# Simple Linear Regression

Grinnell College

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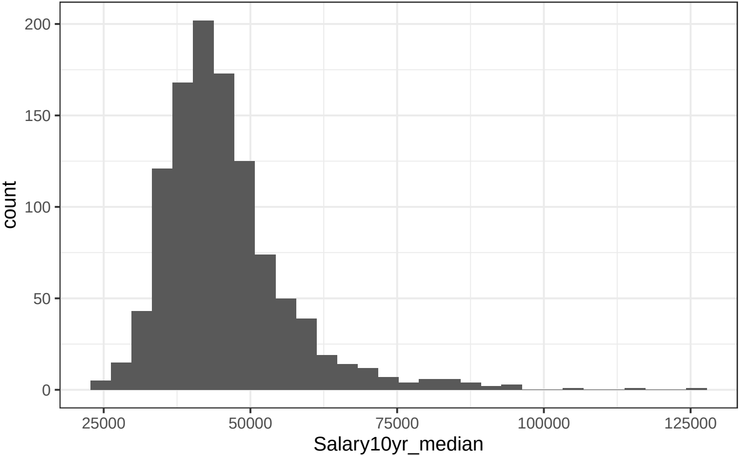
# Review

- Scatterplot descriptions
  - form, strength, direction, outliers
- Pearson's correlation (r)
  - strength and direction of linear relationship for 2 quant. variables
- Spearman's correlation ($\rho$)
  - strength and direction of *monotone* relationship
  - more robust to outliers

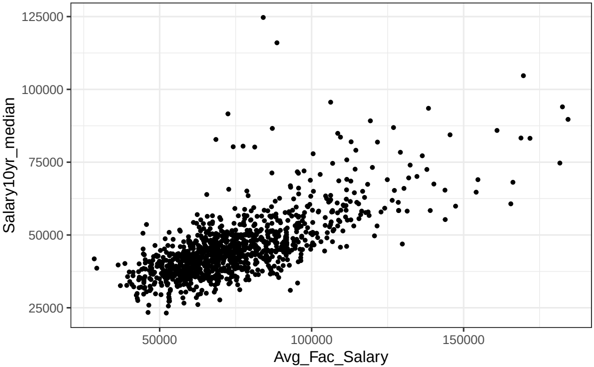
# Motivation

If I asked you to guess your income after ten years, how would you guess?



# Motivation

If I told you my salary, how would you guess your (future) income?



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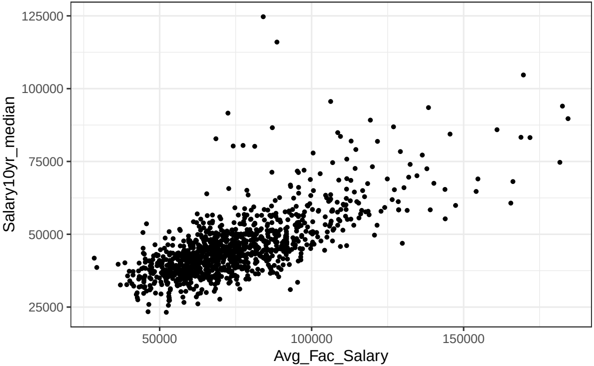
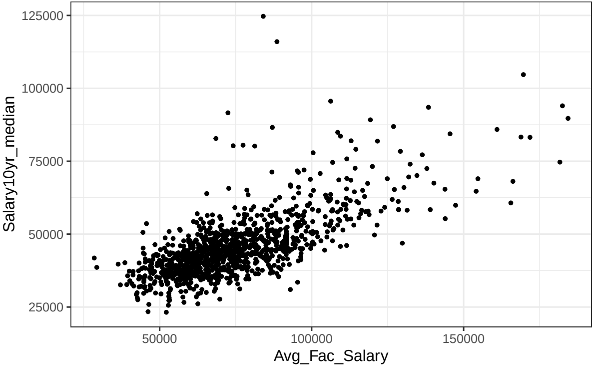



Linear Regression allows us to do this formally

# Correlation, Causation Review

Should you all tell the administration to raise my salary so your future income increases?



# Correlation, Causation Review

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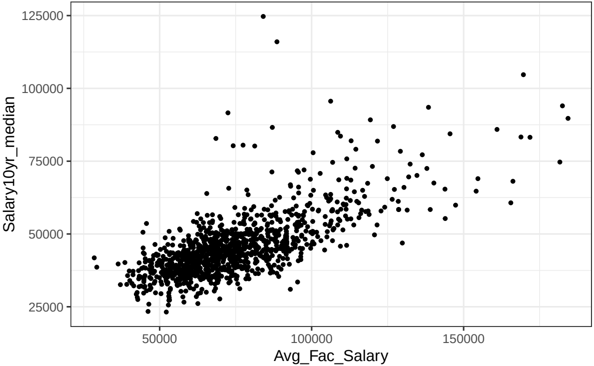



Yes!! But it won't actually increase your income. They are correlated but one doesn't cause the other (at least directly).

# Basic Idea

**Regression** is how we model data; for us it's the "best fit line"

**Two Main Goals**:

- Use the regression/our best fit line(s) to describe the relationship between the explanatory variable(s) and the response variable
  - Science!
- Use the explanatory variable(s) to predict the response variable
  - Machine Learning/AI stuff
  - Business/finance investments
  - Planning around weather

# Notation

- The variable being predicted is the *response* (aka "variable of interest")
  - Usually denoted as y
- the variable we are using to do the prediction/explanation is the *explanatory variable* (aka "covariate" or occasionally "predictor")
  - Usually denoted as x or X
- The estimates themselves are usually denoted with a "hat"
  - $\hat{y}$ is our predicted response
  - $\hat{\beta}_0$ and $\hat{\beta}_1$ are our estimated intercept and slope of the regression line (more in a second)

# Notation Comparison

Statisticians use different symbols to write out a line than what you probably saw in HS algebra

**Algebra**

$y = mx + b$

$m$ = slope: change in y over the change in x (rise / run)

$b$ = intercept: value where the line cross the y-axis

All points fall exactly on the line

**Statistics**

$\hat{y} = \beta_0 + \beta_1 X$

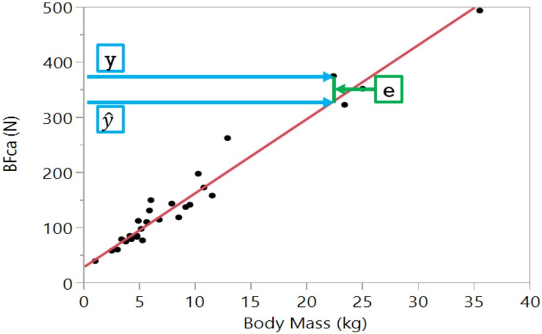
$\beta_1$ = slope

$\beta_0$ = intercept

Not all of our data points will exactly on the line $\rightarrow$ variability

# How it works

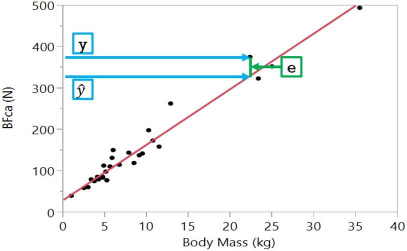
A regression line for the canidae data set predicting bite force (response) using body mass (explanatory)




- y's denote the values of the datapoints for the response variable
- points on the line are predicted values for the y's, denoted as $\hat{y}$
  - $\hat{y}$ are ALWAYS on our best-fit-line
- residual: difference between data and predictions ($e = y - \hat{y}$)

# How it works

The **regression line** is the line that best fits through the data




- Need to define "best"
- Optimality critera: minimizes sum of squared residuals $\sum e_i^2$
- *Least Squares Regression* is another, more explicit name for this

# Some Formulas

- $\hat{y} = \beta_0 + \beta_1 X$ **(regression equation)**
- $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 X$ **(estimated regression equation)**
- $\hat{\beta}_1 = (\frac{s_x}{s_y})r$ (estimated slope)
- $\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}$ (estimated intercept)
- e = y - $\hat{y}$ (**residual**)

# Regression Line vs Estimated Regression Line

What is the difference between these two? Why do we have two?

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# Regression Line vs Estimated Regression Line

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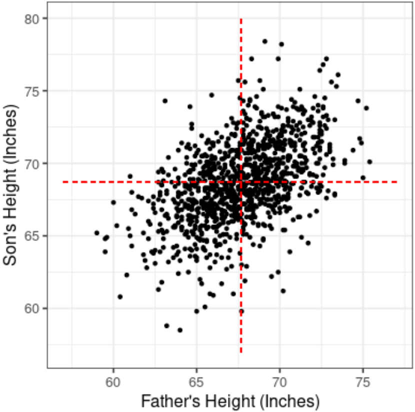
$\beta_0$ and $\beta_1$ are population parameters which means we almost never know them. Instead we have to estimate them using our sample.

Again, ^ (called hat) means estimated

# Pearson's Height Data

| | Mean | Std.Dev. | Correlation ($r_{xy}$) |
|---|---|---|---|
| Father | 67.68 | 2.74 | 0.501 |
| Son | 68.68 | 2.81 | |

| Father | Son |
|---|---|
| 65.0 | 59.8 |
| 63.3 | 63.2 |
| 65.0 | 63.3 |
| 65.8 | 62.8 |
| 61.1 | 64.3 |
| 63.0 | 64.2 |
| ⋮ | ⋮ |



# Pearson's Height Data

We could calculate our regression line using info from this table.

| | Mean | Std.Dev. | Correlation ($r_{xy}$) |
|---|---|---|---|
| Father | 67.68 | 2.74 | 0.501 |
| Son | 68.68 | 2.81 | |

Regression equation:
$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 X$

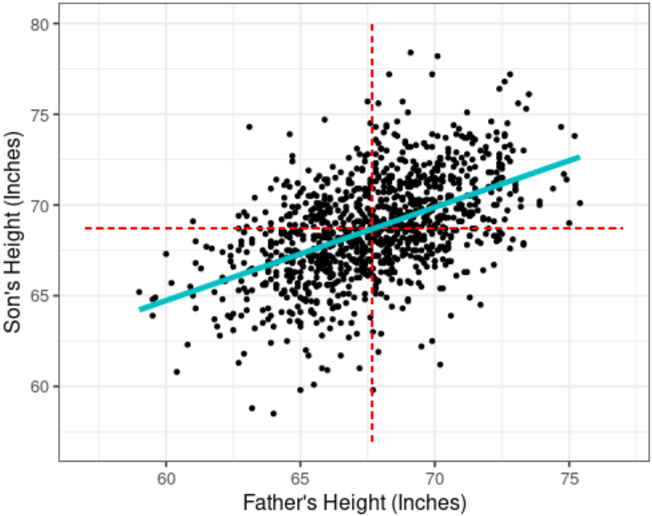
$$\hat{\beta}_1 = (\frac{s_x}{s_y})r$$
$$= (\frac{2.81}{2.74})0.501 = 0.514$$

```
> heights <- read.csv("Pearson.tsv", sep = "\t")
> fit <- lm(Son ~ Father, heights)
> fit

Call:
lm(formula = Son ~ Father, data = heights)

Coefficients:
(Intercept)       Father
     33.893        0.514
```

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1\overline{x}$$
$$= 68.68 - 0.514 * 67.68 = 33.893$$

# Pearson's Height Data – Plot Line

We can make R graph the line on our scatterplot.



# Pearson's Height Data – Prediction

The formula for the regression line

$$\hat{y} = \beta_0 + X\beta_1$$

can be expressed in context of our original data and our estimated values

$$\text{Son}\widehat{\text{'s Hei}}\text{ght} = 33.9 + 0.51 \times \text{Father's Height}$$

*Given* the Father's height, we can predict the son's height using this equation by plugging in a value for the father's height

**Example**: Predict the height of the son for a father with a height of 65in.

$$\text{Son}\widehat{\text{'s Hei}}\text{ght} = 33.9 + 0.51 \times 65.0 = ?$$

# Pearson's Height Data – Prediction

The formula for the regression line

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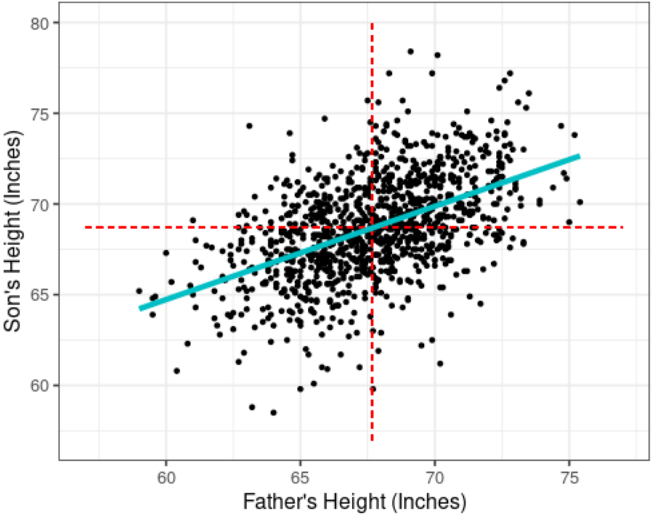
**Example**: Predict the height of the son for a father with a height of 65in.

$$\widehat{\text{Son's Height}} = 33.9 + 0.51 \times 65.0 = 67.30in.$$

# Pearson's Height Data – Prediction

Predicted Son's Height = 67.30 inches for a father with height = 65in

- Check to see if our prediction makes sense on the graph



# Residual

A **Residual** is the difference between an observed value and a prediction

- often labeled as **e** (e for "error", occasionally $\epsilon$)
- e = y - $\hat{y}$

**Interpretation**: the residual tells us whether we have over- or under-predicted the values for the response variable in our data (and by how much)

- positive value $\rightarrow$ under-predicted
- negative value $\rightarrow$ over-predicted
- hard truth $\rightarrow$ I always forget which is which

# Pearson's Height Data – Residual

In our data set, the first father had a height of 65 inches. We can calculate the residual for this father. We predicted the son's height to be 67.30 inches.

$$
\begin{aligned}
e &= y - \hat{y} \\
&= \text{observed value - predicted value} \\
&= 59.8in. - 67.30in. = -7.5in.
\end{aligned}
$$

**Interpretation**: We overpredicted the height of this particular son by 7.5 inches

| Father | Son |
|---|---|
| 65.0 | 59.8 |
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# Next Time

At this point everything we've done in linear regression has only been a mathematical result

- The Best-fit-line is a geometric minimization problem
- We have yet to make assumptions

Next time, we will introduce the assumptions for SLR and then interpretations for the slope and intercept

- Assumptions are wrong -> best fit line is wrong
- ALWAYS check the assumptions before you worry about your interpretations/model's results
- NO INTERPRETATION for $\hat{\beta}_0$ or $\hat{\beta}_1$ is valid if the assumptions are broken (in a meaningful way)