VINCENT WANG-MAŚCIANICA

STRING DIAGRAMS FOR TEXT

Contents

(Acknowledgements will go in a margin note here.)

1
A palette for toy models

1.1 Continuous Relations for iconic semantics

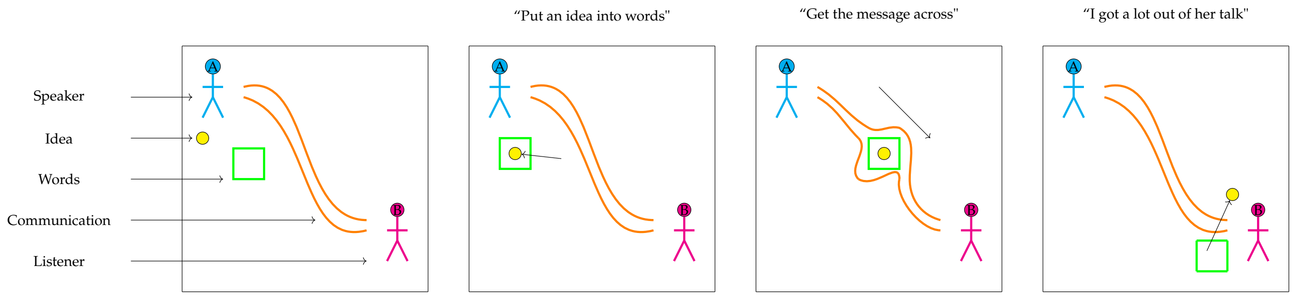


Figure 1.1: Sometimes it is very helpful to illustrate concepts using iconic representations in cartoons. For instance in the *conduit metaphor* CITE , `words` are considered *containers* for `ideas`, and `communication` is considered a *conduit* along which those containers are sent.

The aim of this chapter is to formally paint pictures with words. More verbosely, to formalise cartoon doodles like the one above in a symmetric monoidal category so that we can give semantics to text circuits in terms of graphical, iconic representations – cartoons, in short. To do so, we introduce the category **ContRel** of *continuous relations*, which are a naïve extension of the category **Top** of topological spaces and continuous functions towards continuous relations.

The main reason we prefer **ContRel** to either **Rel** or **Top** for our purposes is that we can diagrammatically characterise set-indexed collections of mutually disjoint open sets as *sticky-spiders*: a generalisation of spiders that interact with idempotents. We can then treat the indexing set as a collection of labels, and an indexed open set as a doodle. Notably, spiders don't exist in cartesian **Top** except for the one-point space, and the spatial structure of open sets doesn't exist in **Rel**.

$$placeholder : stickyspiderlaws$$

But there are all kinds of poorly behaved open sets even on the plane, so enter the next benefit: In **ContRel**, we can diagrammatically characterise the reals as a topological space up to homeomorphism, which gives us a diagrammatic handle on paths and homotopies, mathematical concepts that enable us to diagrammatically characterise when open sets are connected, how they might move and transform continuously in space, and when open sets are contained inside others.

$$realcharacterisation$$

And once we've formalised doodles we'll be able to treat ourselves to cartoons as formal semantics for language and nobody can stop us.

SIDENOTE FOR CATEGORY THEORISTS

The naïve approach I take is to observe that the preimages of functions are precisely relational converses when functions are viewed as relations, so the preimage-preserves-opens condition that defines continuous functions directly translates to the relational case. To the best of my knowledge, the study of **ContRel** is a novel contribution. I venture two potential reasons.

First, it is because and not despite of the naïvity of the construction. Usually, the relationship between **Rel** and **Set** is often understood in sophisticated general methods which are inappropriate in different ways. I have tried applying Kliesli machinery which generalises to "relationification" of arbitrary categories via appropriate analogs of the powerset monad to relate **Top** and **ContRel**, but it is not evident to me whether there is such a monad. The view of relations as spans of maps in the base category should work, since **Top** has pullbacks, but this makes calculation difficult and especially cumbersome when monoidal structure is involved. See Section ref for details.

Second, the relational nature of **ContRel** means that the category has poor exactness properties. Even if the sophisticated machinery mentioned in the first reason do manage to work, relational variants of **Top** are poor candidates for any kind of serious mathematics because they lack many limits and colimits. Since we take an entirely "monoidal" approach, we are able to find and make use of the rich structure of **ContRel** with a different toolkit.

ITINERARY OF THE CHAPTER: First we'll build some intuitions about what continuous relations are by example in Section ref . Before we can start reasoning diagrammatically, we ought to define the category **ContRel** and show it is symmetric monoidal, which will be the work in Section ref . Then we introduce sticky spiders and prove the following theorem:

Theorem 1.1.1.

$$placeholder : thmstatement$$

Finally, in Section ref , we build a vocabulary of topological concepts upon sticky spiders diagrammatically, where the point is to demonstrate sufficient expressivity to reason about whatever we want in principle. We start with the unit interval and isometries, through to rigid motions of shapes in configuration, connectedness and contractibility of shapes via homotopies, until we get to sketching some cognitively primitive relations like parthood, touching, and insideness.

Reminder 1.2.1 (Topological Space)**.** A *topological space* is a pair (X, τ), where X is a set, and $\tau \subset \mathcal{P}(X)$ are the *open sets* of X, such that:

"nothing" and "everything" are open

$$\varnothing, X \in \tau$$

Arbitrary unions of opens are open

$$\{U_i : i \in I\} \subseteq \tau \Rightarrow \bigcup_{i \in I} U_i \in \tau$$

Finite intersections of opens are open $n \in \mathbb{N}$:

$$U_1, \cdots, U_n \in \tau \Rightarrow \bigcap_{1 \cdots, i, \cdots n} U_i \in \tau$$

Reminder 1.2.2 (Relational Converse)**.** Recall that a relation $R : S \to T$ is a subset $R \subseteq S \times T$.

$$R^\dagger : T \to S := \{(t, s) : (s, t) \in R\}$$

Reminder 1.2.3 (Continuous function)**.** A function between sets $f : X \to Y$ is a continuous function between topologies $f : (X, \tau) \to (Y, \sigma)$ if

$$U \in \sigma \Rightarrow f^{-1}(U) \in \tau$$

where f^{-1} denotes the inverse image.

Recall that functions are relations, and the inverse image used in the definition of continuous maps is equivalent to the relational converse when functions are viewed as relations. So we can naïvely extend the notion of continuous maps to continuous relations between topological spaces.

Notation 1.2.4. For shorthand, we denote the topology (X, τ) as X^τ. As special cases, we denote the discrete topology on X as $X^\star$, and the indiscrete topology X°.

The symmetric monoidal structure is that of product topologies on objects, and products of relations on morphisms.

1.2 *Continuous Relations by examples*

Definition 1.2.7 (Continuous Relation)**.** A continuous relation $R : (X, \tau) \to (Y, \sigma)$ is a relation $R : X \to Y$ such that

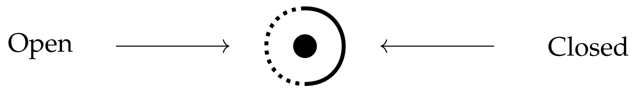
$$U \in \sigma \Rightarrow R^\dagger(U) \in \tau$$

where $\dagger$ denotes the relational converse.

Let's consider three topological spaces and examine the continuous relations between them. This way we can build up intuitions, and prove some tool results in the process.

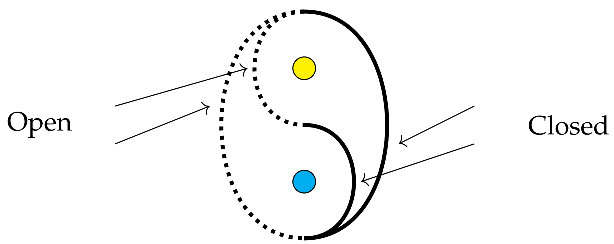
The **singleton space** consists of a single point which is both open and closed. We denote this space $\bullet$. Concretely, the underlying set and topology is

$$(\{\star\}, \{\{\star\}, \varnothing\})$$

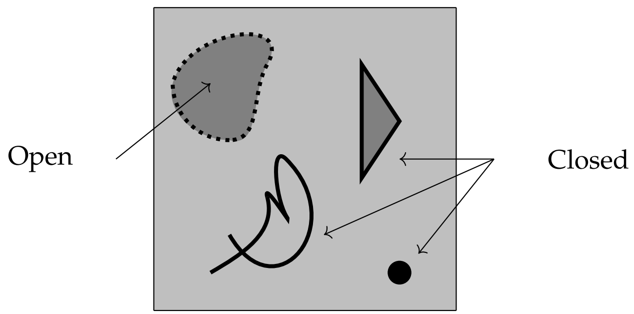


The **Sierpiński space** consists of two points, one of which (in yellow) is open, and the other (in cyan) is closed. We denote this space $\mathcal{S}$. Concretely, the underlying set and topology is:

$$\big(\{0, 1\}, \{\varnothing, \{1\}, \{0, 1\}\}\big)$$



The **unit square** has $[0, 1] \times [0, 1]$ as its underlying set. Open sets are "blobs" painted with open balls. Points, lines, and bounded shapes are closed. We denote this space $\blacksquare$.



$\bullet \to \bullet$: There are two relations from the singleton to the singleton; the identity relation $\{(\bullet, \bullet)\}$, and the empty relation $\varnothing$. Both are topological.

$\bullet \to \mathcal{S}$: There are four relations from the singleton to the Sierpiński space, corresponding to the subsets of $\mathcal{S}$. All of them are topological.

$\mathcal{S} \to \bullet$: There four candidate relations from the Sierpiński space to the singleton, but as we see in Example 1.2.8, not all of them are topological.

Now we need some abstraction. We cannot study the continuous relations between the singleton and the unit square case by case. We discover that continuous relations out of the singleton indicate arbitrary subsets, and that continuous relations into the singleton indicate arbitrary opens.

$\bullet \to \blacksquare$: Proposition 1.2.10 tells us that there are as many continuous relations from the singleton to the unit square as there are subsets of the unit square.

$\blacksquare \to \bullet$: Proposition 1.2.11 tells us that there are as many continuous relations from the unit square to the singleton as there are open sets of the unit square.

There are 16 candidate relations $\mathcal{S} \to \mathcal{S}$ to check. A case-by-case approach won't scale, so we could instead identify the building blocks of continuous relations with the same source and target space.

Given two continuous relations $R, S : X^\tau \to Y^\sigma$, how can we combine them?

Proposition 1.2.13. If $R, S : X^\tau \to Y^\sigma$ are continuous relations, so are $R \cap S$ and $R \cup S$.

Proof. Replace $\square$ with either $\cup$ or $\cap$. For any non-$\varnothing$ open $U \in \sigma$:

$$(R \square S)^\dagger(U) = R^\dagger(U) \square S^\dagger(U)$$

As R, S are continuous relations, $R^\dagger(U), S^\dagger(U) \in \tau$, so $R^\dagger(U) \square S^\dagger(U) = (R \square S)^\dagger(U) \in \tau$. Thus $R \square S$ is also a continuous relation. $\square$

Corollary 1.2.14. Continuous relations $X^\tau \to Y^\sigma$ are closed under arbitrary union and finite intersection. Hence, continuous relations $X^\tau \to Y^\sigma$ form a topological space where each continuous relation is an open set on the base space $X \times Y$, where the full relation $X \to Y$ is "everything", and the empty relation is "nothing".

A topological basis for spaces of continuous relations

Reminder 1.2.5 (Product Topology)**.** We denote the product topology of X^τ and Y^σ as $(X \times Y)^{(\tau \times \sigma)}$. $\tau \times \sigma$ is the topology on $X \times Y$ generated by the basis $\{t \times s : t \in \mathfrak{b}_\tau, s \in \mathfrak{b}_\sigma\}$, where $\mathfrak{b}_\tau$ and $\mathfrak{b}_\sigma$ are bases for τ and σ respectively.

Reminder 1.2.6 (Product of relations)**.** For relations between sets $R : X \to Y, S : A \to B$, the product relation $R \times S : X \times A \to Y \times B$ is defined to be

$$\{((x, a), (y, b)) : (x, y) \in R, (a, b) \in S\}$$

Example 1.2.8 (A noncontinuous relation)**.** The relation $\{(0, \bullet)\} \subset \mathcal{S} \times \bullet$ is not a continuous relation: the preimage of the open set $\{\bullet\}$ under this relation is the non-open set $\{0\}$.

Terminology 1.2.9. Call a continuous relation $\bullet \to X^\tau$ a **state** of X^τ, and a continuous relation $X^\tau \to \bullet$ a **test** of X^τ.

Proposition 1.2.10. States $R : \bullet \to X^\tau$ correspond with subsets of X.

Proof. The preimage $R^\dagger(U)$ of a (non-$\varnothing$) open $U \in \tau$ is $\star$ if $R(\star) \cap U$ is nonempty, and $\varnothing$ otherwise. Both $\star$ and $\varnothing$ are open in $\{\star\}^\bullet$. $R(\star)$ is free to specify any non-$\varnothing$ subset of X. The empty relation handles $\varnothing$ as an open of X^τ. $\square$

Proposition 1.2.11. Tests $R : X^\tau \to \bullet$ correspond with open sets $U \in \tau$.

Proof. The preimage $R^\dagger(\star)$ of $\star$ must be an open set of X^τ by definition **??**. $R^\dagger(\star)$ is free to specify any open set of X^τ. $\square$

Reminder 1.2.12 (Union, intersection, and ordering of relations)**.** Recall that relations $X \to Y$ can be viewed as subsets of $X \times Y$. So it makes sense to speak of the union and intersection of relations, and of partially ordering them by inclusion.

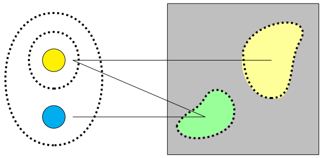


Figure 1.2: Regions of $\blacksquare$ in the image of the yellow point alone will be coloured yellow, and regions in the image of both yellow and cyan will be coloured green:

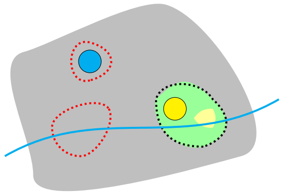


Figure 1.3: Regions in the image of the cyan point alone cannot be open sets by continuity, so they are either points or lines. Points and lines in cyan must be surrounded by an open region in either yellow or green, or else we violate continuity (open sets in red).

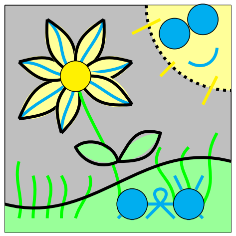


Figure 1.4: A continuous relation $\mathcal{S} \to \blacksquare$: "Flower and critter in a sunny field".

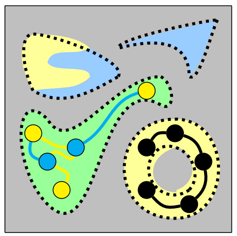


Figure 1.5: A continuous relation $\blacksquare \to \mathcal{S}$: "still math?". Black lines and dots indicate gaps.

Reminder 1.2.15 (Topological Basis)**.** $\mathfrak{b} \subseteq \tau$ is a basis of the topology τ if every $U \in \tau$ is expressible as a union of elements of $\mathfrak{b}$. Every topology has a basis (itself). Minimal bases are not necessarily unique.

Having a tangible topological basis for continuous relations is good for intuition: we can think of breaking down or constructing complex relations to or from simpler parts. Luckily, there do exist nice topological bases for continuous relations!

Definition 1.2.16 (Partial Functions)**.** A **partial function** $X \to Y$ is a relation for which each $x \in X$ has at most a single element in its image. In particular, all functions are special cases of partial functions, as is the empty relation.

Lemma 1.2.17 (Partial functions are a $\cap$-ideal)**.** The intersection $f \cap R$ of a partial function $f : X \to Y$ with any other relation $R : X \to Y$ is again a partial function.

Proof. Consider an arbitrary $x \in X$. $R(x) \cap f(x) \subseteq f(x)$, so the image of x under $f \cap R$ contains at most one element, since $f(x)$ contains at most one element. □

Lemma 1.2.18 (Any single edge can be extended to a continuous partial function)**.** Given any $(x, y) \in X \times Y$, there exists a continuous partial function $X^\tau \to Y^\sigma$ that contains (x, y).

Proof. Let $\mathcal{N}(x)$ denote some open neighbourhood of x with respect to the topology τ. Then $\{(z, y) : z \in \mathcal{N}(x)\}$ is a continuous partial function that contains (x, y). □

Proposition 1.2.19. Continuous partial functions form a topological basis for the space $(X \times Y)^{(\tau \multimap \sigma)}$, where the opens are continuous relations $X^\tau \to Y^\sigma$.

Proof. We will show that every continuous relation $R : X^\tau \to Y^\sigma$ arises as a union of continuous partial functions. Denote the set of continuous partial functions $\mathfrak{f}$. We claim that:

$$R = \bigcup_{F \in \mathfrak{f}} (R \cap F)$$

The $\supseteq$ direction is evident, while the $\subseteq$ direction follows from Lemma 1.2.18. By Lemma 1.2.17, every $R \cap F$ term is a partial function, and by Corollary 1.2.14, continuous. □

$\mathcal{S} \to \mathcal{S}$: We can use Proposition 1.2.19 to write out the topological basis of continuous partial functions, from which we can take unions to find all the continuous relations, which we depict in Figure 1.6.

$\mathcal{S} \to \blacksquare$: Now we use the colour convention of the points in $\mathcal{S}$ to "paint" continuous relations on the unit square "canvas", as in Figures 1.2 and 1.3. So each continuous relation is a painting, and we can characterise the paintings that correspond to continuous relations $\mathcal{S} \to \blacksquare$ in words as follows: Cyan only in points and lines, and either contained in or at the boundary of yellow or green. Have as much yellow and green as you like.

$\blacksquare \to \mathcal{S}$: The preimage of all of $\mathcal{S}$ must be an open set. So the painting cannot have stray lines or points outside of blobs. The preimage of yellow must be open, so the union of yellow and green in the painting cannot have stray lines or points outside of blobs. Point or line gaps within blobs are ok. Each connected blob can contain any colours in any shapes, subject to the constraint that if cyan appears anywhere, then either yellow or green must occur somewhere. Open blobs with no lines or points outside. Yellow and green considered alone is a painting made of blobs with no stray lines or points. If cyan appears anywhere, then either yellow or green have to appear somewhere.

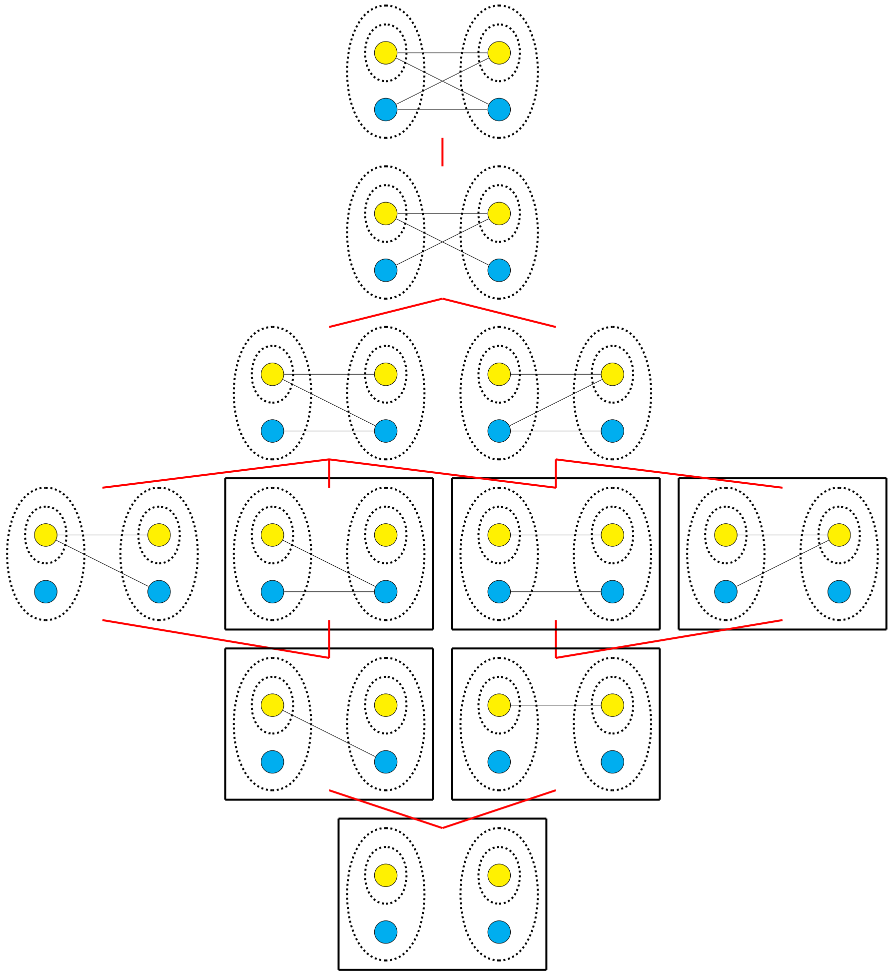


Figure 1.6: Hasse diagram of all continuous relations from the Sierpiński space to itself. Each relation is depicted left to right, and inclusion order is bottom-to-top. Relations that form the topological basis are boxed.

One more example for fun: $[0,1] \to$ ■: We know how continuous functions from the unit line into the unit square look.

Then what are the partial continuous functions? If we understand these, we can obtain all continuous relations by arbitrary unions of the basis. Observe that the restriction of any continuous function to an open set in the source is a continuous partial function. The open sets of $[0,1]$ are collections of open intervals, each of which is homeomorphic to $(0,1)$, which is close enough to $[0,1]$.

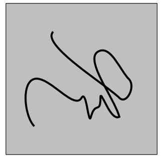


Figure 1.7: continuous functions $[0,1] \to$ ■ follow the naïve notion of continuity: a line one can draw on paper without lifting the pen off the page.

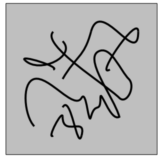


Figure 1.8: So a continuous partial function is `"(countably) many (open-ended) lines, each of which one can draw on paper without lifting the pen off the page."`

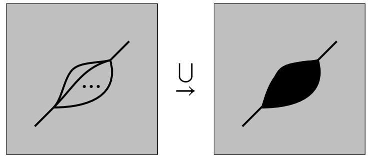


Figure 1.9: We can control the thickness of the brush-stroke, by taking the union of (uncountably) many lines.

Figure 1.10: Assign the visible spectrum of light to $[0,1]$. Colour open sets according to perceptual addition of light, computing brightness by normalising the measure of the open set.

1.3 *The category **ContRel***

Proposition 1.3.1. Continuous functions are always continuous. If $f : X^\tau \to Y^\sigma$ is a continuous function, then it is also a continuous relation.

Proof. Functions are special cases of relations. The relational converse of a function viewed in this way is the same thing as the preimage. □

Corollary 1.3.2. There is a faithful, identity-on-objects monoidal embedding **Top** $\hookrightarrow$ **ContRel**.

Proposition 1.3.3. The **identity relation** $X \to X$ relates anything to itself. It is defined $\{(x,x) : x \in X\} \subseteq X \times X$. The identity relation is always continuous.

Proof. The preimage of any open set under the identity relation is itself, which is open by assumption. The identity relation is also the trivial continuous function from a space to itself, so this also follows from Proposition 1.3.1. □

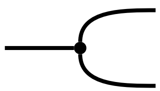


Figure 1.11: The *copy* map $X^\tau \to X^\tau \times X^\tau$, $\{(x, \begin{pmatrix} x \\ x \end{pmatrix}) \mid x \in X\}$.

Proposition 1.3.4. Copy maps are continuous relations.

Proof. For a direct proof, we draw on the fact that given a basis $\mathfrak{b}$ for a topology τ, ordered pairs of $\mathfrak{b}$ form a basis for the product topology $\tau \times \tau$. To show that the preimage of an open in $\tau \times \tau$ is open in τ, we may consider the preimage under the copy map of basis elements of $\tau \times \tau$, which are intersections of pairs of basis elements of τ, and hence definitionally open. By closure of opens under arbitrary unions, all opens of $\tau \times \tau$ have an open preimage in τ. □

Definition 1.3.9 (ContRel). The (purported) category **ContRel** has topological spaces for objects and continuous relations for morphisms.

Proposition 1.3.10 (ContRel is a category). continuous relations form a category **ContRel**.

Proof. Identities: Identity relations, which are always continuous since the preimage of an open U is itself.

Composition: The normal composition of relations. We verify that the composite $X^\tau \xrightarrow{R} Y^\sigma \xrightarrow{S} Z^\theta$ of continuous relations is again continuous as follows:

$$U \in \theta \implies S^\dagger(U) \in \sigma \implies R^\dagger \circ S^\dagger(U) = (S \circ R)^\dagger \in \tau$$

Associativity of composition: Inherited from **Rel**. □

1.3.1 *Symmetric Monoidal structure*

Proposition 1.3.11. $(\mathbf{ContRel}, \bullet, X^\tau \otimes Y^\sigma := (X \times Y)^{(\tau \times \sigma)})$ is a symmetric monoidal category.

Tensor Unit: The one-point space $\bullet$. Explicitly, $\{\star\}$ with topology $\{\varnothing, \{\star\}\}$.

Tensor Product: For objects, $X^\tau \otimes Y^\sigma$ has base set $X \times Y$ equipped with the product topology $\tau \times \sigma$. For morphisms, $R \otimes S$ the product of relations. We show that the tensor of continuous relations is again a continuous relation. Take continuous relations $R : X^\tau \to Y^\sigma$, $S : A^\alpha \to B^\beta$, and let U be open in the product topology $(\sigma \times \beta)$. We need to prove that $(R \times S)^\dagger(U) \in (\tau \times \alpha)$. We may express U as $\bigcup_{i \in I} y_i \times b_i$, where the y_i and b_i are in the bases $\mathfrak{b}_\sigma$ and $\mathfrak{b}_\beta$ respectively. Since for any relations we have that $R(A \cup B) = R(A) \cup R(B)$ and $(R \times S)^\dagger = R^\dagger \times S^\dagger$:

$$\begin{aligned} &(R \times S)^\dagger(\bigcup_{i \in I} y_i \times b_i) \\ &= \bigcup_{i \in I} (R \times S)^\dagger(y_i \times b_i) \\ &= \bigcup_{i \in I} (R^\dagger \times S^\dagger)(y_i \times b_i) \end{aligned}$$

Since each y_i is open and R is continuous, $R^\dagger(y_i) \in \tau$. Symmetrically, $S^\dagger(b_i) \in \alpha$. So each $(R^\dagger \times S^\dagger)(y_i \times b_i) \in (\tau \times \alpha)$. Topologies are closed under arbitrary union, so we are done.

THE NATURAL ISOMORPHISMS ARE INHERITED FROM **REL**. We will be explicit with the unitor, but for the rest, we will only check that the usual isomorphisms from **Rel** are continuous in **ContRel**. To avoid bracket-glut, we will vertically stack some tensored expressions.

UNITORS: The left unitors are defined as the relations $\lambda_{X^\tau} : \bullet \times X^\tau \to X^\tau := \{(\begin{pmatrix} \star \\ x \end{pmatrix}, x) \mid x \in X\}$, and we reverse the pairs to obtain the inverse $\lambda^{-1}_{X^\tau}$. These relations are continuous since the product topology of τ with the singleton is homeomorphic to τ: $U \in \tau \iff (\bullet, U) \in (\bullet \times \tau)$. These relations are evidently inverses that compose to the identity. The construction is symmetric for the right unitors ρ_{X^τ}.

ASSOCIATORS: The associators $\alpha_{X^\tau Y^\sigma Z^\rho} : ((X \times Y) \times Z)^{((\tau\times\sigma)\times\rho)} \to (X \times (Y \times Z))^{(\tau\times(\sigma\times\rho))}$ are inherited from **Rel**. They are:

$$\alpha_{X^\tau Y^\sigma Z^\rho} := \{ ((\begin{pmatrix} x \\ y \end{pmatrix}, z) \, , \, (x, \begin{pmatrix} y \\ z \end{pmatrix})) \mid x \in X \, , \, y \in Y \, , \, z \in Z \}$$

To check the continuity of the associator, observe that product topologies are isomorphic in **Top** up to bracketing, and these isomorphisms are inherited by **ContRel**. The inverse of the associator has the pairs of the relation reversed and is evidently an inverse that composes to the identity.

BRAIDS: The braidings $\theta_{X^\tau Y^\sigma} : (X \times Y)^{\tau\times\sigma} \to (Y \times X)^{\sigma\times\tau}$ are defined:

$$\{(\begin{pmatrix} x \\ y \end{pmatrix} \, , \, \begin{pmatrix} y \\ x \end{pmatrix}) \mid x \in X \, , \, y \in Y\}$$

The braidings inherit continuity from the isomorphisms between $X^\tau \times Y^\sigma$ and $Y^\sigma \times X^\tau$ in **Top**. They inherit everything else from **Rel**

COHERENCES: Since we have verified all of the natural isomorphisms are continuous, it suffices to say that the coherences [] are inherited from the symmetric monoidal structure of **Rel** up to marking objects with topologies.

1.3.2 *Rig category structure*

Definition 1.3.12 (Biproducts and zero objects)**.** A *biproduct* is simultaneously a categorical product and coproduct. A *zero object* is both an initial and a terminal object. **Rel** has biproducts (the coproduct of sets equipped with reversible injections) and a zero object (the empty set).

Proposition 1.3.13. **ContRel** has a zero object.

$$\begin{tikzcd} & X^\tau \arrow[dl, "id"'] \arrow[d, "copy"] \arrow[dr, "id"] & \\ X^\tau & (X \times X)^{(\tau\times\tau)} \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & X^\tau \end{tikzcd}$$

Figure 1.12: An alternative proof of Proposition 1.3.4 follows from Proposition 1.3.1, Corollary 1.3.2, and the definition of the product topology as the coarsest topology that satisfies categorical product for the diagram above.

Figure 1.13: The *everything* state is the relation $\{((\star, x) \mid x \in X\}$, notated as above.

Proposition 1.3.5. The everything states are continuous relations.

Proof. The preimage of any subset of X – in particular the opens – is the whole of the singleton space, which is open. □

Figure 1.14: The *delete* test, $\{((x, \star) \mid x \in X\}$.

Proposition 1.3.6. The delete tests are continuous relations.

Proof. There are only two opens in the singleton space. The preimage of the empty set is the empty set, and the preimage of the singleton is the whole of X; both are opens in X by definition. □

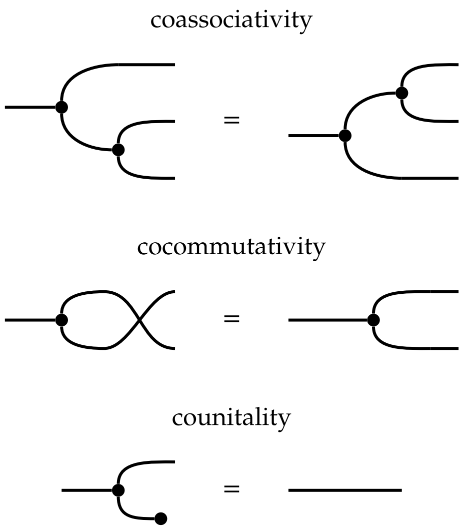


Figure 1.15: Copy and delete satisfy the above properties, expressed as diagrammatic equations.

Proof. As in **Rel**, there is a unique relation from every object to and from the empty set with the empty topology. □

Proposition 1.3.14. **ContRel** has biproducts.

Proof. The biproduct of topologies X^τ and Y^σ is their direct sum topology $(X \sqcup Y)^{(\tau+\sigma)}$ – the coarsest topology that contains the disjoint union $\tau \sqcup \sigma$. As in **Rel**, the (in/pro)jections are partial identities, which are continuous by construction. To verify that it is a coproduct, given continuous relations $R : X^\tau \to Z^\rho$ and $S : Y^\sigma \to Z^\rho$, where the disjoint union $X \sqcup Y$ of sets is $\{x_1 \mid x \in X\} \cup \{y_2 \mid y \in Y\}$, we observe that $R + S := \{(x_1, z) \mid (x, z) \in R\} \cup \{(y_2, z) \mid y \in S\}$ is continuous and commutes with the injections as required. The argument that it is a product is symmetric. □

Remark 1.3.15. Biproducts yield another symmetric monoidal structure which the $\times$ monoidal product distributes over appropriately to yield a rig category. Throughout the chapter we will use $\cup$, but we could have also "diagrammatised" $\cup$ by treating it as a monoid internal to **ContRel** viewed as a symmetric monoidal category with respect to the biproduct. There are at least two diagrammatic formalisms for rig categories that we could have used, CITE and CITE , but it gets too visually complicated. especially when we sometimes take unions over arbitrary indexing sets, which is alright in topology but not depictable as a finite diagram in the $\oplus$-structure. A neat fact that follows is that a topological space is compact precisely when any arbitrarily indexed $\cup$ of tests in the $\times$-structure is *depictable* in the $\oplus$-structure of either diagrammatic calculus for rig categories. **FdHilb** also has a monoidal product notated $\otimes$ that distributes over the monoidal structure given by biproducts $\oplus$. In contrast, we have used $\times$ – the cartesian product notation – for the monoidal product of **ContRel** since that is closer to what is familiar for sets.

1.3.3 Monoidal (co!)closure

Definition 1.3.16 (Closure type)**.** Recall by Proposition 1.2.13 that continuous relations $X^\tau \to Y^\sigma$ form a topological space. Denote this space $(X \times Y)^{(\tau \multimap \sigma)}$

The permissible continuous relations $X^\tau \to Y^\sigma$ are tests on $(X \times Y)^{(\tau \multimap \sigma)}$. **ContRel** is not monoidal closed, because taking a closure type as an input to the evaluator would permit arbitrary subsets of $X \times Y$ as arguments. So what we seek instead is a coevaluation, where the closure type is an output. This is not as straightforward as it is in strongly compact closed **Rel**, where we may use cups and caps for process-state duality (CJ-isomorphism), because in **ContRel** we have cups *but no caps*. However, we may exploit the fact that discrete topologies behave like plain sets See Lemma 1.3.23) and the observation that we may coarsen discrete topologies into target topologies, which is essentially enough to recover monoidal coclosed structure.

Proposition 1.3.17. For any X^τ and Y^σ, $\tau \times \sigma \subseteq \tau \multimap \sigma$; the product topology is coarser than the corresponding closure topology.

Proof. Let $\mathfrak{b}_\tau, \mathfrak{b}_\sigma$ be bases for τ and σ respectively, then $\tau \times \sigma$ has basis $\mathfrak{b}_\tau \times \mathfrak{b}_\sigma$. An arbitrarily element $(t \in \tau, s \in \sigma)$ of this product basis can be viewed as a topological relation $t \times s \subseteq X \times Y$. Every open of $\tau \times \sigma$ is a union of such basis elements, and topological relations are closed under arbitrary union, so we have the (evidently injective) correspondence:

$$\tau \times \sigma \ni \bigcup_{i \in I} (t_i \times s_i) \mapsto \bigcup_{i \in I} (t_i \times s_i) \in \tau \multimap \sigma$$

□

Example 1.3.18 ($\tau \multimap \sigma \not\subseteq \tau \times \sigma$)**.** Recalling Proposition 1.2.10, let $\tau = \{\varnothing, \{\star\}\}$ on the singleton, and σ be an arbitrary nondiscrete topology on base space Y. $(\{\star\} \times Y)^{(\tau \times \sigma)}$ is isomorphic to Y^σ, but $(\{\star\} \times Y)^{(\tau \multimap \sigma)}$ is the isomorphic to the discrete topology $Y^\bullet$. For a more concrete example, consider the Sierpiński space $\mathcal{S}$ again, along with the topological relation $\{(0,0),(1,0),(1,1)\} \subset \mathcal{S} \times \mathcal{S}$; due to the presence of $(0,0)$, this topological relation cannot be formed by a union of basis elements of the product topology, which are:

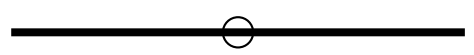
$$\{1\} \times \{1\} = \{(1,1)\}$$

$$\{1\} \times \{0,1\} = \{(1,0),(1,1)\}$$

$$\{0,1\} \times \{1\} = \{(1,1),(0,1)\}$$

$$\{0,1\} \times \{0,1\} = \{(0,0),(0,1),(1,0),(0,1)\}$$

Definition 1.3.19 (Coarsening)**.** Where $\tau \supseteq \rho$ are topologies on X, the identity-on-elements relation $X^\tau \to X^\rho$ is continuous; the relational converse of the identity is the identity, which witnesses opens of ρ as opens of τ. but the converse is not unless $\tau = \rho$. We denote these *coarsening* relations as:

X^τ  X^ρ

Definition 1.3.20 (Pseudo-compare)**.** For any X^τ, the relation $X^\star \times X^\tau \to X^\tau$ defined on objects as

$$\{\begin{pmatrix} x \\ x \end{pmatrix}, x \mid x \in X\}$$

is continuous; the relational converse is the copy map, which sends opens U in τ to $U \times U \in \tau \times \tau \subseteq X \times X$,

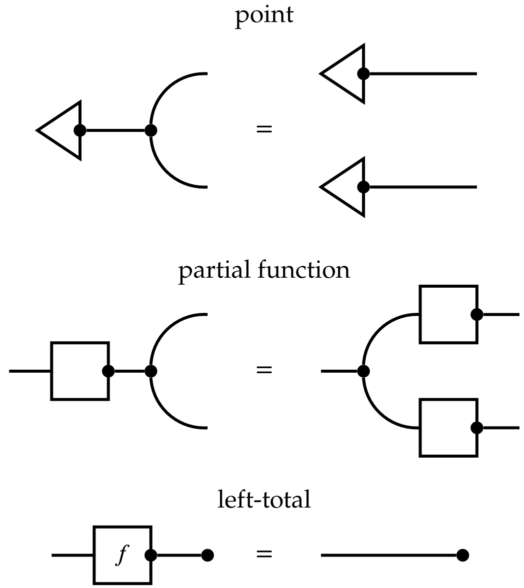


Figure 1.16: Relations that interact with copy and delete are nice, and we notate them with the same black dots as for copy and delete to mark them. States are singletons, or points, when they are copiable. Partial continuous functions are those that commute with copy. Left-total relations are those that commute with delete. Continuous functions are those that satisfy the latter two criteria.

Proposition 1.3.7. The **full relation** $X \to Y$ relates everything to everything. It is all of $X \times Y$. Full relations are always continuous.

Proof. For a direct proof, the preimage of any subset of Y under the full relation is the whole of X, which is open by definition. Alternatively, the full relation is the composite of delete and then everything. □

Figure 1.17: The **empty state** $\bullet \to X^\tau$ and **empty test** relate nothing. The **empty relation** $X^\tau \to Y^\sigma$ is the composite of empty tests and states, and relates nothing: as a relation it is $\varnothing \subset X \times Y$.

Proposition 1.3.8. Empty states, tests, and relations are continuous.

Proof. The preimage of any empty relation is the empty set, which is definitionally open. $\square$

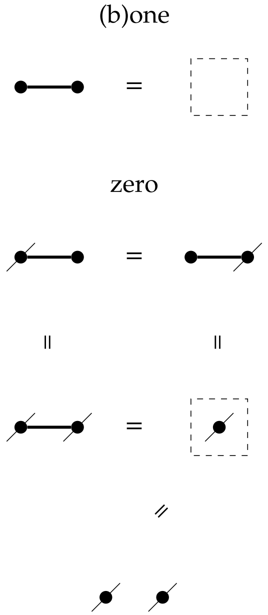
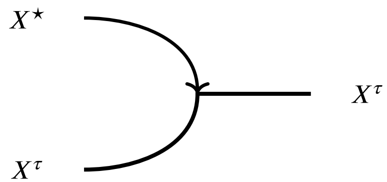


Figure 1.18: There are two scalars: the unit, and the zero scalar. Zero scalars are idempotent, which diagrammatically means that a single zero scalar can copy itself.

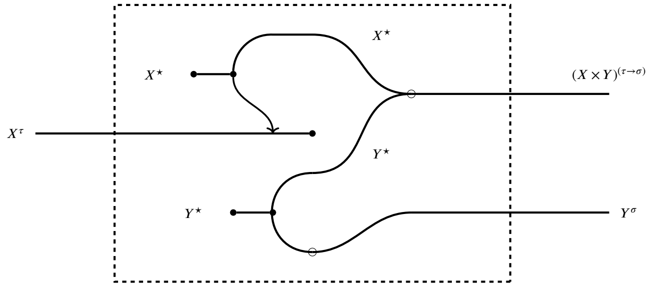
which are opens in $X^\star \times X^\tau$. We denote these *pseudo-compare* maps:



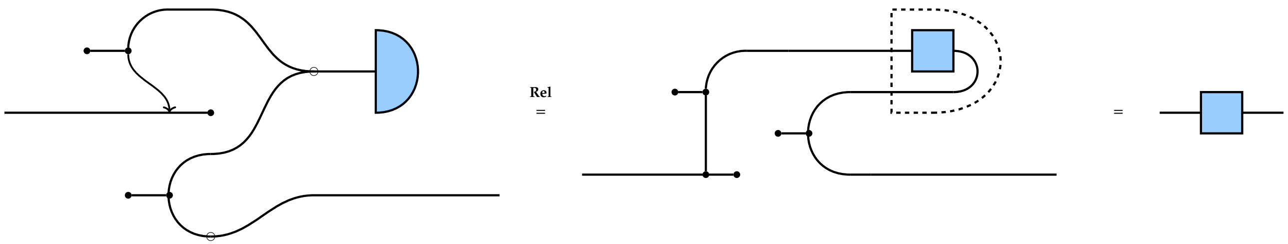
Proposition 1.3.21. Where $\star_X$ and $\star_Y$ are discrete topologies on X and Y, $\star_X \times \star_Y = \star_X \multimap \star_Y$.

Proof. Relations between discrete topologies are just arbitrary relations, and relations are monoidal closed with $X \multimap Y \simeq X \times Y$. $\square$

Proposition 1.3.22 (ContRel is monoidal coclosed). The coevaluation map is:



Proof. The string diagram itself demonstrates that the coevaluation is continuous, since we have demonstrated that **ContRel** is symmetric monoidal, and we know that copy (Prop. 1.3.4), everything (Prop. 1.3.5), coarsenings (Defn. 1.3.19) and pseudo-compares (Defn. 1.3.20) are continuous. What remains to be demonstrated is that the coevaluation behaves like one; i.e. that plugging in a continuous relation expressed as a test into the closure type of the coevaluator recovers that continuous relation. This can be shown diagrammatically by equating the underlying relations on sets in **Rel**.



The first equation is obtained by a few steps. Forgetting topology, we turn all coarsenings into identities in

Rel, and in particular, proposition 1.3.21 deals with the 2-1 coarsening. The expression inside the closure test is obtained by CJ-isomorphism using strong compact closure in **Rel**. The pseudo-compares in **ContRel** becomes an honest compare in **Rel**. The second equation then follows by frobenius. □

THAT'S ALL WE NEED FOR THE DIAGRAMS. The remainder of this section are endnotes for category theorists addressing the question of how **ContRel** relates to **Rel** and **Top**, and some conceptual motivations for topological relations. If none of that interests you, ignore the main body: the margins carry on with diagrammatic facts about **ContRel**.

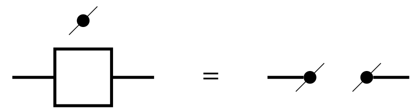


Figure 1.19: There is a zero-morphism for every input-output pair of objects in **ContRel**, which is diagrammatically the composition of the empty test and state. Zero scalars turn any relation into a zero relation. Substituting the zero relation into the LHS of the above equation means that zero relations also spawn zero scalars.

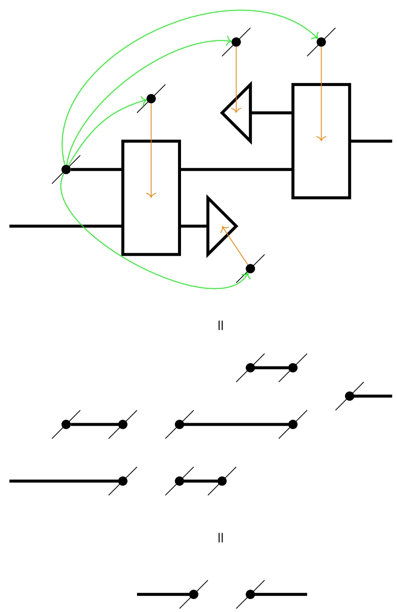


Figure 1.20: So, whenever a zero-process appears in a diagram, it spawns zero scalars which infect all other processes, turning them all into zero-processes. The same holds for whenever a zero-scalar appears; it makes copies of itself to infect all other processes.

1.3.4 Category-theoretic endnotes

ContRel and Rel are related by a free-forgetful adjunction

We provide free-forgetful adjunctions relating **ContRel** to **Rel** by "forgetting topology" and sending sets to "free" discrete topologies. We exhibit a free-forgetful adjunction between **Rel** and **ContRel**.

Lemma 1.3.23 (Any relation R between discrete topologies is continuous)**.** *Proof.* All subsets in a discrete topologies are open. □

Definition 1.3.24 (L: **Rel** → **ContRel**)**.** We define the action of the functor L:

On objects $L(X) := X^\star$, (X with the discrete topology)

On morphisms $L(X \overset{R}{\to} Y) := X^\star \overset{R}{\to} Y^\star$, the existence of which in **ContRel** is provided by Lemma 1.3.23.

Evidently identities and associativity of composition are preserved.

Definition 1.3.25 (R: **ContRel** → **Rel**)**.** We define the action of the functor R as forgetting the topological structure.

On objects $R(X^\tau) := X$

On morphisms $R(X^\tau \overset{S}{\to} Y^\sigma) := X \overset{S}{\to} Y$

Evidently identities and associativity of composition are preserved.

Lemma 1.3.26 ($RL = 1_{\mathbf{Rel}}$)**.** The composite RL (first L, then R) is precisely equal to the identity functor on **Rel**.

Proof. On objects, $FU(X) = F(X^\star) = X$. On morphisms, $FU(X \overset{R}{\to} Y) = F(X^\star \overset{R}{\to} Y^\star) = X \overset{R}{\to} Y$ □

Reminder 1.3.27 (Coarser and finer)**.** Given a set of points X with two topologies X^τ and X^σ, if $\tau \subset \sigma$, we say that τ *is coarser than* σ, or σ *is finer than* τ.

Lemma 1.3.28 (Coarsening is a continuous relation)**.** Let X^σ be coarser than X^τ. The identity relation on underlying points $X^\tau \overset{1_X}{\to} X^\sigma$ is then a continuous relation.

Proof. The preimage of the identity of any open set $U \in \sigma, U \subseteq X$ is again U. By definition of coarseness, $U \in \tau$. □

Proposition 1.3.29 ($L \dashv R$)**.** *Proof.* We verify the triangular identities governing the unit and counit of the adjunction, which we first provide. By Lemma 1.3.26, we take the natural transformation $1_{\mathbf{Rel}} \Rightarrow RL$ we take to be the identity morphism:

$$\eta_X := 1_X$$

The counit natural transformation $LR \Rightarrow 1_{\mathbf{ContRel}}$ we define to be a coarsening, the existence of which in **ContRel** is granted by Lemma 1.3.28.

$$\epsilon_{X^\tau} : X^\star \to X^\tau := \{(x,x) : x \in X\}$$

First we evaluate $L \xrightarrow{L\eta} LRL \xrightarrow{\epsilon L} L$ at an arbitrary object (set) $X \in \mathbf{Rel}$. $L(X) = X^\star = LRL(X)$, where the latter equality holds because LR is precisely the identity functor on **Rel**. For the first leg from the left, $L(\eta_X) = L(1_X) = X^\star \xrightarrow{1_X} X^\star = 1_{X^\star}$. For the second, $\epsilon_{L(X)} = \epsilon_{X^\star} = X^\star \xrightarrow{1_X} X^\star = 1_{X^\star}$. So we have that $L\eta; \epsilon L = L$ as required.

Now we evaluate $R \xrightarrow{\eta R} RLR \xrightarrow{R\epsilon} R$ at an arbitrary object (topological space) $X^\tau \in \mathbf{ContRel}$. $R(X^\tau) = X = RLR(X^\tau)$, where the latter equality again holds because $LR = 1_{\mathbf{Rel}}$. For the first leg from the left, $\eta_{R(X^\tau)} = \eta_X = 1_X$. For the second, $R(\epsilon_{X^\tau}) = R(X^\star \xrightarrow{1_X} X^\tau) = X \xrightarrow{1_X} X = 1_X$. So $\eta R; R\epsilon = R$, as required. □

The usual forgetful functor from **ContRel** to **Loc** has no left adjoint. Just as the forgetful functor from **ContRel** to **Rel** "forgets topology while keeping the points", we might consider a forgetful functor to **Loc** that "forgets points while remembering topology". But we show that there is no such functor that forms a free-forgetful adjunction.

Reminder 1.3.30 (The category **Loc**)**.** CITE A *frame* is a poset with all joins and finite meets satisfying the infinite distributive law:

$$x \wedge (\bigvee_i y_i) = \bigvee_i (x \wedge y_i)$$

A *frame homomorphism* $\phi : A \to B$ is a function between frames that preserves finite meets and arbitrary joins, i.e.:

$$\phi(x \wedge_A y) = \phi(x) \wedge_B \phi(y) \qquad \phi(x \vee_A y) = \phi(x) \vee_B \phi(y)$$

The category **Frm** has frames as objects and frame homomorphisms as morphisms. The category **Loc** is defined to be $\mathbf{Frm}^{\mathrm{op}}$.

Remark 1.3.31. Here are informal intuitions to ease the definition. The lattice of open sets of a given topology ordered by inclusion forms a frame – observe the analogy "arbitrary unions" : "all joins" :: "finite intersections" : "finite meets". Closure under arbitrary joins guarantees a maximal element corresponding to the open set that is the whole space. So frames are a setting to speak of topological structure alone, without referring to a set of underlying points, hence, pointless topology. Observe that in the definition of continuous functions, open sets in the *codomain* must correspond (uniquely) to open sets in the *domain* – so every continuous function induces a frame homomorphism going in the opposite direction that the function does between spaces, hence, to obtain the category **Loc** such that directions align, we reverse the arrows of **Frm**. Observe that continuous relations induce frame homomorphisms in the same way. These observations give us insight into how to construct the free and forgetful functors.

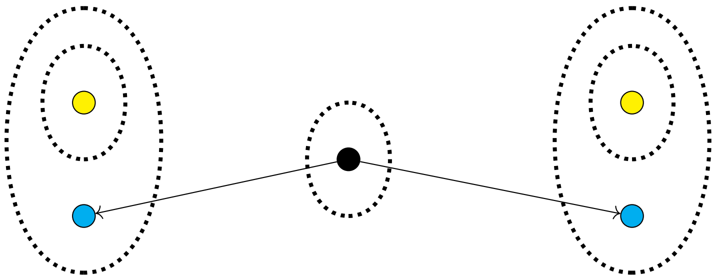
Definition 1.3.32 (U : **ContRel** $\to$ **Loc**)**.** On objects, U sends a topology X^τ to the frame of opens in τ, which we denote $\hat{\tau}$.

On morphisms $R : X^\tau \to Y^\sigma$, the corresponding partial frame morphism $\hat{\tau} \leftarrow \hat{\sigma}$ (notice the direction reversal for **Loc**), we define to be $\{(U_{\in\sigma}, R^\dagger(U)_{\in\tau}) \mid U \in \sigma\}$. We ascertain that this is (1) a function that is (2) a frame homomorphism. For (1), since the relational converse picks out precisely one subset given any subset as input, these pairs do define a function. For (2), we observe that the relational converse (as all relations) preserve arbitrary unions and intersections, i.e. $R^\dagger(\bigcap_i U_i) = \bigcap_i R^\dagger(U_i)$ and $R^\dagger(\bigcup_i U_i) = \bigcup_i R^\dagger(U_i)$, so we do have a frame homomorphism. Associativity follows easily.

Proposition 1.3.33 (U has no left adjoint)**.** *Proof.* Seeking contradiction, if U were a right adjoint, it would preserve limits. The terminal object in **Loc** is the two-element lattice $\bot < \top$, where the unique frame homomorphism to any $\mathcal{L}$ sends $\top$ to the top element of $\mathcal{L}$ and $\bot$ to the bottom element. In **ContRel**, the empty topology $\mathbf{0} = (\varnothing, \{\varnothing\})$ is terminal (and initial). However, $U\mathbf{0}$ is the singleton lattice, not $\bot < \top$ (which is the image under U of the singleton topology). □

This is a rather frustrating result, because U does turn continuous relations into backwards frame homomorphisms on lattices of opens; see Proposition 1.2.13, and note that in the frame of opens associated with a topology, the empty set becomes the bottom element. The obstacle is the fact that the empty topology is both initial and terminal in **ContRel**. We may be tempted to try treating U as a right adjoint going to **Frm** instead, but then the monad induced by the injunction on **Loc** would trivialise: left adjoints preserve colimits, so any putative left adjoint F must send $\bot < \top$ (initial in **Frm** by duality) to the empty toplogy, and the empty topology as terminal object must be sent to the terminal singleton frame, which implies that the monad UF on **Frm** sends everything to the singleton lattice.

Why not Span(**Top**)? One common generalisation of relations is to take spans of monics in the base category []. This actually produces a different category than the one we have defined. Below is an example of a span of monic continuous functions from **Top** that corresponds to a relation that doesn't live in **ContRel**. It is the span with the singleton as apex, with maps from the singleton to the closed points of a two Sierpiński spaces.



Why not a Kleisli construction on **Top**? Another way to view the category **Rel** is as the Kleisli category $K_{\mathcal{P}}$ of the powerset monad on **Set**; that is, every relation $A \to B$ can be viewed as a function $A \to \mathcal{P}B$, and composition works by exploiting the monad multiplication: $A \xrightarrow{f} \mathcal{P}B \xrightarrow{\mathcal{P}g} \mathcal{P}\mathcal{P}C \xrightarrow{\mu_{\mathcal{P}C}} \mathcal{P}C$. So it is reasonable to investigate whether there is a monad T on **Top** such that K_T is equivalent to **ContRel**. We observe that the usual free-forgetful adjunction between **Set** and **Top** sends the former to a full subcategory (of continuous functions between discrete topologies) of the latter, so a reasonable coherence condition we might ask for the putative monad T to satisfy is that it is related to $\mathcal{P}$ via the free-forgetful adjunction. This amounts to asking for the following commutative diagram (in addition to the usual ones stipulating that T and $\mathcal{P}$ are monadic):

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "T"] \arrow[d, bend left] & \mathbf{Top} \arrow[d, bend left] \\
\mathbf{Set} \arrow[r, "\mathcal{P}"] \arrow[u, bend left] \arrow[u, phantom, "\dashv"] & \mathbf{Set} \arrow[u, bend left] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

This condition would be nice to have because it witnesses $K_{\mathcal{P}}$ as precisely K_T restricted to the discrete topologies, so that T really behaves as a conservative generalisation of the notion of relations to accommodate topologies. As a consequence of this condition, we may observe that discrete topologies $X^\star$ must be sent to discrete topologies on their powerset $\mathcal{P}X^\star$. In particular, this means the singleton topology is sent to the the discrete topology on a two-element set; $T\bullet = \mathbf{2}$. This sinks us. We know from Proposition 1.2.11 that the continuous relations $X^\tau \to \bullet$ are precisely the open sets of τ, which correspond to continuous functions into Sierpiński space $X^\tau \to \mathbb{S}$, and $\mathbb{S} \neq \mathbf{2}$.

Where is the topology coming from?

It is category-theoretically natural to ask whether **ContRel** is "giving topology to relations" or "powering up topologies with relations", but we have explored those techniques and it doesn't seem to be that. It is possible that the failure of these regular avenues may explain why I had such difficulty finding **ContRel** in the literature. However, we do have a free-forgetful adjunction between **ContRel** and **Rel**, and if we focus on this, it *is* possible to crack the nut of where topology is coming from with enough machinery; here is one such sketch. Observe that the forgetful functor looks like it could be a kind of fibration, where the elements of the fibre over any set A in **Rel** correspond to all possible topologies on A. Moreover, these topologies may be partially ordered by coarseness-fineness to form a frame (though a considering it a preorder will suffice.) The fibre over a relation $R : A \to B$ is all pairs of topologies τ, σ such that R is continuous between A^τ and B^σ. The crucial observation is that if R is continuous between τ and σ, then R will be continuous for any finer topology in the domain, $\tau \leq \tau'$, and any coarser topology in the codomain $\sigma' \leq \sigma$; that is, the fibre over R displays a boolean-valued profunctor between preorders. So **ContRel** can be viewed as the display category induced by a functor **Rel** $\to \mathcal{C}$, where $\mathcal{C}$ is a category with preorders for objects and boolean-enriched profunctors as morphisms, and the functor encodes topological data by sending sets in **Rel** to preorders of all possible topologies, and relations to profunctors. I have deliberately left this as a sketch because it doesn't seem worth it to view something so simple in such a complex way (I accept the charges of hypocrisy having just used weak n-categories to present TAGs.)

1.4 *Populating space with shapes using sticky spiders*

1.4.1 *When does an object have a spider (or something close to one)?*

Example 1.4.1 (The copy-compare spiders of **Rel** are not always continuous)**.** The compare map for the Sierpiński space is not continuous, because the preimage of $\{0, 1\}$ is $\{(0, 0), (1, 1)\}$, which is not open in the product space of $\mathcal{S}$ with itself.

Reminder 1.4.2 (copy-compare spiders of **Rel**)**.** For a set X, the *copy* map $X \to X \times X$ is defined:

$$\{(x, (x, x)) : x \in X\}$$

the *compare* map $X \times X \to X$ is defined:

$$\{((x, x), x) : x \in X\}$$

These two maps are the (co)multiplications of special frobenius algebras. The (co)units are *delete*:

$$\{(x, \star) : x \in X\}$$

and *everything*:

$$\{(\star, x) : x \in X\}$$

WE CAN USE SPLIT IDEMPOTENTS TO TRANSFORM COPY-SPIDERS FROM DISCRETE TOPOLOGIES TO STICKY-SPIDERS ON OTHER SPACES.

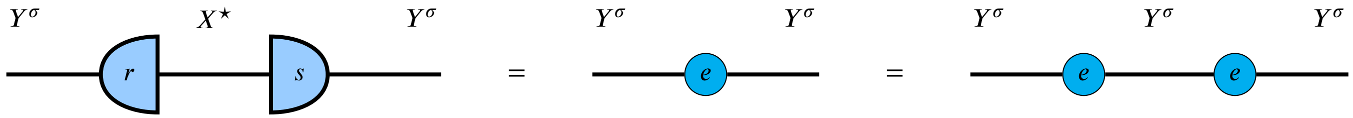
Reminder 1.4.5 (Split idempotents)**.** An **idempotent** in a category is a map $e : A \to A$ such that

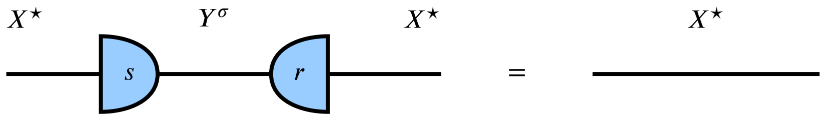
$$A \xrightarrow{e} A \xrightarrow{e} A = A \xrightarrow{e} A$$

A **split idempotent** is an idempotent $e : A \to A$ along with a **retract** $r : A \to B$ and a **section** $s : B \to A$ such that:

$$A \xrightarrow{e} A = A \xrightarrow{r} B \xrightarrow{s} A$$
$$B \xrightarrow{s} A \xrightarrow{r} B = B \xrightarrow{id} B$$

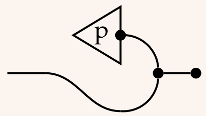
We can graphically express the behaviour of a split idempotent e as follows, where the semicircles for the section and retract r, s form a visual pun.


Y^σ
r
X^⋆
s
Y^σ
=
Y^σ
e
Y^σ
=
Y^σ
e
Y^σ
e
Y^σ

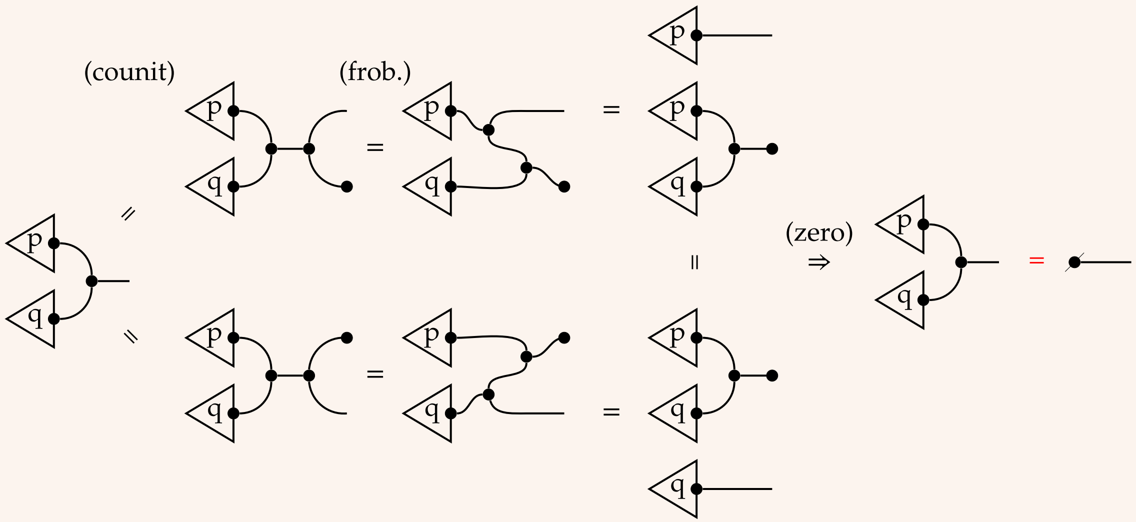

X^⋆
s
Y^σ
r
X^⋆
=
X^⋆

Proposition 1.4.3. The copy map is a spider iff the topology is discrete.

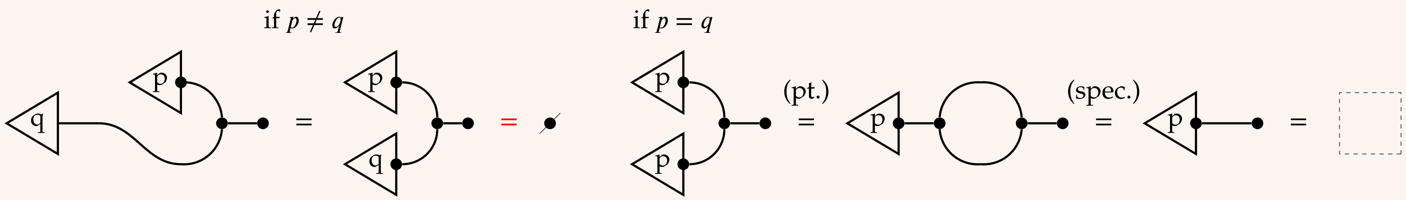
Proof. Discrete topologies inherit the usual copy-compare spiders from **Rel**, so we have to show that when the copy map is a spider, the underlying wire must have a discrete topology. Suppose that some wire has a spider, and construct the following open set using an arbitrary point p:



It will suffice to show that this open set tests whether the input is the singleton $\{p\}$ – when all singletons are open, the topology is discrete. As a lemma, we show that comparing distinct points $p \neq q$ yields the empty state.

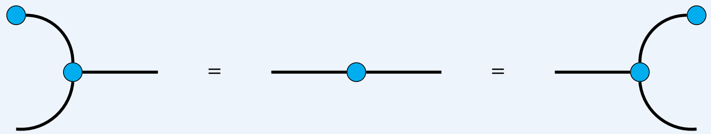


The (zero) implication follows since $p \neq q$ by assumption, so we know that deleting the comparison of p and q cannot be the unit scalar, and so must be the zero scalar, hence the comparison of p and q is the empty state. Now, the following case analysis shows that our open set only contains the point p.

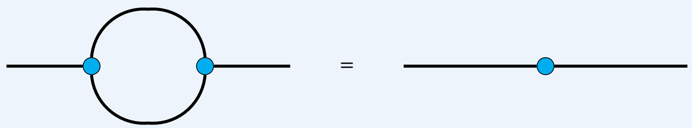


□

Definition 1.4.4 (Sticky spiders)**.** A **sticky spider** (or just an e-spider, if we know that e is a split idempotent), is a spider *except* every identity wire on any side of an equation in Figure **??** is replaced by the idempotent e.



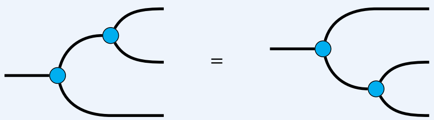
e-(co)unitality



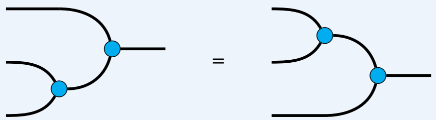
e-special

The desired graphical behaviour of a sticky spider is that one can still coalesce all connected spider-bodies together, but the 1-1 spider "sticks around" rather than disappearing as the identity. This is achieved by the following rules that cohere the idempotent e with the (co)unit and (co)multiplications; they are the same as the usual rules for a special commutative frobenius algebra with two exceptions. First, where an identity wire appears in an equation, we replace it with an idempotent. Second, the monoid and comonoid components freely emit and absorb idempotents. By these rules, the usual proof [] for the normal form of spiders follows, except the idempotent becomes an explicit 1-1 spider, rather than the identity.

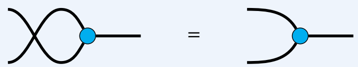
Coassociative



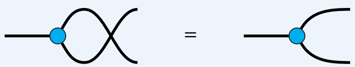
Associative



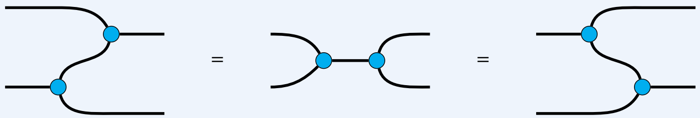
Commutative



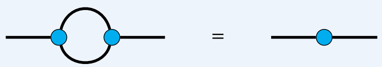
Cocommutative



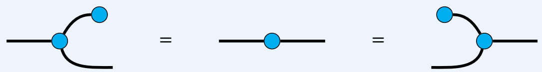
Frobenius



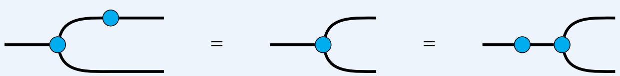
Sticky-special



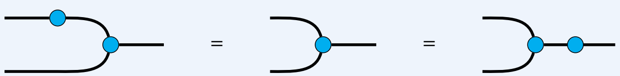
Sticky-(co)unital



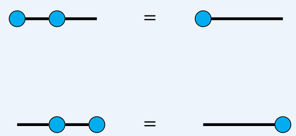
Absorb-comult



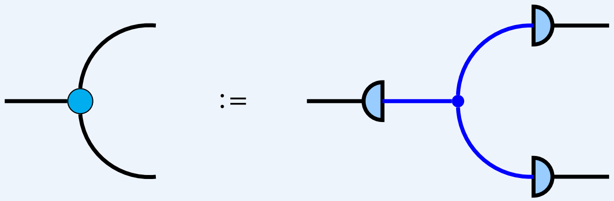
Absorb-mult

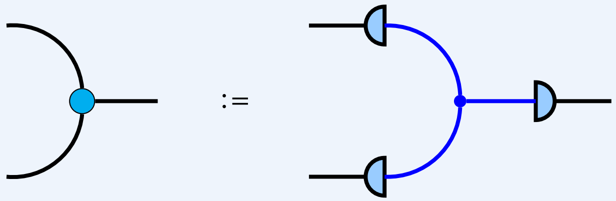


Absorb-(co)unit

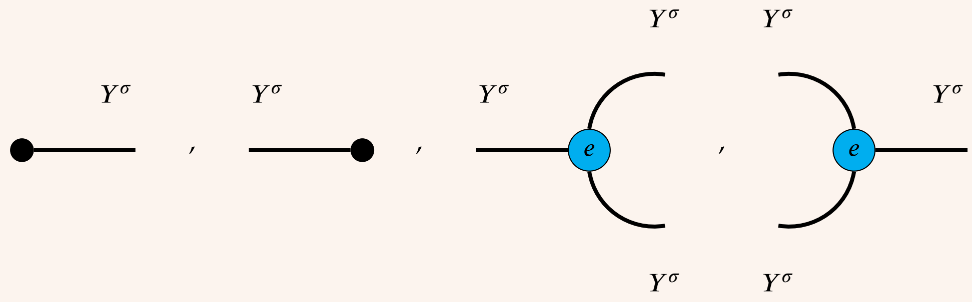


Construction 1.4.6 (Sticky spiders from split idempotents)**.** Given an idempotent $e : Y^\sigma \to Y^\sigma$ that splits through a discrete topology $X^\star$, we construct a new (co)multiplication as follows:





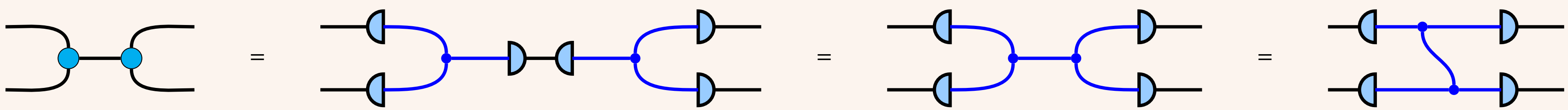
Proposition 1.4.7 (Every idempotent that splits through a discrete topology gives a sticky spider)**.**

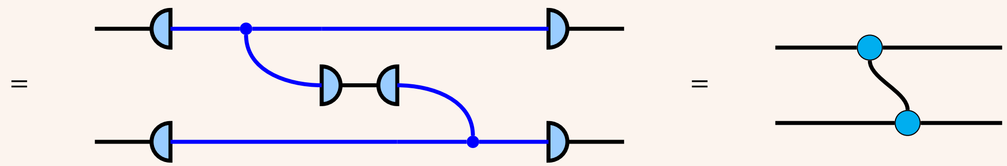


is a sticky spider

We can check that Construction 1.4.6 satisfies the frobenius rules as follows. We only present one equality; the rest follow the same idea.

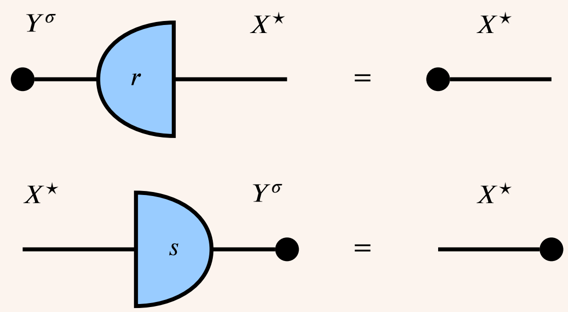
(defn.) (idem.) (frob.)



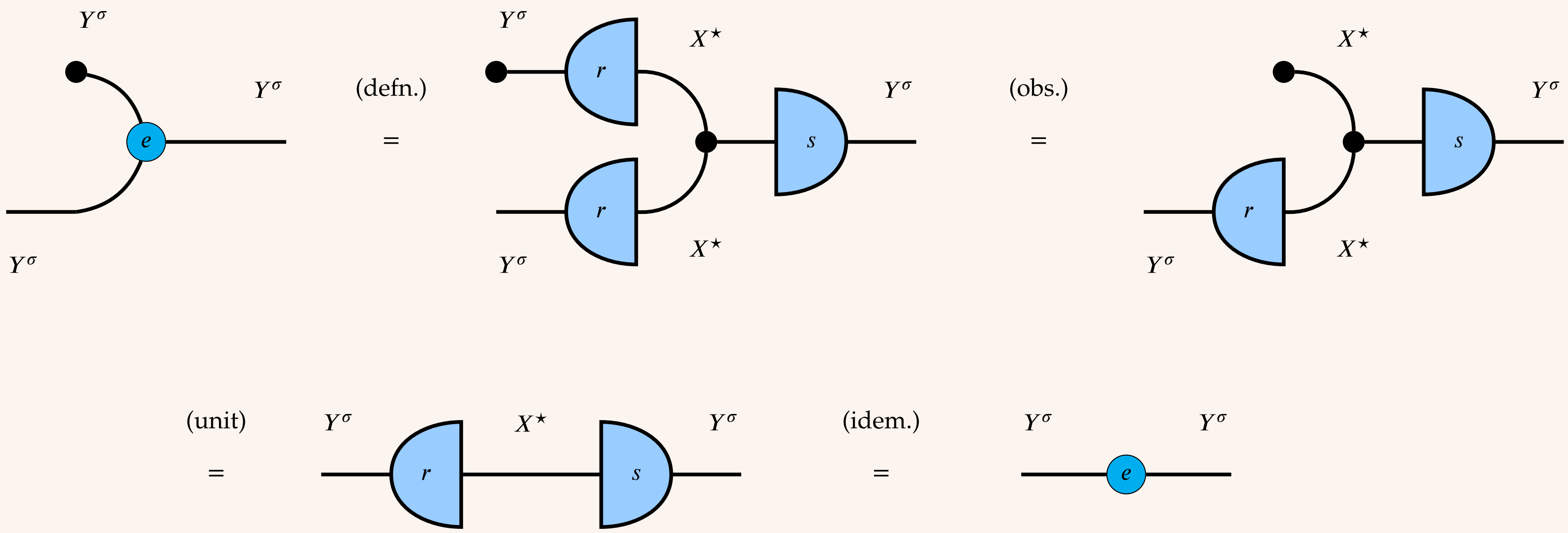


(split) (defn.)

To verify the sticky spider rules, we first observe that since $X^\star \xrightarrow{s} Y^\sigma \xrightarrow{r} X^\star = X^\star \xrightarrow{id} X^\star$, r must have all of $X^\star$ in its image, and s must have all of $X^\star$ in its preimage, so we have the following:



Now we show that e-unitality holds:



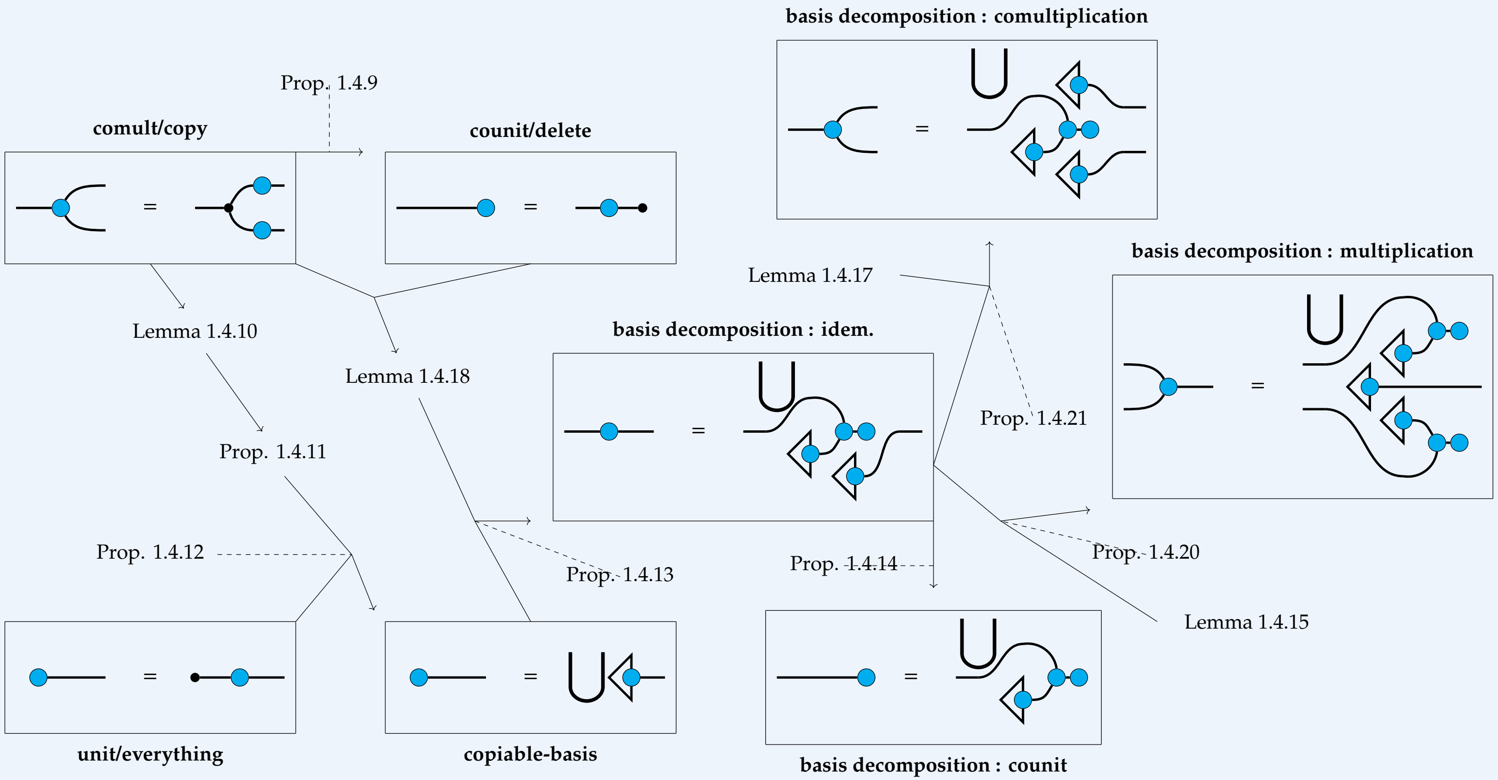
The proofs of e-counitality, and e-speciality follow similarly.

We can prove a partial converse of Proposition 1.4.7: we can identify two diagrammatic equations that tell us precisely when a sticky spider has an idempotent that splits though some discrete topology.

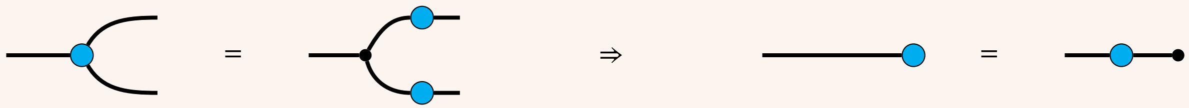
Theorem 1.4.8. A sticky spider has an idempotent that splits through a discrete topology if and only if in addition to the sticky spider equalities, the following relations are also satisfied.

Unit/everything Comult/copy

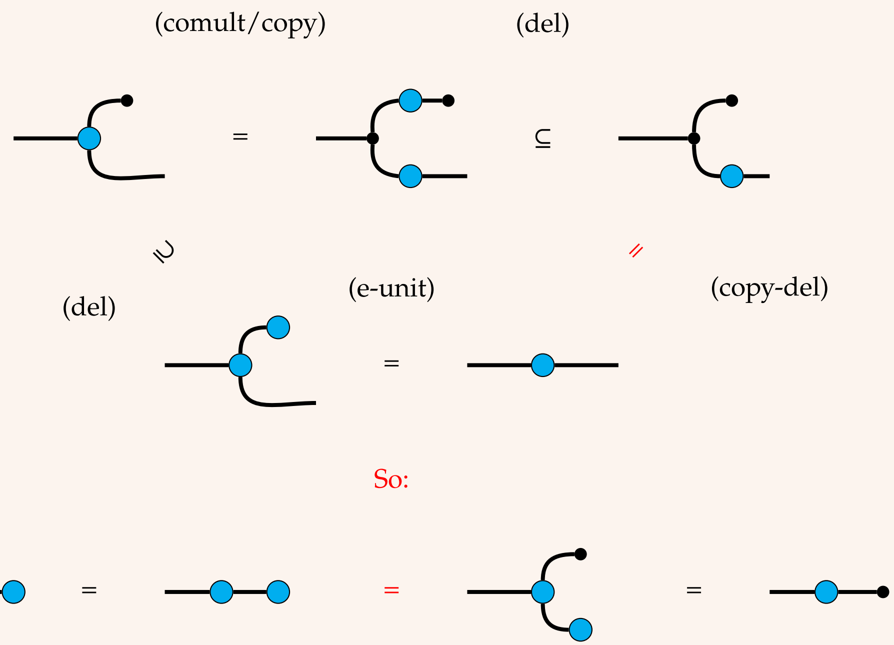
The proof is involved, so here is a map of lemmas and propositions.



Proposition 1.4.9 (comult/copy implies counit/delete)**.**

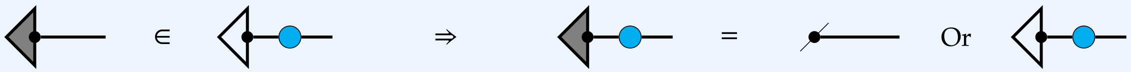


Proof.



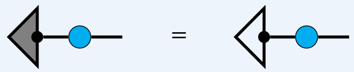
□

Lemma 1.4.10 (All-or-Nothing)**.** Consider the set $e(\{x\})$ obtained by applying the idempotent e to a singleton $\{x\}$, and take an arbitrary element $y \in e(x)$ of this set. Then $e(\{y\}) = \varnothing$ or $e(\{x\}) = e(\{y\})$. Diagrammatically:

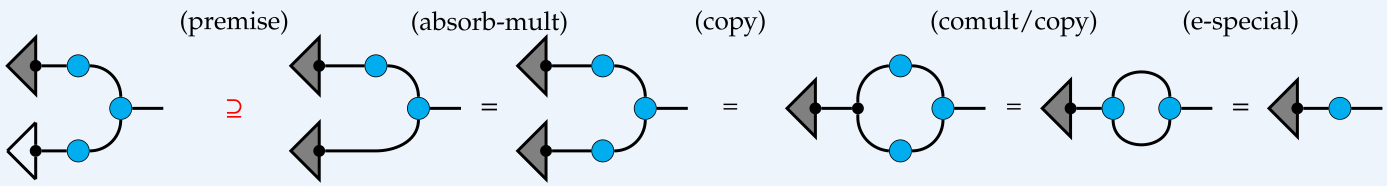


Suppose

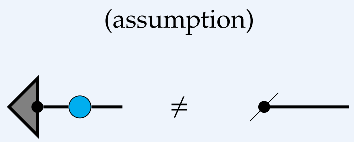
For the claim, we seek:



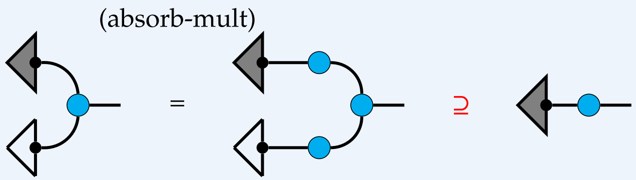
We have the following inclusion:



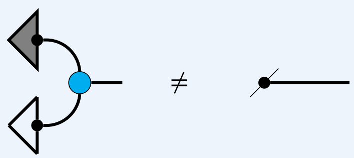
Since:



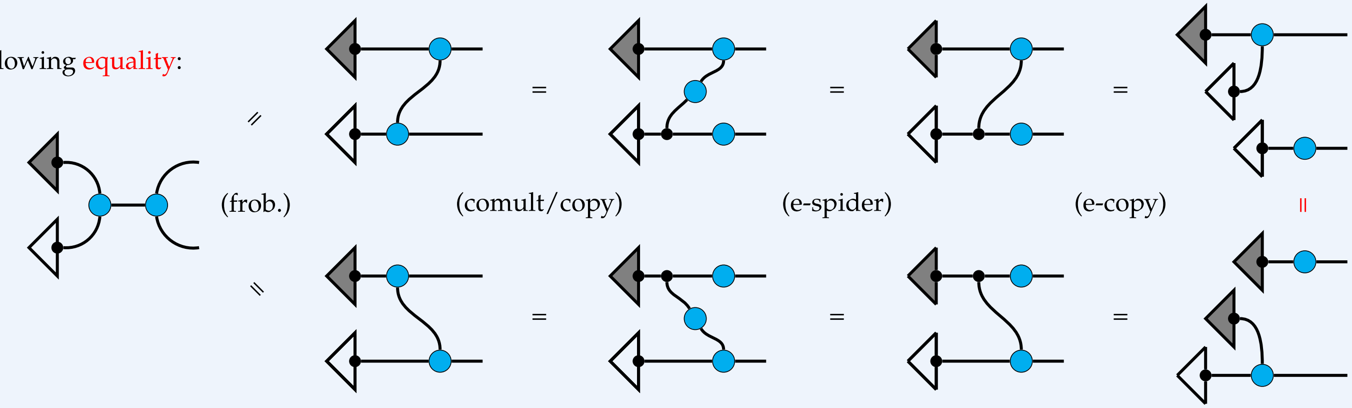
and



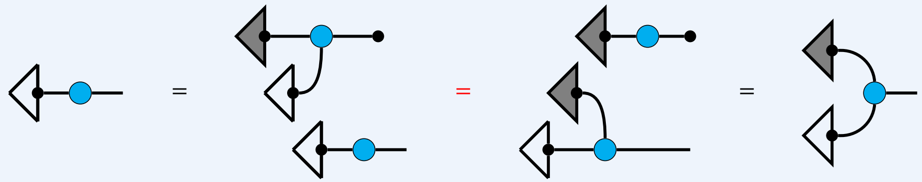
we have that:



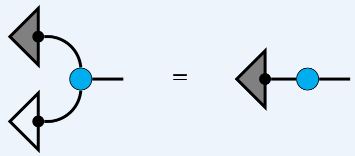
So we have the following equality:



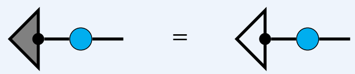
Which implies:



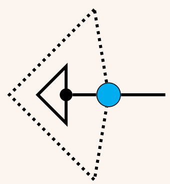
and symmetrically,



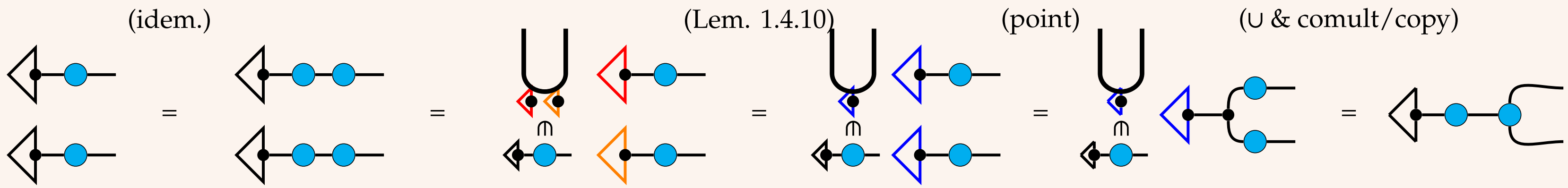
So we have the claim:



Proposition 1.4.11 (e of any point is e-copiable)**.**

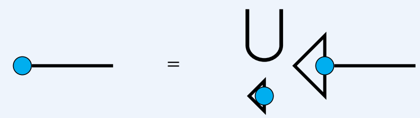


Proof.



□

Proposition 1.4.12 (The unit is the union of all *e*-copiables)**.**

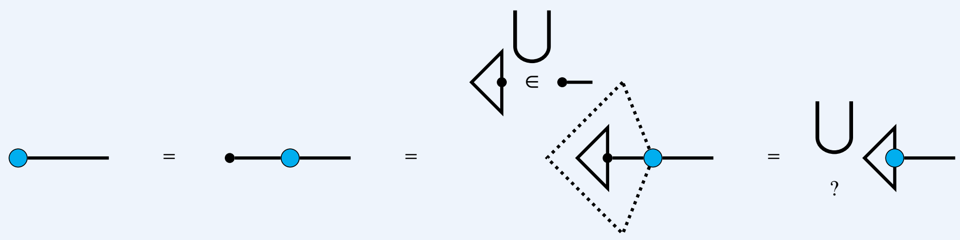


Proof.

The union of *all* *e*-copiables is a subset of the unit.

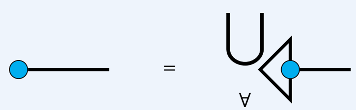
(Lem. 1.4.18) (evr.) (unit/evr.)

The unit is *some* union of *e*-copiables.



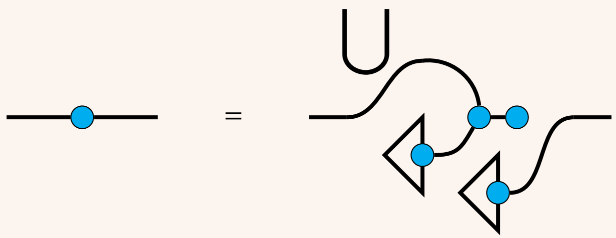
(unit/evr.) (Prop. 1.4.11)

So the containment must be an equality.

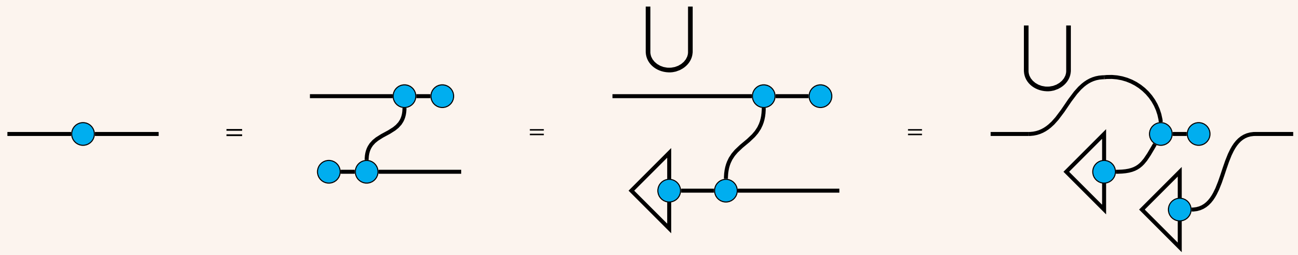


□

Proposition 1.4.13 (e-copiable decomposition of e)**.**



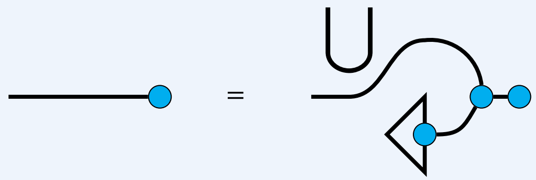
Proof.



(Prop. 1.4.12) (e-copiable)

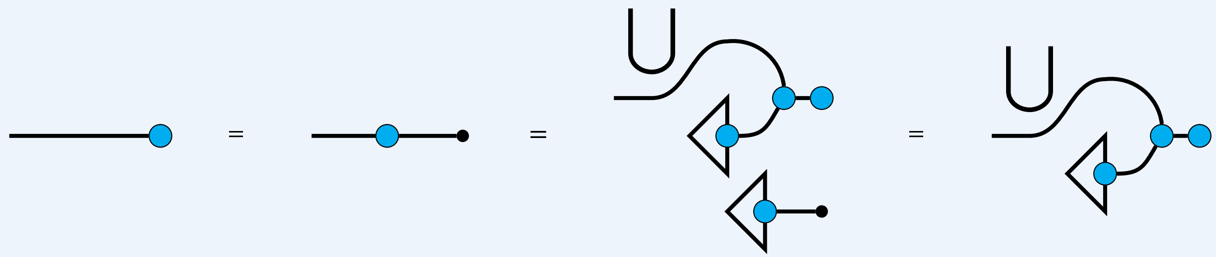
□

Proposition 1.4.14 (*e*-copiable decomposition of counit)**.**



Proof.

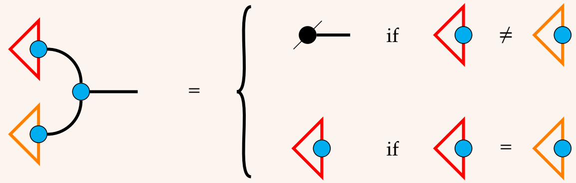
(Prop. 1.4.13)



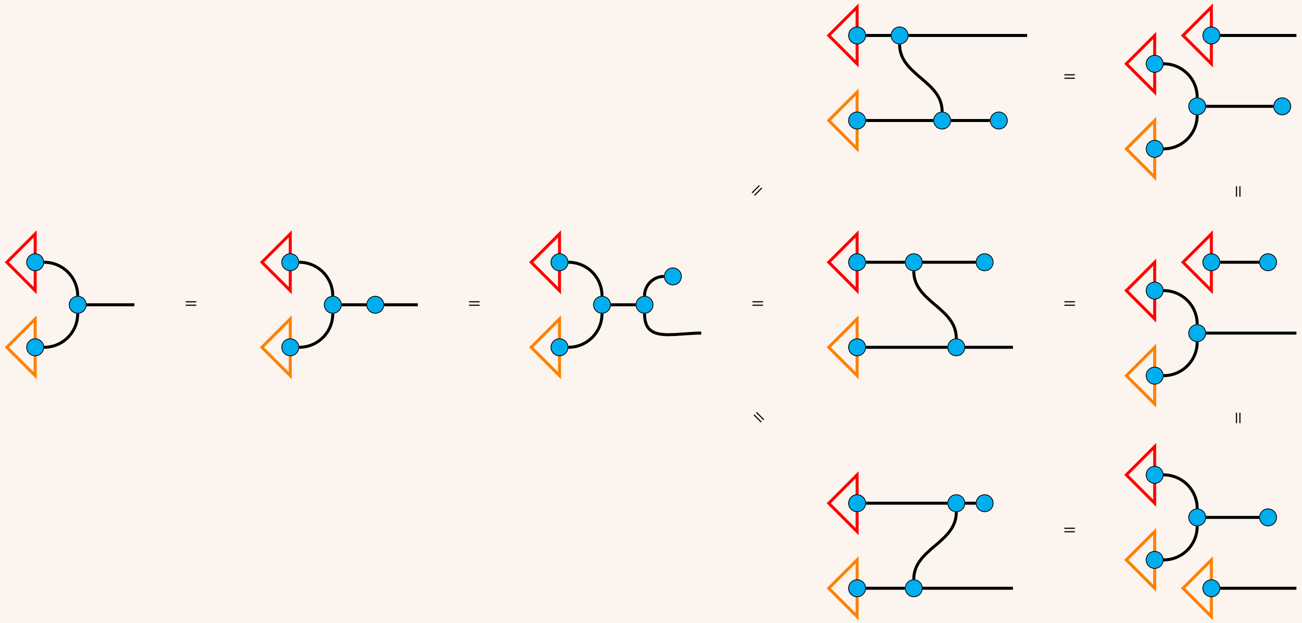
□

The e-copiable states really do behave like an orthonormal basis, as the following Lemmas show.

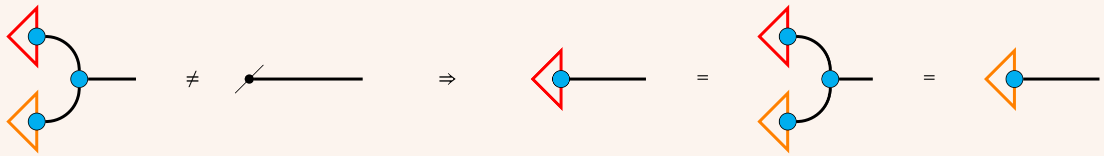
Lemma 1.4.15 (e-copiables are orthogonal under multiplication)**.**



Proof.

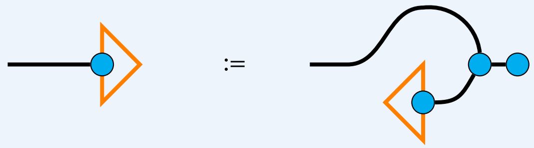


So:

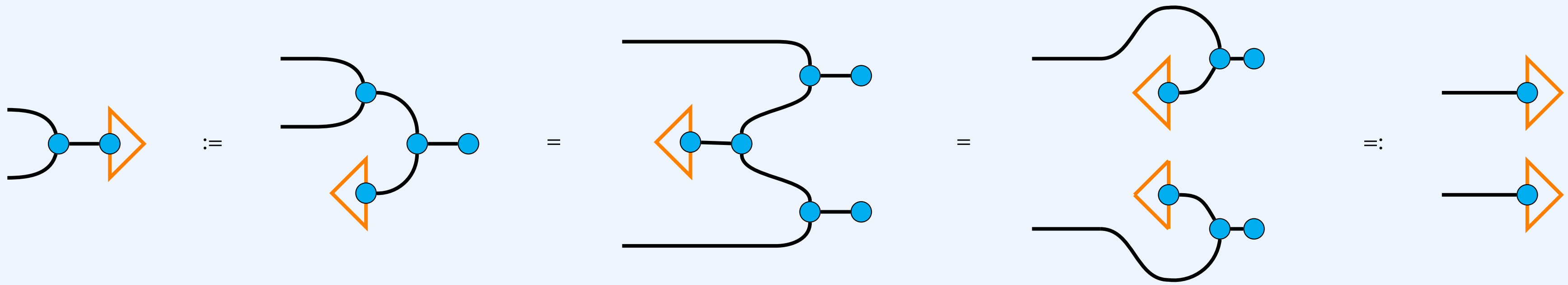


□

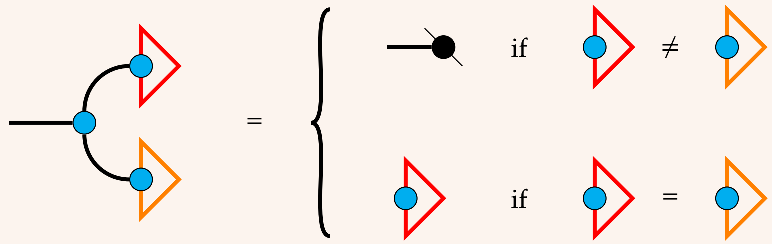
Convention 1.4.16 (Shorthand for the open set associated with an e-copiable)**.** We introduce the following diagrammatic shorthand.



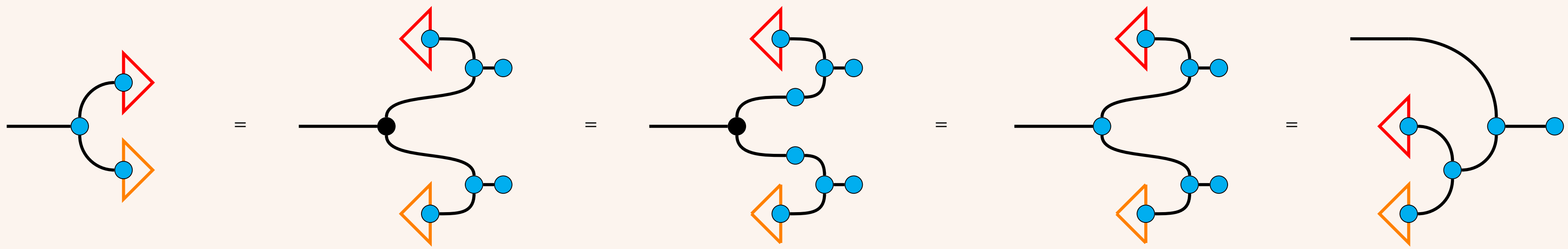
Including the coloured dot is justified, because these open sets are co-copiable with respect to the multiplication of the sticky spider.



Lemma 1.4.17 (Co-match)**.**



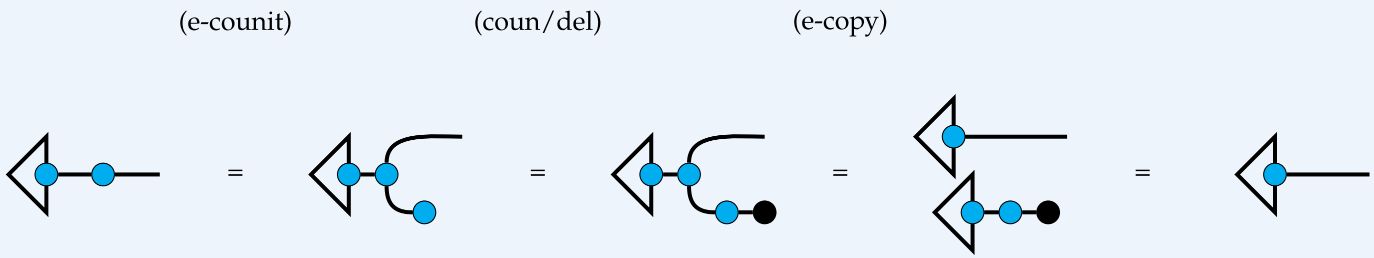
Proof.



The claim then follows by applying Lemma 1.4.15 to the final diagram. □

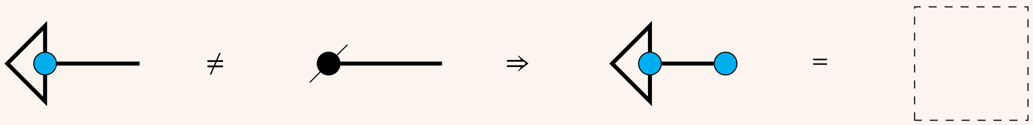
Lemma 1.4.18 (e-copiables are e-fixpoints)**.**

Proof.



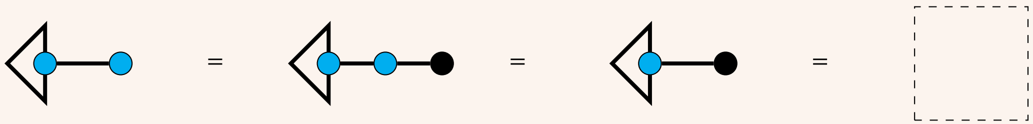
Observe that the final equation of the proof also holds when the initial e-copiable is the empty set. □

Lemma 1.4.19 (*e*-copiables are normal)**.**



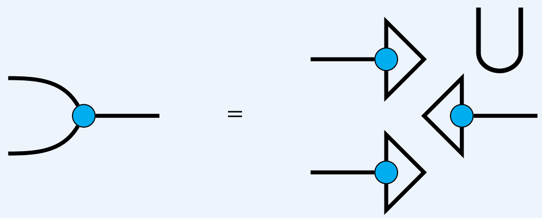
Proof.

(coun/del) (Lem. 1.4.18) (Prem.)

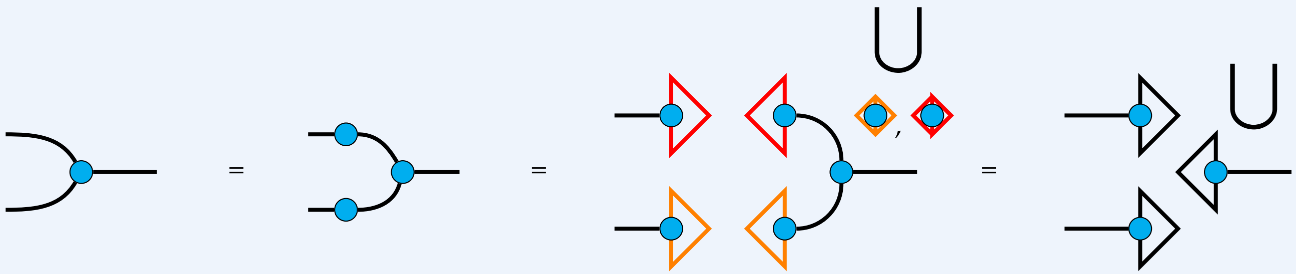


□

Proposition 1.4.20 (*e*-copiable decomposition of multiplication)**.**



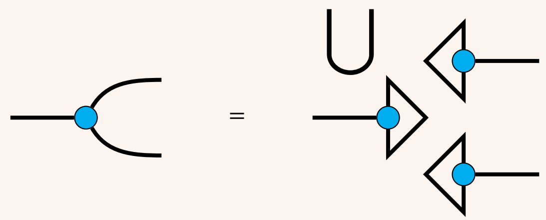
Proof.



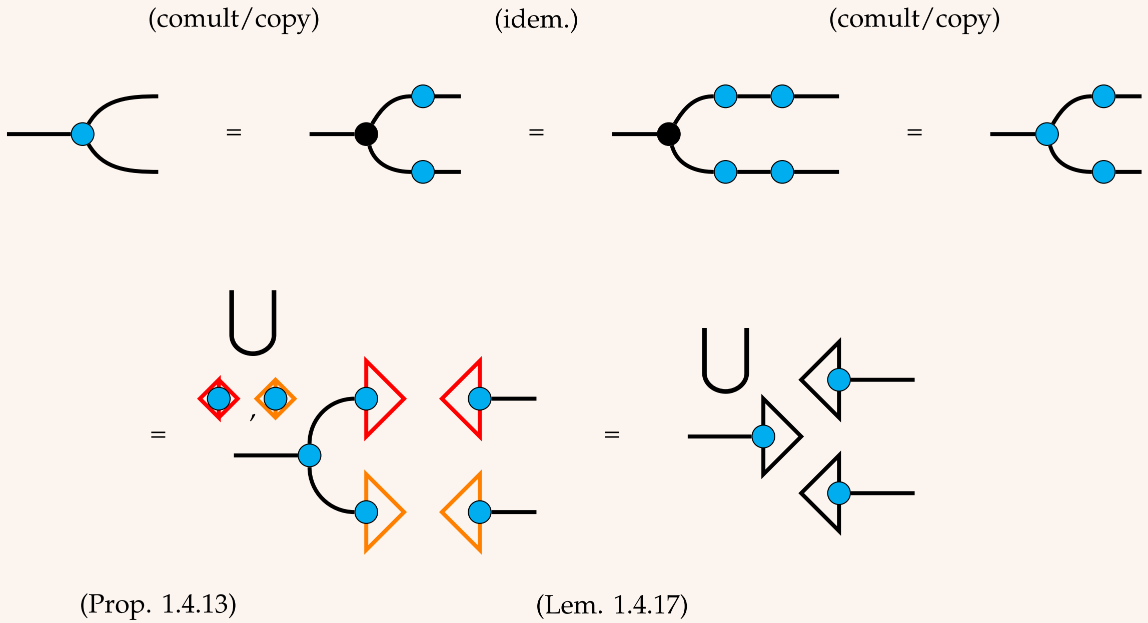
(e-spider) (Prop. 1.4.13) (Lem. 1.4.15)

□

Proposition 1.4.21 (*e*-copiable decomposition of comultiplication)**.**



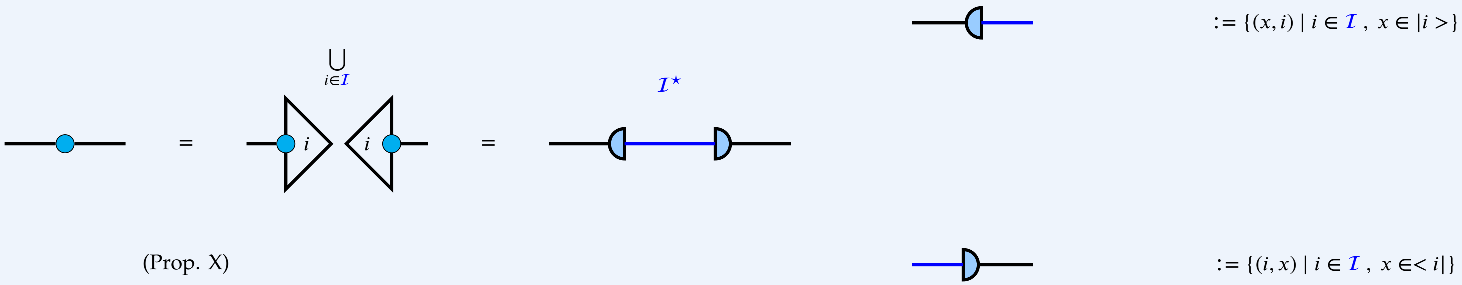
Proof.



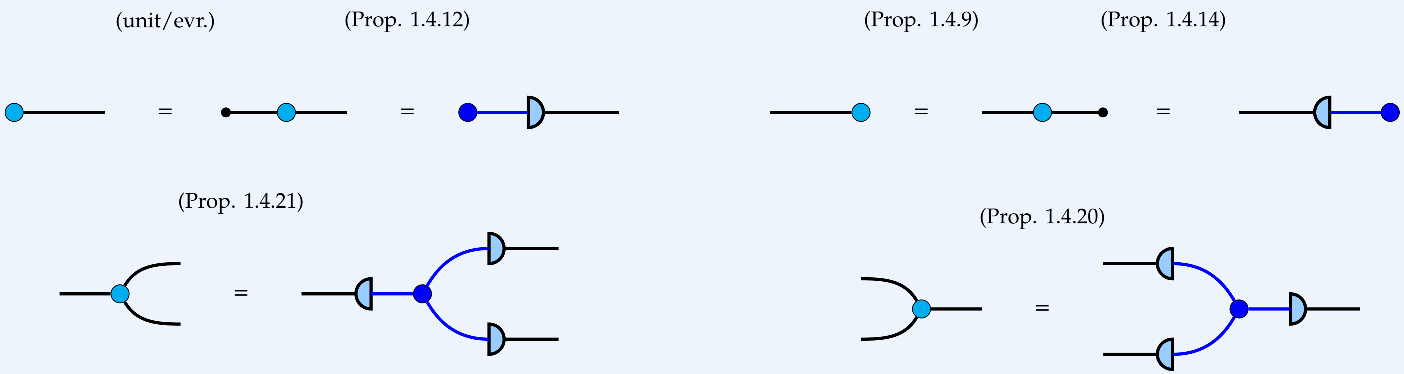
□

Now we can prove Theorem 1.4.8. First a reminder of the claim; we want to show that when given a sticky spider, the following relations hold if and only if the idempotent splits through a discrete topology.

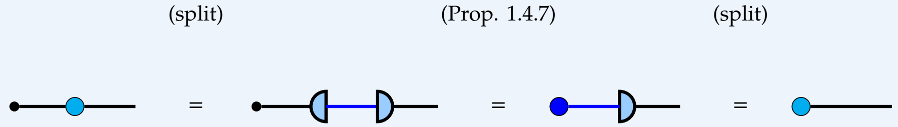
The crucial observation is that the e-copiable decomposition of the idempotent given by Proposition 1.4.13 is equivalent to a split idempotent though the set of e-copiables equipped with discrete topology.



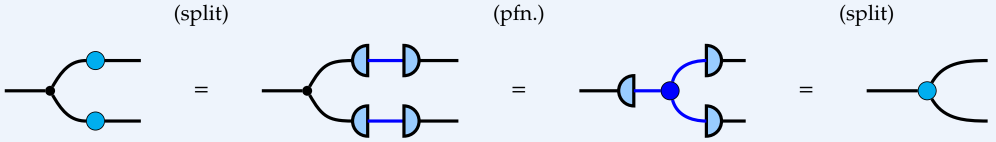
By copiable basis Proposition 1.4.12 and the decompositions Propositions 1.4.14, 1.4.20, 1.4.21, we obtain the only-if direction.



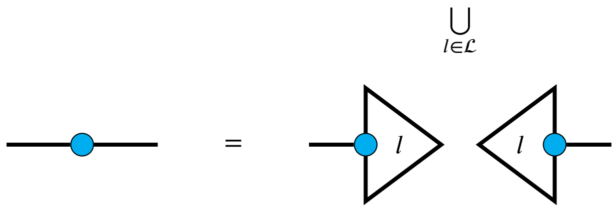
The if direction is an easy check. For the unit/everything relation, we have:



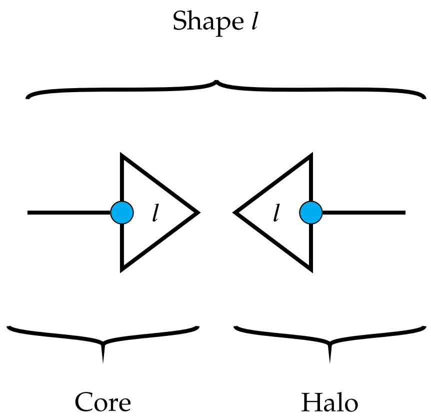
For the counit/delete relation, we observe that for any split idempotent, the retract must be a partial function. To see this, suppose the split idempotent $e = r;s$ is on (X, τ) and the discrete topology is $Y^\star$. Seeking contradiction, if the retract is not a partial function, then there is some point $x \in X$ such that $x \in e(x)$, and the image $I := r(x) \subseteq Y$ contains more than one point, which we denote and discriminate $a, b \in r(x) \subseteq Y$ and $a \neq b$. Because the composite $s;r = 1_Y$ of the section and retract must recover the identity on $Y^\star$, the section s must be total – i.e. the image $s(X) = Y$. So $x \in s(a) \cap s(b)$. Now we have that $(a, x), (b, x) \in s$, and $(x, a), (x, b) \in r$, therefore $(a, b), (b, a) \in s;r$, which by $a \neq b$ contradicts that $s;r$ is the identity relation 1_Y.



Definition 1.4.22 (Labels, shapes, cores, halos)**.** Recall by Proposition 1.4.13 that we can express the idempotent as a union of continuous relations formed of a state and test, for some indexing set of *labels* $\mathcal{L}$.



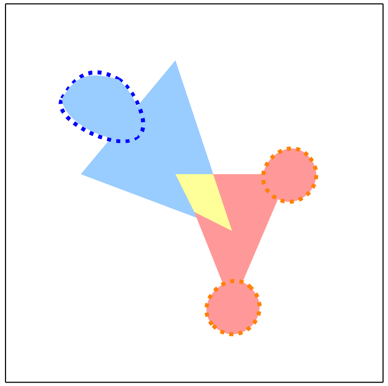
A *shape* is a component of this union corresponding to some arbitary $l \in L$. So we refer to a sticky spider as a labelled collection of shapes. The state of a shape is the *halo* of the shape. The halos are precisely the copiables of the sticky spider. The test of a shape is the *core*. The cores are precisely the cocopiables of the sticky spider.



Proposition 1.4.23 (Core exclusion: Distinct cores cannot overlap)**.** *Proof.* A direct consequence of Lemma 1.4.17. □

Proposition 1.4.24 (Core-halo exclusion: Each core only overlaps with its corresponding halo)**.** *Proof.* Seeking contradiction, if a core overlapped with multiple halos, Lemma 1.4.18 would be violated. □

Proposition 1.4.25 (Halo non-exclusion: halos may overlap)**.** *Proof.* By example:



The two shapes are colour coded cyan and magenta. The halos are two triangles which overlap at a yellow region, and partially overlap with their blobby cores. The cores are outlined in dotted blue and orange respectively. Observe that cores and halos do not have to be simply connected; in this example the core of the magenta shape has two connected components. Viewing these sticky spiders as a process, any shape that overlaps with the magenta core will be deleted and replaced by the magenta triangle, and similarly with the cyan cores and triangle. Any shape that overlaps with both the magenta and cyan cores will be deleted and replaced by the union of the triangles. Any shape that overlaps with neither core will be deleted and not replaced. □

Example 1.4.26 (Analog of quantum venn diagram paradox)**.**

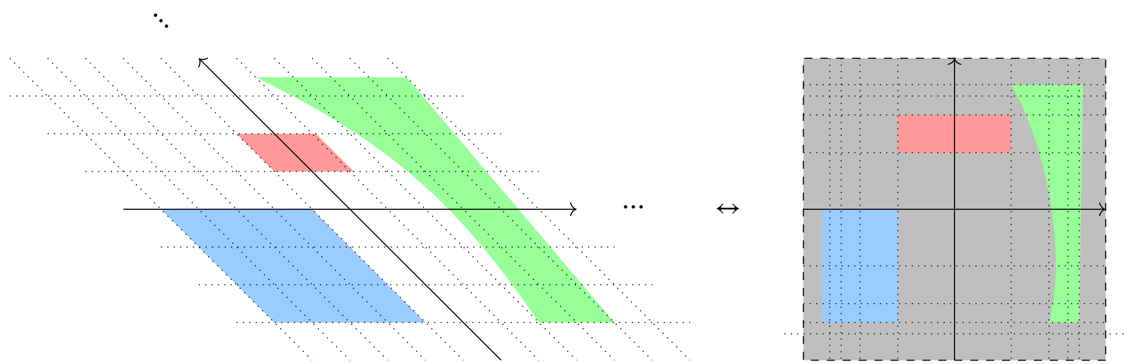
Proposition 1.4.27 (Set-indexed collection of open sets)**.**

1.5 *Topological concepts in flatland via* ***ContRel***

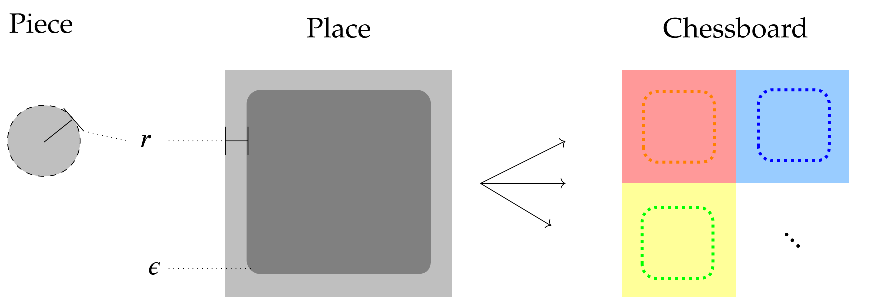
The goal of this section is to demonstrate the use of sticky spiders as formal semantics for the kinds of schematic doodles or cartoons we would like to draw. Throughout we consider sticky spiders on $\mathbb{R}^2$. In Section 1.5.1, we introduce how sticky spiders may be viewed as labelled collections of shapes. In service of defining *configuration spaces* of shapes up to rigid displacement, we diagrammatically characterise the topological subgroup of isometries of $\mathbb{R}^2$ by building up in Sections 1.5.2 and 1.5.3 the diagrammatic presentations of the unit interval, metrics, and topological groups. To further isolate rigid displacements that arise from continuous sliding motion of shapes in the plane (thus excluding displacements that result in mirror-images), in Sections 1.5.4 and 1.5.5 we diagrammatically characterise an analogue of homotopy in the relational setting. Finally, in Sections 1.5.6 and 1.5.7 we build up a stock of topological concepts and study by examples how implementing these concepts within text circuits explains some idiosyncrasies of the theory: namely why noun wires are labelled by their noun, why adjective gates ought to commute, and why verb gates do not.

1.5.1 *Shapes and places*

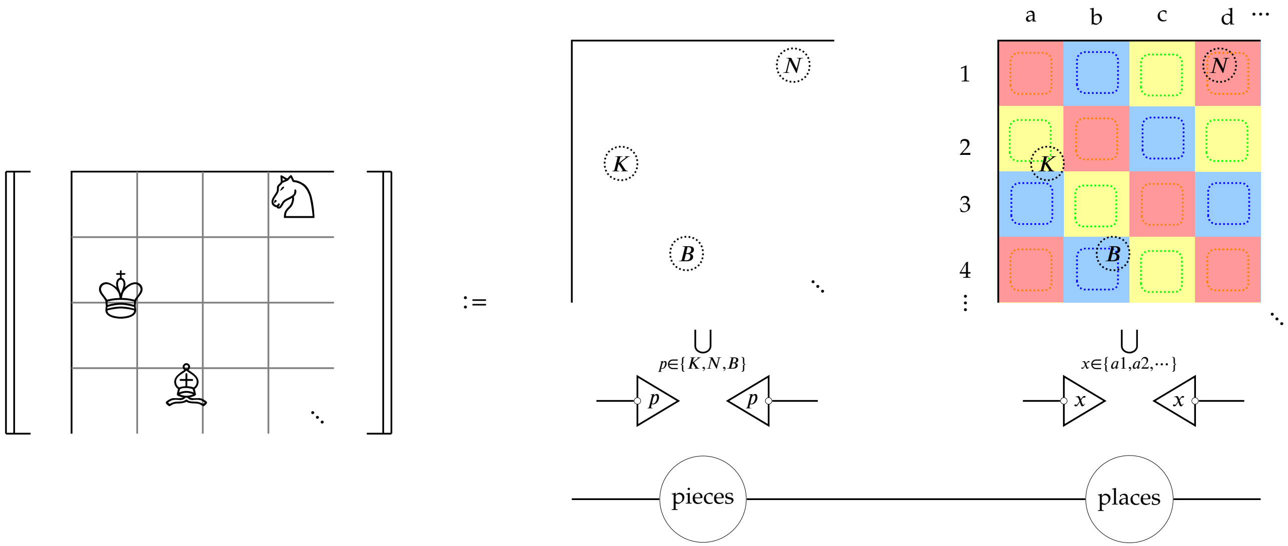
Remark 1.5.1. When we draw on a finite canvas representing all of euclidean space, properly there should be a fishbowl effect that relatively magnifies shapes close to the origin and shrinks those at the periphery, but that is only an artefact of representing all of euclidean space on a finite canvas. Since all the usual metrics are still really there, going forward we will ignore this fishbowl effect and just doodle shapes as we see fit.



Example 1.5.2 (Where is a piece on a chessboard?). How is it that we quotient away the continuous structure of positions on a chessboard to locate pieces among a discrete set of squares? Evidently shifting a piece a little off the centre of a square doesn't change the state of the game, and this resistance to small perturbations suggests that a topological model is appropriate. We construct two spiders, one for pieces, and one for places on the chessboard. For the spider that represents the position of pieces, we open balls of some radius r, and we consider the places spider to consist of square halos (which tile the chessboard), containing a core inset by the same radius r; in this way, any piece can only overlap at most one square. As a technical aside, to keep the core of the tiles open, we can choose an arbitrarily sharp curvature ϵ at the corners.



Now we observe that the calculation of positions corresponds to composing sticky spiders. We take the initial state to be the sticky spider that assigns a ball of radius r on the board for each piece. We can then obtain the set of positions of each piece by composing with the places spider. The composite (pieces;places) will send the king to a2, the bishop to b4, and the knight to d1, i.e. $\langle K| \mapsto \langle a2|$, $\langle B| \mapsto \langle b4|$ and $\langle N| \mapsto \langle d1|$. In other words, we have obtained a process that models how we pass from continuous states-of-affairs on a physical chessboard to an abstract and discrete game-state.



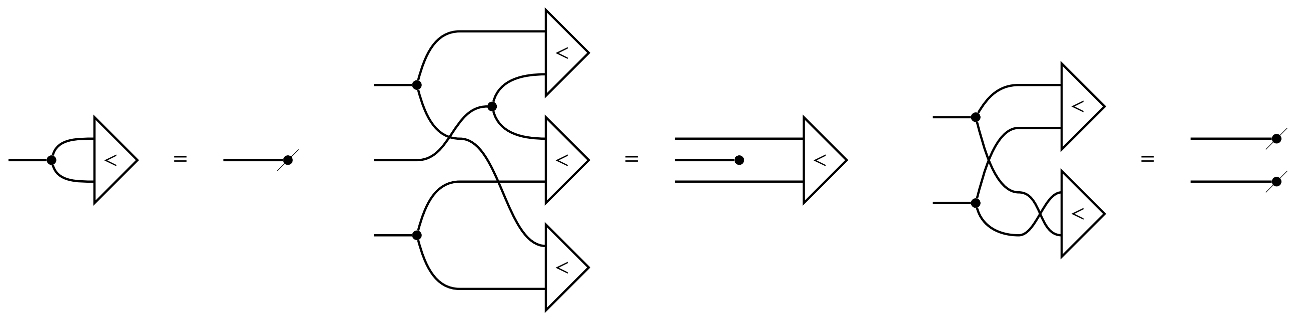
1.5.2 *The unit interval*

To begin modelling more complex concepts, we first need to extend our topological tools. If we have the unit interval, we can begin to define what it would mean for spaces to be connected (by drawing lines between points in those spaces), and we can also move towards defining motion as movement along a line. There are many spaces homeomorphic to the real line. How do we know when we have one of them? The following theorem provides an answer:

Theorem 1.5.3 (Friedman)**.** Let $\big((X,\tau),<\big)$ be a topological space with a total order. If there exists a continuous map $f : X \times X \to X$ such that $\forall a, b_{\in X} : a < f(a,b) < b$, then X is homeomorphic to $\mathbb{R}$. CITE

Figure 1.21: We can express the theorem using diagrammatic equations. First we define a total order $<$ as an open set on $X \times X$ that obeys the usual axiomatic rules:

Antireflexive ($\forall x : x \not< x$) Transitive ($\forall xyz : x < y \,\&\, y < z \Rightarrow x < z$) Antireflexive $\forall xy \neg(x < y \,\&\, y < x)$



Trichotomy $(\forall xy : x < y \vee y < x \vee x = y)$

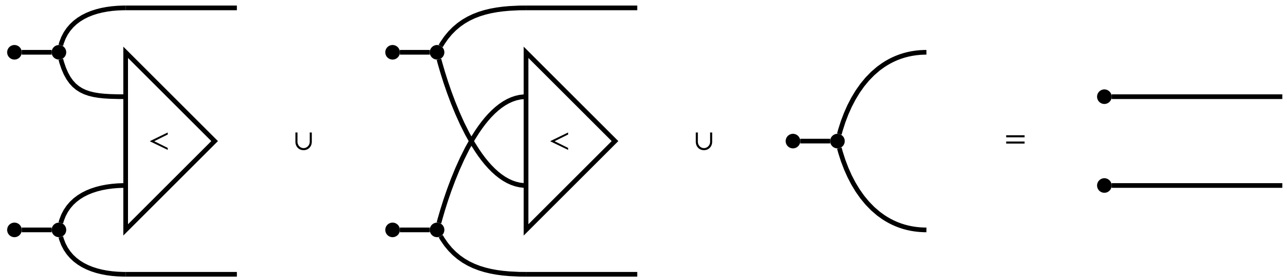
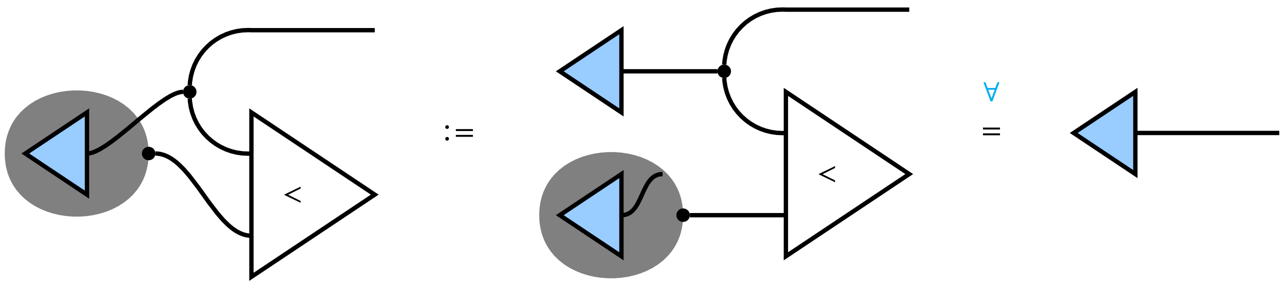


Figure 1.22: Trichotomy requires us to appeal to the rig structure, which is nonstandard for process-theoretic equations, but algebraically permissible. Going forward we will also introduce quantifiers into process-theoretic equations, essentially treating process-equations as we would any other symbolic algebraic specification.

Using the technology in the margins, we can define:

Upper Bound

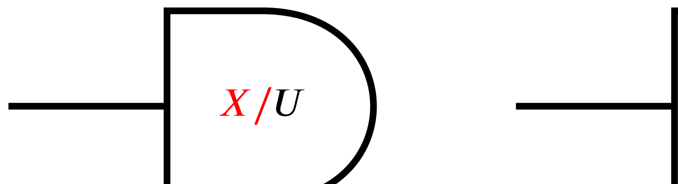


And we can add in further equations governing the upper bound endocombinator to turn it into a supremum, where the lower endpoint is obtained as the supremum of the empty set, and the upper endpoint is the supremum of the whole set.

We can introduce endpoints for open intervals directly by asking for the space X to have points that are less than or greater than all other points. Another method, which we will use here for primarily aesthetic reasons, is to use endocombinators to define suprema. Endocombinators are like functional expressions applied to diagrams. For a motivating example, consider the case when we have a locally indiscrete topology:

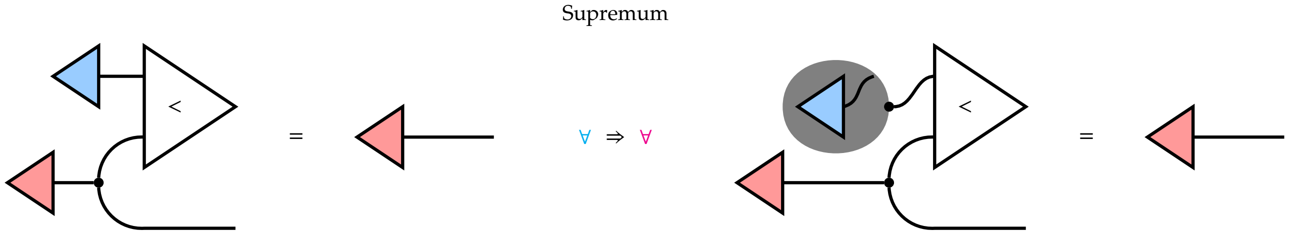
Definition 1.5.4 (Locally indiscrete topology)**.** (X, τ) is *locally indiscrete* when every open set is also closed.

If we know that a topology is locally indiscrete and we are given an open U, we would like to notate the complement X/U – which we know to be open – as any of the following, which only differ up to notation.



Unfortunately, the complementation operation $X/-$ is not in general a continuous relation, hence in the lattermost expression above we resort to using bubbles as a syntactic sugar. Formally, these bubbles are *endocombinators*, the semantics and notation for which we borrow and modify from CITE .

Definition 1.5.5 (Partial endocombinator)**.** In a category C, a *partial endocombinator* on a homset $(C)(A, B)$ is a function $(C)(A, B) \to (C)(A, B)$



Now we can define endpoints purely graphically:

Going forward, we will denote the unit interval using a thick dotted wire.

Simply connected spaces

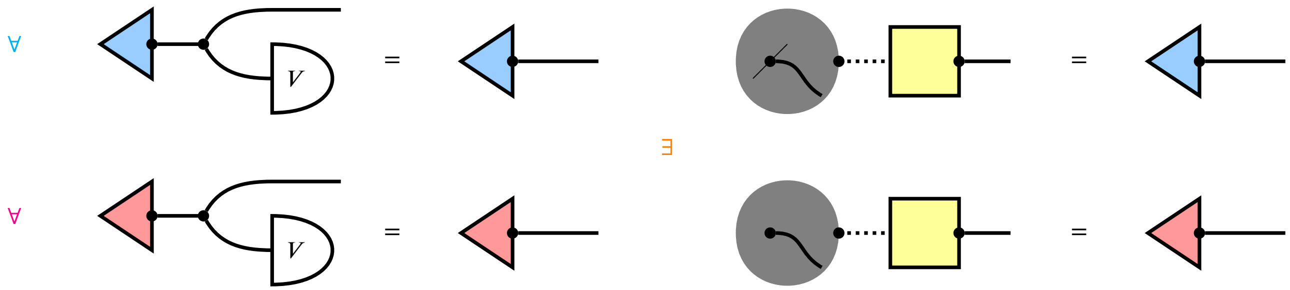
Once we have a unit interval, we can define the usual topological notion of a simply connected space: one where any two points can be connected by a continuous line without leaving the space.

Figure 1.23:

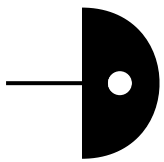
Definition 1.5.6 (Simple connectivity)**.**

In prose: for any pair of points that are within the open V, there exists some continuous function from the unit interval into the space that starts at one of the points and ends at the other.

V is *simply connected* when:



Simple connectivity is a useful enough concept that we will notate simply connected open sets as follows, where the hole is a reminder that simply connected spaces might still have holes in them.



1.5.3 *Displacing shapes*

Static shapes in space are nice, but moving them around would be nicer. So we have to define a stock of concepts to express rigid motion. Rigidity however is a difficult concept to express in topological spaces up to homeomorphism – everyone is well aware of the popular gloss of topology in terms of coffee cups being homeomorphic to donuts. To obtain rigid transformations as we have in Euclidean space, we need to define metrics, and in order to do that, we need addition.

Rigid displacements

Now we return to our sticky spiders. From now we consider sticky spiders on the open unit square, so that we can speak of shapes on a canvas. Now we will try to displace the shapes of a sticky spider. We know

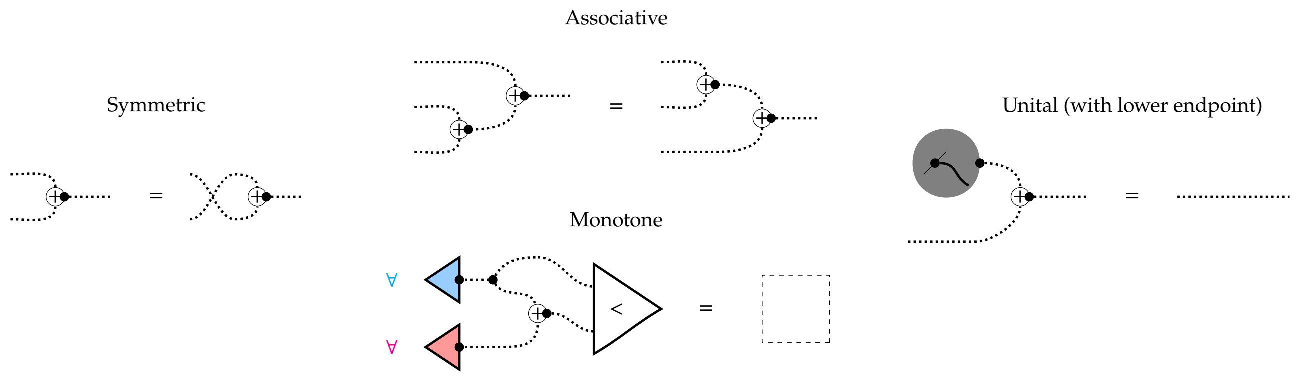


Figure 1.24:

Definition 1.5.7 (Addition)**.**

More precisely, we only need an additive monoid structure on the unit interval. We do not care about obtaining precise values from our metric, and we will not need to subtract distances from each other. All we need to know is that the lower endpoint stands in for "zero distance" – as the unit of the monoid – and that adding positive distances together will give you a larger positive distance deterministically.

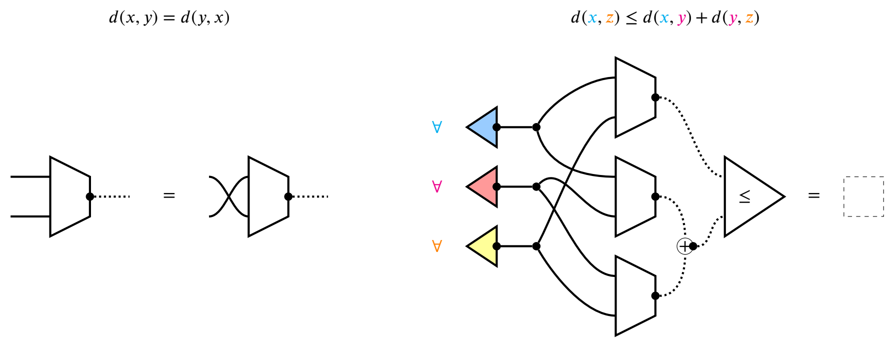


Figure 1.25:

Definition 1.5.8 (Metric)**.**

A metric on a space is a continuous map $X \to \mathbf{R}^+$ to the positive reals that satisfies these axioms.

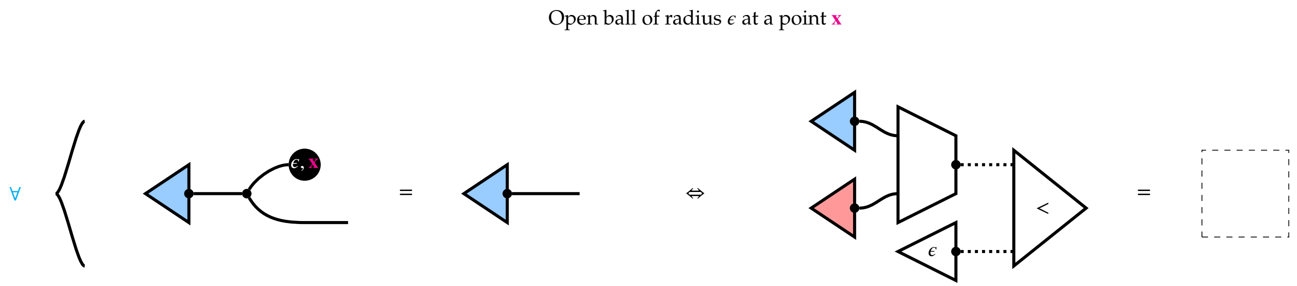


Figure 1.26:

Definition 1.5.9 (Open balls)**.** Once we have metrics, we can define the usual topological notion of open balls. Open balls will come in handy later, and a side-effect which we note but do not explore is that open balls form a basis for any metric space, so in the future whenever we construct spaces that come with natural metrics, we can speak of their topology without any further work.

A topological group G

$\forall \gamma_{\in G} \exists \gamma^{-}_{\in G}$

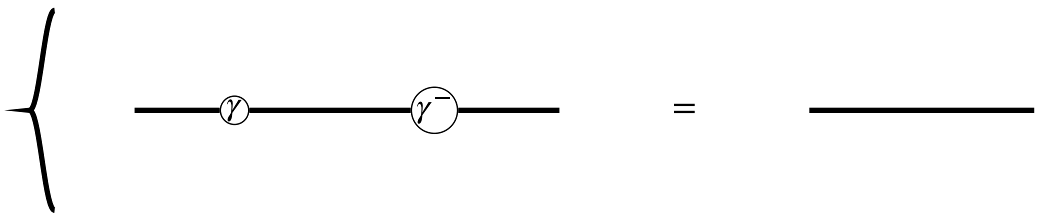


Figure 1.27:

Definition 1.5.10 (Topological groups).

It is no trouble to depict collections of invertible transformations of spaces $X \to X$. A consequence of invertibility and the requirement that the identity transform is a group element forces all transformations in a topological group to be functions.

γ is an *isometry*

$\exists$

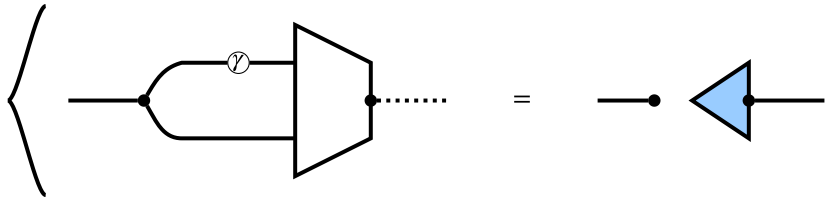


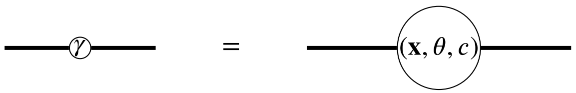
Figure 1.28:

Definition 1.5.11 (Isometry).

But recall that the collections of invertible transformations we are really interested in are the *rigid* ones, the ones that move objects in space without deforming them, i.e. the isometries. Iso-metry means same-distance. A distance preserving transformation is one such that applying the metric pointwise before and after the transformation of a shape gives a fixed value. RHS should be dotted output!

the planar isometries of Euclidean space can be expressed as a translation, rotation, and a bit to indicate the chirality of the shape – as mirror reflections are also an isometry.

Isometries of $\mathbf{R}^2$



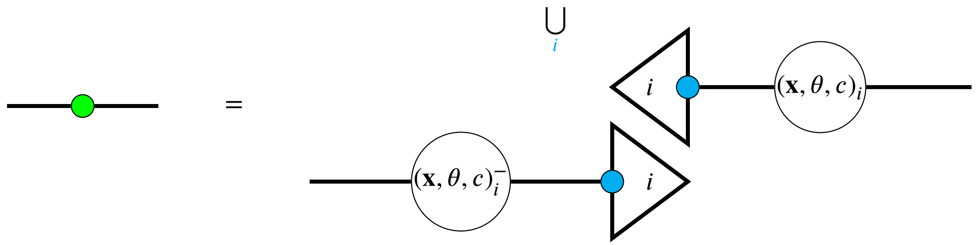
$\mathbf{x} \in \mathbf{R}^2$

$\theta \in S^1 \simeq [0, 2\pi)$

$c \in \{-1, 1\}$

With this in mind, we have the following condition relating different spiders, telling us when one is the same as the other up to rigidly displacing shapes.

Rigid displacement



Chirality leaves us with a wrinkle: in flatland, we do not expect shapes to suddenly flip over. We would like to express just those rigid transformations that leave the chirality of the shape intact, because really we want to only be able to slide the shapes around the canvas, not leave the canvas to flip over. So we go on to define rigid continuous motion in flatland.

1.5.4 Moving shapes

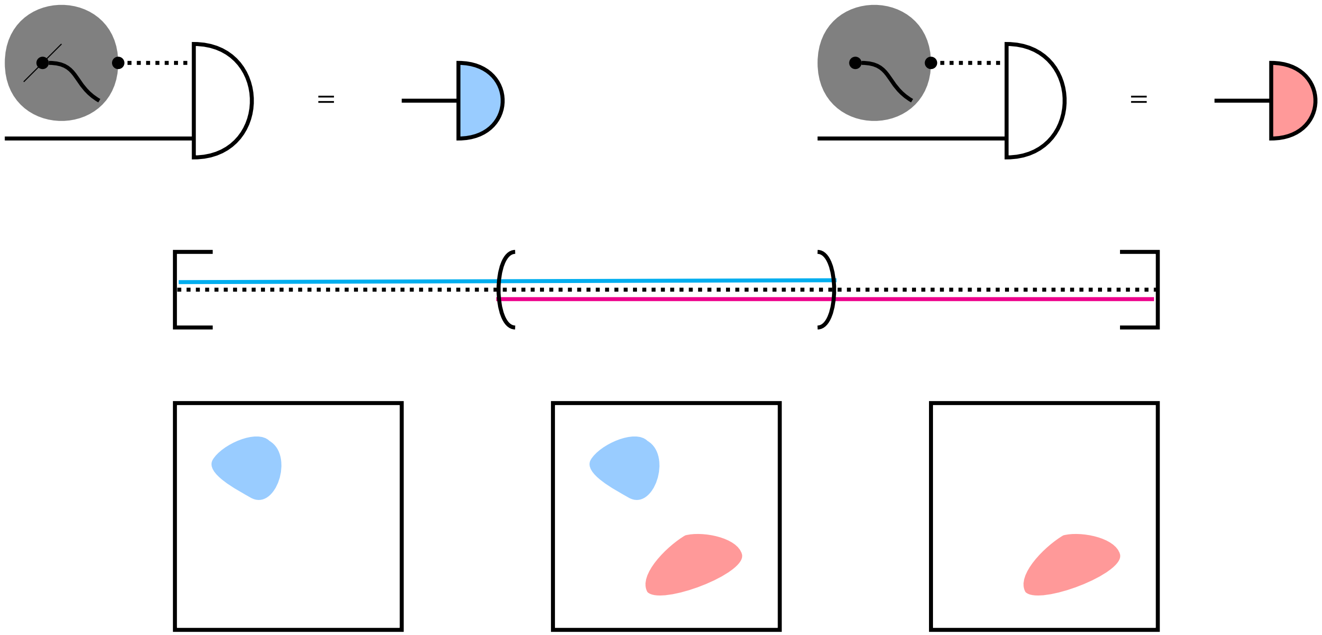
If we want continuous transformations in the plane from the configuration of shapes in one spider to end at the configuration of shapes in another, we ought to define an analogue of *homotopy*: the continuous deformation of one map to another. However, we will have to massage the definition a little to work in our setting of continuous relations.

Homotopy in **ContRel**

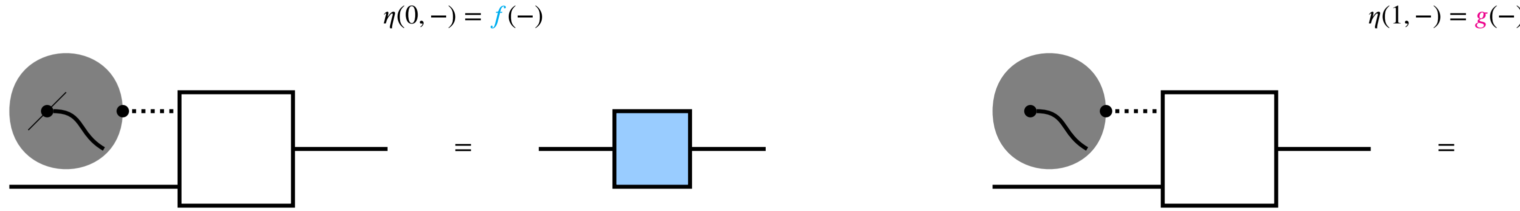
Usually, when we are restricted to speaking of topological spaces and continuous functions, a homotopy is defined:

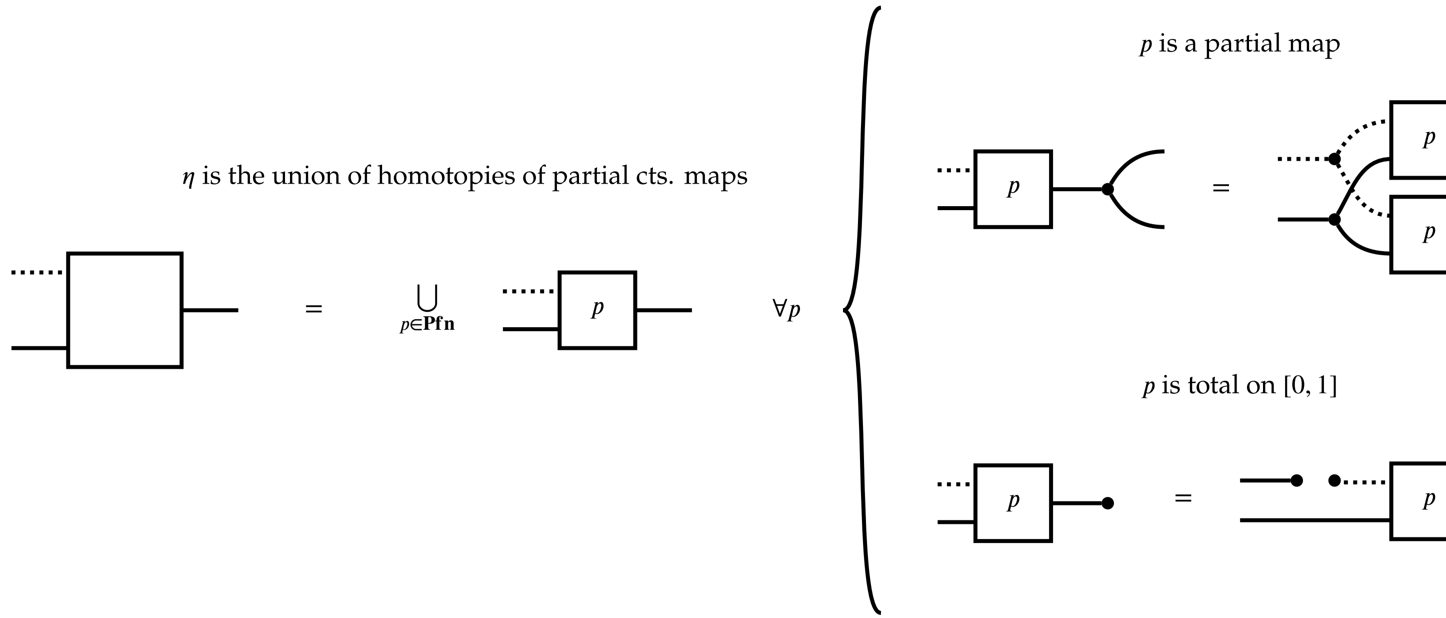
Definition 1.5.12 (Homotopy in **Top**)**.** where f and g are continuous maps $A \to B$, a *homotopy* $\eta : f \Rightarrow g$ is a continuous function $\eta : [0,1] \times A \to B$ such that $\eta(0,-) = f(-)$ and $\eta(1,-) = g(1,-)$.

In other words, a homotopy is like a short film where at the beginning there is an f, which continuously deforms to end the film being a g. Directly replacing "function" with "relation" in the above definition does not quite do what we want, because we would be able to define the following "homotopy" between open sets.

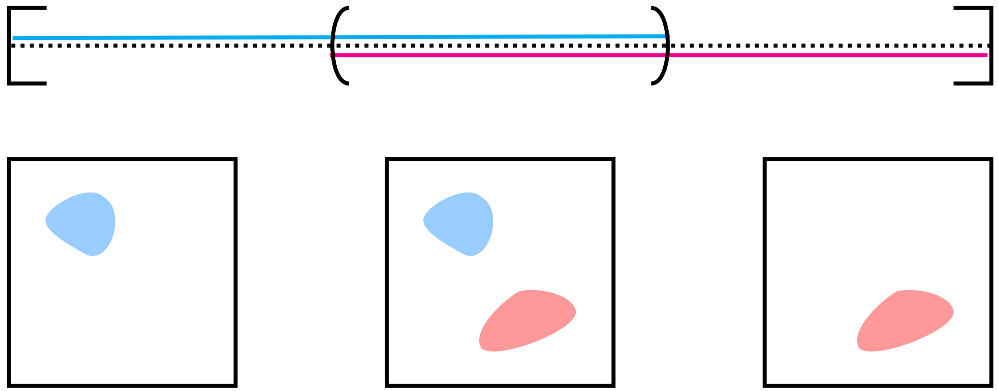


What is happening in the above film is that we have our starting open set, which stays constant for a while. Then suddenly the ending open set appears, the starting open disappears, and we are left with our ending; while *technically* there was no discontinuous jump, this isn't the notion of sliding we want. The exemplified issue is that we can patch together (by union of continuous relations) vignettes of continuous relations that are not individually total on $[0, 1]$. We can patch this problem by asking for homotopies in **ContRel** to satisfy the additional condition that they are expressible as a union of continuous partial maps that are total on the unit interval.

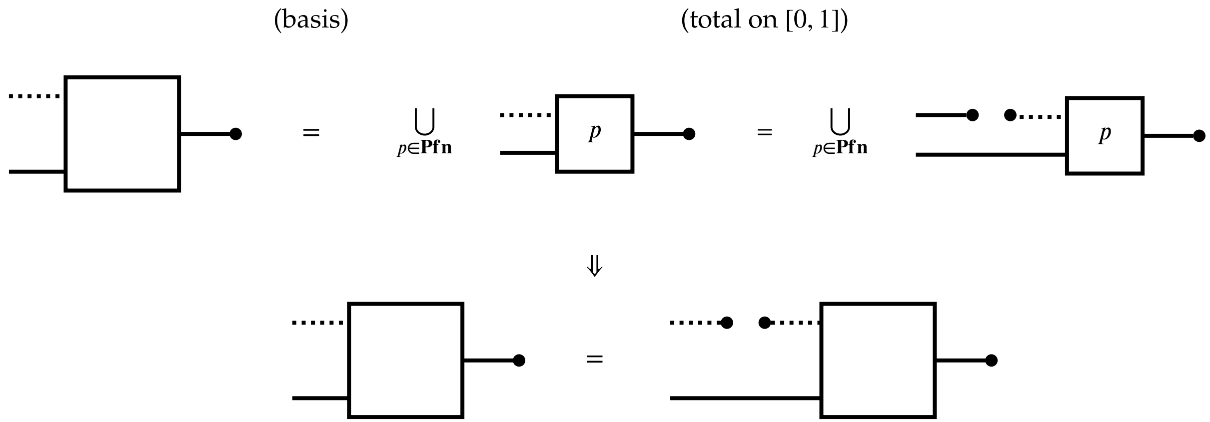




Observe that the second condition asking for decomposition in terms of partial comes for free by Proposition 1.2.19; the constraint of the definition is provided by the first condition, which is a stronger condition than just asking that the original continuous relation be total on I:



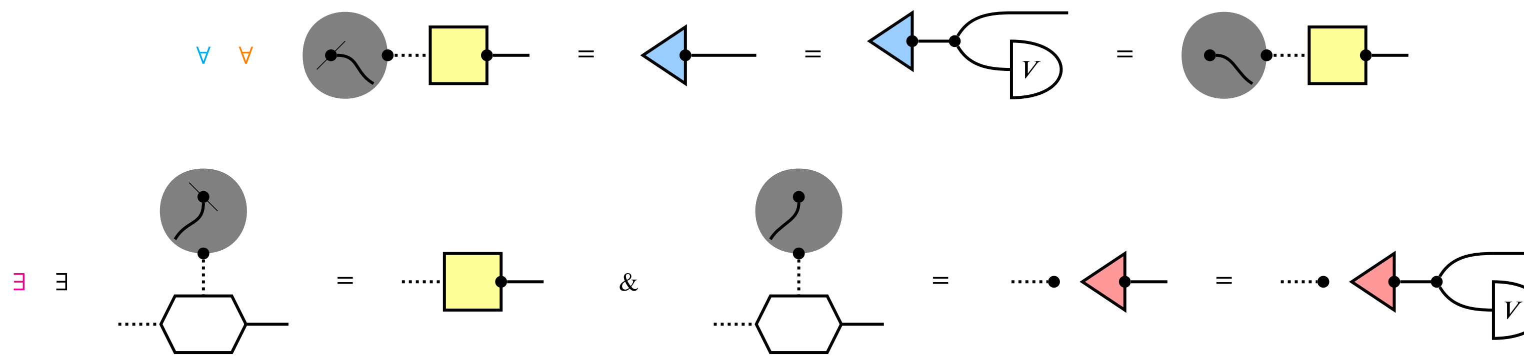
This definition is "natural" in light of Proposition 1.2.19, that the partial continuous functions $A \to B$ form a basis for **ContRel**(A, B): we are just asking that homotopies between partial continuous functions – which can be viewed as regular homotopies with domain restricted to the subspace topology induced by an open set – form a basis for homotopies between continuous relations.



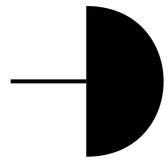
CONTRACTIBLE SPACES

With homotopies in hand, we can define a stronger notion of connected shapes with no holes, which are usually called *contractible*. The reason for the terminology reflects the method by which we can guarantee a shape in flatland has no holes: when any loop in the shape is *contractible* to a point.

V is *contractible* when:



Contractible open sets are worth their own notation too; a solid black effect, this time with no hole.



1.5.5 *Rigid motion*

CONFIGURATION SPACES

1.5.6 *Modelling linguistic topological concepts*

By "linguistic", I mean to refer to the kinds of concepts we use in everyday language. These are concepts that even young children have an intuitive grasp of [], but their formal definitions are difficult to pin down. One such relation modelled here – touching – is in fact a *semantic prime* []: a word that is present in essentially all natural languages that is conceptually primitive, in the sense that it resists definition in simpler terms. It is among the ranks of concepts like *wanting* or *living*, words that are understood by the experience of being

Figure 1.29: Now at last we can define sliding shapes. What we mean by two sticky spiders being relatable by sliding shapes is that we have a homotopy that begins at one and ends at the other, such that every point in between is itself a sticky spider related to the first by rigid displacement.

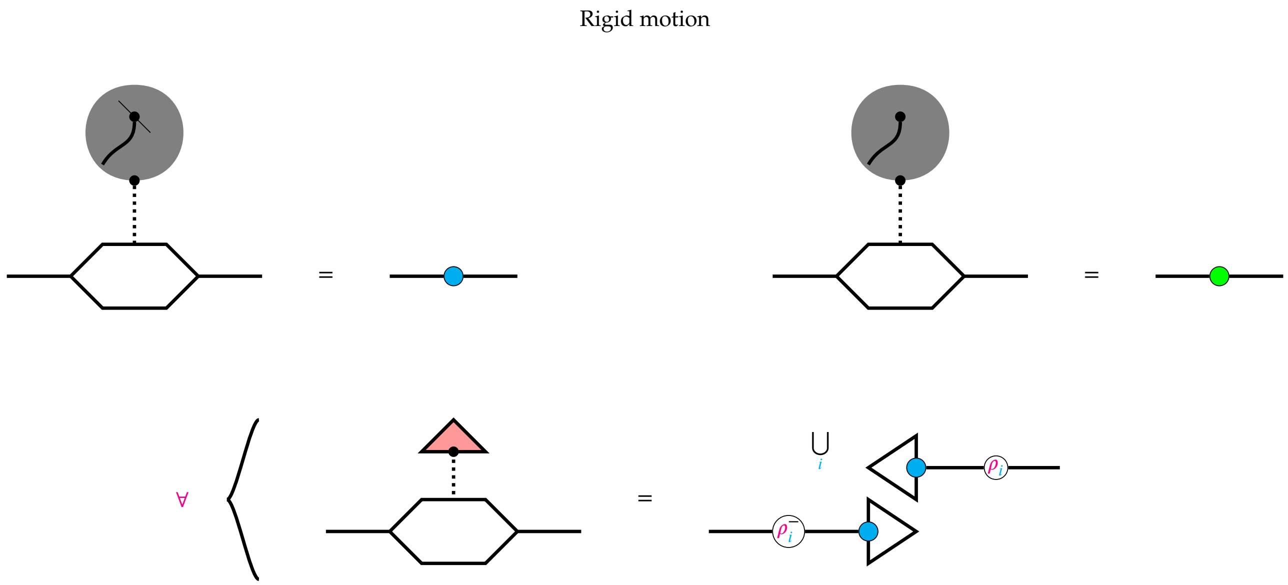
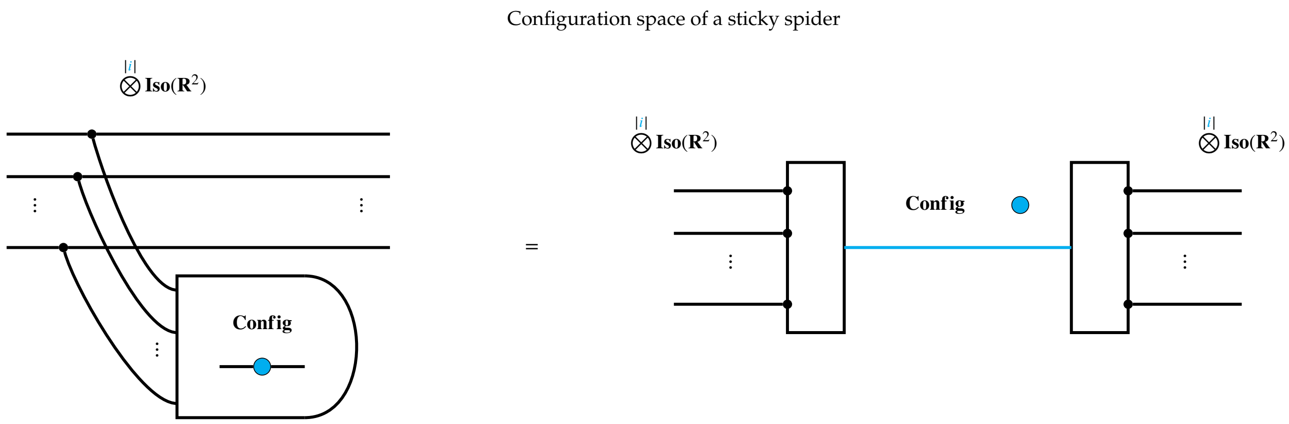


Figure 1.30: We can depict the *configuration space* of shapes that are obtainable by displacing the shapes of a given spider by a split idempotent through the n-fold tensor of rigid transformations – a restriction to the subspace of the largest open set contained in the subset of all valid (with correct chirality) combinations of displacements that yield another spider. Note that as a subspace, the retract of the idempotent is a function. The section and retract pair is not unique; they may for instance encode a choice of basepoint for each shape relative to which the displacement of that shape occurs.



human, rather than by school. As such, I make no claim that these definitions are "correct" or "canonical", just that they are good enough to build upon moving forward.

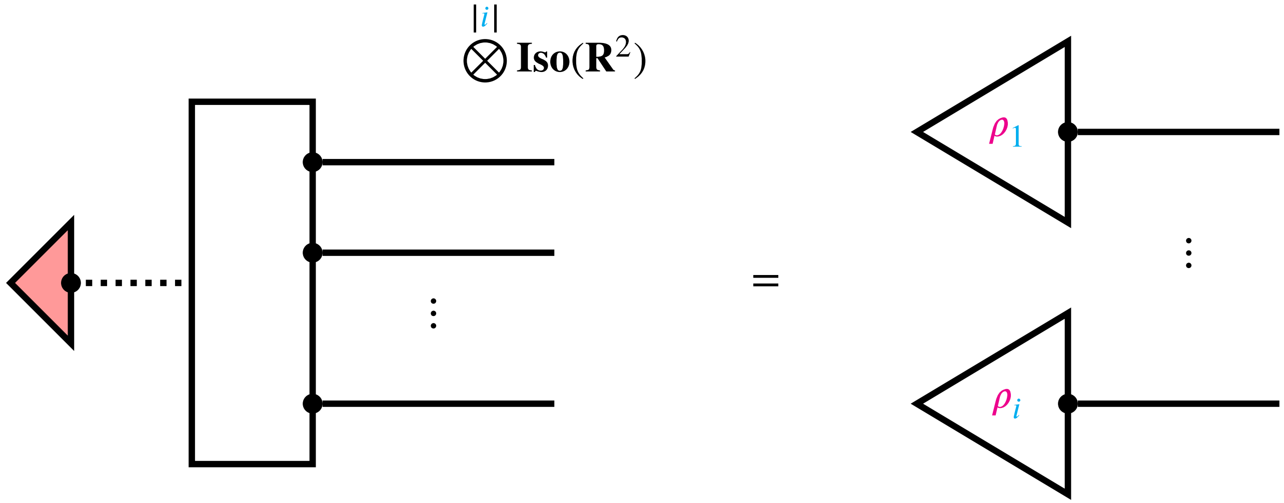


Figure 1.31: Observe that the data of rigid motion on a sticky spider as we have defined above can be captured as a continuous map from the unit interval to rigid transformations: one for each shape in the spider. This is precisely a continuous path in configuration space.

Parthood

Let's say that a "part" refers to an entire simply connected component. Simply connected is already a concept in our toolkit. A shape U is disjoint from another shape V intuitively when we can cover U in a blob with no holes such that the blob has no overlap with V. So, U is a part of V when it is simply connect, wholly contained in V, and there exists a contractible open that is disjoint from V that covers U. Diagrammatically, this is:

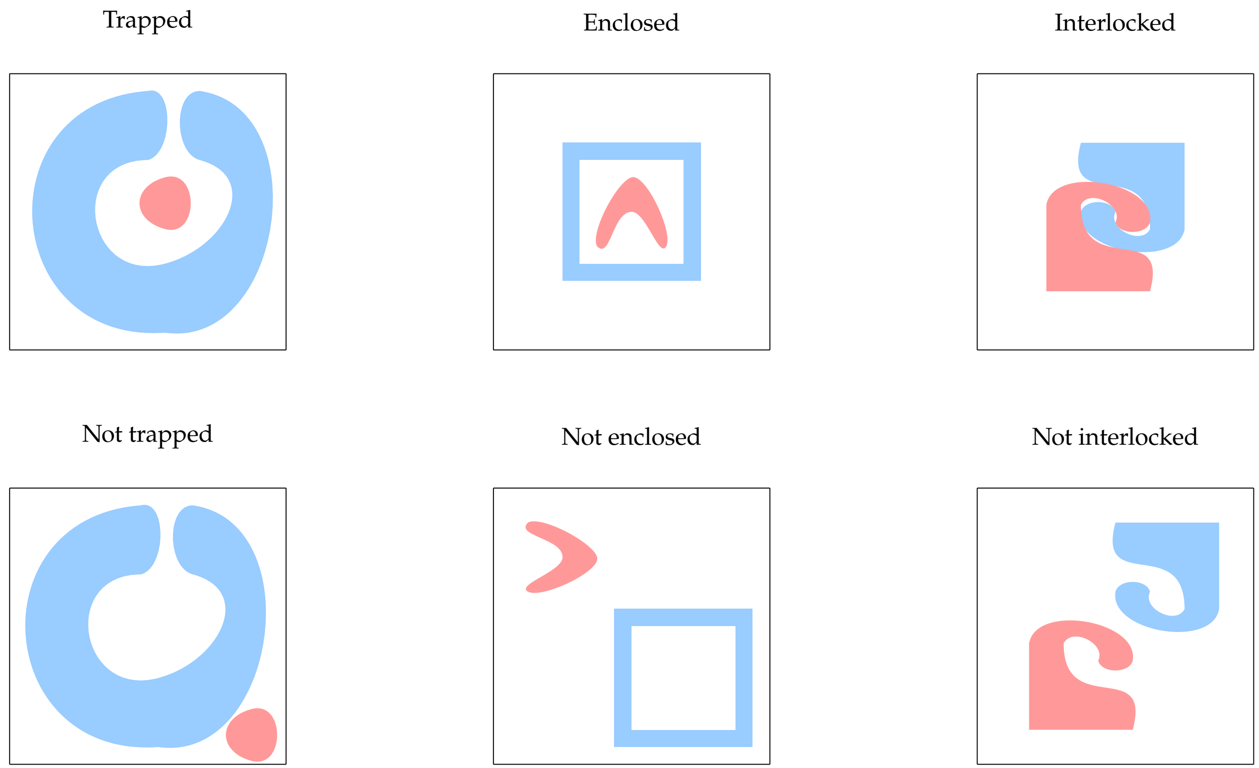
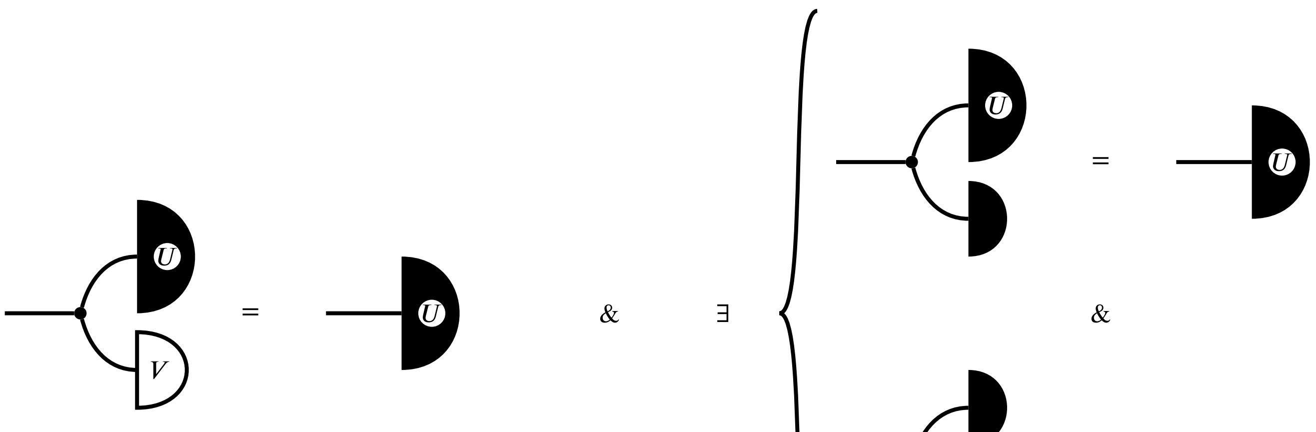


Figure 1.32: What are the connected components of configuration space? Evidently, there are pairs of spiders that are both valid displacements, but not mutually reachable by rigid motion. For example, shapes might *enclose* or *trap* other shapes, or shapes might be *interlocked*. Depicted are some pairs of configurations that are mutually unreachable by rigid transformations. Now we have the conceptual toolkit to begin modelling these concepts in the configuration space of a sticky spider.

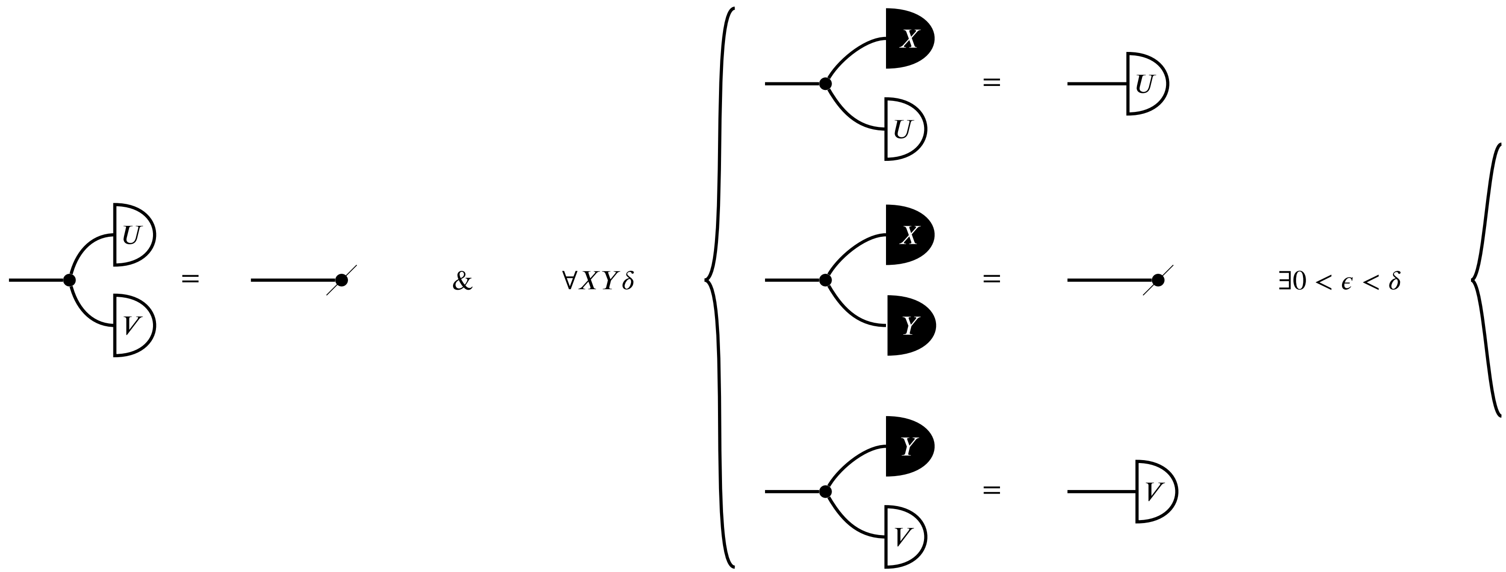
U is *a part of* V



Touching

Let’s distinguish touching from overlap. Two shapes are "touching" intuitively when they are as close as they can be to each other, somewhere; any closer and they would overlap. Let’s assume that we can restrict our attention to the parts of the shape that are touching, and that we can fill in the holes of these parts. At the point of touching, there is an infinitesimal gap – just as when we touch things in meatspace, there is a very small gap between us and the object due to the repulsive electromagnetic force between atoms. To deal with infinitesimals we borrow the $\epsilon - \delta$ trick from mathematical analysis; for any arbitrarily small δ, we can pick an even smaller ball of radius ϵ such that if we stick the ball in the gap, the ball forms a bridge that overlaps the two filled-in shapes, which allows us to draw a continuous line between them. Diagrammatically, this is:

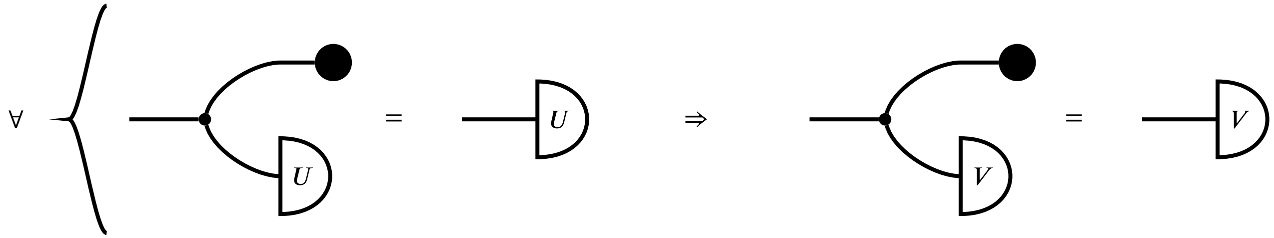
U and V are *touching*



Within

If U surrounds V, or equivalently, if V is within U, then we are saying that leaving V in almost any direction, we will see some of U before we go off to infinity. We can once again use open balls for this purpose, which correspond to possible places you can get to from a starting point $\mathbf{x}$ within a distance ϵ. In prose, we are asking that any open ball that contains all of U must also contain all of V.

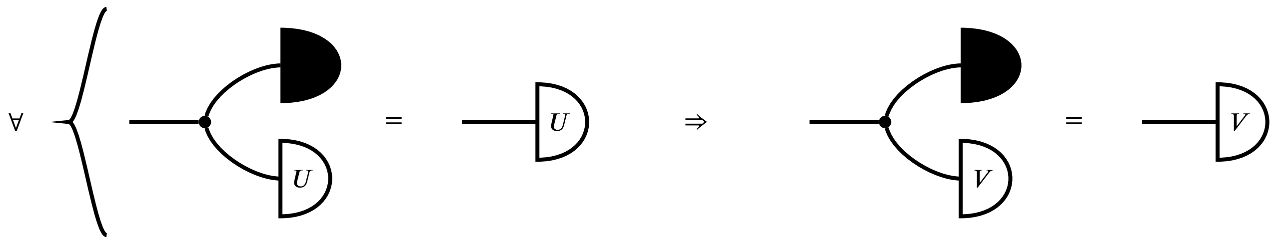
V is *within* U, or U *surrounds* V



Containers and enclosure

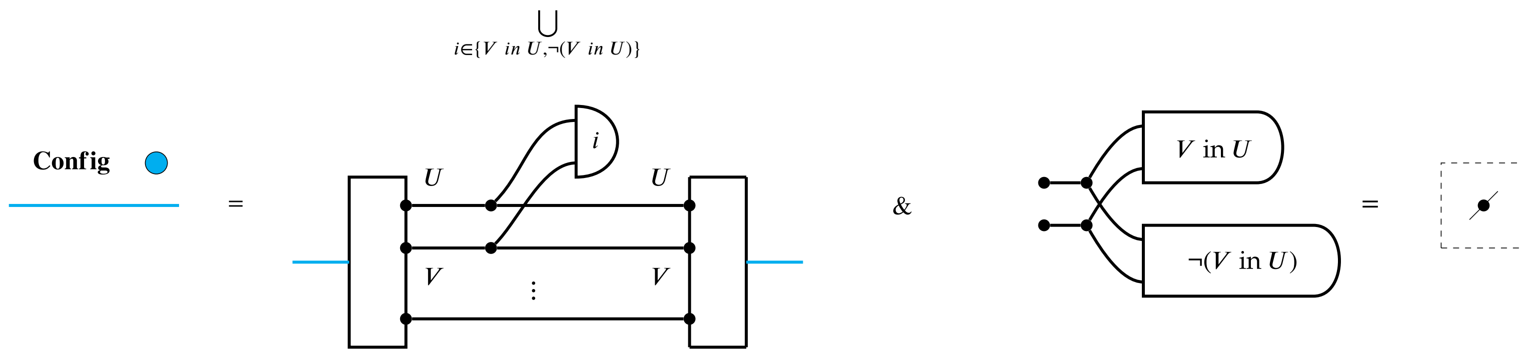
There is a strong version of within-ness, which we will call enclosure. As in when we are in elevators and the door is shut, nothing gets in or out of the container. Intuitively, there is a hole in the container surrounded on all sides, and the contained shape lives within the hole. To give a real-world example, honey lives within a honeycomb cell in a beehive, but whether the honey is enclosed in the cell depends on whether it is sealed off from air with beeswax. So in prose we are asking that any way we fill in the holes of the container with a blob, that blob must cover the contained shape. Diagrammatically, this amounts to levelling up from open balls in our previous definition to contractible sets:

U *encloses* V



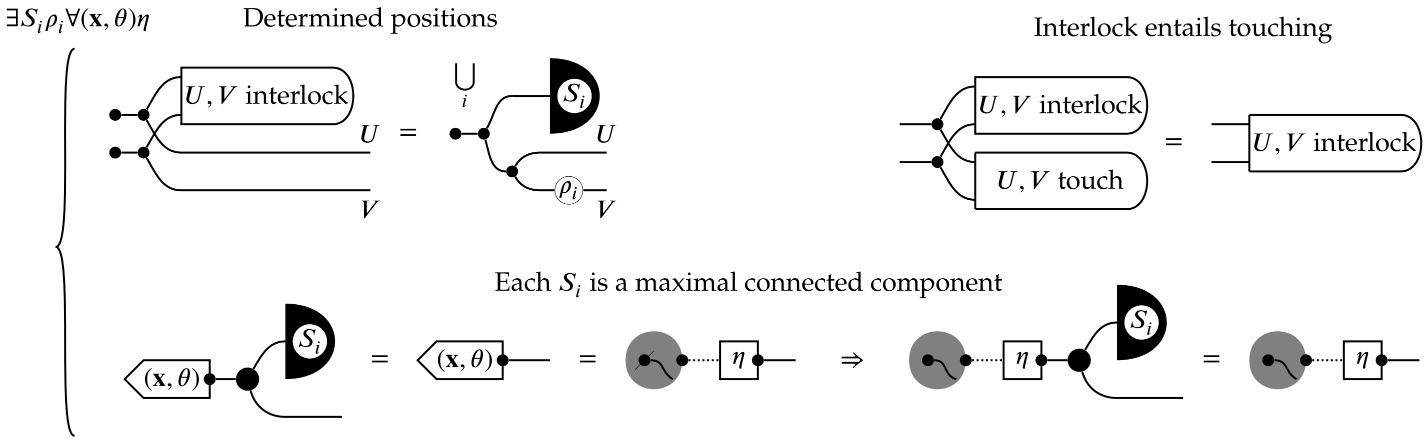
Trapped

There is an intermediate notion between within-ness and enclosure; for instance, standing in the stonehenge you are surrounded by the pillars, but you can always walk away, whereas if the pillars are very close, such as the bars of a jail cell, a human would not be able to leave the trap while still being able to see the outside. The difficulty here is that relative sizes come into play: small animals would still consider it a case of mere within-ness, because they can still walk away between the bars. So we would like to say that no matter how the pair of objects move rigidly, being trapped means that the trapped V stays within U. In other words, that in configuration space, if we forget about all other shapes, we can partition our space of configurations by two concepts, whether V is within U or not, and moreover that these two components is disjoint – i.e. not simply connected – so there is no rigid motion that can allow V to escape from being within U if V starts off trapped inside in U.



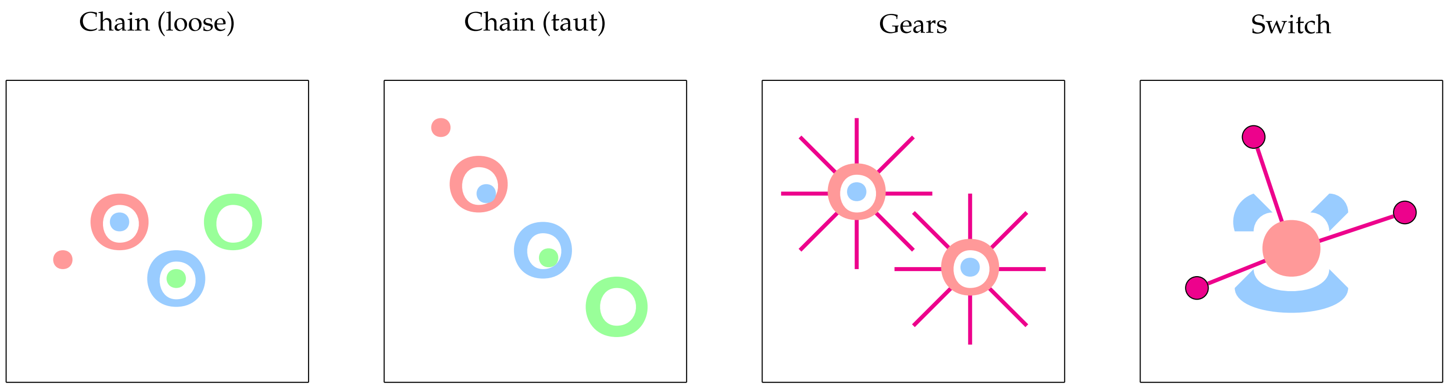
Interlocked

Two shapes might be tightly interlocked without being inside one another. Some potentially familiar examples are plastic models of molecular structure that we encounter in school, metal lids in cold weather that are too tightly hugging the glass jar, or stubborn Lego pieces that refuse to come apart. The commonality of all these cases is that the two shapes must move together as one, unless deformed or broken. In other words, when two shapes are interlocked, knowing the position in space of one shape determines the position of the other, and this determination is a fixed isometry of space. So we only need to specify a range of positions S for the entire subconfiguration of interlocked shapes U and V, and we may obtain their respective positions by a fixed rigid motion ρ. Since objects may interlock in multiple ways, we may have a sum of these expressions. We additionally observe that interlocking shapes should also be touching, which translates to containment inside the touching concept. Finally, we observe that as in the case of entrapment and enclosure, rigid motions are interlocking-invariant, which translates diagrammatically to the constraint that each S, ρ expression is an entire connected component in configuration space.



Constrained motion

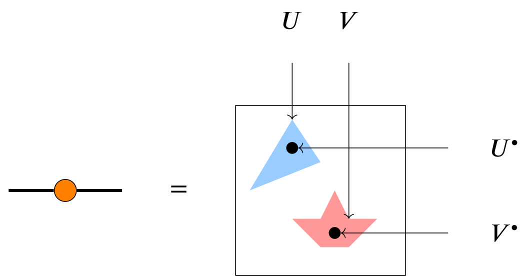
A weaker notion of interlocking is when shapes only imperfectly determine each other's potential displacements, by specifying an allowed range. Here is an understatement: there is some interest in studying how shapes mutually constrain each other's movements in this way.



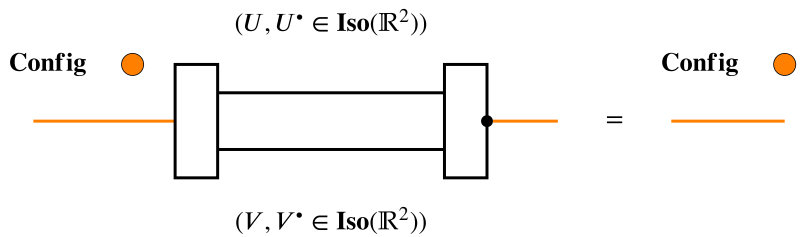
There are as many definitions to go through here as there are potential mechanical models, and among other things, there are mechanically realised clocks [], computers [], and analogues of electric circuits []. So instead, we will allow ourselves to additionally specify open sets as concepts in configuration space that correspond to whatever mechanical concepts we please, and we assure the reader seeking rigour that blueprints exist for all the mechanisms humans have built. Of course in reality mechanical motions are reversible among rigid objects, and directional behaviour is provided by a source of energy, such as gravitational potential, or wound springs. But we may in principle replace these sources of energy by a belt that we choose to spin in one direction – our own arrow of time. We postpone discussion of causal-mechanistic understanding and analogy for a later section.

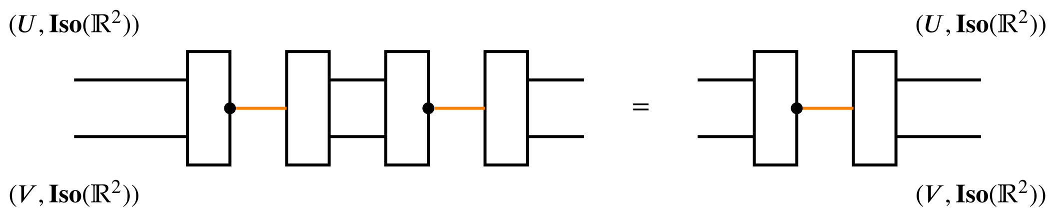
1.5.7 States, actions, manner

Configuration space explains why we label noun wires: each wire in expanded configuration space must be labelled with the shape within the sticky spider it corresponds to so that the section and retract know how to reconstruct the shapes, since each shape may have a different spatial extent.

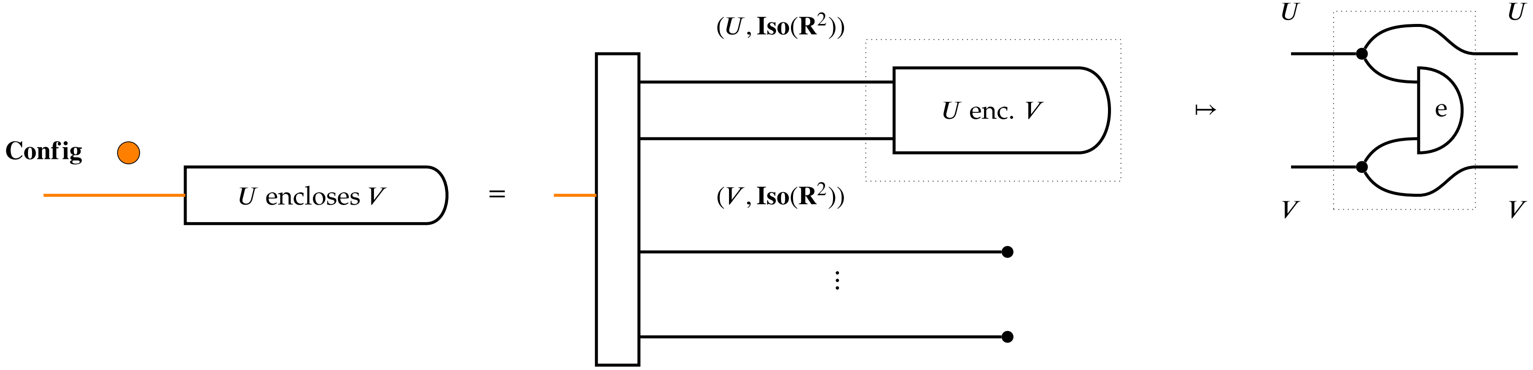


Concretely, for each shape in a sticky spider,
in order to reconstruct the shape,
the data of configuration space
only has to remember a basepoint
(which the isometries act upon)
paired with a label naming the shape
(so that the extent of the shape is reconstructible)

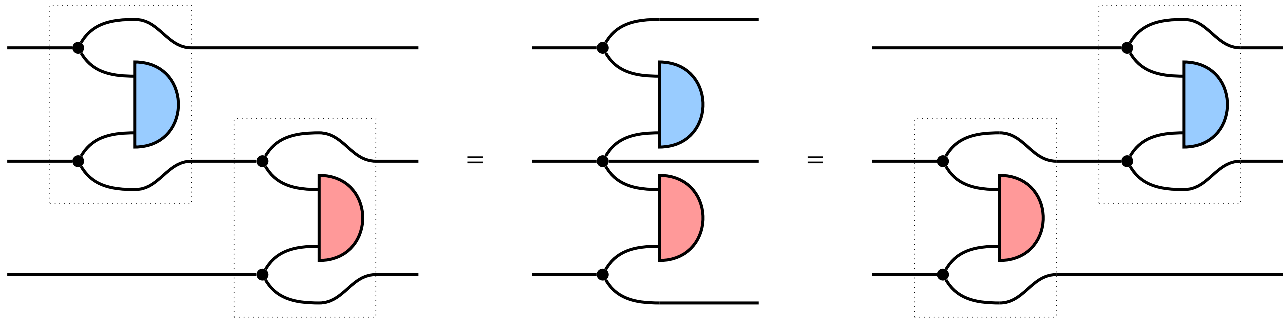




All of the concepts we have defined so far are open sets in configuration space – and for any concept that isn't, we are always free to take the interior of the set; the largest open set contained within the concept. Passing through the split idempotent, we can recast each as a circuit gate using copy maps.

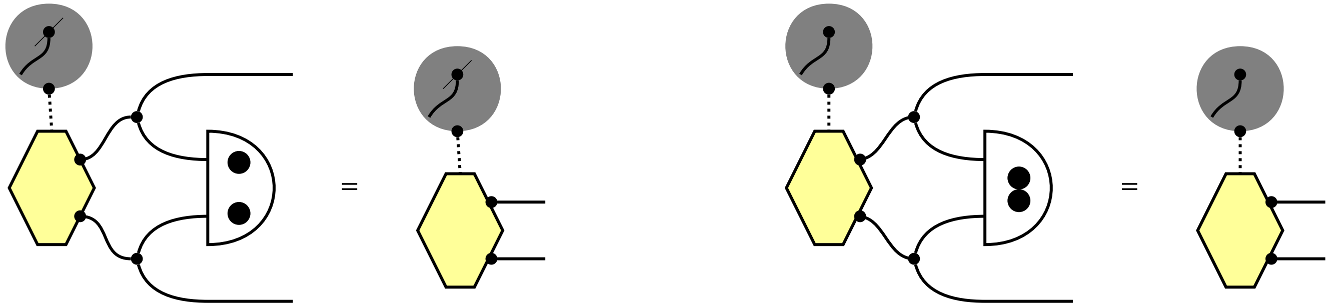


Going forward, we will just label the wires with the names of each shape when necessary. We notice that one feature of this procedure to get gates from open sets is that all gates commute, due to the commutativity of copy.



Moreover, since each gate of this form is a restriction to an open set, the gates are idempotent. So the concepts we have defined so far behave as if describing *states* of affairs in space, as if we adding commuting adjectives to space to elaborate detail. For example, `fast red car`, `fast car that is red`, `car is (red and fast)` all mean the same thing. As we add on progressively more concepts, we get diminishing subspaces of configurations in the intersection of all the concepts. So the natural extension is to ask how states of affairs can change with motion. A simple example is the case of *collision*, where two shapes start off not touching, and then they move rigidly towards one another to end up touching.

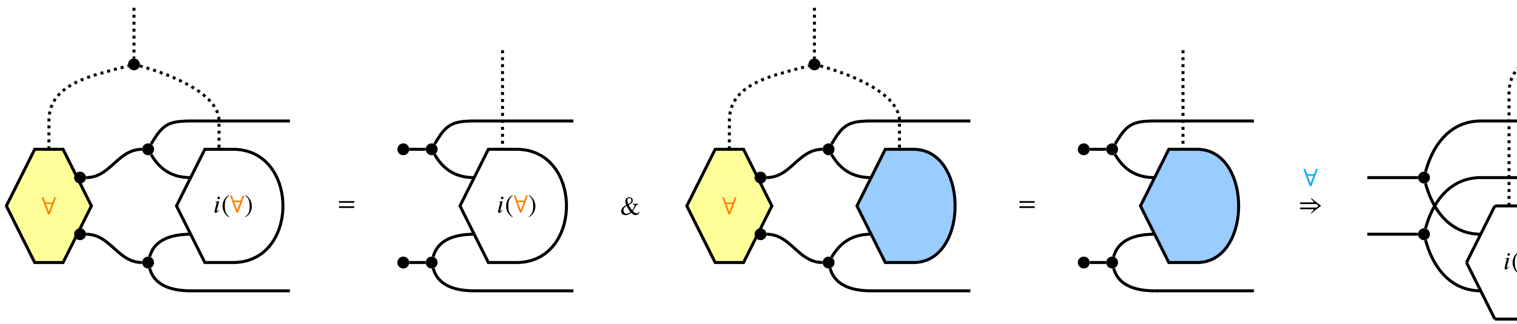
A particular collision trajectory



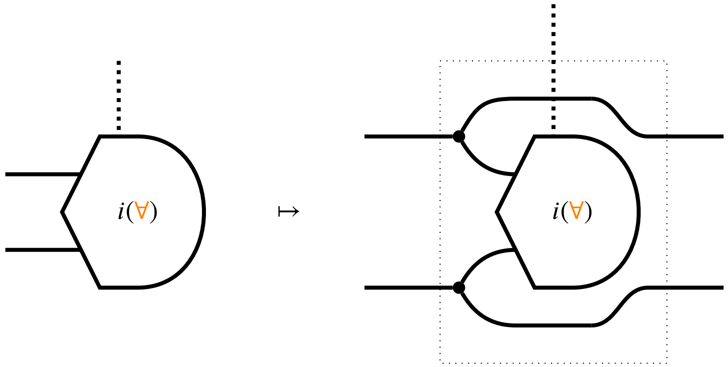
At $t = 0$, the shapes do not touch

At $t = 1$, the shapes touch

Recalling that homotopies between relations are the unions of homotopies between maps, we have a homotopy that is the union of all collision trajectories, which we mark ∀. Now we seek to define the interior i(∀) as the concept of collision; the expressible collection of all particular collisions. But this is not just an open set on the potential configuration of shapes, it is a collection of open sets parameterised by homotopy.

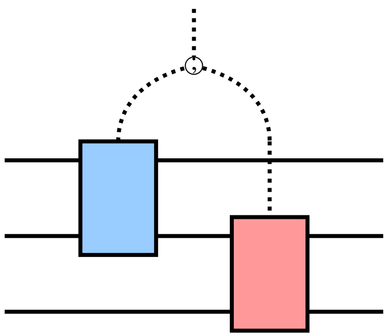


Once we have the open set i(∀) that corresponds to all expressible collisions, we have a homotopy-parameterised gate. Following a similar procedure, we can construct gates of motion that satisfy whatever pre- and postconditions we like.

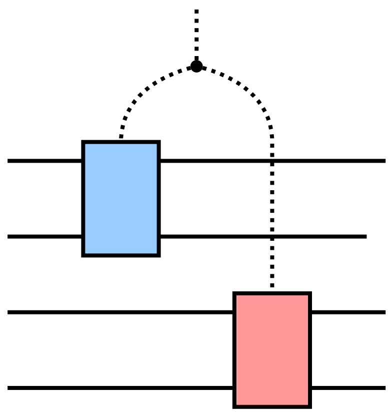


We can compose multiple rigid motions sequentially by a continuous function ; that splits a single unit interval into two: $; \ := \ x \mapsto \begin{cases} (2x, 0) \text{ if } x \in [0, \frac{1}{2}) \\ (1, 2x-1) \text{ if} x \in [\frac{1}{2}, 1] \end{cases}$. The effect of the map is to splice two vignettes of the same length together by doubling their speed, then placing them one after the other. We can achieve the same thing without resorting to units of measurement, because recall by Theorem 1.5.3 and by construction that we have access to a map that selects midpoints for us; we will revisit a string-diagrammatic treatment of homotopy and tenses in a later section. We can also compose multiple motions in parallel by copying the unit interval, allowing it to parameterise multiple gates simultaneously.

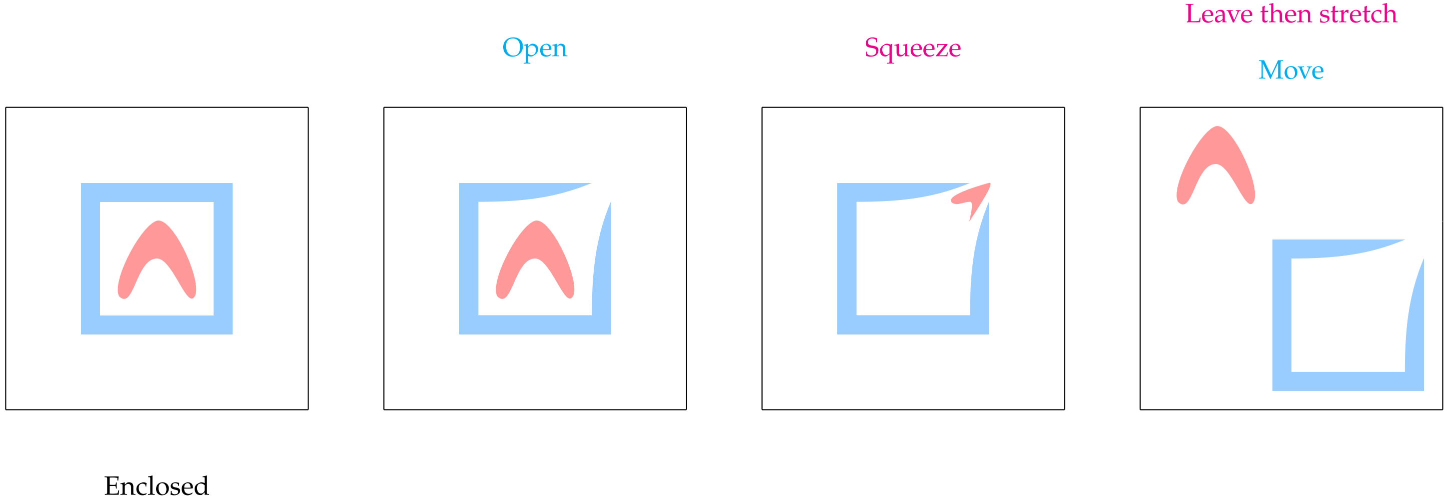
Sequential composition of motions



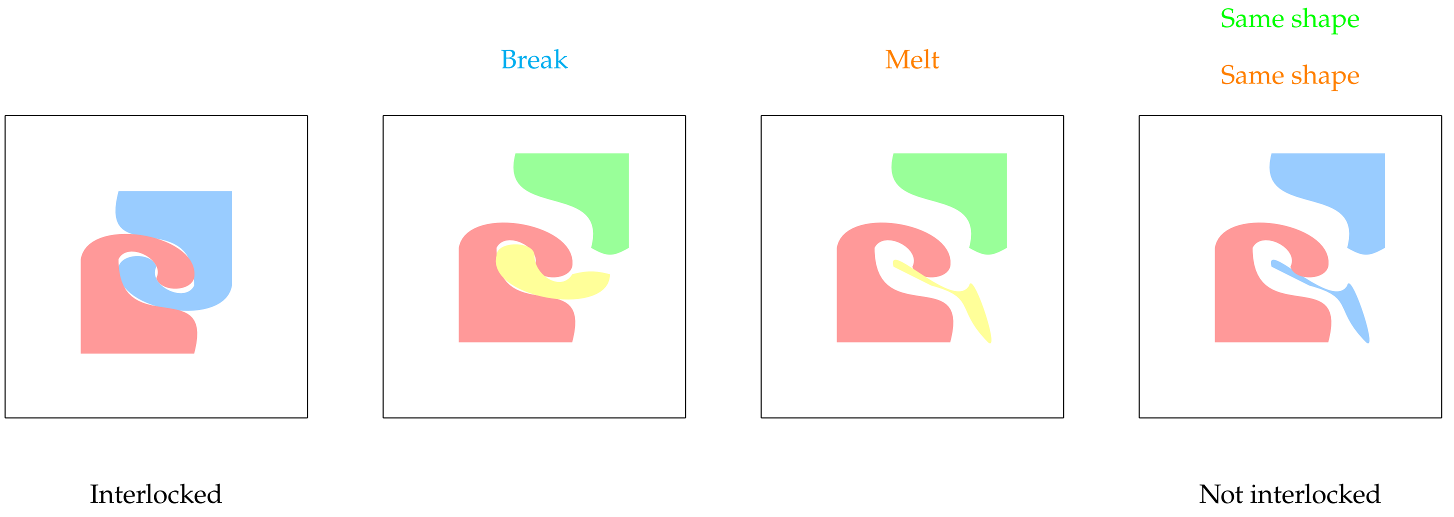
Parallel composition of motions



It is easy to see that the gates can always be rewritten to respect the composition order given by ; and copy, since for any input point at the unit interval the gates behave as restrictions to open sets. These new gates do not generally commute; consider comparing the situation where a tenant moves into one apartment and then another, with the situation where the tenant reverses the order of the apartments. These are different paths, as the postconditions must be different. So now we have noncommuting gates that model *actions*, or verbs. What kinds of actions are there? In our toy setting, in general we can define actions that arbitrarily change states of affairs if we do not restrict ourselves to rigid motions. The trick to doing this is the observation that arbitrary homotopies allow deformations, so our verb gates allow shapes to shrink and open and bend in the process of a homotopy, as long as at the end they arrive at a rigid displacement of their original form.



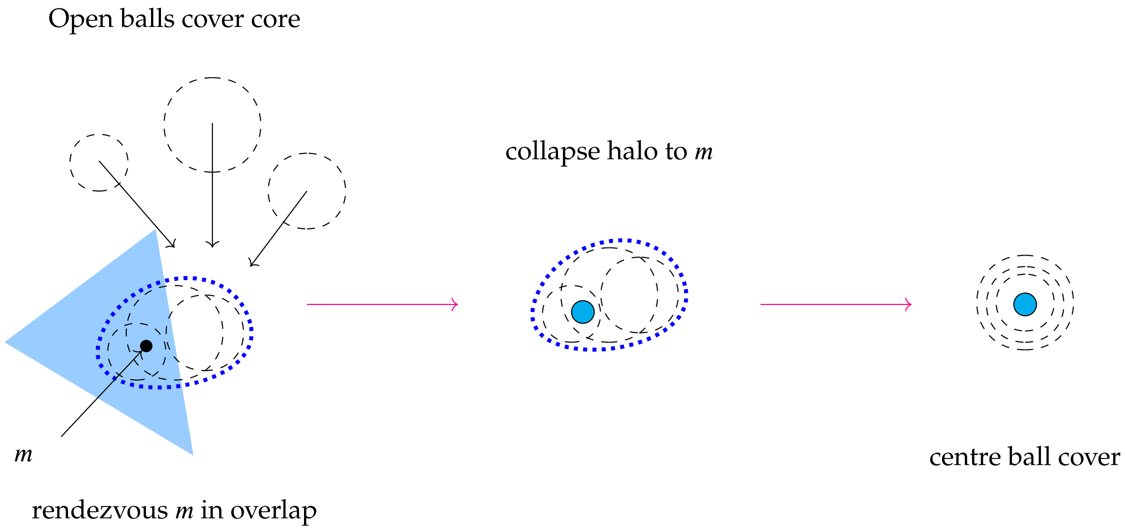
We can further generalise by noting that completely different spiders can be related by homotopy, so we can model a situation where there is a permanent bend, or how a rigid shape might shatter.



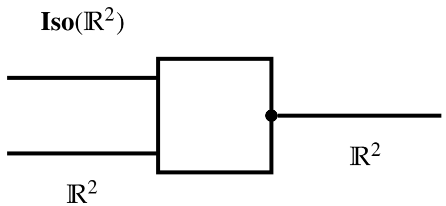
We provide the following construction as a general recipe to construct homotopies between spiders.

Construction 1.5.13 (Morphing sticky spiders with homotopies)**.** We aim to construct homotopies relating (almost) arbitrary sticky spiders. For now we focus on just changing one shape into another arbitrary one. The idea is as follows. First, we need a cover of open balls $\cup\mathcal{J} = T^0$ and $\cup\mathcal{K} = T^1$ of the start and end cores T^0 and T^1 such that each $k \in T^1$ is expressible as a rigid isometry of some core $j \in \mathcal{J}$; this is so we can slide and rearrange open balls comprising T^0 and reconstruct them as T^1. As an intermediate step to eliminate holes and unify connected components, we gather all of the balls at a meeting point m (to be determined shortly.)

Intuitively we can illustrate this process as follows:

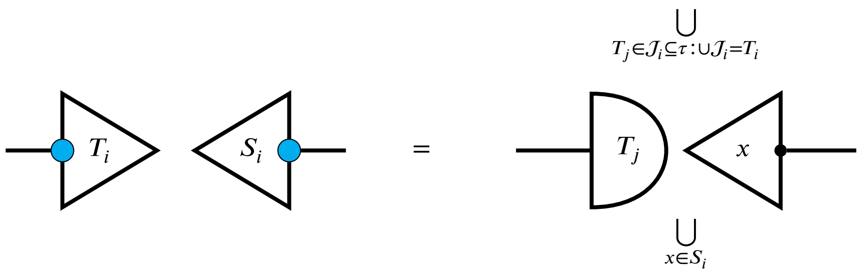


Second, in order to perform the sliding of open balls, we observe that, given a basepoint to act as origin (which we assume is provided by the data of the split idempotent of configuration space) we can express the group action of rigid isometries $\mathbf{Iso}(\mathbf{R}^2)$ on $\mathbf{R}^2$ as a continuous function:



$$\big((\mathbf{a},\theta),\mathbf{b}\big) \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{b}+\mathbf{a}$$

Third, before we begin sliding the open balls, we must ensure that the halo of the shape cooperates. We observe that a given shape i in a sticky spider may be expressed as the union of a family of constant continuous partial functions in the following way. Given an open cover $\mathcal{J}$ such that $\cup\mathcal{J} = T_i$, where T_i is the core of the shape i, each function is a constant map from some $T_j \in \mathcal{J}$ to some point $x \in S_i$, where S_i is the halo of the shape i. For each $T_j \in \mathcal{J}$ and every point $x \in S_i$, the constant partial function that maps T_j to x is in the family.



By definition of sticky spiders, there must exist some point m that is in both the core and the halo: we pick such a point as the rendezvous for the open balls. For each partial map in the family, we provide a homotopy that varies only the image point x continuously in the space to finish at m. Now we can slide the open balls to the rendezvous m. Since homotopies are reversible by the continuous map $t \mapsto (1 - t)$ on the interval, we can perform the above steps for shapes T^0 and T^1 to finish at the same open ball, reversing the process for T^1 and composing sequentially to obtain a finished transformation. The final wrinkle to address is when dealing with multiple shapes. Recalling our exclusion conditions **??** for shapes, it may be that parts of one shape are enclosed in another, so the processes must be coordinated so that there are no overlaps. For example, the enclosing shape must be first opened, so that the enclosed shape may leave. I will keep it an article of faith that such coordinations exist. I struggle to come up with a proof that all spiders $\mathbf{R}^2$ are mutually transformable by homotopy in this (or any other) way, so that will remain a conjecture. But it is clear that a great deal of spiders are mutually transformable; almost certainly any we would care to draw. So this will just be a construction for now.

Going forward, I will consider any linguistic semantics that can be grounded by a mechanical or tabletop model to be formal. The preceding analysis extends to talk of rigid and deforming bodies and the manner, order, and coordination of their movement and interaction three-dimensional Euclidean space. At this point, I have sketched out enough to, in principle, linguistically specify mechanical models. Further, by Example 1.5.2, we have enough technology to speak of locations in space, so we have access to "tabletop semantics": anything that in principle can be represented by counters and meeples in a boardgame, with for instance reserved spaces on the board for health and hunger and whatever else is necessary. Wherever this talk falls short, I consider videogame design to be applied formal semantics, so I permit myself more or less any conceivable interactive world with its own internal logic.

Objection: That is way outside the scope of formal semantics. Insofar as semantics is sensemaking, we certainly are capable of making sense of things in terms of mechanical models and games by means of metaphor, the mathematical treatment of which is concern of Section **??**, so I claim that I am, definitionally, *doing formal semantics for natural language.* Whether or not I'm exceeding the scope of what a linguist might consider formal semantics is ultimately irrelevant, as I am not ultimately concerned with the modal human mechanism. There is maybe also a prejudice that formal semantics must necessarily resolve in some symbolic logic, to which I might charitably respond that I'm working with algebraic system, just not a one-dimensional one. Less charitably, I don't care what these people think.

1.6 Interpreting text circuits in **ContRel**

Recall that there were some mathematically odd choices in conventions for text circuits, such as the labelling of noun wires and some subtleties with respect to interchange of parallel gates and text order. The explanation of those choices was deferred "to semantics", and since we're here now we have to make good. The aim of this section is to now explain those choices through an interpretation of text circuits in **ContRel**.

1.6.1 Sticky spiders: iconic semantics of nouns

Without loss of generality we may consider sticky spiders to be set-indexed collections of disjoint open subsets of an ambient space, i.e. a collection of shapes in space that a labelled from a set of names. So particular sticky spiders will be particular models of a collection of nouns; the indexing set names them, and the corresponding shapes in space are a particular iconic representation of those nouns in space.

1.6.2 Open sets: concepts

Apart from enabling us to paint pictures with words, **ContRel** is worth the trouble because the opens of topological spaces crudely model how we talk about concepts, and the points of a topological space crudely model instances of concepts. We consider these open-set tests to correspond to "concepts", such as redness or quickness of motion. Figure 1.33 generalises to a sketch argument that insofar as we conceive of concepts in (possibly abstractly) spatial terms, the meanings of words are modellable as shared strategies for spatial deixis; absolute precision is communicatively impossible, and the next best thing mathematically requires topology.

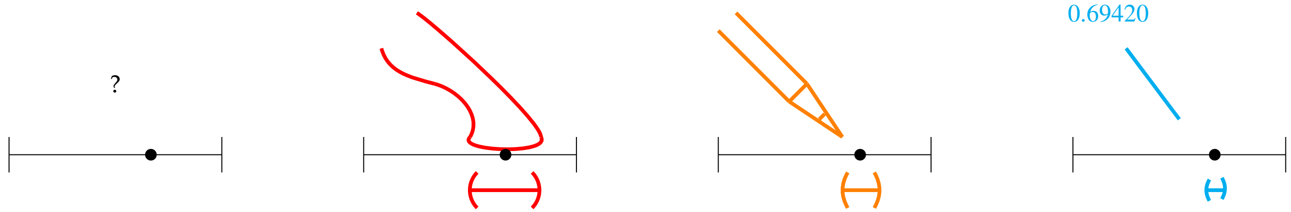


Figure 1.33: Points in space are a useful mathematical fiction. Suppose we have a point on a unit interval. Consider how we might tell someone else about where this point is. We could point at it with a pudgy appendage, or the tip of a pencil, or give some finite decimal approximation. But in each case we are only speaking of a vicinity, a neighbourhood, an *open set in the borel basis of the reals* that contains the point. Identifying a true point on a real line requires an infinite intersection of open balls of decreasing radius; an infinite process of pointing again and again, which nobody has the time to do. In the same way, most language outside of mathematics is only capable of offering successively finer, finite approximations to whatever it is that occurs in the mind or in reality.

This may explain the asymmetry of why tests are open sets, but why are states allowed to be arbitrary subsets? One could argue that states in this model represent what is conceived or perceived. Suppose we have an analog photograph whether in hand or in mind, and we want to remark on a particular shade of red in some uniform patch of the photograph. As in the case of pointing out a point on the real interval, we have successively finer approximations with a vocabulary of concepts: "red", "burgundy", "hex code #800021"... but never the point in colourspace itself. If someone takes our linguistic description of the colour and tries to reproduce it, they will be off in a manner that we can in principle detect, cognize, and correct: "make it a little darker" or "add a little blue to it". That is to say, there are in principle differences in mind that we cannot distinguish linguistically in a finite manner; we would have to continue the process of "even darker" and "add a bit less blue than last time" forever. All this is just the mathematical formulation of a very common observation: sometimes you cannot do an experience justice with words, and you eventually give up with "I guess you just had to be there". Yet the experience is there and we can perform linguistic operations on it, and the states accommodate this.

1.6.3 Configuration spaces: labelled noun wires

Whereas sticky spiders correspond to particular iconic representations, we want to interpret text circuits in the configuration spaces of sticky spiders, so that text process-theoretically restricts, expands, and modifies a space of permissible iconic representations[1]. So it is via configuration spaces that we aim for an iconic semantics of general text. Recall from the definition of configuration spaces via split idempotents REF that the section and retract diagrammatically allow the configuration space wire to open up to guitar strings we can place our circuits in. However, this section and retract pair is not uniquely determined. For example, in the case of configuration spaces of rigid transformations of shapes, one obtains a family of encodings of the configuration space of an n-shape sticky spider in the n-fold tensor of isometry spaces by choosing a basepoint for each shape and rigidly transforming with respect to that basepoint, which on the plane may lead to different reifications of rotating shapes. Interpreting text circuits with respect to a particular split idempotent of configuration space hence gives a good reason to label the noun wires: so that we can remember which space of isometries corresponds to which noun-shape in the absence of knowing how precisely the split idempotent encodes configuration space as the n-fold contribution of spatial possibilities for each noun.

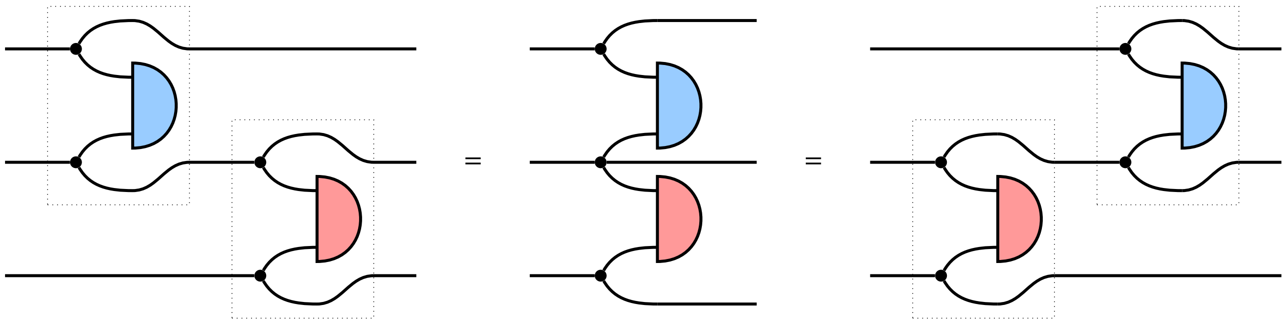
Example 1.6.1 (Differing encodings).

1.6.4 Copy: stative verbs and adjectives

Stative verbs are those that posit an unchanging state of affairs, such as `Bob likes drinking`. Insofar as stative verbs are restrictions of all possible configurations to a permissible subset, they are conceptually similar to adjectives, such as `red car`, which restricts permissible representations in colourspace. When we interpret concepts as open-set tests, **ContRel** conspires in our favour by giving us free copy maps on every wire REF . This allows us to define a family of processes that really behave like stative constructions that merely restrict possibilities. The desirable property we obtain is that in the absence of *dynamic* verbs that posit a change in the state of affairs, stative constructions commute in text: if I'm just telling you static properties of the way

[1] **Top** is symmetric monoidal closed with respect to product, why we just work there from the start? Because **Top** is cartesian monoidal, which in particular means that there is only one test (the map into the terminal singleton topology), and worse, all states are tensor-separable. The latter fact means that we cannot reason natively in diagrams about correlated states, which are extremely useful representing entangled quantum states [dodo], and for reasoning about spatial relations [talkspace] . I'll briefly explain the gist of the analogy in prose because it is already presented formally in the cited works and elaborated in [bobcomp] . The Fregean notion of compositionality is roughly that to know a composite system is equivalent to knowing all of its parts, and diagrammatically this amounts to tensor-separability, which arises as a consequence of cartesian monoidality. Schrödinger suggests an alternative of compositionality via a lesson from entangled states in quantum mechanics: *perfect knowledge of the whole does not entail perfect knowledge of the parts.* Let's say we have information about a composite system if we can restrict the range of possible outcomes; this is the case for the bell-state, where we know that there is an even chance both qubits measure up or both measure down, and we can rule out mismatched measurements. However, discarding one entangled qubit from a bell-state means we only know that the remaining qubit has a 50/50 of measuring up or down, which is the minimal state of information we can have about a qubit. So we have a case where we can know things about the whole, but nothing about its parts. A more familiar example from everyday life is if I ask you to imagine `a cup on a table in a room`. There are many ways to envision or realise this scenario in your mind's eye, all drawn from a restricted set of permissable positions of the cup and the table in some room. The spatial locations of the cup and table are entangled, in that you can only consider the positions of both together. If you discard either the cup or the table from your memory, there are no restrictions about where the other object could be in the room; that is, the meaning of the utterance is not localised in any of the parts, it resides in the entangled whole.

things are, it doesn't matter in what order I tell you the facts because restrictions commute. Recall:



Example 1.6.2 (Adjectives by analysis of configuration spaces)**.**

1.6.5 Homotopies: dynamic verbs and weak interchange

Dynamic verbs are those that posit a change in state, such as `Bob goes home`. We want to model these verbs by homotopies, where the unit interval parameter models time. A nice and diagrammatically immediate property is that dynamic verbs are obstacles to the commutation of stative words.

Example 1.6.3 (Dynamic verbs block stative commutations)**.**

The composition of dynamic verbs in time also explains the weak-interchange subtlety of text diagrams. When we are dealing with dynamic verbs, the order of sentences does make a difference in semantics between `Bob goes home. Bob gets drunk.` and `Bob gets drunk. Bob goes home.` So we have:

Example 1.6.4 (Sequential versus concurrent)**.**

Three short sketches

I want to sketch in passing three mathematically and linguistically interesting avenues to do with playing with unit-interval parameters that I won't explore fully here. The first is to do with the string-diagrammatic algebra of tenses, which can be modelled like so:

$$before$$

Recall that unit intervals come with a $\leq$ relation and that **ContRel** is a rig category with respect to unions. So suppose we model a tense algebra to be generated by the following PROP, to which we also include the ability to take unions:

prop

Using these generators it's easy to create more complex temporal relations string-diagrammatically such as `during`, `after`, `between` and so on. The interesting claim would be that *all* temporal relationships in natural languages are *necessarily* obtained in this way. The argument is one from computational feasibility, and the path to it is unexpected. The gist is that the generators along with union correspond to the axioms of an o-minimal structure via interpreting the copy-delete comonoid as product and projection (via Fox's theorem ref), and o-minimal structures are considered good candidates for so-called "tame topologies" that don't have counterintuitive counterexamples that plague the usual definition of topology ref . It turns out that o-minimality and tameness is a sufficient condition for learnability in a formal sense ref , so there lurks an argument for the canonicity of tensed language on the grounds of computational hardness.

Second, there is a strong but little-known criterion for the goodness of a theory of language called Becker's criterion CITE , which is stated "Any theory (or partial theory) of the English Language that is expounded in the English Language must account for (or at least apply to) the text of its own exposition", and followed by the comment: "Using this handy guideline, you can pretty much wipe your theoretical linguistics shelf clean and start over". Text circuits with iconic semantics in **ContRel** appear to have sufficient structure to satisfy Becker's criterion in some sense. Recall that the generative syntactic formulation of text circuits is defined in terms of weak n-categories with strict unitality and associativity but weak interchange. In **ContRel**, the composition of homotopies satisfies the first two strictness conditions and is close to satisfying weak interchange: for any composite gluing of the unit interval, there is a continuous endo-relation on the real line that achieves arbitrary permutation of the constituent intervals (up to finitely many discontinuities, at the gluing points), since continuous relations are unions of partial continuous functions REF . So modulo a model of weak interchange that is insensitive to finitely many discontinuities, and a daunting Gödel-style self-encoding argument, we could have a (perhaps the only) theory of language that suffices to model itself.

Third, the interaction of stative and dynamic (or, more generally, costates intersected by comonoids versus parameterised morphisms) appears to be a suitable model for categorified hypergraphs, just as monoidal categories are categorified monoids. The gist is that any open hypergraph is representable as a morphism in a hypergraph category CITE , and we may use the special frobenius laws to rewrite such morphisms to only use the comonoid part of the spiders in stative form. The non-commutation of statives past dynamics then justifies the view that the dynamic parameterised morphisms can be viewed going between objects enriched in a hypergraph structure.

hypergist

1.6.6 Coclosure: adverbs and adpositions

1.6.7 Turing objects: sentential complementation