{r^2} + \frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right]. \tag{11.51}$$

In the static case, the first two terms are the monopole and dipole contributions to the multipole expansion for V; the third term, of course, would not be present.

Meanwhile, the vector potential is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t - \mathscr{r}/c)}{\mathscr{r}}\, d\tau'. \tag{11.52}$$

As you'll see in a moment, to first order in r' it suffices to replace $\mathscr{r}$ by r in the integrand:

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}\int \mathbf{J}(\mathbf{r}', t_0)\, d\tau'. \tag{11.53}$$

According to Prob. 5.7, the integral of $\mathbf{J}$ is the time derivative of the dipole moment, so

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_0)}{r}. \tag{11.54}$$

Now you see why it was unnecessary to carry the approximation of $\mathscr{r}$ beyond the zeroth order ($\mathscr{r} \cong r$): $\mathbf{p}$ is *already* first order in r', and any refinements would be corrections of *second* order.

Next we must calculate the fields. Once again, we are interested in the radiation zone (that is, in the fields that survive at large distances from the source), so we keep only those terms that go like $1/r$:

$$\textbf{approximation 3}: \; \text{discard } 1/r^2 \text{ terms in } \mathbf{E} \text{ and } \mathbf{B}. \tag{11.55}$$

For instance, the Coulomb field,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}},$$

coming from the first term in Eq. 11.51, does not contribute to the electromagnetic radiation. In fact, the radiation comes entirely from those terms in which we differentiate the argument t_0. From Eq. 11.48 it follows that

$$\nabla t_0 = -\frac{1}{c}\nabla r = -\frac{1}{c}\hat{\mathbf{r}},$$

and hence

$$\nabla V \cong \nabla\left[\frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right] \cong \frac{1}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)}{rc}\right]\nabla t_0 = -\frac{1}{4\pi\epsilon_0 c^2}\frac{[\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)]}{r}\hat{\mathbf{r}}.$$

Similarly,

$$\nabla\times\mathbf{A} \cong \frac{\mu_0}{4\pi r}[\nabla\times\dot{\mathbf{p}}(t_0)] = \frac{\mu_0}{4\pi r}[(\nabla t_0)\times\ddot{\mathbf{p}}(t_0)] = -\frac{\mu_0}{4\pi rc}[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_0)],$$

while

$$\frac{\partial\mathbf{A}}{\partial t} \cong \frac{\mu_0}{4\pi}\frac{\ddot{\mathbf{p}}(t_0)}{r}.$$

So

$$\boxed{\mathbf{E}(\mathbf{r},t) \cong \frac{\mu_0}{4\pi r}[(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}})\hat{\mathbf{r}} - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})],} \tag{11.56}$$

where $\ddot{\mathbf{p}}$ is evaluated at time $t_0 = t - r/c$, and

$$\boxed{\mathbf{B}(\mathbf{r},t) \cong -\frac{\mu_0}{4\pi rc}[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}].} \tag{11.57}$$

In particular, if we use spherical polar coordinates, with the z axis in the direction of $\ddot{\mathbf{p}}(t_0)$, then

$$\left.\begin{aligned} \mathbf{E}(r,\theta,t) &\cong \frac{\mu_0\ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\theta}}, \\ \mathbf{B}(r,\theta,t) &\cong \frac{\mu_0\ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\phi}}. \end{aligned}\right\}. \tag{11.58}$$

The Poynting vector is

$$\mathbf{S} \cong \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{16\pi^2 c}[\ddot{p}(t_0)]^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}}, \tag{11.59}$$

and the total radiated power is

$$P \cong \int \mathbf{S}\cdot d\mathbf{a} = \frac{\mu_0\ddot{p}^2}{6\pi c}. \tag{11.60}$$

Notice that $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular, transverse to the direction of propagation ($\hat{\mathbf{r}}$), and in the ratio $E/B = c$, as always for radiation fields.

Example 11.2

(a) In the case of an oscillating electric dipole,

$$p(t) = p_0 \cos(\omega t), \quad \ddot{p}(t) = -\omega^2 p_0 \cos(\omega t),$$

and we recover the results of Sect. 11.1.2.

(b) For a single point charge q, the dipole moment is

$$\mathbf{p}(t) = q\mathbf{d}(t),$$

where $\mathbf{d}$ is the position of q with respect to the origin. Accordingly,

$$\ddot{\mathbf{p}}(t) = q\mathbf{a}(t),$$

where $\mathbf{a}$ is the acceleration of the charge. In this case the power radiated (Eq. 11.60) is

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}. \tag{11.61}$$

This is the famous **Larmor formula**; I'll derive it again, by rather different means, in the next section. Notice that the power radiated by a point charge is proportional to the *square* of its *acceleration*.

What I have done in this section amounts to a multipole expansion of the retarded potentials, carried to the lowest order in r' that is capable of producing electromagnetic radiation (fields that go like $1/r$). This turns out to be the electric dipole term. Because charge is conserved, an electric *monopole* does not radiate—if charge were *not* conserved, the first term in Eq. 11.51 would read

$$V_{\text{mono}} = \frac{1}{4\pi\epsilon_0}\frac{Q(t_0)}{r},$$

and we would get a monopole field proportional to $1/r$:

$$\mathbf{E}_{\text{mono}} = \frac{1}{4\pi\epsilon_0 c}\frac{\dot{Q}(t_0)}{r}\hat{\mathbf{r}}.$$

You might think that a charged sphere whose radius oscillates in and out would radiate, but it *doesn't*—the field outside, according to Gauss's law, is exactly $(Q/4\pi\epsilon_0 r^2)\hat{\mathbf{r}}$, regardless of the fluctuations in size. (In the acoustical analog, by the way, monopoles *do* radiate: witness the croak of a bullfrog.)

If the electric dipole moment should happen to vanish (or, at any rate, if its second time derivative is zero), then there is no electric dipole radiation, and one must look to the next term: the one of *second* order in r'. As it happens, this term can be separated into two parts, one of which is related to the *magnetic* dipole moment of the source, the other to its electric *quadrupole* moment. (The former is a generalization of the magnetic dipole radiation we considered in Sect. 11.1.3.) If the magnetic dipole and electric quadrupole contributions vanish, the $(r')^3$ term must be considered. This yields magnetic quadrupole and electric octopole radiation ... and so it goes.

Problem 11.8 Apply Eqs. 11.59 and 11.60 to the rotating dipole of Prob. 11.4. Explain any apparent discrepancies with your previous answer.

Problem 11.9 An insulating circular ring (radius b) lies in the xy plane, centered at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin\phi$, where λ_0 is constant and ϕ is the usual azimuthal angle. The ring is now set spinning at a constant angular velocity ω about the z axis. Calculate the power radiated.

Problem 11.10 An electron is released from rest and falls under the influence of gravity. In the first centimeter, what fraction of the potential energy lost is radiated away?

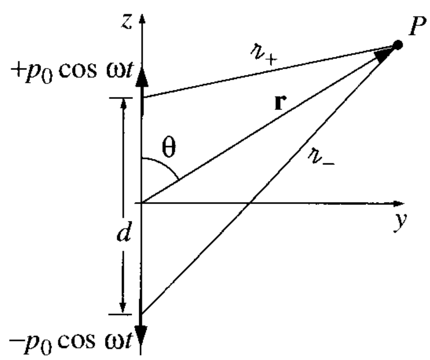

Figure 11.10

! **Problem 11.11** As a model for electric quadrupole radiation, consider two oppositely oriented oscillating electric dipoles, separated by a distance d, as shown in Fig. 11.10. Use the results of Sect. 11.1.2 for the potentials of each dipole, but note that they are *not* located at the origin. Keeping only the terms of first order in d:

(a) Find the scalar and vector potentials.

(b) Find the electric and magnetic fields.

(c) Find the Poynting vector and the power radiated. Sketch the intensity profile as a function of θ.

! **Problem 11.12** A current $I(t)$ flows around the circular ring in Fig. 11.8. Derive the general formula for the power radiated (analogous to Eq. 11.60), expressing your answer in terms of the magnetic dipole moment ($m(t)$) of the loop. [*Answer:* $P = \mu_0 \ddot{m}^2/6\pi c^3$]

11.2 Point Charges

11.2.1 Power Radiated by a Point Charge

In Chapter 10 we derived the fields of a point charge q in arbitrary motion (Eqs. 10.65 and 10.66):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})], \tag{11.62}$$

where $\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$, and

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t). \tag{11.63}$$

The first term in Eq. 11.62 is called the **velocity field**, and the second one (with the triple cross-product) is called the **acceleration field.**

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0 c}[\mathbf{E} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E})] = \frac{1}{\mu_0 c}[E^2 \hat{\boldsymbol{\mathscr{r}}} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{E})\mathbf{E}]. \tag{11.64}$$

However, not all of this energy flux constitutes *radiation*; some of it is just field energy carried along by the particle as it moves. The *radiated* energy is the stuff that, in effect, *detaches* itself from the charge and propagates off to infinity. (It's like flies breeding on a garbage truck: Some of them hover around the truck as it makes its rounds; others fly away and never come back.) To calculate the total power radiated by the particle at time t_r, we draw a huge sphere of radius $\mathscr{r}$ (Fig. 11.11), centered at the position of the particle (at time t_r), wait the appropriate interval

$$t - t_r = \frac{\mathscr{r}}{c} \tag{11.65}$$

for the radiation to reach the sphere, and at that moment integrate the Poynting vector over the surface.[6] I have used the notation t_r because, in fact, this *is* the retarded time for all points on the sphere at time t.

Now, the area of the sphere is proportional to $\mathscr{r}^2$, so any term in $\mathbf{S}$ that goes like $1/\mathscr{r}^2$ will yield a finite answer, but terms like $1/\mathscr{r}^3$ or $1/\mathscr{r}^4$ will contribute nothing in the limit $\mathscr{r} \to \infty$. For this reason only the *acceleration* fields represent true radiation (hence their other name, **radiation fields**):

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} [\boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})]. \tag{11.66}$$

[6]Note the subtle change in strategy here: In Sect. 11.1 we worked from a fixed point (the origin), but here it is more appropriate to use the (moving) location of the charge. The implications of this change in perspective will become clearer in a moment.

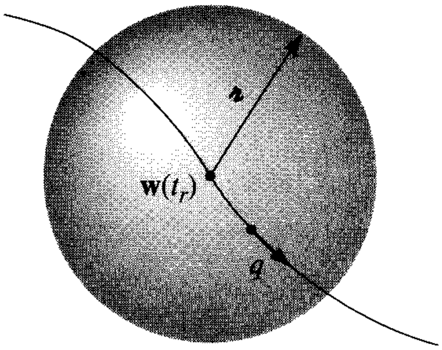

Figure 11.11

The velocity fields carry energy, to be sure, and as the charge moves this energy is dragged along—but it's not *radiation.* (It's like the flies that stay with the garbage truck.) Now $\mathbf{E}_{\text{rad}}$ is perpendicular to $\hat{\boldsymbol{\imath}}$, so the second term in Eq. 11.64 vanishes:

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c} E_{\text{rad}}^2 \, \hat{\boldsymbol{\imath}}. \tag{11.67}$$

If the charge is instantaneously at *rest* (at time t_r), then $\mathbf{u} = c\hat{\boldsymbol{\imath}}$, and

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2 \imath}[\hat{\boldsymbol{\imath}} \times (\hat{\boldsymbol{\imath}} \times \mathbf{a})] = \frac{\mu_0 q}{4\pi \imath}[(\hat{\boldsymbol{\imath}} \cdot \mathbf{a})\hat{\boldsymbol{\imath}} - \mathbf{a}]. \tag{11.68}$$

In that case

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi \imath}\right)^2 [a^2 - (\hat{\boldsymbol{\imath}} \cdot \mathbf{a})^2]\hat{\boldsymbol{\imath}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\imath^2}\right)\hat{\boldsymbol{\imath}}, \tag{11.69}$$

where θ is the angle between $\hat{\boldsymbol{\imath}}$ and $\mathbf{a}$. No power is radiated in the forward or backward direction—rather, it is emitted in a donut about the direction of instantaneous acceleration (Fig. 11.12).

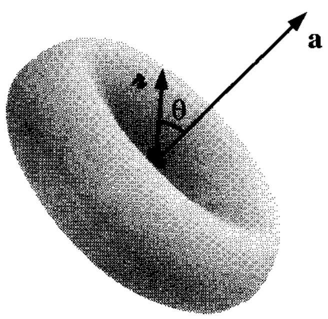

Figure 11.12

The total power radiated is evidently

$$P = \oint \mathbf{S}_{\text{rad}} \cdot d\mathbf{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{\mathscr{r}^2} \mathscr{r}^2 \sin\theta \, d\theta \, d\phi,$$

or

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}.} \tag{11.70}$$

This, again, is the **Larmor formula**, which we obtained earlier by another route (Eq. 11.61).

Although I *derived* them on the assumption that $v = 0$, Eqs. 11.69 and 11.70 actually hold to good approximation as long as $v \ll c$. An exact treatment of the case $v \neq 0$ is more difficult,[7] both for the obvious reason that $\mathbf{E}_{\text{rad}}$ is more complicated, and also for the more subtle reason that $\mathbf{S}_{\text{rad}}$, the rate at which energy passes through the sphere, is *not* the same as the rate at which energy left the particle. Suppose someone is firing a stream of bullets out the window of a moving car (Fig.11.13). The rate N_t at which the bullets strike a stationary target is not the same as the rate N_g at which they left the gun, because of the motion of the car. In fact, you can easily check that $N_g = (1 - v/c)N_t$, if the car is moving towards the target, and

$$N_g = \left(1 - \frac{\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v}}{c}\right) N_t$$

for arbitrary directions (here $\mathbf{v}$ is the velocity of the car, c is that of the bullets—relative to the ground—and $\hat{\boldsymbol{\mathscr{r}}}$ is a unit vector from car to target). In our case, if dW/dt is the rate at which energy passes through the sphere at radius $\mathscr{r}$, then the rate at which energy left the charge was

$$\frac{dW}{dt_r} = \frac{dW/dt}{\partial t_r \partial t} = \left(\frac{\boldsymbol{\mathscr{r}} \cdot \mathbf{u}}{\mathscr{r} c}\right) \frac{dW}{dt}. \tag{11.71}$$

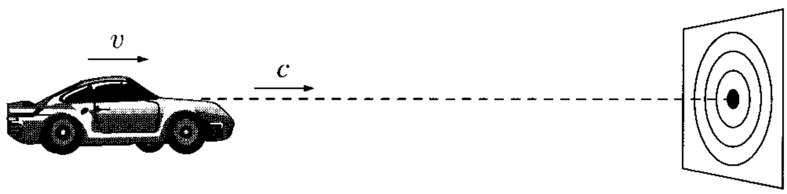

Figure 11.13

[7] In the context of special relativity, the condition $v = 0$ simply represents an astute choice of reference system, with no essential loss of generality. If you can decide how P transforms, you can *deduce* the general (Liénard) result from the $v = 0$ (Larmor) formula (see Prob. 12.69).

(I used Eq. 10.71 to express $\partial t_r/\partial t$.) But

$$\frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}{\mathscr{r}c}=1-\frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{v}}{c},$$

which is precisely the ratio of N_g to N_t; it's a purely geometrical factor (the same as in the Doppler effect).

The power radiated by the particle into a patch of area $\mathscr{r}^2\sin\theta\,d\theta\,d\phi=\mathscr{r}^2\,d\Omega$ on the sphere is therefore given by

$$\frac{dP}{d\Omega}=\left(\frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}{\mathscr{r}c}\right)\frac{1}{\mu_0 c}E_{\text{rad}}^2\mathscr{r}^2=\frac{q^2}{16\pi^2\epsilon_0}\frac{|\hat{\boldsymbol{\mathscr{r}}}\times(\mathbf{u}\times\mathbf{a})|^2}{(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{u})^5},\tag{11.72}$$

where $d\Omega=\sin\theta\,d\theta\,d\phi$ is the **solid angle** into which this power is radiated. Integrating over θ and ϕ to get the total power radiated is no picnic, and for once I shall simply quote the answer:

$$P=\frac{\mu_0 q^2\gamma^6}{6\pi c}\left(a^2-\left|\frac{\mathbf{v}\times\mathbf{a}}{c}\right|^2\right),\tag{11.73}$$

where $\gamma\equiv 1/\sqrt{1-v^2/c^2}$. This is **Liénard's generalization** of the Larmor formula (to which it reduces when $v\ll c$). The factor γ^6 means that the radiated power increases enormously as the particle velocity approaches the speed of light.

Example 11.3

Suppose $\mathbf{v}$ and $\mathbf{a}$ are instantaneously collinear (at time t_r), as, for example, in straight-line motion. Find the angular distribution of the radiation (Eq. 11.72) and the total power emitted.

Solution: In this case $(\mathbf{u}\times\mathbf{a})=c(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a})$, so

$$\frac{dP}{d\Omega}=\frac{q^2c^2}{16\pi^2\epsilon_0}\frac{|\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a})|^2}{(c-\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{v})^5}.$$

Now

$$\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a})=(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})\hat{\boldsymbol{\mathscr{r}}}-\mathbf{a},\quad\text{so }|\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a})|^2=a^2-(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})^2.$$

In particular, if we let the z axis point along $\mathbf{v}$, then

$$\frac{dP}{d\Omega}=\frac{\mu_0 q^2a^2}{16\pi^2c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5},\tag{11.74}$$

where $\beta\equiv v/c$. This is consistent, of course, with Eq. 11.69, in the case $v=0$. However, for very large v ($\beta\approx 1$) the donut of radiation (Fig. 11.12) is stretched out and pushed forward by the factor $(1-\beta\cos\theta)^{-5}$, as indicated in Fig. 11.14. Although there is still no radiation in *precisely* the forward direction, most of it is concentrated within an increasingly narrow cone *about* the forward direction (see Prob. 11.15).

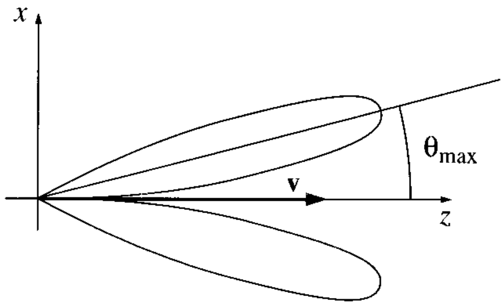

Figure 11.14

The *total* power emitted is found by integrating Eq. 11.74 over all angles:

$$P = \int \frac{dP}{d\Omega}\, d\Omega = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \sin\theta\, d\theta\, d\phi.$$

The ϕ integral is 2π; the θ integral is simplified by the substitution $x \equiv \cos\theta$:

$$P = \frac{\mu_0 q^2 a^2}{8\pi c} \int_{-1}^{+1} \frac{(1-x^2)}{(1-\beta x)^5}\, dx.$$

Integration by parts yields $\frac{4}{3}(1-\beta^2)^{-3}$, and I conclude that

$$P = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}. \tag{11.75}$$

This result is consistent with the Liénard formula (Eq. 11.73), for the case of collinear **v** and **a**. Notice that the angular distribution of the radiation is the same whether the particle is *accelerating* or *decelerating*; it only depends on the *square* of a, and is concentrated in the forward direction (with respect to the velocity) in either case. When a high speed electron hits a metal target it rapidly decelerates, giving off what is called **bremsstrahlung**, or "braking radiation." What I have described in this example is essentially the classical theory of bremsstrahlung.

Problem 11.13

(a) Suppose an electron decelerated at a constant rate a from some initial velocity v_0 down to zero. What fraction of its initial kinetic energy is lost to radiation? (The rest is absorbed by whatever mechanism keeps the acceleration constant.) Assume $v_0 \ll c$ so that the Larmor formula can be used.

(b) To get a sense of the numbers involved, suppose the initial velocity is thermal (around 10^5 m/s) and the distance the electron goes is 30 Å. What can you conclude about radiation losses for the electrons in an ordinary conductor?

Problem 11.14 In Bohr's theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius 5×10^{-11}m, held in orbit by the Coulomb attraction of the proton.

According to classical electrodynamics, this electron should radiate, and hence spiral in to the nucleus. Show that $v \ll c$ for most of the trip (so you can use the Larmor formula), and calculate the lifespan of Bohr's atom. (Assume each revolution is essentially circular.)

Problem 11.15 Find the angle $\theta_{\max}$ at which the maximum radiation is emitted, in Ex. 11.3 (see Fig. 11.14). Show that for ultrarelativistic speeds (v close to c), $\theta_{\max} \cong \sqrt{(1-\beta)/2}$. What is the intensity of the radiation in this maximal direction (in the ultrarelativistic case), in proportion to the same quantity for a particle instantaneously at rest? Give your answer in terms of γ.

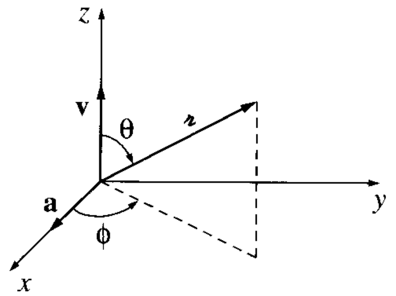

Figure 11.15

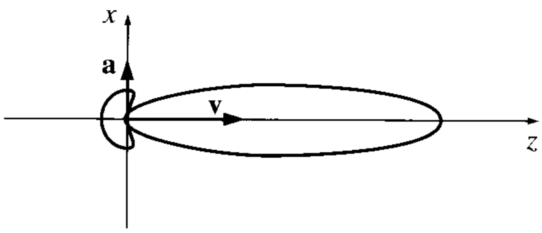

Figure 11.16

! **Problem 11.16** In Ex. 11.3 we assumed the velocity and acceleration were (instantaneously, at least) *collinear.* Carry out the same analysis for the case where they are *perpendicular.* Choose your axes so that $\mathbf{v}$ lies along the z axis and $\mathbf{a}$ along the x axis (Fig. 11.15), so that $\mathbf{v} = v\,\hat{\mathbf{z}}$, $\mathbf{a} = a\,\hat{\mathbf{x}}$, and $\hat{\boldsymbol{\mathscr{r}}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$. Check that P is consistent with the Liénard formula. [*Answer:*

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{[(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi]}{(1-\beta\cos\theta)^5}, \quad P = \frac{\mu_0 q^2 a^2 \gamma^4}{6\pi c}.$$

For relativistic velocities ($\beta \approx 1$) the radiation is again sharply peaked in the forward direction (Fig. 11.16). The most important application of these formulas is to *circular* motion—in this case the radiation is called **synchrotron radiation.** For a relativistic electron the radiation sweeps around like a locomotive's headlight as the particle moves.]

11.2.2 Radiation Reaction

According to the laws of classical electrodynamics, an accelerating charge radiates. This radiation carries off energy, which must come at the expense of the particle's kinetic energy. Under the influence of a given force, therefore, a charged particle accelerates *less* than a neutral one of the same mass. The radiation evidently exerts a force ($\mathbf{F}_{\text{rad}}$) back on the charge—a *recoil* force, rather like that of a bullet on a gun. In this section we'll derive the

radiation reaction force from conservation of energy. Then in the next section I'll show you the actual *mechanism* responsible, and derive the reaction force again in the context of a simple model.

For a nonrelativistic particle ($v \ll c$) the total power radiated is given by the Larmor formula (Eq. 11.70):

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}. \tag{11.76}$$

Conservation of energy suggests that this is also the rate at which the particle *loses* energy, under the influence of the radiation reaction force $\mathbf{F}_{\text{rad}}$:

$$\mathbf{F}_{\text{rad}} \cdot \mathbf{v} = -\frac{\mu_0 q^2 a^2}{6\pi c}. \tag{11.77}$$

I say "suggests" advisedly, because this equation is actually *wrong*. For we calculated the radiated power by integrating the Poynting vector over a sphere of "infinite" radius; in this calculation the *velocity* fields played no part, since they fall off too rapidly as a function of $\mathscr{r}$ to make any contribution. But the velocity fields *do* carry energy—they just don't transport it out to infinity. As the particle accelerates and decelerates energy is exchanged between it and the velocity fields, at the same time as energy is irretrievably radiated away by the acceleration fields. Equation 11.77 accounts only for the latter, but if we want to know the recoil force exerted by the fields on the charge, we need to consider the *total* power lost at any instant, not just the portion that eventually escapes in the form of radiation. (The term "radiation reaction" is a misnomer. We should really call it the *field reaction*. In fact, we'll soon see that $\mathbf{F}_{\text{rad}}$ is determined by the *time derivative* of the acceleration and can be nonzero even when the acceleration itself is instantaneously zero, so that the particle is not radiating.)

The energy lost by the particle in any given time interval, then, must equal the energy carried away by the radiation *plus* whatever extra energy has been pumped into the velocity fields.[8] However, if we agree to consider only intervals over which the system returns to its initial state, then the energy in the velocity fields is the same at both ends, and the only *net* loss is in the form of radiation. Thus Eq. 11.77, while incorrect *instantaneously*, is valid on the *average*:

$$\int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\, dt = -\frac{\mu_0 q^2}{6\pi c}\int_{t_1}^{t_2} a^2\, dt, \tag{11.78}$$

with the stipulation that the *state of the system is identical at t_1 and t_2*. In the case of periodic motion, for instance, we must integrate over an integral number of full cycles.[9] Now, the

[8] Actually, while the total field is the sum of velocity and acceleration fields, $\mathbf{E} = \mathbf{E}_v + \mathbf{E}_a$, the *energy* is proportional to $E^2 = E_v^2 + 2\mathbf{E}_v \cdot \mathbf{E}_a + E_a^2$ and contains *three* terms: energy stored in the velocity fields alone (E_v^2), energy radiated away (E_a^2), and a *cross* term $\mathbf{E}_v \cdot \mathbf{E}_a$. For the sake of simplicity, I'm referring to the *combination* ($E_v^2 + 2\mathbf{E}_v \cdot \mathbf{E}_a$) as "energy stored in the velocity fields." These terms go like $1/\mathscr{r}^4$ and $1/\mathscr{r}^3$, respectively, so neither one contributes to the radiation.

[9] For *non*periodic motion the condition that the energy in the velocity fields be the same at t_1 and t_2 is more difficult to achieve. It is not enough that the instantaneous velocities and accelerations be equal, since the fields farther out depend on v and a at *earlier times*. In *principle*, then, v and a *and all higher derivatives* must be identical at t_1 and t_2. In *practice*, since the velocity fields fall off rapidly with $\mathscr{r}$, it is sufficient that v and a be the same over a brief interval prior to t_1 and t_2.

right side of Eq. 11.78 can be integrated by parts:

$$\int_{t_1}^{t_2} a^2\,dt = \int_{t_1}^{t_2} \left(\frac{d\mathbf{v}}{dt}\right)\cdot\left(\frac{d\mathbf{v}}{dt}\right) dt = \left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d^2\mathbf{v}}{dt^2}\cdot\mathbf{v}\,dt.$$

The boundary term drops out, since the velocities and accelerations are identical at t_1 and t_2, so Eq. 11.78 can be written equivalently as

$$\int_{t_1}^{t_2} \left(\mathbf{F}_{\text{rad}} - \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}\right)\cdot\mathbf{v}\,dt = 0. \tag{11.79}$$

Equation 11.79 will certainly be satisfied if

$$\boxed{\mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}.} \tag{11.80}$$

This is the **Abraham-Lorentz formula** for the radiation reaction force.

Of course, Eq. 11.79 doesn't *prove* Eq. 11.80. It tells you nothing whatever about the component of $\mathbf{F}_{\text{rad}}$ *perpendicular* to $\mathbf{v}$; and it only tells you the *time average* of the parallel component—the average, moreover, over very special time intervals. As we'll see in the next section, there are other reasons for believing in the Abraham-Lorentz formula, but for now the best that can be said is that it represents the *simplest* form the radiation reaction force could take, consistent with conservation of energy.

The Abraham-Lorentz formula has disturbing implications, which are not entirely understood nearly a century after the law was first proposed. For suppose a particle is subject to no *external* forces; then Newton's second law says

$$F_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{a} = ma,$$

from which it follows that

$$a(t) = a_0 e^{t/\tau}, \tag{11.81}$$

where

$$\tau \equiv \frac{\mu_0 q^2}{6\pi m c}. \tag{11.82}$$

(In the case of the electron, $\tau = 6\times 10^{-24}$s.) The acceleration spontaneously *increases* exponentially with time! This absurd conclusion can be avoided if we insist that $a_0 = 0$, but it turns out that the systematic exclusion of such **runaway solutions** has an even more unpleasant consequence: If you *do* apply an external force, the particle starts to respond *before the force acts!* (See Prob. 11.19.) This **acausal preacceleration** jumps the gun by only a short time τ; nevertheless, it is (to my mind) philosophically repugnant that the theory should countenance it at all.[10]

[10]These difficulties persist in the relativistic version of the Abraham-Lorentz equation, which can be derived by starting with Liénard's formula instead of Larmor's (see Prob. 12.70). Perhaps they are telling us that there can be no such thing as a point charge in classical electrodynamics, or maybe they presage the onset of quantum mechanics. For guides to the literature see Philip Pearle's chapter in D. Teplitz, ed., *Electromagnetism: Paths to Research* (New York: Plenum. 1982) and F. Rohrlich, *Am. J. Phys.* **65**. 1051 (1997).

Example 11.4

Calculate the **radiation damping** of a charged particle attached to a spring of natural frequency ω_0, driven at frequency ω.

Solution: The equation of motion is

$$m\ddot{x} = F_{\text{spring}} + F_{\text{rad}} + F_{\text{driving}} = -m\omega_0^2 x + m\tau\dddot{x} + F_{\text{driving}}.$$

With the system oscillating at frequency ω,

$$x(t) = x_0 \cos(\omega t + \delta),$$

so

$$\dddot{x} = -\omega^2 \dot{x}.$$

Therefore

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = F_{\text{driving}}, \tag{11.83}$$

and the damping factor γ is given by

$$\gamma = \omega^2 \tau. \tag{11.84}$$

[When I wrote $F_{\text{damping}} = -\gamma m v$, back in Chap. 9 (Eq. 9.152), I assumed for simplicity that the damping was proportional to the velocity. We now know that *radiation* damping, at least. is proportional to $\ddot{v}$. But it hardly matters: for sinusoidal oscillations *any* even number of derivatives of v would do, since they're all proportional to v.]

Problem 11.17

(a) A particle of charge q moves in a circle of radius R at a constant speed v. To sustain the motion, you must, of course, provide a centripetal force mv^2/R; what *additional* force ($\mathbf{F}_e$) must you exert, in order to counteract the radiation reaction? [It's easiest to express the answer in terms of the instantaneous velocity $\mathbf{v}$.] What power (P_e) does this extra force deliver? Compare P_e with the power radiated (use the Larmor formula).

(b) Repeat part (a) for a particle in simple harmonic motion with amplitude A and angular frequency ω ($\mathbf{w}(t) = A\cos(\omega t)\,\hat{\mathbf{z}}$). Explain the discrepancy.

(c) Consider the case of a particle in free fall (constant acceleration g). What is the radiation reaction force? What is the power radiated? Comment on these results.

Problem 11.18

(a) Assuming (implausibly) that γ is entirely attributable to radiation damping (Eq. 11.84), show that for optical dispersion the damping is "small" ($\gamma \ll \omega_0$). Assume that the relevant resonances lie in or near the optical frequency range.

(b) Using your results from Prob. 9.24, estimate the width of the anomalous dispersion region, for the model in Prob. 9.23.

! **Problem 11.19** With the inclusion of the radiation reaction force (Eq. 11.80), Newton's second law for a charged particle becomes

$$a = \tau\dot{a} + \frac{F}{m},$$

where F is the external force acting on the particle.

(a) In contrast to the case of an *uncharged* particle ($a = F/m$), acceleration (like position and velocity) must now be a *continuous* function of time, even if the force changes abruptly. (Physically, the radiation reaction damps out any rapid change in a.) *Prove* that a is continuous at any time t, by integrating the equation of motion above from $(t - \epsilon)$ to $(t + \epsilon)$ and taking the limit $\epsilon \to 0$.

(b) A particle is subjected to a constant force F, beginning at time $t = 0$ and lasting until time T. Find the most general solution $a(t)$ to the equation of motion in each of the three periods: (i) $t < 0$; (ii) $0 < t < T$; (iii) $t > T$.

(c) Impose the continuity condition (a) at $t = 0$ and $t = T$. Show that you can *either* eliminate the runaway in region (iii) *or* avoid preacceleration in region (i), *but not both.*

(d) If you choose to eliminate the runaway, what is the acceleration as a function of time, in each interval? How about the velocity? (The latter must, of course, be continuous at $t = 0$ and $t = T$.) Assume the particle was originally at rest: $v(-\infty) = 0$.

(e) Plot $a(t)$ and $v(t)$, both for an *uncharged* particle and for a (nonrunaway) charged particle, subject to this force.

11.2.3 The Physical Basis of the Radiation Reaction

In the last section I derived the Abraham-Lorentz formula for the radiation reaction, using conservation of energy. I made no attempt to identify the actual *mechanism* responsible for this force, except to point out that it must be a recoil effect of the particle's own fields acting back on the charge. Unfortunately, the fields of a point charge blow up right at the particle, so it's hard to see how one can calculate the force they exert.[11] Let's avoid this problem by considering an *extended* charge distribution, for which the field is finite everywhere; at the end, we'll take the limit as the size of the charge goes to zero. In general, the electromagnetic force of one part (A) on another part (B) is *not* equal and opposite to the force of B on A (Fig. 11.17). If the distribution is divided up into infinitesimal chunks, and the imbalances are added up for all such pairs, the result is a *net force of the charge* on itself. It is this **self-force**, resulting from the breakdown of Newton's third law within the structure of the particle, that accounts for the radiation reaction.

Lorentz originally calculated the electromagnetic self-force using a spherical charge distribution, which seems reasonable but makes the mathematics rather cumbersome.[12] Because I am only trying to elucidate the *mechanism* involved, I shall use a less realistic model: a "dumbbell" in which the total charge q is divided into two halves separated by

[11] It can be done by a suitable averaging of the field, but it's not easy. See T. H. Boyer, *Am. J. Phys.* **40**, 1843 (1972), and references cited there.

[12] See J. D. Jackson, *Classical Electrodynamics*, 3rd ed., Sect. 16.3 (New York: John Wiley, 1999).

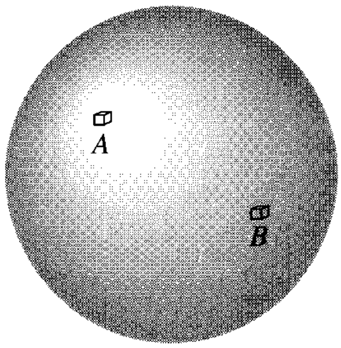

Figure 11.17

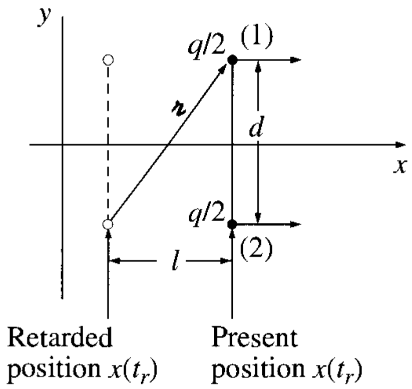

Figure 11.18

a fixed distance d (Fig. 11.18). This is the simplest possible arrangement of the charge that permits the essential mechanism (imbalance of internal electromagnetic forces) to function. Never mind that it's an unlikely model for an elementary particle: in the point limit ($d \to 0$) *any* model must yield the Abraham-Lorentz formula, to the extent that conservation of energy alone dictates that answer.

Let's assume the dumbbell moves in the x direction, and is (instantaneously) at rest at the retarded time. The electric field at (1) due to (2) is

$$\mathbf{E}_1 = \frac{(q/2)}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u} - (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}] \tag{11.85}$$

(Eq. 10.65), where

$$\mathbf{u} = c\,\hat{\boldsymbol{\mathscr{r}}} \quad \text{and} \quad \boldsymbol{\mathscr{r}} = l\,\hat{\mathbf{x}} + d\,\hat{\mathbf{y}}, \tag{11.86}$$

so that

$$\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = c\mathscr{r}, \quad \boldsymbol{\mathscr{r}}\cdot\mathbf{a} = la, \quad \text{and} \quad \mathscr{r} = \sqrt{l^2 + d^2}. \tag{11.87}$$

Actually, we're only interested in the x component of $\mathbf{E}_1$, since the y components will cancel when we add the forces on the two ends (for the same reason, we don't need to worry about the magnetic forces). Now

$$u_x = \frac{cl}{\mathscr{r}}, \tag{11.88}$$

and hence

$$E_{1_x} = \frac{q}{8\pi\epsilon_0 c^2}\frac{(lc^2 - ad^2)}{(l^2 + d^2)^{3/2}}. \tag{11.89}$$

By symmetry, $E_{2_x} = E_{1_x}$, so the net force on the dumbbell is

$$\mathbf{F}_{\text{self}} = \frac{q}{2}(\mathbf{E}_1 + \mathbf{E}_2) = \frac{q^2}{8\pi\epsilon_0 c^2}\frac{(lc^2 - ad^2)}{(l^2 + d^2)^{3/2}}\,\hat{\mathbf{x}}. \tag{11.90}$$

So far everything is exact. The idea now is to expand in powers of d; when the size of the particle goes to zero, all *positive* powers will disappear. Using Taylor's theorem

$$x(t) = x(t_r) + \dot{x}(t_r)(t - t_r) + \frac{1}{2}\ddot{x}(t_r)(t - t_r)^2 + \frac{1}{3!}\dddot{x}(t_r)(t - t_r)^3 + \cdots ,$$

we have,

$$l = x(t) - x(t_r) = \frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^3 + \cdots, \tag{11.91}$$

where $T \equiv t - t_r$, for short. Now T is determined by the retarded time condition

$$(cT)^2 = l^2 + d^2, \tag{11.92}$$

so

$$d = \sqrt{(cT)^2 - l^2} = cT\sqrt{1 - \left(\frac{aT}{2c} + \frac{\dot{a}T^2}{6c} + \cdots\right)^2} = cT - \frac{a^2}{8c}T^3 + (\ )T^4 + \cdots .$$

This equation tells us d, in terms of T; we need to "solve" it for T as a function of d. There's a systematic procedure for doing this, known as **reversion of series,**[13] but we can get the first couple of terms more informally as follows: Ignoring all higher powers of T,

$$d \cong cT \quad \Rightarrow \quad T \cong \frac{d}{c};$$

using this as an approximation for the cubic term,

$$d \cong cT - \frac{a^2}{8c}\frac{d^3}{c^3} \quad \Rightarrow \quad T \cong \frac{d}{c} + \frac{a^2 d^3}{8c^5},$$

and so on. Evidently

$$T = \frac{1}{c}d + \frac{a^2}{8c^5}d^3 + (\ )d^4 + \cdots . \tag{11.93}$$

Returning to Eq. 11.91, we construct the power series for l in terms of d:

$$l = \frac{a}{2c^2}d^2 + \frac{\dot{a}}{6c^3}d^3 + (\ )d^4 + \cdots . \tag{11.94}$$

Putting this into Eq. 11.90, I conclude that

$$\mathbf{F}_{\text{self}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a}{4c^2 d} + \frac{\dot{a}}{12c^3} + (\ )d + \cdots\right]\hat{\mathbf{x}}. \tag{11.95}$$

Here a and $\dot{a}$ are evaluated at the *retarded* time (t_r), but it's easy to rewrite the result in terms of the *present* time t:

$$a(t_r) = a(t) + \dot{a}(t)(t - t_r) + \cdots = a(t) - \dot{a}(t)T + \cdots = a(t) - \dot{a}(t)\frac{d}{c} + \cdots ,$$

[13]See, for example, the *CRC Standard Mathematical Tables* (Cleveland: CRC Press).

and it follows that

$$\mathbf{F}_{\text{self}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a(t)}{4c^2 d} + \frac{\dot{a}(t)}{3c^3} + (\ )d + \cdots\right]\hat{\mathbf{x}}. \tag{11.96}$$

The first term on the right is proportional to the acceleration of the charge; if we pull it over to the other side of Newton's second law, it simply adds to the dumbbell's mass. In effect, the total inertia of the charged dumbbell is

$$m = 2m_0 + \frac{1}{4\pi\epsilon_0}\frac{q^2}{4dc^2}, \tag{11.97}$$

where m_0 is the mass of either end alone. In the context of special relativity it is not surprising that the electrical repulsion of the charges should enhance the mass of the dumbbell. For the potential energy of this configuration (in the static case) is

$$\frac{1}{4\pi\epsilon_0}\frac{(q/2)^2}{d}, \tag{11.98}$$

and according to Einstein's formula $E = mc^2$, this energy contributes to the inertia of the object.[14]

The second term in Eq. 11.96 is the radiation reaction:

$$F_{\text{rad}}^{\text{int}} = \frac{\mu_0 q^2 \dot{a}}{12\pi c}. \tag{11.99}$$

It alone (apart from the mass correction[15]) survives in the "point dumbbell" limit $d \to 0$. Unfortunately, it differs from the Abraham-Lorentz formula by a factor of 2. But then, this is only the self-force associated with the *interaction* between 1 and 2—hence, the superscript "int." There remains the force of *each end on itself.* When the latter is included (see Prob. 11.20) the result is

$$F_{\text{rad}} = \frac{\mu_0 q^2 \dot{a}}{6\pi c}, \tag{11.100}$$

reproducing the Abraham-Lorentz formula exactly. *Conclusion: The radiation reaction is due to the force of the charge on itself*—or, more elaborately, the net force exerted by the fields generated by different parts of the charge distribution acting on one another.

[14] The fact that the *numbers* work out perfectly is a lucky feature of this configuration. If you do the same calculation for the dumbbell in *longitudinal* motion, the mass correction is only *half* of what it "should" be (there's a 2, instead of a 4, in Eq. 11.97), and for a sphere it's off by a factor of 3/4. This notorious paradox has been the subject of much debate over the years. See D. J. Griffiths and R. E. Owen, *Am. J. Phys.* **51**, 1120 (1983).

[15] Of course, the limit $d \to 0$ has an embarrassing effect on the mass term. In a sense, it doesn't matter, since only the *total* mass m is observable; maybe m_0 somehow has a compensating (negative!) infinity, so that m comes out finite. This awkward problem persists in *quantum* electrodynamics, where it is "swept under the rug" in a process known as **mass renormalization**.

Problem 11.20 Deduce Eq. 11.100 from Eq. 11.99, as follows:

(a) Use the Abraham-Lorentz formula to determine the radiation reaction on each end of the dumbbell; add this to the interaction term (Eq. 11.99).

(b) Method (a) has the defect that it *uses* the Abraham-Lorentz formula—the very thing that we were trying to *derive*. To avoid this, smear out the charge along a strip of length L oriented perpendicular to the motion (the charge density, then, is $\lambda = q/L$); find the cumulative interaction force for all pairs of segments, using Eq. 11.99 (with the correspondence $q/2 \to \lambda\, dy_1$, at one end and $q/2 \to \lambda\, dy_2$ at the other). Make sure you don't count the same pair twice.

More Problems on Chapter 11

Problem 11.21 A particle of mass m and charge q is attached to a spring with force constant k, hanging from the ceiling (Fig. 11.19). Its equilibrium position is a distance h above the floor. It is pulled down a distance d below equilibrium and released, at time $t = 0$.

(a) Under the usual assumptions ($d \ll \lambda \ll h$), calculate the intensity of the radiation hitting the floor, as a function of the distance R from the point directly below q. [*Note:* The intensity here is the average power per unit area of *floor*.] At what R is the radiation most intense? Neglect the radiative damping of the oscillator. [*Answer:* $\mu_0 q^2 d^2 \omega^4 R^2 h / 32\pi^2 c (R^2 + h^2)^{5/2}$]

(b) As a check on your formula, assume the floor is of infinite extent, and calculate the average energy per unit time striking the entire floor. Is it what you'd expect?

(c) Because it is losing energy in the form of radiation, the amplitude of the oscillation will gradually decrease. After what time τ has the amplitude been reduced to d/e? (Assume the fraction of the total energy lost in one cycle is very small.)

Problem 11.22 A radio tower rises to height h above flat horizontal ground. At the top is a magnetic dipole antenna, of radius b, with its axis vertical. FM station KRUD broadcasts from this antenna at angular frequency ω, with a total radiated power P (that's averaged, of course, over a full cycle). Neighbors have complained about problems they attribute to excessive

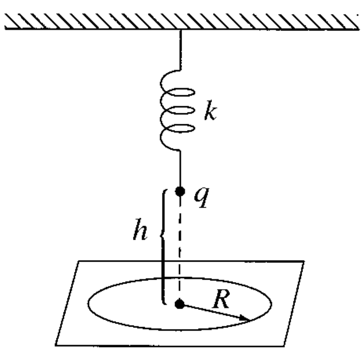

Figure 11.19

radiation from the tower—interference with their stereo systems, mechanical garage doors opening and closing mysteriously, and a variety of suspicious medical problems. But the city engineer who measured the radiation level at the base of the tower found it to be well below the accepted standard. You have been hired by the Neighborhood Association to assess the engineer's report.

(a) In terms of the variables given (not all of which may be relevant, of course), find the formula for the intensity of the radiation at ground level, a distance R from the base of the tower. You may assume that $a \ll c/\omega \ll h$. [*Note:* we are interested only in the *magnitude* of the radiation, not in its *direction*—when measurements are taken the detector will be aimed directly at the antenna.]

(b) How far from the base of the tower *should* the engineer have made the measurement? What is the formula for the intensity at this location?

(c) KRUD's actual power output is 35 kilowatts, its frequency is 90 MHz, the antenna's radius is 6 cm, and the height of the tower is 200 m. The city's radio-emission limit is 200 microwatts/cm^2. Is KRUD in compliance?

Problem 11.23 As you know, the magnetic north pole of the earth does not coincide with the geographic north pole—in fact, it's off by about 11°. Relative to the fixed axis of rotation, therefore, the magnetic dipole moment vector of the earth is changing with time, and the earth must be giving off magnetic dipole radiation.

(a) Find the formula for the total power radiated, in terms of the following parameters: Ψ (the angle between the geographic and magnetic north poles), M (the magnitude of the earth's magnetic dipole moment), and ω (the angular velocity of rotation of the earth). [*Hint:* refer to Prob. 11.4 or Prob. 11.12.]

(b) Using the fact that the earth's magnetic field is about half a gauss at the equator, estimate the magnetic dipole moment M of the earth.

(c) Find the power radiated. [*Answer:* 4×10^{-5} W]

(d) Pulsars are thought to be rotating neutron stars, with a typical radius of 10 km, a rotational period of $10^{-3} s$, and a surface magnetic field of 10^8 T. What sort of radiated power would you expect from such a star? [See J. P. Ostriker and J. E. Gunn, *Astrophys. J.* **157**, 1395 (1969).] [*Answer:* 2×10^{36} W]

Problem 11.24 Suppose the (electrically neutral) yz plane carries a time-dependent but uniform surface current $K(t)\,\hat{\mathbf{z}}$.

(a) Find the electric and magnetic fields at a height x above the plane if

(i) a constant current is turned on at $t = 0$:

$$K(t) = \begin{cases} 0, & t \le 0, \\ K_0, & t > 0. \end{cases}$$

(ii) a linearly increasing current is turned on at $t = 0$:

$$K(t) = \begin{cases} 0, & t \le 0, \\ \alpha t, & t > 0. \end{cases}$$

(b) Show that the retarded vector potential can be written in the form

$$\mathbf{A}(x,t) = \frac{\mu_0 c}{2}\,\hat{\mathbf{z}}\int_0^\infty K\left(t - \frac{x}{c} - u\right)du,$$

and from this determine **E** and **B**.

(c) Show that the total power radiated per unit area of surface is

$$\frac{\mu_0 c}{2}[K(t)]^2.$$

Explain what you mean by "radiation," in this case, given that the source is not localized. [For discussion and related problems, see B. R. Holstein, *Am. J. Phys.* **63**, 217 (1995), T. A. Abbott and D. J. Griffiths, *Am. J. Phys.* **53**, 1203 (1985).]

Problem 11.25 When a charged particle approaches (or leaves) a conducting surface, radiation is emitted, associated with the changing electric dipole moment of the charge and its image. If the particle has mass m and charge q, find the total radiated power, as a function of its height z above the plane. [*Answer:* $(\mu_0 c q^2/4\pi)^3/6m^2 z^4$]

Problem 11.26 Use the duality transformation (Prob. 7.60) to construct the electric and magnetic fields of a magnetic monopole q_m in arbitrary motion, and find the "Larmor formula" for the power radiated. [For related applications see J. A. Heras, *Am. J. Phys.* **63**, 242 (1995).]

Problem 11.27 Assuming you exclude the runaway solution in Prob. 11.19, calculate

(a) the work done by the external force,

(b) the final kinetic energy (assume the initial kinetic energy was zero),

(c) the total energy radiated.

Check that energy is conserved in this process.[16]

Problem 11.28

(a) Repeat Prob. 11.19, but this time let the external force be a Dirac delta function: $F(t) = k\delta(t)$ (for some constant k).[17] [Note that the acceleration is now *discontinuous* at $t = 0$ (though the *velocity* must still be continuous); use the method of Prob. 11.19 (a) to show that $\Delta a = -k/m\tau$. In this problem there are only *two* intervals to consider: (i) $t < 0$, and (ii) $t > 0$.]

(b) As in Prob. 11.27, check that energy is conserved in this process.

! **Problem 11.29** A charged particle, traveling in from $-\infty$ along the x axis, encounters a rectangular potential energy barrier

$$U(x) = \begin{cases} U_0, & \text{if } 0 < x < L, \\ 0, & \text{otherwise.} \end{cases}$$

Show that, because of the radiation reaction, it is possible for the particle to **tunnel** through the barrier—that is: even if the incident kinetic energy is less than U_0, the particle can pass

[16]Problems 11.27 and 11.28 were suggested by G. L. Pollack.

[17]This example was first analyzed by P. A. M. Dirac, *Proc. Roy. Soc.* **A167**, 148 (1938).

through. (See F. Denef *et al.*, *Phys. Rev.* E **56**, 3624 (1997).) [*Hint:* Your task is to solve the equation

$$a = \tau \dot{a} + \frac{F}{m},$$

subject to the force

$$F(x) = U_0[-\delta(x) + \delta(x - L)].$$

Refer to Probs. 11.19 and 11.28, but notice that this time the force is a specified function of x, not t. There are three regions to consider: (i) $x < 0$, (ii) $0 < x < L$, (iii) $x > L$. Find the general solution (for $a(t)$, $v(t)$, and $x(t)$) in each region, exclude the runaway in region (iii), and impose the appropriate boundary conditions at $x = 0$ and $x = L$. Show that the final velocity (v_f) is related to the time T spent traversing the barrier by the equation

$$L = v_f T - \frac{U_0}{m v_f}\left(\tau e^{-T/\tau} + T - \tau\right),$$

and the initial velocity (at $x = -\infty$) is

$$v_i = v_f - \frac{U_0}{m v_f}\left[1 - \frac{1}{1 + \frac{U_0}{m v_f^2}\left(e^{-T/\tau} - 1\right)}\right].$$

To simplify these results (since all we're looking for is a specific example), suppose the final kinetic energy is half the barrier height. Show that in this case

$$v_i = \frac{v_f}{1 - (L/v_f \tau)}.$$

In particular, if you choose $L = v_f \tau/4$, then $v_i = (4/3)v_f$, the initial kinetic energy is $(8/9)U_0$, and the particle makes it through, even though it didn't have sufficient energy to get over the barrier!]

! **Problem 11.30**

(a) Find the radiation reaction force on a particle moving with arbitrary velocity in a straight line, by reconstructing the argument in Sect. 11.2.3 *without* assuming $v(t_r) = 0$. [*Answer:* $(\mu_0 q^2 \gamma^4/6\pi c)(\dot{a} + 3\gamma^2 a^2 v/c^2)$]

(b) Show that this result is consistent (in the sense of Eq. 11.78) with the power radiated by such a particle (Eq. 11.75).

Problem 11.31

(a) Does a particle in hyperbolic motion (Eq. 10.45) radiate? (Use the exact formula (Eq. 11.75) to calculate the power radiated.)

(b) Does a particle in hyperbolic motion experience a radiation reaction? (Use the exact formula (Prob. 11.30) to determine the reaction force.)

[*Comment:* These famous questions carry important implications for the **principle of equivalence**. See T. Fulton and F. Rohrlich, *Annals of Physics* **9**, 499 (1960); J. Cohn, *Am. J. Phys.* **46**, 225 (1978); Chapter 8 of R. Peierls, *Surprises in Theoretical Physics* (Princeton: Princeton University Press, 1979); and the article by P. Pearle in *Electromagnetism: Paths to Research*, ed. D. Teplitz (New York: Plenum Press, 1982).]